

Computational Methods for Linear Integral Equations

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Preface

This book presents numerical methods and computational aspects for linear integral equations. Such equations occur in various areas of applied mathematics, physics, and engineering. The material covered in this book, though not exhaustive, offers useful techniques for solving a variety of problems. Historical information covering the nineteenth and twentieth centuries is available in fragments in Kantorovich and Krylov (1958), Anselone (1964), Mikhlin (1967), Lonseth (1977), Atkinson (1976), Baker (1978), Kondo (1991), and Brunner (1997).

Integral equations are encountered in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queueing theory, and medicine. Most of the boundary value problems involving differential equations can be converted into problems in integral equations, but there are certain problems which can be formulated only in terms of integral equations. A computational approach to the solution of integral equations is, therefore, an essential branch of scientific inquiry.

Overview

The basic terminology and notation are adopted from Porter and Stirling (1993), Baker (1978), and Mikhlin and Smolitskiy (1967). Generally, while discussing eigenvalue problems the terms ‘eigenvalues’ and ‘characteristic values’ are used interchangeably. But in the literature on integral equations we find that these terms

represent different quantities. Some authors use the notation λ for eigenvalues only and others the notation μ for eigenvalues or characteristic values such that $\lambda\mu = 1$. To avoid this ambiguity we have maintained a precise distinction between these two terms by denoting eigenvalues by λ and characteristic values by μ . With this notation the eigenvalues λ of the kernel of an integral equation coincide with those of the corresponding differential equation. Similarly, we have used a distinct notation k for the kernel function and K for the kernel operator.

A general description of the topics covered in the book is as follows. In Chapter 1, besides an outline of the subject of linear integral equations, definitions, and notations used, the concept of the Nyström method is introduced at the outset with some examples. Eigenvalue problems are discussed in Chapter 2 and different methods are presented with examples. Chapters 3 through 6 cover Fredholm and Volterra equations of the second kind, where classical methods, like the expansion method, product-integration method, quadrature method, collocation method, Galerkin methods, and iteration methods are discussed in detail. Singular equations are introduced in Chapter 7, and different methods and iteration schemes to solve them numerically are presented. Weakly singular equations are discussed in Chapter 8 and a variety of methods for their numerical solution is presented. Singular equations of the Cauchy type are studied in Chapter 9 with different methods for their numerical solution. Chapter 10 deals with the application of Whittaker's cardinal function, and the sinc-Galerkin method is presented for solving singular equations. Chapter 11 covers equations of the first kind. Their inherent ill-posedness makes them so special and difficult. Chapter 12 deals with the numerical inversion of Laplace transform. This topic is important in itself as it finds its use in numerous applied and engineering problems. There are four appendixes: Appendix A discusses the subject of numerical integration and presents different useful quadrature rules. Appendix B lists properties and results for orthogonal polynomials. The definitions of and results on Whittaker's cardinal functions are given in Appendix C. Singular integrals, including Cauchy's principal-value (p.v.) and Hadamard's finite-part integrals and their examples are presented in Appendix D. A large number of examples are solved throughout the book with computational results presented in tabular form.

Salient Features

The book is designed as a new source of classical as well as modern topics on the subject of the numerical computation of linear integral equations. We have not only discussed the underlying theory of integral equations and numerical analysis

of numerical integration and convergence but also provided Mathematica® files which become a readily available source of computation and verification of numerical results. The main features of the book are as follows:

1. The notation is kept straightforward; it makes a distinction between the kernel function and the kernel operator as well as between eigenvalues and characteristic values.
2. The Nyström method which forms the basis of the numerical approach is discussed in the very beginning with enough examples where a distinction between the Nyström points and the quadrature points is carefully explained.
3. All quadrature rules used in the book are discussed in detail toward the end (see Appendix A).
4. Eigenvalue problems are treated in a separate chapter in the beginning of the book.
5. Different kinds of linear integral equations are covered in the following order: Fredholm and Volterra equations of the second kind, singular equations, weakly singular equations, singular equations of the Cauchy type, and integral equations of the first kind.
6. Because numerical approximations fail significantly in the neighborhood of singularities of the kernel function, the solution of singular integral equations by the sinc-Galerkin methods, using Whittaker's cardinal function, is presented in a separate chapter.
7. Integral equations of the first kind occupy a special place since they present an inherent ill-posedness that makes their numerical approximations difficult. Some well-known methods to deal with this situation are discussed in detail.
8. The numerical inversion of Laplace transforms is a special case of integral equations of the first kind. Because of the importance of this topic in research and industry, different numerical and computational methods that yield desirable and verifiable results are presented in a separate chapter devoted to this topic, which is usually not considered in books on integral equations.
9. The notation used in the book is presented in the very beginning in tabular form for ready reference.
10. Enough computational details are provided in our examples so that the numerical results can be verified and duplicated.
11. A bibliography toward the end of the book, though not exhaustive, covers more references than cited in the book. The purpose is to provide a sufficiently large repertoire of references for the readers to use if they so desire.
12. A subject index at the end of the book should be very useful.

Intended Readers

The book, written primarily for graduate students, can be used as a textbook or a reference book depending on need and interest. No familiarity with any programming language, like Fortran or C, is needed. A good knowledge of numerical integration and some working knowledge of Mathematica® are required. Since almost all important aspects of the numerical computation of linear integral equations along with numerical methods and auxiliary Mathematica files to generate numerical approximations are contained in this book, it definitely takes the form of a handbook that the interested reader can readily use.

This is basically a graduate-level book. The readers are naturally graduate students and researchers in applied mathematics, physics, engineering, and industry. It is hoped that all such readers will find the handbook very useful.

Computational Aspects

It has been our experience that Mathematica is sufficient to carry out any and all computational aspects of the methods presented in the book. Some computer codes in Fortran are available in the public domain. Most of them deal with the numerical solution of Fredholm integral equations of the second kind. About 130 examples are solved in this book. Mathematica 4.0 has been used to verify most of the solutions. Mathematica files are available at the authors' websites:

www.math.uno.edu/fac/pkythe.html

www.math.uno.edu/fac/ppuri.html

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Notation

A list of the notation and abbreviations used in this book is given below.

a.e.	almost everywhere
\arg	argument of a complex number
$A \times A$	product of sets A and B
$A \setminus B$	complement of a set B with respect to a set A
\overline{A}	closure of a set A
A_m	m th trace of the kernel $k(x, s)$; ($= \int_a^b k_m(x, s) ds$)
\mathbf{A}^T	transpose of a matrix \mathbf{A}
CSK1	Cauchy singular equations of the first kind
CSK2	Cauchy singular equations of the second kind
$C^k(D)$	class of continuous functions with k continuous derivatives on D
$C^\infty(D)$	infinitely differentiable functions on a domain D
$C(\phi, h, x)$	Whittaker's cardinal function (cardinal interpolant of ϕ)
\mathbb{C}	complex plane
\mathbb{C}_∞	extended complex plane
$d_\lambda(x, s)$	Fredholm minor
D	domain
$D(\lambda)$	Fredholm determinant
$\{e_i\}_{i=1}^n$	basis
e.g.	for example
E	error
Eq(s)	equation(s) (when followed by an equation number)
$f(x)$	free term
$\overline{f(x)}$	complex conjugate of a function $f(x)$
FK1	Fredholm equation of the first kind
FK2	Fredholm equation of the second kind
FK3	Fredholm equation of the third kind

$F \star G$	type 1 convolution
$F \circ G$	type 2 convolution
\mathbf{f}	vector
\mathcal{F}	Fourier transform
\mathcal{F}_c	Fourier cosine transform
\mathcal{F}_s	Fourier sine transform
$nF_m(\cdot, \cdot; x)$	generalized hypergeometric function
$g(x, s)$	regular (continuous) part of a kernel
h	step size
H	resolvent operator defined by $L^{-1} = I + \lambda H$
HSK1	first-kind singular equation with Hilbert kernel
HSK2	second-kind singular equation with Hilbert kernel
$H(x)$	Heaviside unit step function
$H_2[\phi]$	Hadamard transform of ϕ
$H_n(x)$	Hermite polynomials of degree n
H^α	Hölder condition for $0 < \alpha \leq 1$
H^1	Lipschitz condition
$H_n(x)$	Hermite polynomial of degree n
i.e.	that is
iff	if and only if
IDE	integro-differential equation
I	identity operator
I_{1N}	one-dimensional sinc-interpolant
I_{2N}	two-dimensional sinc-interpolant
\mathbf{I}	identity matrix
\Im	imaginary part
$J_0(x)$	Bessel's function of first kind and order zero
$k(x, s)$	kernel of an integral equation
$k^*(x, s)$	adjoint (conjugate) kernel ($= \overline{k(x, s)}$)
$k^{[n]}(x, s)$	degenerate (separable) kernel
$k_\lambda(x, s)$	resolvent of the kernel $k(x, s)$
$k_n(x, s)$	n th iterated kernel
k^T	transposed kernel
K	integral operator
K^*	operator associated with k^*
$l_{im}(x)$	(also $l_j(x)$) Lagrange interpolating polynomial
$\text{Lip}[a, b]$	Lipschitz condition on $[a, b]$
L^{-1}	inverse integral operator ($= (I - \lambda K)^{-1}$)
L	operator ($= I - \lambda K$)
L_2	function space
L_p	function space, $p \geq 1$
$L_n(x)$	Laguerre polynomials
$L_n^{(\alpha)}(x)$	generalized Laguerre polynomial

\mathcal{L}	Laplace transform $\mathcal{L}\{f(t)\} = F(s)$
\mathcal{L}^{-1}	inverse Laplace transform
$p_i(x)$	orthogonal polynomials
$\{p_i(x)\}$	set of linearly independent basis functions
P_n	bounded linear projections; interpolation projections
$P_n(x)$	Legendre polynomials of degree n
$P_n^*(x)$	shifted Legendre polynomial
$P_n^{(\nu)}(x)$	Gegenbauer polynomial
$P_n^{\alpha, \beta}$	Jacobi polynomials of degree n
P_N	projection operator
$Q_j(F), Q_j$	quadrature rule $j = 1, 2, \dots, N$
$r[\tilde{\phi}_n]$	residual for the n th approximate solution $\tilde{\phi}_n$
r_σ	spectral radius
\Re	real part
\mathbb{R}^n	Euclidean n -space
\mathbb{R}^+	set of positive real numbers
s	variable of the Laplace transform
s_j	quadrature points
Si	sine integral
S	square $\{(x, s) : x, s \in [a, b] \times [a, b]\}$
$S(m, h)(z)$	sinc function
$S_p^m(\mathbf{z})$	space of splines of order p with knot sequence \mathbf{z}
SK1	singular equation of the first kind
SK2	singular equation of the second kind
SFK1	singular Fredholm equations of the first kind
SFK2	singular Fredholm equations of the second kind
SVK1	singular Volterra equations of the first kind
SVK2	singular Volterra equations of the second kind
T	triangle $\{(x, s) : a \leq s \leq x \leq b\}$
$T_n(x)$	Chebyshev polynomials of the first kind of degree n
$\langle u, v \rangle$	inner product of u and v
(u, v)	inner product with a weight function of u and v
$\prec u, v \succ$	discrete inner product of u and v
U	unit disk $ z < 1$
$U(t - a)$	unit step function
$U_n(x)$	Chebyshev polynomials of the second kind of degree n
$V_n(x)$	interpolating polynomial of degree n
VK1	Volterra equation of the first kind
VK2	Volterra equation of the second kind
VK3	Volterra equation of the third kind
v_j, w_j	weights in quadrature rules
x_i	Nyström points
\mathbf{x}^T	transpose of a matrix \mathbf{x} (§4.1)

\mathcal{X}	Banach space
$Z(x)$	fundamental solution
δ_{ij}	Kronecker delta ($= 0$ if $i \neq j$; $= 1$ if $i = j$)
λ	numerical parameter; also, eigenvalue
λ_i	i th eigenvalue
μ	characteristic value ($= 1/\lambda$)
ν	index
κ	regular (continuous) part of $k(x, s)$; also, index
ξ_j	Chebyshev points; also, Gauss points
$\rho(A)$	resolvent of a set A
$\rho(\mathbf{A})$	condition number of a matrix \mathbf{A}
$\sigma(A)$	resolvent of a set A ; also, spectrum of a matrix A
ϕ	unknown function in an integral equation
$\tilde{\phi}$	approximate value of ϕ
$\tilde{\Phi}$	vector of approximate solutions $\tilde{\phi}$'s; also, blocks
$\psi_i(x)$	complete set of functions; also, i th eigenfunction
∇^2	Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
\mathfrak{f}	Cauchy principal-value (p.v.) integral
\mathfrak{f}'	Hadamard (finite-part) integral
\sum'	summation with the first term halved
\sum''	summation with the first and last terms halved
\otimes	operation defined by $\sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^n a_i(x) \overline{b_i(s)}$

1

Introduction

We discuss in this chapter some basic concepts and present results needed to study the numerical and computational aspects of the solution of integral equations. Proofs of most of the results can be found in standard textbooks on integral equations, real and complex analysis, integral transforms, and numerical analysis. The notation used in this book, although standard, is also presented for clarification.

1.1. Notation and Definitions

Let \mathbb{R}^n denote the Euclidean n -space and \mathbb{R}^+ the set of nonnegative real numbers. The complement of a set B with respect to a set A is denoted by $A \setminus B$ (or $\text{compl}(B)$ if the reference to set A is obvious), the product of the sets A and B by $A \times B$, and the closure of a set A by \bar{A} .

Let \mathbb{C} denote the complex plane. If $a \in \mathbb{C}$ and $r > 0$, then $\{z \in \mathbb{C} : |z - a| < r\}$, $\{z \in \mathbb{C} : |z - a| \leq r\}$, and $\{z \in \mathbb{C} : |z - a| = r\}$ denote an open disk, a closed disk, and a circle, each of radius r and centered at a , respectively. The open unit disk $|z| < 1$ is sometimes denoted by U . A connected open set $A \subseteq \mathbb{C}$ is called a domain (or region). The extended complex plane is denoted by \mathbb{C}_∞ . If D is a region in \mathbb{C}_∞ , then the following statements are equivalent: (a) D is simply connected; (b) $\mathbb{C}_\infty \setminus D$ is connected; and (c) the boundary of D in \mathbb{C}_∞ is connected. Also,

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2} (f_x - i f_y), \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} (f_x + i f_y), \quad (1.1.1)$$

where $z = z + iy$ and $\bar{z} = x - iy$. Let $z = re^{i\theta}$. Then $|z| = r$, $\arg\{z\} = \theta$, and $\arg\{\bar{z}\} = -\arg\{z\}$. Also, if $z_1, z_2 \in \mathbb{C}$, then $|z_1 z_2| = |z_1| |z_2|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$; $\arg\{z_1 z_2\} = \arg\{z_1\} + \arg\{z_2\}$, $\arg\left\{\frac{z_1}{z_2}\right\} = \arg\{z_1\} - \arg\{z_2\}$ provided $z_2 \neq 0$, and $\arg\left\{\frac{1}{z}\right\} = -\arg\{z\}$. A function $f : D \mapsto \mathbb{C}$ is analytic on D iff $\bar{\partial}f = 0$, which is equivalent to the *Cauchy–Riemann equations* for the function $f(z) = u(x, y) + iv(x, y)$:

$$u_x = v_y, \quad u_y = -v_x, \quad (1.1.2)$$

or, in polar form,

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta. \quad (1.1.3)$$

Thus,

$$f'(z) = u_x + i v_x = v_y - i u_y. \quad (1.1.4)$$

The Cauchy–Riemann equations are necessary conditions for $f(z)$ to be analytic on D . However, merely satisfying the Cauchy–Riemann equations alone is not sufficient to ensure the differentiability of $f(z)$ at a point in D . The following results are obvious: $\partial(\log|z|) = \frac{1}{2z}$, $\bar{\partial}(\log|z|) = \frac{1}{2\bar{z}}$, $\overline{\partial f} = \bar{\partial}\bar{f}$, and the Laplacian $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\bar{\partial}\partial = 4\partial\bar{\partial}$. Thus, $\nabla^2 u = 0$, and $\nabla^2 v = 0$.

A real- or complex-valued function f is said to belong to the class $C^k(D)$ if it is continuous together with its k th derivatives, in a domain D , $0 \leq k < \infty$. In this case we shall often write that f is a $C^k(D)$ -function or that f is a C^k -function on D , and a C^0 -function is written as a C -function. The functions f in the class $C^k(D)$, for which all k th derivatives admit continuous continuations in the closure \overline{D} , form the class of functions $C^k(\overline{D})$. The class $C^\infty(D)$ consists of functions f that are infinitely differentiable on D , i.e., continuous partial derivatives of all orders exist. These classes are linear sets. Thus, every linear combination $\lambda f + \mu g$, where λ and μ are arbitrary complex numbers, also belongs to the respective class.

A simple closed curve is generally called a (Jordan) *contour*, but we shall explicitly indicate if a contour is unclosed. The Jordan curve theorem states that if C is a simple closed contour, then $\mathbb{C} \setminus C$ has two components, one called the interior of C , denoted by $\text{Int}(C)$, and the other called the exterior of C , denoted by $\text{Ext}(C)$, each of which has C as its boundary. Thus, if C is a simple contour, then $\text{Int}(C)$ and $\text{Ext}(C) \cup \{\infty\}$ are simply connected regions.

Let $f(z)$ be analytic in a simply connected domain D except for an isolated singularity $z_0 \in D$. Then the *residue* of $f(z)$ at z_0 is given by

$$\operatorname{Res}_{z=z_0} \{f(z)\} = \frac{1}{2i\pi} \int_{C_\varepsilon} f(z) dz, \quad (1.1.5)$$

where C_ε is the circle $|z - z_0| = \varepsilon$ of arbitrarily small radius ε . If the point z_0 is a pole of order n , then

$$\operatorname{Res}_{z=z_0} \{f(z)\} = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]. \quad (1.1.6)$$

If $f(z) = \frac{g(z)}{h(z)}$, where $g(z) \neq 0$ and $h(z)$ has a simple zero at $z = z_0$, i.e., $h(z_0) = 0$ and $h'(z_0) \neq 0$, then

$$\operatorname{Res}_{z=z_0} \{f(z)\} = \frac{g(z_0)}{h'(z_0)}. \quad (1.1.7)$$

CAUCHY'S RESIDUE THEOREM states that if $f(z)$ is analytic in a simply connected domain D except for finitely many points z_j at which f has isolated singularities and if C is a Jordan contour that lies in D and does not pass through any points z_j lying inside D , then

$$\int_C f(z) dz = 2i\pi \sum_{j=1}^n \operatorname{Res}_{z=z_j} \{f(z)\}. \quad (1.1.8)$$

HÖLDER CONDITION. Let $f(t)$ be defined on a Jordan curve Γ (open or closed). If

$$|f(t_1) - f(t_2)| \leq A |t_1 - t_2|^\alpha, \quad 0 < \alpha \leq 1, \quad (1.1.9)$$

for arbitrary points $t_1, t_2 \in \Gamma$ ($t_1 \neq t_2$), where $A > 0$ and α are real constants, then $f(t)$ is said to satisfy the *Hölder condition of order α* , or simply the condition H^α on Γ , denoted by $f(t) \in H^\alpha(\Gamma)$. If $|t_1|$ and $|t_2|$ are sufficiently large, then $f(t) \in H^\alpha(\Gamma)$ if

$$|f(t_1) - f(t_2)| \leq A \left| \frac{1}{t_1} - \frac{1}{t_2} \right|^\alpha, \quad 0 < \alpha \leq 1.$$

This inequality is also known as the Hölder condition at infinity. The condition H^1 is known as the *Lipschitz condition*, and we write $f \in \operatorname{Lip}(\Gamma)$. If $f(t) \in C(\Gamma)$ and $f(t) \in H^\alpha(\Gamma)$, we say that $f(t)$ is *Hölder-continuous* on Γ . If $f \in C(\Gamma)$ and $f \in \operatorname{Lip}(\Gamma)$, then $f(t)$ is said to be *Lipschitz-continuous* on Γ .

FATOU'S LEMMA. Let $f_n : I \mapsto \mathbb{R}$ be nonnegative, extended real-valued, measurable functions defined in an interval I and such that the sequence $\{f_n\}$ converges pointwise to the function $f : I \mapsto \mathbb{R}$. If $\lim_{n \rightarrow \infty} \int_I f_n < \infty$, then f is integrable and

$$\int_I f \leq \liminf_{n \rightarrow \infty} \int_I f_n.$$

In the case of a real function f of a single variable, which are integrable in the sense of Riemann, suppose that $y = f(x)$ is a bounded function on the finite interval $[a, b]$. Partition this interval into n subintervals by the points $a = x_0 < x_1 < \dots < x_n = b$. Let ξ be any point in the i th subinterval: $x_{i-1} \leq \xi_i \leq x_i$. Then form the sum

$$S_n = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1}). \quad (1.1.10)$$

Such a sum is called a *Riemann sum*. Let $\Delta = \max_i (x_i - x_{i-1})$ denote the maximum length (or norm) of the subintervals, and consider a sequence of sums of type (1.1.10), S_1, S_2, \dots , whose respective norms $\Delta_1, \Delta_2, \dots$ are such that $\lim_{m \rightarrow \infty} \Delta_m = 0$. If, for any sequence of this type and any choice of ξ_i , the sequence $\{S_m\}$ has a common limit S , then $f(x)$ is said to have the *Riemann integral* S over the interval $[a, b]$, i.e.,

$$S = \int_a^b f(x) dx. \quad (1.1.11)$$

A bounded function $f(x)$ is Riemann-integrable iff $f(x)$ is continuous almost everywhere. In particular, if $f(x) \in C[a, b]$, it has a Riemann integral. Also, if $f(x)$ is bounded on $[a, b]$ and continuous except for finitely many points of discontinuity, it is Riemann-integrable.

The Schwarz, Hölder, and Minkowski inequalities are, respectively:

$$\left| \int_a^b f(x) g(x) dx \right| \leq \left(\int_a^b |f(x)|^2 dx \right)^{1/2} \left(\int_a^b |g(x)|^2 dx \right)^{1/2}, \quad (1.1.12)$$

$$\left| \int_a^b f(x) g(x) dx \right| \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q}, \quad p > 1, \quad (1.1.13)$$

$$\left(\int_a^b |f(x)+g(x)|^p dx \right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^p dx \right)^{1/p}, \quad p \geq 1, \quad (1.1.14)$$

where $1/p + 1/q = 1$.

BESSEL'S INEQUALITY. Let $f(x)$ be a piecewise continuous function on $[-l, l]$ with the Fourier series $f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left(a_j \cos \frac{j\pi x}{l} + b_j \sin \frac{j\pi x}{l} \right)$.

Then

$$\frac{a_0^2}{2} + \sum_{j=1}^{\infty} (a_j^2 + b_j^2) \leq \frac{1}{l} \int_{-l}^l [f(x)]^2 dx. \quad (1.1.15)$$

IMPROPER INTEGRALS are such that their range or integrands are unbounded. They are defined as the limits of certain proper integrals. Thus, an improper integral over the interval $[0, \infty)$ is defined as

$$\int_0^\infty f(x) dx = \lim_{R \rightarrow \infty} \int_0^R f(x) dx$$

whenever the limit exists. Improper integrals over $[a, \infty)$, $(-\infty, a]$ are similarly defined. Improper integrals over the interval $(-\infty, \infty)$ are defined in two ways: First, there is the usual definition:

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^\infty f(x) dx.$$

Then, there is the other definition:

$$\int_{-\infty}^\infty f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx,$$

provided both limits exist. This definition is also known as *Cauchy's principal value* (or p.v.) of the integral, denoted by $\int_{-\infty}^\infty f(x) dx$. A common p.v. integral

is the Hilbert transform $\int_A^b \frac{f(t)}{t-x} dt$, where $-\infty \leq a < b \leq \infty$, and $a < x < b$.

A sufficient condition for the Hilbert transform to exist over a finite interval $[a, b]$ is that $f(t)$ satisfy a *Lipschitz* or *Hölder* condition in $[a, b]$, i.e., there are constants $A > 0$ and $0 < \alpha \leq 1$ such that for any two points $T_1, t_2 \in [a, b]$ we have $|f(t_1) - f(t_2)| \leq A |t_1 - t_2|^\alpha$.

Hadamard finite-part integrals are defined by

$$\int_a^b (x-a)^{\lambda-n} f(x) dx = \lim_{R \rightarrow a^+} \left[\int_R^b (x-a)^{\lambda-n} f(x) dx - g(R)(R-a)^{\lambda-n+1} \right], \quad (1.1.16)$$

where $-1 < \lambda \leq 0$, $f \in C^n[a, b]$, $f(s) \neq 0$, and $g(R)$ is any function in $C^n[a, b]$ such that the limit exists and diverges for all positive integers n . Cauchy's p.v. integrals of the type

$$\int_a^b \frac{\phi(x)}{x-s} dx$$

and Hadamard's finite-part integrals of the type

$$\int_a^b \frac{\phi(x)}{(x-s)^2} dx$$

are discussed in Appendix D.

EIGENVALUES, EIGENVECTORS, SPECTRUM, AND RESOLVENT SET OF A MATRIX. An *eigenvalue* of a square matrix $A = (a_{ij})$ is a number λ such that the equation $Ax = \lambda x$ has a nonzero solution x . This x is called an *eigenvector* corresponding to that eigenvalue λ . Such eigenvectors and the zero vector form a vector subspace of a finite-dimensional normed space X , which is called the *eigenspace* corresponding to that eigenvalue λ . The set $\sigma(A)$ of all eigenvalues of A is called the *spectrum* of A . Its complement $\rho(A) = \mathbb{C} \setminus \sigma(A)$ in the complex plane is called the *resolvent set* of A . The *spectral radius* $r_\sigma(S)$ is the radius $r_\sigma(S) = \sup_{w \in \sigma(S)} |w|$ of the smallest closed disk centered at the origin of the complex w -plane and containing the spectrum $\sigma(S)$. If S is a linear bounded operator on E , then $r_\sigma(S) = \lim_{n \rightarrow \infty} \sqrt[n]{\|S^n\|}$.

Let $T : X \mapsto X$ be a linear operator on the normed space X of dimension n . Let $e = \{e_1, \dots, e_n\}$ be any basis for X and let $T_e = (\alpha_{ij})$ be the matrix representing T with respect to that basis whose elements are kept in the given order. Then the eigenvalues of the matrix T_e are called *eigenvalues of the operator T_e* , and the spectrum and the resolvent set are similarly defined. A basis result is as follows: All matrices representing a given linear operator $T : X \mapsto X$ on a finite-dimensional normed space relative to various bases for X have the same eigenvalues. Other results are: (i) a linear operator on a finite-dimensional complex normed space $X \neq \{0\}$ has at least one eigenvalue; (ii) the eigenvalues of a Hermitian (self-adjoint) matrix A are real, whereas those of a skew-Hermitian matrix are purely imaginary or zero, and those of a unitary matrix have absolute value 1.

For computational purpose we shall require *quadrature rules*; those most frequently used are discussed in Appendix A.

1.2. Classification

Here we classify integral equations of a single real variable into their kinds and types, provide subclassifications of the kernel, and present some related theorems.

1.2.1. Kinds and Types. To classify integral equations, we denote the unknown function by $\phi(x)$, the kernel of the equation by $k(x, s)$, and the free term, which is assumed known, by $f(x)$. The kernel and the free term are, in general, complex-valued functions of a real variable. The classification is based on three basic features, namely,

- (i) the *kind*, which is determined by the location of the unknown function $\phi(x)$. Thus, *first kind* equations have $\phi(x)$ present only under the integral sign, whereas the *second* and *third* kinds have $\phi(x)$ under the integral sign as well as outside it.
- (ii) the *type*, which depends on the type of the interval of integration. Thus, a *Fredholm* equation has the integration over a finite interval with fixed endpoints, whereas a *Volterra* equation has an indefinite integral.
- (iii) the *singular* equations are those that have improper integrals because of either an infinite interval or an unbounded integrand within the given interval, or both.

We shall denote Fredholm equations of the first, second, and third kinds by FK1, FK2, and FK3, and Volterra equations of similar kinds by VK1, VK2, and VK3, respectively. Thus,

$$\text{FK1 : } \int_a^b k(x, s)\phi(s) ds = f(x), \quad a \leq x \leq b, \quad (1.2.1)$$

$$\text{FK2 : } \phi(x) = f(x) + \lambda \int_a^b k(x, s)\phi(s) ds, \quad a \leq x \leq b, \quad (1.2.2)$$

where $\lambda \neq 0$ is a numerical parameter (in general, complex) which in practical applications defines physical quantities. Eq (1.2.2) can sometimes be represented alternatively as

$$\text{FK2 : } \mu \phi(x) = \mu f(x) + \int_a^b k(x, s)\phi(s) ds, \quad a \leq x \leq b, \quad (1.2.3)$$

where the parameter μ is such that $\lambda\mu = 1$. The Fredholm equation of the third kind is represented by

$$\text{FK3 : } \mu g(x)\phi(x) = \mu f(x) + \int_a^b k(x, s)\phi(s) ds, \quad a \leq x \leq b, \quad (1.2.4)$$

where g is a preassigned function. If $g \neq 0$ in $[a, b]$, then after dividing Eq (1.2.4) by g and absorbing it into f and k , we reduce it to FK2.

Volterra equations differ from Fredholm equations in that their integration is indefinite. Thus,

$$\text{VK1 : } \int_a^x k(x, s)\phi(s) ds = f(x), \quad a \leq x \leq b, \quad (1.2.5)$$

$$\text{VK2 : } \phi(x) = f(x) + \lambda \int_a^x k(x, s)\phi(s) ds, \quad a \leq x \leq b, \quad (1.2.6)$$

or alternatively

$$\text{VK2 : } \mu\phi(x) = f(x) + \int_a^x k(x, s)\phi(s) ds, \quad a \leq x \leq b, \quad (1.2.7)$$

and

$$\text{VK3 : } \mu g(x)\phi(x) = f(x) + \lambda \int_a^x k(x, s)\phi(s) ds, \quad a \leq x \leq b. \quad (1.2.8)$$

As in the Fredholm case, VK3 reduces to VK2.

If $f(x) = 0$ in $[a, b]$, Fredholm or Volterra equations are said to be *homogeneous*; otherwise they are inhomogeneous. The superposition principle applies to linear and homogeneous equations only, i.e., if ϕ_1 and ϕ_2 are both solutions of a homogeneous equation, then their linear combination $c_1\phi_1 + c_2\phi_2$ is also a solution, where c_1 and c_2 are constants. Thus, Fredholm and Volterra equations are *linear integral equations*.

Fredholm equations reduce to Volterra equations if the kernels are such that

$$k(x, s) = 0 \quad \text{for } x \leq s \leq b. \quad (1.2.9)$$

This result, however, does not minimize the differences between the two types of equations. A VK1 is of the form (1.2.5) where the functions $\phi(x)$ and $f(x)$ are assumed to belong to the class $L_2[a, b]$, and the kernel $k(x, s)$ is assumed to be continuous on the square $S = \{(x, s) : x, s \in [a, b] \times [a, b]\}$, or such that $k(x, s)$ is square-integrable on S , i.e.,

$$\int_a^b \int_a^b k^2(x, s) dx ds = B^2 < \infty \quad \text{on } S, \quad (1.2.10)$$

and that $k(x, s) \equiv 0$ for $s > x$.

If $k(x, s)$ and $\phi(x)$ are continuous, the free term $f(x)$ must satisfy the following conditions:

- (a) If $k(a, a) \neq 0$, then $f(a) = 0$,
- (b) If $k(a, a) = k'_x(a, a) = \dots = k_x^{(n-1)}(a, a) = 0$, $0 < |k_x^{(n)}(a, a)| < \infty$, then $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$.
- (c) If $k(a, a) = k'_x(a, a) = \dots = k_x^{(n-1)}(a, a) = 0$, and $k_x^{(n)}(a, a) = \infty$, then $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$.

The kernel $k(x, s)$ is called

- (i) a *difference kernel* if it depends on the difference of its arguments, i.e., if $k(x, s) = k(x - s)$;
- (ii) a *polar kernel* if

$$k(x, s) = \frac{g(x, s)}{(x - s)^\alpha} + h(x, s), \quad 0 < \alpha < 1, \quad (1.2.11)$$

where g and h are bounded on S and $g(x, x) \neq 0$;

- (iii) a *logarithmic kernel* (or a kernel with logarithmic singularity) if

$$k(x, s) = g(x, s) \ln(x - s) + h(x, s), \quad (1.2.12)$$

where g and h satisfy the same conditions as in (ii);

- (iv) a *degenerate kernel* if $k(x, s)$ can be represented in the form

$$k(x, s) = g_1(x)h_1(s) + \dots + g_n(x)h_n(s). \quad (1.2.13)$$

Singular equations involve (a) the interval of integration to be infinite or (b) the kernel to be unbounded in the interval of integration. A further classification of singular equations depends on the kernel. Thus, integral equations that have *kernels with weak singularity* include polar and logarithmic kernels, and such equations are called *weakly singular equations*. Note that an equation with the kernel (1.2.12) can be written in the weakly singular form as

$$k(x, s) = \frac{b(x, s) \log|x - s|}{|x - s|^\varepsilon}, \quad \text{where } b(x, s) = g(x, s) |x - s|^\varepsilon, \quad (1.2.14)$$

provided that the numerator is bounded for $\varepsilon > 0$. An example of a *strongly singular* or *Cauchy singular* kernel is

$$k(x, s) = \frac{g(x, s)}{x - s}, \quad (1.2.15)$$

1. INTRODUCTION

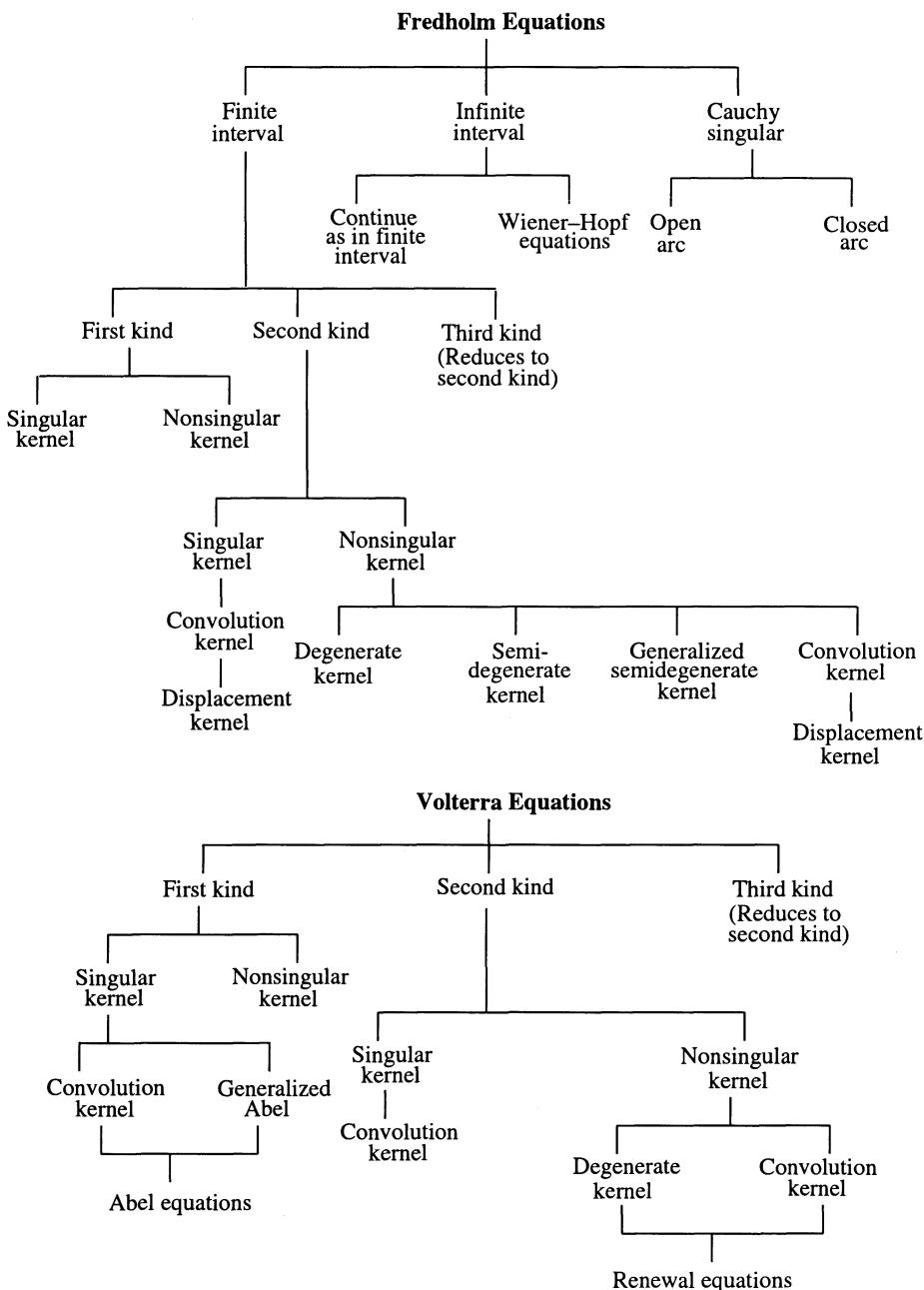


Fig. 1.2.1. Classification of integral equations.

where g is bounded. A special case of Eq (1.2.5) of the form

$$\int_a^x \frac{\phi(s)}{(x-s)^\beta} ds = f(x), \quad 0 < \beta < 1, \quad (1.2.16)$$

is known as a *generalized Abel equation*.

If the function $f(x) \in C^1[a, b]$ and the kernel $k(x, s) \in C^1(S)$, and if $k(x, x) \neq 0$ for $x \in [a, b]$, and if $f(a) = 0$, then there exists a unique solution $\phi(x)$ of Eq (1.2.5). If we introduce the integral operator K defined by

$$(K\phi)(x) = \int_a^b k(x, s) \phi(s) ds, \quad a \leq x \leq b, \quad (1.2.17a)$$

or

$$(K\phi)(x) = \int_a^x k(x, s) \phi(s) ds, \quad a \leq s \leq x \leq b, \quad (1.2.17b)$$

then

$$\begin{aligned} \text{FK1/VK1 : } & K\phi = f, \\ \text{FK2/VK2 : } & \phi = f + \lambda K\phi, \\ \text{FK2/VK2 : } & \mu\phi = f + K\phi. \end{aligned} \quad (1.2.18)$$

A real-valued kernel with the property $k(x, s) = k(s, x)$, $a \leq x, s \leq b$, is said to be *symmetric*; otherwise it is called *nonsymmetric*. The *quadratic form* $\langle K\phi, \phi \rangle$, generated by a kernel $k(x, s)$, is defined by

$$\langle K\phi, \phi \rangle = \int_a^b \int_a^b k(x, s) \phi(x) \phi(s) ds. \quad (1.2.19)$$

A classification for Fredholm and Volterra equations in a tree form is presented in Fig. 1.2.1.

A VK1 can be treated as an FK1 whose kernel $k(x, s) = 0$ for $s > x$. The above definitions of kernels also hold for an FK1 of the form

$$\int_a^b k(x, s) \phi(s) ds = f(x). \quad (1.2.20)$$

For example, an FK1 with weak singularity has a polar kernel (1.2.11) with $0 < \alpha < 1/2$, or a logarithmic kernel (1.2.12). The FK1 (1.2.20) also includes the case when the limits of integration are $-\infty \leq a < b \leq +\infty$, but, as in all cases, the

condition (1.2.10) must be verified to hold. An FK1, with difference kernel and both infinite limits of integration, and of the form

$$\int_{-\infty}^{\infty} k(x-s)\phi(s) ds = f(x), \quad -\infty < x < \infty, \quad (1.2.21)$$

is called an *equation of convolution type of the first kind with a single kernel*. An FK1 of convolution type on the half real axis and of the form

$$\int_0^{\infty} k(x-s)\phi(s) ds = f(x), \quad 0 < x < \infty, \quad (1.2.22)$$

is called a *one-sided equation* or a *Wiener-Hopf equation of the first kind*. An FK1 of convolution type with two kernels k_1 and k_2 is of the form

$$\int_0^{\infty} k_1(x-s)\phi(s) ds + \int_{-\infty}^0 k_2(x-s)\phi(s) ds = f(x), \quad -\infty < x < \infty. \quad (1.2.23)$$

In general, *dual integral equations of the first kind* have the form

$$\begin{aligned} \int_a^{\infty} k_1(x,s)\phi(s) ds &= f_1(x), & a < x < b, \\ \int_a^{\infty} k_2(x,s)\phi(s) ds &= f_2(x), & b < x < \infty, \end{aligned} \quad (1.2.24)$$

where k_1 and k_2 are kernels of each equation in (1.2.24). A special case of this equation is the dual equation of the first kind with difference kernels (of convolution type)

$$\begin{aligned} \int_{-\infty}^{\infty} k_1(x-s)\phi(s) ds &= f(x), & 0 < x < \infty, \\ \int_{-\infty}^{\infty} k_2(x-s)\phi(s) ds &= f(x), & -\infty < x < 0. \end{aligned} \quad (1.2.25)$$

Integral equations of convolution type are sometimes multiplied by the factor $1/\sqrt{2\pi}$.

EXAMPLE 1.2.1. Consider the FK1 of convolution type

$$\int_{-a}^a \ln\left(\frac{A}{|x-s|}\right) \phi(s) ds = f(x).$$

This equation arises in plane elasticity, and its solution using Krein's method (Polyanin and Manzhinov 1998, p.494; Krein 1962) is given by

$$\begin{aligned}\phi(x) = & \frac{1}{2M'(a)} \left[\frac{d}{da} \int_{-a}^a w(s, a) f(s) ds \right] w(x, s) \\ & - \frac{1}{2} \int_{|x|}^a w(x, t) \frac{d}{dt} \left[\frac{1}{M'(t)} \frac{d}{dt} \int_{-t}^t w(s, t) f(s) ds \right] dt \\ & \frac{1}{2} \frac{d}{dx} \int_{|x|}^a \frac{w(t, t)}{M'(t)} \left[\int_{-t}^t w(s, t) f(s) ds \right] dt,\end{aligned}\quad (1.2.26)$$

where

$$M(t) = \int_0^t M(x, t) dx = \left(\ln \frac{2A}{t} \right)^{-1}, \quad w(s, t) = \frac{M(t)}{\pi \sqrt{t^2 - s^2}}. \blacksquare$$

EXAMPLE 1.2.2. For the equation

$$\int_{-a}^a \frac{\phi(s) ds}{|x - s|^\alpha} = f(x), \quad 0 < \alpha < 1,$$

which arises in plane elasticity, the solution is given by formula (1.2.26), where in this case

$$M(t) = \frac{2\sqrt{\pi}}{\alpha \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right)} t^\alpha, \quad w(s, t) = \frac{1}{\pi} \cos\left(\frac{\pi\alpha}{2}\right) (t^2 - s^2)^{(\alpha-1)/2}. \blacksquare$$

1.2.2. Convolution of Kernels. Let \mathcal{R} denote the class of functions $F(x, s)$, defined on the closed square $S = \{a \leq x \leq b, a \leq s \leq b\}$ or on the closed triangle $T = \{a \leq s \leq x \leq b\}$, which are bounded on either domain and continuous there except for discontinuities which are regularly distributed. Then for any two functions $F(x, s)$ and $G(x, s)$, each in \mathcal{R} , the *type 1 convolution* denoted by $F * G$, and the *type 2 convolution* denoted by $F \circ G$, are defined as

$$F * G = \int_s^x F(x, t) G(t, s) dt, \quad (1.2.27)$$

$$F \circ G = \int_a^b F(x, t) G(t, s) dt, \quad (1.2.28)$$

respectively. The former is also called the convolution of finite variables or of first kind, while the latter the convolution on a finite interval or of second kind. The convolution $F \star G$ and $F \circ G$ are each continuous at all points on S or T . They satisfy the laws of combination and of distribution, namely,

$$\begin{aligned} F \star (G \star H) &= (F \star G) \star H, & F \circ (G \circ H) &= (F \circ G) \circ H; \\ F \star (G + H) &= F \star G + F \star H, & (G + H) \star F &= H \star F + G \star H, \\ F \circ (G + H) &= F \circ G + F \circ H, & (G + H) \circ F &= H \circ F + G \circ H, \end{aligned}$$

but the law of commutation does not, in general, apply to the convolution, i.e.,

$$F \star G \neq G \star F, \quad F \circ G \neq G \circ F.$$

However, there are cases where convolution is commutative. Thus, since in the case of a kernel $k(x, s)$ the order is not important, we shall write $k \star k = k^{\star 2}$, and $k \circ k = k^{\circ 2}$, and since $k \star k^{\star 2} = k^{\star 3} = k^{\star 2} \star k$, and $k \circ k^{\circ 2} = k^{\circ 3} = k^{\circ 2} \circ k$, we can define $k^{\star n}$ and $k^{\circ n}$ similarly. Thus,

$$k \star k^{\star n} = k^{\star n} \star k, \quad k \circ k^{\circ n} = k^{\circ n} \circ k;$$

also,

$$\begin{aligned} a_1 k + a_2 k^{\star 2} + a_3 k^{\star 3} + \cdots + a_n k^{\star n} + \cdots, \\ a_1 k + a_2 k^{\circ 2} + a_3 k^{\circ 3} + \cdots + a_n k^{\circ n} + \cdots \end{aligned}$$

are commutative with k provided these series are uniformly convergent. Since the kernel $k(x, s)$ is bounded, say $|k(x, s)| < M$, we have

$$\begin{aligned} |k^{\star 2}| &< (x - s)M^2, \quad |k^{\star 3}| < \frac{(x - s)^2}{2!} M^3, \quad \dots, \quad |k^{\star n}| < \frac{(x - s)^{n-1}}{(n-1)!} M^n, \\ |k^{\circ 2}| &< (b - a)M^2, \quad |k^{\circ 3}| < (b - a)^2 M^3, \quad \dots, \quad |k^{\circ n}| < (b - a)^{n-1} M^n, \end{aligned} \tag{1.2.29}$$

and so on. In particular, if $k(x, s) = 1$, $0 \leq x, s \leq 1$, then

$$\begin{aligned} 1^{\star 2} &= x - s, \quad 1^{\star 3} = \frac{(x - s)^2}{2!}, \quad \dots, \quad 1^{\star n} = \frac{(x - s)^{n-1}}{(n-1)!}, \quad \dots, \\ 1^{\circ 2} &= b - a, \quad 1^{\circ 3} = (b - a)^2, \quad \dots, \quad 1^{\circ n} = (b - a)^{n-1}, \quad \dots. \end{aligned}$$

For another example, let $k(x, s) = xs$, $0 \leq x, s \leq 1$. Then

$$\begin{aligned} k^{\star 2} &= \int_s^x xt \cdot ts dt = \frac{xs(x^3 - s^3)}{3}, \\ k^{\star 3} &= \int_s^x xt \cdot \frac{ts(t^3 - s^3)}{3} dt = \frac{xs(x^3 - s^3)^2}{3 \cdot 6}, \\ k^{\star 4} &= \int_s^x xt \cdot \frac{ts(t^3 - s^3)^2}{3 \cdot 6} dt = \frac{xs(x^3 - s^3)^3}{3 \cdot 6 \cdot 9}, \\ &\dots, \\ k^{\star n} &= \frac{xs(x^3 - s^3)^{n-1}}{3 \cdot 6 \cdot 9 \cdots (3n-3)} = \frac{xs(x^3 - s^3)^{n-1}}{3^{n-1} (n-1)!}; \\ k^{\circ 2} &= \int_0^1 xt \cdot ts dt = \frac{xs}{3}, \\ k^{\circ 3} &= \int_0^1 xt \cdot \frac{ts}{3} dt = \frac{xs}{3^2}, \\ &\dots, \\ k^{\circ n} &= \frac{xs}{3^{n-1}}. \end{aligned}$$

Since in this example $|k| < 1$, we note that $|k^{\star n}| < \frac{(x^3 - s^3)^{n-1}}{(n-1)!}$ and $|k^{\circ n}| < 1$.

NOTES: 1. Using the convolution notation we can represent an FK2 as $(I - \lambda K) \circ \phi = f$, and its adjoint equation as $\phi \circ (I - \lambda K) = f$; similarly, a VK2 can be represented as $(I - \lambda K) \star \phi = f$, and its adjoint equation as $\phi \star (I - \lambda K^*) = f$.

2. According to the definition of convolution of operational calculus (Faltung theorem), the Laplace transform of the convolution of $f_1(t)$ and $f_2(t)$ is defined as $\int_0^t f_1(t-\tau) f_2(\tau) d\tau$, which is the type 1 convolution, i.e.,

$$f_1 \star f_2 = \int_0^t f_1(t-s) f_2(s) ds, \quad (1.2.30)$$

which shows that the Laplace transform of the type 1 convolution is equal to the product of the Laplace transform of each function. A similar definition holds for the Fourier transform, but it leads to the type 2 convolution, namely,

$$f_1 \circ f_2 = \int_{-\infty}^{\infty} f_1(t-s) f_2(s) ds. \quad (1.2.31)$$

1.3. Function Spaces

The set of all functions $f(x)$ of the real variable x on an interval (a, b) , where $-\infty \leq a < b \leq +\infty$, is called the *function space* $L_2(a, b)$, and $f(x)$ is called an L_2 -function if

$$\int_a^b |f(x)|^2 dx < +\infty \quad (1.3.1)$$

in the Lebesgue sense. Two L_2 -functions f and g are *equivalent* if they are equal almost everywhere (a.e.), i.e., at almost all values of $x \in (a, b)$ except for values of Lebesgue measure zero. We write

$$f(x) = g(x) \iff \int_a^b [f(x) - g(x)]^2 dx = 0. \quad (1.3.2)$$

The ‘null’ function $h(x)$, which vanishes a.e. in (a, b) , is equal to the zero function, i.e., $h(x) = 0 \iff \int_a^b h^2(x) dx = 0$. Hence, the set of L_2 -functions forms a complete linear vector space, and with an appropriate norm and inner product the space L_2 is a Hilbert space (a complete normed inner product space). The norm $\|f\|_2$ (the L_2 -norm) of an L_2 -function f is defined as

$$\|f\|_2 = \left\{ \int_a^b |f(x)|^2 dx \right\}^{1/2}. \quad (1.3.3)$$

The suffix 2 shall be suppressed where the context is clear. Note that the condition $\|f\| = 0$ holds for the null function f . Thus,

- (i) $\|f\| \geq 0$,
- (ii) $\|f\| = 0 \iff f = 0$,
- (iii) $\|\lambda f\| = |\lambda| \|f\|$,
- (iv) $\|f + g\| \leq \|f\| + \|g\|$,

the last one being the Schwarz triangle inequality for two L_2 -functions f and g . The *inner product* $\langle f, g \rangle$ of two L_2 -functions f and g are defined as

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx, \quad (1.3.4)$$

where the bar denotes the complex conjugate. The inner product is a scalar-valued function such that

- (i) $\langle f, g \rangle = \overline{\langle g, f \rangle}$,
- (ii) $\langle \alpha f_1 + \beta f_2, g \rangle = \bar{\alpha} \langle f_1, g \rangle + \bar{\beta} \langle f_2, g \rangle$,
- (iii) $\langle f, f \rangle \geq 0$,
- (iv) $\langle f, f \rangle = 0 \iff f = 0$.

The functions f and g are orthogonal if $\langle f, g \rangle = 0$. The norm and inner product are related by

$$\|f\| = \langle f, f \rangle^{1/2}. \quad (1.3.5)$$

The distance $d(f, g)$ between two L_2 -functions f and g is defined as

$$d(f, g) = \|f - g\| = \langle f - g, f - g \rangle^{1/2}, \quad (1.3.6)$$

where d is a metric such that (i) $d(f, g) \geq 0$; (ii) $d(f, g) = 0 \iff f = g$; (iii) $d(f, g) = d(g, f)$; and (iv) $d(f, h) \leq d(f, g) + d(g, h)$, $h \in L_2$.

Fubini's theorem implies the interchange of the order of integration and leads to an important result in simplifying repeated integrals:

$$\int_0^x dt \int_0^t f(t, s) ds = \int_0^x ds \int_s^x f(s, t) dt. \quad (1.3.7)$$

If $k(x, s)$ is an L_2 -kernel and f an L_2 -function, then (i) $\phi(x) = \int k(x, s) \phi(s) ds$ is defined for almost all $x \in [a, b]$; and (ii) $|\phi(x)|^2 \leq \int |k(x, s)|^2 ds \int |\phi(s)|^2 ds$. Hence, by Fubini theorem, $\phi(x) \in L_2$ and

$$\int |\phi(x)|^2 dx \leq \iint |k(x, s)|^2 dx ds \int |\phi(s)|^2 ds,$$

and the linear operator $(K\phi)(x)$ maps an L_2 -function into an L_2 -function. We can, therefore, define the norm $\|K\|$ of a bounded operator K as

$$\|K\| = \sup_{\substack{\phi \in L_2 \\ \phi \neq 0}} \frac{\|K\phi\|}{\|\phi\|}, \quad (1.3.8)$$

which implies that for any L_2 -function ϕ

$$\|K\phi\| \leq \|K\| \cdot \|\phi\|. \quad (1.3.9)$$

Since

$$\|K\phi\| \leq \left[\int_a^b \int_a^b |k(x, s)|^2 dx ds \right]^{1/2} \cdot \|\phi\|,$$

we find the upper bound for $\|K\|$ as

$$\|K\| \leq \left[\int_a^b \int_a^b |k(x, s)|^2 dx ds \right]^{1/2} \equiv \|K\|_E, \quad (1.3.10)$$

where K_E is known as the euclidean norm of K such that

$$\|K\phi\| \leq \|K\|_E \cdot \|\phi\|. \quad (1.3.11)$$

Let K and H be two L_2 linear integral operators defined by

$$(K\phi)(x) = \int_a^b k(x, s)\phi(s) ds, \quad (H\phi)(x) = \int_a^b h(x, s)\phi(s) ds.$$

Then their sum $K + H$ is defined as

$$((K + H)\phi)(x) = \int_a^b (k(x, s) + h(x, s))\phi(s) ds,$$

where $K + H \in L_2$, and

$$\|K + H\| \leq \|K\| + \|H\|. \quad (1.3.12)$$

Similarly, let the product operator $L = KH$ be defined as

$$\begin{aligned} (L\phi)(x) &= (KH\phi)(x) = \int_a^b k(x, s)(H\phi)(s) ds \\ &= \int_a^b k(x, s) \int_a^b h(x, s)\phi(r) dr ds, \end{aligned}$$

which leads to

$$\|L\| \leq \|K\| \cdot \|H\|. \quad (1.3.13)$$

Finally, if

$$\int_a^b k_1(x, s)\phi(s) ds - \int_a^b k_2(x, s)\phi(s) ds = 0$$

for all ϕ , then $K_1 - K_2 = 0$, which implies that the operator relation $K_1 = K_2$ holds. This means that $K_1(x, s) = K_2(x, s)$ a.e. Set $K = K_1 - K_2$. Then for all $\phi \in L_2$,

$$\int_a^b k(x, s)\phi(s) ds = 0 \text{ a.e.} \quad (1.3.14)$$

For fixed x , take $\phi(s) = k(x, s)$. Then from (1.3.14) we have $\int_a^b |k(x, s)|^2 ds = 0$ a.e., that is,

$$\int_a^b \int_a^b |k(x, s)|^2 dx ds = 0,$$

or $\|K\| = 0$, which implies that $k(x, s) = 0$ for almost all x, s .

Note that the kernel $k(x, s)$ is supposed to be such that the integral operator K in (1.2.14) is completely continuous in the space $L_2(a, b)$; this occurs, e.g., if the double integral

$$B^2 = \int_a^b \int_a^b |k(x, s)|^2 dx ds \quad (1.3.15)$$

is finite. We make this assumption in many cases.

Consider the Lebesgue measure on the subset $X = [a, b]$ of the real line. The space $L_p(X)$, $1 \leq p < \infty$, is defined as the set of all measurable real- or complex-valued functions f defined on X , except possibly on a set of measure zero, such that $|f(x)|^p$ is integrable over X . The space $L_p(X)$ becomes a separable complete normed vector space if the functions that are equal up to a set of measure zero are regarded as the same function. The norm is defined by

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}. \quad (1.3.16)$$

The space $L_\infty(X)$ is defined as follows: Consider the measurable functions f defined on X , except possibly a set of measure zero. We say that f is *essentially bounded* if $|f(x)|$ is bounded from above for all x outside a set of measure zero. The *essential supremum* of f is the smallest of these upper bounds, denoted by $\text{ess sup}|f(x)|$. Thus, the space $L_\infty(X)$ consists of all essentially bounded measurable functions defined on X , where we identify functions that are equal almost everywhere. If a norm on $L_\infty(X)$ is defined by

$$\|f\|_\infty = \text{ess sup}|f(x)|, \quad (1.3.17)$$

then it is a complete normed vector space, but it is not separable as in the case of the space $L_p(X)$, $1 \leq p < \infty$. Using Hölder's inequality we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad (1.3.18)$$

which implies that if $1 \leq p_1 < p_2 < \infty$, then $L_\infty \subseteq L_{p_2} \subset L_{p_1} \subseteq L_1$ and if $f \in L_{p_2}$, then

$$\|f\|_{p_1} \leq \|f\|_{p_2} (b-a)^{1/p_1 - 1/p_2}. \quad (1.3.19)$$

If $f \in L_\infty(X)$, then $\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p$. If $b - a = 1$, we have an interesting case; the sequence $\{\|f\|_p\}_{p=1}^\infty$ is then monotonically increasing, and we have

$\|f\|_p \uparrow \|f\|_\infty$. For a complete discussion of the theory of L_p -spaces, see Köthe (1969) and Kreyszig (1978).

1.4. Convergence

We shall now present some results on sequences of functions and their convergence. A sequence of L_2 -functions $\{f_n\}$ converges to f

- (i) *strongly* iff $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$, and
 - (ii) *weakly* iff $\lim_{n \rightarrow \infty} \langle f_n, g \rangle = \langle f, g \rangle$ for all $g \in L_2$. The following basic results hold:
- (a) The sequence $\{f_n\}$ converges (weakly or strongly) in L_2 iff

$$\lim_{m,n \rightarrow \infty} (f_n - f_m) = 0. \quad (1.4.1)$$

In this case we say that $\{f_n\}$ is a *Cauchy* sequence; and

(b) If $\{f_n\}$ converges to f strongly, it also converges weakly to f , but the converse is not true. Thus, with the L_2 -norm we almost always use strong convergence in numerical computations; it is often called *convergence in the mean*. A strongly convergent computation safeguards against divergence at a particular point, such as a point of discontinuity or a badly behaving endpoint. However, convergence in other norms (L^1 , L^∞), which are defined as

$$\begin{aligned} \|f\|_p &= \left[\int_a^b |f(x)|^p dx \right]^{1/p}, \quad p \geq 1, \\ \|f\|_\infty &= \lim_{p \rightarrow \infty} \|f\|_p = \sup_{a \leq x \leq b} |f(x)|, \end{aligned} \quad (1.4.2)$$

is sometimes also important. Note that a weak convergence is very weak for most computations. In fact, let $\{h_n\}$ be an orthonormal sequence of functions. Then the following result holds: For any $f \in L_2$,

$$\sum_{n=0}^{\infty} |\langle h_n, f \rangle|^2 \leq \langle f, f \rangle, \quad (1.4.3)$$

and, as a corollary of (1.4.3),

$$\lim_{n \rightarrow \infty} \langle h_n, f \rangle = 0 \quad \text{for all } f \in L_2. \quad (1.4.4)$$

The inequality (1.4.3) is known as Bessel's inequality. It follows from (1.4.4) that every orthonormal sequence of functions in L_2 converges weakly to zero.

The above results (a) and (b) lead to the problem of expanding a given function in terms of a set of orthonormal functions. An orthonormal sequence $\{h_n\}$ is *complete* if for all n

$$\langle h_n, \phi \rangle = 0 \quad \text{implies that } \phi = 0.$$

In fact, let $\{h_i\}$ be a complete orthonormal sequence, and for any $\phi \in L_2$ let

$$\phi_n = \sum_{i=1}^n \langle h_i, \phi \rangle h_i. \quad (1.4.5)$$

Then the sequence $\{\phi_n\}$ converges strongly to ϕ and

$$\phi = \sum_{i=1}^n a_i h_i, \quad (1.4.6)$$

where

$$a_i = \langle h_i, \phi \rangle. \quad (1.4.7)$$

In practice, since we do not know the limit ϕ , the sequence of functions $\{\phi_n\}$ may yield a sequence of approximate solutions of an equation that may not have an L_2 -solution ϕ .

1.5. Inverse Operator

Consider an equation of the form

$$(I - \lambda K) \phi = f, \quad (1.5.1)$$

where $f \in L_2$ is given, and K is an integral operator from L_2 onto L_2 , which may, or may not, have a solution. Let $L = I - \lambda K$. If Eq (1.5.1) has a solution, it is unique; then L is said to have an *inverse* L^{-1} , defined by the equation

$$\phi = L^{-1} f = (I - \lambda K)^{-1} f, \quad (1.5.2)$$

(1.5.1) for any given f , and L^{-1} is an L_2 -operator. Conversely, if there exists an operator L^{-1} such that ϕ defined by (1.5.2) satisfies Eq (1.5.1), then L^{-1} is the inverse operator of L . Thus, $LL^{-1}f = L\phi = f$, or

$$LL^{-1} = I = L^{-1}L, \quad (1.5.3)$$

where I is the unit operator. For a given value of λ , if the operator L^{-1} satisfying (1.5.2) exists, then it is unique and so is the solution ϕ of Eq (1.5.1). Thus, if for a given $\lambda = \lambda_0$ the inverse operator L^{-1} exists such that (1.5.3) holds, then λ_0 is known as a *regular value* of the operator K . In particular, $\lambda = 0$ is a regular value of every kernel with inverse $L^{-1} \equiv I$.

If $k(x, s) \in L_2$, so does the *adjoint* (Hermitian adjoint, Hermitian conjugate) kernel $k^*(x, s)$, which is defined as

$$k^*(x, s) = \overline{k(s, x)}. \quad (1.5.4)$$

Thus,

- (i) $(k^*)^* = k$,
 - (ii) $\|k^*\| = \|k\|$,
 - (iii) $(\lambda k)^* = \bar{\lambda} k^*$,
 - (iv) $(k_1 + k_2)^* = k_1^* + k_2^*$.
- (1.5.5)

The adjoint operator K^* is associated with the adjoint kernel k^* just as the operator K is with the kernel k . The operator K^* also satisfies the relations (1.5.5). Also, since $(k_1 k_2)^*(x, s) = k_2^* k_1^*(x, s)$, we have

$$(K_1 K_2)^* = K_2^* K_1^*. \quad (1.5.6)$$

Moreover, for the inner product of any L_2 -functions f and g ,

$$\langle f, K^* g \rangle = \overline{\langle g, Kf \rangle}. \quad (1.5.7)$$

Note that the relations (1.5.5)–(1.5.7) resemble those for square matrices and their Hermitian adjoints.

If $K = K^*$, then K is said to be *Hermitian*; if it is real then it is also symmetric. The unit operator I is obviously symmetric, and $I^* = I$. The equation

$$\phi = \bar{f} + \bar{\lambda} K^* \phi \quad (1.5.8)$$

is called the *adjoint equation* of FK2.

THEOREM 1.5.1. If k is an L_2 -kernel, then λ is a regular value of k iff $\bar{\lambda}$ is a regular value of k^* . Also, if $(I - \lambda K)^{-1} = (L^{-1})$, then $(I - \bar{\lambda} K^*)^{-1} = (L^{-1})^*$.

The homogeneous equation

$$(I - \lambda K)\phi = 0 \quad (1.5.9)$$

always has a trivial solution $\phi = 0$. If it has a nontrivial solution ϕ_0 , then it cannot have a unique solution, and λ cannot be a regular value of k . In this case λ is called the *eigenvalue* and $\phi_0(x)$ the corresponding *eigenfunction*. We sometimes write the characteristic equation for a homogeneous FK2 as

$$K\phi = \mu\phi, \quad (1.5.10)$$

where $\mu = 1/\lambda$, defined in (1.2.3), is called a *characteristic value* and ϕ a *characteristic function*. We notice that there is a duality of notation in eigenvalues and characteristic values, but since μ simplifies certain numerical methods, we shall use both λ and μ as the need arises. However, we shall adopt the convention that λ is an eigenvalue of K if Eq (1.5.9) has a nontrivial solution. Thus, every λ is either a regular value or an eigenvalue. The computation of *eigenpairs* (i.e., eigenvalues and corresponding eigenfunctions) is discussed in Section 2.1.

1.6. Nyström System

Consider a quadrature rule

$$Q(F) = \sum_{j=0}^n w_j F(x_j), \quad a \leq x_j \leq b, \quad (1.6.1)$$

for computing the integral $\int_a^b F(x) dx$, where x_j are the quadrature points. If this rule is used to compute the integral occurring in an integral equation, we have

$$\int_a^b k(x, s)\phi(s) ds = \sum_{j=0}^n w_j k(x, s_j) \tilde{\phi}(s_j), \quad (1.6.2)$$

and the integral equation leads to the eigenvalue problem

$$\sum_{j=0}^n w_j k(x, s_j) \tilde{\phi}(s_j) = \tilde{\mu} \tilde{\phi}(x), \quad a \leq x \leq b. \quad (1.6.3)$$

If we can solve Eq (1.6.3), we obtain an approximate eigenvalue $\tilde{\lambda} = 1/\tilde{\mu}$ and an associated eigenfunction $\tilde{\phi}(x)$. The solution of Eq (1.6.3) is found by using the well-known Nyström method, which is as follows: Set $x = x_i$, $i = 0, 1, \dots, n$, in (1.6.3); this yields the system of $(n + 1)$ algebraic equations

$$\sum_{j=0}^n w_j k(x_i, s_j) \tilde{\phi}(s_j) = \tilde{\mu} \tilde{\phi}(x_i), \quad i = 0, 1, \dots, n, \quad (1.6.4)$$

where we shall call the points x_i as the Nyström points, $a \leq x_i \leq b$, and s_j are the above-mentioned quadrature points. The system (1.6.4) can be written in matrix form by setting $\mathbf{k} = [k(x_i, s_j)]$, $\tilde{\Phi} = [\tilde{\phi}(x_0), \tilde{\phi}(x_1), \dots, \tilde{\phi}(x_n)]^T$, and $\mathbf{D} = \text{diag}(w_0, w_1, \dots, w_n)$. Then the system (1.6.4) becomes the matrix eigenvalue problem

$$\mathbf{k} \mathbf{D} \tilde{\Phi} = \tilde{\mu} \tilde{\Phi}, \quad (1.6.5)$$

which, when solved, yields $(n + 1)$ eigenvalues $\tilde{\lambda}_m = 1/\tilde{\mu}_m$, $m = 0, 1, \dots, n$, and the associated eigenfunctions $\tilde{\phi}_m$. Note that the solution of the system (1.6.4) or (1.6.5) depends on the choice of the values of x_i and experience has shown that different choices of x_i and s_j generate different solutions, where the choice of s_j becomes fixed depending on the quadrature rule used. We have not seen any work which optimizes the choice for the Nyström points x_i that will yield the best solution. Although it is not necessary to choose the Nyström points x_i the same as the quadrature points s_j , $i, j = 0, 1, \dots, n$, it is usually done so except in the case when the kernel is not defined at $x = s$ or becomes unbounded there. The advantage of choosing x_i the same points as s_j is that the determinant of the matrix \mathbf{k} in (1.6.5) is symmetric if the kernel is symmetric.

If $\tilde{\mu}_m \neq 0$, the eigenvectors $\tilde{\phi}_m$ may be extended to a function $\tilde{\phi}_m(x)$ by substituting it in Eq (1.6.3), that is,

$$\tilde{\phi}_m(x) = \tilde{\lambda}_m \sum_{j=0}^n w_j k(x, s_j) \tilde{\phi}_m(s_j). \quad (1.6.6)$$

The function $\tilde{\phi}_m(x)$ is called the *Nyström extension* of the eigenvector $\tilde{\phi}_m$. This extension may, however, also be obtained by using a polynomial, piecewise-polynomial, or spline interpolation on the values $\tilde{\phi}_m(s_0), \tilde{\phi}_m(s_1), \dots, \tilde{\phi}_m(s_n)$, but the resulting function may not satisfy Eq (1.6.3) although it must satisfy the system (1.6.4).

EXAMPLE 1.6.1. Consider the eigenvalue problem (Baker 1978, p.171)

$$\int_0^1 k(x, s) \phi(s) ds = \mu \phi(x), \quad 0 \leq x \leq 1,$$

with a symmetric kernel defined for $0 \leq x \leq s \leq 1$ by

$$k(x, s) = \begin{cases} \frac{x}{2}(2-s) & \text{if } x \leq s, \\ \frac{s}{2}(2-x) & \text{if } x \geq s. \end{cases}$$

The exact eigenvalues are the roots of the equation $\sqrt{\lambda} + \tan \sqrt{\lambda} = 0$, with the corresponding eigenfunctions $\phi(x) = \sin(\sqrt{\lambda}x)$. Thus, the first four exact eigenvalues* are $\mu_1 = 0.24296$, $\mu_2 = 0.041426$, $\mu_3 = 0.015709$. Using a 3-point trapezoidal rule on $[0, 1]$ we have $Q(F) = 0.25 F(0) + 0.5 F\left(\frac{1}{2}\right) + 0.25 F(1)$, and choosing $x_0 = s_0 = 0$, $x_1 = s_1 = 1/2$, $x_2 = s_2 = 1$, we find that

$$\mathbf{kD} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3/16 & 1/16 \\ 0 & 1/8 & 1/8 \end{pmatrix},$$

so that the solution of the system (1.6.5) reduces to solving $\mathbf{kD} - \mu = 0$, which gives the approximate characteristic values as $\tilde{\mu}_1 = 1/4$, $\tilde{\mu}_2 = 1/16$, and $\tilde{\mu}_3 = 0$. Note that the eigenvector corresponding to μ_3 is $\tilde{\phi}_3 = [1, 0, 0]^T$. But since $\tilde{\mu}_3 = 0$, we cannot use the Nyström extension formula (1.6.4) to extend the eigenvector $\tilde{\phi}_3$ to a function $\tilde{\phi}_3(x)$. Even if we choose different values for x_0 , say, $x_0 = 1/10$ or $1/100$, we find the same characteristic values as above. If instead of the 3-point trapezoidal rule we use the 4-point trapezoidal, or Simpson's 3/8 rule, the situation remains the same; thus, in the case of the 4-point trapezoidal rule we find that $\tilde{\mu}_1 = 0.013889$, $\tilde{\mu}_2 = 0.123069$, $\tilde{\mu}_3 = 0.0250787$, and $\tilde{\mu}_4 = 0$, whereas in the case of Simpson's 3/8 rule we get $\tilde{\mu}_1 = 0.0245766$, $\tilde{\mu}_2 = 0.0514826$, $\tilde{\mu}_3 = 0.257274$, and $\tilde{\mu}_4 = 0$ (see `nystrom1.nb` for computational details), and again, we cannot use the Nyström extension formula (1.6.4) to extend the eigenvector $\tilde{\phi}_4$ to a function $\tilde{\phi}_4(x)$. This leads us to conclude that in order to extend an eigenvector $\tilde{\phi}_m$ to an eigenfunction $\tilde{\phi}_m(x)$ by the formula (1.6.6), it is necessary that the corresponding characteristic value $\tilde{\mu}_m$ be nonzero. ■

EXAMPLE 1.6.2. Consider the eigenvalue problem with the kernel defined on $(0, 1]$ by

$$k(x, s) = \begin{cases} -\sqrt{xs} \ln s, & x \leq s, \\ -\sqrt{xs} \ln x, & x \geq s. \end{cases}$$

*For the distinction between characteristic values μ and eigenvalues λ , see Section 2.1; for this kernel, see Example 2.1.2.

