KOÇ UNIVERSITY

PHYS 514 - Computational Physics

Term Project

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1 Newtonian Approach

1.1 Derivation of *Lane-Emden Equation*

In this project, the first task is to derive the *Lane-Emden* equation to be able to have a general expression for the *hydrostatic equilibrium*. In the beginning, it is known that:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2} \tag{2}$$

where m, ρ, and p are functions of r. To be able to solve this system of differential equations, the expression for the ρ is required, which is not explicitly given. However, by using the *equation of state (EOS)*, one can easily construct a relation between p and ρ, such that:

$$p = \frac{k_B}{\mu m_H} T\rho$$

Assuming that isentropic relation holds, one can conclude that:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \tag{3}$$

To be able to obtain a unified expression, one can inspect the *mass* variable in the (1), and (2). It is required to calculate the derivative of the (2) to write (2) in terms of (1). Differentiating (2) yields:

$$\frac{d^2p}{dr^2} = \frac{2Gm\rho}{r^3} + \frac{-G\rho}{r^2}\frac{dm}{dr} + \frac{-Gm}{r^2}\frac{d\rho}{dr} \tag{4}$$

Before further processing on the (4), the some terms must be revised, that is, (4) can be rewritten in the following form:

$$\frac{d^2p}{dr^2} = \frac{2}{r}\left(-\frac{dp}{dr}\right) + \frac{d\rho}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + 4\pi G\rho^2 \tag{5}$$

Multiplying (5) with $\frac{r^2}{\rho}$ and rearranging the terms yields:

$$\frac{r^2}{\rho}\frac{d^2p}{dr^2} + 2r\left(\frac{1}{\rho}\frac{dp}{dr}\right) - r^2\frac{d\rho}{dr}\left(\frac{1}{\rho^2}\frac{dp}{dr}\right) = 4\pi G r^2\rho \tag{6}$$

Observing (6), one can easily realize that the left hand side is nothing but the total derivative of the term $\frac{r^2}{\rho}\frac{dp}{dr}$. Hence, in conclusion, the (4), can be written as the following:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = 4\pi G r^2\rho \tag{7}$$

From now on, the finite amount of transformation should be applied on the ρ and r. Firstly, assume that the following relations hold:

$$\rho = \rho_c \theta^n$$

Therefore,

$$p = K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \rightarrow dp = K\rho_c^{1+\frac{1}{n}}(n+1)\,\theta^n d\theta \tag{8}$$

Applying (8) to (7), one can write:

$$\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n}K\rho_c^{1+\frac{1}{n}}(n+1)\,\theta^n\frac{d\theta}{dr}\right) = 4\pi G r^2 \rho_c \theta^n \rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\frac{d\theta}{dr}\right) + \theta^n = 0 \tag{9}$$

Followingly, another transformation that should be applied might be:

$$\alpha^2 = \frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G} \rightarrow r = \alpha\xi \tag{10}$$

Then, (7) can be written as:

$$\left(\frac{1}{\alpha^2}\frac{1}{\xi^2}\right)\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2\alpha^2\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) + \theta^n = 0$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{11}$$

The approximate solution of the (11) via *Wolfram Mathematica* reads:

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \frac{n(5-8n)\xi^6}{15120} + \ldots$$

For $n = 1$, solution can be written as:

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}$$

1.2 Calculation of Total Mass

To be able to calculate the mass of the star for a given radius, one can easily make use of the (1), that is, integration of the (1) for a certain radius interval yields the mass of the subject star. That being said, the following relation can be observed:

$$M = \int_0^R 4\pi r^2 \rho(r) dr$$

After the change of variables,

$$M = \int_0^{\xi_*} 4\pi\alpha^2\xi^2\rho_c\theta^n(\xi)\alpha d\xi$$

where $\xi_* = \frac{R}{\alpha}$

$$M = 4\pi\rho_c\alpha^3 \int_0^{\xi_*} \xi^2\theta^n(\xi)d\xi$$

By using the relation in (11), one can deduce that:

$$M = 4\pi\rho_c\alpha^3 \int_0^{\xi_*} -\xi^2\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi$$

$$M = 4\pi\rho_c\alpha^3{\xi_*}^2\theta' = 4\pi\rho_c\alpha^3{\xi_*}^3\frac{-\theta'}{\xi_*}$$

