Problem Set 1

The power of mathematics is often to change one thing into another,
to change geometry into language.
— *Marcus du Sautoy*

- This problem set is due **at 9:00am** on **February 3, 2025** .
- This problem set comprises 3 problems and an ungraded optional exploration.
- The TAs will provide a detailed document describing how you should submit your PDF and code. Make sure you read it! We suggest that you perform a trial submission prior to the deadline to make sure that everything works for you – you can overwrite that submission with a new one up to the deadline.
- Each solution should start on a new page.
- We will give full credit **only** for correct solutions that are described clearly and convincingly.

Problem 1-1. Shape Up or Ship Out [30 points]

(a) [15 points] In *Disquisitiones Arithmeticae (1798)*, Carl Friedrich Gauss demonstrated that a regular heptadecagon (17-sided polygon) can be constructed using only a compass and straight-edge. The proof uses the identity:

$$\cos\left(\frac{2\pi}{17}\right) = \frac{1}{16}\left(-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}\right) + \frac{1}{8}\left(\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}\right) \tag{1}$$

Using the GeoGebra geometry calculator, and restricting yourself to operations that correspond to compass and straight-edge constructions, replicate Gauss' construction of the heptadecagon. Please submit the link to your construction.

Instructions: On GeoGebra, the following tools are permissible:

- Circle with Center and Through a Point
- Circle with Center and Radius Equaling the Length of a Segment.
- Line, Line Segment, or Ray Through Two Points.
- Angle Bisector and Perpendicular Bisector.
- Intersection of Objects.
- Perpendicular and Parallel Lines Through a Point.

And the following are not:

- Measuring Tools (e.g., Distance, Angle, Area).
- Numeric Input (e.g., Sliders, Direct Coordinates).
- Algebraic Calculations or Explicit Constructions Using Equations.
- Special Curves (e.g., Parabolas, Ellipses).
- Regular Polygon Operator.

(b) [15 points] Given a regular heptadecagon of side length a, describe a precise procedure to construct a square of equal area using only compass and straightedge constructions. Briefly specify the operational, denotational, and axiomatic semantics of this procedure.

Problem 1-2. There's more than one way to slice π [30 points]

Over centuries, mathematicians have discovered several ways to approximate the value of π using *infinite series expansions*. A series expansion is a way to represent π as the sum of an infinite sequence of terms. Each successive term in the series brings the sum closer to the actual value of π; some series converge *faster* than others, depending on their structure. In this problem we will explore three important series expansions for π:

1. In the 14th century, Madhava of Sangamagrama formulated an infinite series to compute the $\tan^{-1}$ function as follows:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{2}$$

   By setting $x = 1$ and $x = \frac{1}{\sqrt{3}}$ in the above equation, one can approximate the value of π by summing the terms of these series.

2. In 1593, François Viète published a way to express the reciprocal of π as the following infinite product of nested radicals:

$$\frac{1}{\pi} = \frac{1}{2}.\frac{\sqrt{2}}{2}.\frac{\sqrt{2+\sqrt{2}}}{2}.\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\ldots = \frac{1}{2}\prod_{n=0}^{\infty}\frac{a_i}{2} \tag{3}$$

   Where $a_0 = \sqrt{2}$ and $a_{i+1} = \sqrt{2 + a_i}$.

3. In 1917, Ramanujan published the following extraordinary series for the reciprocal of π. This result is famously associated with the legend of the goddess Namagiri Amman of Namakkal revealing mathematical insights to Ramanujan in a dream. In 1985, William Gosper used this series to calculate the first 17 million digits of π. The series is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} \tag{4}$$

Using Google Sheets, compute the value of π based on the following series expansions:

1. Madhava's Series $\left(\text{for both } x = 1 \text{ and } x = \frac{1}{\sqrt{3}}\right)$,
2. Viète's Infinite Product, and
3. Ramanujan's Series

For each series, compute the partial sums or products up to a reasonable number of terms. Plot the convergence of each series in a graph. For each series, plot the partial sum (or product) on the y-axis and the number of terms (or iterations) on the x-axis. Compare how quickly each series approaches the true value of π and comment on your observations.

Problem 1-3. Ashoka Logo [40 points]

Draw the following using turtle:

(a) [10 points] Your full name written in English and one other script of your choice (e.g., Arabic, Bengali, Chinese, Devanagari, etc.).

(b) [10 points] The Ashoka University logo.

(c) [20 points] The Koch snowflake fractal. For this one, make sure to clearly specify any assumptions you are making regarding the recursion depth, scale of the figure, or any other parameters relevant to your drawing.

Exploration (Optional). Follow the instructions at the turtledraw GitHub repository to install the package and convert your photos into turtle drawings. Use the tool to convert an image of your choice into turtle commands. Investigate the `svgparse.py` file from the repository and explore how the script parses SVG files and converts them into turtle commands. If you're feeling creative, consider extending the functionality by adding options such as drawing the SVG image in different styles (e.g., dashed lines or dotted patterns) or applying colour when drawing with the turtle module.