An introduction to Bayesian statistics

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What is Bayesian statistics?

A statistical inference framework.

Can be used for estimation, classification, detection, model selection, etc.

Key characteristic: unknown quantities are described as random.

Applications of Bayesian statistics

- A medical application: analyze the disease of a patient.

– **Quantity of interest**: the disease, θ.

– **Observations**: blood samples, temperature, comments by patient, etc.

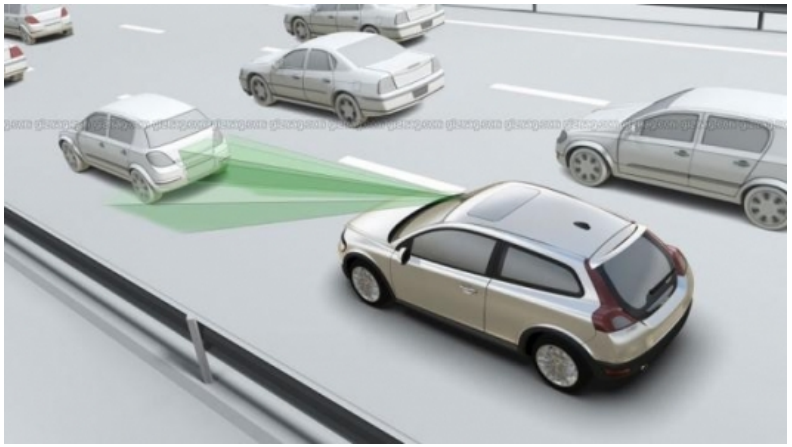
- In Bayesian statistics θ is described as random
  $\rightsquigarrow$ we can make statements like "patient has disease x with 97% probability". (e.g., x = flue)
- **Possible concern:** is the disease random?

Applications of Bayesian statistics

- Modern safety systems for cars rely on the ability to position surrounding vehicles.
- This enables the system to intervene, e.g., by braking, before a collision.

– **Quantity of interest:** relative position and velocity of other cars at time k.

– **Observations**: wheel speeds, radar detections (distance and angle), GPS measurements, etc.



- Bayesian statistics: vehicle motions are modelled statistically
  $\rightsquigarrow$ helps us to rule out unrealistic trajectories.
- **Possible concern:** are the vehicle motions random?

Comparison: Bayes vs Frequentist

- There are **two main strategies** to decision making: **Bayesian and frequentist statistics**.
- In **frequenstist statistics**, the quantities of interest are described as **unknown and deterministic**.

An example: suppose we wish to estimate the height of the eiffel tower. Is the height random or not?

- **Frequentist perspective:** the tower has a certain height and is therefore not random.
- **Bayesian perspective:** we describe our uncertainties in the height stochastically $\Rightarrow$ height is described as random!

Overview of the Bayesian strategy

Suppose we wish to estimate θ given measurements y.

Key steps in a Bayesian method:

1. **Modeling.** Model what we know about θ (using a prior $p(\theta)$) and the measurements y (using a density $p(y|\theta)$).
2. **Measurement update.** Combine what we knew before (the prior) with our measurement (with $p(y|\theta)$, also called the likelihood) to summarize what we know about θ ($p(\theta|y)$).
3. **Decision making.** Given what we know about θ (described by $p(\theta|y)$) and a loss function, we compute *an optimal decision*.

In the upcoming videos we first discuss how to perform step 2) and after that we look at step 3).

Self-assessment questions

Which of the following statements are correct:

- Bayesian methods can be used to solve many types of decision making problems including estimation, detection and classification.
- We can model the height of the Eiffel tower as random only if we think that there are many similar towers with different heights.
- In Bayesian statistics we describe what we know about θ (the quantity of interest) before observing any measurements.

Check all that apply.