Since the kernel is undefined at $x = 0$, we shall use a 3-point trapezoidal rule as in the above example by choosing $x_0 = s_0 = 0$, $x_1 = s_1 = 1/2$, and $x_2 = s_2 = 1$. Then we have

$$\mathbf{k} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\ln 2)/2 & 0 \\ 0 & -(\ln 2)/\sqrt{2} & 0 \end{pmatrix},$$

and the solution of the system (1.6.5) gives $\tilde{\mu}_1 = 0.173287$, $\tilde{\mu}_2 = \tilde{\mu}_3 = 0$. If, in view of the fact that the kernel is not defined at $x = 0$, we opt to choose the Nyström points as $x_0 = 1/100$, $x_1 = 1/2$, and $x_2 = 1$, we still get the same values for $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$. Thus, it makes no difference if we choose a different Nystrom point x_0 . In fact, the point x_0 can be any point in $(0, 1/2)$. Hence, for this problem we cannot extend the eigenvectors $\tilde{\phi}_2$ and $\tilde{\phi}_3$. For computational details see `nystrom2.nb` and also Example 2.1.5. ■

An FK2 of the form $(\mu I - K)\phi = f$, $a \leq x \leq b$, where $k \in C[a, b]$ can be approximated by

$$\mu \tilde{\phi}(x) - \sum_{j=1}^n w_j k(x, s_j) \tilde{\phi}(s_j) = f(x), \quad a \leq x \leq b. \quad (1.6.7)$$

By replacing x by the Nyström points x_i , $i = 1, \dots, n$, we obtain a linear system

$$\mu \tilde{\phi}(x_i) - \sum_{j=1}^n w_j k(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 1, \dots, n, \quad (1.6.8)$$

or in matrix form

$$(\mu \mathbf{I} - \mathbf{kD}) \tilde{\Phi} = \mathbf{f}, \quad (1.6.9)$$

where \mathbf{I} is the identity matrix. Nyström (1930) made the following observation: Each solution $\tilde{\phi}(x)$ of (1.6.7) leads to the solution $\{\tilde{\phi}(x_1), \dots, \tilde{\phi}(x_n)\}$ of (1.6.8). More precisely, to each solution, say, $\{z_1, \dots, z_n\}$ of (1.6.8), there corresponds a unique solution $z(x)$ of (1.6.7) with which it matches at the points s_1, \dots, s_n in the interval $[a, b]$. To see this, define (recall that $\mu = 1/\lambda$)

$$z(x) = \lambda f(x_i) + \lambda \sum_{j=1}^n w_j k(x, s_j) z_j, \quad a \leq x \leq b, \quad (1.6.10)$$

under the assumption that $\{z_1, \dots, z_n\}$ is a solution of the system (1.6.8) or (1.6.9). Then

$$z(x_i) = \lambda f(x_i) + \lambda \sum_{j=1}^n w_j k(x_i, s_j) z_j = z_i, \quad i = 1, \dots, n. \quad (1.6.11)$$

Thus, z_i can be replaced by $\tilde{\phi}(x_i)$ in (1.6.10). It is easy to note that (1.6.10) is a rearrangement of (1.6.7), and $z(x)$ is a solution of (1.6.7). Also, if the two solutions of (1.6.7) match at the points s_1, \dots, s_n , then they match at all points of $[a, b]$ (this can be seen by solving (1.6.7) for $\tilde{\phi}(x)$).

EXAMPLE 1.6.3. Consider the FK2

$$\phi(x) + \int_0^1 \sqrt{xs} \phi(s) ds = \sqrt{x}, \quad 0 \leq x \leq 1,$$

whose exact solution is $\phi(x) = 2\sqrt{x}/3$. We use the 3-point trapezoidal rule with $x_0 = s_0 = 0$, $x_1 = s_1 = 1/2$, and $x_2 = s_2 = 1$, and solve the system $(\mathbf{I} + \mathbf{kD}) \tilde{\Phi} = \mathbf{f}$. The results are $\tilde{\phi}_1 = 0$, $\tilde{\phi}_2 = 0.471405$ and $\tilde{\phi}_3 = 0.666667$, which match the exact solution. For computational details see `nystrom3.nb`. Note that the method used in this example is known as the quadrature method (see Section 4.3). ■

Some GUIDELINES for the success of the Nyström method are as follows:

- (i) Choice of the quadrature rule to compute the solution is very important. We should choose a rule with positive weights w_j , $j = 0, 1, \dots, n$, so that $\mathbf{D} > 0$.
- (ii) A real and symmetric kernel implies a real and symmetric \mathbf{k} , but the product \mathbf{kD} is not symmetric. Assume that $\mathbf{D} > 0$, and consider the matrix $\mathbf{D}^{1/2} \mathbf{kD}^{1/2}$, which is real and symmetric and similar to \mathbf{kD} . Since

$$\left(\mathbf{D}^{1/2} \mathbf{kD}^{1/2} \right) \mathbf{D}^{1/2} \tilde{\Phi} = \tilde{\mu} \mathbf{D}^{1/2} \tilde{\Phi}, \quad (1.6.12)$$

we can start with solving the system $\mathbf{D}^{1/2} \mathbf{kD}^{1/2} \tilde{\Phi} = \tilde{\mu} \tilde{\Psi}$ in the case of eigenvalue problems for eigenvectors $\{\tilde{\Psi}\}$ and then recover $\tilde{\Phi}$ as $\mathbf{D}^{1/2} \tilde{\Psi}$. A similar situation applies to the case of solving an integral equation by the Nyström method.

- (iii) If the kernel $k(x, s)$ is Hermitian, then $\mathbf{D}^{1/2} \mathbf{kD}^{1/2}$ is a Hermitian matrix, and the technique described in (ii) is useful in this case also.
- (iv) If the kernel is nonsymmetric, then the above technique is useless.
- (v) Let the matrix $\mathbf{A} = \mathbf{D}^{1/2} \mathbf{kD}^{1/2}$. Then, in terms of the condition number $\rho(\mathbf{A})$ (see Section A.8) it can be shown that $\rho(\tilde{\mu}) = \rho(\mu)$ (see Baker 1978, p.174). Note that $0 < \rho(\mu) \leq 1$ only for a simple characteristic value μ . If $\rho(\mu)$ is small, then the characteristic value μ is ill-conditioned, so is a multiple characteristic value of a non-Hermitian kernel.
- (vi) If $k(x, s)$ is Hermitian and μ is a simple characteristic value, then $\rho(\mu) = 1$. If \mathbf{A} is Hermitian and $\tilde{\mu}$ is simple, then $\rho(\tilde{\mu}) = 1$.
- (vii) Discretization errors play a significant role in the choice of the quadrature rule. Thus, when the product $k(x, s)\phi(s)$ is sufficiently smooth, the repeated

trapezoidal rule with step size $h = (b - a)/n$ gives discretization errors of order $O(h^2)$. Therefore, it may be advantageous to take smaller values of h .

- (viii) In the case when $k(x, s) \in C^\infty[a, b]$, the eigenfunctions corresponding to nonzero characteristic values are also continuously differentiable. Higher-order Gauss–Legendre rule should be used in such cases.
 - (ix) In the case when $k(x, s)$ has a discontinuity in some derivative, the relative merits of different quadrature rules must be verified.
-

1.7. Other Types of Kernels

Here we distinguish between nondegenerate and degenerate kernels and present some related results.

1.7.1. Nondegenerate Kernels. Some theorems on nondegenerate kernels are stated below; their proofs are available in most standard textbooks, for example, Kanwal (1997).

THEOREM 1.7.1. If $k(x, s)$ satisfies a linear differential equation $L[k(x, s)] = 0$, where $L = D^n + p_1(x)D^{n-1} + \dots + p_n(x)$, $D = \frac{\partial}{\partial x}$, then $f(x)$ must satisfy the same differential equation.

The proof is obvious (see Fox and Goodwin 1953): Assuming that we can interchange the order of integration with respect to s and differentiation with respect to x , then applying the operator L to both sides of Eq (1.2.1), we have

$$Lf(x) = L \left[\int_a^b k(x, s)\phi(s) ds \right] = \int_a^b L[k(x, s)]\phi(s) ds = 0.$$

RIESZ–FISCHER THEOREM. If $\{u_\nu(x)\}$ is an orthonormal system of L_2 functions in the interval $[a, b]$, and $\{h_n\}$ is a sequence such that the series $\sum_{n=1}^{\infty} |h_n|^2$ converges, then there exists a unique function $f(x) \in L_2[a, b]$ such that

$$f(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n h_i u_i(x). \quad (1.7.1)$$

An implication of the Riesz–Fischer theorem is the GENERALIZED FOURIER SERIES

$$\sum_{i=1}^{\infty} h_i u_i(x) \quad (1.7.2)$$

of an arbitrary L_1 -function $f(x)$ in $[a, b]$ with respect to a complete orthonormal system $\{u_\nu(x)\}$, which is convergent in the mean to the function $f(x)$ whose Fourier coefficients satisfy Parseval's identity.

A function $f(x)$ is said to be *orthogonal* to a symmetric kernel $k(x, s)$ if $\int_a^b k(x, s) f(s) ds = 0$ implies that

$$\int_a^b k(s, x) \overline{f(s)} ds = 0. \quad (1.7.3)$$

LEMMA 1.7.1. For a function $h \in L_2[a, b]$ to be orthogonal to a symmetric kernel $k(x, s)$, it is necessary and sufficient that it be orthogonal to all eigenfunctions of the kernel.

HILBERT–SCHMIDT THEOREM. Every function $f(x)$ of the form

$$f(x) = \int_a^b k(s, x) h(s) ds \quad (1.7.4)$$

is equal to $\sum_{n=1}^{\infty} c_n \phi_n(x)$ almost everywhere, where the kernel $\phi_n(x)$ is the orthonormal system of eigenfunctions of the symmetric L_2 -kernel $k(x, s)$, c_n are constants, and $h(x)$ is an L_2 -function.

A symmetric kernel $k(x, s)$ is said to be *closed* if there does not exist a function $h(x)$ with a nonzero norm satisfying the equation

$$\int_a^b \int_a^b k(x, s) \overline{h(x)} h(s) dx ds = 0. \quad (1.7.5)$$

A symmetric kernel $k(x, s)$ is said to be *definite* if there does not exist a function $h(x)$ with a nonzero norm satisfying the equation

$$\int_a^b \int_a^b k(s, x) h(s) ds = 0. \quad (1.7.6)$$

Every definite kernel is closed. A symmetric kernel $k(x, s)$ is said to be *positive* if for every L_2 -function $h(x)$ in $[a, b]$

$$\int_a^b \int_a^b k(x, s) \overline{h(x)} h(s) dx ds \geq 0. \quad (1.7.7)$$

MERCER'S THEOREM. A positive, continuous, and L_2 -kernel $k(x, s) \in L_2[a, b]$ can be expressed as

$$k(x, s) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \overline{\phi_n(s)}}{\lambda_n}, \quad (1.7.8)$$

where the Fourier series in Eq (1.7.8) is uniformly and absolutely convergent with respect to both x and s .

THEOREM 1.7.2. Eq (1.2.1) with a closed symmetric kernel possesses a unique solution in $L_2[a, b]$ if and only if the series of constants

$$\sum_{n=1}^{\infty} |\lambda_n f_n| \quad (1.7.9)$$

converges, where λ_n are the eigenvalues of the kernel $k(x, s)$ and f_n are Fourier coefficients of $f(x)$ with respect to the orthonormal system $\phi_n(x)$ of the eigenfunctions of the kernel, given by

$$f_n = \int_a^b f(s) \overline{\phi_n(s)} ds. \quad (1.7.10)$$

If the kernel $k(x, s) \in L_2[a, b]$ is not symmetric, then the following theorem applies:

SCHMIDT'S THEOREM. Let $\phi_n(x)$ and $\psi_n(x)$ satisfy the following integral equations

$$\phi_n(x) = \lambda_n \int_a^b \overline{k(s, x)} \psi_n(s) ds, \quad \psi_n(x) = \lambda_n \int_a^b k(x, s) \phi_n(s) ds, \quad (1.7.11)$$

respectively. Then every function $f(x)$ having one of the two forms

$$f(x) = \int_a^b k(x, s) h(s) ds \quad \text{or} \quad f(x) = \int_a^b \overline{k(s, x)} h(s) ds, \quad (1.7.12)$$

is the sum of its Fourier series with respect to the system $\psi_n(x)$ in the case of the first form and with respect to the system $\phi_n(x)$ in the case of the second form, where it is assumed that both $k(x, s)$ and $h(x)$ are L_2 -functions.

1.7.2. Degenerate Kernels. The Nyström formula (1.6.4) also provides approximate solutions for equations with *non-Hermitian kernels*. However, the resulting matrix problem becomes more difficult because a non-Hermitian kernel may have complex eigenvalues or it may have no eigenvalue. For example, consider $k(x, s) = g(x)h(s)$, $\langle g, h \rangle = 0$. Then every finite λ is a regular value of the kernel. Since eigenvalues of k (or of k^*) may not exist, we shall introduce *singular values* γ and *singular functions* $\{u, v\}$ of the kernel k , as defined by

$$u = \gamma K v, \quad v = \gamma k^* u, \quad (1.7.13)$$

with the following properties:

- (i) γ is real;
- (ii) $(I - \gamma^2 k k^*) u = 0$, and $(I - \gamma^2 k^* k) v = 0$;
- (iii) If $\{u, v\}$ is a pair of singular functions of k corresponding to the singular value γ , then $\{u, -v\}$ is a pair of singular functions of k corresponding to the singular value $-\gamma$;
- (iv) If $\{u, v\}$ corresponds to γ and $\{u', v'\}$ to γ' and $\gamma^2 \neq (\gamma')^2$, then $\langle u, u' \rangle = 0 = \langle v, v' \rangle$.

Eqs (1.7.13) imply that the singular functions u, v are eigenfunctions of the Hermitian kernels kk^* and k^*k , respectively; in fact, if $k = k^*$, then $u = v = \phi$. In many solutions we shall compute only the singular values and corresponding singular functions, instead of the eigenpairs.

Let $\{a_i(s)\}, \{b_i(s)\}$ be two sequences of linearly independent L_2 -functions. Then a kernel of the form

$$k^{[n]}(x, s) = \sum_{i=1}^n a_i(x) \overline{b_i(s)} \quad (1.7.14)$$

is said to be of *finite rank n*. It is also called a *degenerate* or *separable* kernel. We shall rewrite (1.7.14) by using the notation $k^{[n]} = \sum_{i=1}^n a_i \otimes b_i$. Then

$$\begin{aligned} \alpha k^{[n]} &= \sum_{i=1}^n (\alpha a_i) \otimes b_i = \sum_{i=1}^n a_i \otimes (\bar{\alpha} b_i), \\ k^{*[n]} &= \sum_{i=1}^n b_i \otimes a_i, \\ k^{[n]} \phi &= \sum_{i=1}^n a_i \langle b_i, \phi \rangle a_i, \\ h k^{[n]} &= \sum_{i=1}^n (h a_i) \otimes b_i, \quad k^{[n]} h = \sum_{i=1}^n a_i \otimes (h^* b_i). \end{aligned} \quad (1.7.15)$$

Consider the FK2: $\phi = f + \lambda K\phi$, where $f \in L_2$. Let $\langle b_j, f \rangle = f_j$, $\langle b_j, a_i \rangle = \kappa_{ji}$, $1 \leq i, j \leq n$. Then, if ϕ is an L_2 -solution of the FK2 and $\phi_j = \langle b_j, \phi \rangle$, then

$$\phi_j = f_j + \lambda \sum_{i=1}^n \kappa_{ji} \phi_i. \quad (1.7.16)$$

Conversely, if ϕ_j satisfies (1.7.16), then

$$\phi = f + \lambda \sum_{i=1}^n \phi_i a_i$$

is an L_2 -solution of this FK2 with $\phi_j = \langle b_j, \phi \rangle$. This result reduces the solution of an FK2 with finite-rank kernel to that of a system of algebraic equations (1.7.16).

EXAMPLE 1.7.1. Consider

$$\phi(x) = e^x - \frac{e}{2} + \frac{1}{2} + \frac{1}{2} \int_0^1 \phi(s) ds.$$

We have $k = 1$, $\lambda = 1/2$, $n = 1$, $a_1(s) = 1 = b_1(s)$. Then

$$\begin{aligned} \kappa_{11} &= \int_0^1 a_1 b_1 ds = 1, \\ f_1 &= \int_0^1 \left(e^x - \frac{e}{2} + \frac{1}{2} \right) dx = \frac{1}{2}(e-1), \\ \phi_1 &= f_1 + \frac{1}{2} \cdot 1 \cdot \phi_1, \quad \text{which gives } \phi_1 = 2f_1 = e-1. \end{aligned}$$

Thus, $\phi(x) = e^x - \frac{e}{2} + \frac{1}{2} + \frac{1}{2} \cdot 1 \cdot (e-1) = e^x$. ■

EXAMPLE 1.7.2. Consider the same equation as in Example 1.7.1, but with $n = 2$, $a_1(s) = 1/2 = a_2(s)$, and $b_1(x) = 1 = b_2(x)$. Then

$$\begin{aligned} \kappa_{11} &= \frac{1}{2} = \kappa_{12} = \kappa_{21} = \kappa_{22}, \\ f_1 &= \int_0^1 1 \cdot \left(e^x - \frac{e}{2} + \frac{1}{2} \right) dx = \frac{e-1}{2} = f_2, \\ \phi_1 &= f_1 + \lambda \sum_{i=1}^2 \kappa_{1i} \phi_i = f_1 + \frac{1}{2} (\kappa_{11}\phi_1 + \kappa_{12}\phi_2) = f_1 + \frac{1}{4} (\phi_1 + \phi_2), \end{aligned}$$

which gives $\frac{3}{4} \phi_1 - \frac{1}{4} \phi_2 = f_1$,

$$\phi_2 = f_2 + \lambda \sum_{i=1}^2 \kappa_{2i} \phi_i = f_2 + \frac{1}{2} (\kappa_{21}\phi_1 + \kappa_{22}\phi_2) = f_2 + \frac{1}{4} (\phi_1 + \phi_2),$$

which gives $\frac{3}{4} \phi_2 - \frac{1}{4} \phi_2 = f_2$;

the last two equations yield $\phi_1 = \phi_2 = e - 1$. Hence,

$$\phi(x) = f + \lambda \sum_{i=1}^2 a_i \phi_i = e^x - \frac{e}{2} + \frac{1}{2} + \frac{1}{2} [\frac{1}{2} (e-1) + \frac{1}{2} (e-1)] = e^x. \blacksquare$$

1.8. Neumann Series

If λ is a regular value of K , then an FK2 has the unique solution

$$\phi = (I - \lambda K)^{-1} f = L^{-1} f. \quad (1.8.1)$$

In fact, it can be shown that any given value of λ is a regular value if we can explicitly obtain the inverse operator L^{-1} or the unique solution. Since $\lambda = 0$ is always a regular value, we can formally represent L^{-1} as a power series in λ as

$$L^{-1} = (I - \lambda K)^{-1} = I + \sum_{n=1}^{\infty} \lambda^n K^n. \quad (1.8.2)$$

Alternatively, an FK2 can be solved for ϕ by successive approximations. For $\lambda = 0$ the unique solution is $\phi = f$. Now, define successive approximations ϕ_n of ϕ by setting

$$\phi_0(x) = f(x),$$

$$\phi_1(x) = f(x) + (\lambda K) \phi_0(x) = f(x) + \int_a^b k(x, s) f(s) ds,$$

$$\phi_2(x) = f(x) + (\lambda K) \phi_1(x) = f(x) + (\lambda^2 K^2) f(x)$$

$$= f(x) + \lambda^2 \int_a^b \int_a^b k(x, s_1) k(s_1, s_2) f(s_2) ds_2 ds_1,$$

and so on, or, in general, by the recurrence relation

$$\phi_n(x) = f(x) + (\lambda K) \phi_{n-1}(x) = f(x) + \sum_{i=1}^n \lambda^i u_i(x), \quad (1.8.3)$$

where

$$u_i(x) = \int_a^b \cdots \int_a^b k(x, s_1) k(s_1, s_2) \cdots k(s_{i-1}, s_i) f(s_i) ds_i \dots ds_1. \quad (1.8.4)$$

Although the relation (1.8.3) formally defines a sequence of functions ϕ_n that may or may not converge as $n \rightarrow \infty$, it, however, suggests the following representation for ϕ :

$$\phi = \sum_{i=0}^{\infty} \lambda^i K^i f. \quad (1.8.5)$$

This representation is known as the *Neumann series* for the solution ϕ of an FK2. We would obtain the same series if we were to apply the series (1.8.2) to (1.8.1). Thus, (1.8.2) is also called the Neumann series for the inverse operator L^{-1} . If, for some fixed λ and all f , the series (1.8.5) converges to a solution of FK2, or if the series (1.8.2) converges to an operator satisfying the relation (1.5.2), then λ is a regular value of K . In fact, the Neumann series (1.8.2) for L^{-1} converges strongly if $\|\lambda K\| < 1$. This can be proved as follows: If we truncate the series (1.8.5) after n terms, and denote

$$\phi_n = \sum_{i=0}^n \lambda^i K^i f, \quad (1.8.6)$$

then the recurrence relations

$$\begin{aligned} \phi_0 &= f, \\ \phi_{n+1} &= f + \lambda K \phi_n, \end{aligned} \quad (1.8.7)$$

provide an iterative scheme which is easy to use provided the integral $K \phi_n = \int_a^b k(s, x) \phi_n(s) ds$ can be computed by using a suitable quadrature rule. Let $e_n = \phi_n - \phi$ denote the error due to this truncation. Then subtracting (1.8.7) from the FK2 in (1.2.15), we get $\phi - \phi_{n+1} = \lambda K (\phi - \phi_n)$, or

$$e_{n+1} = \lambda K e_n. \quad (1.8.8)$$

Also, since $\phi_{n+1} - \phi_n = \lambda K (\phi_n - \phi_{n+1}) = e_n - e_{n+1}$, we find that $e_n = e_{n+1} - (\phi_{n+1} - \phi_n)$, which gives

$$\begin{aligned} \|e_n\| &\leq \|e_{n+1}\| + \|\phi_{n+1} - \phi_n\| \\ &\leq \|\lambda K\| \cdot \|e_n\| + \|\phi_{n+1} - \phi_n\|, \end{aligned}$$

or

$$\|e_n\| \leq \frac{\|\phi_{n+1} - \phi_n\|}{1 - \|\lambda K\|}. \quad (1.8.9)$$

This provides an upper bound for the error provided we can estimate $\|\lambda K\|$ and $\|\lambda K\| < 1$. Thus, the above iterative scheme is computable provided $\|\lambda K\| < 1$. For more on the application of a quadrature rule to the above iterative scheme, see Section 6.4.

1.9. Resolvent Operator

Sometimes, instead of the inverse operator $L^{-1} \equiv (I - \lambda K)^{-1}$ (see Section 1.5), the existence of solutions of an FK2 is discussed in terms of the *resolvent operator* H defined by

$$L^{-1} = I - \lambda H, \quad (1.9.1)$$

or, in view of (1.5.3), equivalently by the *resolvent equations*

$$H - K = \lambda K H = \lambda H K. \quad (1.9.2)$$

Thus, we have two choices to develop the theory of the FK2, either in terms of L^{-1} or in terms of H . But it sometimes becomes convenient to introduce H , for example, in the Neumann series (1.8.2), which can be written as a series for H as

$$H_\lambda = K + \sum_{i=1}^{\infty} \lambda^i K^{i+1}. \quad (1.9.3)$$

Note that

$$\lim_{\lambda \rightarrow 0} H_\lambda = K, \quad (1.9.4)$$

and the i th term in (1.9.3) is given by

$$H_i \equiv \lambda^i K^{i+1}, \quad (1.9.5)$$

where K^i for an $i = 1, 2, \dots$ is an integral operator for an FK2 with an L_2 -kernel $k_i(x, s)$, i.e.,

$$(K^i \phi)(x) = \int_a^b k_i(x, s) \phi(s) ds, \quad (1.9.6)$$

and this kernel $k_i(x, s)$ satisfies the operator relation $K^i = K K^{i-1}$, and hence the recurrence relation

$$k_i(s, x) = \int_a^b k(x, r) k_{i-1}(r, x) dr. \quad (1.9.7)$$

Thus, the resolvent operator H has an associated kernel, called the *resolvent kernel* and denoted by $h_\lambda(s, x)$, which is formally represented as the series

$$h_\lambda(s, x) = k(x, s) [I + \sum_{i=1}^{\infty} \lambda^i k_i(x, s)]. \quad (1.9.8)$$

In practice, however, (1.9.7) is not used to compute the inverse operator L^{-1} by reducing the computations to the resolvent operator H . The reason is that the series (1.9.8) is often possibly convergent only for small $|\lambda|$. Also, notice that in the range $\rho(K)$ of regular values of K the resolvent kernel $h_\lambda(s, x)$ is an analytic function of λ for each s, x , and at any point $\lambda_0 \in \rho(K)$ the function $h_\lambda(s, x)$ can be represented by the absolutely convergent power series in $(\lambda - \lambda_0)$.

The resolvent of a kernel $k(x, s)$ shall be denoted by $k_\lambda(x, s)$. The solution of an FK2 of the form (1.2.2) is given by

$$\phi(x) = f(x) + \lambda \int_a^b k_\lambda(x, s) \phi(s) ds, \quad a \leq x, s \leq b, \quad (1.9.9)$$

where

$$k_\lambda(x, s) = \frac{d_\lambda(x, s)}{D(\lambda)}, \quad (1.9.10)$$

$$d_\lambda(x, s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x, s) \lambda^n, \quad D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \lambda^n, \quad (1.9.11)$$

such that

$$A_0(x, s) = k(x, s),$$

$$A_n(x, s) = \underbrace{\int_a^b \cdots \int_a^b}_{n \text{ times}} \begin{vmatrix} k(x, s) & k(x, s_1) & \cdots & k(x, s_n) \\ k(s_1, s) & k(s_1, s_1) & \cdots & k(s_1, s_n) \\ \cdots & \cdots & \cdots & \cdots \\ k(s_n, s) & k(s_n, s_1) & \cdots & k(s_n, s_n) \end{vmatrix} ds_1 \cdots ds_n, \quad (1.9.12)$$

$$B_0 = 1,$$

$$B_n = \underbrace{\int_a^b \cdots \int_a^b}_{n \text{ times}} \begin{vmatrix} k(s_1, s_1) & k(s_1, s_2) & \cdots & k(s_1, s_n) \\ k(s_2, s_1) & k(s_2, s_2) & \cdots & k(s_2, s_n) \\ \cdots & \cdots & \cdots & \cdots \\ k(s_n, s_1) & k(s_n, s_2) & \cdots & k(s_n, s_n) \end{vmatrix} ds_1 \cdots ds_n, \\ n = 0, 1, 2, \dots \quad (1.9.13)$$

The function $d_\lambda(x, s)$ is known as the *Fredholm minor* and $D(\lambda)$ as the *Fredholm determinant*; their series converge for all values of λ , and hence, they are entire (analytic) functions of λ . The resolvent $k_\lambda(x, s)$ is an analytic function of λ for all values of λ except those that are the zeros of $D(\lambda)$. These zeros are the eigenvalues of the FK2 and poles of the resolvent $k_\lambda(x, s)$.

In most problems the formulas (1.9.12) and (1.9.13) are not very convenient to use. However, they are equivalent to the following recurrence relations:

$$\begin{aligned} A_n(x, s) &= B_n k(x, s) - n \int_a^b k(x, t) A_{n-1}(t, s) dt, \\ B_n &= \int_a^b A_{n-1}(t, t) dt. \end{aligned} \quad (1.9.14)$$

EXAMPLE 1.9.1. Compute the resolvent of the kernels (i) $k(x, s) = xe^s$, and (ii) $k(x, s) = x - 2s$, $0 \leq x, s \leq 1$.

Solution of (i): We have $A_0(x, s) = xe^s$; $B_0 = 1$. Now, using (1.9.14), we find that $B_1 = \int_0^1 te^t dt = 1$, and $A_1(x, s) = B_1 xe^s - \int_0^1 xte^{t+s} dt = 0$, and $B_0 = \int_0^1 A_1(t, t) dt = 0$; so $B_n = 0$ for $n \geq 2$. Also, $A_n(x, s) = 0$ for $n \geq 1$. Hence,

$$d_\lambda(x, s) = A_0(x, s) = xe^s, \quad D(\lambda) = B_0 - B_1 \lambda = 1 - \lambda,$$

which give

$$k_\lambda(x, s) = \frac{d_\lambda(x, s)}{D(\lambda)} = \frac{xe^s}{1 - \lambda}.$$

Note that if formulas (1.9.12) and (1.9.13) were used, then

$$\begin{aligned} A_1(x, s) &= \int_0^1 \begin{vmatrix} xe^s & xe^{s_1} \\ s_1 e^s & s_1 e^{s_1} \end{vmatrix} ds_1 = 0, \\ A_2(x, s) &= \int_0^1 \int_0^1 \begin{vmatrix} xe^s & xe^{s_1} & xe^{s_2} \\ s_1 e^s & s_1 e^{s_1} & s_1 e^{s_2} \\ s_2 e^s & s_2 e^{s_1} & s_2 e^{s_2} \end{vmatrix} ds_1 ds_2 = 0, \end{aligned}$$

because in each case the integrands (determinants) are zero. So, $A_n(x, s) = 0$ for $n \geq 1$. Also,

$$\begin{aligned} B_1 &= \int_0^1 k(s_1, s_1) ds_1 = \int_0^1 s_1 e^{s_1} ds_1 = 1, \\ B_2 &= \int_0^1 \int_0^1 \begin{vmatrix} k(s_1, s_1) & k(s_1, s_2) \\ k(s_2, s_1) & k(s_2, s_2) \end{vmatrix} ds_1 ds_2 = 0. \end{aligned}$$

Thus, $B_n = 0$ for $n \geq 2$.

Solution of (ii): We have $B_0 = 1$ and $A_0(x, s) = x - 2s$. Now, using (1.9.14),

$$\begin{aligned} B_1 &= A_0(t, t) dt = \int_0^1 (-t) dt = -\frac{1}{2}, \\ A_1(x, s) &= B_1(x - 2s) - \int_0^1 (x - 2t) A_0(t, s) dt = -x - s + 2xs + \frac{2}{3}, \\ B_2 &= \int_0^1 A_1(t, t) dt = \int_0^1 \left(-2t + 2t^2 + \frac{2}{3} \right) dt = \frac{1}{3}, \\ A_2(x, s) &= B - 2k(x, s) - 2 \int_0^1 k(x, t) A_1(t, s) dt \\ &= \frac{1}{3}(x - 2s) - 2 \int_0^1 (x - 2s) \left(-t - s + 2ts + \frac{2}{3} \right) dt = 0, \\ B_3 &= B_4 = \dots = 0, \quad A_3(x, s) = A_4(x, s) = \dots = 0. \end{aligned}$$

Thus,

$$d_\lambda(x, s) = x - 2s + \lambda \left(x + s - 2xs - \frac{2}{3} \right), \quad D(\lambda) = 1 + \frac{1}{2}\lambda + \frac{1}{6}\lambda^2,$$

and

$$k_\lambda(x, s) = \frac{d_\lambda(x, s)}{D(\lambda)} = \frac{x - 2s + \lambda(x + s - 2xs - 2/3)}{1 + \frac{1}{2}\lambda + \frac{1}{6}\lambda^2}.$$

Note that in (i) the solution of the equation is

$$\phi(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 x e^s f(s) ds, \quad 0 \leq x \leq 1, \quad \lambda \neq 1.$$

In particular, if $f(x) = e^{-x}$, then the solution is

$$\phi(x) = e^{-x} + \frac{\lambda}{1-\lambda} x, \quad 0 \leq x \leq 1, \quad \lambda \neq 1.$$

In (ii) the solution is of the form

$$\phi(x) = f(x) + \frac{\lambda}{1 + \frac{1}{2}\lambda + \frac{1}{6}\lambda^2} \int_0^1 [x - 2s + \lambda(x + s - 2xs - 2/3)] f(s) ds,$$

$$0 \leq x \leq 1,$$

which is valid when $\lambda^2 + 3\lambda + 6 \neq 0$. ■

EXAMPLE 1.9.2. We shall solve the FK2

$$\phi(x) - \lambda \int_0^1 xs \phi(s) ds = f(x), \quad 0 \leq x \leq 1,$$

by the method of successive approximations using its resolvent. Here $k(x, s) = xs$. Define successively

$$k_1(x, s) = xs,$$

$$k_2(x, s) = \int_0^1 (xt)(ts) dt = \frac{xs}{3},$$

$$k_3(x, s) = \int_0^1 (xt) \left(\frac{ts}{3}\right) dt = -\frac{xs}{3^2},$$

$$\dots \dots \dots,$$

$$k_n(x, s) = \frac{xs}{3^{n-1}}.$$

Hence, the resolvent $k + \lambda(x, s)$ is given by

$$k_\lambda(x, s) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x, s) = xs \sum_{n=1}^{\infty} \left(\frac{\lambda}{3}\right)^{n-1} = \frac{3xs}{3-\lambda}, \quad |\lambda| < 3.$$

This yields the solution as

$$\phi(x) = f(x) + \lambda \int_0^1 \frac{3xs}{3-\lambda} f(s) ds, \quad 0 \leq x \leq 1, \lambda \neq 3.$$

In particular, if $f(x) = x$, we get

$$\phi(x) = \frac{3x}{3-\lambda}, \quad 0 \leq x \leq 1, \lambda \neq 3. ■$$

1.10. Fredholm Alternative

There are four Fredholm theorems:

THEOREM 1. If λ is a regular value, then both FK2 and its transposed equation* are solvable for any free term $f(x)$, and both equations have unique solutions. The associated homogeneous equation, with $f(x) = 0$, has only the trivial solution.

THEOREM 2. The nonhomogeneous FK2 (1.2.2) is solvable iff the free term $f(x)$ satisfies the condition

$$\int_a^b f(x) \phi_j^*(x) dx = 0, \quad j = 1, \dots, n,$$

where $\{\phi_j^*(x)\}$ denotes the complete set of linearly independent solutions of the associated transposed equation.

THEOREM 3. If λ is an eigenvalue, then both the homogeneous FK2 and the transposed equation have nontrivial finitely many solutions.

THEOREM 4. An FK2 has at most countably many eigenvalues whose only possible accumulation point is the point at infinity.

These theorems imply the FREDHOLM ALTERNATIVE, which states that either (i) the nonhomogeneous FK2

$$\phi(x) = f(x) + \lambda \int_a^b k(x, s) \phi(s) ds \quad (1.10.1)$$

for a fixed λ and degenerate kernel $k(x, s) = \sum_{i=1}^n a_i(x) b_i(s)$ possesses a unique solution $\phi(x)$ for arbitrary L_2 -functions $f(x)$ and $k(x, s)$; in particular, the trivial solution $\phi(x) = 0$ for $f(x) = 0$; or (ii) the homogeneous FK2

$$\phi(x) = \lambda \int_a^b k(x, s) \phi(s) ds \quad (1.10.2)$$

possesses a finite number r of linearly independent solutions. In the latter case the integral equation will have linearly independent r solutions provided that

$\int_a^b f(s) \bar{\psi}(s) ds = 0$, where $\psi(x)$ is a solution of the homogeneous equation

$$\psi(x) = \bar{\lambda} \int_a^b \bar{k}(x, s) \psi(s) ds, \quad (1.10.3)$$

*The transposed equation is obtained from an FK2 by replacing the kernel $k(x, s)$ by $k(s, x)$.

and a bar over a quantity denotes its complex conjugate.

PROOF. Consider the integral equation (1.10.1) where $k(x, s) = \sum_{i=1}^n a_i(x) b_i(s)$; then define $c_i = \int_a^b b_i(s) \phi(s) ds$, and Eq (1.10.1) reduces to

$$\phi(x) = f(x) + \lambda \sum_{i=1}^n c_i a_i(x). \quad (1.10.4)$$

Now all we need to solve Eq (1.10.1) is to find c_i . If we multiply Eq (1.10.4) by $b_m(x)$ and integrate from a to b , we get

$$c_m = \int_a^b b_m(x) [f(x) + \lambda \sum_{i=1}^n c_i a_i(x)] b_m(x) dx, \quad i = 1, 2, \dots, n. \quad (1.10.5)$$

Define

$$f_i = \int_a^b b_i(x) f(x) dx, \quad a_{mi} = \int_a^b a_i(x) b_m(x) dx, \quad (1.10.6)$$

where f_i and a_{mi} are known constants, and the equations for c_m are given by

$$c_m = f_m + \lambda \sum_{i=1}^n a_{mi} c_i. \quad (1.10.7)$$

This is a system of n linear algebraic equations which determines c_i . The determinant of this system is

$$D(\lambda) = \begin{vmatrix} 1 - \lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1 - \lambda a_{22} & \dots & -\lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1 - \lambda a_{nn} \end{vmatrix}. \quad (1.10.8)$$

$D(\lambda)$ is a polynomial in λ of degree $\leq n$. Since $D(0) = 1$, $D(\lambda)$ is not identically equal to 0. The system of equations (1.10.8) has a unique solution when $D(\lambda) \neq 0$ and these values of λ are known as regular values. In this case the integral equation also has a unique solution. If $f(x) = 0$, then both the system (1.10.8) and the integral equation (1.10.1) have only a trivial solution. Therefore, for $f(x) = 0$ the interesting case arises when $D(\lambda) = 0$. The values of λ for which $D(\lambda) = 0$ are known as the eigenvalues of the system and, consequently, the eigenvalues of the kernel of the integral equation. For these values the system

(1.10.8) has either no solution (thus, the integral equation has no solution) or the system has an infinity of solutions. In the latter case the system is consistent. For the second part, note the property for the system of homogeneous linear equations $c_m = \lambda \sum_{i=1}^n a_{mi} c_i$, according to which the system will have r linearly independent solutions corresponding to each λ for which $D(\lambda) = 0$. Consequently, the integral equation will have r linearly independent solutions corresponding to each λ for which $D(\lambda) = 0$.

Now we should find out the conditions under which the FK2 (1.10.1) has solutions even when $D(\lambda) = 0$. Eq (1.10.1) has a solution whenever the system of equations (1.10.8) has a solution, i.e., $c_m = f_m + \lambda \sum_{i=1}^n a_{mi} c_i$, or $c_m - \lambda \sum_{i=1}^n a_{mi} c_i = f_m$, or in the matrix notation

$$(\mathbf{I} - \lambda \mathbf{A})\mathbf{c} = \mathbf{f} \quad (1.10.9)$$

has a solution. The system (1.10.9) has a nontrivial solution whenever

$$c_m - \lambda \sum_{i=1}^n a_{mi} c_i = 0 \quad (1.10.10)$$

has a nontrivial solution, and f_i is orthogonal to all the solutions of

$$(\mathbf{I} - \bar{\lambda} \mathbf{A}^T) \mathbf{c} = 0, \quad (1.10.11)$$

where $A = a_{ij}$, and $A^T = \bar{a}_{ji}$. The existence of a nontrivial solution of the system (1.10.10) implies that there exist numbers γ_i such that $\gamma_i = \lambda \sum_{i=1}^n a_{ij} \gamma_j$, which yields $\phi(x) = \lambda \sum_{i=1}^n \gamma_i a_i(x)$. This last expression for $\phi(x)$ clearly implies that a solution of the homogeneous integral (1.10.2) exists. Now, consider the solution \mathbf{y} of Eq (1.10.11), which means that $y_i = \bar{\lambda} \sum_{j=1}^n \bar{a}_{mj} y_m$. For the system (1.10.8) to have a solution $\mathbf{y}^T \mathbf{f} = 0$, where \mathbf{y}^T is the transpose of \mathbf{y} , now define $\psi(x) = \sum_{i=1}^n y_i \bar{b}_i(x)$, $y_i = \bar{\lambda} \sum_{j=1}^n \bar{a}_{mj} y_m$, which implies that

$$y_i \bar{b}_i(x) = \bar{\lambda} \sum_{j=1}^n \bar{a}_{mi} y_m \bar{b}_i(x), \text{ or}$$

$$\begin{aligned} \sum_{i=1}^n y_i \bar{b}_i(x) &= \bar{\lambda} \sum_{i=1}^n \sum_{j=1}^n \bar{a}_{mi} y_m \bar{b}_i(x) = \bar{\lambda} \sum_{i=1}^n \sum_{j=1}^n \int_a^b (y_j \bar{b}_j(s) \bar{a}_i(s) ds) \bar{b}_i(x) \\ &= \bar{\lambda} \sum_{i=1}^n \int_a^b \bar{a}_i(s) \bar{b}_i(x) \psi(s) ds, \end{aligned}$$

which gives

$$\psi(x) = \bar{\lambda} \int_a^b \bar{k}(x, s) \psi(s) ds. \quad (1.10.12)$$

The condition $\mathbf{y}^T f = 0$ means that $\sum_{i=1}^n y_i f_i = 0$, i.e., $\sum_{i=1}^n \int_a^b y_i f(s) b_i(s) ds = 0$,

which gives $\int_a^b f(s) \bar{\psi}(s) ds = 0$. Since $\psi(s)$ is any solution of the adjoint equation (1.10.12), the above result is true for all other solutions also. ■

We will discuss eigenvalues of a kernel in the next chapter.

REFERENCES USED: Atkinson (1976), Baker (1978), Davis and Rabinowitz (1984), Delves and Mohamed (1985), Fox and Goodwin (1953), Goldberg (1979), Hackbusch (1995), Hochstadt (1973), Hoffman (1975), Hoheisel (1968), Kanwal (1997), Köthe (1969), Krein (1962), Kreyszig (1978), Nyström (1930), Polyanin and Manzhinov (1998), Porter and Stirling (1993), Smirnov (1964), Tricomi (1957).

2

Eigenvalue Problems

The number μ that appears in an FK2, $(\mu I - K)\phi = f$, $a \leq x \leq b$, is called the *characteristic value* of the kernel $k(x, s)$ or of the integral equation itself if there exists nontrivial solutions of the corresponding homogeneous equation (see §1.5). The number $\lambda = 1/\mu$ is called an *eigenvalue** of the kernel or of the integral equation. The nontrivial solutions are themselves called the *eigenfunctions* of the integral equation corresponding to the eigenvalues λ , and their pair is known as the *eigenpair* of the integral equation.

The concept of eigenvalues and eigenfunctions is central to the theory of integral equations. In the introductory discussion of this topic the existence and uniqueness theorem on nonzero eigenpairs are generally not discussed. Some major results about positive kernels are as follows:

- (1) If the kernel $k(x, s)$ is continuous and positive for $0 \leq x, s \leq 1$, then the homogenous FK2 has a characteristic value μ which is positive, simple, larger in modulus than any other characteristic value, and has an associated characteristic function $\psi(x)$ for $0 \leq x \leq 1$.
- (2) If $k^{(1)}(x, s)$ and $k^{(2)}(x, s)$ are two distinct continuous and positive kernels for $0 \leq x, s \leq 1$ such that $k^{(1)}(x, s) \geq k^{(2)}(x, s)$, then their largest characteristic values satisfy $\mu^{(1)} > \mu^{(2)}$.
- (3) If the kernel $k(x, s)$ is continuous (not necessarily real-valued) on $0 \leq x, s \leq 1$, and if the trace A_m of the m th iterated kernel $k_m(x, s)$ is nonzero for $m \geq 3$, then there exists a nonzero characteristic value of this kernel.

*Mikhlin and Smolitskiy (1967, p.271) note that “the reciprocals of the eigenvalues of the kernel are called the characteristic values or roots of the kernel.” We shall adhere to this terminology, although some authors do not make this distinction.

- (4) If $k(x, s) \neq 0$ is a continuous Hermitian kernel, then it has a nonzero characteristic value.
- (5) If $k(x, s) \geq 0$ is continuous on $0 \leq x, s \leq 1$ and if $A_m \neq 0$ for some $m \geq 1$, then this kernel has a nonzero characteristic value.
- (6) If $k(x, s)$ satisfies a Hölder condition in either variable with exponent $\alpha > 1/2$ and if $A_m \neq 0$ for some $m \geq 1$, then it has at least one nonzero value.
-

2.1. Linear Symmetric Equations

The eigenvalues of symmetric kernels of an FK2 are real and the corresponding eigenfunctions are orthonormal. With any symmetric kernel $k(x, s)$ there is associated a *quadratic form* of the kernel, which is defined in (1.2.19). Sometimes it is also defined by

$$\langle K\phi, \phi \rangle = \int_a^b \int_a^b k(x, s) \overline{\phi(x)} \phi(s) dx ds, \quad (2.1.1)$$

which is real.

The smallest (in absolute value) eigenvalue λ_1 of a symmetric kernel $k(x, s)$ is equal to the minimum value of $|k(x, s)|$ under the condition $\|\phi\|^2 = 1$; the corresponding eigenfunction $\phi_1(x)$ coincides with the function for which this minimum is achieved.

Let $\lambda_1, \dots, \lambda_n$ (arranged in order of increasing absolute value) be the first n eigenvalues of a symmetric kernel $k(x, s)$, and let $\phi_1(x), \dots, \phi_n(x)$ be the corresponding orthonormal eigenfunctions. Then the absolute value of the next eigenvalue $|\lambda_{n+1}|$ is equal to the minimum of $|\langle K\phi_{n+1}, \phi \rangle|$ such that

$$\|\phi\|^2 = 1, \quad \langle \phi, \phi_1 \rangle + \dots + \langle \phi, \phi_n \rangle = 0. \quad (2.1.2)$$

The corresponding eigenfunction $\phi_{n+1}(x)$ is the same as the function for which this minimum is achieved under the condition (2.1.2).

A symmetric kernel is said to be *positive-definite* if its quadratic form has only positive values when $\phi(x)$ does not vanish identically. Because such kernels are of much interest, we study different numerical methods in this section. These methods are based on the above results. First, we consider the following classical methods:

2.1.1. Ritz Method. Choose a sequence of functions $\{\psi_n(x)\}$, called coordinate functions, such that

- (i) $\psi_n(x) \in L_2(a, b)$;
- (ii) $\psi_1(x), \dots, \psi_n(x)$ are linearly independent for all n ; and
- (iii) the sequence $\{\psi_n(x)\}$ is complete in $L_2(a, b)$.

Condition (iii) means that for every function $f(x) \in L_2(a, b)$ and any given $\varepsilon > 0$, there exist a natural number N and coefficients c_1, \dots, c_n such that

$$\|f - \sum_{i=1}^N c_i \psi_i\| < \varepsilon. \quad (2.1.3)$$

Note that the above three conditions are satisfied by any orthonormal polynomial that is complete in $L_2(a, b)$. The Ritz method requires that we set

$$\phi_n = \sum_{j=1}^n a_j \psi_j, \quad (2.1.4)$$

where the coefficients a_j are subject to the condition $\|\phi\|^2 = 1$, which yields

$$\sum_{i,j=1}^n \langle \psi_i, \psi_j \rangle a_i \bar{a}_j = 1. \quad (2.1.5)$$

Then the quadratic form becomes

$$\langle K\phi_n, \phi_n \rangle = \sum_{i,j=1}^n \langle K\psi_i, \psi_j \rangle a_i \bar{a}_j. \quad (2.1.6)$$

Using Lagrange's method, with μ as a Lagrange multiplier, Eq (2.1.6) reduces to a homogeneous linear system of algebraic equations (Nyström system) in the unknown coefficients a_j :

$$\sum_{j=1}^n \left\{ \langle K\psi_i, \psi_j \rangle - \mu \langle \psi_i, \psi_j \rangle \right\} a_j = 0, \quad n = 1, \dots, n. \quad (2.1.7)$$

For brevity we shall use the notation: $p_{ij} = \langle K\psi_i, \psi_j \rangle$ and $q_{ij} = \langle \psi_i, \psi_j \rangle$. Then the system (2.1.7) can be written as

$$\sum_{j=1}^n \left\{ p_{ij} - \mu q_{ij} \right\} a_j = 0, \quad n = 1, \dots, n. \quad (2.1.8)$$

For this system to be consistent, its determinant must be zero, i.e.,

$$\begin{vmatrix} p_{11} - \mu q_{11} & p_{12} - \mu q_{12} & \cdots & p_{1n} - \mu q_{1n} \\ p_{21} - \mu q_{21} & p_{22} - \mu q_{22} & \cdots & p_{2n} - \mu q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} - \mu q_{n1} & p_{n2} - \mu q_{n2} & \cdots & p_{nn} - \mu q_{nn} \end{vmatrix} = 0. \quad (2.1.9)$$

The roots of Eq (2.1.9) yield characteristic values μ 's of the kernel $k(x, s)$. The largest root gives an undersize approximation and the smallest root an oversize approximation of the eigenvalues of the kernel. Once μ is determined, it can be substituted in (2.1.7) to obtain the approximate eigenfunction corresponding to an eigenvalue.

EXAMPLE 2.1.1. Consider the positive-definite symmetric kernel

$$k(x, s) = \begin{cases} s, & x \geq s, \\ x, & x \leq s, \end{cases} \quad a = 0, b = 1.$$

For the coordinate functions we choose $\psi_n(x) = P_n(2x - 1)$, $n = 0, 1, \dots$, where P_n denotes the n th Legendre polynomial. Since these polynomials are orthogonal in the interval $(0, 1)$, we have $\langle P_i, P_j \rangle = 0$ for $i \neq j$, and $\langle P_i, P_i \rangle = 1/(2i + 1)$, where the argument of P is taken as $2x - 1$. Also note that

$$\begin{aligned} \langle K\psi_i, \psi_j \rangle &= \int_0^1 dx \left\{ \int_0^x s\psi_i(s)\psi_j(x) ds + \int_x^1 x\psi_i(s)\psi_j(x) ds \right\} \\ &= \int_0^1 dx \int_0^x s [\psi_i(s)\psi_j(x) + \psi_i(x)\psi_j(s)] ds, \end{aligned} \quad (2.1.10)$$

where formula (1.3.7) is used in the second double integral. Now we shall compute the eigenvalues in two cases:

(i) Take $n = 2$ in (2.1.4), that is, taking two terms

$$\phi_n(x) = \phi_2(x) = a_1 P_0(2x - 1) + a_2 P_1(2x - 1) = a_1 + a_2(2x - 1).$$

Also, $\langle P_0, P_0 \rangle = 1$, $\langle P_0, P_1 \rangle = 0$, $\langle P_1, P_1 \rangle = 1/3$, and using (2.1.10) with the argument of P as $2x - 1$, we find that

$$\langle KP_0, P_0 \rangle = \frac{1}{3}, \quad \langle KP_0, P_1 \rangle = \frac{1}{12} = \langle KP_1, P_0 \rangle, \quad \langle KP_1, P_1 \rangle = \frac{1}{30},$$

and Eq (2.1.9) becomes

$$\begin{vmatrix} 1/3 - \mu & 1/12 \\ 1/12 & 1/30 - \mu/3 \end{vmatrix} = 0,$$

or

$$\frac{\mu^2}{3} - \frac{13}{90}\mu + \frac{1}{240} = 0, \quad (2.1.11)$$

which yields $\mu = \frac{13 \pm \sqrt{124}}{60}$. Thus, the first two approximate eigenvalues of the kernel (2.1.10) are

$$\lambda_1 = \frac{1}{\mu_1} = 2.48596, \quad \lambda_2 = \frac{1}{\mu_2} = 32.18069.$$

Since the integral equation $\phi(x) - \lambda \int_0^1 k(x, s) \phi(s) ds = 0$ is equivalent to the differential equation $\phi'' + \lambda \phi = 0$ with the boundary conditions $\phi(0) = 0, \phi'(1) = 0$, the exact eigenvalues of the kernel (2.1.4) are $(2n-1)^2 \pi^2 / 4$, $n = 1, 2, \dots$. Thus, the first two exact eigenvalues are $\lambda_1 = 2.4674$ and $\lambda_2 = 22.2066$. Notice that the smallest of the roots of Eq (2.1.11) yields an oversize approximate eigenvalue λ_2 . The above results are obtained if the Chebyshev polynomials $T_n(2x-1)$ of the first kind are used instead of Legendre polynomials $P_n(2x-1)$.

(ii) Take $n = 3$ in (2.1.4). Then, taking three terms in (2.1.4), Eq (2.1.9) reduces to

$$\begin{vmatrix} 1/3 - \mu & 1/12 & -1/60 \\ 1/12 & 1/30 - \mu/3 & 0 \\ -1/60 & 0 & 1/210 - \mu/5 \end{vmatrix} = 0,$$

or

$$\frac{\mu^3}{15} - \frac{16}{525}\mu^2 + \frac{1}{700}\mu - \frac{1}{94500} = 0,$$

which yields the first three approximate eigenvalues

$$\lambda_1 = 2.46774, \quad \lambda_2 = 23.3913, \quad \lambda_3 = 109.141.$$

Compare these values with the exact eigenvalues $\lambda_1 = 2.4674$, $\lambda_2 = 22.2066$, and $\lambda_3 = 61.685$. Note that the same results are obtained if the Chebyshev polynomials $T_n(2x-1)$ of the first kind are used instead of the Legendre polynomials $P_n(2x-1)$. For computational details, see `ritz.nb`. Note that in this method the initial eigenvalues approximate the exact eigenvalues very closely; the penultimate eigenvalue gives a rough approximation, and the last one is always far off its exact value. ■

2.1.2. Method of Moments. This method is similar to the Ritz method. Let ψ_1, \dots, ψ_n be the first n coordinate functions; let R_n denote the subspace

spanned by these functions, and let P_n be the operator projecting $L_2(a, b)$ into R_n . Then the Ritz formulas (2.1.4)–(2.1.9) are used to find eigenvalues of the finite-dimensional operator $P_n K P_n$, instead of K as in part (a) above. Thus, we choose the functions $\psi_m = K^{m-1} \omega$, $m = 1, 2, \dots$, as the coordinate functions, where $\omega(x)$ is an arbitrary function such that the $\omega, K\omega, \dots, K^{n-1}\omega$ are linearly independent. Then the eigenvalues of the operator $P_n K P_n$ are computed from the roots of the equation

$$\mu^n + \alpha_{n-1} \mu^{n-1} + \dots + \alpha_0 = 0, \quad (2.1.12)$$

where the coefficients α_j , $j = 0, 1, \dots, n-1$, are evaluated by solving the Nyström system

$$\sum_{j=1}^n \langle \psi_i, \psi_j \rangle \alpha_{j-1} + \langle \psi_i, \psi_j \rangle = 0, \quad i = 1, 2, \dots, n. \quad (2.1.13)$$

The method of moments also applies to asymptotic kernels.

EXAMPLE 2.1.2. Consider the kernel

$$k(x, s) = \begin{cases} \frac{1}{2}(2-s)x, & x \leq s, \\ \frac{1}{2}(2-x)s, & x \geq s, \end{cases} \quad 0 \leq x \leq s \leq 1. \quad (2.1.14)$$

Take $\omega(x) = \sin \pi x$, and $n = 4$. Then

$$\begin{aligned} \psi_1(x) &= \omega = \sin \pi x, \\ \psi_2(x) &= K\omega = \frac{x}{2\pi} + \frac{1}{\pi^2} \sin \pi x, \\ \psi_3(x) &= K^2\omega = \frac{x}{6\pi} \left(1 + \frac{3}{\pi^2}\right) - \frac{x^3}{12\pi} + \frac{\sin \pi x}{\pi^4}, \\ \psi_4(x) &= K^3\omega = \left(\frac{31}{720\pi} + \frac{1}{6\pi^3} + \frac{1}{2\pi^5}\right) x - \frac{(3+\pi^2)x^3}{36\pi^3} + \frac{x^5}{240\pi} + \frac{\sin \pi x}{\pi^6}, \\ \psi_5(x) &= K^4\omega = \frac{1}{30240\pi^7} \left[x (15120 + 5040\pi^2 + 1302\pi^4 + 320\pi^6 \right. \\ &\quad \left. - 7\pi^2 (360 + 120\pi^2 + 31\pi^4) x^2 + 42\pi^4 (3 + \pi^2) x^4 - 3\pi^6 x^6) \right] + \frac{\sin \pi x}{\pi^8}. \end{aligned} \quad (2.1.15)$$

For $n = 4$, Eq (2.1.13) becomes

$$\langle \psi_i, \psi_1 \rangle \alpha_0 + \langle \psi_i, \psi_2 \rangle \alpha_1 + \langle \psi_i, \psi_3 \rangle \alpha_2 + \langle \psi_i, \psi_4 \rangle \alpha_3 + \langle \psi_i, \psi_5 \rangle = 0, \quad (2.1.16)$$

where $i = 1, 2, 3, 4$. This leads to a 4×4 system in four unknowns $\alpha_0, \alpha_1, \alpha_2, \alpha_3$:

$$\begin{bmatrix} \langle \psi_1, \psi_1 \rangle & \langle \psi_1, \psi_2 \rangle & \langle \psi_1, \psi_3 \rangle & \langle \psi_1, \psi_4 \rangle \\ \langle \psi_2, \psi_1 \rangle & \langle \psi_2, \psi_2 \rangle & \langle \psi_2, \psi_3 \rangle & \langle \psi_2, \psi_4 \rangle \\ \langle \psi_3, \psi_1 \rangle & \langle \psi_3, \psi_2 \rangle & \langle \psi_3, \psi_3 \rangle & \langle \psi_3, \psi_4 \rangle \\ \langle \psi_4, \psi_1 \rangle & \langle \psi_4, \psi_2 \rangle & \langle \psi_4, \psi_3 \rangle & \langle \psi_4, \psi_4 \rangle \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} + \begin{Bmatrix} \langle \psi_1, \psi_5 \rangle \\ \langle \psi_2, \psi_5 \rangle \\ \langle \psi_3, \psi_5 \rangle \\ \langle \psi_4, \psi_5 \rangle \end{Bmatrix} = \{0\}. \quad (2.1.17)$$

Now,

$$\begin{aligned} \langle \psi_1, \psi_1 \rangle &= \int_0^1 \psi_1^2 dx = 0.5, & \langle \psi_1, \psi_2 \rangle &= \int_0^1 \psi_1 \psi_2 dx = 0.101321, \\ \langle \psi_1, \psi_3 \rangle &= 0.0238424, & \langle \psi_1, \psi_4 \rangle &= 0.00376145, \\ \langle \psi_1, \psi_5 \rangle &= 0.00139853, & \langle \psi_2, \psi_2 \rangle &= 0.0238424, \\ \langle \psi_2, \psi_3 \rangle &= 0.00576145, & \langle \psi_2, \psi_4 \rangle &= 0.00139853, \\ \langle \psi_2, \psi_5 \rangle &= 0.000339737, & \langle \psi_3, \psi_3 \rangle &= 0.00139853, \\ \langle \psi_3, \psi_4 \rangle &= 0.000339737, & \langle \psi_3, \psi_5 \rangle &= 0.0000825413, \\ \langle \psi_4, \psi_4 \rangle &= 0.000339737, & \langle \psi_4, \psi_5 \rangle &= 0.0000200544. \end{aligned}$$

Note that $\langle \psi_i, \psi_j \rangle = \langle \psi_j, \psi_i \rangle$ for $i \neq j$, and $\langle \psi_i, \psi_j \rangle = \langle \psi_{i-1}, \psi_{j+1} \rangle$ for $i, j \geq 2$. Thus, the system (2.1.17) becomes

$$\begin{bmatrix} 0.5 & 0.1013210 & 0.0238424 & 0.00376145 \\ 0.0238424 & 0.00576145 & 0.00139853 & 0 \\ 0 & 0.00139853 & 0.00033973 & 0 \\ 0 & 0 & 0.00033973 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = - \begin{Bmatrix} 0.00139853 \\ 0.00033973 \\ 0.00008254 \\ 0.00002005 \end{Bmatrix},$$

which yields $\alpha_0 = 6.94578 \times 10^{-7}$, $\alpha_1 = -0.000218735$, $\alpha_2 = 0.0157598$, and $\alpha_3 = -0.304171$. Then Eq (2.1.12) becomes

$$\mu^4 - 0.304171\mu^3 + 0.0157598\mu^2 - 0.000218735\mu + 6.94578 \times 10^{-7} = 0,$$

which has the roots

$$\mu_1 = 0.242963, \quad \mu_2 = 0.0414251, \quad \mu_3 = 0.0152607, \quad \mu_4 = 0.00452214.$$

Thus, the approximate eigenvalues are $\lambda_1 = 1/\mu_1 = 4.11586$, $\lambda_2 = 1/\mu_2 = 24.1399$, $\lambda_3 = 1/\mu_3 = 65.5279$, and $\lambda_4 = 1/\mu_4 = 221.134$, which compare very well with the first three exact eigenvalues $\lambda_1 = 4.11585$, $\lambda_2 = 24.14$, $\lambda_3 = 63.61$. For computational details, see moments.nb. ■

2.1.3. Kellogg's Method. Let $k(x, s)$ be a positive-definite, symmetric kernel and $\omega(x)$ an arbitrary function in $L_2(a, b)$. Then construct a sequence of functions $\{\omega_n(x)\} = K^n \omega$, $n = 1, 2, \dots$, and a sequence of numbers

$$\left\{ \frac{\|\omega_{n-1}\|}{\|\omega_n\|} \right\}. \quad (2.1.18)$$

Let $\phi_1(x), \phi_2(x), \dots$ be orthonormal eigenfunctions of the kernel $k(x, s)$, and let $\lambda_1 \leq \lambda_2 \leq \dots$ be the corresponding eigenvalues. Now, suppose the functions $\omega(x)$ are orthogonal to the eigenfunctions $\phi_1(x), \phi_2(x), \dots$, but not to $\phi_i(x)$. Then the sequence (2.1.18) has the i th eigenvalue λ_i as its limit, and the sequence of functions $\frac{\omega(x)}{\|\omega\|}$ has, for its limit, some linear combinations of the eigenfunctions corresponding to the eigenvalues λ_i . In particular, if the function $\omega(x)$ is *not* orthogonal to the eigenfunction $\phi_1(x)$, the limit of the sequence (2.1.18) is λ . The sequence $\frac{1}{\sqrt[n]{\|\omega_n\|}}$ approaches the same limit as (2.1.18). Under the assumption that $\langle \omega, \omega_1 \rangle \neq 0$, we obtain two approximate formulas for the smallest eigenvalue:

$$\lambda_1 \approx \frac{\|\omega_{n-1}\|}{\|\omega_n\|}, \quad (2.1.19)$$

$$\lambda_1 \approx \frac{1}{\sqrt[n]{\|\omega_n\|}}. \quad (2.1.20)$$

Note that formula (2.1.19) yields an oversize value of λ_1 , and formula (2.1.20) can be effectively used only for very large n . Also, the first eigenfunction is approximately given by

$$\phi_1(x) \approx \frac{\omega_n(x)}{\|\omega_n\|}, \quad (2.1.21.)$$

If the kernel is symmetric but not positive-definite, then (2.1.19) and (2.1.20) yield approximate values of the smallest absolute value $|\lambda_1|$ of the kernel $k(x, s)$. This method is, thus, at best appropriate to obtain very approximate value of λ_1 only.

EXAMPLE 2.1.3. Consider the kernel

$$k(x, s) = \begin{cases} x, & x \leq s, \\ s, & x \geq s, \end{cases}$$

as in Example 2.1.1. Choose $\omega(x) = x$ and $n = 3$. Then

$$\begin{aligned} Kx^n &= \int_0^1 k(x, s) s^n ds = \int_0^x s \cdot s^n ds + \int_x^1 x \cdot s^n ds \\ &= \frac{1}{n+1} \left(x - \frac{x^{n+2}}{n+2} \right). \end{aligned}$$

Also,

$$\begin{aligned}\omega_1(x) &= K\omega = \frac{x}{2} - \frac{x^3}{6}, \\ \omega_2(x) &= K\omega_1 = \int_0^x s\omega_1(s) ds + \int_x^1 x\omega_1(s) ds = \frac{x}{120} (x^2 - 5)^2, \\ \omega_3(x) &= K\omega_2 = -\frac{1}{240} (x-2)x^2(x^2-5)^2, \\ \omega_4(x) &= K\omega_3 = \frac{x^3}{480} (x^3 - 2x^2 - 5x + 10)^2, \\ \omega_5(x) &= K\omega_4 = -\frac{1}{190} (x-2)^3 x^4 (x^2-5)^2.\end{aligned}$$

Note that we have computed $\omega_4(x)$ and $\omega_5(x)$, although they are not needed for this method. Moreover,

$$\begin{aligned}\|\omega_1\| &= \left[\int_0^1 \omega_1^2(x) dx \right]^{1/2} = 0.232311, \\ \|\omega_2\| &= 0.0941448, \\ \|\omega_3\| &= 0.0381554.\end{aligned}$$

Now set $n = 2$ and $n = 3$ in (2.1.19). This gives

$$\lambda_1 \approx \frac{\|\omega_1\|}{\|\omega_2\|} = 2.46759; \quad \text{also,} \quad \lambda_1 \approx \frac{\|\omega_2\|}{\|\omega_3\|} = 2.4674.$$

Note that these are oversized approximations of λ_1 (compare with its value in Example 2.1.1). Now, set $n = 2$ and $n = 3$ in (2.1.20); then

$$\lambda_1 \approx \frac{1}{\sqrt{\|\omega_2\|}} = 3.25913; \quad \text{also,} \quad \lambda_1 \approx \frac{1}{\sqrt[3]{\|\omega_n\|}} = 2.9704,$$

which shows that this formula does not work for smaller values of n . For computational details, see `kellogg1.nb`. ■

2.1.4. Trace Method. The m th trace of the kernel $k(x, s)$ is defined by

$$A_m = \int_a^b k_m(x, s) ds, \tag{2.1.22}$$

where $k_m(x, s)$ denotes the m th iterated kernel, i.e.,

$$(K^m \phi) = \int_a^b k_m(x, s) \phi(s) ds, \quad a \leq s \leq x \leq b.$$

If the kernel is symmetric and satisfies the condition

$$B^2 = \int_a^b \int_a^b |k(x, s)|^2 dx ds, \quad (2.1.23)$$

then its traces, beginning with the second, are finite.

The traces of a kernel are related to its eigenvalues by

$$A_m = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^m}. \quad (2.1.24)$$

This yields approximate formulas for the smallest eigenvalue λ_1 in magnitude as

$$|\lambda_1| \approx \sqrt{\frac{A_{2m}}{A_{2m+2}}}, \quad (2.1.25)$$

$$|\lambda_1| \approx \frac{\sqrt[2m]{r}}{\sqrt[2m]{A_{2m}}}, \quad (2.1.26)$$

where r is the multiplicity of the eigenvalue λ_1 . If λ_1 as well as $-\lambda_1$ are the eigenvalues of a given kernel, then r must be taken as the sum of their multiplicities. Formula (2.1.25) yields an oversize value of $|\lambda_1|$, while (2.1.26) gives its undersize value. For subsequent eigenvalues we have

$$|\lambda_2| \approx \frac{1}{|\lambda_1|} \sqrt{\frac{B_{2m}}{B_{2m+2}}}, \quad (2.1.27)$$

$$|\lambda_2| \approx \frac{1}{|\lambda_1|} \sqrt[2m]{\frac{2}{B_{2m}}}, \quad (2.1.28)$$

$$|\lambda_3| \approx \frac{1}{|\lambda_1^2 \lambda_2|} \sqrt[2m]{\frac{8}{B_{2m}^2 - 2B_{4m}}}, \quad (2.1.29)$$

where $B_{2m} = A_{2m}^2 - A_{4m}$. Note that formula (2.1.27) gives an oversize value and (2.1.28) an undersize value of λ_2 . The even-order traces for a symmetric kernel can be computed by

$$A_{2m} = \int_a^b \int_a^b |k_m(x, s)|^2 dx ds = 2 \int_a^b \int_a^x |k_m(x, s)|^2 ds dx. \quad (2.1.30)$$

This requires only half as many iterations.

EXAMPLE 2.1.4. Consider the same kernel as in Example 2.1.3 or 2.1.1. Note that

$$K_2(x, s) = xs - \frac{x^2 s}{2} - \frac{s^3}{3}, \quad s \leq x;$$

thus,

$$A_2 = \frac{1}{6}, \quad A_4 = 0.00962302.$$

Take $m = 1$ in (2.1.25). Then we get $|\lambda_1| = 2.94275$, and if we take $r = 1$ and $m = 2$ in (2.1.26), then we get $|\lambda_1| = 2.68482$. ■

EXAMPLE 2.1.5. The squares of the zeros of the Bessel's function $J_0(x)$ are the eigenvalues of a symmetric kernel. We shall consider

$$k(x, s) = \begin{cases} -\sqrt{xs} \ln s, & x \leq s, \\ -\sqrt{xs} \ln x, & x \geq s, \end{cases}$$

with $a = 0, b = 1$, and use the trace method to compute the first two real zeros α_1 and α_2 of $J_0(x)$. Note that the second iterated kernel is

$$K_2(x, s) = \frac{\sqrt{xs}}{4} \left[(x^2 + s^2) \ln x + 1 - x^2 \right], \quad x \geq s.$$

Then

$$A_2 = \frac{1}{32}, \quad A_4 = \frac{11}{12288} = 0.000895182,$$

which gives

$$\begin{aligned} B_2 &= \frac{1}{1024} - \frac{11}{12288} = \frac{1}{12288}; \\ \lambda_1 &= \frac{1}{\sqrt[4]{A_4}} = 5.7812549, \\ \lambda_2 &= \frac{1}{\lambda_1} \sqrt{\frac{2}{B_2}} = 27.116489. \end{aligned}$$

Thus, the first two zeros of $J_0(x)$ are approximately given by

$$\alpha_1 = \sqrt{\lambda_1} = 2.404424, \quad \alpha_2 = \sqrt{\lambda_2} = 5.2073495.$$

Compare them with their known values, which are 2.4048 and 5.52, respectively. For computational details, see `trace1.nb` and `trace2.nb`. ■

2.2. Residual Methods

Residual methods, mostly used in differential equations, are also important for numerically solving Fredholm integral equations. The idea is very simple, and we explain it for an FK2, which we write as

$$K\phi \equiv \phi(x) - \lambda \int_a^b k(x, s)\phi(s) ds - f(x) = 0. \quad (2.2.1)$$

Let the approximate solution of this equation be of the form

$$\tilde{\phi}_n(x) = \Phi(x, c_1, \dots, c_n),$$

where the parameters c_1, \dots, c_n are undetermined coefficients. Substituting this approximate solution in Eq (2.2.1), we obtain the residual

$$r[\tilde{\phi}_n] = K\phi - K\tilde{\phi}_n = [\phi(x) - \tilde{\phi}_n(x)] - \lambda \int_a^b k(x, s)[\phi(s) - \tilde{\phi}_n(s)] ds, \quad (2.2.2)$$

where $r[\tilde{\phi}_n]$ will vanish for an exact solution $\phi(x)$. Next, we choose the parameters c_1, \dots, c_n such that the residual $r[\tilde{\phi}_n]$ is as small as possible. The problem, therefore, reduces to that of minimizing the residual. This can, for example, be done by choosing the function $\tilde{\phi}_n(x)$ linearly dependent on c_1, \dots, c_n . Then, after comparing the parameters c_1, \dots, c_n , we obtain the approximate solution $\tilde{\phi}_n(x)$. Since $\lim_{n \rightarrow \infty} \tilde{\phi}_n(x) = \phi(x)$, the accuracy of obtaining an approximate solution depends on n , i.e., we can always find $\tilde{\phi}_n(x)$ within a prescribed accuracy for sufficiently large n . Three methods that use the residual are discussed below.

2.2.1. Collocation Method. The approximate solution $\tilde{\phi}_n(x)$ is constructed as follows: Set

$$\tilde{\phi}_n(x) = \phi_0(x) + \sum_{i=1}^n c_i \phi_i(x), \quad (2.2.3)$$

where $\phi_0, \phi_1, \dots, \phi_n$ are prescribed linearly independent coordinate functions, usually chosen from a complete basis set in $L_2(a, b)$, like $\{x^i\}_{i=1}^\infty$. In particular, we take $\phi_0 = 0$ or $f(x)$. Substituting (2.2.3) into (2.2.1), we get

$$\begin{aligned} r[\tilde{\phi}_n] &= \phi_0(x) + \sum_{i=1}^n c_i \phi_i(x) - f(x) - \lambda \int_a^b k(x, s) [\phi_0(s) + \sum_{i=1}^n c_i \phi_i(s)] ds \\ &= \psi_0(x, \lambda) + \sum_{i=1}^n c_i \psi_i(x, \lambda), \end{aligned} \quad (2.2.4)$$

where

$$\begin{aligned} \psi_0(x, \lambda) &= \phi_0(x) - f(x) - \lambda \int_a^b k(x, s) \phi_0(s) ds, \\ \psi_i(x, \lambda) &= \phi_i(x) - \lambda \int_a^b k(x, s) \phi_i(s) ds. \end{aligned} \quad (2.2.5)$$

In the collocation method the residual $r[\tilde{\phi}_n]$ must be zero at a prescribed set of points x_1, \dots, x_n on the interval $[a, b]$, i.e.,

$$r[\tilde{\phi}_n(x_j)] = 0, \quad j = 1, \dots, n, \quad (2.2.6)$$

where $a \leq x_1 < x_2 < \dots < x_{n-1} < x_n \leq b$. Then (2.2.4) and (2.2.6) yield the following system of linear equations:

$$\sum_{i=1}^n c_i \psi_i(x_j, \lambda) = -\psi_0(x_j, \lambda), \quad j = 1, \dots, n, \quad (2.2.7)$$

which uniquely determines the coefficients c_1, \dots, c_n provided the determinant of this system is nonzero, namely,

$$\Delta(x_j, \lambda) = \begin{vmatrix} \psi_1(x_1, \lambda) & \psi_1(x_2, \lambda) & \cdots & \psi_1(x_n, \lambda) \\ \psi_2(x_1, \lambda) & \psi_2(x_2, \lambda) & \cdots & \psi_2(x_n, \lambda) \\ \dots & \dots & \dots & \dots \\ \psi_n(x_1, \lambda) & \psi_n(x_2, \lambda) & \cdots & \psi_n(x_n, \lambda) \end{vmatrix} \neq 0. \quad (2.2.8)$$

The approximate solution $\tilde{\phi}_n(x)$ is then obtained from (2.2.3).

