PHYS 514 - Computational Physics

Solution Set 8

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1 Problem XXIV

1.1 Construction of Symmetric Matrix for Neumann Boundary Condition

In the *Poisson equation*, the following discretisized form is present:

$$u_{j+1,l} + u_{j-1,l} + u_{j,l+1} + u_{j,l-1} - 4u_{j,l} = h^2\rho_{j,l}$$

where $u_{j,l}$ and $\rho_{j,l}$ represents the grid points and the forcing terms on a uniform rectangular grid, and $j = 0, \ldots, N$ and $l = 0, \ldots, M$.

In the problem construction, the following information is given:

- The effective boundary condition is *Neumann boundary condition*.
- Discretization scheme is such that $x_i = a + \left(i + \frac{1}{2}\right)\frac{b-a}{N}$ for $i = 0, 1, \ldots, N-1$

The first thing that should be done is to observe the behavior of the edges. Because, the intermediate points have always the same structure. For example, the following relation can be observed for edges:

$$u_{j+1,0} + u_{j-1,0} + u_{j,1} + u_{j,-1} - 4u_{j,0} = h^2\rho_{j,0} \tag{1}$$

$$u_{j+1,M-1} + u_{j-1,M-1} + u_{j,M} + u_{j,M-2} - 4u_{j,M-1} = h^2\rho_{j,M-1} \tag{2}$$

$$u_{1,l} + u_{-1,l} + u_{0,l+1} + u_{0,l-1} - 4u_{0,l} = h^2\rho_{0,l} \tag{3}$$

$$u_{N,l} + u_{N-2,l} + u_{N-1,l+1} + u_{N-1,l-1} - 4u_{N-1,l} = h^2\rho_{N-1,l} \tag{4}$$

The fact that the $u_{j,-1}$, $u_{j,M}$, $u_{-1,l}$, and $u_{N,l}$ are unknown can be immediately observed. To be able to have an idea about the $u_{j,-1}$, $u_{j,M}$, $u_{-1,l}$, and $u_{N,l}$, one can easily incorporate the information of the boundary conditions. To be able to simplify the derivation process, the transformation to the one dimensional *dummy variable*, y, might be more feasible.

$$y'_{-\frac{1}{2}} = \frac{y_{-\frac{1}{2}} - y_{-1}}{\frac{h}{2}} = 0$$

$$y'_{-\frac{1}{2}} = \frac{y_{-\frac{1}{2}} - y_{0}}{\frac{h}{2}} = 0$$

$$y_{-\frac{1}{2}} = y_{-1}$$

$$y_{-\frac{1}{2}} = y_0$$

$$\boxed{y_{-1} = y_0}$$

Similarly for the end point, the following relation can be hold:

$$y'_{N-\frac{1}{2}} = \frac{y_{N-\frac{1}{2}} - y_{N-1}}{\frac{h}{2}} = 0$$

$$y'_{N-\frac{1}{2}} = \frac{y_{N-\frac{1}{2}} - y_{N}}{\frac{h}{2}} = 0$$

$$y_{N-\frac{1}{2}} = y_{N-1}$$

$$y_{N-\frac{1}{2}} = y_{N}$$

$$\boxed{y_{N-1} = y_N}$$

Now, it is time to turn back to the (1), (2), (3), and (4). That is, (1), (2), (3), and (4) can be written in the following form:

$$u_{j+1,0} + u_{j-1,0} + u_{j,1} + u_{j,0} - 4u_{j,0} = h^2\rho_{j,0} = u_{j+1,0} + u_{j-1,0} + u_{j,1} - 3u_{j,0}$$

$$u_{j+1,M-1} + u_{j-1,M-1} + u_{j,M-1} + u_{j,M-2} - 4u_{j,M-1} = h^2\rho_{j,M-1} = u_{j+1,M-1} + u_{j-1,M-1} + u_{j,M-2} - 3u_{j,M-1}$$

$$u_{1,l} + u_{0,l} + u_{0,l+1} + u_{0,l-1} - 4u_{0,l} = h^2\rho_{0,l} = u_{1,l} + u_{0,l+1} + u_{0,l-1} - 3u_{0,l}$$

$$u_{N-1,l} + u_{N-2,l} + u_{N-1,l+1} + u_{N-1,l-1} - 4u_{N-1,l} = h^2\rho_{N-1,l} = u_{N-2,l} + u_{N-1,l+1} + u_{N-1,l-1} - 3u_{N-1,l}$$

