Inverse Kinematics (Mathematical Solution)

From (*), we know that:

$$x = c_1 c_{23} L_2 + c_1 c_2 L_1$$

$$y = s_1 c_{23} L_2 + s_1 c_2 L_1$$

Take square of both sides:

$$x^2 + y^2 = \begin{array}{l} \underbrace{\begin{array}{l} c_1^2 c_{23}^2 L_2^2 + \\ s_1^2 c_{23}^2 L_2^2 + \end{array}}_{c_{23}^2 L_2^2} \underbrace{\begin{array}{l} c_1^2 c_2^2 L_1^2 + \\ s_1^2 c_2^2 L_1^2 + \end{array}}_{c_2^2 L_1^2} \begin{array}{l} 2 c_1 c_{23} L_2 c_1 c_2 L_1 + \\ 2 s_1 c_{23} L_2 s_1 c_2 L_1 \end{array} \end{array}$$

$$2 c_1^2 c_{23} c_2 L_1 L_2$$

$$2 s_1^2 c_{23} c_2 L_1 L_2$$

$$\begin{aligned} x^2 + y^2 &= c_{23}^2 L_2^2 + c_2^2 L_1^2 + 2 c_2 c_{23} L_1 L_2 \\ z^2 &= s_{23}^2 L_2^2 + s_2^2 L_1^2 + 2 s_2 s_{23} L_1 L_2 \end{aligned}$$

$$L_2^2 \qquad L_1^2$$

$$x^2 + y^2 + z^2 = L_1^2 + L_2^2 + 2 L_1 L_2 \cos(\theta_2 + \theta_3 - \theta_2)$$

$$\cos\theta_3 = \frac{x^2 + y^2 + z^2 - L_1^2 - L_2^2}{2 L_1 L_2} = K$$

$$\sin\theta_3 = \pm\sqrt{1 - K^2}$$

$$\boxed{\theta_3 = \mathrm{Atan2}\left(-\sqrt{1-K^2}, K\right)}$$

$$z = (s_2c_3 + c_2s_3)L_2 + s_2L_1$$

$$\underbrace{z}_{c} = s_2\underbrace{(c_3L_2 + L_1)}_{b} + c_2\underbrace{(s_3L_2)}_{a}$$

$$\theta_2 = \text{Atan2}(b, a) \mp \text{Atan2}(\sqrt{a^2+b^2-c^2}, c)$$

$$\boxed{\theta_2 = \text{Atan2}(kL_2 + L_1, -\sqrt{1-k^2}L_2) + \text{Atan2}(\sqrt{L_1^2+L_2^2+2kL_1L_2}, z)}$$

Finally,

$$\cos\theta_1 = \frac{x}{c_{23}L_2 + c_2L_1} \qquad \sin\theta_1 = \frac{y}{c_{23}L_2 + c_2L_1}$$

$$\boxed{\theta_1 = \text{Atan2}(y, x)}$$