The zeros of the determinant $\Delta(x_j, \lambda)$ are the approximate eigenvalues $\tilde{\lambda}_j$ of the kernel $k(x, s)$. The corresponding approximate eigenfunctions $\tilde{\phi}_n^{(j)}(x)$ are then given by

$$\tilde{\phi}_n^{(j)}(x) = \sum_{i=1}^n \tilde{c}_i^{(j)} \tilde{\phi}_i(x), \quad (2.2.9)$$

where $\tilde{c}_i^{(j)}$ are the nonzero solutions of the homogeneous system

$$\sum_{i=1}^n \tilde{c}_i^{(j)} \psi_i(x_j, \tilde{\lambda}_j) = 0, \quad j = 1, \dots, n. \quad (2.2.10)$$

EXAMPLE 2.2.1. Consider the FK2

$$\phi(x) - \int_0^1 \frac{s^2}{x^2 + s^2} \phi(s) ds = x \tan^{-1} \frac{1}{x}.$$

We take $\phi_0(x) = 0$, and set $\tilde{\phi}_3(x) = c_1 + c_2 x + c_3 x^2$. Since $\phi_1(x) = 1$, $\phi_2(x) = x$, $\phi_3(x) = x^2$, we get

$$\begin{aligned} \psi_0(x, \lambda) &= -x \tan^{-1} \frac{1}{x}, & \psi_1(x, \lambda) &= x \tan^{-1} \frac{1}{x}; \\ \psi_2(x, \lambda) &= x - \frac{1}{2} + \frac{x^2}{2} \ln \left(1 + \frac{1}{x^2} \right), & \psi_3(x, \lambda) &= 2x^2 - \frac{1}{3} - x^3 \tan^{-1} \frac{1}{x}. \end{aligned}$$

Then the residual is

$$\begin{aligned} r[\tilde{\phi}_3] &= c_1 x \tan^{-1} \frac{1}{x} + c_2 \left[x - \frac{1}{2} + \frac{x^2}{2} \ln \left(1 + \frac{1}{x^2} \right) \right] \\ &\quad + c_3 \left[2x^2 - \frac{1}{3} - x^3 \tan^{-1} \frac{1}{x} \right] - x \tan^{-1} \frac{1}{x}. \end{aligned}$$

Choose three collocation points at $x_1 = 0$, $x_2 = 1/2$, and $x_3 = 1$. Then the system (2.2.7) is

$$\begin{aligned} \begin{bmatrix} 0 & -1/2 & -1/3 \\ (\tan^{-1} 2)/2 & -(ln 5)/8 & 1/6 - (\tan^{-1} 2)/8 \\ \pi/4 & (1 + ln 2)/2 & 5/3 - \pi/4 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} \\ = \begin{Bmatrix} 0 \\ -(\tan^{-1} 2)/2 \\ -\pi/4 \end{Bmatrix}, \end{aligned}$$

which gives $c_1 = -1$, $c_2 = 0 = c_3$. Thus, $\tilde{\phi}_3(x) = -1$. It can be easily verified that this solution is exact. For computational details, see `collocation1.nb`. ■

2.2.2. Least-Squares Method. In this method the coefficients c_1, \dots, c_n in (2.2.4) are determined by minimizing the integral

$$I = \int_a^b \left(r[\tilde{\phi}_n] \right)^2 dx = \int_a^b \left[\psi_0(x, \lambda) + \sum_{i=1}^n c_i \psi_i(x, \lambda) \right]^2 dx. \quad (2.2.11)$$

Thus, we require that $\frac{\partial I}{\partial c_j} = 0$ for $j = 1, \dots, n$, which leads to

$$\frac{1}{2} \frac{\partial I}{\partial c_j} = \int_a^b \psi_j(x, \lambda) \left[\psi_0(x, \lambda) + \sum_{i=1}^n c_i \psi_i(x, \lambda) \right]^2 dx, \quad j = 1, \dots, n, \quad (2.2.12)$$

or the system

$$\begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) & \cdots & a_{1n}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & \cdots & a_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\lambda) & a_{n2}(\lambda) & \cdots & a_{nn}(\lambda) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix} = \begin{Bmatrix} a_{10}(\lambda) \\ a_{20}(\lambda) \\ \vdots \\ a_{n0}(\lambda) \end{Bmatrix}, \quad (2.2.13)$$

where we have used the notation

$$a_{ij} = \int_a^b \psi_i(x, \lambda) \psi_j(x, \lambda) dx. \quad (2.2.14)$$

Since $a_{ij} = a_{ji}$, the matrix of the above system is symmetric; also, if $\phi_0(x) \equiv 0$, then $\psi_0(x, \lambda) = -f(x)$.

The least-squares method can be used to compute approximate eigenvalues of the kernel $k(x, s)$ in the same way as in the collocation method; thus, the zeros of the determinant $|a_{ij}(\lambda)|$ are the approximate eigenvalues $\tilde{\lambda}_j$, and the corresponding eigenfunctions $\tilde{\phi}_n^{(j)}(x)$ are given by (2.2.9).

EXAMPLE 2.2.2. Consider the FK2

$$\phi(x) = \int_{-1}^1 \sinh(x+s) \phi(s) ds + x^2,$$

which has a degenerate kernel (see Section 2.3). The exact solution of this equation is

$$\phi(x) = x^2 + \alpha \sinh x + \beta \cosh x,$$

where

$$\alpha = -\frac{16e^3 (e^2 - 5)}{1 - 34e^4 + e^8} = -0.682084,$$

$$\beta = \frac{4e (e^2 - 5) (1 + 4e^2 - e^4)}{1 - 34e^4 + e^8} = -0.554827.$$

We shall solve it numerically by the least-squares method by taking $\tilde{\phi}_3(x) = c_1 + c_2 x + x^2$; thus, $\phi_1 = 1$, $\phi_2 = x$, and $\phi_3 = x^2$. The following results will be useful:

$$\begin{aligned}\int_{-1}^1 \sinh(x+s) ds &= (e - 1/e) \sinh x = 2.3504 \sinh x, \\ \int_{-1}^1 s \sinh(x+s) ds &= \frac{2}{e} \cosh x = 0.735759 \cosh x, \\ \int_{-1}^1 s^2 \sinh(x+s) ds &= (e - 5/e) \sinh x = 0.878885 \sinh x.\end{aligned}$$

Then from (2.2.5) we get

$$\begin{aligned}\psi_0(x, \lambda) &= -0.878885 \sinh x, & \psi_1(x, \lambda) &= 1 - 2.3504 \sinh x, \\ \psi_2(x, \lambda) &= x - 0.735759 \cosh x, & \psi_3(x, \lambda) &= x^2 - 0.878885 \sinh x,\end{aligned}$$

which yields

$$\begin{bmatrix} 6.4937 & -3.45866 & 2.347 \\ & 2.18969 & -1.29329 \\ \text{sym} & & 1.02832 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1.68033 \\ 0.646647 \\ -0.628325 \end{Bmatrix}.$$

Thus, we have $c_1 = -0.639316$ and $c_2 = -0.714498$, and the approximate solution is

$$\tilde{\phi}_3(x) = x^2 - 0.714498 x - 0.639316.$$

This solution is compared with the exact solution given in Table 2.2.2. For computational details, see `leastsquares1.nb`. ■

Table 2.2.2

x	$\tilde{\phi}_3(x)$	Exact
-1.0	1.07518	0.945442
-0.8	0.572287	0.503717
-0.5	-0.0320675	-0.0202072
-0.4	-0.193517	-0.159641
-0.2	-0.457417	-0.388633
0.0	-0.6393166	-0.5548727
0.2	-0.742216	-0.663289
0.4	-0.765116	-0.719976
0.5	-0.746565	-0.731068
0.8	-0.570915	-0.707808
1.0	-0.353814	-0.657729

2.2.3. Galerkin Method. In this method we seek the approximate solution of Eq (2.2.1) in the form

$$\tilde{\phi}_n(x) = f(x) + \sum_{i=1}^n c_i \phi_i(x), \quad (2.2.15)$$

where $\phi_i(x)$, $i = 1, \dots, n$, are, as before, the prescribed linearly independent coordinate functions. Substituting (2.2.15) into Eq (2.2.1) and taking $\phi_0(x) = 0$ or $f(x)$, we obtain the residual

$$r[\tilde{\phi}_n(x)] = \sum_{j=1}^n c_j \left[\phi_j(x) - \lambda \int_a^b k(x, s) \phi_j(s) ds \right] - \lambda \int_a^b k(x, s) f(s) ds. \quad (2.2.16)$$

Then the coefficients c_1, \dots, c_n are determined from the condition that the residual is orthogonal to each one of the coordinate functions $\phi_1(x), \dots, \phi_n(x)$, i.e., $\langle r[\tilde{\phi}_n], \phi_i \rangle = 0$ for $i = 1, \dots, n$, or

$$\int_a^b r[\tilde{\phi}_n] \phi_i(x) dx = 0, \quad i = 1, \dots, n, \quad (2.2.17)$$

which, in view of (2.2.16), gives the system of equations

$$\sum_{j=1}^n (\alpha_{ij} - \lambda \beta_{ij}) c_j = \lambda \gamma_i, \quad i = 1, \dots, n, \quad (2.2.18)$$

where

$$\begin{aligned} \alpha_{ij} &= \int_a^b \phi_i(x) \phi_j(x) dx, & \beta_{ij} &= \int_a^b \int_a^b k(x, s) \phi_i(x) \phi_j(s) ds dx, \\ \gamma_i &= \int_a^b \int_a^b k(x, s) \phi_i(x) f(s) ds dx. \end{aligned} \quad (2.2.19)$$

The system (2.2.18) uniquely determines the coefficients c_1, \dots, c_n provided that the determinant of this system

$$D(\lambda) = \det [\alpha_{ij} - \lambda \beta_{ij}] \neq 0. \quad (2.2.20)$$

Then the approximate solution $\tilde{\phi}_n(x)$ is given by (2.2.15).

The zeros of $D(\lambda)$ are the approximate eigenvalues $\tilde{\lambda}_m$, $m = 1, \dots, n$, and the corresponding approximate eigenfunctions $\tilde{\phi}_n^{(m)}(x) = \sum_{j=1}^n \tilde{c}_j^{(m)} \phi_i(x)$, where $\tilde{c}_i^{(m)}$ are the roots of the homogeneous equation

$$\sum_{j=1}^n (\alpha_{ij} - \tilde{\lambda} \beta_{ij}) \tilde{c}_j^{(m)} = 0, \quad i = 1, \dots, n. \quad (2.2.21)$$

EXAMPLE 2.2.3. Consider the kernel as in Example 2.1.1:

$$k(x, s) = \begin{cases} s & \text{if } x \geq s, \\ x & \text{if } x \leq s. \end{cases}$$

Then the integral equation is

$$\phi(x) - \lambda \left[\int_0^x s\phi(s) ds + \int_x^1 x\phi(s) ds \right] = 0.$$

We take $\tilde{\phi}_3(x) = c_1 x + c_2 x^2 + c_3 x^3$. Then from (2.2.16) the residual is

$$\begin{aligned} r[\tilde{\phi}_3(x)] &= c_1 x + c_2 x^2 + c_3 x^3 - \lambda \left[\int_0^x s(c_1 s + c_2 s^2 + c_3 s^3) ds \right. \\ &\quad \left. + \int_x^1 x(c_1 s + c_2 s^2 + c_3 s^3) ds \right] \\ &= \frac{x}{60} \left[10c_1(6 + (x^2 - 3)\lambda) + 5c_2(12x + (x^3 - 4)\lambda) \right. \\ &\quad \left. + 3c_3(20x^2 + (x^4 - 5)\lambda) \right]. \end{aligned}$$

Then by (2.2.17) we have

$$\int_0^1 r[\tilde{\phi}_3(x)] x dx = 0, \quad \int_0^1 r[\tilde{\phi}_3(x)] x^2 dx = 0, \quad \int_0^1 r[\tilde{\phi}_3(x)] x^3 dx = 0,$$

which give the system

$$\begin{aligned} c_1 \left(\frac{1}{3} - \frac{2\lambda}{5} \right) + c_2 \left(\frac{1}{4} - \frac{7\lambda}{72} \right) + c_3 \left(\frac{1}{5} - \frac{8\lambda}{105} \right) &= 0, \\ c_1 \left(\frac{1}{4} - \frac{7\lambda}{72} \right) + c_2 \left(\frac{1}{5} - \frac{\lambda}{14} \right) + c_3 \left(\frac{1}{6} - \frac{9\lambda}{160} \right) &= 0, \\ c_1 \left(\frac{1}{5} - \frac{8\lambda}{105} \right) + c_2 \left(\frac{1}{6} - \frac{9\lambda}{160} \right) + c_3 \left(\frac{1}{7} - \frac{2\lambda}{45} \right) &= 0. \end{aligned}$$

Then the roots of the equation

$$D(\lambda) = \begin{vmatrix} 1/3 - 2\lambda/5 & 1/4 - 7\lambda/72 & 1/5 - 8\lambda/105 \\ 1/4 - 7\lambda/72 & 1/5 - \lambda/14 & 1/6 - 9\lambda/160 \\ 1/5 - 8\lambda/105 & 1/6 - 9\lambda/160 & 1/7 - 2\lambda/45 \end{vmatrix} = 0$$

yield the approximate eigenvalues

$$\tilde{\lambda}_1 = 2.46741, \quad \tilde{\lambda}_2 = 22.4012, \quad \tilde{\lambda}_3 = 69.6314.$$

These values can be compared with the exact eigenvalues $\lambda_1 = \frac{\pi^2}{4} = 2.4674011$, $\lambda_2 = \frac{9\pi^2}{4} = 22.2066099$, and $\lambda_3 = \frac{25\pi^2}{4} = 61.6850275$. The computational details are given in `galerkin1.nb`; also see Example 2.1.1. ■

Note that if λ is not an eigenvalue and n is sufficiently large, then Eq (2.2.16) has a unique (approximate) solution of the form (2.2.15). As $n \rightarrow \infty$, this approximate solution approaches the exact solution $\phi(x)$ of FK2 such that in the L_2 -norm

$$\|\phi - \tilde{\phi}_n\| \leq (1 + \varepsilon_n) \|\phi - P_n \phi\|, \quad (2.2.22)$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and P_n denotes the projection operator into the space spanned by the functions $u_1(x), u_2(x), \dots, u_n(x)$.

A generalization of the above method, known as the *Bubnov–Galerkin method*, is as follows: Choose two complete systems $\{u_n(x)\}$ and $\{v_n(x)\}$ such that the functions $u_1(x), u_2(x), \dots, u_n(x)$ and $v_1(x), v_2(x), \dots, v_n(x)$ are linearly independent. Then choose an approximate solution of the form (2.2.15), where the coefficients c_j , $j = 1, \dots, n$, are now determined by the condition that the residual $r[\tilde{\phi}_n(x)]$ be orthogonal to the functions $v_1(x), v_2(x), \dots, v_n(x)$. This yields the following linear system:

$$\begin{aligned} \sum_{i=1}^n c_i \int_a^b \left[u_i(x) - \lambda \int_a^b k(x, s) u_i(s) ds \right] \overline{v_j(x)} dx \\ = \int_a^b f(x) \overline{v_j(x)} dx, \quad j = 1, 2, \dots, n. \end{aligned} \quad (2.2.23)$$

The conditions for convergence of the Bubnov–Galerkin method are as follows: Assuming that the sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ are the bases of $L_2(a, b)$, let U_n and V_n denote the spaces spanned by $u_1(x), u_2(x), \dots, u_n(x)$ and $v_1(x), v_2(x)$,

$\dots, v_n(x)$, respectively, and let P_n be the projection operator into the space U_n . Let $C > 0$ be a constant such that $\|v\| \leq C \|P_n v\|$ for all $v \in V_n$. Then

- (i) the system (2.2.23) has a unique (approximate) solution when n is sufficiently large;
 - (ii) $\tilde{\phi}_n \rightarrow \phi$ in the L_2 -norm;
 - (iii) the estimate (2.2.22) holds for this method as well.
-

2.3. Degenerate Kernels

Since the resolvent kernel $h_\lambda(s, x)$ (see Section 1.9) is represented formally as a power series in λ , as in (1.9.8), and, therefore, cannot be constructed explicitly, the eigenvalues of a degenerate kernel possess the following properties:

- (i) If $L^{-1} \equiv (I - \lambda K)^{-1}$ exists for a given value of λ , then the resolvent also exists for that value of λ , which is therefore a regular value of the kernel;
- (ii) If L^{-1} is singular, then the equation

$$(I - \lambda K) \phi = 0 \quad (2.3.1)$$

has a nontrivial solution, and the corresponding nonzero function $\phi(x)$, given by

$$\phi(x) = \lambda \sum_{i=1}^n a_i(x) b_i(s), \quad (2.3.2)$$

satisfies the homogeneous FK2. Hence, λ is an eigenvalue of $k(x, s)$. For a degenerate kernel $k^{[n]}(x, s) = \sum_{m=0}^n X_m(x) S_m(s)$ of rank n^* (see Section 1.7), every value of λ is either a regular value or an eigenvalue. If we treat Eq (2.3.1) as an eigenvalue problem, then this equation has r nonzero eigenvalues, where r is the rank of K . Thus, a degenerate kernel of rank r has, in general, $r(\leq n)$ nonzero eigenpairs.

However, there exist infinitely many zero eigenvalues of a kernel of finite rank, i.e., there are infinitely many linearly independent functions $\xi_i \neq 0$ such that

$$K \xi_i = 0, \quad \xi_i \neq 0, \quad (2.3.3)$$

*An operator is said to be of finite rank if its image has finite dimension. Degenerate kernels give rise to finite-rank operators. If the functions a_i and b_i are all square-integrable on $[a, b] \times [a, b]$, then the operator K on $L_2[a, b]$, given by $(K\phi)(x) = \int_a^b k(x, s) \phi(s) ds$, has finite rank (see Porter and Stirling 1993).

since for any ξ , we have $K\xi = \sum_{i=1}^n \langle b_i, \xi \rangle a_i$, which implies that Eq (2.3.3) holds only if

$$\langle b_i, \xi \rangle = 0, \quad i = 1, \dots, n. \quad (2.3.4)$$

The finite set of conditions (2.3.4) can, however, be satisfied by infinitely many functions ξ_i .

In the case of degenerate kernels the following result holds (see Smithies 1958, Section 2.3): For any kernel $k(x, s) \in L_2$ and any $\varepsilon > 0$ there exists an L_2 -kernel $k_0(x, s)$ of finite rank such that $\|k - k_0\| < \varepsilon$. Any L_2 -kernel can be closely approximated by a continuous kernel. Thus, any kernel k can be written as

$$k = p + q, \quad (2.3.5)$$

where p is degenerate:

$$p = \sum_{i=1}^n a_i \otimes b_i, \quad (2.3.6)$$

as in (1.7.15), and q is arbitrarily small, i.e., $\|q\| < \varepsilon$. In order to compute eigenvalues λ of the kernel k , let ω denote a large finite number. We consider the spectrum for $|\lambda| < \omega$. Choose $\varepsilon = 1/\omega$, so that $-\omega < \lambda < \omega$ for all λ , and $|\lambda q| = |\lambda| \|q\| < \omega \varepsilon = 1$. Hence, for $|\lambda| < \omega$ there exists a resolvent q_λ for q

$$q_\lambda = q \sum_{i=0}^{\infty} \lambda^i q^i, \quad (2.3.7)$$

such that q_λ satisfies the resolvent equations

$$q_\lambda - q = \lambda q_\lambda q = \lambda q q_\lambda. \quad (2.3.8)$$

Hence, for an FK2 with $|\lambda| < \omega$, let

$$\begin{aligned} z &= f + \lambda q_\lambda f, \quad \langle \phi_i, b_i \rangle = \phi_i, \\ \langle z, b_i \rangle &= z_i, \quad \langle a_i + \lambda q_\lambda a_j, b_i \rangle = h_{ji}. \end{aligned}$$

Then

$$\phi_j = z_j + \lambda \sum_{i=1}^n h_{ji} \phi_i, \quad j = 1, \dots, n. \quad (2.3.9)$$

Also, if ϕ_j satisfies (2.3.9), then the FK2 has a solution

$$\phi = f + \lambda q_\lambda f + \sum_{i=1}^n \phi_i (a_i + \lambda q_\lambda a_i), \quad (2.3.10)$$

where ϕ_i is defined above. This result means that the solution of an FK2 reduces formally to that of the matrix equation (2.3.9). Note that if ϕ satisfies the FK2, then

$$\phi = f + \lambda K\phi = (f + \lambda p\phi) + \lambda q\phi. \quad (2.3.11)$$

Multiplying this equation by q_λ we get, in view of (2.3.8),

$$q_\lambda \phi = q_\lambda (f + \lambda p\phi) + \lambda q_\lambda q\phi = q_\lambda (f + \lambda p\phi) + (q_\lambda - q)\phi,$$

namely,

$$q\phi = \lambda q_\lambda (f + \lambda p\phi),$$

which, when substituted in the FK2, yields

$$\begin{aligned} \phi &= f + \lambda p\phi + \lambda q_\lambda (f + \lambda p\phi) \\ &= f + \lambda q_\lambda f + \lambda (p\phi + \lambda q_\lambda p\phi) \\ &= f + \lambda q_\lambda f + \lambda \sum_{i=1}^n \langle \phi, b_i \rangle \langle a_i + \lambda q_\lambda a_i \rangle \quad \text{from (2.3.6)} \\ &= z + \lambda F\phi, \end{aligned} \quad (2.3.12)$$

where the operator F , defined by

$$F = \sum_{i=1}^n \langle a_i + \lambda q_\lambda a_i \rangle \otimes b_i \equiv p + \lambda q_\lambda p,$$

has a kernel of finite rank n . Hence, we can construct a resolvent of k by writing (2.3.12) in the form

$$(I - \lambda F)\phi = z, \quad (2.3.13)$$

where the vector $\Phi = (\phi_j)$ satisfies the algebraic equations

$$(\mathbf{I} - \lambda \mathbf{F})\Phi = \mathbf{z} = (z_j), \quad \mathbf{F} = (f_{ji}). \quad (2.3.14)$$

Now, let $\mathbf{D} = (\mathbf{I} - \lambda \mathbf{F})^{-1} = (d_{ji})$ be the inverse matrix. Then the resolvent k_λ of k is given by

$$k_\lambda = q + \sum_{i,j=1}^n d_{ji} (a_j + \lambda q_\lambda a_j) \otimes (b_i \bar{\lambda} q_\lambda b_i). \quad (2.3.15)$$

Hence, we conclude that

(i) λ is a regular value of k for $|\lambda| < \omega$ unless $\mathbf{I} - \lambda \mathbf{F}$ is singular. The singular values of the matrix equation (2.3.13) are at most n . If $\det(\mathbf{I} - \lambda \mathbf{F}) = 0$, then

there exists a nontrivial solution of a homogeneous FK2. Thus, for $|\lambda| < \omega$, every value of λ is either a regular value or an eigenvalue.

(ii) Let $\omega \rightarrow \infty$. Then

- (a) Every value of λ is a regular value or an eigenvalue.
- (b) The eigenvalues are at most enumerable and have a zero limit value.
- (c) The resolvent kernel k_λ is meromorphic in λ for all λ .
- (d) If λ is an eigenvalue of k , the homogeneous FK2 has at most finitely many linearly independent L_2 -solutions.

EXAMPLE 2.3.1. Consider the Hermitian kernel

$$k(x, s) = xs - \frac{x^3 s^3}{6}, \quad 0 \leq x, s \leq 1.$$

We solve the modified Nyström equations (1.6.4). Since $k(x, s)$ is degenerate of rank 2, its nonzero characteristic values are the roots of the algebraic equation

$$\mu^2 - \frac{13}{42}\mu - \frac{2}{1575} = 0,$$

which are $\mu_1 = 0.31357$, $\mu_2 = -0.0040495$, whence the eigenvalues are $\lambda_1 = 3.18908$ and $\lambda_2 = 246.94406$.

Note that the kernel is an approximation of

$$\sin xs = xs - \frac{x^3 s^3}{3!} + \frac{x^5 s^5}{5!} - \dots.$$

An approximate solution of the homogeneous FK2 with this degenerate kernel is $\phi(x) = 1 + 0.0000009x - 0.0000002x^3$ with an error of $O(10^6)$, see Kythe (1998, p.195), Kantorovich and Krylov (1958), and Berezin and Zhidkov (1965).

Using Gauss-Legendre quadrature, we find that

For $n = 2$: $\mu_1 = 0.3145$, $\mu_2 = -0.0012267$;

For $n = 3$: $\mu_1 = 0.31357$, $\mu_2 = -0.0040496$;

For $n = 6$: $\mu_1 = 0.31357$, $\mu_2 = -0.0040496$.

Note that Gauss-Legendre quadrature gives accuracy with fewer integration points; for trapezoidal and Simpson quadratures, which converge more slowly, see Delves and Mohamed (1985, p.162). For computational details, see gauss1.nb. ■

EXAMPLE 2.3.2. Consider the homogeneous FK2

$$\mu \phi(x) = \int_0^1 e^{xs} \phi(s) ds, \quad 0 \leq x, s \leq 1.$$

The kernel is Hermitian but nondegenerate. Using the Nyström method with Gauss-Legendre quadrature, we find that the characteristic values are

For $n = 2$: $\mu_1 = 1.3521$,

For $n = 4$: $\mu_1 = 1.3530$, $\mu_4 = 0.000073823$,

For $n = 6$: $\mu_1 = 1.3530$, $\mu_4 = 0.00007638$,

For $n = 8$: $\mu_1 = 1.3530$, $\mu_4 = 0.00007638$, $\mu_7 = 1.6145 \times 10^{-10}$.

Note that Gauss-Legendre quadrature gives accuracy with fewer integration points; for trapezoidal and Simpson quadratures, which converge more slowly, see Delves and Mohamed (1985, p.163). Brakhage (1960) has determined that for $k(x, s) = e^{xs}$,

$$1.3527 < \mu_1 < 1.3534.$$

For computational details, see `gauss2.nb`. ■

EXAMPLE 2.3.3. Consider

$$\mu \phi(x) = \int_{-1}^1 xs^2 \phi(s) ds, \quad -1 \leq s, s \leq 1.$$

The kernel is Hermitian and degenerate of rank 1. It has no nonzero eigenvalues.

Note that $\mu \phi(x) = cx$, where $c = \int_{-1}^1 s^2 \phi(s) ds$. Thus, $\phi(x) = x$ (or a multiple of x) is the only possible eigenfunction, where the corresponding eigenvalue is given by $\lambda = \int_{-1}^1 s^2 \cdot s ds = 0$. Gauss-Legendre quadrature to solve the Nyström system (1.6.4) also gives the zero eigenvalue. For computational details, see `gauss3.nb`. ■

2.4. Replacement by a Degenerate Kernel

The Galerkin method and the method of least squares are special cases of the method of replacing a given kernel by a degenerate kernel. Consider an FK2 with $\lambda = 1$, which we write as

$$\phi(x) - \int_a^b k(x, s) \phi(s) ds = f(x), \quad (2.4.1)$$

and assume that it is solvable for all $f \in L_2(a, b)$. Also assume that the kernel $k(x, s)$ can be partitioned as the sum

$$k(x, s) = h(x, s) + r(x, s), \quad (2.4.2)$$

where $r(x, s)$ has a small L_2 -norm such that

$$\int_a^b \int_a^b |r(x, s)|^2 dx ds < \varepsilon^2, \quad (2.4.3)$$

where $\varepsilon > 0$ is arbitrarily small. Then the equation

$$\psi(x) - \int_a^b h(x, s) \psi(s) ds = f(x) \quad (2.4.4)$$

has a solution for all $f \in L_2(a, b)$ such that $\|\phi - \psi\| = O(\varepsilon)$. If we can find a partition of the type (2.4.2) such that the kernel $h(x, s)$ is degenerate, then Eq (2.4.4) can be reduced to a linear algebraic system. Further, if the kernel $k(x, s)$ satisfies the condition

$$\int_a^b |k(x, s)|^2 ds \leq A = \text{const}, \quad (2.4.5)$$

and the partition is carried in such a manner that $r(x, s)$ satisfies not only the condition (2.4.3) but also the condition

$$\int_a^b |r(x, s)|^2 ds < c\varepsilon, \quad c = \text{const}, \quad (2.4.6)$$

then, besides the above estimate, we have the uniform estimate $|\phi - \psi| = O(\varepsilon)$, which in the L_2 -norm can be written as $\|\phi - \psi\|_c = O(\varepsilon)$. This estimate can be refined as follows: Let $\gamma_k(x, s)$ and $\gamma_h(x, s)$ denote the resolvents of the kernels $k(x, s)$ and $h(x, s)$, respectively, and let $\|r\|$, $\|\gamma_k\|$, and $\|\gamma_h\|$ denote the L_2 -norms of the operators with the corresponding kernels. Then, in view of (2.4.3),

$$\begin{aligned} \|\phi - \psi\| &\leq \|r\| (1 + \|\gamma_k\|) (1 + \|\gamma_h\|) \|f\| \\ &< \varepsilon (1 + \|\gamma_k\|) (1 + \|\gamma_h\|) \|f\|. \end{aligned} \quad (2.4.7)$$

There are six ways in which the partition (2.4.2) can be carried out:

- (i) Choose a sequence $\{u_i(x)\}$, $i = 1, 2, \dots$, of functions that are orthonormal and complete in $L_2(a, b)$. Then we can partition the kernel $k(x, s)$ into a double Fourier series that converges in the mean:

$$k(x, s) = \sum_{i,j=1}^{\infty} A_{ij} u_i(x) u_j(s), \quad (2.4.8)$$

where

$$A_{ij} = \int_a^b \int_a^b k(x, s) \overline{u_i(x)} \overline{u_j(s)} dx ds. \quad (2.4.9)$$

Then take n sufficiently large and set

$$h(x, s) = \sum_{i,j=1}^n A_{ij} u_i(x) u_j(s). \quad (2.4.10)$$

(ii) Take

$$k(x, s) = \sum_{i,j=1}^{\infty} B_{ij} u_i(x) \overline{u_j(s)}, \quad (2.4.11)$$

where

$$B_{ij} = \int_a^b \int_a^b k(x, s) \overline{u_i(x)} u_j(s) dx ds. \quad (2.4.12)$$

Then for sufficiently large n set

$$h(x, s) = \sum_{i,j=1}^n A_{ij} u_i(x) \overline{u_j(s)}. \quad (2.4.13)$$

(iii) If the sequence $\{u_i(x)\}$, $i = 1, 2, \dots$, is complete but not orthonormal in $L_2(a, b)$, so is the sequence $\{u_i(x) u_j(s)\}$, $i, j = 1, 2, \dots$, complete but not orthonormal in $L_2(a, b)$. In this case, set

$$h(x, s) = \sum_{i,j=1}^{\infty} C_{ij} u_i(x) \overline{u_j(s)}, \quad (2.4.14)$$

and choose the coefficients C_{ij} such that

$$\int_a^b \int_a^b |k(x, s) - h(x, s)|^2 dx ds = \min. \quad (2.4.15)$$

This is done by computing the coefficients C_{ij} from the system

$$\begin{aligned} & \sum_{i,j=1}^{\infty} C_{ij} \int_a^b u_i(x) \overline{u_{i'}(x)} dx \int_a^b \overline{u_j(s)} u_{j'}(s) ds \\ &= \int_a^b \int_a^b k(x, s) \overline{u_{i'}(x)} u_{j'}(s) dx ds, \quad i', j' = 1, 2, \dots, n. \end{aligned} \quad (2.4.16)$$

Alternatively, we can also set

$$h(x, s) = \sum_{i,j=1}^{\infty} D_{ij} u_i(x) u_j(s), \quad (2.4.17)$$

and compute the coefficients D_{ij} from the system

$$\begin{aligned} & \sum_{i,j=1}^{\infty} D_{ij} \int_a^b u_i(x) \overline{u_{i'}(x)} dx \int_a^b u_j(s) \overline{u_{j'}(s)} ds \\ &= \int_a^b \int_a^b k(x, s) \overline{u_{i'}(x)} \overline{u_{j'}(s)} dx ds, \quad i', j' = 1, 2, \dots, n. \end{aligned} \quad (2.4.18)$$

(iv) The two methods described in (iii) are special cases of the following general method: Suppose that the two sequences $\{u_i(x)\}$ and $\{v_i(x)\}$, $i = 1, 2, \dots$, are complete but not orthogonal in $L_2(a, b)$. Then set

$$h(x, s) = \sum_{i,j=1}^{\infty} E_{ij} u_i(x) v_j(s), \quad (2.4.19)$$

where the coefficients E_{ij} are chosen such that the condition (2.4.15) is satisfied. Then the coefficients E_{ij} are computed from the system

$$\begin{aligned} & \sum_{i,j=1}^{\infty} E_{ij} \int_a^b u_i(x) \overline{u_{i'}(x)} dx \int_a^b v_j(s) \overline{v_{j'}(s)} ds \\ &= \int_a^b \int_a^b k(x, s) \overline{u_{i'}(x)} \overline{v_{j'}(s)} dx ds. \end{aligned} \quad (2.4.20)$$

(v) If the interval (a, b) is finite and the kernel $k(x, s)$ is continuous in the rectangle $a \leq x, s \leq b$, then approximate the kernel by a polynomial in x and s , and take this polynomial for $h(x, s)$

(vi) See the Galerkin method and the method of least squares as discussed in Sections 5.1 and 9.6, respectively.

EXAMPLE 2.4.1. (Mikhlin and Smolitskiy 1967, pp.291–292) Consider Eq (2.4.1) with $k(x, s) = \sin xs$, $0 < x, s < 1$, and set

$$r(x, s) = \sin xs - xs + \frac{x^3 s^3}{6} = \frac{x^5 s^5}{120} - \dots .$$

Then Eq (2.4.4) becomes

$$\psi(x) - x \int_0^1 s \psi(s) ds + \frac{x^3}{6} \int_0^1 s^3 \psi(s) ds = f(x),$$

whose solution is

$$\psi(x) = f(x) + c_1 x + c_2 x^3, \quad (2.4.21)$$

where

$$c_1 = \int_0^1 s \psi(s) ds, \quad c_2 = -\frac{1}{6} \int_0^1 \int_0^1 s^3 \psi(s) ds.$$

Multiplying (2.4.21) by x and $-x^3/6$ and integrating, we get

$$c_1 = f_1 + \frac{1}{3} c_1 + \frac{1}{5} c_2, \quad c_2 = f_2 - \frac{1}{30} c_1 - \frac{1}{42} c_2,$$

or

$$\frac{2}{3} c_1 - \frac{1}{5} c_2 = f_1, \quad \frac{1}{30} c_1 + \frac{43}{42} c_2 = f_2, \quad (2.4.22)$$

where

$$f_1 = \int_0^1 x f(x) dx, \quad f_2 = -\frac{1}{6} \int_0^1 x^3 f(x) dx.$$

The solution $\psi(x)$ is obtained by solving this system. Note that since $|r(x, s)| < \frac{x^5 s^5}{120}$, we have

$$\|r\| = \max_{0 < x < 1} \int_0^1 |r(x, s)| ds \leq \max_{0 < x < 1} \int_0^1 \frac{x^5 s^5}{120} ds = \frac{1}{720},$$

also, since

$$\begin{aligned} \|k(x, s)\| &= \max_{0 < x < 1} \int_0^1 \sin xs ds = \max_{0 < x < 1} \frac{1 - \cos x}{x} \\ &= 1 - \cos 1 < 1 - \cos \frac{\pi}{3} = \frac{1}{2}, \end{aligned}$$

we get

$$\|\gamma_k\| \leq \frac{\|k\|}{1 - \|k\|} = 1;$$

and

$$\begin{aligned} \|h(x, s)\| &= \max_{0 < x < 1} \int_0^1 \left(xs - \frac{x^3 s^3}{6} \right) ds \\ &= \max_{0 < x < 1} \left(\frac{x}{2} - \frac{x^3}{24} \right) = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}, \end{aligned}$$

which gives

$$\|\gamma_h\| < \frac{\|h\|}{1 - \|h\|} = \frac{11}{13}.$$

Hence, from (2.4.7),

$$\begin{aligned}\|\phi - \psi\| &< \frac{1}{720} \cdot (1+1) \left(1 + \frac{11}{13}\right) \|f\| = 0.0051282 \|f\| \\ &= 0.0051282 \max_{0 \leq x \leq 1} |f(x)|.\end{aligned}$$

Now, to compute $\|\phi - \psi\|$ in the metric of $L_2(0, 1)$, note that in view of (1.2.10), the norm of the integral operator K with kernel $k(x, s)$ in L_2 is at most equal to B . Thus,

$$|r(x, s)| < \frac{x^5 s^5}{120},$$

and

$$\|r\| \leq \frac{1}{120} \left\{ \int_0^1 \int_0^1 x^{10} s^{10} dx ds \right\}^{1/2} = \frac{1}{1320}.$$

Also,

$$\begin{aligned}\|k\| &\leq \left\{ \int_0^1 \int_0^1 \sin^2 xs dx ds \right\}^{1/2} \\ &\leq \left\{ \int_0^1 \int_0^1 x^2 s^2 dx ds \right\}^{1/2} = \frac{1}{3}, \quad \text{since } |\sin xs| < |xs|, \\ \|h\| &\leq \left\{ \int_0^1 \int_0^1 \left(xs - \frac{x^3 s^3}{6}\right) dx ds \right\}^{1/2} = \left(\frac{1}{9} - \frac{1}{75} + \frac{1}{1764}\right)^{1/2} < \frac{1}{3}.\end{aligned}$$

Hence, $\|\gamma_k\| \leq 1/2$, $\|\gamma_h\| < 1/2$, and

$$\|\phi - \psi\| < \frac{1}{1320} \cdot \frac{9}{4} \|f\| = 0.0017045 \|f\|.$$

See Example 2.3.1 also. ■

2.5. Bateman's Method

Besides replacing the given kernel by a degenerate (algebraic) kernel, it is sometimes useful to represent it approximately in the form of the sum of a kernel whose

resolvent is known and a degenerate kernel. A method of obtaining this resolvent, due to Bateman (1922), is as follows: For an FK2 with kernel $k(x, s)$, let the resolvent $k_\lambda(x, s)$ be known so that the solution of this FK2 is given by

$$\phi(x) = f(x) + \lambda \int_a^b k_\lambda(x, s)f(s) ds. \quad (2.5.1)$$

Then for an FK2 with the kernel

$$\kappa(x, s) = \frac{\begin{vmatrix} k(x, s) & f_1(x) & \dots & f_n(x) \\ g_1(s) & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ g_n(s) & a_{n1} & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}, \quad (2.5.2)$$

where f_i and g_i , $i = 1, \dots, n$, are arbitrary functions and a_{ij} arbitrary numbers, the resolvent $k_\lambda(x, s)$ is given by

$$\kappa_\lambda(x, s) = \frac{\begin{vmatrix} k_\lambda(x, s) & \phi_1(x) & \dots & \phi_n(x) \\ \psi_1(s) & a_{11} + \lambda\tau_{11} & \dots & a_{1n} + \lambda\tau_{1n} \\ \dots & \dots & \dots & \dots \\ \psi_n(s) & a_{n1} + \lambda\tau_{n1} & \dots & a_{nn} + \lambda\tau_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} + \lambda\tau_{11} & a_{12} + \lambda\tau_{12} & \dots & a_{1n} + \lambda\tau_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} + \lambda\tau_{n1} & a_{n2} + \lambda\tau_{2n} & \dots & a_{nn} + \lambda\tau_{nn} \end{vmatrix}}, \quad (2.5.3)$$

where

$$\begin{aligned} \phi_i(x) &= f_i(x) + \lambda \int_a^b k_\lambda(x, s)f_i(s) ds, \\ \psi_i(x) &= g_i(x) + \lambda \int_a^b k_\lambda(x, s)g_i(s) ds, \\ \tau_{ij} &= \int_a^b g_i(x)\phi_j(x) dx, \quad i, j = 1, \dots, n. \end{aligned} \quad (2.5.4)$$

Obviously, $\phi_i(x)$ and $\psi_i(x)$ are the solutions of the FK2s

$$\begin{aligned} \phi_i(x) - \lambda \int_a^b k(x, s)\phi_i(s) ds &= f_i(x), \\ \psi_i(x) - \lambda \int_a^b k(x, s)\psi_i(s) ds &= g_i(x), \quad i, j = 1, \dots, n. \end{aligned} \quad (2.5.5)$$

For a proof of this result, see Kantorovich and Krylov (1958, pp.156–158). Two particular cases of this result are the following:

(i) Set $a_{ij} = \delta_{ij}$ (Kronecker delta). Then from (2.5.3)

$$\kappa(x, s) = k(x, s) - \sum_{i=1}^n f_i(x)g_i(s), \quad (2.5.6)$$

and its resolvent is given by

$$\kappa_\lambda(x, s) = \frac{\begin{vmatrix} k_\lambda(x, s) & \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \psi_1(s) & 1 + \lambda\tau_{11} & \lambda\tau_{12} & \dots & \lambda\tau_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \psi_n(s) & \lambda\tau_{n1} & \lambda\tau_{n2} & \dots & 1 + \lambda\tau_{nn} \end{vmatrix}}{\begin{vmatrix} 1 + \lambda\tau_{11} & \lambda\tau_{12} & \dots & 1 + \lambda\tau_{1n} \\ \lambda\tau_{21} & 1 + \lambda\tau_{22} & \dots & \lambda\tau_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda\tau_{n1} & \lambda\tau_{n2} & \dots & 1 + \lambda\tau_{nn} \end{vmatrix}}. \quad (2.5.7)$$

In the case when the kernel $\kappa(x, s)$ is purely algebraic (i.e., $k(x, s) = 0$), we have

$$\kappa(x, s) = - \sum_{i=1}^n f_i(x)g_i(s).$$

Then $k_\lambda(x, s) = 0$ and, from (2.5.5), we get $\phi_i(x) = f_i(x)$ and $\psi_i(x) = g_i(x)$, which yields

$$\tau_{ij} = \int_a^b f_i(x)g_j(x) dx.$$

Hence, the resolvent becomes

$$\kappa_\lambda(x, s) = \frac{\begin{vmatrix} 0 & f_1(x) & \dots & f_n(x) \\ g_1(s) & 1 + \lambda\tau_{11} & \dots & \lambda\tau_{1n} \\ \dots & \dots & \dots & \dots \\ g_n(s) & \lambda\tau_{n1} & \dots & 1 + \lambda\tau_{nn} \end{vmatrix}}{\begin{vmatrix} 1 + \lambda\tau_{11} & \lambda\tau_{12} & \dots & \lambda\tau_{1n} \\ \lambda\tau_{21} & 1 + \lambda\tau_{22} & \dots & \lambda\tau_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda\tau_{n1} & \lambda\tau_{n2} & \dots & 1 + \lambda\tau_{nn} \end{vmatrix}}. \quad (2.5.8)$$

(ii) Consider an FK2 with some kernel $\hat{k}(x, s)$. Choose points x_1, x_2, \dots, x_n and s_1, s_2, \dots, s_n arbitrarily in the interval (a, b) . In (2.5.3) set $k(x, s) = 0$,

$f_i(x) = \hat{k}(x, s_i)$, $g_j(s) = -\hat{k}(x_j, s)$, and $a_{ij} = \hat{k}(x_i, s_j)$. Then, obviously, $k_\lambda(x, s) = 0$, and the kernel $\kappa(x, s)$ becomes

$$\begin{aligned} \kappa(x, s) &= - \frac{\begin{vmatrix} 0 & \hat{k}(x, s_1) & \dots & \hat{k}(x, s_n) \\ \hat{k}(x_1, s) & \hat{k}(x_1, s_1) & \dots & \hat{k}(x_1, s_n) \\ \dots & \dots & \dots & \dots \\ \hat{k}(x_n, s) & \hat{k}(x_n, s_1) & \dots & \hat{k}(x_n, s_n) \end{vmatrix}}{\begin{vmatrix} \hat{k}(x_1, s_1) & \hat{k}(x_1, s_2) & \dots & \hat{k}(x_1, s_n) \\ \hat{k}(x_2, s_1) & \hat{k}(x_2, s_2) & \dots & \hat{k}(x_2, s_n) \\ \dots & \dots & \dots & \dots \\ \hat{k}(x_n, s_1) & \hat{k}(x_n, s_2) & \dots & \hat{k}(x_n, s_n) \end{vmatrix}} \quad (2.5.9) \\ &= \hat{k}(x, s) - \frac{\begin{vmatrix} \hat{k}(x, s_1) & \hat{k}(x, s_2) & \dots & \hat{k}(x, s_n) \\ \hat{k}(x_1, s_1) & \hat{k}(x_1, s_2) & \dots & \hat{k}(x_1, s_n) \\ \dots & \dots & \dots & \dots \\ \hat{k}(x_n, s_1) & \hat{k}(x_n, s_2) & \dots & \hat{k}(x_n, s_n) \end{vmatrix}}{\begin{vmatrix} \hat{k}(x_1, s_1) & \hat{k}(x_1, s_2) & \dots & \hat{k}(x_1, s_n) \\ \hat{k}(x_2, s_1) & \hat{k}(x_2, s_2) & \dots & \hat{k}(x_2, s_n) \\ \dots & \dots & \dots & \dots \\ \hat{k}(x_n, s_1) & \hat{k}(x_n, s_2) & \dots & \hat{k}(x_n, s_n) \end{vmatrix}}. \end{aligned}$$

Note that this kernel is degenerate and coincides with the kernel $\hat{k}(x, s)$ on the lines $x = x_i$, $s = s_i$. In fact, by setting $x = x_i$, $s = s_j$ ($i, j = 1, \dots, n$) in (2.5.9), we find that in the second term the determinant in the numerator will have two identical rows and, hence, vanish. This yields

$$\kappa(x_i, s) = \hat{k}(x_i, s), \quad \kappa(x, s_j) = \hat{k}(x, s_j).$$

This coincidence on $2n$ lines implies that $\kappa(x, s)$ is close to $\hat{k}(x, s)$, and the solution of the FK2 with kernel $\kappa(x, s)$ is close to that of the FK2 with kernel $\hat{k}(x, s)$. But since $\hat{k}(x, s)$ is degenerate, i.e.,

$$\hat{k}(x, s) = \sum_{m=1}^n \alpha_m(x) \beta_m(s),$$

the determinant in the numerator vanishes identically, and therefore in this case $\kappa(x, s) = \hat{k}(x, s)$. Note that if $\hat{k}(x, s)$ contains less than n summands, then the determinant in the denominator should also vanish. But in this case it does not happen; it is generally nonzero, except it vanishes only for some particular choice

of the points x_i and s_j , ($i, j = 1, \dots, n$). Then the resolvent kernel is given by

$$\begin{aligned} k_\lambda(x, s) &= 0, \quad \phi_j(x) = f_j(x) = \hat{k}(x, s_j), \\ \psi_i(s) &= g_i(s) = -\hat{k}(x_i, s), \\ \tau_{ij} &= - \int_a^b \hat{k}(x, s_j) \hat{k}(x_i, s) dx = -\hat{k}_2(x_i, s_j), \end{aligned}$$

where $\hat{k}_2(x, s)$ is the second iterate for $\hat{k}(x, s)$, defined by

$$\hat{k}_2(x, s) = \int_a^b \hat{k}(x, t) \hat{k}(t, s) dt.$$

Thus,

$$\kappa_\lambda(x, s) = \frac{\begin{vmatrix} 0 & \hat{k}(x, s_1) & \dots & \hat{k}(x, s_n) \\ \hat{k}(x_1, s) & \hat{k}(x_1, s_1) - \lambda \hat{k}_2(x_1, s_1) & \dots & \hat{k}(x_1, s_n) - \lambda \hat{k}_2(x_1, s_n) \\ \dots & \dots & \dots & \dots \\ \hat{k}(x_n, s) & \hat{k}(x_n, s_1) - \lambda \hat{k}_2(x_n, s_1) & \dots & \hat{k}(x_n, s_n) - \lambda \hat{k}_2(x_n, s_n) \end{vmatrix}}{\begin{vmatrix} \hat{k}(x_1, s_1) - \lambda \hat{k}_2(x_1, s_1) & \dots & \hat{k}(x_1, s_n) - \lambda \hat{k}_2(x_1, s_n) \\ \hat{k}(x_2, s_1) - \lambda \hat{k}_2(x_2, s_1) & \dots & \hat{k}(x_2, s_n) - \lambda \hat{k}_2(x_2, s_n) \\ \dots & \dots & \dots \\ \hat{k}(x_n, s_1) - \lambda \hat{k}_2(x_n, s_1) & \dots & \hat{k}(x_n, s_n) - \lambda \hat{k}_2(x_n, s_n) \end{vmatrix}}. \quad (2.5.10)$$

We can use (2.5.10) to obtain an approximate solution of an FK2 with kernel $\hat{k}(x, s)$. We also find approximate eigenvalues λ of this kernel by equating to zero the determinant in the denominator of (2.5.10).

EXAMPLE 2.5.1. (Kantorovich and Krylov 1958, p.161) Consider the kernel $\hat{k}(x, s)$ defined in the square $[0, 1] \times [0, 1]$ by

$$\hat{k}(x, s) = \begin{cases} x(s-1) & \text{if } x \leq s, \\ s(x-1) & \text{if } x \geq s. \end{cases}$$

Note that

$$\hat{k}_2(x, s) = \int_0^1 \hat{k}(x, t) \hat{k}(t, s) dt = \begin{cases} \frac{x}{6} (1-s) (2s-x^2-s^2) & \text{if } x \leq s, \\ \frac{s}{6} (1-x) (2x-x^2-s^2) & \text{if } x \geq s. \end{cases}$$

We choose the points x_i and s_j ($i, j = 1, 2, 3, 4, 5$) equally distributed as

$$x_1 = s_1 = \frac{1}{6}, \quad x_2 = s_2 = \frac{1}{3}, \quad x_3 = s_3 = \frac{1}{2}, \quad x_4 = s_4 = \frac{2}{3}, \quad x_5 = s_5 = \frac{5}{6}.$$

Then the denominator of (2.5.10) with $\lambda = 216x$ becomes

$$\begin{vmatrix} 5(1 - 10x) & 4(1 - 19x) & 3(1 - 26x) & 2(1 - 31x) & 1 - 34x \\ 4(1 - 19x) & 8(1 - 16x) & 6(1 - 23x) & 4(1 - 28x) & 2(1 - 31x) \\ 3(1 - 26x) & 6(1 - 23x) & 9(1 - 18x) & 6(1 - 23x) & 3(1 - 26x) \\ 2(1 - 31x) & 4(1 - 28x) & 6(1 - 23x) & 8(1 - 16x) & 4(1 - 19x) \\ 1 - 34x & 2(1 - 31x) & 3(1 - 26x) & 4(1 - 19x) & 5(1 - 10x) \end{vmatrix} = 0.$$

This simplifies to the equation

$$130x^5 - 441x^4 + 488x^3 - 206x^2 + 30x - 1 = 0,$$

with the solution set $\{0.0467458, 0.2, 0.5, 1, 1.64556\}$, which gives the approximate eigenvalues as

$$\lambda_1 = 10.0971, \quad \lambda_2 = 43.2, \quad \lambda_3 = 108, \quad \lambda_4 = 216, \quad \lambda_5 = 355.441.$$

These values can be compared with the exact eigenvalues given by $n^2\pi^2$ for $n = 1, 2, \dots$, which are

$$\lambda_1 = 9.8696, \quad \lambda_2 = 39.4784, \quad \lambda_3 = 88.8264, \quad \lambda_4 = 157.914, \quad \lambda_5 = 246.74.$$

See `bateman.nb` for computational details. This file also computes approximate eigenvalues for $x_1 = s_1 = 0.1, x_2 = s_2 = 0.3, x_3 = s_3 = 0.5, x_4 = s_4 = 0.7$, and $x_5 = s_5 = 0.9$. The results show a high degree of inaccuracy. Somewhat better results can be obtained by choosing a large number of points x_i and s_j in the interval $(0, 1)$. However, the kernel $\hat{k}(x, s)$ has a singularity at $x = s$ because its derivative is discontinuous at $s = x$, which leads to a poor approximation by algebraic kernels. ■

2.6. Generalized Eigenvalue Problem

In the Ritz and Galerkin methods (Sections 2.1, 2.2) we have used expansion methods for the eigenvalue problem. The general case is as follows: An expansion

method for the homogeneous FK2, defined by (1.2.2) with $f(x) = 0$, begins by approximating the solution $\phi(x)$ by the characteristic function $\psi^{(n)}(x)$ by

$$\phi(x) \approx \psi^{(n)}(x) = \sum_{i=1}^n a_i^{(n)} h_i(x), \quad i = 1, \dots, n, \quad (2.6.1)$$

where $\{h_i(x)\}$ is the set of basis functions, and characteristic value μ by $\mu^{(n)}$. Then the residual is defined by

$$r_n(x) = \int_a^b k(x, s) \psi^{(n)}(s) ds - \mu^{(n)} \psi^{(n)}(x). \quad (2.6.2)$$

The unknown coefficients $a_i^{(n)}$, $i = 1, \dots, n$, are determined such that the residual $r_n(x)$ is kept as small as possible. One way to accomplish this is to require that $r_n(x)$ be orthogonal to each basis function $h_i(x)$. Then, using the Galerkin method, we get

$$\int_a^b \overline{h_i(x)} \left[\int_a^b k(x, s) \psi^{(n)}(s) ds - \mu^{(n)} \psi^{(n)}(x) \right] dx = 0, \quad (2.6.3)$$

where the bar denotes the complex conjugate. This, in view of (2.6.2), yields the linear system (in matrix form)

$$\mathbf{L} \mathbf{a}^{(n)} = \mu^{(n)} \mathbf{M} \mathbf{a}^{(n)}, \quad (2.6.4)$$

where

$$\begin{aligned} L_{ij} &= \int_a^b \int_a^b \overline{h_i(x)} k(x, s) h_j(s) dx ds, \\ M_{ij} &= \int_a^b \overline{h_i(x)} h_j(x) dx, \quad i, j = 1, \dots, n. \end{aligned} \quad (2.6.5)$$

This is the generalized eigenvalue problem. If the set $\{h_i(x)\}$ is orthonormal, then $\mathbf{M} = \mathbf{I}$ (identity matrix) and the problem reduces to a simple eigenvalue problem.

The following two results hold for the approximate characteristic value $\tilde{\mu}$ and characteristic function $\tilde{\psi}(x) = \phi(x) + e(x)$, where $e(x)$ is the error term:

(i) Let $k(x, s)$ be a real and symmetric kernel, and let

$$\tilde{\mu} = \frac{\int_a^b \tilde{\psi}(x) \int_a^b k(x, s) \tilde{\psi}(s) ds dx}{\int_a^b |\tilde{\psi}(x)|^2 dx}. \quad (2.6.6)$$

Then $\tilde{\mu} = \mu + O(\|e\|^2)$. The value $\tilde{\mu}$, defined by (2.6.6), is stationary at the solution point.

(ii) Let $\{a_i^{(n)}, \mu_i^{(n)}\}$ be the i th characteristic solution of Eq (2.6.4), where the characteristic values are ordered such that

$$\mu_1^{(n)} \geq \mu_2^{(n)} \geq \cdots \geq \mu_i^{(n)} \geq \cdots \geq \mu_n^{(n)}. \quad (2.6.7)$$

Then $\mu_i^{(n)}$ is the stationary point of the functional (2.6.6). Moreover, if the kernel $k(x, s)$ is positive-definite, the stationary point is a maximum point for each i and the approximate characteristic values satisfy the condition

$$\mu_{i+1}^{(n+1)} \geq \mu_i^{(n)} \geq \mu_1^{(n+1)}, \quad i = 1, \dots, n. \quad (2.6.8)$$

Also, the exact characteristic value $\mu_i \geq \mu_i^{(n)}$ for $i = 1, \dots, n$ for all n . The above method has practical application for Hermitian kernels.

EXAMPLE 2.6.1. Consider the homogeneous FK2 with the Hermitian kernel

$$\mu \phi(x) = \int_0^1 \left(xs - \frac{1}{6} x^3 s^3 \right) \phi(s) ds, \quad 0 \leq x, s \leq 1.$$

Then taking the basis functions $h_i(x) = x^{i-1}$, we find from (2.6.5) that

$$\begin{aligned} L_{ij} &= \int_0^1 \int_0^1 x^{i-1} \left(xs - \frac{1}{6} x^3 s^3 \right) s^{j-1} dx ds \\ &= \frac{1}{(i+1)(j+1)} - \frac{1}{6(i+3)(j+3)}, \\ M_{ij} &= \int_0^1 x^{i-1} x^{j-1} dx = \frac{1}{i+j-1}. \end{aligned}$$

Then

$$\sum_{i,j=1}^n \left[\frac{1}{(i+1)(j+1)} - \frac{1}{6(i+3)(j+3)} - \mu^{(n)} \frac{1}{i+j-1} \right] a_i^{(n)} = 0,$$

which, e.g., for $i, j = 1, 2$, yields $L_{11} = \frac{23}{96}, L_{12} = L_{21} = \frac{19}{120}, L_{22} = \frac{47}{450}$, $M_{11} = 1, M_{12} = M_{21} = \frac{1}{2}, M_{22} = \frac{1}{3}$, and the matrix equation (2.6.4) becomes

$$\begin{bmatrix} 23/96 & 19/120 \\ 19/120 & 47/450 \end{bmatrix} \begin{Bmatrix} a_1(2) \\ a_2(2) \end{Bmatrix} = \mu^{(2)} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{Bmatrix} a_1(2) \\ a_2(2) \end{Bmatrix}.$$

This equation has a nonzero solution $a_i^{(2)} \neq 0$ for $i = 1, 2$ only if the following determinant is zero:

$$0 = \begin{vmatrix} \frac{23}{96} - \mu^{(2)} & \frac{19}{120} - \frac{1}{2}\mu^{(2)} \\ \frac{19}{120} - \frac{1}{2}\mu^{(2)} & \frac{47}{450} - \frac{1}{3}\mu^{(2)} \end{vmatrix} = \frac{\mu^2}{12} - \frac{197\mu}{7200} - \frac{1}{21600},$$

whose nonzero solutions are $\mu^{(2)} = 0.313439, -0.00177245$. This is known as the generalized eigenvalue routine. The values of $\mu_i^{(n)}$ for $n = 2, 3, 4, 5, 6, 7, 8$ are listed in Table 2.6.1.

Table 2.6.1

n	$\mu^{(n)}$
2	0.313439
3	0.313570
4	0.313573
5	0.313573
6	0.313573
7	0.313573
8	0.313573

There are only two nonzero characteristic values out of n real values; they are $\mu = -0.00404958, 0.313573$; thus, the two eigenvalues are $\lambda = 1/|\mu| = 3.1890500078, 246.9391888$ (cf. Example 2.3.1). For computational details, see `seis1.nb`. ■

EXAMPLE 2.6.2. Consider the homogeneous FK2 with Hermitian kernel $k(x, s) = e^{xs}$, $0 \leq x, s \leq 1$, and $\lambda = 1$. For this problem Delves and Mohamed (1985) suggest to compute the matrix \mathbf{L} by using the set of recurrence relations

$$p_1 = e - 1, \quad p_i = e - (i - 1)p_{i-1}, \quad i = 2, \dots, n;$$

$$L_{11} = \int_0^1 \frac{e^x - 1}{x} dx,$$

$$L_{i1} = L_{1i} = p_{i-1} - \frac{1}{i-1}, \quad i = 2, \dots, n,$$

$$L_{ij} = p_{i-1} - (j-1)L_{i-1,j-1}, \quad i, j = 2, \dots, n.$$

The matrix \mathbf{M} with the basis functions x^{i-1} , $i = 1, 2, \dots, n$, turns out to be the

same as that in Example 2.6.1. Notice that

$$L_{11} = \int_0^1 \frac{e^x - 1}{x} dx \approx 1.3179.$$

The values of $\mu_i^{(n)}$ for $n = 2, 3, 4, 5, 6, 8$ are listed in Table 2.6.2, where columns 2, 3, and 4 give the largest, the second-largest, and the fourth-largest values in magnitude; a ‘—’ sign indicates that the value is smaller than 10^{-9} .

Table 2.6.2

n	$\mu^{(n)}$		
2	1.3527	0.10408	—
3	1.3530	0.10597	—
4	1.3530	0.10598	0.000075032
5	1.3530	0.10598	0.000077408
6	1.3530	0.10600	0.00078257
8	15.5763	1.3536	0.32421

Compare these values of μ with those in Example 2.3.2. For computational details, see `seis2.nb`. ■

EXAMPLE 2.6.3. Consider the homogeneous FK2

$$\mu \phi(x) = \int_{-1}^1 x s^2 \phi(s) ds, \quad -1 \leq x, s \leq 1.$$

This kernel has rank 1 with nonzero eigenvalues (see Example 2.3.3). With the basic functions $h_i(x) = x^{i-1}$, the matrices **L** and **M** are given by

$$L_{ij} = \frac{1 + (-1)^i}{i+1} \cdot \frac{1 + (-1)^{j+1}}{j+2},$$

$$M_{ij} = \frac{1 + (-1)^{i+j}}{i+j-1}, \quad i, j = 1, \dots, n.$$

The matrix **L** is singular and of rank 1. Hence, all eigenvalues are zero. For computational details, see `seis3.nb`. ■

2.7. Applications

We shall use eigenfunctions to compute the solution of the FK2 (1.2.2) by a simple method that uses a set of invariant functions in $L_2[a, b]$. In view of the Nyström method, we replace the FK2 by an infinite system of linear algebraic equations of the form

$$\xi_i - \lambda \sum_{j=0}^{\infty} c_{ij} \xi_j = \eta_i, \quad i = 0, 1, 2, \dots, \quad (2.7.1)$$

where ξ_i are the unknowns, λ is a parameter, and η_i and the infinite matrix $C = \{c_{ij}\}$ are assumed to be preassigned. This method is also applicable when the FK2 involves vector functions and variables in higher dimensions (≥ 2) although computation will get involved in such cases. One method of transforming the FK2 into the infinite system (2.7.1) is to use a complete set of functions $G = \{g_0(x), g_1(x), g_2(x), \dots\}$ in $L_2[a, b]$ such that every function $f(x)$ has an expansion of the form

$$f(x) = \sum_{i=0}^{\infty} \eta_i g_i(x), \quad (2.7.2)$$

which converges in some sense in $L_2[a, b]$. If we assume that $\phi(x) = \sum_{i=0}^{\infty} \xi_i g_i(x)$, then the FK2 (1.2.2) becomes

$$\sum_{i=0}^{\infty} \xi_i g_i(x) - \lambda \int_a^b k(x, s) \sum_{j=0}^{\infty} \xi_j g_j(s) ds = \sum_{i=0}^{\infty} \eta_i g_i(x). \quad (2.7.3)$$

Also, suppose that

$$\int_a^b k(x, s) g_j(s) ds = \sum_{i=0}^{\infty} c_{ij} g_i(x), \quad j = 0, 1, 2, \dots. \quad (2.7.4)$$

Then, interchanging the order of integration and summation in (2.7.3) and using (2.7.4), we get

$$\sum_{i=0}^{\infty} \xi_i g_i(x) - \lambda \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_{ij} g_i(x) = \sum_{i=0}^{\infty} \eta_i g_i(x). \quad (2.7.5)$$

Thus, the infinite system of the type (2.7.1) is now obtained by equating the coefficients of $g_i(x)$ on both sides of (2.7.5) for $i = 0, 1, 2, \dots$. If the functions

belonging to the complete set G are assumed to be independent in the sense that each function in this class has a unique expansion of the form (2.7.2), then this construction of the infinite system of the type (2.7.1) is unambiguous. The difficulty with this construction is that it does not solve the FK2 because the infinite system (2.7.1) may itself be very difficult to solve unless the coefficient matrix

$$I - \lambda C = \begin{pmatrix} 1 - \lambda c_{11} & -\lambda c_{12} & -\lambda c_{13} & \dots \\ -\lambda c_{21} & 1 - \lambda c_{22} & -\lambda c_{23} & \dots \\ -\lambda c_{31} & -\lambda c_{32} & 1 - \lambda c_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (2.7.6)$$

has a specific structure. We shall use invariant functions to simplify this structure such that the solution of the system (2.7.1) is easily obtained. A set of functions $G = \{g_0(x), g_1(x), g_2(x), \dots\}$ is said to be *invariant* for the integral equation with kernel $k(x, s)$ if

$$\int_a^b k(x, s) g_j(s) ds = \sum_{m=1}^n \alpha_{jm} g_{p_{jm}}(x), \quad j = 0, 1, 2, \dots, \quad (2.7.7)$$

where n is a *fixed* positive integer, and $p_{j1}, p_{j2}, \dots, p_{jn}$ denote n distinct nonnegative integers, arranged in increasing order (for convenience). The transformation (2.7.7) is of the form (2.7.4) with

$$c_{ij} = \sum_{m=1}^n \delta_{ip_{jm}} \alpha_{mj}, \quad i, j = 0, 1, 2, \dots, \quad (2.7.8)$$

where δ_{ij} is the Kronecker delta, defined by $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$. Thus, the simplification in the structure of the coefficient matrix (2.7.6) is evident from (2.7.7) and (2.7.8). With j fixed, in each column at most n of the c_{ij} will be zero. This leads to the possibility of using the elimination method to solve the system (2.7.1) by developing recursive relations for ξ_i , and thus reducing the infinite system to a finite system of linear algebraic equations (see Examples 2.7.1 and 2.7.2 below).

A useful set of invariant functions for FK2 is the set

$$\Psi = \{\psi_0(x), \psi_1(x), \psi_2(x), \dots\}$$

of eigenfunctions of the kernel $k(x, s)$ such that

$$\int_a^b k(x, s) \psi_i(s) ds = \frac{1}{\lambda_i} \psi_i(x), \quad i = 0, 1, 2, \dots, \quad (2.7.9)$$

where the constants λ_i are the eigenvalues of the kernel. For example, if for

$$f(x) = \sum_{i=0}^{\infty} \eta_i \psi_i(x), \quad (2.7.10)$$

we assume that

$$\phi(x) = \sum_{i=0}^{\infty} \xi_i \psi_i(x), \quad (2.7.11)$$

then we obtain a very simple infinite system

$$\left(1 - \frac{\lambda}{\lambda_i}\right) \xi = \eta_i, \quad i = 0, 1, 2, \dots, \quad (2.7.12)$$

by substituting (2.7.10) and (2.7.11) into the FK2 (1.2.2). In this case the matrix $(I - \lambda C)$, defined by (2.7.6), is a diagonal matrix and the system (2.7.12) is obviously solvable. Rall (1969) proves that if $\lambda \neq \lambda_m$ for each nonnegative integer m , then

$$\xi_i = \frac{\lambda_i \eta_i}{\lambda_i - \lambda}, \quad i = 0, 1, 2, \dots, \quad (2.7.13)$$

are unique solutions of (2.7.12); if $\lambda = \lambda_m$ for some nonnegative integer m , then either $\lambda_m \neq 0$, in which case (2.7.12) has no solution, or $\lambda_m = 0$, in which case one may take $\xi_m = A$, where A is an arbitrary constant, and determine $\xi_0, \xi_1, \dots, \xi_{m-1}, \xi_{m+1}, \dots$ from (2.7.13). Then, in this latter case Eq (2.7.12) has an infinite number of solutions. In fact, this result is a version of the Fredholm alternative (see Section 1.10).

EXAMPLE 2.7.1. Consider the FK2 with $a = 0$, $b = 1$, and the kernel

$$k(x, s) = \begin{cases} s(1-x), & 0 \leq s \leq x, \\ x(1-s), & x \leq s \leq 1. \end{cases}$$

This kernel is the Green's function for the differential operator $L[u] = u''$ with the boundary conditions $u(0) = 0 = u(1)$ (see Courant and Hilbert, 1953, Ch.5, Sections 14 and 15). Since $k(x, s)$ is defined as a piecewise polynomial, we shall use $G = \{1, x, x^2, \dots\}$ as a possible set of invariant functions. Since

$$\begin{aligned} \int_0^1 k(x, s) s^j ds &= \int_0^x s(1-x)s^j ds + x \int_x^1 (1-s)s^j ds \\ &= \frac{x - x^{j+2}}{(j+1)(j+2)}, \end{aligned} \quad (2.7.14)$$

we notice that (2.7.7) is satisfied with $n = 2$, and $p_{j1} = 1$, $p_{j2} = j + 2$, $j = 0, 1, 2, \dots$. Now, we take

$$f(x) = \sum_{i=0}^{\infty} \eta_i x^i, \quad \phi(x) = \sum_{i=0}^{\infty} \xi_i x^i \quad (2.7.15)$$

as convergent power series. Then (2.7.14) and (2.7.15) transform the FK2 (1.2.2) into

$$\sum_{i=0}^{\infty} \xi_i x^i - \lambda \sum_{i=0}^{\infty} \frac{\xi_i x}{(i+1)(i+2)} + \lambda \sum_{i=0}^{\infty} \frac{\xi_i x^{i+2}}{(i+1)(i+2)} = \sum_{i=0}^{\infty} \eta_i x^i. \quad (2.7.16)$$

Equating the coefficients of similar powers of x in (2.7.16), we obtain the system

$$\begin{aligned} \xi_0 &= \eta_0, & \xi_1 - \lambda \sum_{i=0}^{\infty} \frac{\xi_i}{(i+1)(i+2)} &= \eta_1, \\ \xi_2 + \lambda \frac{\xi_0}{1 \cdot 2} &= \eta_2, & \xi_3 + \lambda \frac{\xi_1}{2 \cdot 3} &= \eta_3, & \xi_4 + \lambda \frac{\xi_2}{3 \cdot 4} &= \eta_4, & \dots \end{aligned} \quad (2.7.17)$$

Thus, the coefficient matrix (2.7.6) is given by

$$I - \lambda C = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & 0 & 0 & \dots \\ -\frac{1}{1 \cdot 2} & 1 - \frac{\lambda}{2 \cdot 3} & -\frac{\lambda}{3 \cdot 4} & -\frac{\lambda}{4 \cdot 5} & -\frac{\lambda}{5 \cdot 6} & \dots \\ \frac{\lambda}{1 \cdot 2} & 0 & 1 & 0 & 0 & \dots \\ 0 & \frac{\lambda}{2 \cdot 3} & 0 & 1 & 0 & \dots \\ 0 & 0 & \frac{\lambda}{3 \cdot 4} & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \quad (2.7.18)$$

Since all rows of this matrix after the second have only one off-diagonal element, we have the following recursive relation: For $i = 0, 2, 4, \dots$ (even),

$$\begin{aligned} \xi_0 &= \eta_0, \\ \xi_{2m+2} &= \eta_{2m+2} - \frac{\lambda \xi_{2m}}{(2m+1)(2m+2)}, \quad m = 0, 1, 2, \dots, \end{aligned} \quad (2.7.19)$$

and for $i = 3, 5, 7, \dots$ (odd),

$$\xi_{2m+3} = \eta_{2m+3} - \frac{\lambda \xi_{2m+1}}{(2m+2)(2m+3)}, \quad m = 0, 1, 2, \dots \quad (2.7.20)$$

Solving (2.7.18) and (2.7.19), we get

$$\begin{aligned}\xi_{2m} &= \sum_{j=0}^m (-\lambda)^j \frac{(2m-2j)!}{(2m)!} \eta_{2m-2j}, \\ \xi_{2m+3} &= \sum_{j=0}^m (-\lambda)^j \frac{(2m+3-2j)!}{(2m+3)!} \eta_{2m+3-2j} + \frac{(-\lambda)^{m+1}}{(2m+3)!} \xi_1,\end{aligned}\quad (2.7.21)$$

for $m = 0, 1, 2, \dots$. From (2.7.17),

$$\left(1 - \frac{\lambda}{2 \cdot 3}\right) \xi_1 - \lambda \sum_{m=0}^{\infty} \frac{\xi_{2m+3}}{(2m+4)(2m+5)} = \eta_1 + \lambda \sum_{m=0}^{\infty} \frac{\xi_{2m}}{(2m+1)(2m+2)}. \quad (2.7.22)$$

Now, substituting (2.7.20) and (2.7.21) into (2.7.22), we have

$$\begin{aligned}\left(\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{(2n+1)!}\right) \xi_1 &= \eta_1 + \lambda \sum_{m=0}^{\infty} \frac{1}{(2m+1)(2m+2)} \times \\ &\quad \times \left(\sum_{j=0}^m (-\lambda)^j \frac{(2m-2j)!}{(2m)!} \eta_{2m-2j}\right) \\ &\quad + \lambda \sum_{m=0}^{\infty} \frac{1}{(2m+4)(2m+5)} \times \\ &\quad \times \left(\sum_{j=0}^m (-\lambda)^j \frac{(2m+3-2j)!}{(2m+3)!} \eta_{2m+3-2j}\right).\end{aligned}\quad (2.7.23)$$

Thus, the coefficient of ξ_1 is

$$\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{(2n+1)!} = \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}}. \quad (2.7.24)$$

If we denote the right side of (2.7.23) by $\eta(\lambda)$, then Eq (2.7.23) can be written as

$$\frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \xi_1 = \eta(\lambda). \quad (2.7.25)$$

Hence, we can solve the system (2.7.17) by directly solving (2.7.25). Thus, if $\lambda \neq j^2\pi^2$ for each positive integer j , then the coefficient of ξ_1 is nonzero and Eq (2.7.25) has a unique solution

$$\xi_1 = \frac{\eta(\lambda) \sqrt{\lambda}}{\sin \sqrt{\lambda}}, \quad (2.7.26)$$

and the values of the remaining solutions ξ_i of (2.7.17) can be computed uniquely by (2.7.19) and (2.7.20). But if $\lambda = j^2\pi^2$ for some positive integer j , then either $\eta(j^2\pi^2) \neq 0$, in which case (2.7.25), and hence (2.7.17), has no solution, or $\eta(j^2\pi^2) = 0$, in which case we may take $\xi_1 = A$, an arbitrary constant, and this yields an infinite set of solutions of (2.7.17) by using (2.7.19) and (2.7.20). Here we have assumed that the solutions $\xi_0, \xi_1, \xi_2, \dots$ of the system (2.7.17), if they exist, are the coefficients of a convergent power series (2.7.15).