Finally, the total mass can be read as:

$$\boxed{M = 4\pi\rho_c R^3\frac{-\theta'}{\xi_*}} \tag{12}$$

1.3 Relation Between Mass and Radius

For a group of stars sharing same polytopic EOS, one can write a common relation between mass and radius. The first thing that should be implemented is previous version of the (12):

$$M = 4\pi\rho_c\alpha^3{\xi_*}^2\theta' = 4\pi\rho_c\alpha^3{\xi_*}^3\frac{-\theta'}{\xi_*}$$

Replacing α with (10):

$$M = 4\pi\left[\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\right]^{\frac{3}{2}}\rho_c\xi_*^2\left[-\theta'(\xi_*)\right]$$

Rearranging the terms:

$$M = 4\pi\left[\frac{K(n+1)}{4\pi G}\right]^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}\xi_*^2\left[-\theta'(\xi_*)\right] \tag{13}$$

At the same time, one should consider the *radius* equation, which reads:

$$R = \alpha\xi_* = \left[\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\right]^{\frac{1}{2}}\xi_* = \left[\frac{K(n+1)}{4\pi G}\right]^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}\xi_* \tag{14}$$

Now, to be able to construct a relation between mass and radius assuming that the group of stars share same polytopic properties, the ρ_c term should be eliminated from the equation. Therefore, the second process that should be implemented is to find an expression for the ρ_c, from which (13) :

$$\rho_c = \left[\frac{M}{4\pi}\left[\frac{4\pi G}{K\left(n+1\right)}\right]^{\frac{3}{2}}\frac{1}{\xi_*^2\left(-\theta'(\xi_*)\right)}\right]^{\frac{2n}{3-n}} \tag{15}$$

Plugging (15) into (14):

$$R = \left[\frac{K\left(n+1\right)}{4\pi G}\right]^{\frac{1}{2}}\left[\frac{M}{4\pi}\left[\frac{4\pi G}{K\left(n+1\right)}\right]^{\frac{3}{2}}\frac{1}{\xi_*^2\left(-\theta'(\xi_*)\right)}\right]^{\frac{1-n}{3-n}}\xi_*$$

$$R = \left[\frac{K\left(n+1\right)}{4\pi G}\right]^{\frac{1}{2}}\left[\frac{1}{4\pi}\left[\frac{4\pi G}{K\left(n+1\right)}\right]^{\frac{3}{2}}\frac{1}{\xi_*^2\left(-\theta'(\xi_*)\right)}\right]^{\frac{1-n}{3-n}}\xi_* M^{\frac{1-n}{3-n}}$$

which asserts that:

$$M \propto R^{\frac{3-n}{1-n}}$$

Finally, the constant of proportionality can be written as:

$$M = C(K, n, \xi_*) R^{\frac{3-n}{1-n}} \tag{16}$$

where

$$\boxed{C(K, n, \xi_*) = \frac{\left[\frac{K(n+1)}{4\pi G}\right]^{\frac{n-3}{2-2n}}}{\frac{1}{4\pi}\left[\frac{4\pi G}{K(n+1)}\right]^{\frac{3}{2}}\frac{1}{\xi_*^2(-\theta'(\xi_*))}}\xi_*^{\frac{n-3}{1-n}}} \tag{17}$$

1.4 Special Case for White Dwarfs

White dwarfs are the final stages of the low-mass stars where the pressure-density relation does not governed by the polytopic relation. In the case of white dwarfs, the main mechanism that governs the star is the quantum mechanical effect called *electron degeneracy*. In the case of *electron degeneracy*, the pressure-density relation can be written as:

$$P = C\left[x\left(2x^2-3\right)\left(x^2+1\right)^{\frac{1}{2}} + 3\sinh^{-1}x\right] \tag{18}$$

where $x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}}$

Finally, if the $x << 1$, then the (18) can be reduced to via *series expansion*:

$$P = K_* \rho^{1+\frac{1}{n_*}} \tag{19}$$

where $K_* = \frac{8C}{5D^{\frac{5}{q}}}$ and $n_* = \frac{q}{5-q}$

1.5 Interpreting the Raw Data

First of all, it should be acknowledged that the interpreting of the raw data is a very hard practice. In this section, the subject data will be briefly reviewed and the numerical approachs that is used to process the data will be explained.