In this point, one should note that the vertices will be imposed the boundary conditions from the edges, that is, for example:

$$u_{1,0} + u_{0,0} + u_{0,1} + u_{0,0} - 4u_{j,0} = h^2\rho_{j,0} = u_{1,0} + u_{0,1} - 2u_{j,0}$$

Moreover, the second derivative for the other data point can be easily expressed as in below form:

$$u_{j+1,l} + u_{j-1,l} + u_{j,l+1} + u_{j,l-1} - 4u_{j,l} = h^2\rho_{j,l}$$

for $j = 1, \ldots, N-2$ and $l = 1, \ldots, M-2$.

Using all of the aforementioned information here, one can construct the system matrix in lexicographical order as such:

$$\begin{bmatrix} \mathbf{V} & \mathbf{I} & 0 & 0 & \dots & & & & & \\ \mathbf{I} & \mathbf{J} & \mathbf{I} & 0 & 0 & \dots & & & & \\ 0 & \mathbf{I} & \mathbf{J} & \mathbf{I} & 0 & 0 & \dots & & & \\ 0 & 0 & \mathbf{I} & \mathbf{J} & \mathbf{I} & 0 & 0 & \dots & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & & \dots & & \mathbf{I} & \mathbf{J} & \mathbf{I} \\ 0 & 0 & 0 & 0 & & \dots & & 0 & \mathbf{I} & \mathbf{V} \end{bmatrix}$$

where $\mathbf{V}$ and $\mathbf{J}$ represents:

$$\mathbf{V} = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & & & & & \\ 1 & -3 & 1 & 0 & 0 & \dots & & & & \\ 0 & 1 & -3 & 1 & 0 & 0 & \dots & & & \\ 0 & 0 & 1 & -3 & 1 & 0 & 0 & \dots & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & & \dots & & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & & \dots & & 0 & 1 & -2 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} -3 & 1 & 0 & 0 & \dots & & & & & \\ 1 & -4 & 1 & 0 & 0 & \dots & & & & \\ 0 & 1 & -4 & 1 & 0 & 0 & \dots & & & \\ 0 & 0 & 1 & -4 & 1 & 0 & 0 & \dots & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & & \dots & & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & & \dots & & 0 & 1 & -3 \end{bmatrix}$$