For the homogeneous FK2, we have $f(x) \equiv 0$ and $\eta_0 = \eta_1 = \eta_2 = \dots = 0$ in (2.7.17). If $\lambda = j^2\pi^2$, then $\eta(j^2\pi^2) = 0$, and we take $\xi_1 = A$ arbitrarily. Also, from (2.7.19) and (2.7.20), we have $\xi_{2m} = 0$ and $\xi_{2m+3} = \frac{(-j^2\pi^2)^{m+1}}{(2m+1)!} A_j$ for $m = 0, 1, 2, \dots$. Hence the corresponding solutions are

$$\phi_j(x) = A_j \sum_{m=0}^{\infty} \frac{(-1)^m (j\pi x)^{2m+1}}{j\pi(2m+1)!} = \frac{A_j}{j\pi} \sin j\pi x. \quad (2.7.27)$$

For $j = 1, 2, \dots$, the infinite series $\sum_{j=1}^{\infty} \phi_j(x)$ in (2.7.27) converges. Also, $\phi_j(x)$ are eigenfunctions of the kernel $k(x, s)$ corresponding to the eigenvalues $\lambda_j = j^2\pi^2$. If we take $A_j = j\pi\sqrt{2}$, for $j = 1, 2, \dots$, then the eigenfunctions

$$\psi_i(x) = \sqrt{2} \sin(i+1)\pi x, \quad i = 0, 1, 2, \dots, \quad (2.7.28)$$

are orthonormal; thus, $\int_0^1 \psi_i(x)\psi_j(x) dx = \delta_{ij}$. Thus, using the set

$$\Psi = \{\psi_0(x), \psi_1(x), \dots\}$$

of eigenfunctions (2.7.28), we have solved the given FK2 for the space $L_2[0, 1]$. Also, for $f(x) \in L_2[0, 1]$, the Fourier series $f(x) = \sum_{i=0}^{\infty} \eta_i \psi_i(x)$, where $\eta_i = \int_0^1 f(x)\psi_i(x) dx$, converges in the sense that $\int_0^1 f^2(x) dx = \sum_{i=0}^{\infty} \eta_i^2 < \infty$. Since the quantities $\frac{(i+1)^2\pi^2}{(i+1)^2\pi^2 - \lambda}$, $i = 0, 1, 2, \dots$, are uniformly bounded if λ is not an eigenvalue, we find from (2.7.13) that $\sum_{i=0}^{\infty} \xi_i^2 \leq M \sum_{i=0}^{\infty} \eta_i^2 < \infty$ so that $\phi(x) = \sum_{i=0}^{\infty} \xi_i \psi_i(x) \in L_2[0, 1]$ is a solution of the FK2. ■

EXAMPLE 2.7.2. Let the kernel of a homogeneous FK2 be

$$k(x, s) = \begin{cases} -\ln x, & 0 \leq s \leq x, \\ -\ln s, & x \leq s \leq 1, \end{cases}$$

which is the Green's function for the differential operator $L[u] = (xu')'$ with the boundary conditions $u(0) < \infty, u(1) = 0$ (see Margenau and Murphy 1956). Although this kernel is not a polynomial, we shall still use the set $G = \{1, x, x^2, \dots\}$, which yields

$$\int_0^1 k(x, s) s^j \, ds = \int_0^x (-\ln x) s^j \, ds + \int_x^1 (-\ln s) s^j \, ds = \frac{1 - x^{j+1}}{(j+1)^2}. \quad (2.7.29)$$

By substituting the power series $\phi(x) = \sum_{i=0}^{\infty} \xi_i x^i$ into the homogeneous FK2, we get the system

$$\begin{aligned} \xi_0 - \lambda \sum_{i=0}^{\infty} \frac{\xi_i}{(i+1)^2} &= 0, \\ \xi_1 + \frac{\lambda}{1^2} \xi_0 &= 0, \\ \xi_2 + \frac{\lambda}{2^2} \xi_1 &= 0, \\ \dots &\dots \dots, \end{aligned} \quad (2.7.30)$$

or, since $\xi_j = (-1)^j \frac{\lambda^j}{(j!)^2} \xi_0$, the homogeneous FK2 yields

$$\left(1 - \lambda \sum_{i=0}^{\infty} (-1)^i \frac{\lambda^i}{[(i+1)!]^2} \right) \xi_0 = 0,$$

or

$$\left(\sum_{i=0}^{\infty} (-1)^i \frac{\lambda^i}{(i!)^2} \right) \xi_0 = 0. \quad (2.7.31)$$

The coefficient of ξ_0 in (2.7.31) is the Bessel function of order zero $J_0(2\sqrt{\lambda}) = \sum_{i=0}^{\infty} (-1)^i \frac{\lambda^i}{(i!)^2}$, so that the eigenvalues of $k(x, s)$ are the zeros of $J_0(2\sqrt{\lambda})$. If these zeros are denoted by $\lambda_1, \lambda_2, \dots$, then the corresponding eigenfunctions yield the solution

$$\phi_j(x) = \xi_0 \sum_{i=0}^{\infty} (-1)^i \frac{\lambda_j^i x^i}{(i!)^2} = \xi_0 J_0(2\sqrt{\lambda_j} x), \quad j = 1, 2, \dots. \blacksquare \quad (2.7.32)$$

REFERENCES USED: Baker (1978), Bateman (1922), Berezin and Zhidkov (1965), Brakhage (1960), Chen and Xu (1998), Cochran (1972), Courant and Hilbert (1953), Delves and Mohamed (1985), Hoheisel (1968), Ikebe (1972), Jentzsch (1912), Kantorovich and Krylov (1958), Kohn (1972), Kythe (1998), Logan (1984), Margenau and Murphy (1956), Mikhailov (1970), Mikhlin and Smolitskiy (1967), Polyanin and Manzhirov (1998), Porter and Stirling (1993), Rall (1969), Russell and Shampine (1971, 1976), Smithies (1958).

3

Equations of the Second Kind

In this chapter we consider integral equations of the second kind (FK2 and VK2), with both kernel $k(x, s)$ and free term $f(x)$ continuous on the interval $[a, b]$, and assume that λ is a regular value of the kernel. As we know, the solution $\phi(x)$ is *not* unique when λ is an eigenvalue of the kernel. We present some basic facts about these equations in this chapter, and different methods of obtaining numerical solutions of these and other equations are discussed in this and subsequent chapters.

3.1. Fredholm Equations

We consider an FK2 of the form

$$\phi(x) - \lambda \int_a^b k(x, s)\phi(s) ds = f(x), \quad a \leq x \leq b, \quad (3.1.1)$$

where the kernel $k(x, s)$ and the free term $f(x)$ are continuous on the interval $[a, b]$, and λ is a regular value of the kernel. A practical difficulty arises when λ is an eigenvalue of the kernel or when the regular value of λ is close to an eigenvalue. Therefore, we are required to look into cases when the kernel is badly behaved, e.g., $k(x, s) \notin C^1[a, b]$. Equations of the form (3.1.1) arise in various physical problems, such as potential theory, Dirichlet problem, electrostatics, radiation heat transfer, particle transport problems in astrophysics, and reactor theory. Numerical solutions of this equation by different methods are available and are presented in subsequent sections and chapters of this book. A basic formulation in almost

all of these methods is to transform the integral equation into a system of linear algebraic equations that can be solved by direct or iterative techniques. Sometimes the matrix of the linear system is full and very large; fast solutions have been developed to handle this situation, for instance, by Atkinson (1997), Hackbusch (1985), Reischel (1987, 1989), and Yan (1994). An alternative approach in use for solving Eq (3.1.1) numerically, mostly in problems of radiative heat transfer between gray surfaces (see Siegel and Howell 1992, Ch. 7; and Sparrow and Cess 1970, Ch. 3) is to transform the integral equation into a linear ordinary differential equation that is solved analytically or numerically if the boundary conditions can be prescribed. One of such techniques was proposed by Perlmutter and Siegel (1963) in a problem of radiative heat transfer within a gray circular tube, where the kernel $k(x, s) = k(x - s)$ is smooth and the function $\phi(x)$ represents the (dimensionless) total radiant flux density from the surface of the cylinder at x and s . We look into this method in Section 3.1.2 ahead.

Under certain conditions an FK1 can always be transformed into an FK2 (see Section 11.4.6).

3.1.1. Systems of Integral Equations. A system of integral equations has the form

$$\phi_i(x) - \lambda \sum_{j=1}^N k_{i,j}(x, s)\phi_j(s) ds = f_i(x), \quad i = 1, \dots, N, \quad (3.1.2)$$

which can be written in matrix form with components as functions. This system can be represented as a single integral equation in which $k(x, s)$ and $f(x)$ may even not be continuous on the interval $[a, b]$.

EXAMPLE 3.1.1. The system of integral equations

$$\begin{aligned} \phi_1(x) - \lambda \int_a^b k_{1,1}(x, s)\phi_1(s) ds - \lambda \int_a^b k_{1,2}(x, s)\phi_2(s) ds &= f_1(x), \\ \phi_2(x) - \lambda \int_a^b k_{2,1}(x, s)\phi_1(s) ds - \lambda \int_a^b k_{2,2}(x, s)\phi_2(s) ds &= f_2(x), \end{aligned} \quad (3.1.3)$$

defined for $a \leq x \leq b$, may be represented as

$$\phi(x) - \lambda \int_a^b k(x, s)\phi(s) ds = f(x), \quad a \leq x \leq 2b - a, \quad (3.1.4)$$

where

$$\begin{aligned}\phi(x) &= \begin{cases} \phi_1(x) & \text{for } a \leq x \leq b, \\ \phi_2(x - (b - a)) & \text{for } b \leq x \leq 2b - a; \end{cases} \\ k(x, s) &= \begin{cases} k_{1,1}(x, s) & \text{for } a \leq x, s \leq b, \\ k_{1,2}(x, s - (b - a)) & \text{for } a \leq x \leq b, b \leq s \leq 2b - a, \\ k_{2,1}(x, s - (b - a)) & \text{for } b \leq x \leq 2b - a, a \leq s \leq b, \\ k_{2,2}(x - (b - a), s - (b - a)) & \text{for } b \leq x, s \leq 2b - a; \end{cases} \\ f(x) &= \begin{cases} f_1(x) & \text{for } a \leq x \leq b, \\ f_2(x) & \text{for } b \leq x \leq 2b - a. \end{cases}\end{aligned}$$

Note that the kernel $k(x, s)$ is discontinuous at $x = b$ or $s = b$, and $f(x)$ is discontinuous at $x = b$. ■

To discuss theoretical properties of a system of integral equations we use the single-equation representation, like Eq (3.1.4). But for computational purposes we solve the system (3.1.3) directly by a quadrature rule, e.g., a rule of the form

$$\tilde{\phi}_i(x) - \lambda \sum_{j=0}^N \sum_{m=0}^n w_m k_{i,j}(x, s_m) \tilde{\phi}_j(s_m) = f_i(x), \quad (3.1.5)$$

where $\tilde{\phi}$ denotes the approximate solution, w_m are the weights, and s_j are the nodes (see Appendix A for a discussion of quadrature rules). The rule (3.1.5) computes the values $\tilde{\phi}_i(s_m)$ for $i = 0, 1, \dots, N$ and $m = 0, 1, \dots, n$, where the weights w_m and the nodes s_m depend on i, j . Alternatively, if we use expansion methods (see Chapter 3), we obtain approximations of the form

$$\tilde{\phi}_i(x) = \sum_{j=0}^{n_i} \tilde{a}_{i,j} p_{i,j}(x), \quad i = 0, 1, \dots, N; \quad j = 0, 1, \dots, n_i, \quad (3.1.6)$$

where $p_{i,j}(x)$ are chosen (orthogonal) polynomials or some appropriate functions. In the case of the Galerkin methods (see Chapter 5), we obtain approximations of the form

$$\int_a^b \left\{ \tilde{\phi}_i(x) - \lambda \sum_{j=0}^N \int_a^b k_{i,j}(x, s) \tilde{\phi}_j(s) ds - f_i(x) \right\} \overline{p_{i,m}(x)} dx = 0, \quad (3.1.7)$$

$$i = 0, 1, \dots, N; \quad m = 0, 1, \dots, n_i.$$

Note that Eq (3.1.1) and the equation

$$\phi(\mathbf{x}) - \lambda \int_{a_0}^{b_0} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} k(\mathbf{x}, \mathbf{s}) \phi(\mathbf{s}) ds_0 ds_1 \cdots ds_n = f(\mathbf{x}), \quad (3.1.8)$$

where $\mathbf{x} = [x_0, x_1, \dots, x_n]^T$ and $\mathbf{s} = [s_0, s_1, \dots, s_n]^T$ are the vectors (points) \mathbf{x} and \mathbf{s} in \mathbb{R}^n . Computational methods for the n -dimensional equations of the form (3.1.8) can be constructed from those of Eq (3.1.1). For example, a quadrature rule for Eq (3.1.8) has the form

$$\tilde{\phi}(\sigma_i) - \lambda \sum_{j=0}^N W_j k(\sigma_i, \sigma_j) \tilde{\phi}(\sigma_j) = f(\sigma_i), \quad (3.1.9)$$

with weights W_j , which involves the integration rule

$$\int_{a_0}^{b_0} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(\mathbf{s}) ds_0 ds_1 \cdots ds_n = \sum_{j=0}^N W_j p(\mathbf{s}). \quad (3.1.10)$$

However, Eqs (3.1.2) and (3.1.8) present practical computational problems when n is large, and sometimes it may not be practical to extend the quadrature and other methods for such equations when the number of dimensions or that of the ranges of integration is large. In such cases Monte Carlo methods are effective, but we do not discuss these methods in this book.

Computational methods for Fredholm equations of the second kind (FK2) and singular equations of the second kind (SK2) are presented in the next several chapters. We discuss Volterra equations of the second kind (VK2) in Section 3.2.

3.1.2. Taylor's Series Method. For an approximate solution of Eq (3.1.1) we develop a method based on Taylor's series expansion. First, we take $a = 0$ and $b = 1$, which can be achieved by a simple transformation, assume that a unique solution of Eq (3.1.1) exists, and the kernel is either (i) $k(x, s) = k(x-s) \in C[a, b]$ and decreases rapidly as $(x-s)$ increases from zero; or (ii) of the form $u(x-s) \kappa(x, s)$, where κ is continuous for x and s , and $u(x-s) = O(|x-s|^{-\alpha})$, $0 < \alpha < 1$, is weakly singular. Case (ii) is presented in Section 8.3. In this section we consider case (i) in which a Taylor's polynomial can approximate the function $\phi(s)$, i.e.,

$$\phi(s) \approx \tilde{\phi}(s) = \tilde{\phi}(x) + \tilde{\phi}'(x)(x-s) + \cdots + \frac{\tilde{\phi}^{(n)}(x)}{n!}(x-s)^n, \quad (3.1.11)$$

where $\tilde{\phi}(x)$ is the approximate value of $\phi(x)$. This representation is valid because if $E(s)$ denotes the error between $\phi(s)$ and the expansion (3.1.11), then the contribution of the integral $\int_a^b k(x, s) E(s) ds$ is negligible since $k(x, s)$ decreases

rapidly as $(x - s)$ increases. If we substitute (3.1.11) into Eq (3.1.1), we get

$$\begin{aligned} & \left\{ 1 - \lambda \int_a^b k(x, s) ds \right\} \tilde{\phi}(x) - \left\{ \lambda \int_a^b k(x, s)(x - s) ds \right\} \tilde{\phi}'(x) - \dots \\ & \dots - \left\{ \frac{\lambda}{n!} \int_a^b k(x, s)(x - s)^n ds \right\} \tilde{\phi}^{(n)}(x) \approx f(x), \quad a < x < b. \end{aligned} \quad (3.1.12)$$

Notice that the expressions within the braces in (3.1.12) are functions of x only, and Eq (3.1.12) is a linear ordinary differential equation of order n with dependent variable $\phi(x)$, which can be solved either analytically or numerically provided we have the appropriate number of boundary (and/or initial) conditions. The problem of determining these boundary conditions is difficult; sometimes they can be determined from the physical constraints such as symmetry or thermal balance, as shown by Perlmutter and Siegel (1963) in the thermal radiation problem from gray surfaces. But, in general, it is not possible to determine them.

This method can, however, be modified in such a way that the boundary (initial) conditions are not required. This modification, presented by Ren, Zhang, and Qiao (1999), is “simple yet effective” and leads to an accurate approximate solution of not only an FK2 but also a VK2. Assume that $f \in C^n[a, b]$ and that $k(x, s) = k(x-s) \in C^n[a, b]$ in case (i); or $\kappa(x, s) \in C^{n+1}([a, b] \times [a, b])$ and $u \in C^n(a, b)$ for $0 < s \leq 1$ such that $|u^{(m)}(s)| \leq C s^{-\alpha-m}$ for $m = 0, 1, \dots, n-1$ in case (ii), where $C > 0$ is a constant. These assumptions then imply that $\phi \in C^n[a, b]$ in case (i); or $\phi \in C[a, b] \cap C^n(a, b)$ such that $|\phi(m)(s)| \leq C_0 [s^{-\alpha-m+1} + (1-s)^{-\alpha-m+1}]$ for $0 < s < 1$ and $m = 0, 1, \dots, n$ in case (ii), where $C_0 > 0$ is a constant. Let the solution $\phi(x)$ of Eq (3.1.1) be approximated by the Taylor's polynomial (3.1.11). Then, differentiating both sides of Eq (3.1.1), we find that

$$\begin{aligned} & \phi'(x) - \lambda \int_a^b k'_x(x, s) \phi(s) ds = f'(x), \\ & \vdots \\ & \phi^{(n)}(x) - \lambda \int_a^b k_x^{(n)}(x, s) \phi(s) ds = f^{(n)}(x), \quad a < x < b, \end{aligned} \quad (3.1.13)$$

where $k_x^{(n)}(x, s) = \frac{\partial^n k(x, s)}{\partial x^n}$. After substituting $\phi(x)$ for $\phi(s)$ in the integrals in

(3.1.13) and replacing $\phi(x)$ by $\tilde{\phi}(x)$, we get

$$\begin{aligned} \tilde{\phi}'(x) - \left\{ \lambda \int_a^b k'_x(x, s) ds \right\} \tilde{\phi}(x) &= f'(x), \\ \vdots \\ \tilde{\phi}^{(n)}(x) - \left\{ \lambda \int_a^b k_x^{(n)}(x, s) ds \right\} \tilde{\phi}(x) &= f^{(n)}(x), \quad a < x < b. \end{aligned} \tag{3.1.14}$$

Then substituting for $\tilde{\phi}'(x), \dots, \tilde{\phi}^{(n)}(x)$ from (3.1.14) into Eq (3.1.12), we find that

$$\begin{aligned} &\left\{ \left(1 - \lambda \int_a^b k(x, s) ds \right) - \left(\lambda \int_a^b k(x, s)(x-s) ds \right) \left(\lambda \int_a^b k'_x(x, s) ds \right) - \dots \right. \\ &\quad \left. - \left(\frac{\lambda}{n!} \int_a^b k(x, s)(x-s)^n ds \right) \left(\lambda \int_a^b k_x^{(n)}(x, s) ds \right) \right\} \\ &= f(x) + \left(\lambda \int_a^b k(x, s)(x-s) ds \right) f'(x) + \dots \\ &\quad + \left(\frac{\lambda}{n!} \int_a^b k(x, s)(x-s)^n ds \right) f^{(n)}(x), \end{aligned}$$

which gives

$$A_n(x) \tilde{\phi}(x) = F_n(x), \tag{3.1.15}$$

where

$$\begin{aligned} A_n(x) &= 1 - G_0^{(0)}(x) - \sum_{j=1}^n G_j^{(0)}(x) G_0^{(j)}(x), \\ G_p^{(m)}(x) &= \frac{\lambda}{p!} \int_a^b k_x^{(m)}(x, s) (x-s)^p ds, \\ G_0(x) &= f(x), \quad A_0(x) = 1 - G_0^{(0)}(x), \\ F_n(x) &= f(x) + \sum_{j=1}^n G_j^{(0)}(x) f^{(j)}(x), \quad F_0(x) = f(x). \end{aligned} \tag{3.1.16}$$

Hence, $\tilde{\phi}(x)$ can be computed directly from formula (3.1.15).

EXAMPLE 3.1.2. (Ren et al. 1999) Consider Eq (3.1.1) with $\lambda = \alpha/\pi$, and

$k(x, s) = [\alpha^2 + (x - s)^2]^{-1}$, that is,*

$$\phi(x) - \frac{\alpha}{\pi} \int_0^1 \frac{\phi(s)}{\alpha^2 + (x - s)^2} ds = f(x), \quad 0 \leq x \leq 1.$$

We solve this equation for $\alpha = 2$ and for a choice of $f(x)$ that gives the exact solution $\phi(x) = e^{2x}$. The results are given in Table 3.1.2 for $n = 0, 1, 2$.

Table 3.1.2

x	$\tilde{\phi}_0(x)$	$\tilde{\phi}_1(x)$	$\tilde{\phi}_2(x)$	$\phi(x)$
0.0	0.6405	0.7987	1.0204	1.0
0.1	0.8892	1.0499	1.2525	1.2214
0.2	1.1985	1.3501	1.5303	1.4918
0.3	1.5812	1.7073	1.8673	1.8221
0.4	2.0528	2.1308	2.2813	2.2255
0.5	2.6317	2.6317	2.7968	2.7183
0.6	3.3403	3.2235	3.4465	3.3201
0.7	4.2053	3.9220	4.2736	4.0552
0.8	5.2590	4.7470	5.3347	4.9530
0.9	6.5404	5.7224	6.7033	6.0497
1.0	8.0969	6.8775	8.4740	7.3891

The results are quite good except near the endpoints. For computational details, see `ren.nb`. An example for case (ii) where $k(x, s)$ is weakly singular is discussed in Section 8.2. ■

3.2. Volterra Equations

*This is a special case of Love's equation (Love 1949) that arises in electrostatics and is of the form

$$\phi(x) - \frac{d}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{d^2 + (x - s)^2} = 1, \quad -1 \leq x \leq 1.$$

See Examples 4.2.3 and 4.3.2.

We shall consider a VK2 of the form

$$\phi(x) - \lambda \int_0^x k(x, s) \phi(s) ds = f(x), \quad 0 \leq x \leq X, \quad (3.2.1)$$

which has a more general form

$$\phi(x) - \lambda \int_0^x g(x, s; \phi(s)) ds = f(x), \quad 0 \leq x \leq X. \quad (3.2.2)$$

This general form arises from the differential equation $\phi'(x) = g(x; \phi(x))$, $0 \leq x \leq X$, subject to the initial condition $\phi(0) = c$, and has the solution $\phi(x) = \int_0^x g(x; \phi(s)) ds + c$. The lower limit of integration in a Volterra equation need not be zero; the interval of integration can be $[a, X']$, where a is finite, i.e.,

$$\phi(x) - \lambda \int_a^x g(x, s; \phi(s)) ds = f(x), \quad a \leq x \leq X',$$

which, by setting $G(x) = \phi(x + a)$, reduces to

$$G(x) - \lambda \int_0^x g(x, s; G(s)) ds = f(x + a), \quad 0 \leq x \leq X' = X - a.$$

Although Eqs (3.2.1) and (3.2.2) are valid for $0 \leq x \leq X < \infty$, and the solution $\phi(x)$ is sought for this interval, the upper limit X may also be $+\infty$; in such a case Eq (3.2.1) or (3.2.2) can be transformed into an FK2 in terms of X . For example, consider the case of Eq (3.2.2), which is rewritten as

$$\phi(x) - \lambda \int_0^X \psi(x, s; \phi(s)) ds = f(x), \quad 0 \leq x \leq X, \quad (3.2.3)$$

where

$$\psi(x) = \begin{cases} g(x, s; \phi) & \text{if } x \geq s, \\ 0 & \text{if } x < s. \end{cases}$$

In the case of linear Eq (3.2.1), the equation corresponding to (3.2.3) is also linear and a quadrature method can be applied to numerically compute the approximate solution $\tilde{\phi}(x)$, which leads to the Nyström system of equations

$$\tilde{\phi}(x_i) - \lambda \sum_{j=0}^i w_j k(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, n, \quad (3.2.4)$$

where s_j are the quadrature points and x_i the Nyström points (see Section 1.6). Note that if $s_0 = 0$, then the sum in (3.2.4) is zero when $i = 0$. In the scheme (3.2.4), the local truncation errors are given by

$$\int_0^{x_i} k(x_i, s) \phi(s) ds - \sum_{j=0}^i w_j k(x_i, s_j) \tilde{\phi}(s_j) \quad \text{for } i = 0, 1, \dots, n.$$

Thus, in order to control truncation errors we use a numerical integration formula over the interval $[0, x_i]$ instead of the one over $[0, X]$ and set integral zero in $(x_i, X]$.

3.2.1. Quadrature Methods. If we set $x = 0$ in (3.2.1), we get $\phi(0) = f(0)$. Choose a step size $h > 0$ and positive numbers m and N such that $Nh = X$, $m \leq N$. Then we set $x = mh$ in (3.2.1), which gives

$$\phi(mh) - \lambda \int_0^{mh} k(mh, s)\phi(s) ds = f(mh). \quad (3.2.5)$$

Now, if we use a general quadrature rule of the form $\int_0^{mh} F(s) ds = \sum_{j=0}^m w_{mj} F(jh)$,

then Eq (3.2.5) reduces to

$$\tilde{\phi}(mh) - \lambda \sum_{j=0}^m w_{mj} k(mh, jh) \tilde{\phi}(jh) = f(mh), \quad (3.2.6)$$

where we set $m = 1, 2, \dots, N$ and use the initial value $\tilde{\phi}(0) = \phi(0) = f(0)$. This gives the system of equations

$$\begin{aligned} \tilde{\phi}(0) &= f(0), \\ -\lambda w_{10} k(h, 0) \tilde{\phi}(0) + (1 - \lambda w_{11} k(h, h)) \tilde{\phi}(h) &= f(h), \\ -\lambda \{w_{20} k(2h, 0) \tilde{\phi}(0) + w_{21} k(2h, h) \tilde{\phi}(h)\} + (1 - \lambda w_{22} k(2h, 2h)) \tilde{\phi}(2h) \\ &= f(2h), \\ &\vdots \\ -\lambda \sum_{j=0}^{N-1} w_{Nj} K(Nh, jh) \tilde{\phi}(jh) + (1 - \lambda w_{NN} k(Nh, Nh)) \tilde{\phi}(Nh) &= f(Nh), \end{aligned} \quad (3.2.7)$$

which can be solved step-by-step by forward substitution. The accuracy of this scheme depends on the choice of h , the smoothness of $k(x, s)$ and $f(x)$, and the quadrature rule chosen. For example, for the repeated trapezoidal rule the system (3.2.6) becomes

$$\tilde{\phi}(mh) - \lambda h \sum_{j=0}^m {}'' k(mh, jh) \tilde{\phi}(jh) = f(mh), \quad m = 1, 2, \dots, \quad (3.2.8)$$

$$\tilde{\phi}(0) = f(0),$$

which has a local discretization error of order $O(h^2)$ as $h \rightarrow 0$.

EXAMPLE 3.2.1. Consider Eq (3.2.1) with $k(x, s) \equiv 1$, i.e.,

$$\phi(x) - \lambda \int_0^x \phi(s) ds = f(x), \quad 0 \leq x \leq X,$$

whose solution by using the Neumann series (Section 1.9) is given by

$$\phi(x) = f(x) + \lambda \int_0^x e^{\lambda(x-s)} f(s) ds. \quad (3.2.9)$$

Now, using (3.2.8), we have $\tilde{\phi}(0) = f(0)$, and

$$\tilde{\phi}(mh) - \lambda h \sum_{j=0}^m \tilde{\phi}(jh) = f(mh), \quad m = 1, 2, \dots .$$

If we write the differences for $m = r$ and $m = r + 1$ and subtract one from the other, we obtain the difference equation

$$(1 - \lambda h/2) \tilde{\phi}((r+1)h) - (1 + \lambda h/2) \tilde{\phi}(rh) = f((r+1)h) - f(rh) \equiv \Delta f(rh), \quad (3.2.10)$$

for $r = 0, 1, 2, \dots$. Thus, the solution of the VK2 is given by the recurrence relation

$$\tilde{\phi}(rh) = \gamma^m f(0) + (1 - \lambda h/2)^{-1} \sum_{s=0}^{m-1} \gamma^{m-s-1} s \Delta f(sh), \quad (3.2.11)$$

where $\gamma = (1 + \lambda h/2) / (1 - \lambda h/2)$. Since λ is given, choose h such that $h \neq 2/\lambda$. It can be easily shown that the error in the scheme (3.2.11) is of the order $O(h^2)$: Assume f''' is continuous. Then integrating (3.2.9) by parts, we get

$$\phi(x) = f(0) e^{\lambda x} + \int_0^x e^{\lambda(x-s)} f'(s) ds.$$

Set $x = mh$, where h is fixed. Then replacing the integral by the repeated midpoint rule, we obtain

$$\phi(mh) = f(0) e^{\lambda mh} + h \sum_{s=0}^{m-1} e(\lambda(mh - sh - h/2)) f'((s + 1/2)h) + O(h^2). \quad (3.2.12)$$

If we write $f'((s + 1/2)h) = \Delta f(sh)/h+$ (h^2), $(1 - \lambda h/2)^{-1} = e^{\lambda h/2} + O(h^2)$, and $\gamma = (1 + \lambda h/2)(1 - \lambda h/2)^{-1} = 1 + \lambda h + \lambda^2 h^2/2 + O(h^3) = e^{\lambda h} + O(h^3)$, so that $\gamma^m = e^{\lambda mh} + O(h^2)$ as $h \rightarrow 0$, and $(1 - \lambda h/2)^{-1} \gamma^{m-s-1} = e^{\lambda(m-s-1/2)h} + O(h^2)$ for $s = 0, 1, \dots, (m-1)$ as $h \rightarrow 0$, then Eq (3.2.11) yields

$$\begin{aligned} \phi(mh) &= \gamma^m f(0) + (1 - \lambda h/2)^{-1} \sum_{s=0}^{m-1} \gamma^{m-s-1} \Delta f(sh) + O(h^2) \\ &= \tilde{\phi}(mh) + O(h^2). \end{aligned}$$

As an example, take $\lambda = 12$, and $f(x) = 13e^{-x} - 12$ (Baker 1978, p.766). The exact solution is $\phi(x) = e^{-x}$. The results obtained by using (3.2.11) are given in Table 3.2.1(a).

Table 3.2.1(a)

x	$h = 1/5$	$h = 1/10$	$h = 1/20$	Exact
0.0	1.0	1.0	<u>1.0</u>	1.0
0.2	0.782501	0.830397	<u>0.820860</u>	0.818730
0.4	1.039184	0.866543	<u>0.697396</u>	0.670320
0.6	-3.532981	3.696213	0.872328	0.548811
0.8	45.3291739	50.814155	4.298881	0.449328
1.0	-493.326695	806.210343	46.161041	0.367879

For computational details, see `volterra1.nb`. These results are not good at all, but they show signs of improvement as h decreases where only three underlined values are acceptable. On the other hand, if we take $\lambda = -12$ and $f(x) = 13e^x - 12$, so that $\phi(x) = e^x$, the situation becomes a little better (see `volterra2.nb`). Why does this method behave like this? The answer lies in the values of $\phi(mh)$ that satisfy the difference equation (3.2.11). In order to find the effect of perturbation, e.g., in $\tilde{\phi}(0)$, let the error there be ε_0 . Then the value of $\tilde{\phi}(h)$ is obtained as $\gamma\varepsilon_0\tilde{\phi}(h)$, and that of $\tilde{\phi}(2h)$ as $\gamma^2\varepsilon_0\tilde{\phi}(2h)$; thus, a perturbation of ε_0 in $\tilde{\phi}(0)$ grows to a perturbation of $\gamma^m\varepsilon_0$ in $\tilde{\phi}(mh)$, and it grows with m if $|\gamma| > 1$. Now, with $\lambda = 12$ and $h = 1/5$, the value of $\gamma = -11$, and thus the values of $\tilde{\phi}(mh)$ exhibit an oscillatory behavior of $(-11)^m$. The output in Table 3.2.1(a) shows instability.

Table 3.2.1(b)

x	λ	$h = 1/5$	$h = 1/10$	$h = 1/20$	Exact
0.0	± 1	1.0	1.0	1.0	1.0
0.2	1	0.819402	0.818899	0.818773	0.81873
	-1	1.38322	1.32681	1.22136	1.22140
0.4	1	0.671689	0.670662	0.670406	0.67032
	-1	1.8434	1.71595	1.49174	1.49182
0.6	1	0.550935	0.549342	0.548944	0.54881
	-1	2.39796	2.18217	1.82199	1.82211
0.8	1	0.452292	0.450069	0.449514	0.44932
	-1	3.06815	2.74339	2.22536	2.22554
1.0	1	0.371803	0.368859	0.368124	0.36787
	-1	3.87993	3.42144	2.71804	2.71828 ■

But since $|\gamma| \rightarrow 1$ as $h \rightarrow 0$, the effect of perturbation decreases, and the scheme become stable, i.e., it does not display instability. This feature is reflected in the cases of $\lambda = \pm 1$, which shows that γ approaches 1 as h decreases to zero. The output in Table 3.2.1(b) for $\lambda = 1$ and $f(x) = 2e^{-x}x - 1$ for which $\phi(x) = e^{-x}$, and for $\lambda = -1$ and $f(x) = 2e^x - 1$ for which $\phi(x) = e^x$, respectively, exhibits the stability. For computational details, see `volterra3a.nb`, `volterra3b.nb`.

EXAMPLE 3.2.2. (Baker 1978, p.784) Consider the following VK2s:

- (a) $\phi(x) + \int_0^x (x-s) \cos(x-s) \phi(s) ds = \cos x$,
 for which $\phi(x) = (1 + 2 \cos \sqrt{3}x) / 3$;
- (b) $\phi(x) - \int_0^x \sin(x-s) \phi(s) ds = x$, for which $\phi(x) = x + x^3/6$;
- (c) $\phi(x) - \int_0^x (x-s) \phi(s) ds = \sin x$, for which $\phi(x) = (\sin x + \sinh x) / 2$;
- (d) $\phi(x) - 2 \int_0^x \cos(x-s) \phi(s) ds = e^x$, for which $\phi(x) = e^x (1+x)^2$;
- (e) $\phi(x) - \int_0^x \phi(s) ds = \cos x$, for which $\phi(x) = (e^x + \cos x + \sin x) / 2$;
- (f) $\phi(x) + \int_0^x \cosh(x-s) \phi(s) ds = \sinh x$,
 for which $\phi(x) = \frac{2}{\sqrt{5}} \sinh(\sqrt{5}x/2) e^{-x/2}$.

Table 3.2.2

x	0.1	0.2	0.3	0.4	0.5
Eq (a)	0.985054 (0.990025)	0.960464 (0.960398)	0.912116 (0.912007)	0.846486 (0.846298)	0.76548 (0.76524)
Eq (b)	0.1 (0.100167)	0.201331 (0.201333)	0.304496 (0.3045)	0.41066 (0.410667)	0.520824 (0.500251)
Eq (c)	0.0998334 (0.547419)	0.2 (0.589368)	0.300016 (0.625428)	0.400079 (0.65524)	1.22140 (0.678504)
Eq (d)	1.44908 (1.33726)	1.7906 (1.75882)	2.3156 (2.28126)	2.9687 (2.92398)	2.9687 (3.70962)
Eq (e)	1.15264 (1.1)	1.20733 (1.20007)	1.30736 (1.30036)	1.40988 (1.40115)	1.51253 (1.50286)
Eq (f)	0.105439 (0.0953212)	0.193663 (0.182479)	0.281064 (0.263081)	0.357128 (0.338518)	0.435247 (0.410001) ■

Using the formula (3.2.7) and repeated Simpson's rule with the 3/8-rule and the weights w_{mj} given in Table A.1.5, our results for $h = 1/10$ are presented in Table 3.2.2, where the exact values are shown within parentheses under each computed value. For computational details, see `volterra4.nb`.

The choice of λh is important for the stability of the solution. Also, the role of γ is significant since it is an approximation of $e^{2\lambda h}$ that goes to 1 as $\lambda h \rightarrow 0$. A perturbation in $\tilde{\phi}(0)$ increases if $|\gamma| > 1$, but the rate of increase in error propagation may be small if $|\gamma|$ is small. The error also depends on the range $[0, X]$ over which the solution is sought.

3.2.2. Block-by-Block Method. A block-by-block method is a step-by-step method for computing values of $\tilde{\phi}(0), \tilde{\phi}(h), \tilde{\phi}(2h), \dots$ in sequence. If we write $\tilde{\Phi}_p = [\tilde{\phi}_{pq}, \tilde{\phi}_{pq+1}, \dots, \tilde{\phi}_{pq+q-1}]^T$ for some $p \geq 0, q \geq 1$, where $\tilde{\phi}_j = \tilde{\phi}(jh)$, then by grouping successive values together, we can regard the step-by-step computation of each value as a block-by-block technique for computing sequentially $\tilde{\Phi}_0, \tilde{\Phi}_1, \dots$, i.e., we break down the computation of a block of values that consists of the elements of the block $\tilde{\Phi}_p$ into successive computations of each of its elements. Thus, the successive vectors $\tilde{\Phi}_p$ contain blocks of q values, say, $\tilde{\phi}_{pq}, \tilde{\phi}_{pq+1}, \dots, \tilde{\phi}_{pq+q-1}$, that are computed block-by-block, which is also known as a *q-element block method*. A block-by-block method for computing $\tilde{\Phi}_0, \tilde{\Phi}_1, \dots$ by solving the equations

$$\mathbf{F}_p (\tilde{\Phi}_0, \tilde{\Phi}_1, \dots, \tilde{\Phi}_p) = \mathbf{0}, \quad p = 0, 1, \dots, \quad (3.2.13)$$

has an error that is bounded in norm, i.e., if $\hat{\Phi}_0 = \tilde{\Phi}_0 + \mathbf{z}_0$, and

$$\mathbf{F}_p (\tilde{\Phi}_0, \tilde{\Phi}_1, \dots, \tilde{\Phi}_p - \mathbf{z}_p) = \mathbf{0},$$

then

$$\|\hat{\Phi}_p - \tilde{\Phi}_p\| \leq A \varepsilon, \quad p = 0, 1, 2, \dots, \quad (3.2.14)$$

where A is a positive constant whenever $\|\mathbf{z}_p\| \leq \varepsilon$, $p \geq 0$. Thus, the error propagation is independent of the norm. In fact, in a block-by-block method it propagates with "at most linear growth" if, in some norm, $\|\hat{\Phi}_p - \tilde{\Phi}_p\| \leq A(p+1)\varepsilon$ for $p \geq 0$. This implies that a block-by-block (or a step-by-step) scheme is stable.

EXAMPLE 3.2.3. Consider Eq (3.2.13) with $k(x, s) \equiv 1$, as in Example 3.2.1. We shall use the step-by-step methods, as in Examples 3.2.1 and 3.2.2 (all

parts), using the weights given in Table A.1.2. Set $\alpha = -\lambda h$; then the array of coefficients on the left side of Eq (3.2.7) can be written with partitions as

$$\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline & \alpha/2 & 1+\alpha/2 & & & \\ \hline & \alpha/3 & 4\alpha/3 & 1+\alpha/3 & 0 & \\ & \alpha/3 & 4\alpha/3 & 5\alpha/6 & 1+\alpha/2 & \\ \hline & \alpha/3 & 4\alpha/3 & 2\alpha/3 & 4\alpha/3 & 1+\alpha/3 & 0 \\ & \alpha/3 & 4\alpha/3 & 2\alpha/3 & 4\alpha/3 & 5\alpha/6 & 1+\alpha/2 \end{array}$$

The values of $\tilde{\phi}(0), \tilde{\phi}(h), \tilde{\phi}(2h), \dots$ are associated in blocks that consist of adjacent function values. For example, we may take $\tilde{\Phi}_0 = [\tilde{\phi}(0), \tilde{\phi}(h)]^T$, $\tilde{\Phi}_1 = [\tilde{\phi}(2h), \tilde{\phi}(3h)]^T$, and so on. Let \mathbf{I} denote the identity matrix of order 2, and denote

$$\mathbf{a} = \alpha \begin{pmatrix} 2/3 & 4/3 \\ 2/3 & 4/3 \end{pmatrix}, \quad \mathbf{a}_0 = \alpha \begin{pmatrix} 1/3 & 4/3 \\ 1/3 & 4/3 \end{pmatrix}, \quad \mathbf{b} = \alpha \begin{pmatrix} 1/3 & 0 \\ 5/6 & 1/2 \end{pmatrix};$$

then, from the above partitions, we have

$$(\mathbf{I} + \mathbf{b}) \tilde{\Phi}_p = -\mathbf{a} (\tilde{\Phi}_{p-1} + \cdots + \tilde{\Phi}_1) - \mathbf{a}_0 \tilde{\Phi}_0 + \mathbf{f}_p, \quad p \geq 1, \quad (3.2.15)$$

where $\mathbf{f}_0 = [f(2ph), f((2p+1)h)]^T$. Writing (3.2.15) for $p+1$ and subtracting (3.2.15) from it, we get

$$(\mathbf{I} + \mathbf{b}) \tilde{\Phi}_{p+1} = (\mathbf{I} + (\mathbf{b} - \mathbf{a})) \tilde{\Phi}_p + \Delta \mathbf{f}_p, \quad (3.2.16)$$

where $\Delta \mathbf{f}_p = \mathbf{f}_{p+1} - \mathbf{f}_p$. Since \mathbf{b} is lower triangular, the matrix $(\mathbf{I} + \mathbf{b})$ is nonsingular if $\lambda h \neq 2$ and $\lambda h \neq 3$, and in particular, if $\lambda < 0$ or if $0 \leq \lambda h < 2$. In the case when $(\mathbf{I} + \mathbf{b})^{-1}$ exists, Eq (3.2.16) can be written as the recurrence relation

$$\tilde{\Phi}_{p+1} = \mathbf{m} \tilde{\Phi}_p + \boldsymbol{\delta}_p, \quad p \geq 0, \quad (3.2.17)$$

where $\mathbf{m} = (\mathbf{I} + \mathbf{b})^{-1} (\mathbf{I} + (\mathbf{b} - \mathbf{a}))$ is a square matrix, and $\boldsymbol{\delta}_p = (\mathbf{I} + \mathbf{b})^{-1} \Delta \mathbf{f}_p$. If $r(\mathbf{m}) = \lim_{n \rightarrow \infty} \|\mathbf{m}\|^{1/n}$ is the spectral radius of \mathbf{m} , then the block-by-block method is said to be *damped* iff $r(\mathbf{m}) \leq 1$. Moreover, for block stability a necessary condition is that $r(\mathbf{m}) \leq 1$ and a sufficient condition is that $\|\mathbf{m}\| \leq 1$

(we have “strict stability” if $\|\mathbf{m} < 1$). For proof of these conditions, see Baker (1978, p.800). To compute $r(\mathbf{m})$, we write the equation $\mathbf{m} \mathbf{x} = \mu \mathbf{x}$ in the form

$$\mu (\mathbf{I} + \mathbf{b}) \mathbf{x} = (\mathbf{I} + (\mathbf{b} - \mathbf{a})) \mathbf{x},$$

and solve the equation $\det |(\mathbf{I} + (\mathbf{b} - \mathbf{a})) - \mu (\mathbf{I} + \mathbf{b})| = 0$ for μ , which gives $\mu = 1$ and $\mu = \frac{6 - 7\lambda h + 3\lambda^2 h^2}{6 + 5\lambda h + \lambda^2 h^2}$ (see `block1.nb`). Thus, $r(\mathbf{m}) \geq 1$ for all λ . This means that the block-by-block method has no error propagation that is bounded in norm. For block stability, we solve $r(\mathbf{m}) = 1$, i.e., $|\mu| < 1$, which gives

$$-2 \leq \frac{2\lambda h(6 + \lambda h)}{(2 - \lambda h)(3 - \lambda h)} \leq 0. \quad (3.2.18)$$

If $\lambda < 0$, then from inequalities (3.2.18) we find that $|\lambda| \leq 6$, and we have block stability if $|\lambda h| < 6$ as well as if $\lambda h = -6$ for the blocks $\tilde{\Phi}_0, \tilde{\Phi}_1, \dots$, in the norm associated with \mathbf{m} .

If we use the weights from Table A.1.5, then the array of coefficients on the left of Eq (3.2.7) is given by

1	0								
$\alpha/2$	$1 + \alpha/2$								
$\alpha/3$	$4\alpha/3$	$1 + \alpha/3$	0						
$3\alpha/8$	$9\alpha/8$	$9\alpha/8$	$1 + 3\alpha/8$						
$\alpha/3$	$4\alpha/3$	$2\alpha/3$	$4\alpha/3$	$1 + \alpha/3$	0				
$\alpha/3$	$4\alpha/3$	$17\alpha/24$	$9\alpha/8$	$9\alpha/8$	$1 + 3\alpha/8$				
$\alpha/3$	$4\alpha/3$	$2\alpha/3$	$4\alpha/3$	$2\alpha/3$	$4\alpha/3$	$1 + \alpha/3$	0		
$\alpha/3$	$4\alpha/3$	$2\alpha/3$	$4\alpha/3$	$17\alpha/24$	$9\alpha/8$	$9\alpha/8$	$1 + \alpha/3$		

where we have set $\alpha = -\lambda h$. The block computations in this case get a little complicated and instead of Eq (3.2.15) we obtain

$$(\mathbf{I} + \mathbf{b}) \tilde{\Phi}_{p+1} + \mathbf{c} \tilde{\Phi}_p + \mathbf{a} \left(\tilde{\Phi}_{p-1} + \cdots + \tilde{\Phi}_1 \right) + \mathbf{a}_0 \tilde{\Phi}_0 = \mathbf{f}_{p+1}, \quad p \geq 0, \quad (3.2.19)$$

where

$$\mathbf{a} = \alpha \begin{pmatrix} 2/3 & 4/3 \\ 2/3 & 4/3 \end{pmatrix}, \quad \mathbf{b} = \alpha \begin{pmatrix} 1/3 & 0 \\ 9/8 & 3/8 \end{pmatrix},$$

$$\mathbf{c} = \alpha \begin{pmatrix} 2/3 & 4/3 \\ 17/24 & 9/8 \end{pmatrix}, \quad \mathbf{a}_0 = \alpha \begin{pmatrix} 1/3 & 4/3 \\ 1/3 & 4/3 \end{pmatrix},$$

which leads to the recurrence relation

$$(\mathbf{I} + \mathbf{b}) \tilde{\Phi}_{p+1} - (\mathbf{I} + \mathbf{b} - \mathbf{c}) \tilde{\Phi}_p + (\mathbf{a} - \mathbf{c}) \tilde{\Phi}_{p-1} = \Delta \mathbf{f}_p. \quad (3.2.20)$$

This is not of the same form as (3.2.17). But if we set

$$\mathbf{m} = \left(\begin{array}{c|c} \mathbf{I} + \mathbf{b} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array} \right)^{-1} \left(\begin{array}{c|c} \mathbf{I} + \mathbf{b} - \mathbf{c} & \mathbf{c} - \mathbf{a} \\ \hline \mathbf{I} & \mathbf{0} \end{array} \right), \quad (3.2.21)$$

then the relation (3.2.21) becomes of the same form as (3.2.17), where it is assumed that $(\mathbf{I} + \mathbf{b})^{-1}$ exists. Then, as above, it can be shown that $r(\mathbf{m})$ depends on $\alpha = \lambda h$. ■

Note that the above two partitions of the coefficients, as shown, cover three blocks, namely,

$$\tilde{\Phi}_0 = [\tilde{\phi}(0), \tilde{\phi}(h)], \quad \tilde{\Phi}_1 = [\tilde{\phi}(2h), \tilde{\phi}(3h)], \quad \text{and} \quad \tilde{\Phi}_2 = [\tilde{\phi}(4h), \tilde{\phi}(5h)],$$

but by adding more rows it can be extended to subsequent blocks as far as we require. It is easy to see that the matrix equation for the block $\tilde{\Phi}_0$ consists of the first two equations of the system (3.2.7), that for the block $\tilde{\Phi}_1$ the first four equations of the system (3.2.7), and so on. Thus, in this sense this method appears to be somewhat artificial. But the idea of block stability and error propagation that is bounded in norm are very realistic. What we have emphasized is that the condition $r(\mathbf{m}) < 1$ leads to bounded-in-norm error propagation while $r(\mathbf{m}) = 1$ signifies the possibility of linear error growth. Other examples and details can be found in Baker (1978).

REFERENCES USED: Abdalkhani (1993), Atkinson (1997), Baker (1978), Brunner (1983, 1984), Brunner and van der Houwen (1986), Choi and Churchill (1985), Delves and Mohamed (1985), Hackbusch (1985), Kershaw (1982), Krylov, Lugin and Yanavich (1963), Linz (1985), Love (1949), Lubich (1983, 1985), Perlmutter and Siegel (1963), Reischel (1987, 1989), Ren, Zhang and Qiao (1999), Siegel and Howell (1992), Sparrow and Cess (1970), Yan (1994).

4

Classical Methods for FK2

Most of the computational methods for the approximate solution of an integral equation can be regarded as “expansion methods.” Although the quadrature rule solves an FK2 of the form $\phi(x) - \lambda (K\phi)(x) = f(x)$ and yields an approximate solution $\tilde{\Phi}$, which we take as a vector with functional values $\tilde{\phi}(x_0), \tilde{\phi}(x_1), \dots, \tilde{\phi}(x_n)$. These values are used in the Nyström methods, discussed in Section 1.6, to yield the approximation $\tilde{\Phi}(x)$. We present in this and the next chapter some of these methods including the following methods: quadrature, expansion, collocation, product-integration, and Galerkin, to solve Fredholm equations of the second kind.

4.1. Expansion Method

An expansion method for an FK2 of the form

$$\phi(x) - \lambda \int_a^b k(x, s)\phi(s) ds = f(x), \quad a \leq x \leq b, \quad (4.1.1)$$

is basically a method in which the approximate solution $\tilde{\phi}(x)$ is obtained as a linear combination of the form $\sum_{j=0}^n a_j p_j(x)$ of prescribed functions $p_j(x)$, $j = 0, 1, \dots, n$, which are generally taken as one of the orthogonal polynomials. Some important orthogonal polynomials and their properties are listed in Appendix B.

We use one of the quadrature rules to obtain the approximate solution $\tilde{\phi}$ as a linear combination of the form

$$\tilde{\phi}(x) = \sum_{j=0}^n a_j k(x, s_j) + f(x), \quad \text{where } a_j = \lambda w_j \tilde{\phi}(s_j).$$

For example, we can use the Chebyshev polynomials $T_j(x)$ of the first kind and match them with the vector $\tilde{\phi}(s_0), \tilde{\phi}(s_1), \dots, \tilde{\phi}(s_n)]^T$, where the nodes s_j and the polynomials $T_j(x)$, $j = 0, 1, \dots, n$, match with each other. If we employ the Clenshaw–Curtis quadrature rule (see Appendix A, rule Q_{10}), we have $s_j = \cos(j\pi/n)$, and this leads to the approximation

$$\tilde{\phi}(x) = \sum_{m=0}^n \tilde{a}_m T_m(x), \quad (4.1.2)$$

where

$$\tilde{a}_m = \frac{2}{n} \sum_{i=1}^{n+1} \tilde{\phi}\left(\cos \frac{j\pi}{n}\right) T_m\left(\cos \frac{j\pi}{n}\right), \quad m = 0, 1, \dots, n. \quad (4.1.3)$$

EXAMPLE 4.1.1. Consider the FK2 (Young 1954a,b, El-gendi 1969)

$$\phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{\phi(s)}{1 + (x-s)^2} ds = 1, \quad -1 \leq x \leq 1.$$

For $n = 8$ El-gendi (1969) obtained

$$\begin{aligned} \tilde{a}_0 &= 1.415185, & \tilde{a}_2 &= 0.049385, & \tilde{a}_4 &= -0.001048, \\ \tilde{a}_6 &= -0.000231, & \tilde{a}_8 &= 0.0000195, & \tilde{a}_1 &= \tilde{a}_3 = \tilde{a}_5 = \tilde{a}_7 = 0. \end{aligned}$$

This suggests that the solution is symmetric about s -axis, and we can take odd coefficients $\tilde{a}_{2m+1} = 0$. Thus, we can solve only $[n/2]$ equations in the $[n/2]$ unknowns \tilde{a}_{2m} , and obtain the solution $\tilde{\phi}$ at different equally spaced $(n+1)$ values of x in the interval $[-1, 1]$. For example, if we take $n = 10$, we solve only 5 equations in 5 unknowns \tilde{a}_{2m} , $m = 0, 1, 2, 3, 4$, and get the solution at 11 equally spaced points in $[-1, 1]$, as given in Table 4.1.1.

Table 4.1.1

x	0	± 0.2	± 0.4	± 0.6	± 0.8	± 1
$\tilde{\phi}(x)$	0.65740	0.66151	0.67390	0.69448	0.72248	0.75570

For computational details, see `expansion1.nb`. ■

4.2. Product-Integration Method

In Section A.3 we present quadrature rules for integrals when the integrand is the product of two functions. Accordingly, in the case of an integral equation we get

$$\int_a^b k(x, s)\phi(s) ds = \sum_{j=0}^n v_j k(x, s_j) \phi(s_j), \quad (4.2.1)$$

where the weights v_j are defined for various cases in Section A.3. The above rule is exact when $\phi(x)$ is a polynomial of a certain degree on the interval $[a, b]$ or when $\phi(x)$ is a piecewise polynomial of a certain degree on the subintervals $[a, a+h], [a+2h], \dots, [b-h]$. The approximation (4.2.1) permits us to apply the product-integration rule to an FK2 of the form (4.1.1), which yields the approximation

$$\tilde{\phi}(x_i) - \lambda \sum_{j=0}^n v_j k(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, n, \quad (4.2.2)$$

or in matrix form

$$(\mathbf{I} - \lambda \mathbf{KD}) \tilde{\Phi} = \mathbf{f}, \quad (4.2.3)$$

where \mathbf{I} is the identity matrix, $\mathbf{k} = (k(x_i, s_j))_{ij}$, and $\mathbf{D} = \text{diag}\{v_0, v_1, \dots, v_n\}$. This method is exact for $\int_{-1}^1 k(x, s) \phi(s) ds$ if $k(x, s)$ is continuous and quadratic in each subinterval of $[-1, 1]$. In the case when the kernel $k(x, s)$ is badly behaved at $s = x$, we use the rule (A.3.21), or its modifications.

EXAMPLE 4.2.1. Consider the same FK2 as in Example 4.1.1:

$$\phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{\phi(s)}{1 + (x-s)^2} ds = 1, \quad -1 \leq x \leq 1.$$

This equation in its general form

$$\phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{d}{d^2 + (x-s)^2} \phi(s) ds = f(x), \quad |x| \leq 1,$$

was investigated by Young (1954a,b) by the product-integration method. His results are as follows:

x	0	$\pm 1/5$	$\pm 2/5$	$\pm 3/5$	$\pm 4/5$	± 1
$\tilde{\phi}(x)$	0.65741	0.66151	0.67389	0.69448	0.72249	0.75572

Compare these with the values of $\tilde{\phi}(x)$ in Example 4.1.1. The values of $\tilde{\phi}(x)$ using the trapezoidal rule with step size h are given below.

x	0	$\pm 1/5$	$\pm 2/5$	$\pm 3/5$	$\pm 4/5$	± 1
$h = 0.1$	0.65787	0.66197	0.67432	0.69481	0.72261	0.75553
$h = 0.05$	0.65752	0.66163	0.67340	0.69546	0.72252	0.75567
$h = 0.025$	0.6574	0.66154	0.67392	0.69450	0.72249	0.75571
$h = 0.0125$	0.65742	0.66152	0.67389	0.69449	0.72249	0.75572

This method gives better results for the amount of work involved. ■

EXAMPLE 4.2.2. If the kernel is defined as $k(x, s) = \min\{x, s\}$ on $[0, 1]$, that is,

$$k(x, s) = \begin{cases} x & \text{if } x \leq s, \\ s & \text{if } x \geq s, \end{cases}$$

we consider the FK2

$$\phi(x) + \int_0^1 k(x, s) \phi(s) ds = 1 + x e, \quad 0 \leq x \leq 1,$$

which has the exact solution $\phi(x) = e^x$. We partition the interval $[0, 1]$ into n subintervals of size $h = 1/n$ and take $x_i = i h$, $i = 0, 1, \dots, n$. The approximation

$$\int_0^1 k(x, s) \phi(s) ds = \sum_{j=0}^n v_j \phi(s_j)$$

is then computed for each subinterval $[i h, (i+1)h]$, $i = 0, 1, \dots, n-1$. We find v_j , $j = 0, 1, \dots, n$, from

$$v_0 = \frac{1}{6} h^2 \min\{1, j-1\},$$

$$v_{j-1} = \begin{cases} (\min\{i, j\} - 1) h^2 & \text{if } i \neq j, \\ \left(j - \frac{7}{6}\right) h^2 & \text{if } j \neq 1 \text{ and } j \neq n+1, \end{cases}$$

$$v_n = \frac{1}{6} h^2 \min\{3i-3, 3n-1\}.$$

The values of $\tilde{\phi}$ are given in Table 4.2.2.

Table 4.2.2

x	$h = 0.05$	$h = 0.1$	Exact
0.0	1.00000	1.00000	1.00000
0.2	1.22135	1.22121	1.22142
0.4	1.49173	1.49147	1.49182
0.6	1.82199	1.82163	1.82211
0.8	2.22540	2.22498	2.22554
1.0	2.71813	2.71769	2.71828

For computational details, see `productmin.nb`. ■

EXAMPLE 4.2.3. Consider Love's equation

$$\phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{d}{d^2 + (x-s)^2} \phi(s) ds = f(x), \quad -1 \leq x \leq 1,$$

with $d = 1$ (see Example 4.3.2; also see Young (1954a,b), who uses a different method). Young's results are as follows:

x	0	± 0.2	± 0.4	± 0.6	± 0.8	± 1
$\tilde{\phi}(x)$	0.65741	0.66151	10.67389	0.69448	0.72249	0.75572

Baker (1978) uses the trapezoidal rule with step size h and obtained these results:

h	0	$x = \pm 0.2$	$x = \pm 0.4$	$x = \pm 0.6$	$x = \pm 0.8$	$x = \pm 1$
1/10	0.65787	0.66197	10.67432	0.69481	0.72249	0.75572
1/20	0.65752	0.66163	10.67340	0.69546	0.72252	0.75567
1/40	0.65744	0.66154	10.67392	0.69450	0.72249	0.75571
1/80	0.65742	0.66152	10.67389	0.69449	0.72249	0.75572 ■

EXAMPLE 4.2.4. Consider an FK2 of the form

$$\phi(x) + \lambda \int_0^1 \kappa(x, s) g(s) \phi(s) ds = f(x), \quad 0 \leq x \leq 1,$$

where $g(x) > 0$ for $0 \leq x \leq 1$ and $\frac{\partial \kappa(x, s)}{\partial s}$ has a discontinuity at $x = s$. An example is given by $\kappa(x, s) = \min\{x, s\}$, $0 \leq x \leq s \leq 1$, and $g(x) = e^x$. Because of the discontinuity in $\frac{\partial \kappa}{\partial s}$ in this problem, we do not use the product quadrature (4.5.8) but instead a quadrature of the form

$$\int_0^1 \min\{x, s\} \phi(s) ds = \sum_{j=0}^n v_j(x) \phi(jh), \quad (4.2.4)$$

and then find the solution of the system

$$\tilde{\phi}(ih) + \lambda \sum_{j=0}^n v_j(ih) g(jh) \tilde{\phi}(jh) = f(ih), \quad (4.2.5)$$

where we have $\lambda = 1$, $f(x) = 1.25 e^{2x} - 0.5 e^2 x - 0.25$, so that the equation has the exact solution $\phi(x) = e^x$. Baker (1978, p.385) has obtained the values of $\tilde{\phi}(x)$ given in Table 4.2.4, which are compared with the exact values. The results show an error of order $O(h^2)$.

Table 4.2.4

h	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
1/10	1.2334	1.5158	1.8561	2.2665	2.7612
1/20	1.2245	1.4978	1.8306	2.2355	2.7292
1/40	1.2221	1.4933	1.8242	2.2280	2.7209
Exact	1.2214	1,4918	1.8221	2.2255	2.71828 ■

4.3. Quadrature Method

Consider an FK2 of the form (4.1.1), where λ is a regular value of the kernel $k(x, s)$, and the functions $k(x, s)$ and $f(x)$ are at least piecewise continuous on $[a, b]$. Suppose that we have chosen a quadrature rule $Q(F) = \sum_{j=0}^n w_j F(s_j)$ for an approximate evaluation of the integral $\int_a^b F(s) ds$, where w_j are the weights

and s_j the nodes taken on the interval $[a, b]$ (see Appendix A). Then we can replace Eq (4.1.1) by

$$\tilde{\phi}(x) - \lambda \sum_{j=0}^n w_j k(x, s_j) \tilde{\phi}(s_j) = f(x), \quad (4.3.1)$$

where $\tilde{\phi}(x)$ denotes the approximate solution of Eq (4.1.1). Now, by using the Nyström method (Section 1.6), we find the solution $\tilde{\phi}(x)$ of Eq (4.3.1) by setting $x = x_i, i = 0, 1, \dots, n$, which replaces Eq (4.3.1) by

$$\tilde{\phi}(x_i) - \lambda \sum_{j=0}^n w_j k(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, n. \quad (4.3.2)$$

In matrix notation, if we denote the approximate solution by the vector

$$\tilde{\Phi} = [\tilde{\phi}(x_0), \tilde{\phi}(x_1), \dots, \tilde{\phi}(x_n)]^T,$$

then it satisfies the system

$$(\mathbf{I} - \lambda \mathbf{kD}) \tilde{\Phi} = \mathbf{f}, \quad (4.3.3)$$

where $\mathbf{k} = [k(x_i, s_j)]^T$, $\mathbf{D} = \text{diag}(w_0, w_1, \dots, w_n)$, and $\mathbf{f} = [f_0, f_1, \dots, f_n]^T$. Since we have assumed that λ is a regular value of the kernel, the system (4.3.3) has a unique solution. However, for a choice of a quadrature rule the matrix $(\mathbf{I} - \lambda \mathbf{kD})$ in the system (4.3.3) may become singular. But we can be certain that under certain restrictions on λ , $k(x, s)$, and $f(x)$, the matrix $(\mathbf{I} - \lambda \mathbf{kD})$ will remain nonsingular provided the rule $Q(F)$ that we have chosen yields a “sufficiently accurate” quadrature. These restrictions are discussed ahead in “Notes.”

EXAMPLE 4.3.1. (Baker 1978, p. 357) Consider Eq (4.1.1) with $\lambda = 1$ and

$$k(x, s) = \begin{cases} x(1-s) & \text{if } 0 \leq x \leq s \leq 1, \\ s(1-x) & \text{if } 0 \leq s \leq x \leq 1. \end{cases}$$

Note that $\lambda \neq j^2\pi^2, j = 0, 1, 2, \dots$, is a regular value of the kernel (see Example 3.5.1). Assume that $f(x) \in C^2[0, 1]$. Under these conditions the solution $\phi(x)$ is unique. For the quadrature rule we choose the trapezoidal rule Q_2 with step size $h = 1/n$:

$$Q_2(F) = \sum_{j=0}^{n+1} h F(jh). \quad (4.3.4)$$

Then Eq (4.3.2) becomes

$$\tilde{\phi}(ih) - h \sum_{j=0}^{n+1}'' k(ih, jh) \tilde{\phi}(jh) = f(ih), \quad i = 0, 1, \dots, n. \quad (4.3.5)$$

The matrix $(\mathbf{I} - \mathbf{KD})$ is nonsingular since $\lambda = 1$ and $\lambda \notin \left\{ 4h^{-2} \sin^2(\pi jh/2) \right\}$ for $j = 1, 2, \dots, (n-1)$, so that λ is a regular value. This yields a unique solution $\tilde{\phi} = [\tilde{\phi}(0), \tilde{\phi}(h), \dots, \tilde{\phi}(1)]^T$. It can be shown that $\tilde{\phi}(0) = f(0)$, $\tilde{\phi}(1) = f(1)$, and $h^{-2} \delta^2 \tilde{\phi}(jh) + \lambda \tilde{\phi}(jh) = h^{-2} \delta^2 f(jh)$ for $j = 1, 2, \dots, n$, where δ is the central difference operator defined by $\delta \phi(jh) = \tilde{\phi}((j+1/2)h) - \tilde{\phi}((j-1/2)h)$. Thus, $\delta^2 \tilde{\phi}(jh) = \tilde{\phi}((j+1)h) - 2\tilde{\phi}(jh) + \tilde{\phi}((j-1)h)$. ■

EXAMPLE 4.3.2. Consider Love's equation

$$\phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{d}{d^2 + (x-s)^2} \phi(s) ds = f(x), \quad -1 \leq x \leq 1,$$

which occurs in electrostatics. For $d = -1$, we have $k(x, s) = -\frac{1}{\pi[1 + (x-s)^2]}$, $f(x) = 1$; using the trapezoidal rule, we get

$$\tilde{\phi}(ih-1) - h \sum_{j=0}^{n+1}'' \frac{\tilde{\phi}(jh-1)}{\pi[1 + (i-j)^2 h^2]} = 1,$$

where $h = 2/n$, $i = 0, 1, \dots, n$. The results are given in Table 4.3.2.

Table 4.3.2

n	8	16	32	64
$x = \pm 1.0$	1.63639	1.63887	1.63949	1.63964
$x = \pm 0.75$	1.74695	1.75070	1.75164	1.75187
$x = \pm 0.5$	1.83641	1.84089	1.84201	1.84229
$x = \pm 0.25$	1.89332	1.89804	1.89922	1.89952
$x = 0.0$	1.91268	1.91744	1.91863	1.91839

For computational details, see quadrature2.nb. Baker (1978, p.358) uses the Romberg scheme to obtain the following results, where the underlines indicate similar values:

	9 points	17 points	33 points	65 points
$x = \pm 1.0$	1.63673	1.63969	<u>1.63970</u>	<u>1.63970</u>
$x = \pm 0.75$	1.75183	1.75196	<u>1.75195</u>	<u>1.75195</u>
$x = \pm 0.5$	1.84215	1.84239	<u>1.84238</u>	<u>1.84238</u>
$x = \pm 0.25$	1.89969	<u>1.89961</u>	<u>1.89961</u>	<u>1.89961</u>
$x = 0.0$	1.91934	<u>1.91903</u>	<u>1.91903</u>	<u>1.91903</u> ■

NOTES: (i) In the case of a DEGENERATE KERNEL where, e.g., $k^{[m]}(x, s) = \sum_{m=0}^N X_m(x) S_m(s)$ (see Sections 1.7 and 2.3), the exact solution of Eq (4.1.1) is of the form

$$\phi(x) = f(x) + \lambda \sum_{m=0}^N a_m X_m(x), \quad (4.3.6)$$

where $(\mathbf{I} - \lambda \mathbf{A}) \mathbf{a} = \mathbf{b}$, with

$$A_{ij} = \int_a^b X_j(x) S_i(x) dx, \quad b_i = \int_a^b f(x) S_i(x) dx. \quad (4.3.7)$$

Similarly, the solution of Eq (4.3.1) is of the form

$$\tilde{\phi}(x) = f(x) + \lambda \sum_{m=0}^N \tilde{a}_m X_m(x), \quad (4.3.8)$$

where $(\mathbf{I} - \lambda \tilde{\mathbf{A}}) \tilde{\mathbf{a}} = \tilde{\mathbf{b}}$, with

$$\begin{aligned} \tilde{A}_{ij} &= \sum_{m=0}^N w_m X_j(s_m) S_i(s_m), \\ b_i &= \sum_{m=0}^N w_m f(s_m) S_i(s_m). \end{aligned} \quad (4.3.9)$$

Note that in order to keep the quantities $\|\mathbf{A} - \tilde{\mathbf{A}}\|$ and $\|\mathbf{b} - \tilde{\mathbf{b}}\|$ small, we must choose the quadrature rule $Q(F)$ according to the behavior of both $k(x, s)$ and $f(x)$. But if the system $(\mathbf{I} - \lambda \mathbf{A}) \mathbf{a} = \mathbf{b}$ becomes ill conditioned, it may not be possible to compute \mathbf{a} . This may happen for certain choices of λ and $k(x, s)$ which in turn impart ill-conditioning to the whole problem. If this is the case, we will not be able to compute $\tilde{\phi}(x)$.

(ii) If the kernel $k(x, s)$ or the function $f(x)$ is known or suspected to be badly behaved, we can use the METHOD OF DEFERRED CORRECTION (see next section). With this approach the accuracy of the initial approximation $\tilde{\phi}(x)$ does not remain important. As a rule, we choose a quadrature rule Q for which we can “predict” the truncation error

$$t(x) = \phi(x) - \lambda \sum_{j=0}^n w_j k(x, s_j) \phi(s_j) - f(x). \quad (4.3.10)$$

This means that if we want to find an accurate initial solution $\tilde{\phi}$, we should choose a quadrature rule such that

$$\|t\|_\infty = \sup_{0 \leq i \leq n} |t(s_i)| \text{ is small.}$$

Whether this condition is satisfied depends on the properties of the quadrature rule as well as on the properties of the differentiability of $\phi(x)$. Since

$$\phi(x) = f(x) + \lambda \int_a^b k_\lambda(x, s) f(s) ds,$$

where $k_\lambda(x, s)$ is the resolvent kernel (see Section 1.9), any bad behavior of $f(x)$ shall influence the solution $\phi(x)$.

EXAMPLE 4.3.3. Consider the FK2 with $\lambda = -1$, $k(x, s) = \sqrt{xs}$, and $f(x) = \sqrt{x}$, $0 \leq x \leq 1$ (see Example 1.6.3). The exact solution is $\phi(x) = 2\sqrt{x}/3$. Although $\phi'(x)$ is unbounded at $x = 0$, we can use the trapezoidal rule Q_2 to obtain an accurate approximate solution $\tilde{\phi}$ because the bad behavior of $k(x, s) = \sqrt{xs}$ and that of $f(x) = \sqrt{x}$ annihilate each other. The results are given in Example 1.6.3. For computational details, see `nystrom3.nb`.

4.4. Deferred Correction Methods

The deferred correction method uses either Romberg’s or Gregory’s scheme to obtain the approximate solution $\tilde{\phi}(x)$ of the FK2 (4.1.1). We first discuss the Romberg scheme. Suppose we have applied the trapezoidal rule with step size $h = (b - a)/n$ to obtain the approximate solution $\tilde{\phi}(x) \equiv \tilde{\phi}_h(x)$ by solving the system

$$\tilde{\phi}_h(a + ih) - \lambda h \sum_{j=0}^{n+1}'' w_j k(a + ih, a + jh) \tilde{\phi}_h(a + jh) = f(a + ih), \quad (4.4.1)$$

for $i = 0, 1, \dots, n$ and for $h \in \{h_0, h_0/2, h_0/4\}$, say. Now, if we compute the vectors $\tilde{\Phi}_h$ for $h = h_0, h_0/2, h_0/4$, we obtain

3 estimates of each value of $\phi(a), \phi(a + h_0), \phi(a + 2h_0), \dots, \phi(b)$,
 2 estimates of each value of $\phi(a + h_0/2), \phi(a + 3h_0/2), \dots, \phi(b - h_0/2)$, and
 1 estimate of each value of $\phi(a + h_0/4), \phi(a + 3h_0/4), \dots, \phi(b - h_0/4)$.

