PHYS 514 - Computational Physics

Term Project

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1 Derivation of ***Lane-Emden Equation***

In this project, the first task is to derive the *Lane-Emden* equation to be able to have a general expression for the *hydrostatic equilibrium.* In the beginning, it is known that:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2} \tag{2}$$

where m, ρ, and p are functions of r. To be able to solve this system of differential equations, the expression for the ρ is required, which is not explicitly given. However, by using the *equation of state (EOS)*, one can easily construct a relation between p and ρ, such that:

$$p = \frac{k_B}{\mu m_H} T\rho$$

Assuming that isentropic relation holds, one can conclude that:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \tag{3}$$

To be able to obtain a unified expression, one can inspect the *mass* variable in the (1), and (2). It is required to calculate the derivative of the (2) to write (2) in terms of (1). Differentiating (2) yields:

$$\frac{d^2p}{dr^2} = \frac{2Gm\rho}{r^3} + \frac{-G\rho}{r^2}\frac{dm}{dr} + \frac{-Gm}{r^2}\frac{d\rho}{dr} \tag{4}$$

Before further processing on the (4), the some terms must be revised, that is, (4) can be rewritten in the following form:

$$\frac{d^2p}{dr^2} = \frac{2}{r}\left(-\frac{dp}{dr}\right) + \frac{d\rho}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + 4\pi G\rho^2 \tag{5}$$

Multiplying (5) with $\frac{r^2}{\rho}$ and rearranging the terms yields:

$$\frac{r^2}{\rho}\frac{d^2p}{dr^2} + 2r\left(\frac{1}{\rho}\frac{dp}{dr}\right) - r^2\frac{d\rho}{dr}\left(\frac{1}{\rho^2}\frac{dp}{dr}\right) = 4\pi G r^2 \rho \tag{6}$$

Observing (6), one can easily realize that the left hand side is nothing but the total derivative of the term $\frac{r^2}{\rho}\frac{dp}{dr}$. Hence, in conclusion, the (4), can be written as the following:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = 4\pi G r^2 \rho \tag{7}$$

From now on, the finite amount of transformation should be applied on the ρ and r. Firstly, assume that the following relations hold:

$$\rho = \rho_c \theta^n$$

Therefore,

$$p = K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \rightarrow dp = K\rho_c^{1+\frac{1}{n}}(n+1)\,\theta^n d\theta \tag{8}$$

Applying (8) to (7), one can write:

$$\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n}K\rho_c^{1+\frac{1}{n}}(n+1)\,\theta^n\frac{d\theta}{dr}\right) = 4\pi G r^2\rho_c\theta^n \rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\frac{d\theta}{dr}\right) + \theta^n = 0 \tag{9}$$

Followingly, another transformation that should be applied might be:

$$\alpha^2 = \frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G} \rightarrow r = \alpha\xi$$

Then, (7) can be written as:

$$\left(\frac{1}{\alpha^2}\frac{1}{\xi^2}\right)\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2\alpha^2\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) + \theta^n = 0$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{10}$$

The approximate solution of the (10) via *Wolfram Mathematica* reads:

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \frac{n(5-8n)\xi^6}{15120} + \ldots$$

For $n = 1$, solution can be written as:

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}$$

1.1 Calculation of Total Mass

To be able to calculate the mass of the star for a given radius, one can easily make use of the (1), that is, integration of the (1) for a certain radius interval yields the mass of the subject star. That being said, the following relation can be observed:

$$M = \int_0^R 4\pi r^2\rho(r)dr$$

After the change of variables,

$$M = \int_0^{\Xi} 4\pi\alpha^2\xi^2\rho_c\theta^n(\xi)\alpha d\xi$$

where $\Xi = \frac{R}{\alpha}$

$$M = 4\pi\rho_c\alpha^3 \int_0^{\Xi} \xi^2\theta^n(\xi)d\xi$$

By using the relation in (10), one can deduce that:

$$M = 4\pi\rho_c\alpha^3 \int_0^{\Xi} \xi^2 \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi$$

$$M = 4\pi\rho_c\alpha^3\Xi^2\theta' = 4\pi\rho_c\alpha^3\Xi^3\frac{\theta'}{\Xi}$$

Finally, the total mass can be read as:

$$\boxed{M = 4\pi\rho_c R^3\frac{\theta'}{\Xi}}$$