PHYS 514 - Computational Physics

Term Project

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Contents

1 Newtonian Approach

1.1 Derivation of *Lane-Emden Equation*

In this project, the first task is to derive the *Lane-Emden* equation to be able to have a general expression for the *hydrostatic equilibrium*. In the beginning, it is known that:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2} \tag{2}$$

where m, ρ, and p are functions of r. To be able to solve this system of differential equations, the expression for the ρ is required, which is not explicitly given. However, by using the *equation of state (EOS)*, one can easily construct a relation between p and ρ, such that:

$$p = \frac{k_B}{\mu m_H} T\rho$$

Assuming that isentropic relation holds, one can conclude that:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \tag{3}$$

To be able to obtain a unified expression, one can inspect the *mass* variable in the (1), and (2). It is required to calculate the derivative of the (2) to write (2) in terms of (1). Differentiating (2) yields:

$$\frac{d^2p}{dr^2} = \frac{2Gm\rho}{r^3} + \frac{-G\rho}{r^2}\frac{dm}{dr} + \frac{-Gm}{r^2}\frac{d\rho}{dr} \tag{4}$$

Before further processing on the (4), the some terms must be revised, that is, (4) can be rewritten in the following form:

$$\frac{d^2p}{dr^2} = \frac{2}{r}\left(-\frac{dp}{dr}\right) + \frac{d\rho}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + 4\pi G\rho^2 \tag{5}$$

Multiplying (5) with $\frac{r^2}{\rho}$ and rearranging the terms yields:

$$\frac{r^2}{\rho}\frac{d^2p}{dr^2} + 2r\left(\frac{1}{\rho}\frac{dp}{dr}\right) - r^2\frac{d\rho}{dr}\left(\frac{1}{\rho^2}\frac{dp}{dr}\right) = 4\pi G r^2 \rho \tag{6}$$

Observing (6), one can easily realize that the left hand side is nothing but the total derivative of the term $\frac{r^2}{\rho}\frac{dp}{dr}$. Hence, in conclusion, the (4), can be written as the following:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = 4\pi G r^2 \rho \tag{7}$$

From now on, the finite amount of transformation should be applied on the ρ and r. Firstly, assume that the following relations hold:

$$\rho = \rho_c \theta^n$$

Therefore,

$$p = K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \rightarrow dp = K\rho_c^{1+\frac{1}{n}}(n+1)\,\theta^n d\theta \tag{8}$$

Applying (8) to (7), one can write:

$$\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n}K\rho_c^{1+\frac{1}{n}}(n+1)\,\theta^n\frac{d\theta}{dr}\right) = 4\pi G r^2\rho_c\theta^n \rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\frac{d\theta}{dr}\right) + \theta^n = 0 \tag{9}$$

Followingly, another transformation that should be applied might be:

$$\alpha^2 = \frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G} \rightarrow r = \alpha\xi \tag{10}$$

Then, (7) can be written as:

$$\left(\frac{1}{\alpha^2}\frac{1}{\xi^2}\right)\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2\alpha^2\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) + \theta^n = 0$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{11}$$

The approximate solution of the (11) via *Wolfram Mathematica* reads:

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \frac{n(5-8n)\xi^6}{15120} + \ldots$$

For $n = 1$, solution can be written as:

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}$$

1.2 Calculation of Total Mass

To be able to calculate the mass of the star for a given radius, one can easily make use of the (1), that is, integration of the (1) for a certain radius interval yields the mass of the subject star. That being said, the following relation can be observed:

$$M = \int_0^R 4\pi r^2\rho(r)dr$$

After the change of variables,

$$M = \int_0^{\xi_*} 4\pi\alpha^2\xi^2\rho_c\theta^n(\xi)\alpha d\xi$$

where $\xi_* = \frac{R}{\alpha}$

$$M = 4\pi\rho_c\alpha^3 \int_0^{\xi_*} \xi^2\theta^n(\xi)d\xi$$

By using the relation in (11), one can deduce that:

$$M = 4\pi\rho_c\alpha^3 \int_0^{\xi_*} -\xi^2\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi$$

$$M = 4\pi\rho_c\alpha^3{\xi_*}^2\theta' = 4\pi\rho_c\alpha^3{\xi_*}^3\frac{-\theta'}{\xi_*}$$