Suppose that

$$\tilde{\phi}_h(a + ih) = \phi(a + ih) + \sum_{m=1}^N h^{2m} p_m(a + ih) + O(h^{2N+1}), \quad (4.4.2)$$

where the basis functions $p_m(x)$, $m = 1, 2, \dots, N$, are independent of h . Then, combining the estimates of the values of $\phi(a_i h_0)$ and using the deferred corrections that are similar to those of Romberg's scheme (as in Baker 1978, p.362ff.), we get

$$\begin{aligned} \tilde{\phi}_{h_0}^{[1]}(a + ih_0) &= \frac{4\tilde{\phi}_{h_0/2}(a + ih_0) - \tilde{\phi}_{h_0}(a + ih_0)}{3}, \\ \tilde{\phi}_{h_0}^{[2]}(a + ih_0) &= \frac{16\tilde{\phi}_{h_0/2}^{[1]}(a + ih_0) - \tilde{\phi}_{h_0}^{[1]}(a + ih_0)}{15}, \end{aligned} \quad (4.4.3)$$

and so on. This gives

$$\begin{aligned} \phi(a + ih_0) - \tilde{\phi}(a + ih_0) &= O(h_0^2), \\ \phi(a + ih_0) - \tilde{\phi}^{[1]}(a + ih_0) &= O(h_0^4), \\ \phi(a + ih_0) - \tilde{\phi}^{[2]}(a + ih_0) &= O(h_0^6), \end{aligned}$$

where the order terms are uniform in i as h_0 decreases.

If the integrand $k(x, s)\phi(s)$ in the FK2 is sufficiently differentiable with respect to s , a deferred correction method can be used to obtain the approximate solution $\tilde{\phi}$ by using GREGORY'S SCHEME. Suppose that an initial approximation $\tilde{\phi}(x) = \tilde{\phi}^{(0)}(x)$ is found after using a quadrature rule, say a repeated trapezoidal rule with step size $h = (b - a)/n$. Then we should compute only the vector $\tilde{\Phi}^{(0)} = [\tilde{\phi}^{(0)}(a), \tilde{\phi}^{(0)}(a + h), \dots, \tilde{\phi}^{(0)}(b)]^T$, where

$$\tilde{\phi}_h^{(0)}(a + ih) - \lambda h \sum_{j=0}^n k''(a + ih, a + jh) \tilde{\phi}_h^{(0)}(a + jh) = f(a + ih), \quad (4.4.4)$$

for $i = 0, 1, \dots, n$ (compare this with (4.4.1)). If we use Gregory's scheme instead of the repeated trapezoidal rule, we get

$$\tilde{\phi}_h(a + ih) - \lambda \left\{ h \sum_{j=0}^n k''(a + ih, a + jh) \tilde{\phi}_h(a + jh) + \delta_i(\tilde{\phi}) \right\} = f(a + ih), \quad (4.4.5)$$

for $i = 0, 1, \dots, n$, where $\delta_i(\tilde{\phi})$ is Gregory's correction term to the trapezoidal rule, defined by

$$\begin{aligned}\delta_i(\tilde{\phi}) &= h \sum_{m=1}^p c_m^* \left\{ \nabla^m \vartheta_i(\tilde{\phi}; b) + (-1)^m \Delta^m \vartheta_i(\tilde{\phi}; a) \right\}, \quad p \leq n, \\ \nabla^m \vartheta_i(\tilde{\phi}; b) &= \Delta^m \vartheta_i(x) \Big|_{x=b}, \\ \nabla^m \vartheta_i(\tilde{\phi}; a) &= \Delta^m \vartheta_i(x) \Big|_{x=a}, \\ \vartheta_i(x) &= \vartheta_i(\tilde{\phi}; x) = k(a + ih, x) \tilde{\phi}(x).\end{aligned}\tag{4.4.6}$$

This scheme uses the forward and backward differences Δ^m and ∇^m , respectively, defined for $m \geq 1$ by

$$\begin{aligned}\Delta^m \vartheta(x) &= \Delta^{m-1} \vartheta(x + h) - \Delta^{m-1} \vartheta(x), \\ \nabla^m \vartheta(x) &= \Delta^m \vartheta(x - mh),\end{aligned}$$

such that

$$\Delta^m \vartheta(x) = \nabla^m \vartheta(x + mh) = \sum_{j=0}^m (-1)^{j+m} \binom{m}{j} \vartheta(x + jh),$$

where

$$\binom{m}{j} = \frac{m!}{(m-j)!j!}$$

are the binomial coefficients, and the coefficients c_m^* in (4.4.6) satisfy a well-defined relation* (see Henrici 1964, p.252) where $c_1^* = \frac{1}{2}$, $c_2^* = -\frac{1}{12}$, $c_3^* = -\frac{1}{24}$, $c_4^* = -\frac{19}{720}$, $c_5^* = -\frac{3}{160}$, $c_6^* = -\frac{863}{160480}$, and so on. Thus, we can write the

*These coefficients are defined by the recursive formula

$$\begin{aligned}c_0^* &= 1, \\ c_m^* &= -\left[\frac{1}{2} c_{m-1}^* + \cdots + \frac{1}{m+1} c_0^* \right], \quad m = 1, 2, \dots.\end{aligned}$$

trapezoidal rule with Gregory's correction as

$$\begin{aligned} \int_a^b F(x) ds &\approx \frac{h}{2} \left[F(a) + 2F(a+h) + 2F(a+2h) + \dots + 2F(b-h) \right. \\ &\quad \left. + F(b) \right] - \frac{h}{12} (\nabla F(b) - \Delta F(a)) - \frac{h}{24} (\nabla^2 F(b) - \Delta^2 F(a)) \\ &\quad - \frac{19h}{720} (\nabla^3 F(b) - \Delta^3 F(a)) - \frac{3h}{160} (\nabla^4 F(b) - \Delta^4 F(a)) \\ &\quad - \frac{863h}{60480} (\nabla^5 F(b) - \Delta^5 F(a)), \end{aligned} \quad (4.4.7)$$

where $F(b) = F(x)|_{x=b}$, and so on.

Now, we can regard the value of $\tilde{\phi}^{(0)}$ obtained from (4.4.4) with Gregory correction (4.4.5) as the zeroth corrected approximation, which can be improved further to $\tilde{\phi}^{(1)}$ by computing

$$\tilde{\phi}_h^{(1)}(a+ih) - \lambda h \sum_{j=0}^n k(a+ih, a+jh) \tilde{\phi}_h^{(1)}(a+jh) = f(a+ih) + \lambda \delta_i(\tilde{\phi}^{(0)}). \quad (4.4.8)$$

In fact, if we write $\mathbf{d}^{(1)} = \tilde{\Phi}^{(1)} - \tilde{\Phi}^{(0)}$, we get

$$d_i^{(1)} - \lambda h \sum_{j=0}^n k(a+ih, a+jh) d_j^{(1)} = \lambda \delta_i(\tilde{\phi}^{(0)}), \quad (4.4.9)$$

which in matrix notation becomes

$$(\mathbf{I} - \lambda \mathbf{kD}) \mathbf{d}^{(1)} = \lambda \delta(\tilde{\Phi}^{(0)}). \quad (4.4.10)$$

Once $\mathbf{d}^{(1)}$ is computed, we obtain $\tilde{\Phi}^{(1)}$ from the relation $\tilde{\Phi}^{(1)} = \tilde{\Phi}^{(0)} + \mathbf{d}^{(1)}$. This scheme of computing $\tilde{\Phi}^{(1)}$ from $\tilde{\Phi}^{(0)}$ is known as DEFERRED CORRECTION (see Baker 1978, p.368).

An ITERATIVE DEFERRED CORRECTION SCHEME is as follows: Once we have computed $\tilde{\Phi}^{(1)}$ by the above correction scheme, we compute the vector $\tilde{\Phi}^{(1)}$, $i > 1$, such that

$$(\mathbf{I} - \lambda \mathbf{kD}) \mathbf{d}^{(i)} = \lambda \delta(\tilde{\Phi}^{(i-1)}), \quad (4.4.11)$$

where

$$\delta(\tilde{\Phi}^{(i-1)}) = [\delta_0(\tilde{\Phi}^{(i-1)}), \delta_1(\tilde{\Phi}^{(i-1)}), \dots, \delta_n(\tilde{\Phi}^{(i-1)})]^T,$$

and $\delta_j(\tilde{\Phi}^{(i-1)})$ is computed by substituting $\tilde{\Phi}^{(i-1)}$ for $\tilde{\Phi}$ in (4.4.6). Note that in (4.4.11) $\mathbf{k} = [k(a + ih, a + jh)]$, $\mathbf{D} = \text{diag}\left(\frac{h}{2}, h, \dots, h, \frac{h}{2}\right)$, and \mathbf{I} is the identity matrix. To solve (4.4.11), we compute $\mathbf{d}^{(i)} = \tilde{\Phi}^{(i)} - \tilde{\Phi}^{(i-1)}$ at each stage for $i > 1$, such that

$$(\mathbf{I} - \lambda \mathbf{kD}) \mathbf{d}^{(i)} = \lambda \delta(\mathbf{d}^{(i-1)}), \quad i > 1, \quad (4.4.12)$$

which yields

$$\tilde{\Phi}^{(i)} = \tilde{\Phi}^{(i-1)} + \mathbf{d}^{(i)} = \mathbf{d}^{(1)} + \dots + \mathbf{d}^{(i)} + \tilde{\Phi}^{(0)}, \quad (4.4.13)$$

where we compute $\delta(\mathbf{d}^{(i)})$ for each next stage. This leads to the final desired approximation $\tilde{\Phi}^{(p)}$.

The above approximation scheme can also be constructed by using the Lagrangian form for $\delta_i(\tilde{\phi})$; thus, in formula (4.4.6) we write

$$\delta_i(\tilde{\phi}) = h \sum_{j=0}^p \Omega_j^{[p]} \left\{ \vartheta_i(a + jh) + \vartheta_i(b - jh) \right\}, \quad (4.4.14)$$

where $\vartheta_i(x) = \vartheta_i(\tilde{\phi}; x)$, and some values of the coefficients $\Omega_j^{[p]}$ are tabulated below.

p	$j = 0$	$j = 1$	$j = 2$	$j = 3$
1	-1/12	1/12		
2	-1/2	1/6	-1/24	
3	-109/720	177/720	-87/720	19/720

Note that in the case when the matrix $(\mathbf{I} - \lambda \mathbf{kD})$ is ill conditioned, the vector $\mathbf{d}^{(i-1)}$, $i = 1, 2, \dots$, may be computed by Gaussian elimination followed by the above iterative scheme. Another advantage of this iterative scheme is that it foretells if there is any ill-conditioning in the integral equation.

EXAMPLE 4.4.1. Let $k(x, s) = e^{xs}$, $f(x) = 1 - \frac{e^x - 1}{x}$, $0 \leq x \leq 1$, and $\lambda = 1$. The exact solution is $\phi(x) = 1$. The trapezoidal rule with Gregory's

correction is used. Baker (1978, p.370) has the following results:

(a) By a single deferred correction, the maximum errors are:

h	Trapezoidal	1 difference	2 differences	3 differences
1/8	2.42(-3)	2.97(-4)	4.22(-6)	1.33(-6)
1/16	6.0(-4)	3.81(-5)	3.63(-7)	4.54(-8)
1/32	1.50(-4)	4.85(-6)	2.81(-8)	1.81(-9)
1/64	3.75(-5)	6.13(-7)	2.42(-9)	6.86(-10)

(b) By iterative deferred correction, the maximum errors $\|\Phi - \tilde{\Phi}^{(i)}\|_\infty$ are:

h	Trapezoidal	1 iteration	2 iterations
1/8	1.6(-3)	2.2(-4)	1.0(-5)
1/16	4.0(-4)	3.0(-5)	7.6(-7)
1/32	1.1(-4)	1.2(-5)	1.4(-5)
1/64	3.7(-5)	2.0(-5)	2.0(-5)

No improvements over 2 iterations are obtained in 3, 4, or 5 iterations.

(c) By an iterative correction scheme using 5 differences (i.e., $p = 5$ in (4.4.6)), the values of $\tilde{\phi}$ are:

x	$h = 1/4$	$h = 1/8$	$h = 1/16$	Exact
± 1.0	1.64135	1.64011	1.63979	1.63970
± 0.75	1.75445	1.75258	1.75210	1.75195
± 0.5	1.84537	1.84313	1.84257	1.84238
± 0.25	1.90176	1.90040	1.89982	1.89962
± 0.0	1.92220	1.91982	1.91923	1.91903

These results are from Baker (1978, p.370ff). ■

EXAMPLE 4.4.2. (Baker 1978, p. 372) Consider

$$\phi(x) - \frac{1}{\pi} \int_0^1 \frac{\phi(s) ds}{1 + (x-s)^2} = 1.$$

Using the trapezoidal rule with step size $h = 1/32$ and the Gregory correction, the results are given in Table 4.4.2, where “iter(s)” stands for the number of iterations.

Table 4.4.2

x	trapezoidal	1 iter	2 iters	3 iters	4 iters	5 iters
0.0	1.918932	1.919033	1.919032	1.919031	1.919031	1.919031
0.25	1.899516	1.899616	1.899615	1.899615	1.899615	1.8996151
0.5	1.842291	1.842385	1.842384	1.842384	1.84238	1.8423847
0.75	1.751876	1.751954	1.751954	1.751995	1.751954	1.7519547
1.0	1.639643	1.639694	1.6396952	1.639695	1.639695	1.6396952 ■

4.5. A Modified Quadrature Method

If $k(x, s)$ is badly behaved at $x = s$, we can use the following method to reduce the error in the quadrature method: Write Eq (4.1.1) as

$$\phi(x) - \lambda \phi(x) \int_a^b k(x, s) ds - \lambda \int_a^b k(x, s) [\phi(s) - \phi(x)] ds = f(x), \quad (4.5.1)$$

and use a quadrature rule $Q(F)$ to get

$$\hat{\phi}(x) [1 - \lambda A(x)] - \lambda \sum_{j=0}^n w_j k(x, s_j) [\hat{\phi}(s_j) - \hat{\phi}(x)] = f(x), \quad (4.5.2)$$

where

$$A(x) = \int_a^b k(x, s) ds. \quad (4.5.3)$$

Hence,

$$\hat{\phi}(x) [1 - \lambda \Delta(x)] - \lambda \sum_{j=0}^n w_j k(x, s_j) \hat{\phi}(s_j) = f(x), \quad (4.5.4)$$

where

$$\Delta(x) = A(x) - \sum_{j=0}^n w_j k(x, s_j). \quad (4.5.5)$$

If we set $x = x_i$, $i = 0, 1, \dots, n$ in (4.5.5), we obtain by the Nyström method

$$\hat{\phi}(x_i) [1 - \lambda \Delta(x_i)] - \lambda \sum_{j=0}^n w_j k(x_i, s_j) \hat{\phi}(s_j) = f(x_i), \quad (4.5.6)$$

which in matrix notation becomes

$$(\mathbf{I} - \lambda (\mathbf{\Delta} - \mathbf{kD})) \tilde{\Phi} = \mathbf{f}, \quad (4.5.7)$$

where $\mathbf{\Delta} = \text{diag}(\Delta(x_0), \Delta(x_1), \dots, \Delta(x_n))$. In this method it is important that $A(x)$ is computed accurately. Note that this modified quadrature method is useful when $k(x, s)$ is discontinuous at $x = s$ and also when $k(x, s)$ is weakly singular (see Chapter 8). It is found that $\hat{\phi}$ gives better approximations than $\tilde{\phi}$ in the above cases.

EXAMPLE 4.5.1. Consider Love's equation (see Example 4.3.2). Here

$$A(x) = \int_{-1}^1 k(x, s) ds = -\frac{1}{\pi} \tan^{-1} \left(\frac{2}{x^2} \right).$$

Use the trapezoidal rule with $h = 2/n$. The values of $\hat{\phi}$ at $x = (-1)(0.25)1$ are presented in Table 4.5.1.

Table 4.5.1

	$n = 8$	$n = 16$	$n = 32$	$n = 64$
$x = \pm 1.0$	1.63838	1.63937	1.63961	1.63967
$x = \pm 0.75$	1.751071	1.75164	1.75188	1.75194
$x = \pm 0.5$	1.84135	1.84213	1.84232	1.84237
$x = \pm 0.25$	1.89874	1.89940	1.89956	1.89960
$x = 0.0$	1.91821	1.91883	1.91898	1.91902 ■

A further MODIFICATION is possible in the case when $k(x, s) = g(s) \kappa(x, s)$ in (4.1.1), where $g(s)$ is badly behaved; e.g., $g(s) = \sqrt{s}$ so that $g'(0)$ is unbounded. Then we use a quadrature rule for products (Section A.3), i.e.,

$$\int_a^b g(s) p(s) ds = \sum_{j=0}^n v_j p(s_j). \quad (4.5.8)$$

While a quadrature rule produces a system of the form (4.3.2), the product rule (4.5.8) yields a system of the form

$$\tilde{\phi}(x_i) - \lambda \sum_{j=0}^n v_j \kappa(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, n. \quad (4.5.9)$$

If $g(x) > 0$, then it is possible to develop a Gaussian quadrature to approximate the integral (4.5.8). However, this modified method is successful if the functions $f(x)$ and $\kappa(x, s)$ are smooth. Otherwise the product-integration method discussed in the next section is better to use.

4.6. Collocation Methods

We have noted that in the Galerkin, Rayleigh–Ritz and least-squares methods the problem of finding an approximate solution of the FK2 (4.1.1) leads to a system of infinite equations in infinitely many unknowns, which we reduce to a finite system in finitely many unknowns. However, the collocation method consists of obtaining the approximate solution $\tilde{\phi}$ in the form of a (finite) linear combination of the form $\tilde{\phi}(x) = \sum_{j=0}^n a_j p_j(x)$, where $p_j(x)$ are finitely many linearly independent basis functions, and the coefficients $a_j, j = 0, 1, \dots, n$, are chosen such that

$$\tilde{\phi}(x_i) - \lambda \int_a^b k(x_i, s) \tilde{\phi}(s) ds = f(x_i), \quad (4.6.1)$$

where $x_i, i = 0, 1, \dots, n$, are the nodes (properly chosen collocation points) on the interval $[a, b]$. Thus, the equations to determine the coefficients a_0, a_1, \dots, a_n are

$$\sum_{j=0}^n \left\{ p_j(x_i) - \lambda g_j(x_i) \right\} a_j = f(x_i), \quad i = 0, 1, \dots, n, \quad (4.6.2)$$

where

$$g_j(x_i) = \int_a^b k(x_i, s) p_j(s) ds. \quad (4.6.3)$$

The system (4.6.2) can be written in matrix form as

$$(\mathbf{B} - \lambda \mathbf{A}) \mathbf{a} = \mathbf{f}, \quad (4.6.4)$$

where

$$B_{ij} = p_j(x_i), \quad A_{ij} = g_j(x_i),$$

$$\mathbf{a} = [a_0, a_1, \dots, a_n]^T, \quad \mathbf{f} = [f(s_0), f(s_1), \dots, f(s_n)]^T.$$

Now, it may so happen that a particular choice of the collocation points x_i may make the matrix $\mathbf{B} - \lambda \mathbf{A}$ singular even if λ is a regular value of the kernel. Since we have assumed that $\tilde{\Phi} = [\tilde{\phi}(s_0), \tilde{\phi}(s_1), \dots, \tilde{\phi}(s_n)]^T$, where $\tilde{\phi}(x) = \sum_{i=0}^n a_i p_i(x)$, and \mathbf{a} satisfies Eq (4.6.4), then $\mathbf{B}\mathbf{a} = \tilde{\Phi}$, and if \mathbf{B} is nonsingular, then Eq (4.6.4) implies that

$$(\mathbf{I} - \lambda \mathbf{AB}^{-1}) \tilde{\Phi} = \mathbf{f}, \quad (4.6.5)$$

which, when solved, will yield the values of $\tilde{\phi}(x_i)$ directly. So now we have a choice between systems (4.6.4) and (4.6.5) to obtain a solution of the FK2; one is better than the other depending on various situations pertaining to the behavior of the kernel and the free term. Another question is about the choice of the basis functions $p_i(x)$, which are chosen in the same manner as before; namely, if $f(x)$ is badly behaved, we may take $p_0(x) = f(x)$. But if $p_0(x)$ is linearly dependent on $p_1(x), \dots, p_n(x)$, the matrix $\mathbf{B} - \lambda \mathbf{A}$ may turn out to be ill conditioned, which will result in incorrect approximations for the coefficients a_i . In fact, the choice of the basis functions $p_i(x)$ and of the collocation points x_i , $i = 0, 1, \dots, n$, affects the condition number of the system (4.6.4), and these two choices must be matched with each other (see Section A.8). If nothing else works, choose p_i or x_i or both such that the matrix \mathbf{B} remains well conditioned. Since we have assumed $\tilde{\phi}(x) = \sum_{i=0}^n a_i p_i(x)$, some choices of $p_i(x)$ are better than others in practice. Some guidelines for the choice of the basis functions $p_i(x)$ and the collocation points s_i are as follows:

- (i) If two sets of basis functions, say $\{p_i^{(1)}(x)\}$ and $\{p_i^{(2)}(x)\}$, each span the same space, either can be chosen.
- (ii) The Chebyshev polynomials $T_i(x)$ of the first kind which contain monomials x^n , $n = 0, 1, \dots$, make a good choice for the basis function $p_i(x)$.
- (iii) Since the polynomial interpolation of a continuous function at equally spaced collocation points may give poor approximations as the degree of the polynomial increases, we should not choose the collocation points s_i as equally spaced points on $[a, b]$ if we plan to use polynomials for the function $p_i(x)$. Thus, in the case when $p_i(x)$ is chosen as Chebyshev polynomials $T_i(x)$ of the first kind or as Legendre polynomials $P_i(x)$, when shifted to the interval $[a, b]$ (which in most problems is $[-1, 1]$), we should choose the zeros of $T_{n+1}(x)$ or of $P_{n+1}(x)$, or the extrema of $T_n(x)$, shifted to $[a, b]$, as the collocation points x_i . For a discussion on the choice of x_i , see Kadner (1967). Some particular cases are as follows:

- (a) If $p_i(x)$ is chosen as an algebraic polynomial of degree i on the interval $[-1, 1]$, and if x_0, x_1, \dots, x_n are precisely the zeros of the Legendre polynomial $P_{n+1}(x)$, then $\|K - K^{[n]}\|_2 \rightarrow 0$ as $n \rightarrow \infty$, where $K^{[n]}$ is a degenerate kernel of rank n (see Section 2.3).

(b) If $p_i(x)$ is chosen as Chebyshev polynomials $T_i(x)$, or if $p_i(x) = c_i T_i(x)$, $c_i \neq 0$, and if x_0, x_1, \dots, x_n are zeros of $T_{n+1}(x)$ or the extrema of $T_n(x)$, then $\|K - K^{[n]}\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ provided that $k(x, s) \in H^\alpha[-1, 1]$.

(c) Let $p_i(x)$ be defined by

$$p_{i,n}(x) = \begin{cases} 1 & \text{if } a + ih \leq x \leq a + (i+1)h, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 0, 1, \dots, n-1$, and

$$p_n(x) = \begin{cases} 1 & \text{if } b - h \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|K - K^{[n]}\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ provided that $k(x, s) \in C[a, b]$ for $a \leq x \leq s \leq b$. For a proof of these results, see Baker 1978, p.342.

(iv) For the following two choices of $p_i(x)$ in terms of Chebyshev polynomials $T_i(x)$ on the interval $[-1, 1]$, the inverse \mathbf{B}^{-1} is explicitly known:

(a) If

$$\begin{aligned} p_0(x) &= \frac{1}{2}, \\ p_i(x) &= T_i(x), \quad i = 0, 1, \dots, n, \end{aligned}$$

then the collocation points $x_j = \cos \frac{(2j+1)\pi}{2(n+1)}$, $j = 0, 1, \dots, n$, and

$$(\mathbf{B}^{-1})_{ij} = \frac{2}{n+1} T_i(x_j), \quad i, j = 0, 1, \dots, n.$$

(b) If

$$\begin{aligned} p_0(x) &= \frac{1}{2}, \\ p_i(x) &= T_i(x), \quad i = 0, 1, \dots, n-1, \\ p_n(x) &= \frac{1}{2} T_n(x), \end{aligned}$$

then the collocation points $x_j = \cos(j\pi/n)$, $j = 0, 1, \dots, n$, and

$$(\mathbf{B}^{-1})_{ij} = \frac{2}{n} T_i(x_j) c_i, \quad i, j = 0, 1, \dots, n,$$

where $c_0 = c_n = 1/2$ and $c_1 = c_2 = \dots = c_{n-1} = 1$.

(v) Polynomial approximations do not yield good results if $\phi(x)$ is badly behaved. In such a case we may use piecewise polynomial or spline approximations. Equally spaced collocation points in spline approximations are desirable if the spline knots coincide with the collocation points.

(vi) If $\phi(x)$ is suspected to be periodic, then choose $p_i(x)$ as periodic functions.

(vii) If the functions $p_i(x)$ are orthogonal, or if the vector components $p_i(x_j)$, $i, j = 0, 1, \dots, n$, are orthogonal, where x_0, x_1, \dots, x_n are the collocation points, then the sensitivity to errors and ill-conditioning effects remain more satisfactory (Mikhlin 1971, Delves and Walsh 1974). This is so because of the fact that with such a choice the inverse \mathbf{B}^{-1} of the matrix \mathbf{B} in the system (4.6.4) is well conditioned.

EXAMPLE 4.6.1. Consider the FK2

$$\phi(x) - \frac{1}{2} \int_{-1}^1 |x-s| \phi(s) ds = e^x. \quad (4.6.6)$$

Take $p_i(x) = T_i(x)$, $i = 0, 1, \dots, n$, and $x_j = \cos \frac{(2j+1)\pi}{2(n+1)}$, $j = 0, 1, \dots, n$, as collocation points. To compute the functions $g_i(x_j)$ approximately, we should determine the functions

$$g_i(x) = \int_{-1}^1 |x-s| T_i(s) ds$$

analytically, which yields

$$\begin{aligned} g_0(x) &= \frac{3}{2} + \frac{1}{2} T_2(x), \\ g_1(x) &= \frac{1}{12} T_3(x) - \frac{3}{4} T_1(x), \\ g_2(x) &= \frac{1}{24} T_4(x) - \frac{1}{3} T_2(x) - \frac{3}{8}, \\ g_m(x) &= \frac{T_{m+2}(x)}{2(m+1)(m+2)} - \frac{T_m(x)}{m^2-1} + \frac{T_{m-2}(x)}{2(m-1)(m-2)} \\ &\quad + \frac{1-(-1)^m}{m^2-1} T_1(x) - \frac{1+(-1)^m}{m^2-4}, \quad m \geq 3. \end{aligned}$$

The coefficients a_i are computed with these choices as

For $n = 3$: 4.212715, 8.35344(-1), 1.135396, 7.31161(-2);

For $n = 7$: 4.203920, 8.35644(-1), 1.134557, 7.45274(-2), 2.817511(-2), 1.44452(-3), 2.75829(-4), 1.16240(-5).

The values of $\tilde{\phi}$ using the trapezoidal rule with $h = 1/25$ are

$$\begin{aligned}\tilde{\phi}(-1) &= 4.45897, & \tilde{\phi}(-0.8) &= 3.86002, & \tilde{\phi}(-0.6) &= 3.43406, \\ \tilde{\phi}(-0.4) &= 3.16799, & \tilde{\phi}(-0.2) &= 3.05604, & \tilde{\phi}(0) &= 3.09969, \\ \tilde{\phi}(0.2) &= 3.30798, & \tilde{\phi}(0.4) &= 3.69820, & \tilde{\phi}(0.6) &= 4.29687, \\ \tilde{\phi}(0.8) &= 5.14134, & \tilde{\phi}(1) &= 6.28172.\end{aligned}$$

The maximum error for $n = 3$ is $3.5(-2)$, and for $n = 7$ it is $1.5(-6)$. ■

NOTES: 1. In Example 4.6.1, $k(x, s) = |x - s|$, $0 \leq x, s \leq 1$, and the derivative $\frac{\partial k}{\partial s}$ is discontinuous at $x = s$. So we set $k_1(x, s) = s - x$ and $k_2(x, s) = x - s$. But which of these two kernels k_1 and k_2 to choose is rather difficult.

2. Another example is provided by the kernel $k(x, s) = \frac{1}{|x - s| + 1/2}$, $0 \leq x, s \leq 1$. Then if we take $k_1(x, s) = \frac{1}{s - x + 1/2}$ and $k_2(x, s) = \frac{1}{x - s + 1/2}$, both kernels remain unbounded at certain points on the interval $[0, 1]$

4.7. Elliott's Modification

Elliott (1963) has modified the collocation method by replacing the kernel $k(x, s)$ by a degenerate kernel $k^{[m]}$ (see Section 2.4) on the interval $[-1, 1]$, where Chebyshev polynomials $T_i(x)$, $i = 0, 1, \dots, n$, are used for the basis functions $p_i(x)$. Thus, we take the degenerate kernel as

$$k^{[m]}(x, s) = \sum_{i=0}^n b_i(x) T_i(s), \quad (4.7.1)$$

where

$$b_i(x) = \frac{2}{m} \sum_{j=0}^n k \left(x, \cos \frac{j\pi}{m} \right) \cos \frac{ij\pi}{m}. \quad (4.7.2)$$

This method then determines the coefficients a_0, a_1, \dots, a_n such that

$$\tilde{\phi} \left(\cos \frac{i\pi}{n} \right) - \lambda \int_{-1}^1 k^{[m]} \left(\cos \frac{i\pi}{n}, s \right) \tilde{\phi}(s) ds = f \left(\cos \frac{i\pi}{n} \right), \quad i = 0, 1, \dots, n, \quad (4.7.3)$$

where

$$\tilde{\phi}(x) = \frac{a_0}{2} + \sum_{i=1}^n a_i T_i(x). \quad (4.7.4)$$

This leads to a linear system of $(n + 1)$ algebraic equations for the unknowns a_0, a_1, \dots, a_n in terms of the values of $k \left(\cos \frac{i\pi}{n}, \cos \frac{j\pi}{n} \right)$ and $f \left(\cos \frac{i\pi}{n} \right)$, $i, j = 0, 1, \dots, n$.

A further modification by Elliott and Warne (1967) deals with kernels that are badly behaved, i.e., they have a discontinuity either in the kernel itself or in the derivative at $x = s$. This discontinuity is treated by replacing the kernel $k(x, s)$ by an approximate kernel $k^{[m]}(x, s)$, defined by

$$k^{[m]}(x, s) = \begin{cases} k^{[m,1]}(x, s) = \sum_{i=0}^m b_i^{[1]}(x) T_i(x) & \text{for } x \leq s, \\ k^{[m,2]}(x, s) = \sum_{i=0}^m b_i^{[2]}(x) T_i(x) & \text{for } x > s. \end{cases} \quad (4.7.5)$$

For brevity we write $k^{[m,1]}$ as $k^{[1]}$ and $k^{[m,2]}$ as $k^{[2]}$. Note that both $k^{[1]}$ and $k^{[2]}$ are defined on the square $-1 \leq x, s \leq 1$. This method is successful only if both $k^{[1]}$ and $k^{[2]}$ are smooth on this square. This is explained in Examples 4.7.1 and 4.7.2.

EXAMPLE 4.7.1. For the kernel in Example 4.6.1, we use Elliott and Warne's modification and set $k^{[1]}(x, s) = -(x - s)$ and $k^{[2]}(x, s) = x - s$ for $m \geq 1$, and choose the collocation points as $s_j = \cos(j\pi/n)$, $j = 0, 1, \dots, n$. Then the values of a_j are as follows:

For $n = 3$: 4.19646, 8.36677(-1), 1.15731, 7.45397(-2).

The maximum error in $\tilde{\phi}(x)$ for $n = 3$ is 5.8(-2); for $n = 7$ it is 2.5(-6); and for $n = 15$ it is 1.1(-13). ■

EXAMPLE 4.7.2. Consider Love's equation with $d = -1$ (Example 4.3.2), and apply Elliott and Warne's modification with $m = n$, where m is the degree of the approximation of the degenerate kernel $k^{[m]}(x, s)$ as a polynomial in s , and n is the degree of $\tilde{\phi}(x) = \sum_{i=0}^n a_i T_i(x)$. The following results are obtained:

For $n = 4$, the coefficients a_{2i} are: 3.550319, -1.41968(-1), 7.045497(-3);

For $n = 8$, the coefficients a_{2i} are: 3.548891, -1.400434(-1), 4.963829(-3), 3.72087(-4), -4.212621(-5);

For $n = 16$, the coefficients a_{2i} are: $3.548890, -1.400431(-1), 4.961994(-3), 3.762979(-4), -4.368691(-5), -1.623246(-6), 4.966225(-7), -6.190306(-9), -5.041943(-9)$.

The solution $\tilde{\phi}$ is shown in Table 4.7.1.

Table 4.7.1

n	0	$x = \pm 0.25$	$x = \pm 0.5$	$x = \pm 0.75$	$x = \pm 1$
4	1.924173	1.903125	1.842621	1.750588	1.640237
8	1.919038	1.899618	1.842378	1.751938	1.639696
16	1.919032	1.899615	1.842385	1.751953	1.639695

NOTE. If Love's equation with $d = 0.1$ is considered, the above satisfactory results no longer hold. Baker (1978, p.405) has noticed that in this case the coefficients a_{2i} for $n = 32$ decrease very slowly; they are $1.13, 7.95(-2), 4.24(-2), 1.93(-2), 6.54(-3), 9.65(-4), -6.69(-4), \dots, -1.89(-6)$.

This results in a very poor solution for this case, namely, for $n = 32$ we get $\tilde{\phi}(0) = 5.144(-1)$, $\tilde{\phi}(\pm 0.25) = 5.155(-1)$, $\tilde{\phi}(\pm 0.5) = 5.199(-1)$, $\tilde{\phi}(\pm 0.75) = 5.343(-1)$, and $\tilde{\phi}(\pm 1) = 7.125(-1)$,

which can be compared with the desired results in Table 4.7.1. ■

REFERENCES USED: Baker (1978), Davis and Rabinowitz (1984), Delves and Walsh (1974), El-gendi (1969), Elliott (1967), Elliott and Warne (1967), Henrici (1964), Kadner (1967), Kalandiya (1973), Mikhlin (1971), Mushkhelishvili (1992), Polyanin and Manzhirov (1998), Young (1954a,b).

5

Variational Methods

The variational formulation of boundary value problems originates from the fact that weighted variational methods provide approximate solutions of such problems. Variational methods for solving boundary value problems are based on the techniques developed in the calculus of variations. They deal with the problem of minimizing a functional, and thus reducing the given problem to solving a system of algebraic equations. Conversely, a boundary value problem can be formulated as a minimizing problem. The methods developed in solving these boundary value problems are also applicable to integral equations. We discuss them in this chapter.

5.1. Galerkin Method

The classical Galerkin method for an FK2

$$\phi(x) - \lambda \int_a^b k(x, s)\phi(s) ds = f(x), \quad a \leq x \leq b, \quad (5.1.1)$$

consists of seeking an approximate solution of the form

$$\tilde{\phi}(x) = \sum_{i=0}^n \tilde{a}_i p_i(x), \quad (5.1.2)$$

which is a linear combination of prescribed, linearly independent basis functions $\{p_i(x)\}$ defined on $[a, b]$. Substituting (5.1.2) into (5.1.1), we find that

$$\tilde{\phi}(x) - \lambda \int_a^b k(x, s) \tilde{\phi}(s) ds - f(x) = r(x), \quad (5.1.3)$$

where $r(x)$ is the residual such that $r(x) = 0$ for $\tilde{\phi}(x) = \phi(x)$. Our goal is to compute $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n$ such that $r(x) \equiv 0$. However, in general, it is not possible to choose such \tilde{a}_i , $i = 0, 1, \dots, n$. Different conditions on $r(x)$ lead to different equations for \tilde{a}_i . This method has its origin with the numerical solution of differential equations, and Collatz (1966) calls $r(x)$ an “error.” The method described here contains the collocation method as a special case. It aims at making $r(x)$ as small as possible.

Thus, the Galerkin method requires that the functions

$$q_i(x) = \int_a^b k(x, s) p_i(s) ds \quad (5.1.4)$$

and the coefficients \tilde{a}_i , $i = 0, 1, \dots, n$, be obtained by solving the system of algebraic equations

$$(\mathbf{B} - \lambda \mathbf{A}) \tilde{\mathbf{a}} = \mathbf{c}, \quad (5.1.5)$$

for the unknowns $\tilde{\mathbf{a}} = [\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n]^T$. To define the other quantities in Eq (5.1.5), we use the following notation for any two functions u and v :

$$\langle u, v \rangle = \int_a^b u(x) \overline{v(x)} dx, \quad (5.1.6)$$

$$(u, v) = \int_a^b w(x) u(x) \overline{v(x)} dx, \quad (5.1.7)$$

where $w(x) > 0$ for $0 < x < b$ is a known weight function, and a discrete version

$$\prec u, v \succ = \sum_{j=0}^N w_j u(s_j) \overline{v(s_j)}, \quad (5.1.8)$$

where $w_j > 0$ and $a \leq s_j \leq b$ for $j = 0, 1, \dots, N$, $N \geq n$. Note that (5.1.6) is the inner product of u and v as defined in Section 1.3, and the weights w_j and the nodes s_j are chosen to correspond to a quadrature rule for product integration (see Section A.3).

The CLASSICAL GALERKIN METHOD requires that we compute \tilde{a}_i , $i = 0, 1, \dots, n$, such that $\langle r, p_i \rangle = 0$. This is accomplished by setting

$$\begin{aligned} B_{ij} &= \langle p_i, p_j \rangle, \\ A_{ij} &= \langle p_i, q_j \rangle, \\ c_i &= \langle p_i, f \rangle, \quad i, j = 0, 1, \dots, n, \end{aligned} \quad (5.1.9)$$

where q_j are defined by (5.1.4); this yields the elements of the matrices \mathbf{B} , \mathbf{A} , and \mathbf{c} , respectively, occurring in (5.1.5). This method is equivalent to a Rayleigh–Ritz method in the case when the kernel $k(x, s)$ is Hermitian.

A first generalization of the classical Galerkin method consists of requiring that $(r, p_i) = 0$ for $i = 0, 1, \dots, n$. In this case we set

$$\begin{aligned} B_{ij} &= (p_i, p_j), \\ A_{ij} &= (p_i, q_j), \\ c_i &= (p_i, f), \quad i, j = 0, 1, \dots, n, \end{aligned} \tag{5.1.10}$$

to obtain the elements of the matrices \mathbf{B} , \mathbf{A} , and \mathbf{c} , respectively, that occur in the system (5.1.5). A second generalization consists of requiring that $\prec r, p_i \succ = 0$ for $i = 0, 1, \dots, n$. In this case we set

$$\begin{aligned} B_{ij} &= \prec p_i, p_j \succ, \\ A_{ij} &= \prec p_i, q_j \succ, \\ c_i &= \prec p_i, f \succ, \quad i, j = 0, 1, \dots, n, \end{aligned} \tag{5.1.11}$$

to obtain the elements of the matrices \mathbf{B} , \mathbf{A} , and \mathbf{c} , respectively, that occur in the system (5.1.5).

The classical METHOD OF MOMENTS which is a Galerkin method requires that we choose $(n + 1)$ linearly independent functions $h_i(x)$, $i = 0, 1, \dots, n$, and compute $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n$ such that $\langle r, h_i \rangle = 0$ for $i = 0, 1, \dots, n$, by setting

$$\begin{aligned} B_{ij} &= \langle h_i, p_j \rangle, \\ A_{ij} &= \langle h_i, q_j \rangle, \\ c_i &= \langle h_i, f \rangle, \quad i, j = 0, 1, \dots, n, \end{aligned} \tag{5.1.12}$$

which yield the elements of the matrices \mathbf{B} , \mathbf{A} , and \mathbf{c} , respectively, that occur in the system (5.1.5). Note that (5.1.9) or (5.1.10) is a special case of (5.1.12) where we replace $h_i(x)$ by $p_i(x)$ or by $w(x) p_i(x)$, respectively. Similarly, a discrete version of the method of moments is obtained if we replace $\langle \cdot \rangle$ by $\prec \cdot \succ$ in (5.1.12). Although each case mentioned above computes the elements of the matrices \mathbf{B} , \mathbf{A} , and \mathbf{c} , a unique solution of the system (5.1.5) may not be guaranteed.

A RAYLEIGH–RITZ METHOD, mentioned above, is a minimization method. It is useful in the case when $\lambda k(x, s) = \overline{\lambda k(s, x)}$. This method requires that the solution $\phi(x)$ of the FK2 (5.1.1) minimize the functional

$$I[p] = \langle p - \lambda K p, p \rangle - 2 \Re \{ \langle p, f \rangle \}, \tag{5.1.13}$$

where $(Kp)(x) = \int_a^b k(x, s)p(s) ds$. Thus, we need to compute

$$\tilde{\phi}(x) = \sum_{j=0}^n \tilde{a}_j p_j(x), \quad (5.1.14)$$

so that the functional $I[\tilde{\phi}]$ is minimized over all possible choices of $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n$. This implies that

$$\frac{\partial}{\partial \tilde{a}_j} I[\tilde{\phi}] = 0, \quad j = 0, 1, \dots, n, \quad (5.1.15)$$

which reduces to the system of equations (5.1.5), encountered in the classical Galerkin method, to determine $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n$. In view of the condition (5.1.15), the Rayleigh–Ritz method is called a variational method.

The computational aspect of the Galerkin method is more involved than that of the collocation method, although both methods involve the solution of the system (5.1.4). The difference lies in the fact that in the Galerkin method the elements B_{ij} are integrals, and A_{ij} are the integrals of a product of functions one of which $q_i(x)$ is itself obtained from an integral. The choice of the basis functions $p_i(x)$ should be made to ensure that the matrix \mathbf{B} is diagonal. In computing A_{ij} Baker (1978, p.409) notes that a faster computation is achieved if we use an adaptive multiple integration technique (see Section A.3), whereas Delves and Walsh (1974) suggest that we should use the same quadrature rule for computing the elements of the matrices \mathbf{A} and \mathbf{B} .

EXAMPLE 5.1.1. Consider an FK2 with $\lambda = 1$,

$$k(x, s) = \begin{cases} x(1-s) & \text{if } 0 \leq x \leq s \leq 1, \\ s(1-x) & \text{if } 0 \leq s \leq x \leq 1, \end{cases}$$

and $f(x) = x \sin 1$. The exact solution is $\phi(x) = \sin x$. In the classical Galerkin method we seek an approximate solution of the form

$$\tilde{\phi}(x) = \sum_{i=0}^n \tilde{a}_i T_i^*(x),$$

where $T_i^*(x) = T_i(2x - 1)$ are the shifted Chebyshev polynomials of the first kind. Note that we have chosen $p_i(x) = T_i^*(x)$. To determine the coefficients \tilde{a}_i , $i = 0, 1, \dots, n$, we proceed with computing the inner products (5.1.8) and then solve the system (5.1.5). The approximate solution with $n = 4$ is $\tilde{\phi}(x) = 0.15799 x + 0.00506183 x^2 - 0.182099 x^3 + 0.0190105 x^4$. This example occurs in Baker (1978, p.411). For computational details see `galerin10.nb`.

EXAMPLE 5.1.2. To solve the FK2 with $\lambda = -1$,

$$k(x, s) = \begin{cases} (x-s)^2 - 1 & \text{if } |x-s| \leq 1, \\ 0 & \text{if } |x-s| > 1, \end{cases}$$

and $f(x) = 1/(1+x^2)$ by the Rayleigh-Ritz method, we first notice that since $f(x)$ is symmetric, we should choose the basis functions $p_i(x)$ such that $p_i(-x) = p_i(x)$. Also, since the kernel is Hermitian, and $\mathbf{B}^T = \mathbf{B}$ and $\mathbf{A}^T = \mathbf{A}$, we should compute only on and above the diagonal of the matrices \mathbf{B} and \mathbf{A} .

For $n = 1$, we take $p_0(x) = 1, p_1(x) = x^2$. Then if $p_i(-x) = p_i(x)$, we have

$$\begin{aligned} q_i(x) &= \int_{-1}^1 k(x, s)p_i(s) ds = q_i(-x) \\ &= \int_{x-1}^1 [(x-s)^2 - 1] p_i(s) ds \quad \text{for } x \geq 0, \end{aligned}$$

which gives

$$\begin{aligned} q_0(x) &= \int_{x-1}^1 [(x-s)^2 - 1] p_0(s) ds = -\frac{4}{3} + x^2 - \frac{x^3}{3}, \\ q_1(x) &= \int_{x-1}^1 [(x-s)^2 - 1] p_1(s) ds = -\frac{4}{15} - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^5}{30}. \end{aligned}$$

Also,

$$\begin{aligned} B_{00} &= \langle p_0, p_0 \rangle = 2, \quad B_{01} = B_{10} = \langle p_0, p_1 \rangle = \frac{2}{3}, \quad B_{11} = \langle p_1, p_1 \rangle = \frac{2}{5}; \\ A_{00} &= \langle q_0, p_0 \rangle = 2 \int_0^1 \left(\frac{4}{3} + x^2 - \frac{x^3}{3} \right) dx = -\frac{13}{6}, \\ A_{01} &= \langle q_0, p_1 \rangle = 2 \int_0^1 x^2 \left(\frac{4}{3} + x^2 - \frac{x^3}{3} \right) dx = -\frac{3}{5}, \\ A_{10} &= \langle q_1, p_0 \rangle = 2 \int_0^1 \left(\frac{4}{15} - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^5}{30} \right) dx = -\frac{3}{5}, \\ A_{11} &= \langle q_1, p_1 \rangle = 2 \int_0^1 x^2 \left(\frac{4}{15} - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^5}{30} \right) dx = -\frac{5}{24}, \end{aligned}$$

and

$$\begin{aligned} c_1 &= \langle f, p_0 \rangle = \int_{-1}^1 \frac{1}{1+x^2} dx = \frac{\pi}{2}, \\ c_2 &= \langle f, p_0 \rangle = \int_{-1}^1 \frac{x^2}{1+x^2} dx = 2 - \frac{\pi}{2}. \end{aligned}$$

Hence, the system (5.1.5) for \tilde{a}_0 and \tilde{a}_1 is

$$\left(\begin{bmatrix} 2 & 2/3 \\ 2/3 & 2/5 \end{bmatrix} - \begin{bmatrix} -13/6 & -3/5 \\ -3/5 & -5/24 \end{bmatrix} \right) \begin{Bmatrix} \tilde{a}_0 \\ \tilde{a}_1 \end{Bmatrix} = \begin{Bmatrix} \pi/2 \\ 2 - \pi/2 \end{Bmatrix},$$

or

$$\begin{bmatrix} 25/6 & 19/15 \\ 19/15 & 73/120 \end{bmatrix} \begin{Bmatrix} \tilde{a}_0 \\ \tilde{a}_1 \end{Bmatrix} = \begin{Bmatrix} \pi/2 \\ 2 - \pi/2 \end{Bmatrix},$$

which gives $\tilde{a}_0 = 0.442781$, $\tilde{a}_1 = -0.216416$, and thus, $\tilde{\phi}(x) = 0.442781 - 0.216416 x^2$. For computational details, see `rayleigh1.nb`. ■

EXAMPLE 5.1.3. Consider the FK2 with $\lambda = -1$,

$$k(x, s) = \begin{cases} x(1-s) & \text{if } 0 \leq x \leq s \leq 1, \\ s(1-x) & \text{if } 0 \leq s \leq x \leq 1, \end{cases}$$

and $f(0) \neq 0$, $f(1) \neq 0$. If we choose $p_i(x) = \sin i\pi x$, $i = 0, 1, \dots, n$, which is a complete set in $L_2[0, 1]$, the approximate solution is

$$\tilde{\phi}(x) = \sum_{i=0}^n \tilde{a}_i \sin i\pi x,$$

which vanishes at $x = 0$ and $x = 1$, but since we require $\phi(0) = f(0) \neq 0$ and $\phi(1) = f(1) \neq 0$, we should take

$$\tilde{\phi}(x) = g(x) + \sum_{i=0}^n \tilde{a}_i \sin i\pi x,$$

which shall lead to a correct solution for a prescribed function $f(x)$. ■

For additional material on classical Galerkin methods, see Kythe (1995, 1996).

5.2. Ritz–Galerkin Methods

A general theory of the Ritz–Galerkin methods is presented by considering interpolations and approximation formulas for the solution of an FK2 on the interval $[a, b]$. A unified discussion of these methods is available in Ikebe (1972). We first present the general theory and then four cases of interpolating projections (or operators) for the approximate solution of linear or nonlinear functional equations that arise from integral equations.

The theory becomes simple if we assume that the integral operator K is completely continuous. We shall consider an FK2 of the form (1.2.18) with $\lambda = 1$, that is,

$$(I - K) \phi = f \quad (5.2.1)$$

in a Banach space \mathcal{X} , and choose a sequence of bounded linear projections $\{P_n : n = n_1, n_2, \dots\}$, where each P_n maps \mathcal{X} onto an n -dimensional subspace M_n of \mathcal{X} . Then the FK2 (5.2.1) can be approximated by the *Ritz–Galerkin approximate equation*

$$(I - P_n K) \phi_n = P_n f, \quad (5.2.2)$$

which is an equation in the subspace M_n , such that its solution ϕ_n is an approximate solution of the given FK2, under the assumption that $\|\phi_n - \phi\| \rightarrow 0$ as $n \rightarrow \infty$, where appropriate conditions are imposed on $\{P_n\}$ to ensure this convergence. Eq (5.2.2) is then solved by reducing it to the *Ritz–Galerkin system*

$$(I - B_n) \{c_i\} = \{f_i\}, \quad (5.2.3)$$

which is a matrix equation of order n . The matrix B_n is a full matrix: Each of its elements is, in general, nonzero. Moreover, if K is a symmetric integral operator, then the matrix B_n need not be symmetric. But if P_n represents a Fourier projection (see case (iv) below in Section 5.3) with weight 1, then the symmetry of K implies that of the matrix B_n . The elements of the matrix B_n are computed by integration. However, since all such integrations cannot be computed exactly, we are required to use sufficiently accurate quadrature rules for integration. Also note that the spaces $C[a, b]$ and $L_p[a, b]$, $1 \leq p < \infty$, have bases, but the space $L_\infty[a, b]$ does not since it is not separable. Basic facts of the Ritz–Galerkin method are contained in the following theorem (Ikebe 1972):

THEOREM 5.2.1. Assume that $(I - K)^{-1}$ exists and hence is bounded. Let the projections P_n satisfy the following conditions (a)–(d):

- (a) $\|P_n K g - K g\| \rightarrow 0$ for every $g \in \mathcal{X}$;
- (b) $\|P_n f - f\| \rightarrow 0$;
- (c) $\sup_n \|P_n\| < \infty$;
- (d) $\|P_n g - g\| \rightarrow 0$ for every $g \in \mathcal{X}$.

Then we have

- (i) $\|P_n K - K\| \rightarrow 0$;
- (ii) $(I - P_n K)^{-1} = [I - (I - K)^{-1} (P_n K - K)^{-1}] (I - K)^{-1}$ exists for all sufficiently large n ;
- (iii) $\|(I - P_n K)^{-1} - (I - K)^{-1}\| \leq \frac{\|(I - K)^{-1}\|^2 \cdot \|P_n K - K\|}{1 - \|(I - K)^{-1}\| \cdot \|P_n K - K\|}$;
- (iv) $\phi_n - \phi = (I - P_n K)^{-1} (P_n \phi - \phi) = (I - P_n K)^{-1} [(P_n K - K) \phi + P_n f - f] \rightarrow 0$.

In this theorem we have assumed that $(I - K)^{-1}$ exists, which can be proved if we use the condition that $\|(I - P_n K)^{-1} (P_n K - K)\| < 1$ for some n , because then, by the Banach theorem, $\left(I + (I - P_n K)^{-1} (P_n K - K)\right)^{-1}$ exists, and hence,

$$(I - K)^{-1} = \left(I + (I - P_n K)^{-1} (P_n K - K)\right)^{-1} (I - P_n K)^{-1}. \quad (5.2.4)$$

PROOF. We shall prove (i) by contradiction. Let $\sup_{f \in S} \|P_n K g - K g\| \equiv \alpha_n \rightarrow 0$, where S denotes the unit sphere of \mathcal{X} . Since K is continuous and compact, the set $\tilde{K} = \overline{K(S)}$ is compact in S , where \tilde{K} denotes the closure of $K(S)$ in \mathcal{X} . Then $\alpha_n = \sup_{s \in \tilde{K}} \|P_n s - s\|$. Suppose $\{\alpha_n\}$ does not converge to zero. Then there would exist an $\varepsilon > 0$ and a sequence $\{s_j\}$ in \tilde{K} such that

$$\|P_n s_j - s_j\| \geq \varepsilon > 0 \quad \text{for } j = j_1, j_2, \dots,$$

and such that $s_j \rightarrow s \in \tilde{K}$. But then

$$\|P_n s_j - s_j\| \leq \|P_n\|_{\mathcal{X}} \|s_j - s\| + \|P_n s - s\| + \|s - s_j\|,$$

and the right side converges to zero in view of the conditions (a)–(c). Hence, $\|P_n s_j - s_j\|_{\mathcal{X}} \rightarrow 0$, which is a contradiction.

(ii) Choose N_0 so large that for all $n \geq N_0$ the inequality

$$\| (I - K)^{-1} (P_n K - K) \| < 1$$

is true, in view of (i). Then, by (5.2.4), $(I - P_n K)^{-1}$ exists for all $n \geq N_0$.

(iii) This follows from part (i) and the identity

$$(I - P_n K)^{-1} - (I - K)^{-1} = [I - (I - K)^{-1} (P_n K - K)]^{-1} \cdot (I - K)^{-1} (P_n K - K) (I - K)^{-1}.$$

(iv) The result is easily verified. ■

Now, we shall reduce the Ritz–Galerkin approximate equation (5.2.2) to an equivalent matrix form. Thus, we choose a basis $\{e_i\}_{i=1}^n$ from M_n , and for every $g \in \mathcal{X}$ we write $P_n g$ uniquely as

$$P_n g = \sum_{i=1}^n e_i^*(g) \cdot e_i, \quad (5.2.5)$$

where $\{e_i^*(g)\}_{i=1}^n$ are real numbers depending linearly and continuously on g . Thus, e_i^* , $i = 1, \dots, n$, may be regarded as bounded linear functionals on \mathcal{X} . We can also rewrite (5.2.5) as

$$P_n = \sum_{i=1}^n e_i e_i^*, \quad (5.2.6)$$

where the product $e_i e_i^*$, $i = 1, \dots, n$, must be taken as an operator in \mathcal{X} in the sense that

$$(e_i e_i^*) g = e_i [e_i^*(g)] = e_i^*(g) \cdot e_i, \quad (5.2.7)$$

where

$$e_i e_j^* = \delta_{ij}, \quad i, j = 1, \dots, n. \quad (5.2.8)$$

Let ϕ_n be a solution of the Ritz–Galerkin approximate equation (5.2.2). Then $\phi_n \in M_n$, and

$$\phi_n = P_n \phi_n = \sum_{i=1}^n c_i e_i. \quad (5.2.9)$$

The coefficients c_i are determined by applying e_i^* to Eq (5.2.2), which yields

$$c_i - \sum_{j=1}^n e_i^* K e_j \cdot c_j = e_i^* f, \quad i = 1, \dots, n, \quad (5.2.10)$$

or, in matrix form,

$$(I - B_n) \begin{Bmatrix} c_1 \\ \vdots \\ c_n \end{Bmatrix} = \begin{Bmatrix} e_1^* f \\ \vdots \\ e_n^* f \end{Bmatrix}, \quad (5.2.11)$$

$$(B_n)_{ij} = e_i^* K e_j. \quad (5.2.12)$$

The matrix B_n is called the *Ritz–Galerkin matrix* and the system (5.2.11) as the *Ritz–Galerkin system*.

Ikebe (1972) proves that if λ is an eigenvalue of B_n and $c = \{c_i\}$ is a corresponding eigenvector, then λ is also an eigenvalue of $P_n K$ and $\sum_{i=1}^n c_i e_i$ is a corresponding eigenvector. Conversely, if λ is a nonzero eigenvalue of $P_n K$ and g a corresponding eigenvector, then $g \in M_n$ and λ is an eigenvalue of B_n with a corresponding eigenvector $[e_1^* g, \dots, e_n^* g]^T$, where T denotes the transpose of the matrix. Also, if $(I - P_n k)^{-1}$ exists, so does $(I - B_n)^{-1}$, and the spectral radius of B_n is at most $\|P_n K\|$, which means that if $\|K\| < 1$ and $\|P_n\| = 1$, then the spectral radius of B_n is less than 1. The system (5.2.11) can be solved by an iterative method, for example, by the point-Jacobi method, which yields

$$c^{(j+1)} = B_n c^{(j)} + (e_i^* f), \quad j = 0, 1, \dots, \quad (5.2.13)$$

where $C^{(0)}$ is an arbitrary n -vector, say $c^{(0)} = 0$. Other iterative methods, like the Gauss–Seidel or the SOR (successive overrelaxation) method (see Varga 1962, Young 1954a,b) may be used. The Ritz–Galerkin system (5.2.11) is asymptotically stable (Ikebe 1972).

5.3. Special Cases

We shall study certain cases of bounded linear projections P_n and determine conditions that justify their use in the Ritz–Galerkin method.

CASE (I). PIECEWISE LINEAR INTERPOLATORY PROJECTIONS IN $C[a, b]$. Consider Eq (5.2.1) in $\mathcal{X} = C[a, b]$, and assume that the kernel $k(x, s)$ is continuous in both x and s , $a \leq x \leq s \leq b$; that $(I - K)^{-1}$ exists in $C[a, b]$, and $f \in C[a, b]$. For any positive integer $n > 1$, partition the interval $[a, b]$ by n nodes

(points) such that $a = x_1 < x_2 < \dots < x_n = b$. For each $g \in C[a, b]$, let $P_n g$ denote the polygonal (piecewise linear) function that agrees with g at each x_i and is linear in each subinterval $[x_i, x_{i+1}]$, $i = 1, \dots, n - 1$. Then

$$P_n = \sum_{i=1}^n e_i \hat{x}_i, \quad P_n g = \sum_{i=1}^n g(x_i) \cdot e_i, \quad (5.3.1)$$

where e_i is the i th cardinal function, also known as the chapeau function, i.e., $e_i(x)$ is the piecewise linear function taking the value 1 at x_i and 0 at all other nodes x_j , and \hat{x}_i is the evaluation functional corresponding to the node x_i (see Fig. 5.3.1).

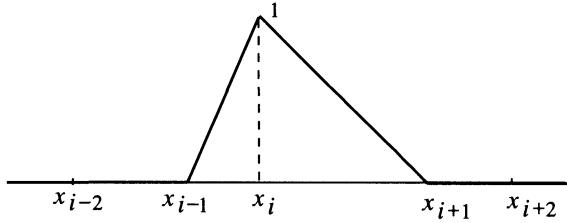


Fig. 5.3.1. Chapeau function $e_i(x)$.

Let

$$\delta_n = \max_{2 \leq i \leq n} |x_{i-1} - x_i| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then, obviously, $\|P_n g - g\| \rightarrow 0$ as $n \rightarrow \infty$. Condition (d) of Theorem 5.2.1 is satisfied, and the Ritz–Galerkin method with the projections P_n in (5.3.1) converges. The Ritz–Galerkin system (5.2.11) in this case becomes

$$(I - B_n) \begin{Bmatrix} c_1 \\ \vdots \\ c_n \end{Bmatrix} = \begin{Bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{Bmatrix}, \quad (5.3.2)$$

where

$$(B_n)_{ij} = \hat{x}_i K e_j = \int_a^b k(x_i, s) e_j(s) ds = \int_a^b k(x_i, s) e_j(s) ds. \quad (5.3.3)$$

The solution of Eq (5.3.2) yields $\{c_i\}$. In view of Theorem 5.2.1, the error in this approximation is given by

$$\phi_n - \phi = (I - P_n K)^{-1} (P_n \phi - \phi), \quad \phi = (I - K)^{-1} f. \quad (5.3.4)$$

Note that in (5.3.4) the term $(P_n\phi - \phi)$ is the interpolation error for ϕ , and this error is bounded by $0.5 \cdot \|\phi''\| \delta_n^2$ if $\phi \in C^2[a, b]$ and by the modulus of continuity $\omega(\phi; \delta_n)$ if $\phi \in C[a, b]$ (see Cheney 1966). This results in slow convergence $\phi_n \rightarrow \phi$.

CASE 2. AVERAGING PROJECTIONS IN $L_\infty[a, b]$. Consider Eq (5.2.1) in $\mathcal{X} = L_\infty[a, b]$ and assume that (a) the integral operator K is completely continuous in $L_\infty[a, b]$, (b) $K\phi \in C[a, b]$ for all $\phi \in L_\infty[a, b]$, (c) $(I - K)^{-1}$ exists in $C[a, b]$, (d) $f \in L_\infty[a, b]$, and (e) $(I - K)^{-1}$ exists in $L_\infty[a, b]$. Conditions (a) and (b) hold if the kernel $k(x, s) \in C[a, b]$. Also, conditions (a) and (c) imply that $\lambda = 1$ is not an eigenvalue of $k(x, s)$. To see this, note that if this were not the case, then $g = Kg$ for some nonzero $g \in L_\infty[a, b]$, and by (b) we would have $Kg \in C[a, b]$; hence, $g \in C[a, b]$. But this is impossible since, by (c), +1 is not an eigenvalue of $k(x, s)$ as a map of $C[a, b]$ onto itself. Now, for any positive integer $n > 1$, choose $(n+1)$ nodes $a = x_0 < x_1 < \dots < x_n = b$, as before. Let $\Delta_i = (x_{i-1}, x_i]$, and $|\Delta_i| = x_i - x_{i-1}$, $i = 1, \dots, n-1$, such that

$$\delta_n = \max_{1 \leq i \leq n} |\Delta_i| \rightarrow 0. \quad (5.3.5)$$

Define piecewise averaging projections P_n of $L_\infty[a, b]$ into itself by

$$P_n g(x) = \frac{1}{|\Delta_i|} \int_{\Delta_i} g(s) ds. \quad (5.3.6)$$

Then $e_i^* e_j = \delta_{ij}$, $i, j = 1, \dots, n$, and P_n can be written as

$$P_n = \sum_{i=1}^n e_i e_i^*. \quad (5.3.7)$$

Note that P_n is a bounded linear projection of $L_\infty[a, b]$ onto M_n spanned by $\{e_1, \dots, e_n\}$. If we assume that $\|P_n f - f\| \rightarrow 0$, then conditions (a)–(c) of Theorem 5.2.1 are satisfied (for a proof, see Ikebe 1972). Thus, the convergence $\phi_n \rightarrow \phi$ holds for the projections P_n defined by (5.3.7). The Ritz–Galerkin system (5.2.11) in this case is

$$(I - B_n) \{c_i\} = \{\bar{f}_i\}, \quad (5.3.8)$$

where

$$(B_n)_{ij} = e_i^* K e_j = \frac{1}{|\Delta_i|} \int_{\Delta_i} \int_{\Delta_i} k(x, s) ds dx, \quad (5.3.9)$$

and

$$\bar{f}_i = \frac{1}{|\Delta_i|} \int_{\Delta_i} f(s) ds. \quad (5.3.10)$$

The approximate solution ϕ_n of the Ritz–Galerkin approximate equation in this case is given by the step function that takes the value c_i at the nodes x_i , where $\{c_i\}$ is the solution of the Ritz–Galerkin system (5.3.8). The error $\phi_n - \phi$ is given by

$$\|\phi_n - \phi\| \leq \| (I - P_n K)^{-1} \| \|P_n \phi - \phi\| \leq (I - P_n K)^{-1} \|\omega(\phi; \delta_n)\|. \quad (5.3.11)$$

CASE (III). PIECEWISE CONSTANT INTERPOLATORY PROJECTION IN $L_\infty[a, b]$. Under the same assumption as in case (ii), choose a point t_i from each Δ_i . For each $g \in L_\infty[a, b]$, assume that $|g(x)| \leq \|g\|$ everywhere on $[a, b]$, and let $P_n g$ denote the step function that takes the value $g(t_i)$ on each Δ_i , $i = 1, \dots, n$. Then, as in case (ii), the operator P_n is a bounded linear projection of $L_\infty[a, b]$ onto M_n spanned by $\{e_1, \dots, e_n\}$, where each e_i denotes the step function defined by

$$e_i(x) = \begin{cases} 1 & \text{if } x \in \Delta_i, \\ 0 & \text{otherwise.} \end{cases} \quad (5.3.12)$$

Thus, $\|P_n g\| \leq \|g\|$ for each $g \in L_\infty[a, b]$, and hence, $\|P_n\| = 1$ for all n . Assume that $\|P_n f - f\| \rightarrow 0$. Then conditions (a)–(c) of Theorem 5.2.1 are satisfied and the Ritz–Galerkin method converges. The Ritz–Galerkin system (5.2.11) in this case becomes

$$(I - B_n) \{c_i\} = \{f(t_i)\}, \quad (5.3.13)$$

where

$$(B_n)_{ij} = \int_{\Delta_i} k(t_j, s) ds. \quad (5.3.14)$$

CASE (IV). FOURIER PROJECTIONS IN $L_{2,\rho}[a, b]$, where $L_{2,\rho}[a, b]$ denotes the real Hilbert space of all real square-integrable functions on $[a, b]$ with respect to a measurable weight function $\rho(x)$ such that (a) $\rho(x) > 0$ everywhere on $[a, b]$, and (b) $\int_a^b \rho(x) dx < \infty$. The inner product in $L_{2,\rho}[a, b]$ is defined by

$$\langle g_1, g_2 \rangle = \int_a^b \rho(x) g_1(x) g_2(x) dx.$$

Consider the FK2 (5.2.1), and assume that (c) $k(x, s) = \sqrt{\tilde{k}(x, s)}$, where

$$\int_a^b \rho(x) \int_a^b |\tilde{k}(x, s)|^2 ds dx < \infty;$$

(d) $(I - K)^{-1}$ exists in $L_{2,\rho}[a, b]$; and (e) $f \in L_{2,\rho}[a, b]$. Then the integral operator K is completely continuous in $L_{2,\rho}[a, b]$ (for a proof, see Ikebe, 1972). Now, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $L_{2,\rho}[a, b]$. We can, for example, take the basis $\{1, x, x^2, \dots\}$, or if $[a, b] = [-1, 1]$ and $\rho(x) = 1$ we take the Legendre polynomials, and if $\rho(x) = (1 - x^2)^{-1/2}$ we obtain the Chebyshev polynomials $T_n(x)$ of the first kind. Define P_n as the n th-order Fourier projection by

$$P_n g = \sum_{i=1}^n \langle e_i, g \rangle e_i, \quad g \in L_{2,\rho}[a, b], \quad (5.3.15)$$

or

$$P_n = \sum_{i=1}^n e_i e_i^*, \quad (5.3.16)$$

where $e_i^* g = \langle e_i, g \rangle$ for $g \in L_{2,\rho}[a, b]$. Thus, $\|P_n g - g\| \rightarrow 0$ because the basis $\{e_1, \dots, e_n\}$ is complete. Since by Bessel's inequality (see Kantorovich and Akilov 1964, p.94)

$$\|P_n g\|^2 = \sum_{i=1}^n |\langle e_i, g \rangle|^2 \leq \|g\|^2,$$

and $\|P_n g\| = \|g\|$ for $g \in M_n$, we have $\|P_n\| = 1$ for all n . Thus, in view of Theorem 5.2.1, the Ritz-Galerkin method in this case converges, and the Ritz-Galerkin system (5.2.11) in this case becomes

$$(I - B_n) \{c_i\} = \{\hat{f}_i\}, \quad (5.3.17)$$

where

$$\begin{aligned} (B_n)_{ij} &= e_i^* K e_j = \langle e_i, K e_j \rangle = \int_a^b \rho(x) e_i(x) \int_a^b k(x, s) ds, \\ \hat{f}_i &= e_i^* f = \int_a^b \rho(x) e_i(x) f(x) dx. \end{aligned} \quad (5.3.18)$$

The error estimate in this case is given by

$$\begin{aligned} \|\phi_n - \phi\| &= \|(I - P_n K)^{-1} (P_n \phi - \phi)\| \\ &\leq \|(I - P_n K)^{-1}\| \cdot \sqrt{\sum_{i=n+1}^{\infty} |\langle e_i, \phi \rangle|^2}. \end{aligned} \quad (5.3.19)$$

CASE (V). PIECEWISE POLYNOMIAL INTERPOLATORY PROJECTIONS. These projections can be used if the solution ϕ of Eq (5.2.1) is sufficiently small. Otherwise the use of such high-order interpolation may yield poor approximations ϕ_n . Among interpolations in this case, spline interpolations can be used, and, in general, $\|P_n\| > 1$ if P_n is a nonlinear piecewise polynomial interpolation. Schurer and Cheney (1968) show that $\|P_n\| < (1 + 3\sqrt{3})/4 \approx 1.548$ for all n in the case of equally-spaced nodes for periodic cubic spline operators in $C[0, 1]$.

5.4. Fredholm–Nyström System

Consider an FK2 in $C[a, b]$, where the kernel $k(x, s)$ and the free term $f(x)$ are continuous, and $\phi \in C[a, b]$. Choose n nodes as the points $x_1 < x_2 < \dots, x_n$ in $[a, b]$ and n functions e_1, \dots, e_n from the class $C[a, b]$ such that

$$e_i(x_j) = \delta_{ij}, \quad i, j = 1, \dots, n. \quad (5.4.1)$$

Let P_n be an interpolation projection (operator) defined by

$$\begin{aligned} P_n &= \sum_{i=1}^n e_i \hat{x}_i, \\ \hat{x}_i &= g(x_i), \quad g \in C[a, b], \quad i = 1, \dots, n. \end{aligned} \quad (5.4.2)$$

This means that for every $g \in C[a, b]$, $P_n g(x) = \sum_{i=1}^n g(x_i) e_i(x)$ represents a unique linear combination of e_1, \dots, e_n that matches with the values of g at the nodes x_1, \dots, x_n , respectively. The interpolatory quadrature rule is then defined by

$$\int_a^b g(x) dx \approx \int_a^b P_n g(x) dx = \sum_{i=1}^n w_i g(x_i), \quad g \in C[a, b], \quad (5.4.3)$$

where the weights w_i are given by

$$w_i = \int_a^b e_i(x) dx, \quad i = 1, \dots, n. \quad (5.4.4)$$

If this quadrature rule is applied to the FK2 (5.2.1), we obtain

$$\phi(x) - \sum_{i=1}^n w_i k(x, s_j) g(s_j) \approx f(x). \quad (5.4.5)$$

This approximation is similar to that obtained by the Nyström method (see Section 1.6, Eq (1.6.4)). If we set $x = x_i$, $i = 1, \dots, n$, in (5.4.5), we obtain the linear system

$$\tilde{c}_i - \sum_{i=1}^n w_i k(x-i, s_j) \tilde{c}_j = f(x_i), \quad i = 1, \dots, n, \quad (5.4.6)$$

or, in matrix form,

$$(I - \tilde{B}_n) \{\tilde{c}_i\} = \{\tilde{f}(x_i)\}, \quad (5.4.7)$$

where \tilde{c}_i is an approximation of $\phi(x_i)$ for $i = 1, \dots, n$, and

$$(B_n)_{ij} = w_i k(x_i, s_j) = \int_a^b e_j(s) ds \cdot k(x_i, s_j). \quad (5.4.8)$$

The system (5.4.7) is known as the Fredholm–Nyström system.

REFERENCES USED: Baker (1978), Cheney (1966), Collatz (1966), Delves and Walsh (1974), Ikebe (1972), Kantorovich and Akilov (1964), Kythe (1995, 1996), Schurer and Cheney (1968), Varga (1962), Young (1954a,b).

6

Iteration Methods

In this chapter we discuss some iteration methods that arise quite naturally. For smaller systems the iterative methods are sometimes faster than the Gaussian elimination method. However, the iterative methods get very complicated to program, and the Gaussian method is still preferred. It seems that the Nyström method with product integration (Section 4.2) is a better numerical method. Iterative methods for singular equations are presented in Section 7.8 and those for the FK1/VK1 in Section 11.7.

