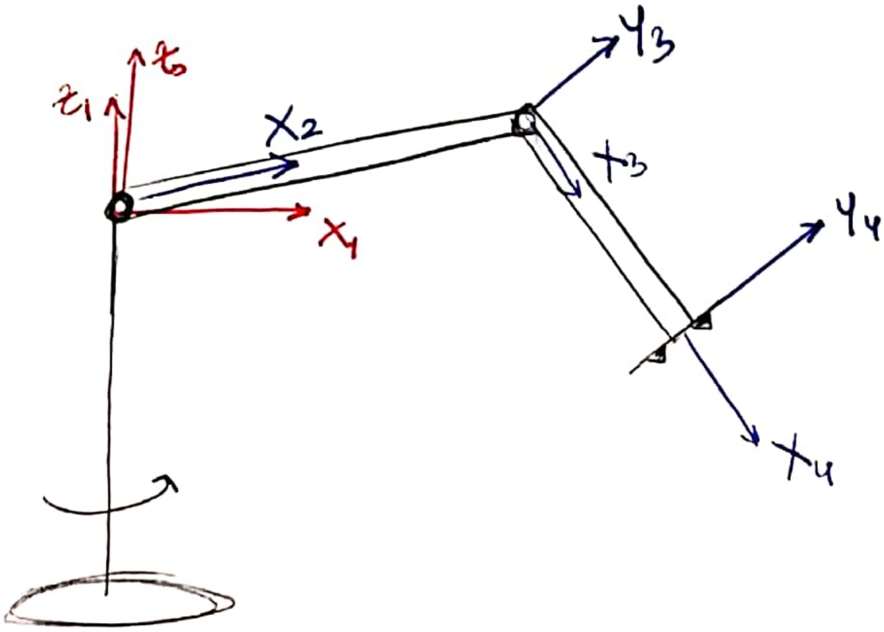
Forward Kinematics



DH Table

	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	0
2	$\pi/2$	0	θ_2	0
3	0	L_1	θ_3	0
4	0	L_2	0	0

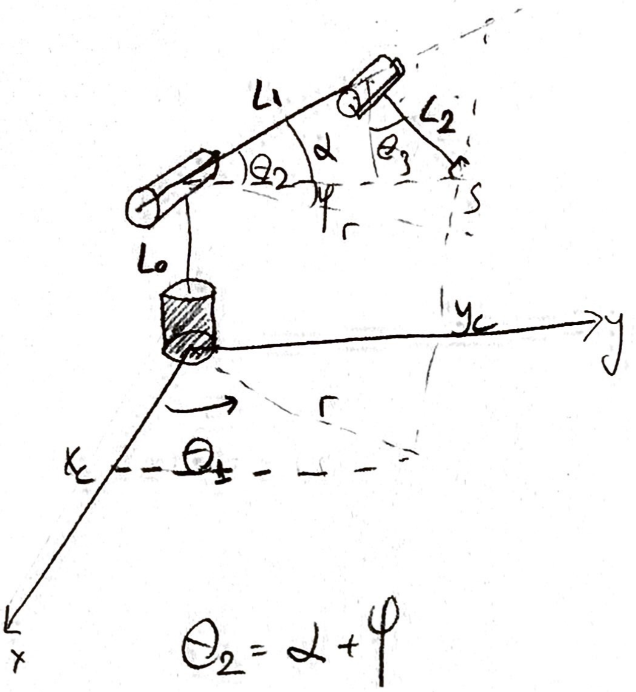
$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} c_3 & -s_3 & 0 & L_1 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{4}T = \begin{bmatrix} c_1c_2c_3 - c_1s_2s_3 & -c_1c_2s_3 - c_1s_2c_3 & s_1 & (c_1c_2c_3 - c_1s_2s_3)L_2 + c_1c_2L_1 \\ s_1c_2c_3 - s_1s_2s_3 & -s_1c_2s_3 - s_1s_2c_3 & -c_1 & (s_1c_2c_3 - s_1s_2s_3)L_2 + s_1c_2L_1 \\ s_2c_3 + c_2s_3 & -s_2s_3 + c_2c_3 & 0 & (s_2c_3 + c_2s_3)L_2 + s_2L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{4}T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & \underbrace{c_1c_{23}L_2 + c_1c_2L_1}_{x} \\ s_1c_{23} & -s_1s_{23} & -c_1 & \underbrace{s_1c_{23}L_2 + s_1c_2L_1}_{y} \\ s_{23} & c_{23} & 0 & \underbrace{s_{23}L_2 + s_2L_1}_{z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (*)
$$

