PHYS 514 - COMPUTATIONAL PHYSICS

Term Project

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Date: January 9, 2021

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1 Derivation of *Lane-Emden Equation*

In this project, the first task is to derive the *Lane-Emden* equation to be able to have a general expression for the *hydrostatic equilibrium*. In the beginning, it is known that:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2} \tag{2}$$

where m, ρ, and p are functions of r. To be able to solve this system of differential equations, the expression for the ρ is required, which is not explicitly given. However, by using the *equation of state (EOS)*, one can easily construct a relation between p and ρ, such that:

$$p = \frac{k_B}{\mu m_H} T\rho$$

Assuming that isentropic relation holds, one can conclude that:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \tag{3}$$

To be able to obtain a unified expression, one can inspect the *mass* variable in the (1), and (2). It is required to calculate the derivative of the (2) to write (2) in terms of (1). Differentiating (2) yields:

$$\frac{d^2p}{dr^2} = \frac{2Gm\rho}{r^3} + \frac{-G\rho}{r^2}\frac{dm}{dr} + \frac{-Gm}{r^2}\frac{d\rho}{dr} \tag{4}$$

Before further processing on the (4), the some terms must be revised, that is, (4) can be rewritten in the following form:

$$\frac{d^2p}{dr^2} = \frac{2}{r}\left(-\frac{dp}{dr}\right) + \frac{d\rho}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + 4\pi G\rho^2 \tag{5}$$

Multiplying (5) with $\frac{r^2}{\rho}$ and rearranging the terms yields:

$$\frac{r^2}{\rho}\frac{d^2p}{dr^2} + 2r\left(\frac{1}{\rho}\frac{dp}{dr}\right) - r^2\frac{d\rho}{dr}\left(\frac{1}{\rho^2}\frac{dp}{dr}\right) = 4\pi G r^2 \rho \tag{6}$$

Observing (6), one can easily realize that the left hand side is nothing but the total derivative of the term $\frac{r^2}{\rho}\frac{dp}{dr}$. Hence, in conclusion, the (4), can be written as the following:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = 4\pi G r^2 \rho \tag{7}$$

From now on, the finite amount of transformation should be applied on the ρ and r. Firstly, assume that the following relations hold:

$$\rho = \rho_c \theta^n$$

Therefore,

$$p = K\rho_c^{1+\frac{1}{n}} \theta^{n+1} \rightarrow dp = K\rho_c^{1+\frac{1}{n}} (n+1) \theta^n d\theta \tag{8}$$

Applying (8) to (7), one can write:

$$\frac{d}{dr}\left(\frac{r^2}{\rho_c \theta^n} K\rho_c^{1+\frac{1}{n}} (n+1) \theta^n \frac{d\theta}{dr}\right) = 4\pi G r^2 \rho_c \theta^n \rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\frac{d\theta}{dr}\right) + \theta^n = 0 \tag{9}$$

Followingly, another transformation that should be applied might be:

$$\alpha^2 = \frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G} \rightarrow r = \alpha\xi \tag{10}$$

Then, (7) can be written as:

$$\left(\frac{1}{\alpha^2}\frac{1}{\xi^2}\right)\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2 \xi^2 \alpha^2 \frac{1}{\alpha}\frac{d\theta}{d\xi}\right) + \theta^n = 0$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{11}$$

The approximate solution of the (11) via *Wolfram Mathematica* reads:

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \frac{n(5-8n)\xi^6}{15120} + \ldots$$

For $n = 1$, solution can be written as:

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}$$

1.1 Calculation of Total Mass

To be able to calculate the mass of the star for a given radius, one can easily make use of the (1), that is, integration of the (1) for a certain radius interval yields the mass of the subject star. That being said, the following relation can be observed:

$$M = \int_0^R 4\pi r^2 \rho(r) dr$$

After the change of variables,

$$M = \int_0^{\xi_*} 4\pi\alpha^2\xi^2\rho_c\theta^n(\xi)\alpha d\xi$$

where $\xi_* = \frac{R}{\alpha}$

$$M = 4\pi\rho_c\alpha^3 \int_0^{\xi_*} \xi^2\theta^n(\xi)d\xi$$

By using the relation in (11), one can deduce that:

$$M = 4\pi\rho_c\alpha^3 \int_0^{\xi_*} -\xi^2\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi$$

$$M = 4\pi\rho_c\alpha^3{\xi_*}^2\theta' = 4\pi\rho_c\alpha^3{\xi_*}^3\frac{-\theta'}{\xi_*}$$

Finally, the total mass can be read as:

$$\boxed{M = 4\pi\rho_c R^3\frac{-\theta'}{\xi_*}} \tag{12}$$

1.2 Relation Between Mass and Radius

For a group of stars sharing same polytopic EOS, one can write a common relation between mass and radius. The first thing that should be implemented is previous version of the (12):

$$M = 4\pi\rho_c\alpha^3{\xi_*}^2\theta' = 4\pi\rho_c\alpha^3{\xi_*}^3\frac{-\theta'}{\xi_*}$$

Replacing α with (10):

$$M = 4\pi\left[\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\right]^{\frac{3}{2}}\rho_c\xi_*^2\left[-\theta'(\xi_*)\right]$$

Rearranging the terms:

$$M = 4\pi\left[\frac{K(n+1)}{4\pi G}\right]^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}\xi_*^2\left[-\theta'(\xi_*)\right] \tag{13}$$

At the same time, one should consider the *radius* equation, which reads:

$$R = \alpha\xi_* = \left[\frac{K\rho_c^{\frac{1}{n}-1}(n+1)}{4\pi G}\right]^{\frac{1}{2}}\xi_* = \left[\frac{K(n+1)}{4\pi G}\right]^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}\xi_* \tag{14}$$

Now, to be able to construct a relation between mass and radius assuming that the group of stars share same polytopic properties, the ρ_c term should be eliminated from the equation. Therefore, the second process that should be implemented is to find an expression for the ρ_c, from which (13) :

$$\rho_c = \left[\frac{M}{4\pi} \left[\frac{4\pi G}{K\left(n+1\right)} \right]^{\frac{3}{2}} \frac{1}{\xi_*^2 \left(-\theta'(\xi_*)\right)} \right]^{\frac{2n}{3-n}} \tag{15}$$

Plugging (15) into (14):

$$R = \left[\frac{K\left(n+1\right)}{4\pi G} \right]^{\frac{1}{2}} \left[\frac{M}{4\pi} \left[\frac{4\pi G}{K\left(n+1\right)} \right]^{\frac{3}{2}} \frac{1}{\xi_*^2 \left(-\theta'(\xi_*)\right)} \right]^{\frac{1-n}{3-n}} \xi_*$$

$$R = \left[\frac{K\left(n+1\right)}{4\pi G} \right]^{\frac{1}{2}} \left[\frac{1}{4\pi} \left[\frac{4\pi G}{K\left(n+1\right)} \right]^{\frac{3}{2}} \frac{1}{\xi_*^2 \left(-\theta'(\xi_*)\right)} \right]^{\frac{1-n}{3-n}} \xi_* M^{\frac{1-n}{3-n}}$$

which asserts that:

$$M \approx R^{\frac{3-n}{1-n}}$$

Finally, the constant of proportionality can be written as:

$$M = C(K, n, \xi_*) R^{\frac{3-n}{1-n}}$$

where

$$\boxed{C(K, n, \xi_*) = \frac{\left[\frac{K(n+1)}{4\pi G}\right]^{\frac{n-3}{2-2n}}}{\frac{1}{4\pi}\left[\frac{4\pi G}{K(n+1)}\right]^{\frac{3}{2}} \frac{1}{\xi_*^2(-\theta'(\xi_*))}} \xi_*^{\frac{n-3}{1-n}}}$$