6.1. Simple Iterations

Consider an FK2:

$$\phi(x) - \lambda \int_a^b k(x, s) \phi(s) ds = f(x). \quad (6.1.1)$$

The method of simple iterations that leads to the solution of an FK2 through the resolvent of its kernel is as follows: Assume that the solution of Eq (6.1.1) is in the form of a series in powers of λ :

$$\phi(x) = \phi_0(x) + \lambda \phi_1(x) + \lambda^2 \phi_2(x) + \dots. \quad (6.1.2)$$

Then substituting (6.1.2) into (6.1.1) yields

$$\begin{aligned} \phi_0(x) + \lambda \phi_1(x) + \lambda^2 \phi_2(x) + \dots &= f(x) \\ + \lambda \int_a^b k(x, s) [\phi_0(s) + \lambda \phi_1(s) + \lambda^2 \phi_2(s) + \dots] ds. \end{aligned}$$

By equating the coefficients of similar powers of λ on both sides, we get

$$\begin{aligned}\phi_0(x) &= f(x), \\ \phi_1(x) &= \int_a^b k(x, s)\phi_0(s) ds, \\ \phi_2(x) &= \int_a^b k(x, s)\phi_1(s) ds, \\ &\dots \quad \dots \\ \phi_{n+1}(x) &= \int_a^b k(x, s)\phi_n(s) ds.\end{aligned}\tag{6.1.3}$$

From Eqs (6.1.3) we can successively determine all the functions $\phi_1(x), \phi_2(x), \dots$. In fact, if we introduce the iterated kernels

$$\begin{aligned}k_1(x, s) &= k(x, s), \\ k_2(x, s) &= \int_a^b k(x, t)k_1(t, s) dt, \\ k_3(x, s) &= \int_a^b k(x, t)k_2(t, s) dt, \\ &\dots \quad \dots \\ k_{n+1}(x, s) &= \int_a^b k(x, t)k_n(t, s) ds,\end{aligned}\tag{6.1.4}$$

then

$$\begin{aligned}\phi_0(x) &= f(x), \\ \phi_n(x) &= \int_a^b k_n(x, s)f(s) ds, \quad n = 1, 2, \dots\end{aligned}\tag{6.1.5}$$

Now we can write the series (6.1.2) as

$$\begin{aligned}\phi(x) &= f(x) + \lambda \int_a^b k_1(x, s)f(s) ds + \lambda^2 \int_a^b k_2(x, s)f(s) ds + \dots \\ &= f(x) + \lambda \int_a^b [k_1(x, s) + \lambda k_2(x, s) + \dots] f(s) ds \\ &= f(x) + \lambda \int_a^b k_\lambda f(s) ds,\end{aligned}\tag{6.1.6}$$

where k_λ which denotes the series in the square brackets is known as the *resolvent* of Eq. (6.1.1) (see Section 1.9):

$$k_\lambda = \sum_{j=1}^{\infty} \lambda^{j-1} k_j(x, s).\tag{6.1.7}$$

Assuming that the kernel $k(x, s)$ is bounded, i.e., $|k(x, s)| \leq M$, it can be proved that the series (6.1.2), (6.1.6), and (6.1.7) are all uniformly convergent, and hence, $\phi(x)$ will be the solution of Eq (6.1.1) provided that λ is sufficiently small; more precisely, λ must satisfy the condition

$$|\lambda| < \frac{1}{M(b-a)}. \quad (6.1.8)$$

In fact, the estimate (6.1.8) holds if M is not an upper bound of $k(x, s)$ but is any number such that any one of the following inequalities are satisfied (see Kantorovich and Krylov 1958, footnote on p.112):

$$\begin{aligned} \int_a^b |k(x, s)| ds &< M(b-a), \\ \sqrt{\int_a^b \int_a^b k^2(x, s) dx ds} &\leq M(b-a), \\ |k_n(x, s)| &< M^n(b-a)^{n-1}. \end{aligned} \quad (6.1.9)$$

If λ satisfies the inequality (6.1.8), then the series (6.1.2) can be used to determine an approximate solution. If the kernel has a weak singularity, i.e., if $k(x, s) = \frac{A(x, s)}{|x-s|^\alpha}$, $|A(x, s)| \leq M = \text{const}$, $0 < \alpha < 1$, then the sequence $\{\phi_n(x)\}$ converges if

$$|\lambda| < \frac{1-\alpha}{2^\alpha M(b-a)^{1-\alpha}}. \quad (6.1.10)$$

In many cases when the integrals in (6.1.3) can be computed exactly, the series (6.1.2) will converge to the exact solution $\phi(x)$ at least as a geometrical progression with ratio $|\lambda|M(b-a)$. The error in truncating the series (6.1.2) after n terms can be evaluated as follows: Assuming that $|f(x)| \leq C$, we obtain in succession

$$|\phi_n(x)| < CM^n(b-a)^n, \quad (6.1.11)$$

and the remainder $R_n(x)$ of the series (6.1.2) after n terms is

$$\begin{aligned} R_n(x) &= |\lambda^n \phi_n(x) + \lambda^{n+1} \phi_{n+1}(x) + \dots| \\ &< |\lambda|^n CM^n(b-a)^n + |\lambda|^{n+1} CM^{n+1}(b-a)^{n+1} + \dots \\ &= \frac{|\lambda|^n CM^n(b-a)^n}{1 - |\lambda|CM(b-a)}. \end{aligned} \quad (6.1.12)$$

In practice, however, the integrals in (6.1.3) cannot be computed exactly in most cases. In such cases we use the technique of approximate integration. Suppose we choose a certain quadrature formula of the form

$$\int_a^b F(x) dx = \sum_{j=1}^n w_j F(x_j) + \varepsilon, \quad (6.1.13)$$

where w_j and x_j are real numbers that are constant for the given interval of integration depending on the quadrature formula used, and ε is the error term, and introduce the notation:

$$k_{ij} = k(x_i, x_j), \quad \phi_n^{(i)} = \phi_n(x_i), \quad \phi^{(i)} = \phi(x_i), \quad f^{(i)} = f(x_i),$$

then denoting the approximate values of ϕ by $\tilde{\phi}$ and ignoring the error term $\varepsilon(x)$ for now, we obtain from (6.1.3)

$$\begin{aligned} \phi_0^{(i)} &= f^{(i)}, \\ \phi_1^{(i)} &= \int_a^b k(x_i, s) \phi_0(s) ds \approx \sum_{j=1}^m w_j k_{ij} \phi_0^{(j)}, \\ \tilde{\phi}_1^{(i)} &= \sum_{j=1}^m w_j k_{ij} \phi_0^{(j)}, \\ \tilde{\phi}_{n+1}^{(i)} &= \sum_{j=1}^m w_j k_{ij} \tilde{\phi}_n^{(j)}. \end{aligned} \quad (6.1.14)$$

The computations are carried out for $n = 1, 2, \dots$ until the computed values become so small that they must be ignored. Thus, we obtain the approximate values

$$\tilde{\phi}^{(i)} = \tilde{\phi}_0^{(i)} + \lambda \tilde{\phi}_1^{(i)} + \lambda^2 \tilde{\phi}_2^{(i)} + \dots \quad (6.1.15)$$

That this series converges uniformly under the condition (6.1.8) can be verified by recalling that $|\phi_0^{(i)}| = |f| \leq C$, and by showing that $|\tilde{\phi}_1^{(i)}| \leq |\lambda| CM(b-a)$ (see Kantorovich and Krylov 1958, p.114). The error in the approximate solution $\tilde{\phi}$ can be evaluated by using the computed quantities $\tilde{\phi}^{(i)}$ in (6.1.5); thus,

$$\tilde{\phi}(x) = f(x) + \lambda \sum_{j=1}^m w_j k(x, x_j) \tilde{\phi}^{(j)}. \quad (6.1.16)$$

A computational scheme to determine the error is discussed in Section 6.3.

In the next section we discuss in detail the application of quadrature formulas in evaluating the approximate solution.

6.2. Quadrature Formulas

If we replace the integral in Eq (6.1.1) by a quadrature formula of the form (6.1.13) where $w_j \geq 0$ and $\sum_{j=1}^n w_j = b - a$, then Eq (6.1.1) becomes

$$\phi(x) - \lambda \sum_{j=1}^n w_j k(x, x_j) \phi(x_j) = f(x) + \lambda \varepsilon_j. \quad (6.2.1)$$

If we set $x = x_i$, $i = 1, 2, \dots, n$, and neglect $\varepsilon_j = \varepsilon(x_j)$, we obtain the system of algebraic equations for the unknown approximate values of $\tilde{\phi}(x_i)$ as

$$\tilde{\phi}(x_i) - \lambda \sum_{j=1}^n w_j k(x_i, x_j) \tilde{\phi}(x_j) = f(x_i), \quad i = 1, 2, \dots, n, \quad (6.2.2)$$

or

$$\begin{aligned} & \tilde{\phi}(x_1) [1 - \lambda w_1 k(x_1, x_1)] - \lambda \tilde{\phi}(x_2) w_2 k(x_1, x_2) - \dots \\ & \quad - \lambda \tilde{\phi}(x_n) w_n k(x_1, x_n) = f(x_1), \\ & - \lambda \tilde{\phi}(x_1) w_1 k(x_1, x_1) + \tilde{\phi}(x_2) [1 - \lambda w_2 k(x_2, x_2)] - \dots \\ & \quad - \lambda \tilde{\phi}(x_n) w_n k(x_2, x_n) = f(x_2), \\ & \vdots \\ & - \lambda \tilde{\phi}(x_1) w_1 k(x_n, x_1) - \lambda \tilde{\phi}(x_2) w_2 k(x_n, x_2) - \dots \\ & \quad + \tilde{\phi}(x_n) [1 - \lambda w_n k(x_n, x_n)] = f(x_n). \end{aligned} \quad (6.2.3)$$

Solve this system and substitute the values of $\tilde{\phi}(x_1), \dots, \tilde{\phi}(x_n)$ into Eq (6.2.1). This will yield the required approximate solution of Eq (6.1.1). Any one of the quadrature formulas, mentioned in Sections A.1 and A.2, can be used for this method.

EXAMPLE 6.2.1. (Mikhlin and Smolitskiy 1967, p.286) The plane interior Dirichlet problem for the region lying inside a sufficiently small contour C is

governed by the equation

$$\phi(t) - \frac{1}{\pi} \int_C \frac{\cos(\nu, r)}{r} \phi(\tau) d\sigma = f(t), \quad (6.2.4)$$

where t and τ are the parameters that determine the position of points on the contour C ; r the distance between points corresponding to values of these parameters; ν the exterior normal to C at τ ; $d\sigma$ the arc length element; $\phi(t)$ the unknown function; and $f(t)$ the prescribed function. Eq (6.2.4) is obtained if a harmonic function is sought in the form of a potential of a double layer with density $\phi(\tau)$ (see Kythe 1996, p.21). Let C be defined as the ellipse $x = a \cos t$, $y = b \sin t$, $a = 5$, $b = 3$, and let $f(t) = x^2 + y^2 = 25 - 16 \sin^2 t$. Then

$$\begin{aligned} \int_C \frac{\cos(\nu, r)}{r} \phi(\tau) d\sigma &= -\frac{b}{2a} \int_{-\pi}^{\pi} \frac{\phi(\tau)}{1 - e^2 \cos^2 \frac{t+\tau}{2}} d\tau \\ &= -\frac{3}{10\pi} \int_{-\pi}^{\pi} \frac{\phi(\tau)}{0.68 - 0.32 \cos(t+\tau)} d\tau, \end{aligned}$$

where $e = 4/5$ is the eccentricity of the ellipse. Thus, Eq (6.2.4) reduces to

$$\phi(\tau) + \frac{3}{10\pi} \int_{-\pi}^{\pi} \frac{\phi(\tau)}{0.68 - 0.32 \cos(t+\tau)} d\tau = 25 - 16 \sin^2 t. \quad (6.2.5)$$

Note that the unknown function $\phi(t)$ is periodic of period π , i.e.,

$$\phi(-t) = \phi(\pi - t) = \phi(t).$$

For quadrature we shall use the trapezoidal rule with 12 intervals of equal length $\pi/6$ from $-\pi$ to π (see Fig.6.2.1).

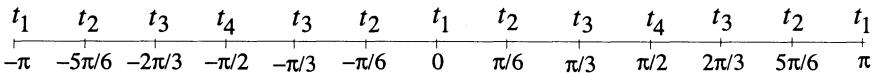


Fig. 6.2.1.

Because of the periodic nature of $\phi(t)$, we adopt the notation: $\phi(0) = y_1$, $\phi(\pi/6) = y_2$, $\phi(\pi/3) = y_3$, $\phi(\pi/2) = y_4$. This leads to a system of four equations:

$$\begin{aligned} 1.18889 y_1 + 0.352697 y_2 + 0.311355 y_3 + 0.147059 y_4 &= 25, \\ 0.176349 y_1 + 1.34457 y_2 + 0.323407 y_3 + 0.153678 y_4 &= 21, \\ 0.155678 y_1 + 0.323407 y_2 + 1.34457 y_3 + 0.176349 y_4 &= 13, \\ 0.147059 y_1 + 0.311355 y_2 + 0.352697 y_3 + 1.18889 y_4 &= 9, \end{aligned}$$

which yields

$$y_1 = 16.0294, y_2 = 12.2647, y_3 = 14.7353, y_4 = 0.970595. \quad (6.2.6)$$

The function $\phi(t)$ can now be constructed by expanding it as a Fourier cosine series which in view of its periodicity can be written as

$$\phi(t) = \sum_{n=0}^{\infty} a_n \cos 2nt. \quad (6.2.7)$$

Taking the first four terms of this series and using the values of y_1, y_2, y_3, y_4 from (6.2.6), we find that

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= y_1, \\ a_0 + \frac{a_1}{2} - \frac{a_1}{2} - a_3 &= y_2, \\ a_0 - \frac{a_1}{2} - \frac{a_1}{2} + a_3 &= y_3, \\ a_0 - a_1 + a_2 - a_3 &= y_4, \end{aligned}$$

which, with the above values of y_1, \dots, y_4 , yield $a_0 = 8.5, a_1 = 7.5294, a_2 = 0, a_3 = 1.18424 \times 10^{-15}$. Ignoring the value of a_3 , which is very small, we obtain

$$\phi(t) \approx 8.5 + 7.5294 \cos 2t. \quad (6.2.8)$$

This matches with the exact solution (see Mikhlin and Smolitsky 1967, p.287)

$$\phi(t) = \frac{17}{2} + \frac{128}{17} \cos 2t = 8.5 + 7.5294117 \cos 2t.$$

For computational details with the trapezoidal rule, see `dirichlet1.nb`. The results obtained from using other quadrature rules are presented in the following table.

Table 6.2.1

Rule	a_0	a_1	a_2	a_3
Exact solution	8.5	7.5294117	0.0	0.0
Trapezoidal ($n = 12$)	8.5	7.529	0.0	$1.18424(-15)$
Rectangular ($n = 12$)	8.5	7.5294	0.0	$1.18424(-15)$
Simpson's ($n = 12$)	8.29185	8.68164	-1.23377	0.62602
Gauss-Legendre ($n = 12$)	8.573	7.10315	-0.001688	-0.0662033
Weddle ($n = 12$)	8.50009	7.48573	0.0108996	0.0021237

Note that the Gauss–Legendre formula shows poor results simply because we have approximated the values of $\phi(t)$ in the integrand by those of $\phi(t_i)$ (which are y_1, y_2, y_3 , and y_4 , respectively); this situation is avoided in the symmetric problem solved in Example 6.5.1. For computational details, see `dirichlet2.nb`, `dirichlet3.nb`, `dirichlet4.nb`, `dirichlet5.nb` and `dirichlet6.nb`. ■

6.3. Error Analysis

Instead of the system (6.2.2), we consider the corresponding homogeneous system (with $f(x_i) = 0, i = 1, 2, \dots, n$), i.e.,

$$\tilde{\phi}(x_i) - \lambda \sum_{j=1}^n w_j k(x_i, x_j) \phi(x_j) = 0, \quad i = 1, 2, \dots, n. \quad (6.3.1)$$

The determinant of this system is

$$D(\lambda) = \begin{vmatrix} 1 - \lambda w_1 k(x_1, x_1) & -\lambda w_2 k(x_1, x_2) & \dots & -\lambda w_n k(x_1, x_n) \\ -\lambda w_1 k(x_2, x_1) & 1 - \lambda w_2 k(x_2, x_2) & \dots & -\lambda w_n k(x_2, x_n) \\ \dots & \dots & \dots & \dots \\ -\lambda w_1 k(x_n, x_1) & -\lambda w_2 k(x_n, x_2) & \dots & 1 - \lambda w_n k(x_n, x_n) \end{vmatrix}. \quad (6.3.2)$$

We solve the equation $D(\lambda) = 0$ and determine those values of λ for which the system (6.3.1) has a solution $\tilde{\phi}(x_i) \neq 0$. Note that these values of λ approximate eigenvalues, which can be used to find approximate eigenfunctions from (6.3.1). However, other methods of evaluating eigenpairs are discussed in Chapter 2.

To find the error, let D_{ij} denote the cofactor of the element in the i th row and j th column in the determinant (6.3.2). Then the approximate solution of the system (6.2.2) is given by

$$\tilde{\phi}(x_i) = \frac{\sum_{j=1}^n D_{ij} f(x_j)}{D(\lambda)}, \quad i = 1, 2, \dots, n. \quad (6.3.3)$$

We shall estimate how much these approximate values $\tilde{\phi}(x_i)$ differ from the exact values $\phi(x_i), i = 1, \dots, n$. Assuming that the functions $k(x, s)$ and $f(x)$ have a certain finite number p of continuous derivatives, i.e., $k \in C^p(a, b)$ and $f \in$

$C^p(a, b)$, the function ϕ shall also belong to the class $C^p(a, b)$. Let

$$\begin{aligned} H^{(0)} &= \text{ub}_{a \leq x \leq b} |\phi(x)|, & H^{(q)} &= \text{ub}_{a \leq x \leq b} |\phi^{(q)}(x)|, \\ N^{(0)} &= \text{ub}_{a \leq x \leq b} |f(x)|, & N^{(q)} &= \text{ub}_{a \leq x \leq b} |f^{(q)}(x)|, \\ M_x^{(q)} &= \text{ub}_{a \leq s \leq x \leq b} \left| \frac{d^q}{dx^q} k(x, s) \right|, & M_y^{(q)} &= \text{ub}_{a \leq s \leq x \leq b} \left| \frac{d^q}{ds^q} k(x, s) \right|, \\ M^{(0)} &= \text{ub}_{a \leq s \leq x \leq b} |k(x, s)|, & q &= 1, 2, \dots, p. \end{aligned} \quad (6.3.4)$$

Then, by using Leibniz' rule, the derivatives of the integrand in Eq (6.1.1) are evaluated as

$$\begin{aligned} \left| \frac{d^q}{dx^q} [k(x, s)\phi(s)] \right| &\leq H^{(0)} M_s^{(q)} c_q^1 H^{(1)} M_s^{(q-1)} + \dots + c_q^{(q-1)} H^{(q-1)} M_s^{(1)} \\ &\quad + H^{(q)} M^{(0)} \\ &\equiv T^{(q)} \quad (\text{say}). \end{aligned} \quad (6.3.5)$$

If E denotes the maximum modulus of the error $\varepsilon(x)$ in (6.1.13) when we replace the integral $\int_a^b k(x, s)\phi(s) ds$ by the sum, i.e., when we take

$$\int_a^b k(x, s)\phi(s) ds - \sum_{j=1}^n w_j k(x, x_j) \phi(x_j) = \varepsilon(x), \quad (6.3.6)$$

so that $|\varepsilon(x)| \leq E$ and $|\varepsilon_j| = |\varepsilon(x_j)| \leq E$, then E can be easily evaluated from (6.3.5) in the form $E \leq d_n T^{(q)}$, where d_n is a factor of the formula; thus, for example, for the trapezoidal rule:

$$E \leq \frac{(b-a)^3}{12(n-1)^2} T^{(2)}; \quad (6.3.7a)$$

and for the Gauss-Legendre rule:

$$E \leq \frac{(b-a)^{2n+1}}{2n+1} \left[\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1) \cdots (2n)} \right]^2 \frac{T^{(2n)}}{1 \cdot 2 \cdot 3 \cdots (2n)}. \quad (6.3.7b)$$

In view of (6.2.1), the values of the exact solution $\phi(x)$ at the points x_1, x_2, \dots, x_n satisfy the system of equations

$$\phi(x_i) - \lambda \sum_{j=1}^n w_j k(x_i, x_j) \phi(x_j) = f(x_i) + \lambda \varepsilon_j. \quad (6.3.8)$$

Then in (6.3.3) we use $f(x_j) + \lambda \varepsilon_j$ instead of $f(x_j)$ and get

$$\phi(x_i) = \frac{\sum_{j=1}^n D_{ij} [f(x_j) + \lambda \varepsilon_j]}{D(\lambda)}, \quad i = 1, 2, \dots, n. \quad (6.3.9)$$

If we compare (6.3.3) and (6.3.9), we find that

$$|\phi(x_i) - \tilde{\phi}(x_i)| = \left| \frac{\sum_{j=1}^n D_{ij} \lambda \varepsilon_j}{D(\lambda)} \right| \leq E |\lambda| \frac{\sum_{j=1}^n |D_{ij}|}{D(\lambda)} \leq E |\lambda| B, \quad (6.3.10)$$

where B is the upper bound of the ratio $\frac{\sum_{j=1}^n |D_{ij}|}{D(\lambda)}$, i.e.,

$$\frac{\sum_{j=1}^n |D_{ij}|}{D(\lambda)} \leq B. \quad (6.3.11)$$

Since the approximate value of $\phi(x)$ is $\tilde{\phi}(x)$, we have

$$\tilde{\phi}(x) = f(x) + \lambda \sum_{j=1}^n w_j k(x, x_j) \tilde{\phi}(x_j). \quad (6.3.12)$$

If we subtract (6.3.12) from (6.2.1), we get

$$\begin{aligned} |\phi(x) - \tilde{\phi}(x)| &\leq |\lambda| \sum_{j=1}^n w_j |k(x, x_j)| \cdot |\phi(x_j) - \tilde{\phi}(x_j)| + E |\lambda| \\ &\leq E |\lambda| [1 + |\lambda| B M^{(0)} (b - a)], \end{aligned} \quad (6.3.13)$$

because $\sum_{j=1}^n w_j = b - a$. The quantity E in the bound (6.3.13) is known only in terms of $T^{(q)}$, which involves all the quantities defined in (6.3.4). However, to avoid this situation, first we differentiate Eq (6.3.1) q -times:

$$\phi^{(q)}(x) = \lambda \int_a^b \left[\frac{d^q}{dx^q} k(x, s) \right] \phi(s) ds + f^{(q)}(x).$$

This, in view of (6.3.4), implies that

$$H^{(q)} \leq H^{(0)} |\lambda| (b - a) M_x^{(q)} + N^{(q)}, \quad q = 0, 1, 2, \dots . \quad (6.3.14)$$

Thus, from (6.3.5), and using (6.3.14), we get

$$\begin{aligned} T^{(q)} &= H^{(0)} M_s^{(q)} + c_q^1 H^{(1)} M_s^{(q-1)} + \dots + H^{(q)} M^{(0)} \\ &\leq N^{(0)} M_s^{(q)} + c_q^1 N^{(1)} M_s^{(q-1)} + \dots + N^{(q)} M^{(0)} \\ &\quad + |\lambda| (b - a) \left[M^{(0)} M_s^{(q)} + c_q^1 M_x^{(1)} M_s^{(q-1)} + \dots + M_x^{(q)} M^{(0)} \right] H^{(0)} \\ &\equiv P_q + Q_q H^{(0)}, \end{aligned} \quad (6.3.15)$$

where P_q and Q_q are known. As noted above, the estimate of the error E is then of the form

$$E \leq d_n T^{(q)} \leq d_n (P_q + Q_q H^{(0)}). \quad (6.3.16)$$

Hence, using (6.3.16), the bounds in (6.3.13) become

$$|\phi(x) - \tilde{\phi}(x)| \leq \left[1 + |\lambda| B M^{(0)} (b - a) \right] |\lambda| d_n (P_q + Q_q H^{(0)}). \quad (6.3.17)$$

Let S denote the upper bound of $|\tilde{\phi}(x)|$. Then, from (6.3.17)

$$|\phi(x)| \leq S + \left[1 + |\lambda| B M^{(0)} (b - a) \right] |\lambda| d_n (P_q + Q_q H^{(0)}),$$

which, in view of (6.3.4), implies that

$$H^{(0)} \leq S + \left[1 + |\lambda| B M^{(0)} (b - a) \right] |\lambda| d_n (P_q + Q_q H^{(0)}),$$

or, collecting terms in $H^{(0)}$, we find an estimate for $H^{(0)}$ as

$$H^{(0)} \leq \frac{S + \left[1 + |\lambda| B M^{(0)} (b - a) \right] |\lambda| d_n (P_q + Q_q H^{(0)})}{1 - \left[1 + |\lambda| B M^{(0)} (b - a) \right] |\lambda| d_n (P_q + Q_q H^{(0)})}, \quad (6.3.18)$$

provided that the denominator on the right side is positive. This is, in fact, the case, since d_n is very small, the other quantities B , $M^{(0)}$, P_q , Q_q , and $|\lambda|$ are also small. In the case when the denominator is positive, the solution $\phi(x)$ will be bounded, which implies that the given value of λ is not an eigenvalue, because, otherwise we could add to $\phi(x)$ an arbitrary solution of the homogeneous equation, which could be taken arbitrarily large, and then the bound in (6.3.18) would not hold.

If we substitute the estimate of $H^{(0)}$ from (6.3.18) into (6.3.17), we will finally obtain an estimate of $|\phi(x) - \tilde{\phi}(x)|$ that involves only known quantities.

We remark in passing that the quantity S can be evaluated directly from (6.3.12). However, an easier method to evaluate S is to note that (6.3.3) and (6.3.11) yield

$$|\tilde{\phi}(x_j)| \leq B N^{(0)}.$$

Also, in view of (6.3.12),

$$\begin{aligned} |\tilde{\phi}(x_j)| &\leq |f(x)| + |\lambda| \sum_{j=1}^n w_j |k(x, x_j)| |\tilde{\phi}(x_j)| \\ &\leq N^{(0)} + |\lambda| BM^{(0)} (b-a) N^{(0)}, \end{aligned} \quad (6.3.19)$$

which can be used for S in the estimate (6.3.18).

EXAMPLE 6.3.1. (Kantorovich and Krylov 1958, pp.108–111) Consider the equation

$$\phi(x) - \frac{1}{2} \int_0^1 e^{xs} \phi(s) ds = 1 - \frac{e^x - 1}{2x}. \quad (6.3.20)$$

We use the Gauss–Legendre quadrature for the interval $(0, 1)$. The system (6.2.3) with $n = 2$ is

$$\begin{aligned} [1 - \lambda w_1 k(x_1, x_1)] \tilde{\phi}(x_1) - \lambda w_2 k(x_1, x_2) \tilde{\phi}(x_2) &= f(x_1), \\ -\lambda w_1 k(x_1, x_1) \tilde{\phi}(x_1) + [1 - \lambda w_2 k(x_2, x_2)] \tilde{\phi}(x_2) &= f(x_2). \end{aligned}$$

Note that $\lambda = 0.5$, $w_1 = w_2 = 0.5$, and $x_1 = 0.2113248654$, $x_2 = 0.7886751346$ (see Abramowitz and Stegun 1968, p.921). Then the determinant of this system is

$$D(\lambda) = \begin{vmatrix} 0.738582 & -0.29534 \\ -0.29534 & 0.544334 \end{vmatrix} = 0.307424.$$

The determinants D_1 and D_2 are

$$\begin{aligned} D_1 &= \begin{vmatrix} 0.443242 & -0.29534 \\ 0.238927 & 0.544334 \end{vmatrix} = 0.307404, \\ D_2 &= \begin{vmatrix} 0.738582 & 0.443242 \\ -0.29534 & 0.238927 \end{vmatrix} = 0.307374. \end{aligned}$$

Thus, $\tilde{\phi}(x_1) = D_1/D(\lambda) = 0.999935$, $\tilde{\phi}(x_2) = D_2/D(\lambda) = 0.999839$, and the approximate solution of Eq (6.3.20) is

$$\tilde{\phi}(x) = 1 + 0.25 \left[0.999935 e^{x_1 x} + 0.999839 e^{x_2 x} \right] - \frac{e^x - 1}{2x}.$$

The exact solution is $\phi(x) = 1$. Comparing these solutions at the points 0, x_1 , x_2 , and 1, we find that the difference $|\phi(x) - \tilde{\phi}(x)|$ is 0.000056, 0.000065, 0.000161 and 0.000301, respectively, which imply that the errors are of the order $O(10^{-3})$.

We now compute a precise estimate of the error E . The quantity B has the bound $(0.738582 + 0.29534)/2 = 3.36318$; so we may take $B = 3.4$. The free term $f(x)$ and its derivatives, expanded in series, are

$$\begin{aligned} f(x) &= \frac{1}{2} - \frac{x}{4} - \frac{x^2}{12} - \frac{x^3}{48} - \frac{x^4}{240} - \frac{x^5}{1440} - \frac{x^6}{10080} - \dots, \\ f'(x) &= -\left(\frac{1}{4} + \frac{x}{6} + \frac{x^2}{16} + \frac{x^3}{60} + \frac{x^4}{288} + \frac{x^5}{1680} + \dots\right), \\ f''(x) &= -\left(\frac{1}{6} + \frac{x}{8} + \frac{x^2}{20} + \frac{x^3}{72} + \frac{x^4}{336} + \dots\right), \\ f'''(x) &= -\left(\frac{1}{8} + \frac{x}{10} + \frac{x^2}{24} + \frac{x^3}{84} + \dots\right), \\ f^{(4)}(x) &= -\left(\frac{1}{10} + \frac{x}{12} + \frac{x^2}{28} + \dots\right). \end{aligned}$$

Then, considering the bounds for $f, f', f'', f''', f^{(4)}$ in the interval $(0, 1)$, we may take

$$N^{(0)} = 0.5 = N^{(1)}, \quad N^{(2)} = 0.36, \quad N^{(3)} = 0.28, \quad N^{(4)} = 0.22.$$

The kernel and its derivatives are

$$k(x, s) = e^{xs}, \quad k_x^{(i)} = s^i e^{xs}, \quad k_s^{(i)} = x^i e^{xs}, \quad i = 1, 2, 3, 4.$$

Then for their bounds in the interval $(0, 1)$ we may take

$$M^{(0)} = M_x^{(1)} = \dots = M_x^{(4)} = M_s^{(1)} = \dots = M_s^{(4)} = e.$$

Hence,

$$\begin{aligned} P_4 &= N^{(0)} M_s^{(4)} + c_4^1 N^{(1)} M_s^{(3)} + c_4^2 N^{(2)} M_s^{(2)} + c_4^3 N^{(3)} M_s^{(1)} \\ &\quad + N^{(4)} M^{(0)} < 8e < 22, \\ Q_4 &= |\lambda| (b-a) \left[M^{(0)} M_s^{(4)} + c_4^1 M_x^{(1)} M_s^{(3)} + c_4^2 M_x^{(2)} M_s^{(2)} \right. \\ &\quad \left. + c_4^3 M_x^{(3)} M_s^{(1)} + M_x^{(4)} M^{(0)} \right] = 8e^2 < 60. \end{aligned}$$

Then $T^{(4)} = P_4 + Q_4 H^{(0)}$, where $H^{(0)}$ denotes the upper bound of $\phi(s)$ on the interval $[0, 1]$. Next, from (6.3.7b)

$$E = d_2 T^{(4)} = \frac{1}{5} \left[\frac{1 \cdot 2}{3 \cdot 4} \right]^2 \frac{T^{(4)}}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{1}{4320} T^{(4)},$$

thus, $d_2 = \frac{1}{4320}$. From (6.3.21) we compute the upper bound S of $|\tilde{\phi}(x)|$ as

$$\begin{aligned} S &= 0.25 (e^{x_1} + e^{x_2}) + \lim_{x \rightarrow 0} \left(1 - \frac{e^x - 1}{2x} \right) \\ &= 0.85936 + 0.5 = 1.35936, \end{aligned}$$

so we may take $S = 1.4$. Then substituting these values in (6.3.18), we find that

$$H^{(0)} \leq \frac{1.4 + [e(3.4)(1)(0.5) + 1] \cdot (0.5)(1/4320)(22)}{1 - [e(3.4)(1)(0.5) + 1] \cdot (0.5)(1/4320)(60)} = 1.46911.$$

Hence, we get the difference

$$\begin{aligned} |\phi(x) - \tilde{\phi}(x)| &= |\lambda| [1 + |\lambda| BM^{(0)}(b-a)] d_2 (P_4 + Q_4 H^{(0)}) \\ &= 0.5 [1 + (0.5)(3.4)(e)(1)] \frac{1}{4320} (22 + (60)(1.46911)) = 0.07166. \end{aligned}$$

Thus, as we have seen earlier, the error is really of the order 10^{-3} , which is considerably lower than the above estimate of .07166. The above estimate, although lengthy, has definitely established that $\lambda = 1/2$ is not an eigenvalue of Eq (6.3.20), and, therefore, this equation has a unique solution. For computational details, see `error1.nb`.

If we take $n = 4$, we find that $D(\lambda) = 0.305786$, and $D_1 = D_2 = D_3 = D_4 = 0.30586$, thus, $\tilde{\phi}_1 = \tilde{\phi}_2 = \tilde{\phi}_3 = \tilde{\phi}_4 = 1$. The approximate solution is

$$\begin{aligned} \tilde{\phi}(x) &= 1 + 0.25 \left[e^{0.0694318442x} + e^{0.3300094782x} + e^{0.669905218x} \right. \\ &\quad \left. + e^{0.9305681556x} \right] - \frac{e^x - 1}{2x}. \end{aligned}$$

The error at $x = 0, 1$ is the same as for $n = 2$, viz., 0.000056 and 0.000301, respectively. The rest of the analysis also matches that of the case $n = 2$. Similarly, for $n = 8$ we get $D(\lambda) = 0.305786$, and the remaining data are the same as in case the case $n = 4$. For computational details for $n = 4$ and 8, see `error2.nb` and `error3.nb`. ■

NOTES:

1. The choice of quadrature formula is important. Higher-order quadratures can be used in cases where the integrand is regular, i.e., the functions $k(x, s)$ and $\phi(s)$ have a certain number of derivatives. Moreover, if both $k(x, s)$ and $f(x)$ are regular, then the function $\phi(s)$ will also be regular, since, e.g., if both $k_x^{(n)}(x, s)$ and $f^{(n)}(x)$ exist, then differentiation of Eq (6.1.1) n -times with respect to x gives

$$\phi^{(n)}(x) = \lambda \int_a^b k_x^{(n)}(x, s) \phi(s) ds + f^{(n)}(x).$$

2. It may happen that there is no regularity in some problems as they are posed. In such cases we can contrive the regularity as follows:

(i) If the kernel is regular but the free term has a singularity, then we introduce a new unknown function $\psi(x) = \phi(x) - f(x)$, and then Eq (6.1.1) becomes

$$\psi(x) - \lambda \int_a^b k(x, s) \psi(s) ds = \lambda \int_a^b k(x, s) f(s) ds, \quad (6.3.21)$$

which is an equation of the same type as (6.3.1) but with a regular free term.

(ii) If the kernel has a singularity for $s = x$ (this often happens), then the kernel $k(x, s)$ and its derivative $k_s(x, s)$ are discontinuous at $s = x$. In this case, before replacing Eq (6.1.1) by a system of linear equations, we must first transform it into the following form:

$$\phi(x) \left[1 - \lambda \int_a^b k(x, s) ds \right] - \lambda \int_a^b k(x, s) [\phi(s) - \phi(x)] ds = f(x). \quad (6.3.22)$$

Now, the second integral on the left side vanishes at $s = x$ because of the presence of the factor $[\phi(s) - \phi(x)]$, and so the integrand is more regular than $k(x, s)\phi(s)$ in Eq (6.1.1). Thus, the integral in (6.3.22) can be replaced by a quadrature of the form (6.1.13), which in this case was not possible earlier. The first integral $\int_a^b k(x, s) ds$ on the left side can be computed directly and easily; suppose that it has the value $g(x_i)$ at a point x_i , $i = 1, \dots, n$. Then Eq (6.3.22) yields the system of equations

$$\begin{aligned} \tilde{\phi}(x_i) \left[1 - \lambda g(x_i) \right] - \lambda \sum_{j=1}^n w_j k(x_i, x_j) [\tilde{\phi}(x_j) - \tilde{\phi}(x_i)] &= f(x_i), \\ i = 1, \dots, n, \end{aligned} \quad (6.3.23)$$

which, as above, can be solved by using a suitable quadrature formula.

(iii) If the kernel $k(x, s)$ becomes unbounded at $s = x$ (as in the case of singular integral equations; see Chapter 7), the i th term on the left side of (6.3.23) becomes meaningless. In this case we must then replace the i th term by an expression that is obtained by applying an interpolation formula to the function $k(x_i, s) [\phi(s) - \phi(x_i)]$. Thus, if we use linear interpolation, we must replace $k(x_i, x_i) [\tilde{\phi}(x_i) - \tilde{\phi}(x_i)]$ by

$$\begin{aligned} & \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} k(x_i, x_{i-1}) [\tilde{\phi}(x_{i-1}) - \tilde{\phi}(x_i)] \\ & + \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} k(x_i, x_{i+1}) [\tilde{\phi}(x_{i+1}) - \tilde{\phi}(x_i)]. \end{aligned} \quad (6.3.24)$$

3. The method of Section 6.2 can be used to determine the approximate values of eigenpairs. Thus, instead of finding the solution of the system (6.2.2), we must consider the corresponding homogeneous system

$$\tilde{\phi}(x_i) - \lambda \sum_{j=1}^n w_j k(x_i, x_j) \tilde{\phi}(x_j) = 0, \quad i = 1, \dots, n. \quad (6.3.25)$$

The determinant of this system $D(\lambda)$ is the same as (6.3.2). We solve the equation $D(\lambda) = 0$ and find those values of λ for which the system (6.3.25) has a nontrivial solution $\{\tilde{\phi}(x_i)\}$. These values of λ are then the approximate eigenvalues. The approximate eigenfunctions can then be determined by forming respective independent solutions of the system (6.3.25) corresponding to these eigenvalues.

4. Even in the case of a symmetric kernel $k(x, s) = k(s, x)$, it may happen that the system of equations (6.3.25) is asymmetric. If in this case solving this system becomes difficult, we must make the matrix of this system symmetric by multiplying the i th equation of the system by $\sqrt{w_i}$, and introduce, instead of $\tilde{\phi}(x_i)$, the new unknowns $\zeta_i = \sqrt{w_i} \tilde{\phi}(x_i)$.

5. A different scheme for error analysis can be constructed by using the resolvent (6.1.7). If we use the notation: $k_{ij}^{(n)} = k_n(x_i, x_j)$, and let $\tilde{k}_{ij}^{(n)}$ denote their approximate values, then we successively compute the following quantities:

$$\begin{aligned} \tilde{k}_{ij}^{(1)} &= k_{ij}, \\ \lambda^{n+1} \tilde{k}_{ij}^{(n+1)} &= \sum_{m=1}^n \lambda w_m k_{im} (\lambda^n \tilde{k}_{mj}^{(n)}). \end{aligned} \quad (6.3.26)$$

This leads to a computational algorithm where the resolvent (6.1.7) is approximated by

$$(\tilde{k}_\lambda)_{ij} = \tilde{k}_{ij}^{(1)} + \lambda \tilde{k}_{ij}^{(2)} + \lambda^2 \tilde{k}_{ij}^{(3)} + \dots, \quad (6.3.27)$$

and the approximate solution $\tilde{\phi}(x)$ is given by the formula

$$\tilde{\phi}_i = \lambda \sum_{j=1}^n w_j \left(\tilde{k}_\lambda \right)_{ij} f^{(j)} + f^{(i)}. \quad (6.3.28)$$

6. Consider a VK1:

$$\int_a^x k(x, s) \phi(s) ds = f(x), \quad f(a) = 0. \quad (6.3.29)$$

This equation reduces to a VK2 if the kernel $k(x, s)$ and the free term $f(x)$ are continuously differentiable and $k(x, s) \neq 0$. However, we can avoid this reduction if we apply a quadrature formula. In fact, assume that the quadrature formula contains the value of the integrand at the end of the interval of integration, such that

$$\int_a^{x_n} F(x) dx = \sum_{m=0}^n w_m^{(n)} F(x_m), \quad w_m^{(n)} \neq 0. \quad (6.3.30)$$

Then we can replace (6.3.29) by the system

$$\sum_{m=0}^n w_m^{(n)} k(x_n, s_m) \phi(x_m) = f(x_n), \quad (6.3.31)$$

which recursively computes the values of $\phi(x_m)$ at all mesh points of the quadrature formula, except for $x_0 = a$, where at this point we have

$$\phi(a) = \frac{f'(a)}{k(a, a)}.$$

Thus, the system (6.3.31) becomes

$$\sum_{m=0}^n w_m^{(n)} k(x_n, s_m) \frac{f'(x_m)}{k(x_m, x_m)} = f(x_n), \quad (6.3.32)$$

6.4. Iterative Scheme

In Section 1.8 we proved that the iterative scheme (1.8.7) converges strongly to the solution of FK2 if $\|\lambda K\| < 1$. In fact, suppose that we approximate the integral

in the FK2 by any one of the quadrature rules Q_j (listed in Sections A.1 and A.2) for the interval $[a, b]$ with weights w_i and base points s_i , as in (6.1.13). Then the integral equation takes the approximate form

$$\tilde{\phi}(x) = f(x) + \lambda \sum_{i=1}^n w_i k(x, s_i) \phi(s_i), \quad (6.4.1)$$

and the approximate iterative scheme (1.8.7) becomes

$$\tilde{\phi}_{n+1} = f(x) + \lambda \sum_{i=1}^n w_i k(x, s_i) \phi_n(s_i). \quad (6.4.2)$$

Note that this scheme is not as simple as it looks. The first three iterations in (6.4.2) are

$$\begin{aligned} \tilde{\phi}_0(x) &= f(x), \\ \tilde{\phi}_1(x) &= f(x) + \lambda \sum_{i=1}^n w_i k(x, s_i) f(s_i), \\ \tilde{\phi}_2(x) &= f(x) + \lambda \sum_{i=1}^n w_i k(x, s_i) \left\{ f(s_i) + \lambda \sum_{j=1}^n w_j k(s_i, s_j) f(s_j) \right\}. \end{aligned}$$

However, this shows that in order to compute $\tilde{\phi}_n(s_i)$, (i) we need only the functions $f(s_i)$ and $k(s_i, s_j)$, and (ii) successive iterates $\tilde{\phi}_n(x)$ need to be computed at points s_i for the iteration process to continue. The iterations can be written in matrix form as

$$\tilde{\Phi}_{n+1} = \mathbf{f} + \lambda \mathbf{k} \tilde{\Phi}_n, \quad (6.4.3)$$

where for $i, j = 1, \dots, n$,

$$(\tilde{\phi}_n)_i = \phi_n(s_i), \quad f_i = f(s_i), \quad k_{ij} = w_i k(s_i, s_j). \quad (6.4.4)$$

Repeating the iterations, we find that the inequality (1.8.9) still holds, and thus, the iterations (6.4.2) converge to the solution $\tilde{\Phi}_Q$ of the equation

$$(\mathbf{I} - \lambda \mathbf{K}) \tilde{\Phi}_Q = \mathbf{f} \quad (6.4.5)$$

if $\|\lambda \mathbf{K}\| < 1$. Note that this solution $\tilde{\Phi}_Q$ is obtained strictly under a quadrature rule Q . If the truncation error vector \mathbf{e}_n is defined as $\mathbf{e}_n = \tilde{\Phi}_Q - \tilde{\Phi}_n$, then

$$\|\mathbf{e}_n\| \leq \frac{\|\tilde{\Phi}_{n+1} - \tilde{\Phi}_n\|}{1 - \|\lambda \mathbf{K}\|}. \quad (6.4.6)$$

Eq (6.4.5) depends on the quadrature rule Q used in the computation. Suppose we solve this equation exactly. Let us denote the truncation error by $e_Q = \phi(x) - \phi_Q(x)$, where the suffix Q refers to the quadrature rule used. Then ϕ_Q satisfies the integral equation

$$\phi_Q(x) = f(x) + \lambda \int_a^b k(x, s) \phi_Q(s) ds - \varepsilon_s(\lambda k(x, s) \phi_Q(s)), \quad (6.4.7)$$

where ε_s , regarded as a function of s for fixed x , denotes the error functional for the rule Q operating on $\lambda k(x, s) \phi_Q(s)$. Then subtracting Eq (6.4.7) from the FK2 in (1.2.18), we get the equation

$$e_Q(x) = \lambda \int_a^b k(x, s) e_Q(s) ds + \varepsilon_s(\lambda k(x, s) \phi_Q(s)). \quad (6.4.8)$$

But this equation itself is an FK2 and, therefore, has the solution

$$\begin{aligned} e_Q(x) &= (I - \lambda K)^{-1} \varepsilon_s(\lambda k(x, s) \phi_Q(s)) \\ &= (I + \lambda H) \varepsilon_s(\lambda k(x, s) \phi_Q(s)), \end{aligned}$$

which yields

$$\begin{aligned} \|e_Q\| &\leq (1 + \|\lambda H\|) \left\| \varepsilon_s(\lambda k(x, s) \phi_Q(s)) \right\| \\ &\leq \frac{\left\| \varepsilon_s(\lambda k(x, s) \phi_Q(s)) \right\|}{1 - \|\lambda K\|} \quad \text{if } \|\lambda K\| < 1. \end{aligned} \quad (6.4.9)$$

This bound on the truncation error e_Q consists of two factors. The first one is $(1 + \|\lambda H\|)$ which depends only on the FK2 and not on the rule Q used in computation; this factor measures the sensitivity of the integral equation to small changes in the integral operator K or the free term f . The other factor $\left\| \varepsilon_s(\lambda k(x, s) \phi_Q(s)) \right\|$ depends on the kernel as well as the solution. It measures the error in integrating the function $\lambda k(x, s) \phi_Q(s)$ for fixed x and variable s . If this error is small for all $s \in [a, b]$, then this factor will be small. A suitable choice of the quadrature rule can make this factor small. We now present some examples to compute the truncation error e_Q .

EXAMPLE 6.4.1. Consider

$$\phi(x) = (1 - \alpha/2)x - \alpha + \alpha \int_0^1 (x + 3s) \phi(s) ds.$$

The exact solution is $\phi(x) = x$. The integrand on the right side is clearly quadratic in s ; hence, we choose any quadrature rule of degree at least 2 to compute the exact solution (Table 6.4.1; for the rule Q_j used, see Sections A.1 and A.2).

Table 6.4.1. Comparison of Values of $|e_Q|$

Rule	Ex. 6.4.1	Ex. 6.4.2	Ex. 6.4.3	Ex. 6.4.4(a)	Ex. 6.4.4(b)
$Q_1, n = 1$	1.125	2.52491	1.85914	169.502	8.24084
$Q_1, n = 4$	0.53215	1.15264	0.86491	76.5028	4.01067
$Q_1, n = 10$	0.205	0.43878	0.33009	28.4763	1.57409
$Q_1, n = 20$	0.10125	0.21582	0.16238	13.8991	0.78142
$Q_1, n = 40$	0.05031	0.10702	0.08052	6.86482	0.38922
$Q_2, n = 5$	0.02	0.05774	0.04248	5.42406	0.08864
$Q_2, n = 10$	0.005	0.0142	0.01064	1.35601	0.02377
$Q_2, n = 20$	0.00125	0.00354	0.00266	0.339	0.00628
$Q_2, n = 40$	0.00031	0.00088	0.00066	0.08475	0.00164
$Q_4, n = 5$	0.3	0.354	0.638	27.1203	2.7427
$Q_4, n = 7$	0.21426	0.24776	0.45617	19.3716	1.96286
$Q_4, n = 9$	0.16667	0.19181	0.35488	15.0668	1.52802
$Q_4, n = 11$	0.13636	0.15668	0.29038	12.3274	1.25083
$Q_4, n = 17$	0.08823	0.10125	0.1879	7.97656	0.8099
$Q_4, n = 33$	0.02675	0.10558	0.0349	4.91973	0.1604
$Q_7, n = 5$	0.01	0.02929	0.02119	2.71203	0.0411
$Q_7, n = 10$	0.0025	0.00713	0.0053	0.678	0.0112
$Q_7, n = 20$	0.00052	0.00177	0.00133	0.1695	0.00299
$Q_5, n = 7$	0.2002	0.4453	0.3635	24.5223	1.6108
$Q_5, n = 13$	0.1148	0.2504	0.2101	14.7898	0.9074
$Q_5, n = 19$	0.0803	0.1738	0.1474	10.5257	0.6309
$Q_5, n = 25$	0.0617	0.133	0.1135	8.1813	0.4835
$Q_5, n = 31$	0.0515	0.10774	0.92228	6.6619	0.3919
$Q_5, n = 37$	0.0422	0.0905	0.0777	5.6272	0.3294
$Q_5, n = 43$	0.03645	0.07805	0.06714	4.8703	0.28418
$Q_8, n = 2$	$8.98(-12)$	0.2098	0.04739	$3.95(-15)$	0.0368
$Q_8, n = 4$	$3.7(-12)$	0.0001	$4.01(-7)$	$7.69(-9)$	0.00214
$Q_9, n = 4$	0.125	2.0249	0.2469	133.861	12.7007
$Q_6, n = 4$	0.0	2.049	0.00009	134.209	12.7308
$Q_6, n = 5$	0.0	2.049	0.00005	134.209	12.7311 ■

EXAMPLE 6.4.2. Consider

$$\phi(x) = \cos x + 2\alpha x + \alpha \int_0^\pi xs \phi(s) ds.$$

The exact solution is $\phi(x) = \cos x$. Since the integrand on the right side is an analytic function of the arguments, no quadrature rule of finite degree will compute the exact solution. We need a rule of higher degree to achieve higher accuracy (see Table 6.4.1; for the rule Q_j used, see Sections A.1 and A.2).

EXAMPLE 6.4.3. Consider

$$\phi(x) = e^x - \frac{e^{x+1} - 1}{x + 1} + \int_0^1 e^{xs} \phi(s) ds.$$

The exact solution is $\phi(x) = e^x$. This equation has smooth integrand and free term, and yet higher-order rules provide better accuracy (see Table 6.4.1).

EXAMPLE 6.4.4. Consider

$$\begin{aligned}\phi(x) &= f(x) + \int_0^\pi (xs)^\beta \phi(s) ds, \\ f(x) &= x^\alpha - \frac{\pi^{\alpha+\beta+1}}{\alpha + \beta + 1} x^\beta.\end{aligned}$$

The exact solution is $\phi(x) = x^\alpha$. The integrand has analytic “difficulties” in the kernel and free term. With noninteger values of α and β , we find slow convergence (see Table 6.4.1 for (a) $\alpha = \beta = 3/2$, which shows a bad behavior except with tangential and Gauss–Legendre quadratures, and (b) $\alpha = 1/2, \beta = 3/4$; also see Delves and Mohamed 1985, pp. 90–91, for a similar problem).

The results for e_Q for these four examples are given in Table 6.4.1, and their computational details can be found in `ex641.nb`, `ex642.nb`, `ex643.nb`, `ex644a.nb`, and `ex644b.nb`. These examples show that convergence of the quadrature error e_Q to zero depends not only on the quadrature rule being of higher order but also on the smoothness of the integrand $k(x, s) \phi(s)$ as a function of s with higher continuous derivatives. In fact, if $f(x) \in C^p[a, b]$ and $k(x, s) \in C^p[a, b] \times C^p[a, b]$, then $\phi(x) \in C^p[a, b]$ for $p \geq 1$. To prove this result (by induction), consider the FK2

$$\phi(x) = f(x) + \lambda \int_a^b k(x, s) \phi(s) ds,$$

which on differentiation with respect to x gives

$$\phi'(x) = f'(x) + \lambda \int_a^b \frac{\partial}{\partial x} k(x, s) \phi(s) ds.$$

Since we know that by hypothesis $\frac{\partial}{\partial x} k(x, s)$ is continuous for $p \geq 1$, so the integral $\int_a^b \frac{\partial}{\partial x} k(x, s) \phi(s) ds$ is a continuous function of x . Hence, since $f'(x)$ is continuous, so is $\phi'(x)$. Let $q \leq p$. Then for any q ,

$$\phi^{(q)}(x) = f^{(q)}(x) + \lambda \int_a^b \frac{\partial^q}{\partial x^q} k(x, s) \phi(s) ds$$

which, by induction, proves that $\phi \subset C^p[a, b]$.

6.5. Krylov–Bogoliubov Method

We present this method by using as an example the problem of a double-layer potential which is as follows: It is known that the function $u = \ln r$, where $r = \sqrt{(x - x')^2 + (y - y')^2}$ denotes the distance between two points $P(x, y)$ and $Q(x', y')$, is a harmonic function and satisfies the Laplace equation

$$\nabla^2 \ln r \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \ln r = 0.$$

The derivatives of this function with respect to x' and y' are also harmonic, and thus, the derivative of this function along any constant direction L with respect to x' and y' is also harmonic.

Let α denote the angle formed by the vector \overrightarrow{PQ} and L . Then $\frac{\partial \ln r}{\partial L} = \frac{1}{r} \frac{\partial r}{\partial L} = \frac{\cos \alpha}{r}$, which is, therefore, a harmonic function. The Dirichlet problem is defined as follows: Let a finite region D in \mathbb{R}^2 be bounded by a simple contour $C : \{x = x(s), y = y(s), 0 \leq s \leq s_0\}$, where the functions $x(s)$ and $y(s)$ have continuous first derivatives that do not vanish simultaneously. Then this problem seeks to find a function u , harmonic in the region D , such that $u = f(s)$ on the boundary contour C , where $f(s)$ is a preassigned continuous function of the arc length s . Take any point $Q(x', y')$ on C and let α denote the angle between the (outward) normal at Q and the vector \overrightarrow{PQ} (Fig. 6.5.1). Let $\phi(s)$ denote an arbitrary continuous function. Then the integral

$$u(x, y) = \int_0^{s_0} \frac{\cos \alpha}{r} \phi(s) ds$$

is a harmonic function of x and y in the region D . In fact,

$$\nabla^2 u = \int_0^{s_0} \nabla^2 \left(\frac{\cos \alpha}{r} \right) \phi(s) ds = 0,$$

since $\phi(s)$ does not depend on x and y , and $\frac{\cos \alpha}{r}$ is harmonic. The function u is known as a double-layer potential, and $\phi(s)$ is called its density (see Kythe 1996, pp.21–22).

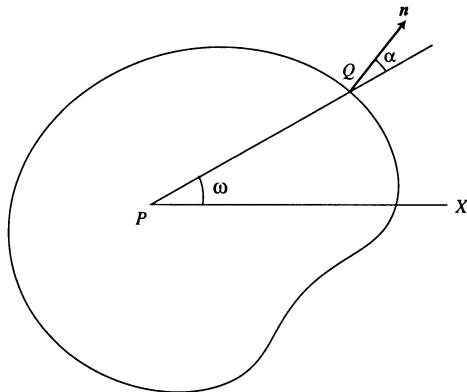


Fig. 6.5.1.

To derive another expression for $u(x, y)$, let ω denote the angle that the vector \overrightarrow{PQ} makes with the x -axis. If the position of the point Q varies along the contour C , the angle ω is a function of r , and $d\omega$ is the angle subtended at the point P by an element ds of the arc length, where $d\omega = \frac{\cos \alpha ds}{r}$, since the projection of ds on the perpendicular to the vector \overrightarrow{PQ} is $ds \cos \alpha$. Thus, we have

$$u(x, y) = \int_0^{s_0} \frac{\cos \alpha}{r} \phi(s) ds = \int_C \phi(s) d\omega = \int_0^{s_0} \frac{d\omega}{ds} \phi(s) ds. \quad (6.5.1)$$

Note that ω is a function of s and of x and y also, i.e., $\omega = \omega(s, x, y)$, and s_0 denotes the total length of the contour C , with the following values:

$$\int_0^{s_0} \frac{d\omega}{ds} ds = \begin{cases} 2\pi & \text{if } P \text{ lies inside } D, \\ \pi & \text{if } P \text{ lies on } C, \\ 0 & \text{if } P \text{ lies outside } C. \end{cases}$$

Thus, the Dirichlet problem reduces to that of finding the density $\phi(s)$ such that the function (6.5.1) attains the value $f(s)$ as it approaches the contour C . Suppose that the point $P(x, y)$ lying inside D tends to a point $P_0(x(t), y(t))$ on the contour C corresponding to an arc length equal to t . Then

$$\begin{aligned} \lim_{P \rightarrow P_0} u(x, y) &= \lim_{P \rightarrow P_0} \int_0^{s_0} \phi(s) d\omega \\ &= \lim_{P \rightarrow P_0} \left[\int_0^{s_0} [\phi(s) - \phi(t)] \frac{d\omega(s, x, y)}{ds} ds \right. \\ &\quad \left. + \phi(t) \int_0^{s_0} \frac{d\omega(s, x, y)}{ds} ds \right] \\ &= \int_0^{s_0} [\phi(s) - \phi(t)] \frac{d\omega(s, t)}{ds} ds + 2\pi \phi(t) \\ &= \int_0^{s_0} \phi(s) \frac{d\omega(s, t)}{ds} ds + \pi \phi(t), \end{aligned} \tag{6.5.2}$$

since $\int_0^{s_0} \phi(t) d\omega = \pi \phi(t)$; here, for the sake of brevity, we have written $\omega(s, t) = \omega(s, x(t), y(t))$. Thus, we get

$$\pi \phi(t) + \int_0^{s_0} \phi(s) \frac{d\omega}{ds} ds = f(t), \tag{6.5.3}$$

which can be rewritten as an FK2:

$$\phi(t) - \lambda \int_0^{s_0} k(s, t) \phi(s) ds = \frac{1}{\pi} f(t), \tag{6.5.4}$$

where $\lambda = 1$, and the kernel $k(s, t)$ simplifies to

$$k(s, t) = -\frac{1}{\pi} \frac{d\omega}{ds} = -\frac{1}{\pi} \frac{d}{ds} \tan^{-1} \left(\frac{y(s) - y(t)}{x(s) - x(t)} \right). \tag{6.5.5}$$

Since $\lambda = 1$ is not an eigenvalue of Eq (6.5.4), this equation has a unique solution for any free term $f(t)$.

Now, we shall find an approximate solution of Eq (6.5.4). Note that the distribution of eigenvalues of this equation is important for this task. First, it can be shown that all eigenvalues are real. Also, $\lambda = -1$ is an eigenvalue since any constant $\phi(s) = c$ satisfies the corresponding homogeneous equation, i.e., $\pi c - \int_0^{s_0} c \frac{d\omega(s, t)}{ds} ds = 0$. Lastly, there are no eigenvalues in the interval

($-1, 1$) (see Goursat 1964, vol. III, Part 2, p.172). We shall solve Eq (6.5.4) numerically by the Krylov–Bogoliubov method, which is a modification of the method of successive approximations. We shall approximate the integral on the left side of Eq (6.5.4) by the following sum:

$$\begin{aligned} & \frac{1}{\pi} \sum_{j=1}^n \phi(s_j) \int_{s_j - \Delta s/2}^{s_j + \Delta s/2} \frac{d\omega(s, t)}{ds} ds \\ &= \frac{1}{\pi} \sum_{j=1}^n \phi(s_j) [\omega(s_j + \Delta s/2, t) - \omega(s_j - \Delta s/2, t)] \\ &= \frac{1}{\pi} \sum_{j=1}^n \phi(s_j) \Delta\omega(s_j, t), \end{aligned} \quad (6.5.6)$$

where Δs is the n th part of the interval $(0, s_0)$, i.e., $\Delta s = \frac{s_0}{n}$, and s_j are equidistant points separated by Δs , namely,

$$s_1 = \frac{\Delta s}{2}, \quad s_2 = \frac{3\Delta s}{2} \quad \dots, \quad s_n = \frac{(2n-1)\Delta s}{2} = s_0 - \frac{\Delta s}{2}.$$

This approximation reduces the integral on the left side of (6.5.4) into integrals over equal subintervals $(s_j - \Delta s/2, s_j + \Delta s/2)$, on each of which the function $\phi(s)$ takes its mean value precisely at the midpoint of the subinterval. Thus, we replace the integral in (6.5.4) by a sum, and successively set $t = s_1, s_2, \dots, s_n$ in the equation obtained. This leads to a system of n equations

$$\tilde{\phi}(s_j) + \frac{1}{\pi} \sum_{j=1}^n \tilde{\phi}(s_j) \Delta_j \omega(s_j, s_i) = f(s_i), \quad i = 1, \dots, n, \quad (6.5.7)$$

where, for brevity, we have put

$$\Delta_j \omega(s_j, s_i) = \int_{s_j - \Delta s/2}^{s_j + \Delta s/2} \frac{d\omega(s, s_i)}{ds}.$$

Eq (6.5.7) determines the approximate values $\tilde{\phi}$ of $\phi(t)$ at the points s_j .

Let $e_i = \phi(s_i) - \tilde{\phi}(s_i)$ denote the error in this approximation method. Then the numbers e_i satisfy the system of equations

$$e_i + \frac{1}{\pi} \sum_{j=1}^n \Delta_j \omega(s_j, s_i) e_i = R(s_i), \quad (6.5.8)$$

where the residual

$$R(t) = \frac{1}{\pi} \left[\sum_{j=1}^n \phi(s_j) \Delta_j \omega(s_j, t) - \int_0^{s_0} \phi(s) \frac{d\omega(s, t)}{ds} ds \right].$$

Let R^* denote the maximum of the residual $R(t)$. Then $|e_i| = |\phi(t) - \tilde{\phi}(t)| \leq (1 + 2\pi R_0) R^*$, where R_0 is the maximum radius of curvature of the contour C (see Kantorovich and Krylov 1956, p.132).

EXAMPLE 6.5.1. We will solve the Dirichlet problem on the ellipse $x = a \cos s, y = b \sin s, a \geq b, 0 \leq s \leq 2\pi$, and determine a function $u(x, y)$, harmonic within the ellipse, which on the boundary of the ellipse becomes a pre-assigned function $f(t)$, i.e., $u(a \cos t, b \sin t) = f(t)$. This problem reduces to that of determining a double-layer potential $\phi(x, y) = \int_0^{2\pi} \frac{d\omega(s, x, y)}{ds} \phi(s) ds$, which satisfies Eq (6.5.4), with $\lambda = 1$ and factoring out $-1/\pi$, that is,

$$\phi(t) + \frac{1}{\pi} \int_0^{2\pi} k(s, t) \phi(s) ds = \frac{1}{\pi} f(t),$$

where

$$\begin{aligned} k(s, t) &= \frac{d}{ds} \tan^{-1} \frac{y(s) - y(t)}{x(s) - x(t)} = \frac{d}{ds} \tan^{-1} \left(\frac{b \sin s - b \sin t}{a \cos s - a \cos t} \right) \\ &= \frac{d}{dt} \tan^{-1} \frac{b}{a} \frac{-2 \cos \left(\frac{s+t}{2} \right) \sin \left(\frac{s-t}{2} \right)}{2 \sin \left(\frac{s+t}{2} \right) \sin \left(\frac{s-t}{2} \right)} \\ &= \frac{d}{dt} \tan^{-1} \left(\frac{b}{a} \cot \frac{s+t}{2} \right) \\ &= \frac{1}{2} \frac{ab}{a^2 \sin^2 \frac{s+t}{2} + b^2 \cos^2 \frac{s+t}{2}} \\ &= \frac{ab}{a^2 - b^2} \cdot \frac{1}{\beta - \cos(s+t)}, \end{aligned} \tag{6.5.9}$$

where $\beta = (a^2 + b^2)/(a^2 - b^2)$. We shall assume that $f(t)$ is symmetric with respect to the coordinate axes, i.e., $f(-t) = f(t)$, and $f\left(\frac{\pi}{2} + t\right) = f\left(\frac{\pi}{2} - t\right)$. Then the required function $\phi(t)$ will also satisfy the same symmetry conditions.

We shall divide the interval $(0, 2\pi)$ into 16 subintervals of equal length $\pi/8$ each, as shown in Fig. 6.5.2:

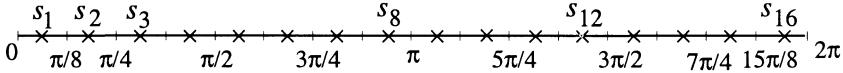


Fig. 6.5.2.

Then, because of symmetry, we have

$$\begin{aligned} f(s_1) &= f(s_8) = f(s_9) = f(s_{16}), & \phi(s_1) &= \phi(s_8) = \phi(s_9) = \phi(s_{16}), \\ f(s_2) &= f(s_7) = f(s_{10}) = f(s_{15}), & \phi(s_2) &= \phi(s_7) = \phi(s_{10}) = \phi(s_{15}), \\ f(s_3) &= f(s_6) = f(s_{11}) = f(s_{14}), & \phi(s_3) &= \phi(s_6) = \phi(s_{11}) = \phi(s_{14}), \\ f(s_4) &= f(s_5) = f(s_{12}) = f(s_{13}), & \phi(s_4) &= \phi(s_5) = \phi(s_{12}) = \phi(s_{13}). \end{aligned}$$

Now, we apply Gauss–Legendre quadrature for $n = 8$ on the interval $(0, \pi)$. Notice that the points s_5, s_6, s_7, s_8 are symmetric with respect to the points s_1, s_2, s_3, s_4 . The Gauss points in radians for the interval $(0, \pi)$ are

$$s_1 = 0.0623765, \quad s_2 = 0.319396, \quad s_3 = 0.745292, \quad s_4 = 1.28266,$$

(see `gausspoints8(0, π).nb`). Next, for the interval $(\pi, 2\pi)$ we take the same Gauss–Legendre quadrature for $n = 8$ and use the Gauss points for s_9, \dots, s_{16} instead. Since the Gauss formula for the integral is

$$\frac{1}{\pi} \int_0^{2\pi} k(s, t) \phi(s) ds = \sum_{j=1}^{16} w_j k(s_j, t) \phi(s_j),$$

the weights w_j , because of the factor $1/\pi$, remain the same as those for the interval $(0, 1)$, namely,

$$\begin{aligned} w_1 &= w_8 = w_9 = w_{10} = 0.0506142681, \\ w_2 &= w_7 = w_{10} = w_{15} = 0.1111905172, \\ w_3 &= w_6 = w_{11} = w_{14} = 0.1568533229, \\ w_4 &= w_5 = w_{12} = w_{13} = 0.1813418917, \end{aligned}$$

see Abramowitz and Stegun, 1968, p.921. But, due to axial symmetry, this system of 16 equations reduces to one set of 4 equations (the first four), that is,

$$\tilde{\phi}(s_j) + \sum_{i=1}^4 B_{ij} \tilde{\phi}(s_i) = \frac{1}{\pi} f(s_i), \quad i = 1, 2, 3, 4, \quad (6.5.10)$$

where from (6.5.9),

$$\begin{aligned} B_{ij} &= w_j \frac{ab}{a^2 - b^2} \left[\frac{1}{\beta - \cos(s_j + s_i)} + \frac{1}{\beta - \cos(\pi - s_j + s_i)} \right. \\ &\quad \left. + \frac{1}{\beta - \cos(\pi + s_j + s_i)} + \frac{1}{\beta - \cos(2\pi - s_j + s_i)} \right] \\ &= w_j \frac{ab}{a^2 - b^2} \left[\frac{1}{\beta - \cos(s_j + s_i)} + \frac{1}{\beta + \cos(s_j - s_i)} \right. \\ &\quad \left. + \frac{1}{\beta + \cos(s_j + s_i)} + \frac{1}{\beta - \cos(s_j - s_i)} \right]. \end{aligned} \quad (6.5.11)$$

After solving the system (6.5.10) for $\tilde{\phi}(s_1), \tilde{\phi}(s_2), \tilde{\phi}(s_3), \tilde{\phi}(s_4)$, we obtain the approximate solution as

$$\tilde{\phi}(t) = \frac{1}{\pi} f(t) - \frac{ab}{a^2 - b^2} \sum_{j=1}^4 \frac{w_j}{\beta \mp \cos(t \pm s_j)} \tilde{\phi}(s_j). \quad (6.5.12)$$

As an example, take $a = 1$, $b = 1/2$, yielding $ab/(a^2 - b^2) = 2/3$, and $\beta = 5/3$; also, assuming that we are seeking for a solution $u(x, y)$ that satisfies the condition $u = (x^2 + y^2)/2$ on the boundary of the ellipse, we have

$$f(t) = \frac{1}{2} \left(\cos^2 t + \frac{1}{4} \sin^2 t \right) = \frac{1}{16} (5 + 4 \cos 2t).$$

Then the system (6.5.10) becomes

$$\begin{pmatrix} 1 + B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & 1 + B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & 1 + B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & 1 + B_{44} \end{pmatrix} \begin{Bmatrix} \tilde{\phi}(s_1) \\ \tilde{\phi}(s_2) \\ \tilde{\phi}(s_3) \\ \tilde{\phi}(s_4) \end{Bmatrix} = \frac{1}{\pi} \begin{Bmatrix} f(s_1) \\ f(s_2) \\ f(s_3) \\ f(s_4) \end{Bmatrix},$$

whose solution is

$\tilde{\phi}(s_1) = 0.103034$, $\tilde{\phi}(s_2) = 0.0928595$, $\tilde{\phi}(s_3) = 0.054038$, and $\tilde{\phi}(s_4) = 0.00469757$ (see `doublelayer.nb`). The exact solution is

$$\phi(t) = \frac{1}{160\pi} (25 + 27 \cos 2t).$$

Thus, we have the exact values

$$\begin{aligned} \phi(0) &= 0.103461, & \phi(s_1) &= 0.103033, \\ \phi(\pi/8) &= 0.087718, & \phi(s_2) &= 0.092859, \\ \phi(\pi/4) &= 0.0497359, & \phi(s_3) &= 0.0540399, \\ \phi(\pi/2) &= -0.00397887, & \phi(s_4) &= 0.00469609. \blacksquare \end{aligned}$$

REFERENCES USED: Abramowitz and Stegun (1968), Atkinson (1976), Delves and Mohamed (1985), Goursat (1964), Kantorovich and Krylov (1956), Kythe (1996), Mikhlin and Smolitsky (1967).

7

Singular Equations

We have so far considered cases where the kernel $k(x, s)$ and the free term $f(x)$ are continuous on $[a, b]$. The term “singular” has been used in different connotations. Delves and Mohamed (1985) use it to mean any kind of lack of analyticity in an integral equation. However, they distinguish between the following types of singular integral equations: (i) those with a semi-infinite or infinite range; (ii) those with a discontinuous derivative in either the kernel or the free term; and (iii) those with either an infinite or nonexisting derivative of some finite order. Any combination of these situations may also occur in an equation. Singular equations arise in the areas of potential problems, Dirichlet problems, radiative equilibrium, and many others.

7.1. Singularities in Linear Equations

We discuss the following cases in this section:

(i) The case of jump singularities: The functions $k(x, s)$ and $f(x)$ are piecewise continuous, with jump discontinuities only along lines parallel to the coordinate axes. In this case the integral equation can, in general, be reformulated as a system of equations like Eq (1.2.2) (see §1.2). If the free term $f(x)$ is badly behaved, it will produce bad behavior in the solution $\phi(x)$ and, in general, propagate its form and nature of singularities into the solution $\phi(x)$. A method to remove this type of singularities is to write $\phi(x) = \psi(x) + f(x)$ and solve the equation

$$\psi(x) = \lambda \int_a^b k(x, s)\psi(s) ds + \lambda \int_a^b k(x, s)f(s) ds. \quad (7.1.1)$$

Another method is to use a single function $\gamma(x)$ that has the “bad” characteristics of $f(x)$ that are propagated into $\phi(x)$, and set $\phi(x) = \psi(x) + \gamma(x)$, such that $f(x) - \gamma(x)$ becomes well behaved. Then solve

$$\psi(x) - \lambda \int_a^b k(x, s)\psi(s) ds = [f(x) - \gamma(x)] + \lambda \int_a^b k(x, s)\gamma(s) ds, \quad (7.1.2)$$

by using a quadrature rule to approximate $\tilde{\psi}(x)$, which in turn yields $\tilde{\phi}(x)$. This method is successful only if $\int_a^b k(x, s)\psi(s) ds$ or $\int_a^b k(x, s)\gamma(s) ds$ are smooth functions of x , and especially if $k(x, s)$ is smooth.

- (ii) In the case when the bad behavior of $f(x)$ is because of a discontinuity in a derivative, a high-order accuracy from a quadrature rule may not generally be obtained since $\phi(x)$ may be affected by such behavior of $f(x)$.
- (iii) Discontinuities in the kernel $k(x, s)$ that lie on lines parallel to the s -axis pass on to the solution $\phi(x)$. An example for this situation is

$$k(x, s) = \frac{g(x, s)}{|x - x_1|^\alpha}, \quad 0 < \alpha < 1, \quad x_1 \in [a, b], \quad (7.1.3)$$

where $g(x, s)$ is continuous. In this case a quadrature method or a collocation method is generally not applicable, but Ritz–Galerkin methods may provide an approximation. Since the kernel is unbounded at $x = x_1$, the product-integration method may be needed to complete the solution. Similarly, the case of the kernel of the form $k(x, s) = \frac{g(x, s)}{|s - s_1|^\alpha}$, $0 \leq \alpha < 1$, needs the product-integration method as well as expansion methods, with a suitable choice of orthogonal polynomials $p_i(x)$.