Bayes' rule – a first example

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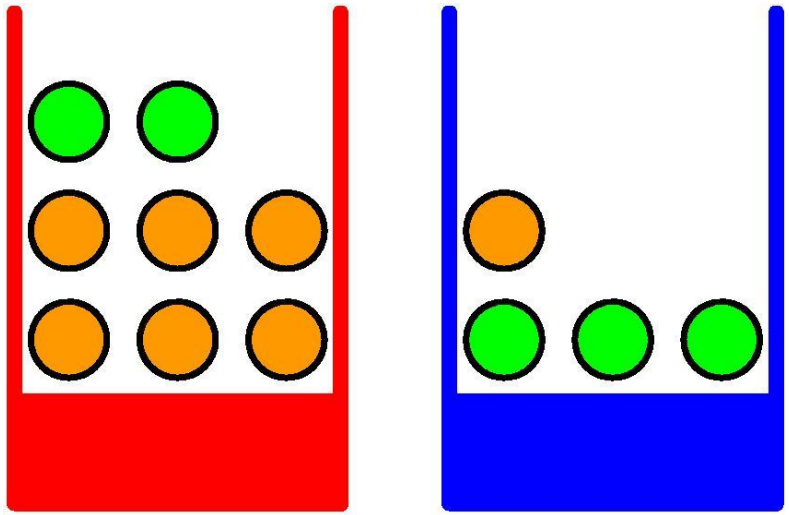
Bayes' rule: a first example

- A box is selected at random (prob. 1/2, 1/2). From that box we pick a fruit.



- If fruit is orange, what is probability that box is red?

Probability theory

- Bayesian statistics is simple!
  We only need two rules:
  1. Conditional probability (product rule)

$$\Pr\{y, \theta\} = \Pr\{y|\theta\} \Pr\{\theta\}$$

  2. The law of total probability (sum rule)

$$\Pr\{y\} = \sum_{\theta} \Pr\{y, \theta\} \qquad \text{discrete variables}$$

$$p(y) = \int_{\theta} p(y, \theta)\, d\theta \qquad \text{continuous variables}$$

Probability theory – Bayes' rule

- Bayes' rule is a consequence of conditional probability,

$$p(y|\theta)p(\theta) = p(\theta|y)p(y).$$

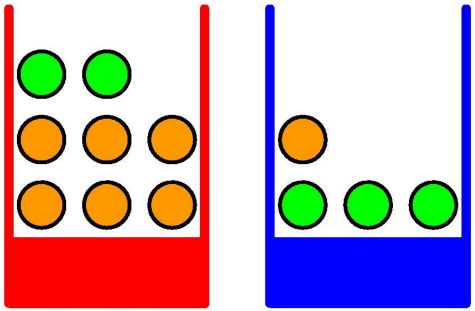
Bayes' rule

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

- Usage of Bayes' rule: express a relation of interest, $p(\theta|y)$, in terms of the relation that we know $p(y|\theta)$.
- Note that $p(y) = \int_\theta p(y|\theta)p(\theta)\, d\theta$.

Bayes' rule: a first example

- Let $\theta \in \{r, b\}$ be color of box, and $y \in \{o, a\}$ be the fruit.



- Bayes' rule gives

$$\Pr\{\theta = r \mid y = o\} = \frac{\Pr\{y = o \mid \theta = r\} \cdot \Pr\{\theta = r\}}{\Pr\{y = o\}}$$

$$\Pr\{y = o, \theta = r\} + \Pr\{y = o, \theta = b\}$$

$$= \frac{3}{4} \cdot \frac{1}{2} \qquad \frac{1}{4} \cdot \frac{1}{2}$$

$$\Rightarrow \Pr\{\theta = r \mid y = o\} = \frac{3}{4}$$

Building blocks of Bayesian models

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Priors, posteriors and likelihoods

- We are interested in an unknown parameter

$$\theta$$

  such that $\theta \in \Theta$.

- **Common problem types** are *estimation* (e.g., $\Theta = \mathbb{R}^n$) and *detection* problems (e.g., $\Theta = \{-1, 1\}$).