Finally, the total mass can be read as:

$$\boxed{M = 4\pi\rho_c R^3\frac{-\theta'}{\xi_*}} \tag{12}$$

1.3 Relation Between Mass and Radius

For a group of stars sharing same polytopic EOS, one can write a common relation between mass and radius. The first thing that should be implemented is previous version of the (12):

$$M = 4\pi\rho_c\alpha^3{\xi_*}^2\theta' = 4\pi\rho_c\alpha^3{\xi_*}^3\frac{-\theta'}{\xi_*}$$

Replacing α with (10):

$$M = 4\pi\left[\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\right]^{\frac{3}{2}}\rho_c\xi_*^2\left[-\theta'(\xi_*)\right]$$

Rearranging the terms:

$$M = 4\pi\left[\frac{K(n+1)}{4\pi G}\right]^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}\xi_*^2\left[-\theta'(\xi_*)\right] \tag{13}$$

At the same time, one should consider the *radius* equation, which reads:

$$R = \alpha\xi_* = \left[\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\right]^{\frac{1}{2}}\xi_* = \left[\frac{K(n+1)}{4\pi G}\right]^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}\xi_* \tag{14}$$

Now, to be able to construct a relation between mass and radius assuming that the group of stars share same polytopic properties, the ρ_c term should be eliminated from the equation. Therefore, the second process that should be implemented is to find an expression for the ρ_c, from which (13) :

$$\rho_c = \left[\frac{M}{4\pi} \left[\frac{4\pi G}{K\,(n+1)} \right]^{\frac{3}{2}} \frac{1}{\xi_*^2\,(-\theta'(\xi_*))} \right]^{\frac{2n}{3-n}} \tag{15}$$

Plugging (15) into (14):

$$R = \left[\frac{K\,(n+1)}{4\pi G} \right]^{\frac{1}{2}} \left[\frac{M}{4\pi} \left[\frac{4\pi G}{K\,(n+1)} \right]^{\frac{3}{2}} \frac{1}{\xi_*^2\,(-\theta'(\xi_*))} \right]^{\frac{1-n}{3-n}} \xi_*$$

$$R = \left[\frac{K\,(n+1)}{4\pi G} \right]^{\frac{1}{2}} \left[\frac{1}{4\pi} \left[\frac{4\pi G}{K\,(n+1)} \right]^{\frac{3}{2}} \frac{1}{\xi_*^2\,(-\theta'(\xi_*))} \right]^{\frac{1-n}{3-n}} \xi_* M^{\frac{1-n}{3-n}}$$

which asserts that:

$$M \approx R^{\frac{3-n}{1-n}}$$

Finally, the constant of proportionality can be written as:

$$M = C(K, n, \xi_*) R^{\frac{3-n}{1-n}}$$

where

$$\boxed{C(K, n, \xi_*) = \frac{\left[\frac{K(n+1)}{4\pi G} \right]^{\frac{n-3}{2-2n}}}{\frac{1}{4\pi} \left[\frac{4\pi G}{K(n+1)} \right]^{\frac{3}{2}} \frac{1}{\xi_*^2(-\theta'(\xi_*))}} \xi_*^{\frac{n-3}{1-n}}}$$

1.4 Interpreting the Raw Data

First of all, it should be acknowledged that the interpreting of the raw data is a very hard practice. In this section, the subject data will be briefly reviewed and the numerical approachs that is used to process the data will be explained.

1.4.1 White Dwarf Data

The white dwarf data consists of the mass and radius information of the 378 stars. The mass and radius informations are given in the form of *solar mass* and $log(g)$, respectively.

$$\frac{d\rho}{dr} = \frac{\frac{dp}{dr}}{\frac{dp}{dx}\frac{dx}{d\rho}}$$