- (iv) The case when the kernel $k(x, s)$ is weakly singular and has the form

$$k(x, s) = \frac{g(x, s)}{|x - s|^\alpha}, \quad 0 < \alpha < 1, \quad (7.1.4)$$

where $g(x, s)$ is continuous on $[a, b]$, is very important and a lot of research is available on this topic (see Chapter 8). For a weakly singular kernel it is sometimes possible to reformulate Eq (7.1.1) so that the kernel becomes at least continuous. It is known that if $k(x, s)$ has the form (7.1.4) with $0 < \alpha < 1/2$, and $f \in C[a, b]$, then the second iterated kernel $k_2(x, s)$ is bounded and continuous (for a proof see Goursat 1964, Mikhlin 1960). Suppose that $(I - \lambda K)^{-1}$ and $(I + \lambda K)^{-1}$ both

exist, i.e., both λ and $-\lambda$ are regular values of Eq (7.1.1). Then Eq (7.1.1) yields the equation

$$\phi(x) - \lambda^2 \int_a^b k_2(x, s)\phi(s) ds = f(x) + \lambda \int_a^b k(x, s)f(s) ds. \quad (7.1.5)$$

Although $k_2(x, s)$ is continuous on $[a, b]$, we may find some difficulty in computing $\int_a^b k(x, s)f(s) ds$. However, it is possible to determine a value ν for which $(I + \lambda K + \nu K_2)^{-1}$ exists. Then Eq (7.1.1) becomes equivalent to

$$\begin{aligned} \phi(x) - \int_a^b \left\{ (\lambda^2 - \nu) k_2(x, s) + \lambda \nu k_3(x, s) \right\} \phi(s) ds \\ = f(x) + \lambda \int_a^b k(x, s)f(s) ds + \nu \int_a^b k_2(x, s)f(s) ds, \end{aligned} \quad (7.1.6)$$

where $k_3(x, s) = \int_a^b k(x, t)k_2(t, s) dt$ is the third iterated kernel. For this equation the kernel $k(x, s)$ is bounded for $0 < \alpha < 1/2$, but $k_2(x, s)$ is unbounded for $1/2 \leq \alpha < 1$. The process can, however, be repeated.

EXAMPLE 7.1.1. (Baker 1978, p.530) Consider Eq (7.1.1) with $k(x, s) = |x - s|^{-1/2}$, $0 \leq x \leq 1$, and let λ be a real regular value. Then

$$\phi(x) - \lambda^3 \int_0^1 k_3(x, s)\phi(s) ds = (I - \lambda \omega K)(I - \lambda \omega^2 K)f(x),$$

where $\omega = e^{2i\pi/3}$. The kernel $k_3(x, s)$ is continuous. The right side is obtained directly by replacing ν by λ^2 in (7.1.6). ■

EXAMPLE 7.1.2. (Kondo 1991, p.271) Consider the FK2

$$\phi(x) - \lambda \int_1^\infty \frac{1}{xs^2} \phi(s) ds = \frac{1}{x}.$$

We will use the resolvent kernel $k_\lambda(x, s)$ defined by (1.9.8) to obtain the solution $\phi(x)$, where

$$\begin{aligned} k_\lambda(x, s) &= k(x, s) + \lambda k_1(x, s) + \lambda^2 k_2(x, s) + \dots \\ &= \frac{1}{xs^2} \left\{ 1 + \frac{\lambda}{2} + \left(\frac{\lambda}{2} \right)^2 + \dots \right\} \\ &= \frac{1}{xs^2} \frac{2}{2 - \lambda}, \quad |\lambda| < 2. \end{aligned}$$

The solution, given by (1.9.9), is

$$\phi(x) = f(x) + \lambda \int_1^\infty k_\lambda(x, s) f(s) ds = \frac{2}{2 - \lambda} \frac{1}{x},$$

which satisfies the given equation. Note that $\lambda = 2$ is the eigenvalue and it has infinitely many eigenfunctions of the form C/x , where C is an arbitrary constant. ■

7.2. Fredholm Theorems

Consider a singular equation of the second kind (SK2) of the form

$$(I + \lambda K) \phi(x) = f(x), \quad (7.2.1)$$

where K is the singular operator. Then the following Fredholm theorems hold.

THEOREM 7.2.1. Eq (7.2.1) has finitely many solutions $\{\phi_1(x), \dots, \phi_n(x)\}$.

THEOREM 7.2.2. A necessary and sufficient condition for a solution of Eq (7.2.1) to exist is that

$$\int_{\Gamma} f(x) \psi_j(x) dx = 0, \quad (7.2.2)$$

where $\{\psi_1(x), \dots, \psi_m(x)\}$ is a maximal finite set of linearly independent solutions of the transposed homogeneous singular equation $(K^T \psi)(x) = 0$. Note that if the functions $f(x)$ and $\psi_j(x)$, $j = 1, \dots, m$, are complex-valued, then the condition (7.2.2) is *not* an orthogonality relation for these functions.

THEOREM 7.2.3. The difference between the number n of linearly independent solutions of the homogeneous singular equation $(K\phi)(x) = 0$ and the number m of linearly independent solutions of the transposed homogeneous singular equation $(K^T \psi)(x) = 0$ depend only on the characteristic part of the operator K and are equal to its index* ν , that is, $n - m = \nu$.

COROLLARY 7.2.3. The number of linearly independent solutions of characteristic equations is maximal among all singular equations with a given index ν . In particular,

*For the definition of index ν , see Muskhelishvili (1992, p.90) and Polyanin and Manzhirov (1998, p.638).

- (i) If $\nu > 0$, then the homogeneous singular equation $(K_s \phi)(x) = 0$ has ν linearly independent solutions.
- (ii) If $\nu \leq 0$, then the homogeneous singular equation $(K\phi)(x) = 0$ has only the trivial solution.
- (iii) If $\nu \geq 0$, then the nonhomogeneous singular equation $(K_s \phi)(x) = f(x)$ has, in general, ν linearly independent solutions for an arbitrary function $f(x)$.
- (iv) If $\nu < 0$, then the nonhomogeneous singular equation $(K_s \phi)(x) = f(x)$ is solvable iff $f(x)$ satisfies the $(-\nu)$ conditions

$$\int_{\Gamma} \Psi_j(x) f(x) dx = 0, \quad j = 1, 2, \dots, -\nu,$$

where Γ is a simple closed contour, $\Psi_j(x) = x^{j-1}/Z(x)$, and $Z(x)$ denotes the fundamental solution of the homogeneous equation (see Muskhelishvili 1992, p. 90).

7.3. Modified Quadrature Rule

A quadrature rule of the form

$$\int_a^b f(s) ds = \sum_{j=0}^n w_j f(s_j), \quad a \leq s_j \leq b, \quad j = 0, 1, \dots, n, \quad (7.3.1)$$

can be used to compute an approximate solution $\tilde{\phi}(x)$ of Eq (7.1.4) as follows: Write

$$\left\{ 1 - \lambda \int_a^b \frac{g(x, s)}{|x - s|^\alpha} ds \right\} \phi(x) - \lambda \int_a^b \frac{g(x, s)}{|x - s|^\alpha} [\phi(s) - \phi(x)] ds = f(x). \quad (7.3.2)$$

If $\phi \in H^\beta[a, b]$, where $\beta > \alpha$, the integrand in the second term on the left side tends to zero continuously as $x \rightarrow s$. The quadrature rule is then applicable and yields the approximate values $\tilde{\phi}(s_i)$, $i = 0, 1, \dots, n$, where

$$\left\{ 1 - \lambda G(s_i) \right\} \tilde{\phi}(s_i) - \lambda \sum_{\substack{j=0 \\ j \neq i}}^n \frac{g(s_i, s_j)}{|s_i - s_j|^\alpha} w_j [\tilde{\phi}(s_j) - \tilde{\phi}(s_i)] = f(s_i), \quad (7.3.3)$$

where

$$G(x) = \int_a^b \frac{g(x, s)}{|x - s|^\alpha} ds. \quad (7.3.4)$$

The values $\tilde{\phi}(s_i)$, $i = 0, 1, \dots, n$, will then define a function $\tilde{\phi}(x)$ by piecewise constant or piecewise linear interpolation if $a < s_0$ or $b > s_n$, or by polynomial or spline interpolation.

EXAMPLE 7.3.1. (Baker 1978, p.535) Let $k(x, s) = \ln|x + s|$, and define $g(x, s) = |x - s|^\alpha \ln|x + s|$, $0 < \alpha < 1$, and $g \in C[0, 1]$. Then $k(x, s) = \frac{g(x, s)}{|x - s|^\alpha} = \ln|x + s|$. The kernel $k(x, s)$ is weakly singular and square-integrable, i.e., $k(x, s) \in L_2[0, 1]$. For the SK2

$$\phi(x) - \int_0^1 \ln|x + s| \phi(s) ds = f(x), \quad 0 \leq x \leq 1, \quad (7.3.5)$$

we can use the modified quadrature rule (7.3.3) with

$$G(x) = \int_0^1 \ln|x + s| ds = (x + 1) \ln(x + 1) - x \ln x - 1.$$

The exact solution of Eq (7.3.5) is $\phi(x) = x$ if we take

$$f(x) = x + \frac{x^2}{2} \ln x - \frac{x^2}{4} - \frac{(x+1)^2}{2} \ln(x+1) + \frac{(x+1)^2}{4},$$

which corresponds to $G(x)$. This modified quadrature rule is applied using the

- (a) $(n + 1)$ -point repeated mid-point rule,
- (b) n -times repeated trapezoidal rule, and
- (c) repeated Simpson's rule with $(n + 1)$ points.

The results are given in Table 7.3.1 for different values of n (Baker 1978, p.535).

Table 7.3.1

n	(a)	(b)	(c)
2	1.16(-2)	5.52(-2)	1.16(-2)
4	4.27(-3)	1.37(-2)	3.45(-3)
8	1.34(-3)	3.43(-3)	9.74(-4)
16	4.31(-4)	1.05(-3)	2.64(-4)
32	1.33(-4)	3.14(-3)	6.95(-5)
64	3.98(-5)	9.18(-5)	1.80(-5)
128	1.16(-5)	2.64(-5)	4.58(-6)

The error seems to be of the order $O(h^2 \ln h)$. For computational details, see `modified1.nb`. ■

EXAMPLE 7.3.2. Consider the FK2

$$\phi(x) - \int_0^1 \ln|x-s| \phi(s) ds = f(x), \quad 0 \leq x \leq 1, \quad (7.3.6)$$

whose exact solution is $\phi(x) = x$ for

$$f(x) = x - \frac{1}{2} \left\{ x^2 \ln x + (1+x^2) \ln(1-x) - \left(x + \frac{1}{2} \right) \right\}.$$

Using the modified rule (7.3.3) with the repeated Simpson's rule, we compute $\tilde{\phi}(x)$ for different step sizes h . The results are given in Table 7.3.2 (see Baker 1978, p.537).

Table 7.3.2.

$h \setminus x$	0.0	0.25	0.5	0.75	1.0
1/2	-1.3(-2)		3.6(-12)		1.3(-2)
1/4	-3.6(-3)	7.5(-5)	3.6(-12)	-7.5(-5)	3.6(-3)
1/8	-1.0(-3)	1.9(-5)	7.3(-12)	-1.9(-5)	1.0(-4)
1/16	-2.7(-4)	3.1(-6)	-1.5(-11)	-3.1(-6)	2.7(-4)
1/32	-7.1(-5)	4.4(-7)	7.3(-12)	-4.4(-7)	7.1(-5)
1/64	-1.8(-5)	5.8(-8)	-2.2(-11)	-5.8(-8)	1.8(-5)

The values at $x = 0.5$ are correct up to the level of roundoff error; the maximum error occurs at $x = 0$ and $x = 1$, and the rate of decrease is that of order $O(h^2 \ln h)$ as h decreases. The errors at the points $x = 0.25$ and $x = 0.75$ decrease faster than the above order. ■

7.4. Convolution-Type Kernels

Some special cases of the convolution-type kernels of singular integral equations, which are of the form $k(x, s) = k(x - s)$, are as follows:

(a) If the interval of integration is $[a, \infty)$, we can use the Fourier transforms to an FK1 or FK2, provided we assume that $f(x) < 0$ for $x < a$. Consider an FK1

$$\int_a^\infty k(x-s) \phi(s) ds = f(x). \quad (7.4.1)$$

The method consists of multiplying both sides of Eq (7.4.1) by $\cos \pi\mu(x - t)$ and integrating with respect to x from a to ∞ ; thus,

$$\int_a^\infty \cos \pi\mu(x - t) f(x) dx = \int_a^\infty \cos \pi\mu(x - t) f(x) dx \int_a^\infty k(x - s) \phi(s) ds.$$

Let $x - s = \tau$. Then by replacing the variable of integration x by τ on the right side we get

$$\begin{aligned} \int_a^\infty \cos \pi\mu(x - t) f(x) dx &= \int_0^\infty \cos \pi\mu(\tau + s - t) k(\tau) d\tau \int_a^\infty \phi(s) ds \\ &= \int_0^\infty [\cos \pi\mu\tau \cos \pi\mu(s - t) - \sin \pi\mu\tau \sin \pi\mu(s - t)] k(\tau) d\tau \int_a^\infty \phi(s) ds. \end{aligned} \quad (7.4.2)$$

Define the Fourier cosine and sine transforms of $k(x - s) = k(\tau)$ as

$$\begin{aligned} \mathcal{F}_c k(\tau) &\equiv \tilde{k}_c(\mu) = \int_0^\infty k(\tau) \cos \pi\mu\tau d\tau, \\ \mathcal{F}_s k(\tau) &\equiv \tilde{k}_s(\mu) = \int_0^\infty k(\tau) \sin \pi\mu\tau d\tau. \end{aligned} \quad (7.4.3)$$

Then

$$\begin{aligned} \int_a^\infty \cos \pi\mu(x - t) f(x) dx &= \tilde{k}_c(\mu) \int_a^\infty \cos \pi\mu(s - t) \phi(s) ds \\ &\quad - \tilde{k}_s(\mu) \int_a^\infty \sin \pi\mu(s - t) \phi(s) ds, \\ \int_a^\infty \sin \pi\mu(x - t) f(x) dx &= \tilde{k}_c(\mu) \int_a^\infty \sin \pi\mu(s - t) \phi(s) ds \\ &\quad + \tilde{k}_s(\mu) \int_a^\infty \cos \pi\mu(s - t) \phi(s) ds, \end{aligned} \quad (7.4.4)$$

which yield

$$\begin{aligned} &\int_a^\infty \cos \pi\mu(s - t) \phi(s) ds \\ &= \int_a^\infty \frac{\tilde{k}_c(\mu) \cos \pi\mu(x - t) + \tilde{k}_s(\mu) \sin \pi\mu(x - t)}{\tilde{k}_s^2(\mu) + \tilde{k}_c^2(\mu)} f(x) dx, \end{aligned} \quad (7.4.5)$$

or

$$\begin{aligned} & \int_a^\infty \cos \pi \mu(s-t) \phi(s) ds \\ &= \int_a^\infty \frac{\tilde{k}_c(\mu) \cos \pi \mu(s-x) + \tilde{k}_s(\mu) \sin \pi \mu(s-x)}{\tilde{k}_s^2(\mu) + \tilde{k}_c^2(\mu)} f(s) ds. \end{aligned} \quad (7.4.6)$$

If $f(x)$ satisfies the Dirichlet condition for $x > a$ and if $\int_a^\infty |f(x)| dx < \infty$, then

$$\int_0^\infty d\mu \int_0^\infty \cos \pi \mu(x-s) f(s) ds = \begin{cases} f(x) & \text{if } x > a, \\ 0 & \text{if } x < a, \end{cases} \quad (7.4.7)$$

and we obtain the solution

$$\phi(x) = \begin{cases} \int_0^\infty d\mu \int_a^\infty \frac{\tilde{k}_c(\mu) \cos \pi \mu(s-x) + \tilde{k}_s(\mu) \sin \pi \mu(s-x)}{\tilde{k}_s^2(\mu) + \tilde{k}_c^2(\mu)} f(s) ds \\ \quad \text{if } x > a, \\ 0 \quad \text{if } x < a. \end{cases} \quad (7.4.8)$$

In the case when $a = 0$, Eq (7.4.1) is known as the *Wiener–Hopf equation*.

(b) If the interval of integration is (a, b) , then for the FK1 $\int_a^b k(x-s) \phi(s) ds = f(x)$, we find in an analogous manner the solution

$$\phi(x) = \int_{-\infty}^\infty d\eta \int_a^b \frac{\tilde{k}_c(\eta) \cos \pi \mu(s-x) + \tilde{k}_s(\eta) \sin \pi \mu(s-x)}{\tilde{k}_s^2(\eta) + \tilde{k}_c^2(\eta)} f(s) ds, \quad (7.4.9)$$

where

$$\tilde{k}_{c,s}(\eta) = \begin{cases} \int_{-\infty}^\infty k(\tau) \begin{cases} \cos \pi \eta \tau & \text{if } a < x < b, \\ \sin \pi \eta \tau & \text{if } x < a, b > x. \end{cases} d\tau \\ 0 \end{cases} \quad (7.4.10)$$

(c) For an FK2 of the form

$$\phi(x) - \lambda \int_a^\infty k(x-s) \phi(s) ds = f(x), \quad (7.4.11)$$

the solution is

$$\phi(x) = \begin{cases} \int_0^\infty d\mu \int_a^\infty \frac{\tilde{k}_c(\mu) \cos \pi\mu(s-x) + \tilde{k}_s(\mu) \sin \pi\mu(s-x)}{\tilde{k}_s^2(\mu) + \tilde{k}_c^2(\mu) - \lambda [\tilde{k}_s^2 + \tilde{k}_c^2]} f(s) ds \\ \quad \text{if } x > a, \\ 0 \quad \text{if } x < a. \end{cases} \quad (7.4.12)$$

(d) If the interval of integration is $(-\infty, \infty)$, we can apply the convolution theorem of the Fourier integral. Thus, for an FK1 of the form $\int_{-\infty}^\infty k(x-s)\phi(s) ds = f(x)$, let $\tilde{\phi}(u)$, $\tilde{k}(u)$, and $\tilde{f}(u)$ denote the Fourier transform of $\phi(x)$, $k(x)$, and $f(x)$, respectively, and set $1/\tilde{k}(u) = \tilde{\kappa}(u)$. Then the solution of the FK1 is

$$\phi(x) = \int_{-\infty}^\infty \tilde{\kappa}(x-s)f(s) ds. \quad (7.4.13)$$

In the case of an FK2 of the form (7.4.11) set $-\frac{\tilde{k}(u)}{1 - \lambda \tilde{k}(u)}$; then the solution is given by

$$\phi(x) = f(x) - \lambda \int_{-\infty}^\infty k_\lambda(x-s)f(s) ds. \quad (7.4.14)$$

EXAMPLE 7.4.1. Consider the FK2 of the form (7.4.1) on the interval $(0, \infty)$. Divide the domain $x > 0, s > 0$ into four parts, marked by dotted lines $x = 1$ and $s = 1$, and let k_{11} , k_{12} , k_{21} , and k_{22} denote the values of the kernel $k(x-s)$ in these parts. Then the singular equation (7.4.1) becomes

$$\begin{aligned} \phi(x) - \lambda \left\{ \int_0^1 k_{11}(x,s)\phi(s) ds + \int_1^\infty k_{21}(x,s)\phi(s) ds \right. \\ \left. + \int_0^1 k_{12}(x,s)\phi(s) ds + \int_1^\infty k_{22}(x,s)\phi(s) ds \right\} = f(x). \end{aligned}$$

For the integral over the interval $(1, \infty)$, set $1/s = \sigma$; then, for example,

$$\int_1^\infty k_{21}(x,s)\phi(s) ds = \int_0^1 k_{21}\left(x, \frac{1}{\sigma}\right) \phi\left(\frac{1}{\sigma}\right) \frac{d\sigma}{\sigma^2}.$$

We then choose the nodes (lattice points) as in Fig. 7.4.1, say, and solve the system of four simultaneous equations. For example, consider $k(x,s) = e^{-2(x+s)}$ and $f(x) = \frac{1}{x}$. We put

$$\frac{1}{x} = \xi, \quad \frac{1}{s} = \sigma, \quad \phi\left(\frac{1}{\xi}\right) = \Phi(\xi), \quad f\left(\frac{1}{\xi}\right) = F(\xi), \quad (7.4.15)$$

and obtain

$$\Phi(\xi) - \int_0^1 \exp \left[-2 \left(\frac{1}{\xi} + \frac{1}{\sigma} \right) \right] \Phi(\sigma) \frac{d\sigma}{\sigma^2} = \frac{\xi}{\xi + 1}.$$

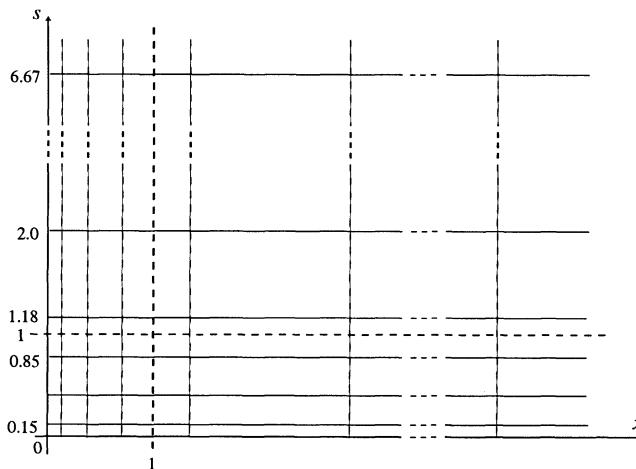


Fig. 7.4.1.

Taking the four representative points (ξ, σ) as in Table 7.4.1, we have

$$\begin{aligned} 4.7\phi_1 + 3.4\phi_2 + 1.5\phi_3 + 1.2\phi_4 &= 9.0(+9), \\ 11.1\phi_1 + 7.1\phi_2 + 3.2\phi_3 + 2.4\phi_4 &= 3.0(+3), \\ 4.6\phi_1 + 3.4\phi_2 + 1.5\phi_3 + 1.1\phi_4 &= 4.0(+2), \\ 14.4\phi_1 + 10.5\phi_2 + 5.4\phi_3 + 3.6\phi_4 &= 5.0(+1). \end{aligned}$$

The solution is given in Table 7.4.1. For computational details, see `singular1.nb`.

Table 7.4.1

ξ	$x = 1/\xi$	$\phi(x)$
0.8973	1.1145	−1.79(−10)
0.5938	1.6841	−7.87(−10)
0.4062	2.4615	−4.13(−10)
0.1029	9.7371	1.04(−10)

7.5. Volterra-Type Singular Equations

A Volterra-type singular equation with an infinite range of integration has the form

$$\phi(x) - \int_{-\infty}^x k(x, s)\phi(s) ds = f(x), \quad -\infty < x < \infty, \quad (7.5.1)$$

where $k(x, s) = 0$ for $x < s$. This equation can be written as (see Kondo 1991, p.278)

$$\phi(x) - \int_x^\infty k(x, s)\phi(s) ds = f(x), \quad (7.5.2)$$

which is another form of a Volterra-type singular equation of the second kind (SVK2). The iterated kernel for this equation is defined by

$$k_n(x, s) = \int_x^\infty k_{n-1}(x, t)k(t, s) ds, \quad (7.5.3)$$

which is obtained by the type 1 convolution $k^{*n} = k^{*(n-1)} * k$ (see Section 1.2). As in Section 1.9, the solution of Eq (7.5.1) or (7.5.2) is given by

$$\phi(x) = f(x) - \int_{-\infty}^x k_\lambda(x, s)f(s) ds \quad (7.5.4)$$

or

$$\phi(x) = f(x) - \int_x^\infty k_\lambda(x, s)f(s) ds, \quad (7.5.5)$$

respectively, provided the integrals on the right side converge, meaning they are uniformly bounded. These solutions are unique.

The METHOD OF SUCCESSIVE APPROXIMATIONS consists of using the change of variables (7.3.15). Then Eq (7.5.1) becomes

$$\Phi(\xi) + \int_0^\xi k\left(\frac{1}{\xi}, \frac{1}{\sigma}\right) \frac{\Phi(\sigma)}{\sigma^2} d\sigma = F(\xi), \quad (7.5.6)$$

which has a singular kernel. Then the following result holds.

THEOREM 7.5.1. (Kondo 1991, p.278) In the singular equation (7.5.1) (or (7.5.2)), if (i) $f(x)$ and $k(x, s)$ are bounded and continuous, (ii) $\int_{-\infty}^x |k(x, s)| ds$

(or $\int_x^\infty |k(x, s)| ds$) exists, and (iii) there exists a value x_0 such that $\int_{-\infty}^x |k(x, s)| ds < N < 1$ always holds for all $x < x_0$, then the successive approximation solutions converge, and the solution of Eq (7.5.1) (or Eq (7.5.2)) exists and is unique.

EXAMPLE 7.5.2. (Kondo 1991, p.282) Consider the singular equation

$$\phi(x) + \int_{-\infty}^x \frac{1}{s^2} \left(\frac{1}{x} - \frac{1}{s} \right) \phi(s) ds = \frac{1}{x}, \quad (7.5.7)$$

which, using the substitutions (7.4.15), becomes

$$\Phi(\xi) - \int_0^\xi \sigma^2 \frac{(\xi - \sigma)}{\sigma^2} \Phi(\sigma) d\sigma = \xi,$$

and its solution is $\Phi(\xi) = \frac{1}{2} (e^\xi - e^{-\xi}) = \sinh \xi$, which gives the solution of Eq (7.5.7) as $\phi(x) = \sinh(1/x)$. ■

Note that for an FK2 of the form (1.2.2) if

$$\int_a^b |k(x, s)| ds \leq q(b - a), \quad (7.5.8)$$

where $q > 0$, then the iterated kernels exist almost everywhere on $[a, b]$ and the series (1.9.8) converges for $|\lambda| < 1/q$ a.e. on $[a, b]$. Let $k_\lambda(x, s)$ denote the resolvent kernel. Then if condition (7.5.8) holds and $\int_a^b |k(x, s)| ds$ is bounded, the solution of the FK2 is given by

$$\phi(x) = f(x) + \lambda \int_a^b k_\lambda(x, s) f(s) ds. \quad (7.5.9)$$

For a proof, see Hille and Tamarkin (1930). This result also holds if we replace condition (7.5.8) by the condition $\int_a^b |k(x, s)| dx \leq q(b - a)$ and assume instead that $\int_a^b f(s) ds$ is bounded. In the case when $f(x)$ is not continuous, we do not expect the solution $\phi(x)$ to be continuous. But if $f(x)$ is continuous, the above conditions on the boundedness of the integrals are not necessary, and the solution is definitely continuous.

The kernel of the solution can be represented in terms of the Fredholm determinant by

$$k_\lambda(x, s) = \frac{D_\lambda(x, s)}{D(\lambda)}, \quad (7.5.10)$$

where

$$\begin{aligned} D_\lambda(x, s) &= k(x, s) \\ &+ \sum_{\nu=1}^{\infty} (-)^{\nu} \frac{\lambda^\nu}{\nu!} \int_a^b \int_a^b \cdots \int_a^b k \begin{pmatrix} x & s_1 & s_2 & \dots, s_\nu \\ s & s_1 & s_2 & \dots, s_\nu \end{pmatrix} ds_1 ds_2 \dots ds_n \end{aligned} \quad (7.5.11)$$

is the Fredholm minor (or minor of the kernel), and $D(\lambda)$ is defined by (1.10.8). For example, if $k(x, s) = xs$, $0 \leq x \leq s \leq 1$, then

$$k \begin{pmatrix} s_1 & s_2 & \dots, s_\nu \\ s_1 & s_2 & \dots, s_\nu \end{pmatrix} = 0, \quad D(\lambda) = 1 - \lambda \int_0^1 s^2 ds = 1 - \frac{\lambda}{3} \quad \text{for } \nu \geq 2.$$

Since all terms in the minor are zero for $\nu \geq 1$, we have $D_\lambda(x, s) = xs$, and for $\lambda \neq 3$ the solution of the FK2 with $f(x) = ax$ is

$$\phi(x) = ax + \frac{\lambda}{1 - \lambda/3} \int_0^1 xsf(s) ds = \frac{3ax}{3 - \lambda}.$$

This method is due to Poincaré (1910).

7.6. Convolution Methods

As we have seen, the most important case of all Volterra-type equations is when the kernel $k(x, s)$ becomes infinitely large at $x = s$, i.e., when the kernel has a factor of the form $(x - s)^{\alpha-1}$, $0 < \alpha < 1$. The solution method involves an iterated kernel in such cases.

7.6.1. Convolution of an SVK1. Let the kernel $k(x, s)$ be defined in the closed triangle domain $T = \{a \leq s \leq x \leq b\}$ and have the form

$$k(x, s) = \frac{(x - s)^{\alpha-1}}{\Gamma(\alpha)} \kappa(x, s), \quad \alpha > 0. \quad (7.6.1)$$

Such a kernel is said to be of *order* α . If $\kappa(x, s)$ is bounded and continuous and $\kappa(x, x) \neq 0$, then $\kappa(x, s)$ is called the *nonsingular factor* of the kernel and $\kappa(x, x)$ its *diagonal*. The kernel $k(x, s)$ is bounded for $\alpha \geq 1$. Hence, we examine only the case $0 < \alpha < 1$. We compute the type 1 convolution (see Section 1.2) of a singular kernel of order α with that of order β , which is defined by $h(x, s) = \frac{(x-s)^{\beta-1}}{\Gamma(\alpha)} \vartheta(x, s)$; thus,

$$k * h = \int_s^x \frac{(x-t)^{\alpha-1} (t-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \kappa(x, t) \vartheta(t, s) dt. \quad (7.6.2)$$

By setting $\frac{x-t}{x-s} = y$, we find that the convolution (7.6.2) becomes

$$\begin{aligned} k * h &= \frac{(x-s)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \kappa(x, x - (x-s)y) \vartheta(x - (x-s)y, s) \\ &\quad \times y^{\alpha-1} (1-y)^{\beta-1} dy \\ &= \frac{(x-s)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} b(x, s), \end{aligned} \quad (7.6.3)$$

where

$$\begin{aligned} b(x, s) &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \kappa(x, x - (x-s)y) \vartheta(x - (x-s)y, s) \\ &\quad \times y^{\alpha-1} (1-y)^{\beta-1} dy. \end{aligned} \quad (7.6.4)$$

Hence we conclude that

- (i) if the order of the singular kernel is less than 1, all operation rules of type 1 convolution, listed in Section 1.2, are valid.
- (ii) The order of the convolution is equal to the sum of the original orders; in the above case it is $\alpha + \beta$, as is obvious from (7.6.3).
- (iii) The diagonal of the convolution is the product of the original function. Note the property of the gamma function $\int_0^1 y^{\alpha-1} (1-y)^{\beta-1} dy = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) + \Gamma(\beta)}$, which we use in the sequel.
- (iv) If $|\kappa| < m_1$ and $|\vartheta| < m_2$, then

$$|k * h| < m_1 m_2 \frac{(x-s)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)}. \quad (7.6.5)$$

Note that since $1^{\star n} = \frac{(x-s)^{n-1}}{(n-1)!} = \frac{(x-s)^{n-1}}{\Gamma(n)}$ and since $1^{\star m} * 1^{\star n} = 1^{\star m+n}$ for all m, n (see Section 1.2), we can deduce that (Kondo 1991, p.310)

$$1^{\star \alpha} = \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}. \quad (7.6.6)$$

Hence, we can represent (7.6.3) by

$$k * h = 1^{\star \alpha+\beta} b(x, s). \quad (7.6.7)$$

7.6.2. Convolution of an SVK2. Consider an SVK2 of the form

$$\phi(x) - \int_0^x k(x, s)\phi(s) ds = f(x), \quad (7.6.8)$$

where the kernel $k(x, s)$ is of the form (7.6.1). If $0 < \alpha < 1$, then from (7.6.5) we have

$$|k^{\star n}| < m^n \frac{(x-s)^{n\alpha-1}}{\Gamma(n\alpha)}. \quad (7.6.9)$$

Thus, if the integer $n > 1/\alpha$, the iterated kernel becomes bounded and continuous, and the series $H \equiv (k + k^{\star 2} + k^{\star 3} + \dots + k^{\star n} + \dots)$ is uniformly convergent. Then the reciprocal theorem $k + H = k * H = H * k$ holds, and the solution of Eq (7.6.8) is given by

$$\phi(x) = f(x) - \int_0^x H(x, s) f(s) ds. \quad (7.6.10)$$

7.6.3. Abel's Equation. This equation is a Volterra-type equation of the first kind with a singular kernel and has the form

$$\int_0^x \frac{\phi(s)}{(x-s)^\alpha} ds = f(x), \quad 0 < \alpha < 1, \quad (7.6.11)$$

where the kernel is defined by

$$k(x, s) = \frac{1}{(x-s)^\alpha} = 1^{\star 1-\alpha} \Gamma(1-\alpha), \quad (7.6.12)$$

where we use the representation (7.6.6). Hence, we can write Eq (7.6.11) in the convolution notation as

$$\Gamma(1-\alpha) 1^{\star 1-\alpha} * \phi = f. \quad (7.6.13)$$

If we multiply both sides of (7.6.13) by $\Gamma(\alpha)$ $1^{*\alpha}$, we get

$$1 \star \phi = \frac{\Gamma(\alpha) \sin \alpha \pi}{\pi} 1^{*\alpha} \star f,$$

or

$$\int_0^x \phi(s) ds = \frac{\sin \alpha \pi}{\pi} \int_0^x (x-s)^{\alpha-1} f(s) ds, \quad (7.6.14)$$

which, when operated by $\frac{d}{dx}$ on both sides, gives the solution

$$\phi(x) = \frac{\sin \alpha \pi}{\pi} \frac{d}{dx} \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} ds. \quad (7.6.15)$$

Also, integrating (7.6.15) by parts, we obtain

$$\phi(x) = \frac{\sin \alpha \pi}{\pi} \frac{d}{dx} \left\{ \frac{f(0)}{x^{-\alpha}} + \int_0^x \frac{f'(s)}{(x-s)^{-\alpha}} ds \right\}, \quad (7.6.16)$$

where for $\alpha = 1/2$ we find from (7.6.16) that

$$\phi(x) = \frac{1}{\pi} \int_0^x \frac{f(s)}{\sqrt{x-s}} ds. \quad (7.6.17)$$

The Laplace transform technique can also be applied to the case of the singular kernel of the form (7.6.12). If we write $k(x, s) \equiv k(x)$, then applying the Laplace transform to both sides of (7.6.12), we get

$$\mathcal{L}\{k(x)\} \equiv \bar{k}(p) = \Gamma(1 - \alpha) p^{\alpha-1},$$

where p is the variable of the transform and the bar over a variable represents its Laplace transform. Thus, Abel's equation (7.6.11) is transformed into

$$\Gamma(1 - \alpha) p^{\alpha-1} \bar{\phi}(p) = \bar{f}(p),$$

or

$$\bar{\phi}(p) = \frac{\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{1}{p^{\alpha-1}} \bar{f}(p) = \frac{\sin \alpha \pi}{\alpha} \frac{1}{p^{\alpha-1}} \bar{f}(p),$$

which, on inversion, gives

$$\phi(x) = \frac{\sin \alpha \pi}{\alpha} \frac{d}{dx} \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} ds,$$

which is (7.6.15). For Abel's equation, also see Section 11.6.

7.6.4. Sonine's Equation. This equation is an SVK1 and has the form

$$K \star \phi = f, \quad (7.6.18)$$

where the kernel operator K is defined by

$$K = 1^{*\alpha} \star (I + a_1 1^* + a_2 1^{*2} + \dots); \quad (7.6.19)$$

here a_1, a_2, \dots are constants. We multiply both sides of (7.6.18) by $1^{*1-\alpha} \star (I + b_1 1^* + b_2 1^{*2} + \dots) \equiv Q$ from the right, which gives

$$1^{*\alpha} \star (I + a_1 1^* + a_2 1^{*2} + \dots) \star \phi \star Q = f \star Q,$$

or

$$1 \star \phi = f \star Q. \quad (7.6.20)$$

This means that

$$\phi(x) = \frac{d}{dx} \int_0^x Q(x-s) f(s) ds, \quad (7.6.21)$$

or

$$\phi(x) = f(0)Q(x) + \int_0^x f'(s)Q(x-s) ds. \quad (7.6.22)$$

The Laplace transform technique can also be applied to Sonine's equation with kernel (7.6.1). In this case we have

$$k(x) \equiv k(x, s) = x^{-\alpha} (1 + a_1 x + a_2 x^2 + \dots + a_n x^n), \quad 0 < \alpha < 1,$$

which, after we apply the Laplace transform, gives

$$\begin{aligned} \bar{k}(p) &= \Gamma(1 - \alpha) p^{\alpha-n-1} \left[p^n + (1 - \alpha)a_1 p^{n-1} + (1 - \alpha)(2 - \alpha)a_2 p^{n-2} \right. \\ &\quad \left. + \dots + (1 - \alpha)(2 - \alpha) \dots (n - \alpha)a_n \right]. \end{aligned}$$

Now, on inversion the solution can be written as

$$\phi(x) = \frac{\sin \alpha \pi}{\pi} F_1 \star f, \quad (7.6.22')$$

where

$$\begin{aligned} F_1(x) &= x^{\alpha-1} + \frac{z_1^{n-\alpha}}{A'(z_1)} \Gamma_\alpha(z_1 x) + \frac{z_2^{n-\alpha}}{A'(z_2)} \Gamma_\alpha(z_2 x) + \dots + \frac{z_n^{n-\alpha}}{A'(z_n)} \Gamma_\alpha(z_n x), \\ A(z) &= z^n + (1 - \alpha)a_1 z^{n-1} + (1 - \alpha)(2 - \alpha)a_2 z^{n-2} \\ &\quad + \dots + (1 - \alpha)(2 - \alpha) \dots (n - \alpha)a_n, \end{aligned}$$

$A'(z_i)$ is the value of $\frac{dA}{dz} \Big|_{z=z_i}$, $i = 1, 2, \dots, n$; z_n are the simple zeros of $A(z)$; and Γ_α is an incomplete gamma function defined by $\Gamma_\alpha(x) = \int_0^x s^{\alpha-1} e^{-s} ds$.

7.6.5. Logarithmic Kernel. A Volterra equation of the first kind of the form $K \star \phi = f$ with kernel

$$k(x, s) = \log(x - s) + c, \quad c = \text{const}, \quad (7.6.23)$$

is called a singular equation with a logarithmic kernel, which becomes unbounded at $x = s$. Consider the type 1 convolution

$$1^{*\alpha} \star 1^{*\beta} = 1^{*\alpha+\beta},$$

that is,

$$\int_s^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} dt = \frac{(x-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}. \quad (7.6.24)$$

Differentiating both sides of (7.6.24) with respect to α and then integrating with respect to β , we get

$$\int_s^x \frac{\partial}{\partial \alpha} \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} dt \int_\beta^\infty \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} d\eta = -\frac{(x-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

where η is the integration variable. Set $\alpha = \beta = 1$, and we have

$$\int_s^x [\log(x-t) - \gamma] dt \int_0^\infty \frac{(t-s)^\beta}{\Gamma(\beta+1)} d\beta = -(x-s), \quad (7.6.25)$$

where η is replaced by $\beta+1$, and $\gamma = \Gamma'(1) = 0.5772156649$ is Euler's constant. Now, if we set

$$G_1(x-s) = \int_0^\infty \frac{(t-s)^\beta}{\Gamma(\beta+1)} d\beta,$$

then we can write (7.6.22) as

$$K \star G_1 = -1^{*2}. \quad (7.6.26)$$

Also, multiplying both sides of (7.6.23) by G_1 from the right and using (7.6.26), we get

$$-1^{*2} \star \phi = G_1 \star f,$$

or

$$\phi(x) = -\frac{d^2}{dx^2} \int_0^x G_1(x-s)f(s) ds. \quad (7.6.27)$$

The Laplace transform technique can be applied to the logarithmic kernel (7.6.23), which gives

$$\bar{k}(p) = -\frac{1}{p} (\log p + c);$$

then

$$\bar{\phi}(p) = -\frac{p}{\log p + c} \bar{f}(p) = -p^2 \left\{ \frac{1}{p(\log p + c)} \bar{f}(p) \right\}.$$

If we put

$$\bar{G}_1(p) = \frac{1}{p(\log p + c)} = \int_0^\infty \frac{1}{p^{\beta+1}} d\beta,$$

on inversion we find that

$$G_1(x) = \int_0^\infty \frac{x^\beta}{\Gamma(\beta+1)} d\beta,$$

since $\mathcal{L}\left\{\frac{x^\beta}{\Gamma(\beta+1)}\right\} = \frac{1}{p^{\beta+1}}$. Then on inversion we find that

$$\phi(x) = -\frac{d^2}{dx^2} \int_0^x G_1(x-s)f(s) ds,$$

which is the same as (7.6.27).

7.7. Asymptotic Methods for Log-Singular Equations

Consider an FK1 of the form

$$\int_{-1}^1 k\left(\frac{x-s}{\lambda}\right) \phi(s) ds = f(x), \quad -1 < x < 1, \quad (7.7.1)$$

where the parameter $\lambda > 0$. Assume that the kernel $k(x, s)$ is an even function that is continuous with respect to x , has a logarithmic singularity as $x \rightarrow \infty$, and decays exponentially as $x \rightarrow -\infty$. Such equations arise in boundary value problems in continuum mechanics with mixed boundary conditions. Let $f' \in H^\alpha$

for $\alpha > 1/2$. Then the solution $\phi(x)$ of Eq (7.7.1) exists, is unique in the class of functions satisfying the Hölder condition for any $\lambda > 0$, and is of the form (see Mushkhelishvili 1992; and also (9.2.2))

$$\phi(x) = \frac{g(x)}{\sqrt{1-x^2}}, \quad (7.7.2)$$

where $g \in C[-1, 1]$ such that $g(\pm 1) \neq 0$ (in general, although $g(\pm 1)$ can be zero in exceptional cases for some special value of λ). Hence, Eq (7.7.1) is bounded as $x \rightarrow \pm 1$. This result shall be used in developing the asymptotic solution in case 2 below when $\lambda \rightarrow 0$. Now, we consider two cases:

CASE 1. SOLUTION FOR LARGE λ . Let the kernel be represented by

$$k(x) = \ln|x| \sum_{n=0}^{\infty} a_n |x|^n + \sum_{n=0}^{\infty} b_n |x|^n, \quad a_0 \neq 0. \quad (7.7.3)$$

This representation for the kernel is valid for Eq (7.7.1) as $x \rightarrow 0$; it implies that in Eq (7.7.1) there exist two totally different large-scale parameters λ and $\ln \lambda$ as $\lambda \rightarrow \infty$, where the latter parameter is known as “quasi-constant”; it grows much slower than the former (e.g., $\lambda = 100$ yields $\ln \lambda = 4.6$, and $\lambda = 1000$ gives $\ln \lambda = 6.9$). We drop all those terms in Eq (7.7.1) which decay as $\lambda \rightarrow \infty$. Then, in view of (7.7.3), for the zeroth approximation ($n = 0$) we have

$$\int_{-1}^1 (a_0 \ln|x-s| - a_0 \ln \lambda + b_0) \phi_0(s) ds = f(x), \quad -1 \leq x \leq 1. \quad (7.7.4)$$

To find an asymptotic solution of Eq (7.7.1) as $\lambda \rightarrow \infty$, we first consider the auxiliary equation

$$\int_{-1}^1 \kappa(x-s, \lambda, \beta) \phi(s) ds = f(x), \quad -1 \leq x \leq 1, \quad (7.7.5)$$

where

$$\kappa(x, \lambda, \beta) = (\ln|x| - \beta) \sum_{n=0}^{\infty} \frac{a_n}{\lambda^n} |x|^n + \sum_{n=0}^{\infty} \frac{b_n}{\lambda} |x|^n, \quad (7.7.6)$$

which has two parameters λ and β , and seek for its solution in the form of a regular asymptotic expansion in negative powers of λ , for fixed β , i.e.,

$$\phi(x, \lambda, \beta) = \sum_{n=0}^{\infty} \lambda^n \phi_n(x, \beta) + O(\lambda^{-n}). \quad (7.7.7)$$

Upon substituting (7.7.7) into (7.7.5), we obtain the recurring integral equations

$$\int_{-1}^1 (a_0 \ln|x-s| - a_0 \beta + b_0) \phi_n(s, \beta) ds = g_n(x, \beta), \quad (7.7.8)$$

which is of the form (7.7.4) and yields the solutions $\phi_n(x, \beta)$ successively, where the right sides depend only on the previously determined successive solutions $\phi_0, \phi_1, \dots, \phi_{n-1}$. Note that for $\beta = \ln \lambda$, Eq (7.7.5) coincides with Eq (7.7.1), where the kernel is represented by (7.7.3). Hence, by using (7.7.7) and (7.7.8) with $\beta = \ln \lambda$, we obtain the required asymptotic solution of Eq (7.7.1).

CASE 2. SOLUTION FOR SMALL λ . In the limiting case $\lambda \rightarrow 0$ we have singularities at the endpoints $x = \pm 1$. We shall consider the following auxiliary system of two equations, corresponding to the endpoints $x = -1$ and $x = 1$, respectively:

$$\begin{aligned} \int_{-1}^{\infty} k\left(\frac{x-s}{\lambda}\right) \phi_1(s) ds - \int_{-\infty}^{-1} k\left(\frac{x-s}{\lambda}\right) \phi_2(s) ds &= f_1(x), \quad -1 \leq x < \infty, \\ \int_{-\infty}^1 k\left(\frac{x-s}{\lambda}\right) \phi_2(s) ds - \int_1^{\infty} k\left(\frac{x-s}{\lambda}\right) \phi_1(s) ds &= f_2(x), \quad -\infty \leq x \leq 1, \end{aligned} \quad (7.7.9)$$

where the functions f_1 and f_2 satisfy the conditions:

$$\begin{aligned} f_1(x) + f_2(x) &= f(x), \quad -1 \leq x \leq 1, \\ f_1(x) &= O(e^{\alpha_1 x}) \quad \text{as } x \rightarrow \infty, \alpha_1 > 0, \\ f_2(x) &= O(e^{\alpha_2 x}) \quad \text{as } x \rightarrow -\infty, \alpha_2 > 0. \end{aligned} \quad (7.7.10)$$

In view of the first condition we shall seek the asymptotic solution $\phi(x)$ as the sum of two solutions of Eq (7.7.9), i.e., $\phi(x) = \phi_1(x) + \phi_2(x)$, $-1 \leq x \leq 1$. The last two conditions in (7.7.10) provide the conditions on ϕ_1 and ϕ_2 as

$$\begin{aligned} \phi_1(x) &= O(e^{-\beta_1 x}) \quad \text{as } x \rightarrow \infty, \beta_1 > 0, \\ \phi_2(x) &= O(e^{-\beta_2 x}) \quad \text{as } x \rightarrow \infty, \beta_2 > 0. \end{aligned} \quad (7.7.11)$$

Since the kernel $k(x)$ is an even function, the free term $f(x)$ in Eq (7.7.1) is an even or odd function. Thus, in the system (7.7.9) we set

$$f_1(x) = \pm f_2(-x), \quad \phi_1(x) = \pm \phi_2(-x). \quad (7.7.12)$$

If we use the change of variables

$$z = \frac{x+1}{\lambda}, \quad \sigma = \frac{s+1}{\lambda}, \quad \omega(\sigma) = \phi(s), \quad F(z) = \frac{f_1(x)}{\lambda},$$

the system (7.7.9) reduces to

$$\int_0^\infty k(z - \sigma) \omega(\sigma) d\sigma = F(z) \pm \int_{2/\lambda}^\infty k(2/\lambda - z - \sigma) \omega(\sigma) d\sigma, \quad 0 \leq z < \infty, \quad (7.7.13)$$

which, by using the first relation in (7.7.8), leads to the asymptotic estimate

$$I(\omega) = \int_{2/\lambda}^\infty k(2/\lambda - z - \sigma) \omega(\sigma) d\sigma = O\left(e^{-2\beta_1/\lambda}\right). \quad (7.7.14)$$

Hence, for small λ we use the iterative scheme

$$\int_0^\infty k(z - \sigma) \omega_n(\sigma) d\sigma = F(z) \pm I(\omega_{n-1}) \quad (7.7.15)$$

to solve Eq (7.7.13) by the method of successive approximations (Section 5.1), where we can omit the integral $I(\omega_0)$. Eq (7.7.15) is a Wiener–Hopf equation of the first kind that can be solved in closed form (see Section 7.4; also Noble 1958). Thus, using (7.7.12) and changing back to the original variables the leading terms of the asymptotic expansion of the solution $\phi = \phi_1 + \phi_2$ of Eq (7.7.1) as $\lambda \rightarrow 0$ are of the form

$$\phi(x) = \omega_1\left(\frac{1+x}{\lambda}\right) \pm \omega_2\left(\frac{1-x}{\lambda}\right), \quad (7.7.16)$$

where $\omega_1 = \omega_1(\sigma)$ is the solution of Eq (7.7.15) with $n = 1$ and $\omega_0 \equiv 0$.

7.8. Iteration Methods

The Nyström form of the product-integration method when applied to the FK2 of the form

$$(I - \lambda K) \phi(x) = f(x), \quad a \leq x \leq b, \quad (7.8.1)$$

yields the linear system (4.2.2), which we rewrite as

$$\phi(x_{in}) = f(x_{in}) + \lambda \sum_{j=0}^n w_{jn} k(x_{in}, s_{jn}) \phi(s_{jn}), \quad (7.8.2)$$

where the w_{jn} are the weights of the quadrature rule used. This system can be written in the operator form at the points $s_{n,i}$, $i = 1, \dots, m$, with both discretizations (m and n , $m \neq n$) as

$$\begin{aligned} (I - \lambda K_n) \phi_n &= f, \\ (I - \lambda K_m) \phi_m &= f, \end{aligned} \quad (7.8.3)$$

which yield

$$(I - \lambda K_n) \phi_m = f + \lambda (K_m - K_n) \phi_m. \quad (7.8.4)$$

We will assume that the inverse operator $(I - \lambda K)^{-1}$ exists and use it to solve Eq (7.8.1) by the iteration method. Now we will discuss the following three iterative schemes.

7.8.1. Atkinson's Scheme. (Atkinson 1976) Assume $m > n$. Then from Eq (7.8.4) define the iterations by

$$(I - \lambda K_n) \phi_{m,p+1} = f + \lambda (K_m - K_n) \phi_{m,p}, \quad p \geq 0, \quad (7.8.5)$$

where $\phi_{m,0}$ is given, and p denotes the number of iterates. Subtracting (7.8.5) from (7.8.4) and then operating by $(I - \lambda K_n)^{-1}$, we get

$$\phi_m - \phi_{m,p+1} = B_{mn} (\phi_m - \phi_{m,p}), \quad p \geq 0, \quad (7.8.6)$$

where

$$B_{mn} = \lambda (I - \lambda K_n)^{-1} (K_m - K_n). \quad (7.8.7)$$

By considering the convergence of (7.8.5), Atkinson (1976, p.146) has shown that $\|B_{mn}^2\| < 1$ for all sufficiently large n . Also, if b_n denotes a bound for the spectral radius of B_{mn} , i.e., a bound for the magnitude of the maximum characteristic value of B_{mn} , then

$$b_n^2 \geq \frac{\|\phi_m - \phi_{m,p+2}\|}{\|\phi_m - \phi_{m,p}\|}, \quad p \geq 0. \quad (7.8.8)$$

If we denote

$$r_{m,p} = f - (I - \lambda K_m) \phi_{m,p}, \quad (7.8.9)$$

then the iteration formula (7.8.5) can be written as

$$\phi_{m,p+1} = \phi_{m,p} + \lambda (I - \lambda K_n)^{-1} r_{m,p}, \quad p \geq 0. \quad (7.8.10)$$

This scheme appears to be a residual correction scheme, where $(I - \lambda K_n)^{-1}$ behaves like an approximate inverse of $(I - \lambda K_m)$. Thus, the *algorithm* for this scheme is as follows:

1. Compute

$$\lambda r_{m,p}(x) = f(x) - \phi_{m,p}(x) + \lambda \sum_{j=1}^m w_{jm} k(x, s_{jm}) \phi_{m,p}(s_{jm})$$

for $x = x_{in}$, $i = 1, 2, \dots, n$, and $x = x_{im}$, $i = 1, 2, \dots, m$;

2. Define $\delta_{p+1} = \lambda (I - \lambda K_n)^{-1} r_{m,p}$;
3. Get $\delta_{p+1}(x_{in})$, $i = 1, 2, \dots, n$, by solving the system

$$\delta_{p+1}(x_{in}) - \lambda \sum_{j=1}^m w_{jm} k(x_{in}, s_{jn}) \delta_{p+1}(s_{jn}) = \lambda r_{m,p}(x_{in});$$

4. Compute

$$\delta_{p+1}(x) = \lambda r_{m,p}(x) + \lambda \sum_{j=1}^n w_{jn} k(x, s_{jn}) \delta_{p+1}(s_{jn})$$

for $x = x_{im}$, $i = 1, 2, \dots, m$;

5. Stop the iteration when

$$\max_{1 \leq i \leq m} |\phi_{m,p+1}(x_{im}) - \phi_{m,p}(x_{im})| \leq \varepsilon,$$

where ε is a preassigned parameter (tolerance).

In this scheme, ignoring the multiplications for the weights, the number of multiplications per iterate is $(m+n)^2 + 2m + n$.

7.8.2. Brakhage's Scheme. (Brakhage 1960) Brakhage considered the problem of solving an SK2 but he felt that solving $(I - \lambda K_n) \delta = \lambda r$ may not be a good idea. This scheme starts from (7.8.4) in which from (7.8.3) we substitute $\phi_m = f + \lambda K_m \phi_m$. This gives

$$(I - \lambda K_m) \phi_m = f + \lambda (K_m - K_n) f + \lambda^2 (K_m - K_n) K_m \phi_m. \quad (7.8.11)$$

Then define the iterations by

$$(I - \lambda K_m) \phi_{m,p+1} = f + \lambda (K_m - K_n) f + \lambda^2 (K_m - K_n) K_m \phi_{m,p}, \quad p \geq 0, \quad (7.8.12)$$

where $\phi_{m,0}$ is given. Subtracting (7.8.12) from (7.8.11) and then multiplying by $(I - \lambda K_n)^{-1}$, we get the Brakhage scheme*

$$\begin{aligned} \phi_m - \phi_{m,p+1} &= C_{mn} (\phi_m / \lambda - \phi_{m,p}), \quad p \geq 0, \\ C_{mn} &= \lambda^2 (I - \lambda K_n)^{-1} (K_m - K_n) K_m. \end{aligned} \quad (7.8.13)$$

*Brakhage (1960) writes this scheme with $\lambda = 1$ as $\phi_{m,p+1} = \phi_{m,p} + r_{m,p} + \delta$, where $r_{m,p} = f - (I - K_m) \phi_{m,p}$ and $\delta = (I - K_n)^{-1} K_m r_{m,p}$.

This scheme converges for sufficiently large n . Atkinson (1976, p.145) gives an equivalent form of formula (7.8.13) as

$$\begin{aligned}\phi_{m,p+1} &= \phi_{m,p} + [I + (I - \lambda K_n)^{-1} K_m] r_{m,p}, \\ \lambda r_{m,p} &= f - (I - \lambda K_m) \phi_{m,p}, \quad p \geq 0.\end{aligned}\tag{7.8.14}$$

Comparing this with Atkinson's scheme, once we obtain $\phi_{m,p}$, we define $r_{m,p}$ in an analogous manner as in Atkinson's scheme. Then we have

$$(I - \lambda K_m) (\phi_m - \phi_{m,p}) = r_{m,p},\tag{7.8.15}$$

$$\phi_m = \phi_{m,p} + \lambda (I - \lambda K_m)^{-1} r_{m,p}.\tag{7.8.16}$$

Formula (7.8.16) is approximated by replacing $(I - \lambda K_m)^{-1}$ by $(I - \lambda K_n)^{-1}$, which yields

$$\phi_{m,p+1} = \phi_{m,p} + \lambda (I - \lambda K_n)^{-1} r_{m,p}.\tag{7.8.17}$$

In this scheme we first regularize (7.8.15) and then at each step solve the correction equation for γ_{p+1} , which is

$$(I - \lambda K_n) \gamma_{p+1} = \lambda K_m r_{m,p}.\tag{7.8.18}$$

Atkinson (1976) notes that although formulas (7.8.13) and (7.8.14) are mathematically equivalent, computationally they are not, because while the number of operations in the former is proportional to m^3 , in the latter it is proportional to m^2 . Thus, formula (7.8.14) is computationally better. The *algorithm* for implementing this scheme is as follows: Given $\phi_{m,p}(x_{im})$, $i = 1, 2, \dots, m$,

1. Compute

$$\lambda r_{m,p}(x) = f(x) - \phi_{m,p}(x) + \lambda \sum_{j=1}^m w_{jm} k(x, s_{jm}) \phi_{m,p}(s_{jm})$$

for $x = x_{in}$, $i = 1, 2, \dots, n$;

2. Define γ_{p+1} as in (7.8.18), and solve for $\gamma_{p+1}(s_{in})$ from the system

$$\begin{aligned}\gamma_{p+1}(s_{in}) - \lambda \sum_{j=1}^n w_{jn} k(s_{in}, s_{jn}) \gamma_{p+1}(s_{jn}) \\ = \lambda \sum_{j=1}^m w_{jm} k(s_{in}, s_{jm}) r_{m,p}(s_{jm}), \quad i = 1, 2, \dots, n;\end{aligned}$$

3. Compute

$$\gamma_{p+1}(x) = \lambda \sum_{j=1}^m w_{jm} j(x, s_{jm}) r_{m,p}(s_{jm}) + \lambda \sum_{j=1}^n w_{jn} k(x, s_{jn}) \gamma_{p+1}(s_{jn})$$

for $x = s_{im}$, $i = 1, 2, \dots, m$;

4. Compute

$$\phi_{m,p+1}(x) = \phi_{m,p}(x) + \lambda r_{m,p}(x) + \lambda \gamma_{p+1}(x)$$

for $x = s_{im}$, $i = 1, 2, \dots, m$. Note that in this scheme there is no need to compute $\phi_{m,p+1}(x)$ for $x = s_{in}$, $i = 1, 2, \dots, n$.

5. Stop the iteration as in Atkinson's scheme.

In this scheme, ignoring the multiplications for the weights and those for computing $k(x, s)$, the number of multiplications per iterate is $2(m + n) + n^2 + 3m$, which is about twice as many multiplications as in Atkinson's scheme.

7.8.3. Atkinson's Direct Scheme: (Atkinson 1976) This scheme is a more direct approach to solve Eq (7.8.1), which, as in Brakhage's scheme, can be written as

$$(I - \lambda K_n) \phi = f + \lambda (K - K_n) \phi + \lambda (K - K_n) K \phi, \quad (7.8.19)$$

or equivalently as

$$\phi = \hat{\phi}_n + G_n \phi, \quad (7.8.20)$$

where

$$\begin{aligned} \hat{\phi}_n &= (I - \lambda K_n)^{-1} [f + \lambda (K - K_n) f], \\ G_n &= \lambda^2 (I - \lambda K_n)^{-1} (K - K_n) K. \end{aligned} \quad (7.8.21)$$

Define

$$\phi_{p+1} = \hat{\phi}_n + G_n \phi_p, \quad p \geq 0, \quad (7.8.22)$$

where ϕ_0 is given. Subtracting (7.8.22) from (7.8.20), we get

$$\phi - \phi_{p+1} = G_n (\phi - \phi_p), \quad p \geq 0. \quad (7.8.23)$$

Although formula (7.8.22) is exactly the same as (7.8.16) with K_m replaced by K , it cannot be computationally carried out because it demands that we know $\phi_p(x)$ a priori for all x in order to compute $\phi_{p+1}(x)$. Therefore, we modify this formula by using a variant M_m for K_m and another variant ψ for ϕ , by defining

$$F_n = \lambda^2 (I - \lambda K_n)^{-1} (K - K_n) M_m, \quad (7.8.24)$$

and by writing the iteration scheme as

$$\psi_{p+1} = \hat{\phi}_n + F_n \psi_p, \quad p \geq 0, \quad (7.8.25)$$

which determines F_n . Note that the notation ψ_{p+1} , ψ_p is used to suggest that the iteration rule for K_m need not be the same as that used for K_n . The scheme (7.8.25) converges if $\|F_n\| < 1$. Now define

$$E_n(x, s) = \int_a^b k(x, t) k(t, s) dt - \sum_{j=1}^n w_{jn} k(x, s_{jn}) k(s_{jn}, s). \quad (7.8.26)$$

Then at each step we solve

$$(I - \lambda K_n) z_{p+1} = \lambda^2 (K - K_n) K_m \psi_p, \quad (7.8.27)$$

where z is a computational variant of ψ . The *algorithm* is as follows:

1. Solve the system

$$\begin{aligned} z_{p+1}(s_{in}) - \lambda \sum_{j=1}^n w_{jn} k(s_{in}, s_{jn}) z_{p+1}(s_{jn}) \\ = \lambda \sum_{j=1}^m w_{jm} E_n(s_{in}, s_{jm}) \psi_p(s_{jm}) \quad \text{for } i = 1, 2, \dots, n; \end{aligned}$$

2. Compute

$$z_{p+1}(x) = \lambda \sum_{j=1}^n w_{jn} k(x, s_{jn}) z_{p+1}(s_{jn}) + \lambda \sum_{j=1}^m w_{jm} E_n(x, s_{jm}) \psi_p(s_{jm})$$

for $x = s_{im}$, $i = 1, 2, \dots, m$;

3. Compute

$$\psi_{p+1}(x) = \hat{\phi}_n(x) + z_{p+1}(x)$$

for $x = s_{im}$, $i = 1, 2, \dots, m$. We need not compute $\psi_{p+1}(x)$ for $x = s_{in}$, $i = 1, 2, \dots, n$.

This scheme is not easier to implement computationally than Brakhage's scheme. If we ignore the multiplications of weights, the total number of multiplications per iterate in this scheme is $(m+n)(m+n+1)$, plus $m(m+n)$ computations for $E_n(x, s)$, where each computation consists of one integration and n multiplications. All this comes to $O(m^2n)$ multiplications per iterate, together with one

numerical integration. In this regard this scheme does not appear to be better than the previous two schemes unless $E_n(x, s)$ can be approximated by a single algorithm. This is provided by a function $E_{mn}(x, s)$, which is an analog to $E_n(x, s)$, depends on both m and n , and is defined by

$$E_{mn}(x, s) = \sum_{j=1}^m w_{jm} k(x, s_{jm}) k(s_{jm}, s) - \sum_{j=1}^n w_{jn} k(x, s_{jn}) k(s_{jn}, s). \quad (7.8.28)$$

Then we have

$$(K_m - K_n) K_n z(x) = \sum_{j=1}^m w_{jm} E_{mn}(x, s_{jm}) z(s_{jm}), \quad (7.8.29)$$

which has the same operator as used in (7.8.12).

As mentioned before, all three schemes can be used with the product-integration form of the Nyström method.

EXAMPLE 7.8.1. (Brakhage 1960, Atkinson 1976) Consider the FK2

$$\phi(x) - \lambda \int_0^a \frac{\phi(s) ds}{1 + (x-s)^2} = f(x), \quad 0 \leq x \leq a,$$

where $f(x)$ is chosen such that $\phi(x) \equiv 1$. The operators K_n are defined by Simpson's rule with $n+1$ quadrature points. Atkinson (1976, p.154–155) provides extensive data for the number of iterates ranging from 2 to 11, with $n = 4, 16, 32$ and $m = 8, 16, 64, 128$ for different values of λ and a . He concludes that (i) this direct scheme is most accurate for well-behaved cases, namely, $a = 1$; (ii) Atkinson's scheme becomes better for large values of λ and converges for a smaller value of n than in Brakhage's and Atkinson's direct schemes; (iii) Brakhage's scheme is better suited for positive values of λ , while Atkinson's is superior for large values of λ . ■

EXAMPLE 7.8.2. (Atkinson 1976, p.156) Consider the FK2

$$\phi(x) - \lambda \int_0^1 k_\rho(x+s)\phi(s) ds = f(x), \quad 0 \leq x \leq 1,$$

where

$$k_\rho(\theta) = \frac{1 - \rho^2}{1 - 2\rho \cos(2\pi\theta) + \rho^2} = 1 + 2 \sum_{j=1}^{\infty} \rho^j \cos(2j\pi\theta), \quad 0 \leq \rho \leq 1.$$

The eigenpairs are $\{\lambda_j = \rho^{-j}, \cos(2j\pi x)\}$ for $j = 0, 1, \dots$ and $\{\lambda_j = -\rho^{-j}, \cos(2j\pi x)\}$ for $j = 1, 2, \dots$. This integral equation is a reformulation of the Dirichlet problem $\nabla^2 u = 0$ on an ellipse; see Example 6.5.1. The operators K_n are defined using the midpoint rule (see Section A.1). Taking $\lambda = 1.25$, we consider two cases: (i) $\phi(x) = 1$, and (ii) $\phi(x) = \sin \frac{2\pi x}{\mu + \rho}$, $\mu = 1/\lambda$, the ratio $r_{m,p} = \frac{\|\phi_{m,p} - \phi_{m,p+1}\|}{\|\phi_{m,p+1} - \phi_{m,p+2}\|}$ for $p = 0, 1, \dots, N-2$, is computed for schemes of Section 7.8.2 and 7.8.3. This ratio is found to be stable for these two schemes, while for Atkinson's scheme this ratio becomes quite oscillatory, which appears to be contrary to the expectations in iteration methods. This oscillatory behavior turns out to be (a) periodic (with period 2) for the function $\phi(x)$ of case (i) with $\lambda = -1.0$, or $\phi(x)$ of case (ii) with $\lambda = 100/99 = 1.010101\dots$; (b) of damped oscillations, eventually becoming stable, for functions $\phi(x)$ of both cases with $a = 10$, $\lambda = -100$, $n = 32$, and $m = 128$; and (c) apparently random oscillation for $\phi(x)$ of case (i) with $\lambda = 100/99$, $n = 32$ and $m = 64$. Atkinson's scheme performs very well for both cases of the function $\phi(x)$ as well as for $\phi(x)$ of case (i) with $\lambda = -1.0$. The difference between Atkinson's and our notation is his use of the symbol λ , which in our notation stands for the characteristic value μ (see beginning of Chapter 2). ■

7.8.4. Kelley's Algorithm. A fast, multilevel algorithm proposed by Kelley (1995) turns out to be half as costly as Brakhage's scheme. This algorithm, based on the discretization of integral equations by composite Gauss rules, is applicable to both linear and nonlinear equations. It is known that given a quadrature rule $Q(F)$ with weights w_j and defined at the quadrature points s_j , $j = 1, \dots, N$, the sequence of operators $\{K_m\}$ bounded on $[0, 1]$ and defined by

$$(K_m \phi)(x) = \sum_{j=1}^N w_j k(x, s_j) \phi(s_j)$$

is collectively compact and converges strongly to the operator K (Anselone 1971). This implies that (i) there exists an n_0 such that for $m \geq n_0$ the operator $(I - \lambda K_m)^{-1}$ is nonsingular and $(I - \lambda K_m)^{-1}$ converges uniformly to $(I - \lambda K)^{-1}$, and (ii) there is an n_0 such that for $m \geq n \geq n_0$ the operator

$$A_{mn} = I + (I - \lambda K_n)^{-1} K_m \quad (7.8.30)$$

satisfies the condition $\|I - A_{mn} (I - \lambda K_m)^{-1}\| \leq \rho$ for any $\rho > 0$. This is a central estimate in Brakhage's scheme because the operator A_{mn} is related to the correction equation (7.8.18) by

$$\gamma_{p+1} = (A_{mn} - I) r_{m,p}. \quad (7.8.31)$$

Symbolically, the two-level iterations in Brakhage's scheme, described in terms of transition from a current state to a new state, are represented as

$$\phi_{\text{new}} = \phi_{\text{current}} - A_{mn} \left[(I - \lambda K_m) \phi_{\text{current}} - f \right].$$

But in the notation of Brakhage's scheme, we can write this as

$$\phi_{m,p+1} = \phi_{m,p} - A_{mn} \left[(I - \lambda K_m) \phi_{m,p} - f \right], \quad (7.8.32)$$

which, in view of (7.8.31), is an alternate form of Eq (7.8.18). Kelley (1995) has noted that $(I - \lambda K_n)^{-1}$ alone will not make an effective approximate inverse, but since the strong convergence of $(I - \lambda K_m)^{-1}$ is not in the operator norm, the effect of convergence of iterations with $(I - \lambda K_n)^{-1}$ is not very efficient. Kelley's algorithm assumes that the quadrature rule at level m is a composite Gauss rule with q_m subintervals I_{rm} , $r = 1, \dots, q_m$, and with $c_m + 1 \geq 1$ points used in each subinterval. Hence, $N = q_m(c_m + 1)$. Consider a sequence of (degenerate) operators $K^{[m]}$ defined by

$$(K^{[m]} \phi)(x) = \int_0^1 k^{[m]}(x, s) \phi(s) ds, \quad (7.8.33)$$

where

$$k^{[m]}(x, s) = \sum_{x_{im}, s_{jm} \in I_{rm}} k(x_{im}, s_{im}) l_{im}(x) l_{jm}(s)$$

is a piecewise polynomial on $[0, 1] \times [0, 1]$ defined for x and s as interior points of the subinterval I_{rm} , and $l_{im}(x)$ is the Lagrange interpolating polynomial of degree c_m on the subinterval I_{rm} , defined by

$$l_{im}(x) = \prod_{\substack{j \neq i \\ x_{jm} \in I_{rm}}} \frac{x - x_{jm}}{x_{im} - x_{jm}}.$$

Let V^m denote the subspace of L^∞ , consisting of piecewise polynomials $k^{[m]}(x, s)$ of degree $c_m + 1$ on each of the m subintervals I_{rm} , and assume that $V^m \subset V^n$ for $m \leq n$. In order to obtain the two-level iteration, we seek to compute an approximate solution $\tilde{\phi}_m \in V^m$ of the SK2 by using the iteration (7.8.17), which is simply a Richardson iteration using $(I - \lambda K_n)^{-1}$ as an approximate inverse for $(I - \lambda K_n)$. Thus, if we write (7.8.17) in an equivalent form as

$$\phi_{m,p+1} = \phi_{m,p} - (I - \lambda K_n)^{-1} ((I - \lambda K_m) \phi_{m,p} - f), \quad (7.8.34)$$

by using the value of $r_{m,p}$ from (7.8.14), then the additional computation of the action of K_m that was needed in computing A_{mn} in (7.8.30) is not required in (7.8.34). Thus, since $V^m \subset V^{m+1}$, the kernel k is continuous, $(I - \lambda K)$ is nonsingular, and the sequence of composite Gauss rules of the form $\int_0^1 F(s) ds = \sum_{j=1}^{N_m} w_{jm} F(x_{jm})$ applies for $N_m = q_m(c_m + 1)$; so the iteration, given by

$$\phi_{m,p+1} = \phi_{m,p} - \left(I - \lambda K_n^{[m]} \right)^{-1} \left(\phi_{m,p} - K_{p+1}^{[m]} \phi_{m,p} - f \right), \quad (7.8.35)$$

converges to $\tilde{\phi} = (I - \lambda K)^{-1} f$ for sufficiently large p . Note that the computation of $\phi_{m,p+1}$ requires only a single evaluation of the action of $K_{p+1}^{[m]}$, which is the computation of $K_{p+1}^{[m]} \phi_{m,p}$ on the right side of (7.8.35). The *algorithm* for this method is as follows:

1. Start by solving $\phi_n = \lambda K_n \phi_n - f$;
2. Given $\phi_{m,p}$, first compute $r_{m,p} = f - \left(I - \lambda K_{p+1}^{[m]} \right) \phi_{m,p}$;
3. Next, solve $\psi - \lambda K_n^{[m]} \psi = r_{m,p}$ for all m ;
4. Finally, compute $\phi_{m,p+1} = \phi_{m,p} - \psi$.

The details for step 3 are as follows: Note that since $K_m^{[m]} = K_m^{[m]} \Pi_m = P_m K_m^{[m]} \Pi_m$ for all m , solve the equation in step 3 by first computing $\Pi_m \psi$ by solving

$$\Pi_n \psi - K_n^{[m]} \Pi_n \psi = \Pi_n r_{m,p}, \quad (7.8.36)$$

and then recovering ψ by using the interpolation $\psi = r_{m,p} + K_n^{[m]} \Pi_n \psi$. The solution of (7.8.36) can be reduced to solving the system

$$\phi(x_{in}) - \lambda \sum_{j=1}^{N_n} w_{jn} k(x_{in}, s_{jn}) = (P_n f)(x_{in}) = f(x_{in}). \quad (7.8.37)$$

The multilevel form of this algorithm refines the mesh as the iteration progresses. A comparison of this scheme with Brakhage's scheme shows that here we use $(I - \lambda K_n^{[m]})^{-1}$ as an approximate inverse of $(I - \lambda K_n)$ for sufficiently large n . All computations in the two examples given below involve the midpoint rule on $[0, 1]$ with $N_m = q_m = 2^m$, and $x_{im} = (i - 1/2)/N_m$, and $w_{ij} = 1/N_m$.