- The observed data, y, is distributed as

$$y \sim p(y|\theta),$$

  where p is a known distribution.

Priors, posteriors and likelihoods

- Since data y is observed, we often view $p(y|\theta)$ as a function of θ,

$$l(\theta|y) = p(y|\theta),$$

where $l(\theta|y)$ is called the likelihood function.
Note: the likelihood function is *not* a density w.r.t. θ.

- In Bayesian statistics we have a prior distribution $p(\theta)$ on θ.
Prior means *earlier*, or before, and $p(\theta)$ describes what we know *before* observing y.

- One objective in Bayesian statistics is to compute the posterior

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} \propto l(\theta|y)p(\theta)$$

Posterior means *after* and $p(\theta|y)$ describes what we know *after observing* y.

Priors, posteriors and likelihoods

- We summarize this as

$$\text{posterior} \propto \text{likelihood} \times \text{prior}.$$

- Given the posterior, $p(\theta|y)$ we can answer, e.g.,
  - What is the most probable θ?
  - What is the probability that $\theta \in \mathcal{A}$?
  - What is the posterior mean of θ?

- We can also minimize expected costs in a decision theoretic manner (see later videos).

Example: scalar in Gaussian noise

- Suppose we observe

$$y = \theta + v, \qquad v \sim \mathcal{N}(0, \sigma^2)$$

such that $p(y|\theta) = \mathcal{N}(y; \theta, \sigma^2) \propto \exp\{-(y-\theta)^2/(2\sigma^2)\}$.

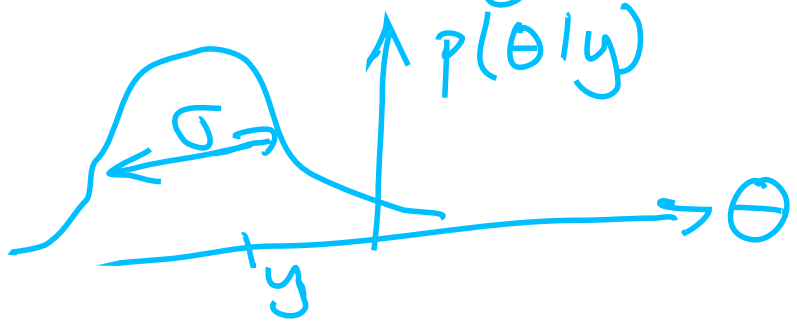
- A common noninformative prior on θ is

$$p(\theta) \propto 1.$$

- The posterior is $p(\theta|y) \propto p(\theta)p(y|\theta)$

$\propto 1 \cdot \exp(-(y-\theta)^2/2\sigma^2) \propto \mathcal{N}(\theta; y, \sigma^2)$

$\Rightarrow p(\theta|y) = \mathcal{N}(\theta; y, \sigma^2)$



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- A common noninformative prior on θ is

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- The posterior is

$$\begin{aligned} p(\theta|y) &\propto p(y|\theta)p(\theta) \propto \exp\{-(y-\theta)^2/(2\sigma^2)\} \\ &\propto \mathcal{N}(\theta; y, \sigma^2) \end{aligned}$$

$$\Rightarrow p(\theta|y) = \mathcal{N}(\theta; y, \sigma^2)$$

Bayesian approach to sensor fusion

- Suppose we collect measurements from **two types of sensors**:

  $$y_1 \quad \text{and} \quad y_2,$$

  say a laser scanner and a camera.

- **Bayesian sensor fusion** is simply Bayesian statistics. As always we seek the posterior distribution:

  $$p(\theta | \underbrace{y_1, y_2}_{y}) \propto p(\theta) p(\underbrace{y_1, y_2}_{y} | \theta).$$

- It is often reasonable to assume that

  $$p(y_1, y_2 | \theta) \approx p(y_1 | \theta) p(y_2 | \theta),$$

  i.e., that measurements are **conditionally independent**.