1.2 Solution of Poisson Equation

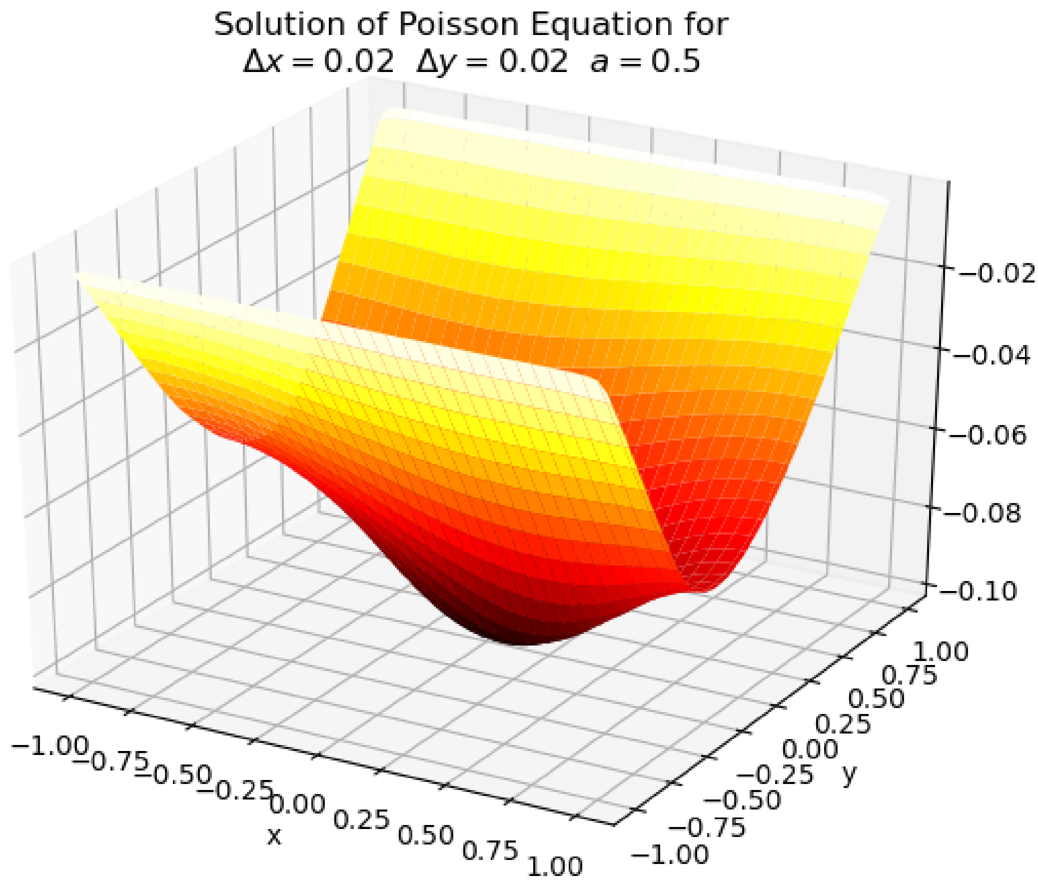

Figure 1: Solution of Poisson equation for $a = 0.5$

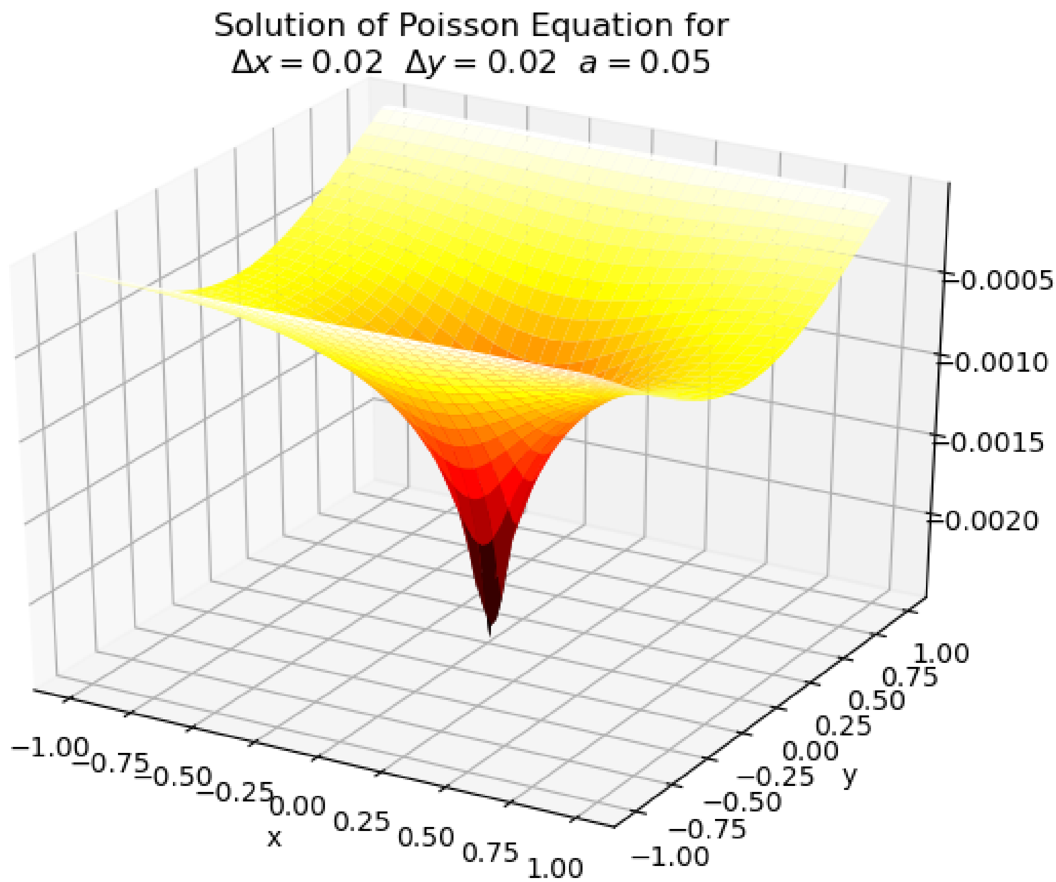

Figure 2: Solution of Poisson equation for $a = 0.05$