1.5.1 White Dwarf Data

The white dwarf data consists of the mass and radius information of the 378 stars. The mass and radius informations are given in the form of *solar mass* and *log*(g), respectively. To be able to work in SI units, some conversion must be done. The most trivial conversion is possibly the one that involves the solar mass. The solar mass can be converted to the *kilogram* counterpart via using the following formulation:

$$1 \; solar \; mass = 1.98847 \times 10^{30} \; kg$$

Then, to be able to find the radius in *meters*, one can make use of the *Newton' Law of Gravitation*, which reads:

$$g = 10^L S$$

where L is the radius in *log*(g), and $S = 0.01$ is the scaler factor that convert the g value in CGS to SI. It is known that:

$$g = \frac{GM}{R^2}$$

Then, the radius R in *meters*, can be written as:

$$R = \sqrt{\frac{GM}{g}}$$

1.6 How to find K_* and q?

The first thing that should be done is to make use of the (16) and (19), since they include the variables that are inherent in *White Dwarf* data and the simplification that makes the numerical processes easier. Note that the (19) is only valid for $x << 1$, which forces to limit the data that will be used in the numerical process. For the time being, accept that the data that will be used has the following property:

$$M \leqslant M_{threshold}$$

The effect of the different variations of the $M_{threshold}$ is discussed in the Section 2. To be able to find K_* and q, one can construct a *curve-fitting* pipeline, which takes into account (16) and (19), since the (19) is the direct implication of the fact that *Lane Emden* equation can be considered valid. In the beginning, it can be assumed that the mass and radius can be related via the following imaginary function:

$$M = f(R, K_*, q) \tag{20}$$

The (20) can be expanded in two different ways, which ultimately determines the fitting pipeline.

1.6.1 Alternative 1

The first alternative can be the fact that mass can be related to the radius in an exponential form with a scaling factor, that is being inspired by the (16):

$$M = AR^{\frac{3-n}{1-n}}$$

where the $n = \frac{q}{q-5}$. In this form, one can easily implement a *non-linear curve-fitting* procedure to find constants A and q. Once the A is found, a little bit of algebraic manipulation is required to find the K_*. From (17), it is known that:

$$A = \frac{\left[\frac{K_*(n+1)}{4\pi G}\right]^{\frac{n-3}{2-2n}}}{\frac{1}{4\pi}\left[\frac{4\pi G}{K_*(n+1)}\right]^{\frac{3}{2}}\frac{1}{\xi_*^2(-\theta'(\xi_*))}}\xi_*^{\frac{n-3}{1-n}}$$

Assume that:

$$B = \left[\frac{n+1}{4\pi G}\right]^{\frac{-2n}{2-2n}}\xi_*^{\frac{-n-1}{1-n}}4\pi\left(-\theta'(\xi_*)\right)$$

Then, after some algebraic manipulations, one can obtain:

$$K_* = \left(\frac{A}{B}\right)^{\frac{n-1}{n}}$$

Here, note that the solution of the IVP (*Lane Emden equation*) is required to obtain ξ_* and $\theta'(\xi_*)$, where ξ_* refers to the ξ value when $r = R$. All in all, this alternative is very cheap in terms of the required operations in one step, and it is very simple in terms of the mathematical equation that is imposed on the *non-linear curve-fitting* subroutine.

1.6.2 Alternative 2

The second alternative is to use the (16) directly, which reads:

$$M = \left[\frac{\left[\frac{K_*(n+1)}{4\pi G}\right]^{\frac{n-3}{2-2n}}}{\frac{1}{4\pi}\left[\frac{4\pi G}{K_*(n+1)}\right]^{\frac{3}{2}} \frac{1}{\xi_*^2(-\theta'(\xi_*))}} \xi_*^{\frac{n-3}{1-n}} \right] R^{\frac{3-n}{1-n}} \tag{21}$$

To (21), one one can easily implement a *non-linear curve-fitting* procedure to find constants K_* and q. However, the negative side of this scheme is that it requires a solution of an IVP (*Lane Emden equation*) in each step, which can be computationaly very demanding.

2 Results and Discussion