Self-assessment

The posterior distribution is $p(\theta|y) \propto p(y|\theta)p(\theta)$.
It is also true that:

- The normalization factor is **not always unique**?
- The posterior $p(\theta|y)$ can **always be uniquely determined** from the fact that $\int p(\theta|y)\, d\theta = 1$?
- The posterior distribution can **only be uniquely determined if** it is proportional to a well known distribution, e.g., a Gaussian.

Only one statement is correct.

Bayesian Decision Theory

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Bayesian decision principle

- How can we use $p(\theta|y)$ to make decisions?
- **Examples** of decision problems
  - How to invest money.
  - Select medicine to give to a patient
  - Estimate a parameter vector (may represent temperature, distance, etc).

Basic principle of Bayesian decision theory

- Minimize expected loss

  or, equivalently,

- Maximize expected utility.

Decision theory – a toy example

- A student wants to decide whether to take a course or not.
- Suppose $\theta \in \{\text{good course, fair course, bad course}\}$ and

	good course	fair course	bad course
$\Pr\{\theta \mid y\}$	0.3	0.3	0.4

- If the loss function is

	good course	fair course	bad course
Taking course	0	5	10
Not taking course	20	5	0

should he/she then take the course?

Decision theory – a toy example

- We can easily compute the expected loss for the two possible decisions.
  - Takes the course:
    $0 \times 0.3 + 5 \times 0.3 + 10 \times 0.4 = 5.5$.
  - Does not take the course:
    $20 \times 0.3 + 5 \times 0.3 + 0 \times 0.4 = 7.5$.

- It is thus better to take the course. (Of course, this conclusion has nothing to do with this course.)

Minimum posterior expected loss

- We often study loss functions

$$C(\theta, \hat{\theta})$$

  instead of utilities. (Typically, $C \geq 0$.)
- Let $\hat{\theta}$ denote an estimate of θ.

Optimal Bayesian decisions

Minimize the posterior expected loss

$$\hat{\theta} = \arg\min_a \mathbb{E}\left\{ C(\theta, a) \middle| y \right\}$$

where $\mathbb{E}\left\{ C(\theta, a) \middle| x \right\} = \int_\Theta C(\theta, a) p(\theta \middle| y)\, d\theta$

- Note: y is given (fixed) and θ is random.

Self-assessment

To make an optimal Bayesian decision it is sufficient to know:

- The prior, $p(\theta)$, the likelihood, $p(y|\theta)$, and a loss function $C(\theta, a)$.
- The likelihood, $p(y|\theta)$, and a loss function $C(\theta, a)$.
- The posterior distribution, $p(\theta|y)$, and a loss function, $C(\theta, a)$.

Check all statements that apply.

Comparison: Bayes vs Frequentist

Frequentist	Bayes
θ is fixed and unknown $\Rightarrow \theta$ is deterministic	Uncertainties in θ are described stochastically $\Rightarrow \theta$ is random
Maximum likelihood (ML) most famous estimator $\hat{\theta}_{ML} = \arg\max_\theta l(\theta \mid y)$	Minimum mean square error and maximum a posteriori estimators, e.g. $\hat{\theta}_{MAP} = \arg\max_\theta p(\theta) l(\theta \mid y)$
Study performance by averaging over y for fixed θ	Make decisions conditioned on the observation y.

- Note 1: most Bayesians also study frequentist performance.
- Note 2: many frequentists agree that parameters may be random in some situations.

Cost functions in Bayesian decision theory

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Bayesian decision theory – summary

- Bayesian decision theory relies on
  1. Likelihood: $p(y|\theta)$
  2. Prior distribution: $p(\theta)$
  3. Loss function: $C(\theta, a)$.