Inverse Kinematics (Geometric Solution)



$$\theta_1 = \operatorname{atan2}(y_c, x_c)$$

$$r = \sqrt{x_c^2 + y_c^2}$$

$$s = z_c - L_0$$

$$r^2 + s^2 = L_1^2 + L_2^2 + 2 L_1 L_2 \cos(90 - \theta_3)$$

$$\sin\theta_3 = \frac{r^2 + s^2 - L_1^2 - L_2^2}{2 L_1 L_2} = D$$

$$\cos\theta_3 = -\sqrt{1 - D^2}$$

↳ this only valid for elbow up solution since the limitations of joints

$$\theta_3 = \operatorname{atan2}(D, -\sqrt{1-D^2})$$

$$\theta_2 = \alpha + \varphi$$

$$\varphi = \operatorname{atan2}(s, r)$$

$$\alpha = \operatorname{atan2}(L_2 \cos\theta_3, L_1 + L_2 \sin\theta_3)$$

$$\theta_2 = \operatorname{atan2}(s, r) + \operatorname{atan2}(L_2 \cos\theta_3, L_1 + L_2 \sin\theta_3)$$

Inverse Kinematics (Mathematical Solution)

From (*), we know that:

$$x = c_1 c_{23} L_2 + c_1 c_2 L_1$$

$$y = s_1 c_{23} L_2 + s_1 c_2 L_1$$

Take square of both sides:

$$2 c_1^2 c_{23} c_2 L_1 L_2$$

$$x^2 + y^2 = \underbrace{c_1^2 c_{23}^2 L_2^2 + s_1^2 c_{23}^2 L_2^2}_{c_{23}^2 L_2^2} + \underbrace{c_1^2 c_2^2 L_1^2 + s_1^2 c_2^2 L_1^2}_{c_2^2 L_1^2} + 2 c_1 c_{23} L_2 c_1 c_2 L_1 + 2 s_1 c_{23} L_2 s_1 c_2 L_1$$

$$2 s_1^2 c_{23} c_2 L_1 L_2$$

$$x^2 + y^2 = \underbrace{c_{23}^2 L_2^2}_{} + c_2^2 L_1^2 + 2 c_2 c_{23} L_1 L_2$$

$$z^2 = s_{23}^2 L_2^2 + s_2^2 L_1^2 + 2 s_2 s_{23} L_1 L_2$$

$$\underbrace{c_{23}^2 L_2^2 + s_{23}^2 L_2^2}_{L_2^2} \qquad \underbrace{c_2^2 L_1^2 + s_2^2 L_1^2}_{L_1^2}$$

$$x^2 + y^2 + z^2 = L_1^2 + L_2^2 + 2 L_1 L_2 \cos(\theta_2 + \theta_3 - \theta_2)$$

$$\cos\theta_3 = \frac{x^2 + y^2 + z^2 - L_1^2 - L_2^2}{2 L_1 L_2} = K$$

$$\sin\theta_3 = \pm\sqrt{1 - K^2}$$

$$\boxed{\theta_3 = \mathrm{Atan2}\left(-\sqrt{1-K^2},\ K\right)}$$

$$z = (s_2 c_3 + c_2 s_3) L_2 + s_2 L_1$$

$$\underbrace{z}_{c} = s_2 \underbrace{(c_3 L_2 + L_1)}_{b} + c_2 \underbrace{(s_3 L_2)}_{a}$$

$$\theta_2 = \mathrm{Atan2}(b, a) \mp \mathrm{Atan2}\left(\sqrt{a^2+b^2-c^2},\ c\right)$$

$$\boxed{\theta_2 = \mathrm{Atan2}\left(kL_2 + L_1,\ -\sqrt{1-k^2}\, L_2\right) + \mathrm{Atan2}\left(\sqrt{L_1^2 + L_2^2 + 2kL_1L_2},\ z\right)}$$

Finally,

$$\cos\theta_1 = \frac{x}{c_{23} L_2 + c_2 L_1} \qquad \sin\theta_1 = \frac{y}{c_{23} L_2 + c_2 L_1}$$

$$\boxed{\theta_1 = \mathrm{Atan2}(y, x)}$$