- Combining likelihood and prior gives posterior

$$p(\theta|y) \propto p(y|\theta)p(\theta).$$

Posterior and loss gives decisions

$$\hat{\theta} = \arg\min_a \int_\Theta C(\theta, a)p(\theta|y)\, d\theta.$$

The quadratic loss function

- **Example:** parameter estimation, $\theta \in \Theta = \mathbb{R}^n$.
- Most common loss function is the **quadratic loss**

$$C(\theta, a) = \left\|\theta - a\right\|_2^2 = (\theta - a)^T(\theta - a)$$

- Let: $\bar{\theta} = \mathbb{E}\left\{\theta \middle| y\right\}$, $\mathbf{P} = \mathrm{Cov}\{\theta|y\} = \mathbb{E}\left\{(\theta - \bar{\theta})(\theta - \bar{\theta})^T \middle| y\right\}$
- **Optimal estimator:**

$$\mathbb{E}\{C(\theta,a)|y\} = \mathbb{E}\{(\theta-a)^T(\theta-a)|y\}$$

$$= \mathbb{E}\{(\theta-\bar{\theta}+\bar{\theta}-a)^T(\theta-\bar{\theta}+\bar{\theta}-a)|y\} = \mathbb{E}\{(\theta-\bar{\theta})^T(\theta-\bar{\theta})|y\}$$

(zero mean, determ.)

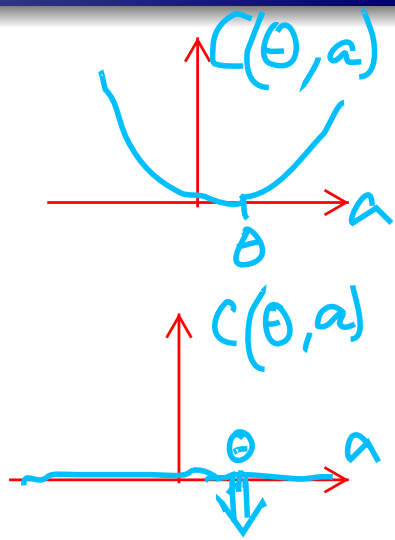
$$+ \mathbb{E}\{(\theta-\bar{\theta})^T|y\}(\bar{\theta}-a) + 0 + (\bar{\theta}-a)^T(\bar{\theta}-a)$$

$(=0)$ $\qquad \mathrm{tr}\{P\}$

$$\hat{\theta} = \underset{a}{\mathrm{argmin}}\, \mathbb{E}\{C(\theta,a)|y\} = \bar{\theta}$$

This is the **minimum mean squared error** (MMSE) estimator, often denoted $\hat{\theta}_{\mathrm{MMSE}}$.

The $0-1$ loss function



- **Example:** parameter estimation, $\theta \in \Theta = \mathbb{R}^n$.
- Another common choice is the $0-1$ **loss** function

$$C(\theta, a) = -\delta(\theta - a),$$

$\delta(\cdot)$ is the Dirac's delta function.

- **Optimal estimator:** $\mathbb{E}\{C(\theta, a)\big|y\} = \mathbb{E}\{-\delta(\theta - a)\big|y\}$

$$= -\int p(\theta\big|y)\delta(\theta - a)\, d\theta = -p(\theta\big|y)\Big|_{\theta=a}$$

$$\Rightarrow \quad \hat{\theta} = \arg\min_a -p(\theta\big|y)\Big|_{\theta=a}$$

$$= \arg\max_\theta p(\theta\big|y)$$

This is the **maximum a posteriori** (MAP) estimator, often denoted $\hat{\theta}_{MAP}$.

Self-assessment

Suppose $p(\theta|y) = \mathcal{N}(\theta; \bar{\theta}, \mathbf{P})$. The MMSE and MAP estimators are, respectively,

- $\bar{\theta} + \text{tr}\{\mathbf{P}\}$ and $\bar{\theta}$.
- $\bar{\theta}$ and $\mathcal{N}(\theta; \bar{\theta}, \mathbf{P})$.
- $\bar{\theta}$ and $\bar{\theta}$.
- $\bar{\theta} + \text{tr}\{\mathbf{P}\}$ and $\mathcal{N}(\theta; \bar{\theta}, \mathbf{P})$.

Only one statement is correct.