EXAMPLE 7.8.3. (Kelley 1995) One of the examples solved is the SK2

$$\phi(x) - \lambda \int_0^1 \frac{\phi(s) ds}{1 + (x - s)^2} = f(x), \quad 0 \leq x \leq 1,$$

with $\lambda = 1$ and $\lambda = 100$. The results obtained show a residual of at most $0.3(-7)$ with $\lambda = 1$ and of $0.32(-4)$ with $\lambda = 100$, both for $N = 1024$. ■

EXAMPLE 7.8.4. (Kelley 1995) The other example deals with the nonlinear equation

$$H(x) = \left(I - \frac{c}{2} \int_0^1 \frac{xH(s)}{x+s} ds \right)^{-1},$$

which is the Chandrasekhar H -equation (Chandrasekhar 1950). This equation has two solutions for $0 < c < 1$, only one of which is physically valid. There is a simple singularity (Kelley 1982) and, therefore, the solution $\phi(x)$ has a singular Fréchet derivative when $c = 1$. The solution also has a logarithmic singularity at $x = 0$, which lends only first-order accuracy. The solutions for $c = 0.5$ and $c = 0.99$ are given, with a residual of $0.94(-7)$ in the former value of c and of $0.17(-6)$ in the latter, with $N = 1024$. Computations in both examples are done for $N = 2^3, \dots, 2^{10}$ using the computer code GMRES (Saad and Schultz 1986) running SUN OS 4.1 in SUNFortran 1.3.1. Further details can be found in Kelley (1982, 1995). ■

7.9. Singular Equations with the Hilbert Kernel

We now discuss singular equations with the Hilbert kernel of both the first and second kinds.

7.9.1. First-Kind Equations. The simplest form of a singular equation of the first kind with the Hilbert kernel (HSK1) is

$$\frac{1}{2\pi} \int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) \phi(s) ds = f(x), \quad 0 \leq x \leq 2\pi, \quad (7.9.1)$$

where $\phi(x)$ is the unknown function on $0 \leq x \leq 2\pi$, $\cot\left(\frac{s-x}{2}\right)$ is the Hilbert kernel, and $f(x)$ is the free term defined on $0 \leq x \leq 2\pi$. To solve this equation, we need the additional condition

$$\int_0^{2\pi} \phi(x) dx = 0. \quad (7.9.2)$$

Since $\int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) dx = 0$, if we integrate (7.9.1) with respect to x from 0

to 2π , we find that the solvability condition for Eq (7.9.1) to have a solution is

$$\int_0^{2\pi} f(x) dx = 0. \quad (7.9.3)$$

This means that Eq (7.9.1) will have a solution only if condition (7.9.3) is satisfied. Now, to determine this solution we use the solution of an SK1C (see Section 13.1), where Γ is taken as the unit circle with its center at the origin. If we write the SK1C (7.9.2) as

$$\frac{1}{\pi} \oint_{\Gamma} \frac{\phi_1(s)}{s-x} ds = f_1(x), \quad (7.9.4)$$

which has the solution

$$\phi_1(x) = -\frac{1}{\pi} \oint_{\Gamma} \frac{f_1(s)}{s-x} ds, \quad (7.9.5)$$

and set $x = e^{i\theta}$ and $s = e^{i\sigma}$, we find a relation between the Cauchy and Hilbert kernels given by

$$\frac{ds}{s-x} = \frac{1}{2} \cot\left(\frac{\sigma-\theta}{2}\right) d\sigma + \frac{i}{2} d\sigma. \quad (7.9.6)$$

Thus, substituting (7.9.6) into both Eq (7.9.4) and the solution (7.9.5), and changing the variables $\phi(\theta) = \phi_1(x)$ and $f(\theta) = f_1(x)$, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \cot\left(\frac{\sigma-\theta}{2}\right) \phi(\sigma) d\sigma + \frac{i}{2\pi} \int_0^{2\pi} \phi(\sigma) d\sigma = f(x), \quad (7.9.7)$$

$$\phi(\sigma) = -\frac{1}{2\pi} \int_0^{2\pi} \cot\left(\frac{\sigma-\theta}{2}\right) f(\sigma) d\sigma - \frac{i}{2\pi} \int_0^{2\pi} f(\sigma) d\sigma. \quad (7.9.8)$$

Since Eq (7.9.1) with the additional requirement (7.9.2) coincides with Eq (7.9.7), the solution of Eq (7.9.1) is given by (7.9.8), which, in view of the solvability condition (7.9.3), becomes

$$\phi(\sigma) = -\frac{1}{2\pi} \int_0^{2\pi} \cot\left(\frac{\sigma-\theta}{2}\right) f(\sigma) d\sigma. \quad (7.9.9)$$

Eqs (7.9.1) and its solution (7.9.9), together with conditions (7.9.2) and (7.9.3), are known as the *Hilbert inversion formula*.

An examples of HSK1 is

$$\int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) \phi(s) ds = f(x), \quad 0 \leq x \leq 2\pi,$$

where f satisfies condition (7.9.3). Its solution is

$$\phi(x) = -\frac{1}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) f(s) ds + C,$$

where C is an arbitrary constant. This solution implies that $\int_0^{2\pi} \phi(s) ds = 2\pi C$. This equation and its solution form a Hilbert transform pair in symmetric form.

7.9.2. Second-Kind Equations. A singular equation of the second kind with the Hilbert kernel (HSK2) is of the form

$$a(x) \phi(x) + \frac{1}{2\pi} \int_0^{2\pi} g(x, s) \cot\left(\frac{s-x}{2}\right) \phi(s) ds = f(x), \quad (7.9.10)$$

where the functions $a(x)$, $g(x, s)$, and $f(x)$ are known, and all satisfy the Hölder condition on the interval $[0, 2\pi]$, with $g(x, s)$ satisfying this condition in both variables. If we write the kernel of Eq (7.9.10) as

$$\begin{aligned} g(x, s) \cot\left(\frac{s-x}{2}\right) &= [g(x, s) - g(x, x)] \cot\left(\frac{s-x}{2}\right) \\ &\quad + g(x, x) g(x, s) \cot\left(\frac{s-x}{2}\right), \end{aligned}$$

and use the notation

$$\begin{aligned} g(x, x) &= -b(x), \\ [g(x, s) - g(x, x)] \cot\left(\frac{s-x}{2}\right) &= 2\pi k(x, s), \end{aligned} \quad (7.9.11)$$

then Eq (7.9.10) is transformed into

$$a(x) \phi(x) - \frac{b(x)}{2\pi} \int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) \phi(s) ds + \int_0^{2\pi} k(x, s) \phi(s) ds = f(x). \quad (7.9.12)$$

This is an equivalent form of an HSK2 that is often used. It is obvious from Eq (7.9.12) that the function $b(x)$ satisfies the Hölder condition on the interval $[0, 2\pi]$, and the kernel $k(x, s)$ satisfies the Hölder condition in both variables on the same interval except at points $x = s$, where we have the estimate

$$|k(x, s)| < \frac{A}{|s-x|^\alpha}, \quad (7.9.13)$$

with A as a positive constant, and $0 \leq \alpha < 1$. The functions $a(x)$ and $b(x)$ are called the coefficients of the HSK2, the function $\cot\left(\frac{s-x}{2}\right)$ the kernel of this equation, and $f(x)$ the free term. The first two terms on the left side of Eq (7.9.12) are called the *characteristic part*, the third term the *regular part*, and the function $k(x, s)$ the *kernel of the regular part* of the equation. Also, Eq (7.9.12) without its regular part, namely, the equation

$$a(x) \phi(x) - \frac{b(x)}{2\pi} \int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) \phi(s) ds = f(x), \quad (7.9.14)$$

is called the *characteristic equation* associated with the HSK2 (7.9.12).

An example of HSK2 is

$$A \phi(x) - \frac{B}{2\pi} \int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) \phi(s) ds = f(x), \quad 0 \leq x \leq 2\pi,$$

where, without loss of generality, we assume that $A^2 + B^2 = 1$. It has the solution

$$\phi(x) = A f(x) + \frac{B}{2\pi} \int_0^{2\pi} \cot\left(\frac{s-x}{2}\right) f(s) ds + \frac{B^2}{2\pi A} \int_0^{2\pi} f(s) ds.$$

7.10. Finite-Part Singular Equations

We apply the product integration method to equations of the form

$$\phi(x) = f(x) + \int_a^b \frac{\hat{k}(x, s)}{(x-s)^\alpha} \phi(s) ds, \quad a < x < b, \alpha > 1, \quad (7.10.1)$$

where the integration is in the finite-part or Hadamard sense (see Appendix D; also Davis and Rabinowitz 1984, pp. 11–13). Equations of the above form are strongly singular and are encountered in fracture mechanics, gas radiation, and fluid flow (Kaya and Erdogan 1987). As before, we partition the interval $[a, b]$ into n parts by $a = s_1 < s_2 < \dots < s_{n+1} = b$. On the subinterval $[s_j, s_{j+2}]$ we interpolate $\hat{k}(x, s)\phi(s)$ by a quadratic polynomial. Then we use Nyström method by taking $x = s_i$, $i = 1, \dots, n+1$, and approximate the integral by a sum of the form

$\sum_{j=1}^n w_{ij}(x) \hat{k}(s_i, s_j) \phi(s_j)$, as in (6.2.1), where in this case the weights w_{ij} are given by (see Bertram and Ruehr 1992)

$$\begin{aligned} w_{i0} &= 2n - \frac{1}{x} + n^2 \left(x - \frac{3}{2n} \right) \ln \frac{\left| \frac{2}{n} - x \right|}{x}, \\ w_{i,2j} &= 4n + 3n \ln \left| \frac{j}{n} - x \right| + n^2 \left(x - \frac{2j+3}{n} \right) \\ &\quad - n^2 \left(x - \frac{2j-3}{2n} \right) \ln \left| \frac{j-2}{n} - x \right|, \quad j = 1, \dots, n-1, \\ w_{i,2j+1} &= -4n + 2n^2 \left(x - \frac{j}{n} \right) \ln \frac{\left| x - \frac{j-1}{n} \right|}{\left| x - \frac{j+1}{n} \right|}, \quad j = 1, \dots, n-1, \\ w_{i,n} &= 2n + \frac{1}{x-1} + n^2 \left(x - \frac{2n-3}{2n} \right) \ln \frac{|1-x|}{\left| \frac{n-2}{n} - x \right|}. \end{aligned}$$

This leads to a linear $(2n-1) \times (2n-1)$ system that can be solved to yield approximate values $\tilde{\phi}$ at s_2, \dots, s_n .

EXAMPLE 7.10.1. (Bertram and Ruehr 1992) We shall consider the case when $\alpha = 2$. Then $p(x, s) = \frac{\hat{k}(x, s)}{(x-s)^2}$, and with $a = 0$, $b = 1$ Eq (7.10.1) becomes

$$\phi(x) = f(x) + \int_0^1 \frac{\phi(s)}{(x-s)^2} ds. \quad (7.10.2)$$

Table D.7 gives the Hadamard transform $H_2[\phi]$ defined by

$$H_2[\phi] = \int_0^1 \frac{\phi(s)}{(x-s)^2} ds \quad (7.10.3)$$

for functions $\phi(x)$ that exhibit different types of singularities, as in (1)–(3) where $\phi(x) = x^\beta(1-x)^\gamma$ for rational β and γ ; in (4) internal (jump) singularities; in (5)–(6) singularities (jump) in the first derivative, and those in (7) in the second derivative; we have also included a smooth function $\phi(x) = \sin \pi x$ in (8). Note that in the finite-part (Hadamard) sense the infinite values of poles and logarithms should be discarded in the second column of this table, and according to a convention in finite-part interpretation we take $\ln(0) = 0$.

Let $f = \phi - H_2[\phi]$. Then we can determine the error for any approximate solution $\tilde{\phi}$. Bertram and Ruehr (1992) have postulated that for sufficiently large n the error at the n th approximation is given by

$$e_n = |\phi(x) - \tilde{\phi}_n(x)| = \frac{A}{n^p} + \frac{B}{n^{p+1}} + O\left(\frac{1}{n^{p+2}}\right), \quad (7.10.4)$$

where n is the number of subintervals of $[a, b]$ used in the above method. ■

REFERENCES USED: Anselone (1971), Atkinson (1976), Baker (1978), Bertram and Ruehr (1992), Brakhage (1960), Chandrasekhar (1950), Davis and Rabinowitz (1967, 1984), Delves and Mohamed (1985), Goursat (1964), Hille and Tamarkin (1930), Kaya and Erdogan (1987), Kelley (1982, 1995), Kondo (1991), Mikhlin (1960), Mushkhelishvili (1992), Noble (1958), Poincaré (1910), Polyanin and Manzhirov (1998), Saad and Schultz (1986).

8

Weakly Singular Equations

In this chapter we discuss some methods for finding approximate solutions of weakly singular integral equations of the Fredholm and Volterra types.

8.1. Weakly Singular Kernel

For a weakly singular kernel of the form

$$k(x, s) = \frac{g(x, s)}{|x - s|^\alpha}, \quad 0 < \alpha < 1, \quad (8.1.1)$$

the iterated kernels of more than a certain degree become bounded. We shall show that the type 2 convolution $k \circ k$ is bounded (for a definition of type 2 convolution see Section 1.2). Suppose that $k_1(x, s)$ and $k_2(x, s)$ are two distinct weakly singular kernels such that

$$k_1(x, s) = \frac{g_1(x, s)}{|x - s|^{\alpha_1}}, \quad k_2(x, s) = \frac{g_2(x, s)}{|x - s|^{\alpha_2}}, \quad 0 < \alpha_{1,2} < 1.$$

Then their type 2 convolution is

$$k_1 \circ k_2 = \int_a^b \frac{g_1(x, t) g_2(t, s)}{|x - t|^{\alpha_1} |t - s|^{\alpha_2}} dt, \quad (8.1.2)$$

which, if $|g_1(x, s)| \leq m_1$ and $|g_2(x, s)| \leq m_2$, yields the bound

$$|k_1 \circ k_2| \leq m_1 m_2 \int_a^b \frac{dt}{|x - t|^{\alpha_1} |t - s|^{\alpha_2}}. \quad (8.1.3)$$

Now, if $\alpha_1 + \alpha_2 < 1$ and $x < s$, the integral on the right side of (8.1.3) is

$$\begin{aligned} \int_a^x \frac{dt}{|x-t|^{\alpha_1} |t-s|^{\alpha_2}} &< \int_a^x \frac{dt}{|x-t|^{\alpha_1+\alpha_2}} = \frac{|x-a|^{-(\alpha_1+\alpha_2-1)}}{\alpha_1 + \alpha_2 - 1}, \\ \int_x^s \frac{dt}{|x-t|^{\alpha_1} |t-s|^{\alpha_2}} &< \max \left\{ \int_x^s \frac{dt}{|t-s|^{\alpha_1+\alpha_2}}, \int_x^s \frac{dt}{|x-t|^{\alpha_1+\alpha_2}} \right\} \\ &= -\frac{1}{\alpha_1 + \alpha_2 - 1} \max \left\{ |t-a|^{-(\alpha_1+\alpha_2-1)}, |t-b|^{-(\alpha_1+\alpha_2-1)} \right\}, \\ \int_s^b \frac{dt}{|x-t|^{\alpha_1} |t-s|^{\alpha_2}} &< \int_s^b \frac{dt}{|t-s|^{\alpha_1+\alpha_2}} = -\frac{|b-s|^{-(\alpha_1+\alpha_2-1)}}{\alpha_1 + \alpha_2 - 1}, \end{aligned}$$

and since these integrals are bounded, so is $k_1 \circ k_2$.

But if $\alpha_1 + \alpha_2 > 1$, we set $\frac{x-t}{x-s} = y$, which yields

$$\int_a^b \frac{dt}{|x-t|^{\alpha_1} |t-s|^{\alpha_2}} < \frac{1}{|x-s|^{\alpha_1+\alpha_2-1}} \int_{-\infty}^{\infty} \frac{dy}{|y|^{\alpha_1} |1-y|^{\alpha_2}},$$

which is obviously bounded, and so is $k_1 \circ k_2$.

For a weakly singular kernel the resolvent is given by

$$k_\lambda(x, s) = \frac{D_\lambda(x, s)}{D(\lambda)}, \quad (8.1.4)$$

and the solution of the SK2 with a weakly singular kernel of the form (8.1.1) is given by

$$\phi(x) = f(x) + \lambda \int_a^b k_\lambda(x, s) f(s) ds. \quad (8.1.5)$$

For weakly singular equations the quadrature rules described in Section A.1 are not applicable because $k(x, s)$ is unbounded if we assume that $g(x, s) \neq 0$ in (7.1.4). However, the product-integration method can be used successfully for such equations. A modified quadrature method described in Section A.3 can be extended to solve weakly singular equations (see Section 8.4). Since $k(x, s)$ is weakly singular, we have unbounded values in Eq (A.3.19). Now, set

$$\left\{ 1 + \lambda \Delta^*(s_j) \right\} \tilde{\phi}(s_i) - \lambda \sum_{\substack{j=0 \\ j \neq i}} v_j k(s_i, s_j) \tilde{\phi}(s_j) = f(s_i), \quad (8.1.6)$$

where

$$\Delta^*(s_i) = \sum_{\substack{j=0 \\ j \neq i}} v_j k(s_i, s_j) - \int_a^b k(s_i, s) ds. \quad (8.1.7)$$

This means that we have set $k(x, s) \{\phi(s) - \phi(x)\} = 0$ at the singularity $x = s$, as we can see from (A.3.15). Finally, from the computational point of view, the Galerkin or collocation methods are applicable to weakly singular equations (see Baker 1978, p.532).

8.2. Taylor's Series Method

This method has been discussed in Section 3.1.2. We shall use formula (3.1.15) to solve a weakly singular equation. Although the method has some merits, it sometimes gives good approximate solutions for nonsingular equations except near the endpoints. The following example shows that this method is not suitable for weakly singular equations.

EXAMPLE 8.2.1. (Ren et al. 1999) Consider the weakly singular FK2

$$\phi(x) - \lambda \int_0^1 k(x, s) \phi(s) ds = f(x), \quad 0 < x < 1,$$

with $\lambda = 0.1$, and $k(x, s) = (x - s)^{-1/3}$, where $f(x)$ is chosen such that $\phi(x) = x^2(1 - x)^2$. The solution for $n = 0$ is defined by $\tilde{\phi}_0(x) = f(x)/A_0(x)$, and the results are given in Table 8.2.1. Although the exact solution $\phi(x) = 0$ at the endpoints $x = 0, 1$, the approximate solution $\tilde{\phi}_0(x)$ is undefined at these points. Also, since the approximations for $n \geq 1$ require derivatives that are hypersingular, we will not obtain accurate approximate solutions for $n \geq 1$. For computational details, see `renweakly.nb`.

Table 8.2.1

x	$\tilde{\phi}_0(x)$	$\phi(x)$
0.0	—	0.0
0.1	0.0089014	0.0081
0.2	0.0216403	0.0256
0.3	0.0352472	0.0441
0.4	0.0440893	0.0576

Table 8.2.1 Continued

0.5	0.0584076	0.0625
0.6	0.0503979	0.0576
0.7	0.0416969	0.0441
0.8	0.0188141	0.0256
0.9	0.00264399	0.0081
1.0	—	0.0 ■

8.3. L_p -Approximation Method

The L_p -approximation method is related to other well-known methods for solving singular integral equations by the notion that the minimization of the L_2 -norm of the residue in other norms is equivalent to a weighted Galerkin method. We consider an SK2 of the form

$$\phi(x) - \int_a^b k(x, s) \phi(s) ds = f(x), \quad a \leq x \leq b, \quad (8.3.1)$$

where the kernel $k(x, s)$ is singular. Let $\{\psi_0, \psi_1, \dots, \psi_N\}$ denote a set of linearly independent trial functions. The solution $\phi(x)$ can be approximated in a linear (or nonlinear) way; in the linear case, e.g., this can be done by a finite series

$$\phi(x) = \sum_{j=0}^N \alpha_j \psi_j(x), \quad (8.3.2)$$

where $\{\alpha_0, \alpha_1, \dots, \alpha_N\} \equiv A$ is the set of $(n + 1)$ coefficients to be determined. Then the residual $r(A, x)$ is defined by

$$r(A, x) = \sum_{j=0}^N \left[\psi_j(x) + \int_a^b k(x, s) \psi_j(s) ds \right] \alpha_j - f(x), \quad a \leq x \leq b. \quad (8.3.3)$$

As in the Galerkin method, we must now choose the coefficients α_j in such a manner that the residual function becomes small in some sense. This is accomplished by minimizing the L_p -norm of the residual function with respect to the coefficients α_j . Thus, the approximation problem reduces to that of determining

$$\min_A \left(\int_a^b |r(A, x)|^p dx \right)^{1/p}. \quad (8.3.4)$$

Suppose that the coefficients \tilde{A} are obtained as a result of this minimization problem. Then \tilde{A} will provide the approximate solution $\tilde{\phi}$ of the SK2 (8.3.1) which can be written in the form of the series (8.3.2). Two computational steps are required in this method: (a) a procedure to compute \tilde{A} , and (b) a numerical integration procedure to compute the L_p -norm. Some general remarks about these two procedures are as follows: (i) The initial value in the minimization process (a) is chosen from previously computed values using smaller N ; (ii) when moving from a smaller to a larger value of p , the solution \tilde{A} of (8.3.4) computed for smaller p makes a good initial guess in solving (8.3.4) for larger p ; and (iii) for procedure (b) we shall use an adaptive integration technique so that the zeros can be determined numerically; for this purpose de Klerk, Eyre and Venter (1995) use a modification of the standard adaptive integration routine QAGE from the Quadpack subroutine package (see Piessens, de Donker-Kapenga, Überhuber and Kahaner 1983), as developed by Laurie and Venter (1993) for automatic quadrature of integrands of the form $|f(x)|$. This computer code evaluates the zeros of the function which are then used as quadrature points.

Other details of these procedures will be explained in terms of some examples given below. However, certain guiding principles are as follows: Employ different values of p in the approximation; while constructing the approximating function, make a good choice of the trial functions so that they should have the same characteristics, as far as possible, as the unknown solution $\phi(x)$; in this respect choose functions that can represent, for instance, the free term or some unknown asymptotic behavior of the analytic solution $\phi(x)$.

Note that in the case of an FK2 and its related linear combination of basis functions of the type (8.3.2), the L_2 -norm of the residual function is equivalent to the Galerkin method (Section 5.2).

EXAMPLE 8.3.1. (Kaneko and Xu 1991; de Klerk, Eyre and Venter 1995)
Consider the SK2

$$\frac{3\sqrt{2}}{4} \phi(x) - \int_0^1 \frac{\phi(s) ds}{\sqrt{|x-s|}} = f(x), \quad 0 \leq x \leq 1,$$

with $f(x) = 3(x(1-x))^{3/4} - 3\pi(1+4x(1-x))/8$; it has the solution $\phi(x) = 2\sqrt{2}(x(1-x))^{3/4}$ for $x \in [0, 1]$. We take the approximating function as (8.3.2), which gives the residual function

$$r(A, x) = \sum_{j=0}^N \alpha_j \left[\frac{3\sqrt{2}}{4} x^j - G_j(x) \right] - f(x), \quad (8.3.5)$$

where

$$G_j(x) = \int_0^1 \frac{s^j \, ds}{\sqrt{|x-s|}}. \quad (8.3.6)$$

The integral in (8.3.6) can be computed either numerically or analytically. Note that analytical integration saves a considerable amount of computational time. Thus, in this case we have the recurrence relation

$$\begin{aligned} G_0(x) &= 2(\sqrt{x} + \sqrt{1-x}), \\ G_j(x) &= \frac{2}{2j+1} (jx G_{j-1}(x) + \sqrt{1-x}), \quad j = 1, \dots, N. \end{aligned} \quad (8.3.7)$$

Thus, the problem reduces to that of minimizing the integral $\left(\int_0^1 |r(A, x)|^p \, dx \right)^{1/p}$, where $r(A, x)$ is given by (8.3.5). Tables 8.3.1(a) and (b) give the best results, as computed by Kaneko and Xu (1991), in the L_p -norm for different values of p .

Table 8.3.1(a). Coefficients of $F(A, x)$

N	$p = 1$	$p = 2$	$p = 4$	$p = 8$	$p = 16$
1	0.729875	0.743128	0.781953	0.823502	0.853554
	0.000001	0.000000	0.000000	0.000000	0.000000
2	0.073169	0.037902	0.005074	-0.013321	-0.021027
	3.8687774	4.067628	4.249036	4.347200	4.386523
	-3.868776	-4.067625	-4.249036	-4.347200	-4.386523

Table 8.3.1(b). Minimum L_p -norms of $r(A, x)$

N	$p = 1$	$p = 2$	$p = 4$	$p = 8$	$p = 16$
1	0.036689	0.079033	0.134422	0.189447	0.228436
2	0.005923	0.008215	0.010976	0.013203	0.014582 ■

8.4. Product-Integration Method

In some problems the integral in an FK2 is singular in the sense that one or more derivatives of $k(x, s)$ have singularity within the interval of integration. In such a case we write

$$k(x, s) = u(x, s) \kappa(x, s), \quad (8.4.1)$$

where $u(x, s)$ is singular in (a, b) and $\kappa(x, s)$ is a regular function of its arguments. Here we assume that this factorization of $k(x, s)$ is feasible; otherwise the method does not work. In fact, u need not be unbounded at $x = s$ to create a singularity; u can be a function like $u(x, s) = \sqrt{x - s}$, where all derivatives with respect to s are unbounded at $x = s$, and p is thus singular in the above sense. Two examples of such integral equations are of the form

$$\phi(x) = f(x) + \lambda \int_a^b \frac{\kappa(x, s)}{|x - s|^\alpha} \phi(s) ds, \quad a < x < b, \quad 0 < \alpha < 1, \quad (8.4.2)$$

$$\phi(x) = f(x) + \lambda \int_a^b \ln|x - s| \kappa(x, s) \phi(s) ds, \quad a < x < b. \quad (8.4.3)$$

Thus, let the FK2 be written as

$$\phi(x) = f(x) + \int_a^b u(x, s) \kappa(x, s) \phi(s) ds. \quad (8.4.4)$$

Then to approximate the integral in (8.4.4) we use the *product-integration* method; the basic ideas about implementation of this method are already presented in Section 4.2, but because of the singularity occurring in the kernel this method is adapted to this situation by using the quadrature rules of Section A.3 in the above two cases as follows.

CASE 1. When $k(x, s) = |x - s|^{-\alpha}$, $0 < \alpha < 1$, we take $\kappa(x, s) = 1$, and partition the interval $[a, b]$ into subintervals by choosing quadrature points $s_j = a + jh$, $h = (b - a)/n$, $j = 0, 1, \dots, n$. However, the points s_j need not be equally spaced. Then use the quadrature rule

$$\int_a^b k(x, s) \phi(s) ds = \sum_{j=0}^n v_j \phi(s_j), \quad (8.4.5)$$

where the weights v_j can be chosen as described in Section A.3. This rule is exact if the function $\phi(x)$ is either piecewise constant in $[a, b]$, i.e., constant in each subinterval $[s_j, s_{j+1}]$, $j = 0, 1, \dots, n$, or continuous and piecewise linear in $[a, b]$, or a smooth function, i.e., a polynomial of degree n on $[a, b]$. Once we decide on the choice of the weights v_j , we apply the Nyström method and solve the system of algebraic equations

$$\tilde{\phi}(x_i) - \lambda \sum_{j=0}^n v_j \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, n. \quad (8.4.6)$$

EXAMPLE 8.4.1. Consider the case of the weakly singular equation with the kernel $k(x, s) = |x - s|^{-1/2}$, for which we use the quadrature rule

$$\int_a^b \frac{1}{|x - s|^{1/2}} F(s) ds = \sum_{j=0}^n v_j(x_i) F(s_j),$$

where $F(x)$ is any function defined on $[a, b]$. This rule is an extension of the repeated trapezoidal rule, where the weights are taken as

$$\begin{aligned} v_0 &= \frac{1}{h} \int_{s_0}^{s_1} (s_1 - s) k(x_m, s) ds, \\ v_j &= \frac{1}{h} \left\{ \int_{s_{j-1}}^{s_j} (s - s_{j-1}) k(x_m, s) ds \right. \\ &\quad \left. + \int_{s_j}^{s_{j+1}} (s_{j+1} - s) k(x_m, s) ds \right\}, \quad j = 1, 2, \dots, n-1, \\ v_n &= \frac{1}{h} \int_{s_{n-1}}^{s_n} (s - s_{n-1}) k(x_m, s) ds, \end{aligned} \tag{8.4.7}$$

where $x_m = a + mh$. Thus, in this example in order to compute v_j for $j = 0, 1, \dots, n$, we are required to compute

$$\frac{1}{h} \int_{s_{j-1}}^{s_j} \frac{(s_j - s)}{|s_m - s|^{1/2}} ds \quad \text{and} \quad \frac{1}{h} \int_{s_{j-1}}^{s_j} \frac{(s - s_{j-1})}{|s_m - s|^{1/2}} ds.$$

8.4.1. Atkinson's Method. Atkinson (1976) suggests the following method to compute the above integrals: Take $s = s_{j-1} + ht$, $0 < t < 1$; then these integrals reduce to

$$h^{1/2} \int_0^1 \frac{1-t}{|q-t|^{1/2}} dt \quad \text{and} \quad h^{1/2} \int_0^1 \frac{t}{|q-t|^{1/2}} dt,$$

respectively, where $q = m - j + 1$, $-n + 1 \leq q \leq n$. Hence, we notice that these n^2 weights v_j are reduced to the above integrals with respect to t over $[0, 1]$, which can be computed analytically:

$$\begin{aligned} v_0 &= h^{1/2} \int_0^1 \frac{1-t}{|m-t|^{1/2}} dt, \\ v_j &= h^{1/2} \int_0^1 \frac{1-t}{|q-t|^{1/2}} dt, \quad j = 1, 2, \dots, n-1, \\ v_n &= h^{1/2} \int_0^1 \frac{1-t}{|m-n+1-t|^{1/2}} dt. \end{aligned} \tag{8.4.8}$$

Note that the weights v_j can be similarly evaluated for the logarithmic kernel $\ln|x - s|$ or any other difference kernel $k(x - s)$. ■

EXAMPLE 8.4.2. (Baker 1978, p.541) Consider the equation

$$\phi(x) - \lambda \int_{-1}^1 |x - s|^{-1/2} \phi(s) ds = f(x), \quad -1 \leq x \leq 1, \quad (8.4.9)$$

with $\lambda = 2/3$ and $f(x) = (1 - x^2)^{3/4} - \frac{\pi}{2\sqrt{2}}(2 - x^2)$. The exact solution is $\phi(x) = (1 - x^2)^{3/4}$, which is not differentiable at $x = \pm 1$. We choose $s_j = jh - 1$, $h = 2/n$; then using the weights (8.4.8) we solve the system (8.4.6) and obtain the following results for $n = 5$:

x	± 1.0	± 0.6	± 0.2
Error	-2.55(-2)	2.25(-2)	1.27(-1)

The rate of convergence is slow. The maximum error is as follows: 6.18(-2) for $n = 10$; 2.3(-2) for $n = 20$; 1.22(-2) for $n = 30$; 7.66(-3) for $n = 40$; 5.48(-3) for $n = 50$; 4.25(-3) for $n = 60$; and 3.42(-3) for $n = 70$. ■

Phillips (1969) has used a polynomial collocation method for solving Eq (8.4.9); he has calculated the accuracy both analytically and theoretically and obtained an error of about 3×10^{-2} with a polynomial of degree 7 (by solving a linear system of 8 equations) and an error of order $O(10^{-3})$ by solving a system of 30 equations.

CASE 2. When $k(x, s) = \ln|x - s|$, we may take, as in case 1, $\kappa(x, s) = 1$ and use the above method to solve the FK2 (8.1.3). But instead, following Atkinson (1966), we choose $u(x, s) = |x - s|^{-1/2}$ and $\kappa(x, s) = |x - s|^{1/2}k(x, s)$. Note that this choice of $\kappa(x, s)$ is continuous but not smooth for $0 \leq x, s \leq 1$, and $u(x, s)$ becomes unbounded at $s = x$. However, such a choice enables us to investigate the behavior of the method under bad computational conditions. Moreover, the method becomes very effective if we use trapezoidal rule (8.4.6) with weights defined by (8.4.8), that is,

$$\int_a^b \frac{1}{|x - s|^{1/2}} F(s) ds = \sum_{j=0}^n v_j F(s_j),$$

which leads to the system of equations

$$\tilde{\phi}(x_i) - \lambda \sum_{j=0}^n v_j \kappa(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, n. \quad (8.4.10)$$

EXAMPLE 8.4.3. Consider Eq (8.4.3), where $\lambda = 1$, and

$$f(x) = x - \frac{1}{2} \left\{ x^2 \ln x + (1-x^2) \ln(1-x) - x - \frac{1}{2} \right\}.$$

The exact solution is $\phi(x) = x$. The system (8.4.10) becomes

$$\tilde{\phi}(x_i) - \sum_{j=0}^n v_j |x_i - s_j|^{1/2} \ln |x_i - s_j| \tilde{\phi}(s_j) = f(x_i),$$

or

$$(\mathbf{I} - \boldsymbol{\kappa}\mathbf{D}) \tilde{\Phi} = \mathbf{f},$$

where $\boldsymbol{\kappa} = (\kappa(x_i, s_j))_{ij}$, and $\mathbf{D} = \{v_0, v_1, \dots, v_n\}$. Using the 4-point trapezoidal rule we get $\tilde{\phi}_1 = 3.47916$, $\tilde{\phi}_2 = 0 = \tilde{\phi}_3$, and $\tilde{\phi}_4 = -4.74689$, which are very bad results. For computational details see `logkernel11.nb`. ■

EXAMPLE 8.4.4. (Baker 1978, p.549) Let $k(x, s) = Y_0(|x - s|)$, where $Y_0(x)$ is the Bessel function of the second kind and order zero (also known as Weber's function of order zero). Note that

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi} \left[\ln \left(\frac{x}{2} \right) + \gamma \right] J_0(x) + \frac{2}{\pi} \left\{ \frac{x^2}{2^2} - \frac{x^4}{2^2 4^2} \left(1 + \frac{1}{2} \right) \right. \\ &\quad \left. + \frac{x^6}{2^2 4^2 6^2} \left(1 + \frac{1}{2} + \frac{1}{3} \right) - \dots \right\}, \end{aligned}$$

where $J_0(x)$ is the Bessel function of the first kind and order zero. The function $Y_0(x)$ is unbounded at $x = 0$, so we take $u(x, s) = |x - s|^{-\alpha}$ and $\kappa(x, s) = |x - s|^\alpha Y_0(|x - s|)$. ■

EXAMPLE 8.4.5. (Baker 1978, p.550) Consider an FK2 of the form (1.2.2) with $a = 0$ and $b = L$, and suppose that the kernel $k(x, s)$ is expressed in terms of complete elliptic integrals of the first and second kinds, i.e., $k(x, s) = P(x, s) \ln |x - s| + Q(x, s)$, where $P(x, s) \in C[0, L]$ and $Q(x, s)$ has a singularity at $x = s$. The free term $f(x)$ is such that $\phi \in C[0, L]$. Baker (1978, p.559)

mentions a work by Kershaw who solved this equation by using the quadrature rules.

$$\begin{aligned} \int_0^L \ln|x-s| F(s) ds &= \sum_{j=0}^n v_{1,j} F(s_j), \\ \int_0^L Q(x, s) F(s) ds &= \sum_{j=0}^n v_{2,j} F(s_j), \quad s_j = jh, \quad h = L/n, \end{aligned} \quad (8.4.11)$$

where each rule is exact if $F(x)$ is piecewise polynomial. Then the approximate solution of the given FK2 is obtained by solving the system

$$\tilde{\phi}(x_i) - \lambda \sum_{j=0}^n \left\{ v_{1,j} P(x_i, s_j) + v_{2,j} \right\} \tilde{\phi}(s_j) = f(x_i). \blacksquare \quad (8.4.12)$$

The preceding examples lead to a general case when the kernel $k(x, s)$ can be represented by

$$k(x, s) = \sum_{m=0}^M k(x, s, m), \quad (8.4.13)$$

where $k(x, s, m)$ can be split as

$$k(x, s, m) = u(x, s, m) \kappa(x, s, m), \quad (8.4.14)$$

such that $\kappa(x, s, m) \in C[a, b]$ and $u(x, s, m)$ are weakly singular. Then we use the quadrature rule

$$\int_a^b u(x, s, m) F(s) ds = \sum_{j=0}^n v_{m,j} F(s_j), \quad (8.4.15)$$

which leads to the approximation formula

$$\tilde{\phi}(x_i) - \lambda \sum_{m=0}^M \sum_{j=0}^n v_{m,j} \kappa(x_i, s_j, m) \tilde{\phi}(s_j) = f(x_i). \quad (8.4.16)$$

Note that, in general, we will get different approximate values $\tilde{\phi}$ from different choices of the kernels $k(x, s, m)$ because its representation (8.4.13) is not explicit. Also, each function $k(x, s, m)$ may be written in many different ways as the product $u(x, s, m) \kappa(x, s, m)$.

EXAMPLE 8.4.6. (Atkinson 1966) Consider

(a) $k(x, s) = |x - s|^{-1/2} \kappa(x, s)$, where $\kappa(x, s) = |x - s|^{1/2} \ln |\cos x - \cos s| \in C[0, \pi]$, since if $\cos x = \cos s$, then $x = s + 2m\pi$, $m = 0, 1, 2, \dots$, and when $0 \leq x, s \leq \pi$, this implies $x = s$. This case satisfies all conditions except $\frac{\partial}{\partial s} \kappa(x, s)$ is unbounded at $x = s$. The results obtained, though poor, are given in Table 8.4.6a.

Table 8.4.6a

n	$\phi_n(\pi/2)$	Error
4	0.59065	0.28
8	0.44853	0.13
16	0.38764	0.073
32	0.35347	0.38

(b) $k(x, s) = \ln |\cos x - \cos s|$, $0 \leq x, s \leq \pi$, which we can write in the form (8.4.13) with $M = 4$ as

$$\begin{aligned}\kappa_0(x, s) &= \ln \left\{ \frac{2 \sin \frac{x-s}{2}}{x-s} \right\}, \\ \kappa_1(x, s) &= \ln \left\{ \frac{2 \sin \frac{x-s}{2}}{(x+s)(2\pi-x-s)} \right\}, \\ \kappa_2(x, s) &= \kappa_3(x, s) = \kappa_4(x, s) = 1, \\ u_0(x, s) &= u_1(x, s)1, \\ u_2(x, s) &= \ln |x-s|, \\ u_3(x, s) &= \ln(2\pi-x-s), \\ u_4(x, s) &= \ln(x+s).\end{aligned}$$

Thus,

$$\begin{aligned}k(x, s) &= \ln |\cos x - \cos s| = \ln \left\{ \frac{\sin \frac{x-s}{2} \sin \frac{x+s}{2}}{\left(\frac{x-s}{2}\right)(x+s)(2\pi-x-s)} \right\} \\ &\quad + \ln |x-s| + \ln(2\pi-x-s) + \ln(x+s) \\ &= \sum_{m=1}^4 u_m(x, s) \kappa_m(x, s),\end{aligned}\tag{8.4.17}$$

where, as defined above, $u_m(x, s)$ are badly behaved whereas $\kappa_m(x, s)$ are regular. The exact solution of this equation is

$$\phi(x) = \frac{1}{1 + \pi \ln 2} \approx 0.314704298033644.$$

Using trapezoidal rule, Atkinson (1967) found the results given in Table 8.4.6b.

Table 8.4.6b

n	$\phi_n(0)$	Error	Relative error
2	0.3051691	0.009535	3.029
4	0.3122181	0.002486	7.9(-1)
8	0.3140722	0.000632	2.0(-1)
16	0.3145453	0.000159	5.0(-2)
32	0.3146644	0.000039	1.267(-2)

Using Simpson's rule, Atkinson (1967) found the results given in Table 8.4.6c.

Table 8.4.6c

n	$\phi_n(0)$	Error	Relative error
2	0.32449048976	0.0002138	6.7(-2)
4	0.31468781777	0.00001648	5.2(-3)
8	0.31470316978	0.000001129	3.5(-4)
16	0.31470422550	0.000000073	2.3(-4)
32	0.31470429345	0.000000005	1.5(-6) ■

NOTES: The representation (8.4.13) for the kernel $k(x, s)$ leads to an error of the order $O(h^3)$, but the error increases if $\phi(x)$ is badly behaved, or if one of the factors $\kappa_m(x, s)$ has a badly behaving derivative. Atkinson achieves an error of $O(h^4)$ in the case of the kernel of Example 8.4.6. An error analysis for the product-integration method is provided by de Hoog and Weiss (1973b), who establish that the maximum local truncation errors at quadrature points are $O(h^4 \ln h)$. In practice, the product-integration formulas mentioned above are not easy to use. Atkinson (1972) has suggested a *hybrid method*, which is as follows: First, notice that the integrand with a singular kernel of the form (8.1.4) makes computational integration very difficult only when s is close to x . If we remove the disk $|s-x| \leq \delta$, $\delta > 0$, from the interval of integration $[a, b]$, then any one of the quadrature rules given in Sections A.1 and A.2 can be used to approximate the solution over the

interval $[a, s - \delta] \cup [s + \delta, b]$, where the quality of results will depend on δ . The integral over the remaining interval $(s - \delta, s + \delta)$ can then be handled by the product-integration methods, where we can take care of the singularity at $x = s$. For example, suppose that $s_0 = a$, $s_n = b$, and $s_j = s_{j-1} + h$; then set $\delta = h$ and use the repeated trapezoidal rule, say, which gives

$$\begin{aligned} \tilde{\phi}(x_i) - \lambda \frac{h}{2} & \sum_{\substack{j=0 \\ j \neq i \\ j \neq i-1}} \left\{ k(x_i, s_j) \tilde{\phi}(s_j) + k(x_i, s_{j+1}) \tilde{\phi}(s_{j+1}) \right\} \\ & - \lambda (1 - \delta_{i0}) \int_{x_{i-1}}^{x_i} \frac{1}{|x_i - s|^\alpha} \left\{ \frac{s - x_{i-1}}{h} g(x_i, x_i) \tilde{\phi}(x_i) \right. \\ & \quad \left. + \frac{x_i - s}{h} g(x_i, s_{i-1}) \tilde{\phi}(s_{i-1}) \right\} \\ & - \lambda (1 - \delta_{i0}) \int_{x_i}^{x_{i+1}} \frac{1}{|x_i - s|^\alpha} \left\{ \frac{s - x_i}{h} g(x_i, s_{i+1}) \tilde{\phi}(s_{i+1}) \right. \\ & \quad \left. + \frac{x_{i+1} - s}{h} g(x_i, s_i) \tilde{\phi}(s_{j+1}) \right\} \\ & + \frac{x_{i+1} - s}{h} g(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, n, \end{aligned} \tag{8.4.18}$$

where δ_{ij} is the Kronecker delta, and $k(x, s)$ is defined by (8.1.4).

8.4.2. Generalization. Delves and Mohamed (1985, p.101) use the Lagrange polynomial approximation and approximate the regular (“smooth”) part $\kappa(x, s)$ of the integrand with step size h , and n even. Then the regular part $\kappa(x, s)$ of the integrand can be approximated over each interval $[s_{2j}, s_{2j+2}]$, $j = 0, 1, \dots, n$, by a second-degree Lagrange interpolation polynomial interpolating at the points $s_{2j}, s_{2j+1}, s_{2j+2}$ in the form of the trapezoidal rule which has the first and the last terms halved:

$$\begin{aligned} \int_a^b u(x_i, s) \kappa(x_i, s) \phi(s) ds &= \sum_{j=0}^n \int_{s_{2j}}^{s_{2j+2}} u(x_i, s) \left\{ \frac{(s_{2j+1} - s)(s_{2j+2} - s)}{2h^2} \right. \\ & \quad \times \kappa(x_i, s_{2j}) \phi(s_{2j}) + \frac{(s - s_{2j})(s_{2j+2} - s)}{h^2} \kappa(x_i, s_{2j+1}) \phi(s_{2j+1}) \\ & \quad \left. + \frac{(s - s_{2j})(s - s_{2j+1})}{2h^2} \kappa(x_i, s_{2j+2}) \phi(s_{2j+2}) \right\} ds \\ &= \sum_{j=0}^n w_{ij} \kappa(x_i, s_j) \phi(s_j), \end{aligned} \tag{8.4.19}$$

where the weights w_{ij} are given by

$$\begin{aligned} w_{i0} &= \frac{1}{2h^2} \int_{s_0}^{s_2} u(x_i, s) (s_1 - s) (s_2 - s) ds = \beta_1(x_i), \\ w_{i,j+1} &= \frac{1}{h^2} \int_{s_{2j}}^{s_{2j+1}} u(x_i, s) (s - s_{2j}) (s_{2j+2} - s) ds = 2\gamma_{j+1}(x_i), \\ w_{i,2j} &= \frac{1}{2h^2} \int_{s_{2j-2}}^{s_{2j}} u(x_i, s) (s - s_{2j-2}) (s - s_{2j+1}) ds \\ &\quad + \frac{1}{2h^2} \int_{s_{2j}}^{s_{2j+2}} u(x_i, s) (s_{2j+1} - s) (s_{2j+2} - s) ds = \alpha_j(x_i) + \beta_{j+1}(x_i), \\ w_{i,n} &= \frac{1}{2h^2} \int_{s_{n-2}}^{s_n} u(x_i, s) (s - s_{n-2}) (s - s_{n-1}) ds = \alpha_{n/2}(x_i), \end{aligned} \tag{8.4.20}$$

$$\begin{aligned} \alpha_j(x_i) &= \frac{1}{2h^2} \int_{s_{2j-2}}^{s_{2j}} u(x_i, s) (s - s_{2j-2}) (s - s_{2j-1}) ds, \\ \beta_j(x_i) &= \frac{1}{2h^2} \int_{s_{2j-2}}^{s_{2j}} u(x_i, s) (s_{2j-1} - s) (s_{2j} - s) ds, \end{aligned} \tag{8.4.21}$$

$$\gamma_j(x_i) = \frac{1}{2h^2} \int_{s_{2j-2}}^{s_{2j}} u(x_i, s) (s - s_{2j-2}) (s_{2j} - s) ds.$$

EXAMPLE 8.4.7. Consider

$$\phi(x) = f(x) + \int_0^1 \ln|x-s| \phi(s) ds, \quad 0 \leq x \leq 1. \tag{8.4.22}$$

Here $u(x, s) = \ln|x-s|$ and $\kappa(x, s) = 1$. Using the change of variable $s = s_{2j-2} + th$, $0 \leq t \leq 2$, we find that

$$\begin{aligned} \alpha_j(x_i) &= \frac{h}{3} \ln h + \frac{h}{2} \int_0^2 t(t-1) \ln|i+2j+2-t| dt \\ &= \frac{h}{3} \ln h + \frac{h}{2} [\psi_2(m) - \psi_1(m)], \\ \beta_j(x_i) &= \frac{h}{3} \ln h + \frac{h}{2} \int_0^2 (1-t)(2-t) \ln|i-2j+2-t| dt \\ &= \frac{h}{3} \ln h + \frac{h}{2} [2\psi_0(m) - 3\psi_1(m) + \psi_2(m)], \\ \gamma_j(x_i) &= \frac{2h}{3} \ln h + \frac{h}{2} \int_0^2 t(2-t) \ln|i-2j+2-t| dt \\ &= \frac{2h}{3} \ln h + \frac{h}{2} [2\psi_1(m) - \psi_2(m)], \end{aligned} \tag{8.4.23}$$

where $m = i - 2j + 2$. Using integration by parts, we have

$$\begin{aligned}\psi_0(m) &= m \ln |m| - (m-2) \ln |m-2| - 2, \\ \psi_1(m) &= \frac{1}{2} [(m-2)^2 \ln |m-2| - m^2 \ln |m|] + \frac{1}{4} [m^2 - (m-2)^2] \\ &\quad + m\psi_0(m), \\ \psi_2(m) &= \frac{1}{3} [m^3 \ln |m| - (m-2)^3 \ln |m-2|] + \frac{1}{9} [(m-2)^2 - m^3] \\ &\quad + m[2\psi_1(m) - m\psi_0(m)].\end{aligned}\tag{8.4.24}$$

For computational details, see `log1.nb`. The advantage of these formulas is in directly computing the weights w_{ij} in (8.4.19). This example is also considered by Atkinson (1976, pp.106–122), where the Nyström formula (Section 1.6) is used with $\phi(x) = e^x$. The norm of the integral operator is

$$\|K\| = 1 + \ln 2 \approx 1.69314718.$$

The results are given in Table 8.4.7.

Table 8.4.7

n	μ	Error
10	1.0	1.16(−3)
20	1.0	2.78(−4)
40	1.0	6.8(−5)
10	−1.0	5.56(−3)
20	−1.0	1.42(−3)
40	−1.0	3.57(−4) ■

8.4.3. Atkinson's Modification. This method is useful in the case when it is not possible to express the kernel $k(x, s)$ as the product (8.4.11) or (8.4.14). In such a situation the weights must then be computed numerically. The number of integrations must also be kept to a minimum. Thus, according to Atkinson (1976, p.117), let n be even, $h = (b-a)/n$, $s_j = a+jh$, and for each x_i , $i = 0, 1, \dots, n$, we shall assume that j belongs to either the class G_i or the class B_i as follows:

$$j \in \begin{cases} G_i & \text{if } k(x_i, s) \text{ is “well behaved” for } s_{2j-2} \leq s \leq s_{2j}, \\ B_i & \text{otherwise for } j = 1, \dots, [n/2]. \end{cases}$$

Then

$$\begin{aligned} K\phi(x_i) &= \sum_{j=1}^{[n/2]} \int_{s_{2j-2}}^{s_{2j}} k(x_i, s) \phi(s) ds \\ &= \left\{ \sum_{j \in G_i} + \sum_{j \in B_i} \right\} \int_{s_{2j-2}}^{s_{2j}} k(x_i, s) \phi(s) ds. \end{aligned} \quad (8.4.25)$$

Hence, for $j \in G_i$ we use

$$\begin{aligned} \int_{s_{2j-2}}^{s_{2j}} k(x_i, s) \phi(s) ds &\approx \frac{h}{3} \left[k(x_i, s_{2j-2}) \phi(s_{2j-2}) + 4k(x_i, s_{2j-1}) \phi(s_{2j-1}) \right. \\ &\quad \left. + k(x_i, s_{2j}) \phi(s_{2j}) \right], \end{aligned} \quad (8.4.26)$$

and for $j \in B_i$ we use

$$\int_{s_{2j-2}}^{s_{2j}} k(x_i, s) \phi(s) ds \approx \int_{s_{2j-2}}^{s_{2j}} k(x_i, s) \tilde{\phi}(s) ds, \quad (8.4.27)$$

where $\tilde{\phi}(s)$ is the quadratic interpolating polynomial to $\phi(s)$ at the points s_{2j-2} , s_{2j-1} , s_{2j} defined by

$$\begin{aligned} \tilde{\phi}(s) &= \frac{(s - s_{2j-1})(s - s_{2j}) \phi(s_{2j-2})}{2h^2} - \frac{(s - s_{2j-2})(s - s_{2j}) \phi(s_{2j-1})}{h^2} \\ &\quad + \frac{(s - s_{2j-2})(s - s_{2j-1}) \phi(s_{2j})}{2h^2}. \end{aligned} \quad (8.4.28)$$

Then, combining (8.4.25)–(8.4.28), we get

$$K_n \phi(x_i) = \sum_{j=0}^n w_{ij} \phi(s_j) \approx K\phi(x_i), \quad i = 0, 1, \dots, n, \quad (8.4.29)$$

where w_{ij} are the new weights to be computed. These numerical integral operators $K_n \phi(s)$ can be defined for s between the nodes by using suitable combinations of $\{K_n \phi(x_i)\}$. The linear system that approximates $(\mu - K)\phi = f$ in this case is

$$\mu z_i - \sum_{j=0}^n w_{ij} z_j = f(x_i), \quad i = 0, 1, \dots, n, \quad (8.4.30)$$

where $z_i = \phi_n(x_i)$.

The definition (8.4.29) for $K_n \phi(x_i)$ uses exact integration of $k(s - i, s)$ in those cases where it is really necessary. This ensures numerical integration is kept to a minimum. In most cases it is very easy to determine the sets G_i and B_i . For many kernels there exists an infinite singularity at $x = s$, and elsewhere $k(x, s)$ is well behaved. In such cases we use a small but fixed $\varepsilon \geq 0$ and let $j \in B_i$ if the distance from x_i to the interval $[s_{2j-2}, s_{2j}]$ is less than or equal to ε ; otherwise we take $j \in G_i$.

EXAMPLE 8.4.8. (Atkinson 1976, p.119) Consider the FK2

$$2\phi(x) - \int_0^1 \phi(s) \ln|x - s| ds = f(x), \quad 0 \leq x \leq 1,$$

where, for the sake of easy verification of numerical approximations, we choose $f(x)$ such that $\phi(s) = s^3$. Then an exact integration yields

$$\begin{aligned} f(x) = & 2x^3 + \frac{1}{48} [3 + 4x + 6x^2 + 12x^3 + 12x^4 \ln|1-x| - 12 \ln|x-1|] \\ & - \frac{1}{4} x^4 \ln|-x|. \end{aligned}$$

Note that $\lim_{x \rightarrow 0} \int_0^1 s^3 \ln|x - s| ds = -1/16 = -0.0625$, and $\lim_{x \rightarrow 1} \int_0^1 s^3 \ln|x - s| ds = -25/48 \approx -0.520833$. The computed values of $\int_0^1 s^3 \ln|x - s| ds$ for $x = 0(.1)1$ are given in Table 8.4.8, where the entry “—” means “indeterminate”; the details using Simpson’s rule for $n = 2, 4, 8, 16, 32$ are given in log2.nb.

Table 8.4.8

x	Exact	$n = 2$	$n = 4$	$n = 8$	$n = 16$	$n = 32$
0.0	-0.06250	—	—	—	—	—
0.1	-0.09872	-0.09391	-0.09832	-0.09877	-0.09867	-0.09872
0.2	-0.14250	-0.13752	-0.14335	-0.14162	-0.14245	-0.14233
0.3	-0.19638	-0.19356	-0.19114	-0.19445	-0.19703	-0.19572
0.4	-0.26213	-0.27702	-0.24805	-0.26305	-0.25938	-0.26269
0.5	-0.33995	— ∞				
0.6	-0.42743	-0.34459	-0.39657	-0.43782	-0.41684	-0.42854
0.7	-0.51796	-0.33478	-0.55929	-0.48335	-0.52073	-0.51011
0.8	-0.59557	-0.36857	-0.58355	-0.55348	-0.60657	-0.58535
0.9	-0.63624	-0.46012	-0.47999	-0.64289	-0.60322	-0.64179
1.0	-0.52083	— ∞ ■				

8.3.4. Asymptotic Expansions. de Hoog and Weiss (1973b) establish a generalized Euler–Maclaurin formula for product integration based on piecewise Lagrangian interpolation, and they derive asymptotic expansions for three cases of the function $u(x, s)$: (a) when $u(x, s)$ is regular; (b) when $u(x, s)$ is singular and of the form $|x - s|^\alpha$, $0 < \alpha < 1$, as in (8.4.2); and (c) when $u(x, s)$ is of the form $\ln|x - s|$, as in (8.4.3). Special cases of (b) and (c), i.e., $|\vartheta - s|^\alpha$ and $\ln|\vartheta - s|$, $0 \leq \vartheta \leq 1$, are also considered. Note that case (i) belongs to Section 4.2, so we shall present the formulas for cases (b) and (c) here.

Consider an SK2 of the form (8.4.4) with $a = 0$, $b = 1$, where the kernel is represented as

$$k(x, s) = \sum_{j=1}^n u_j(x, s) \kappa_j(x, s), \quad (8.4.31)$$

where the functions $u_j(x, s)$ and $\kappa_j(x, s)$ satisfy the following conditions:

- (i) $\kappa_j(x, s) \in C[0, 1]$;
- (ii) $\int_0^1 |u_j(x, s)| ds < +\infty$; and
- (iii) $\lim_{|x_1 - x_2| \rightarrow 0} \int_0^1 |u_j(x_1, s) - u_j(x_2, s)| ds = 0$ uniformly in x_1 and x_2 .

Special cases of $u_j(x, s)$ are mentioned above in (b) and (c). Therefore, we consider the special case $k(x, s) = u(x, s)\kappa(x, s)$. Choose the quadrature points for the interval $[0, 1]$ as $0 \leq s_1 < s_2 < \dots < s_n \leq 1$, and let $x_i = ih$ for $i = 0, 1, \dots, m$, $h = 1/m$. We denote $x_{ij} = x_i + s_j h$ for $j = 1, 2, \dots, n$. Define

$$\omega(x) = \prod_{j=1}^n (x - s_j) \quad (8.4.32)$$

and the Lagrangian interpolating polynomials

$$l_j(x) = \frac{\omega(x)}{\omega'(s_j)(x - s_j)}, \quad j + 1, 2, \dots, n. \quad (8.4.33)$$

Then the computational scheme for the approximate solution $\tilde{\phi}$ of the FK2 is given by

$$\tilde{\phi}_{ij} = f(x_{ij}) + \lambda \sum_{l=0}^{m-1} \sum_{p=1}^n W_{lp} \kappa(x_{ij}, s_{lp}) \tilde{\phi}_{lp}, \quad (8.4.34)$$

for $i = 0, 1, \dots, m - 1$; $j = 1, 2, \dots, n$, where

$$W_{lp}(x) = \int_{x_l}^{x_{l+1}} u(x, s) l_p \left(\frac{s - x_l}{h} \right) ds, \quad (8.4.35)$$

and $\tilde{\phi}_{ij}$ denotes the numerical approximation to $\phi(x_{ij})$. Atkinson (1967) shows that if λ is a regular value of the kernel, then (8.4.34) has a unique solution for sufficiently small h and $\max_{\substack{0 \leq i \leq m-1 \\ 1 \leq j \leq n}} |\phi(x_{ij}) - \tilde{\phi}_{ij}| = O(E)$, where the error

$$E = \max_{\substack{0 \leq i \leq m-1 \\ 1 \leq j \leq n}} \left| \sum_{l=0}^{m-1} \sum_{p=1}^n W_{lp} \kappa(x_{ij}, s_{lp}) \tilde{\phi}_{lp} - \int_0^1 k(x_{ij}, s) ds \right|.$$

The special cases for the error term are as follows:

When $u(x, s) = |\vartheta - x|^\alpha$, $0 < \vartheta < 1$, or $u(x, s) = |x - s|^\alpha$, $0 < \alpha < 1$,

$$E = O \left(h^n \int_0^1 \omega(s) ds \right) + O(h^{n+1-\alpha});$$

when $u(x, s) = \ln |\vartheta - x|$,

$$E = O \left(h^n \int_0^1 \omega(s) ds \right) + O(h^{n+1});$$

when $u(x, s) = \ln |x - s|$,

$$E = O \left(h^n \int_0^1 \omega(s) ds \right) + O(h^{n+1} \ln h).$$

The asymptotic expansions technique developed by Lyness and Ninham (1967) has been used to develop corresponding expansions for the two cases when $u(x, s)$ is singular. Thus, for the case when $u(x, s) = |x - s|^\alpha$ we have

$$\begin{aligned} h \sum_{l=0}^{m-1} |x_l + \vartheta h - s_{ij}|^\alpha \tilde{\phi}(x_l + \vartheta h) &= \int_0^1 |s - x_{ij}|^\alpha \\ &+ \sum_{q=0}^p \frac{h^{q+1}}{q!} \left\{ \hat{\zeta}(-q, \vartheta) \frac{d^q}{dx^q} (|x - x_{ij}|)^\alpha \phi(s) ds \Big|_{x=0} \right. \\ &\quad \left. + (-1)^q \hat{\zeta}(-q, 1 - \vartheta) + \frac{d^q}{dx^q} (|x - x_{ij}|)^\alpha \phi(s) ds \Big|_{x=1} \right\} \\ &+ \sum_{q=0}^p \frac{h^{q+1-\alpha}}{q!} \left[\hat{\zeta}(\alpha - q, \vartheta - s_j) + (-1)^q \hat{\zeta}(\alpha - q, 1 + s_j - \vartheta) \right] \phi^q(x_{ij}) \\ &+ O \left(x_{ij}^1 - \alpha \hat{h}^{p+1} \right) + O \left((1-x)^{1-\alpha} \hat{h}^{p+1} \right), \quad 0 < x_{ij} < 1, p \geq 0, \end{aligned} \tag{8.4.36}$$

where $\hat{h} = h/x_{ij}$, and $\hat{\zeta}(\alpha, n)$ is the periodic zeta function defined by $\hat{\zeta}(\alpha, n) = \zeta(\alpha, \hat{\vartheta})$, $0 < \vartheta < 1$, such that $\vartheta - \hat{\vartheta}$ is an integer, and $\zeta(\alpha, x)$ is the generalized Riemann zeta function. Again, for the case when $u(x, s) = \ln|x - s|$, a formula similar to (8.4.36) holds except that the term $|x - x_{ij}|^\alpha$ is replaced by $\ln|x - x_{ij}|$ and the last two error terms are replaced by $O(\ln x_{ij} \bar{h}^{p+1}) + O(\ln(1 - x_{ij}) \bar{h}^{p+1})$, where $\bar{h} = h/(1 - x_{ij})$.

EXAMPLE 8.4.9. de Hoog and Weiss (1973b) consider the SK2

$$\phi(x) = 1 + \int_0^\pi \sum_{j=1}^4 u_j(x, s) \kappa_j(x, s) \phi(s) ds, \quad 0 \leq x \leq \pi,$$

where

$$\kappa_1(x, s) = \left\{ \frac{\sin((x-s)/2)}{((x-s)/2)} \right\} + \ln \left\{ \frac{\sin((x+s)/2)}{((x+s)(2\pi-x-s))} \right\},$$

$$\begin{aligned} u_2(x, s) &= \ln|x - s|, & u_3(x, s) &= \ln(2\pi - x - s), \\ u_4(x, s) &= \ln(x + s), & u_1 &= \kappa_2 + \kappa_3 = \kappa_4 = 1. \end{aligned}$$

Compare this example with Example 8.4.6, where Atkinson (1967) applies the product Simpson's rule. Although the convergence rate was observed to be of the order $O(h^4)$, only $O(h^3)$ convergence was established. ■

The above results are also valid for an VK2 with singular kernels. Linz (1969) applies a product Simpson's rule to the equation

$$\phi(x) = f(x) + \int_0^x \frac{g(x, s, \phi(s))}{\sqrt{x-s}} ds, \quad x \geq 0,$$

and estimates convergence of the order $O(h^3)$. According to de Hoog and Weiss (1973b) the correct order is "three and a half."

8.5. Splines Method

A weakly singular equation of the form (8.3.4) has a unique solution $\phi(x) \in C[0, 1]$ iff the spectrum $\sigma(K) \neq 1$. One of the main difficulties with weakly singular

equations is that even if $f \in C[0, 1]$ and $\kappa(x, s) \in C^{n+1}([0, 1] \times [0, 1])$, its solution $\phi(x)$ may not belong to $C^1[0, 1]$. However, an exception is the case when ϕ is in the class $C^n[0, 1]$ only under the condition that f be in $C^n[0, 1]$, κ be in the class $C^{n+1}([0, 1] \times [0, 1])$, and both be periodic (Kussmaul and Werner 1968), or if the singular equation is of the Kirkwood–Riseman type.* It is also known (Schneider 1979, 1981) that if $f \in C^n[a, b]$, $\kappa \in C^{n+1}([a, b] \times [a, b])$, then $\phi \in H^\alpha[a, b]$ and $\phi \in C^n[a, b]$; also, if $0 < \alpha < 1$, then there exists a $\beta \in [\alpha, n]$ and if $\alpha = 1$ for any $\varepsilon \in (0, 1)$, then there exists a $\beta \in [1 - \varepsilon, n]$, such that

$$|\phi^{(j)}(s)| \leq \begin{cases} C_j (s-a)^{\beta_j} & \text{if } a < s \leq (a+b)/2, \\ C_j (b-s)^{\beta-j} & \text{if } (a+b)/2 \leq s < b, \end{cases} \quad (8.5.1)$$

where C_j , $j = 1, 2, \dots, n$, are Hölder constants.

8.5.1. From SK2 to IDE. We shall transform Eq (8.3.4) into an integro-differential equation (IDE) using integration by parts. We assume throughout this section that $a = 0$, $b = 1$, and $\kappa(x, s) \equiv 1$ for the sake of simplicity; a generalization to any other κ is routine. First, integration by parts gives

$$\int_0^1 u(x, s) \phi(s) ds = u_1(x, 1) \phi(1) - \int_0^1 u_1(x, s) \phi'(s) ds, \quad (8.5.2)$$

where $\frac{\partial u_1}{\partial s} = u(x, s)$. Substituting (8.5.2) into (8.3.4) (with $\kappa(x, s) = 1$), we get

$$\phi(x) + \int_0^1 u_1(x, s) \phi'(s) ds = f(x) + u_1(x, 1) \phi(1). \quad (8.5.3)$$

Note that since u is singular, u_1 has a “mild” singularity on the line $s = x$, i.e., $u_1 \in C([0, 1] \times [0, 1])$, but the partial derivatives of u_1 are singular on the line $s = x$. In the case of the logarithmic kernel $u(x, s) = l|x - s|$, as in Eq (8.3.3), u_1 has a removable discontinuity along $s = x$, and thus, by continuously extending u_1 to $s = x$, we may assume in this case that $u_1 \in C([0, 1] \times [0, 1])$.

*This equation has the form $\phi(x) + \lambda \int_{-1}^1 \frac{\phi(s) ds}{|x - s|^\alpha} = f(x)$, $-1 \leq x \leq 1$, such that the kernel is represented as $|x - s|^{-\alpha} = \sum_{n=0}^{\infty} a_n(\nu) P_n^{(\nu)}(x) P_n^{(\nu)}(s)$, $\nu = (\alpha - a)/2$, where $P_n^{(\nu)}(x)$ are the Gegenbauer polynomials defined on $-1 \leq x \leq 1$ by $P_n^{(\nu)}(x) = \frac{(-1)^n \Gamma(\nu + 1)}{2^n \Gamma(n + \nu + 1)} (1 - x^2)^{-\nu} \frac{d^n}{dx^n} (1 - x^2)^{n+\nu}$. It occurs in the study of hydrodynamic interaction between elements of a polymer chain in solution (see Auer and Gardner 1955, p.1545).

It can be easily proved that Eqs (8.3.4) and (8.5.3) are equivalent; for a proof see Kaneko and Xu (1991). If $\frac{\partial}{\partial x} u_1(x, s)$ exists and has no singularity, then we differentiate (8.5.3) with respect to x , to obtain

$$\phi'(x) + \int_0^1 \frac{\partial}{\partial x} u_1(x, s) \phi'(s) ds = f'(x) + \left[\frac{d}{dx} u_1(x, 1) \right] \phi(1). \quad (8.5.4)$$

In this equation we set $u_2(x, s) = \frac{\partial}{\partial x} u_1(x, s) = \int_0^s \frac{\partial}{\partial x} u(x, t) dt$, $y(x) = \phi'(x)$, $F(x) = f'(x) + \left[\frac{d}{dx} u_1(x, 1) \right] \phi(1)$; then we have

$$y(x) + \int_0^1 u_2(x, s) \phi(s) ds = F(s), \quad 0 \leq x \leq 1. \quad (8.5.5)$$

This represents a required FK2, and there are many computational methods to numerically solve it. Note that the reduction of (8.3.4) to (8.5.5) is valid only for the kernel that is in the class C^1 with respect to x and has a weak singularity with respect to s .

8.5.2. *B*-Spline Method. (Schneider 1979, Kaneko and Xu 1991) Using the differential operator $(D\phi)(x) \equiv \phi'(x)$ and the integral operator $(U_1\phi)(x) \equiv \int_0^1 u_1(x, s) \phi(s) ds$, we can write Eq (8.5.3) as

$$(I + U_1 D) \phi = \hat{f},$$

where $\hat{f} = f(x) + u_1(x, 1) \phi(1)$. Assuming that ϕ satisfies the conditions (8.5.1) for some positive β and n , let $q = p/\beta$ for some positive integers p (which is the order of *B*-splines defined below), and choose a set of breakpoints $\{x_0, x_1, \dots, x_m\}$ for $m \geq 1$ by setting

$$\begin{aligned} x_0 &= 0, \\ x_1 &= \left(\frac{1}{m} \right)^q, \\ x_j &= j^q x_1 \quad \text{for } j = 2, 3, \dots, [m/2] - 1, \\ x_{m-j} &= 1 - x_1 \quad \text{for } j = 0, 1, \dots, [m/2] - 1, \\ x_{m/2} &= \frac{1}{2} \quad \text{if } m \text{ is even.} \end{aligned} \quad (8.5.6)$$

Given a continuity condition vector $\mathbf{c} = (c_i)$, $i = 1, \dots, m$, we form the knot sequence $\mathbf{z} = (z_i)$, $i = 1, \dots, M$, depending on \mathbf{c} such that the parameter c_i of \mathbf{c} at the breakpoints x_i produces $(p - c_i)$ -times repeated x_i in \mathbf{z} for $i = 1, \dots, m - 1$; thus, $z_1 = \dots = z_p = x_1$ and $z_{m-p+1} = \dots = z_M = x_n$, where $M = 2p + \sum_{i=1}^{m-1} (p - c_i)$. Take $N = p + \sum_{i=1}^{m-1} (p - c_i)$, so that $M - N = p$. Let $\{B_j\}_{j=1}^N$ denote the normalized splines of order p associated with the knot sequence \mathbf{z} . Then we replace $\phi(x)$ in (8.5.3) by $\phi_N(x) = \sum_{i=1}^N a_i B_i(x)$, which gives

$$\sum_{i=1}^N a_i \left[B_i(x) + \int_0^1 u_1(x, s) B'_i(s) ds \right] = f(x) + u_1(x, 1) \sum_{i=1}^N a_i B_i(1). \quad (8.5.7)$$

Now, choose a set of collocation points $\{\xi_j\}$, $j = 1, \dots, N$, such that $\det|B_i(\xi_j)| \neq 0$. Then using the collocation method, we reduce the system (8.5.7) to

$$\sum_{i=1}^N a_i \left[B_i(\xi_j) - u_1(\xi_j, 1) B_i(1) + \int_0^1 u_1(\xi_j, s) B'_i(s) ds \right] = f(\xi_j), \quad (8.5.8)$$

for $j = 1, 2, \dots, N$. Kaneko and Xu (1991) establish convergence of this method. To write Eqs (8.5.8) in operator form, let P_n define projections of $[0, 1]$ onto $\text{span}[B_i]_{i=1}^N$ such that $(P_n \phi)(x) = \sum_{i=1}^N \gamma_i B_i(x)$, where the coefficients γ_i are chosen such that $\sum_{i=1}^N \gamma_i B_i(\xi_j) = \phi(\xi_j)$ for each $j = 1, \dots, N$. Then Eqs (8.5.8) can be represented in operator form as

$$(I + P_N U_1 D) \phi_N = P_n \hat{f}. \quad (8.5.9)$$

Assuming that $P_n \phi \rightarrow \phi$ for each $\phi \in C[0, 1]$ as $N \rightarrow \infty$, the inverse operator $(I + P_N U_1 D)^{-1}$ exists for sufficiently large N and, according to Kaneko and Xu (1991),

$$\|\phi - \phi_N\| \leq \|(I + P_N U_1 D)^{-1}\| \|\phi - P_n \phi\| = O(m^{-p}), \quad (8.5.10)$$

where p is the order of splines.

The system of equations (8.5.8) needs some explaining before solving it. Since each B_i is a spline of order p , B'_i will be a spline of order $p - 1$, and the integrand

$u_1(\xi_j, s) B'_i(s)$ is a function with mild singularity at ξ_j . For example, if $u(x, s) = |x - s|^{-1/2}$, then

$$u_1(x, s) = \begin{cases} 2(\sqrt{\xi_j} - \sqrt{\xi_j - s}) & \text{if } s \leq \xi_j, \\ 2(\sqrt{\xi_j} - \sqrt{s - \xi_j}) & \text{if } s \geq \xi_j. \end{cases}$$

In practice, however, we approximate the integrals in (8.5.8). But for functions with mild singularity a quadrature rule based on a piecewise polynomial approximation over a set of uniformly spaced knots does *not* give a satisfactory result. Therefore, we use the following result due to de Boor (1973): Assume that a function $g \in C^p[a, b]$ except at finitely many points in $[a, b]$ near which $|g^{(p)}|$ is monotone and $\int_a^b |g^{(p)}(x)|^{1/p} dx < +\infty$. Then for a knot sequence $\mathbf{z} = \{a = x_0 < x_1 < \dots < x_m = b\}$ the order of approximation of g from the space $S_p^m(\mathbf{z})$ of splines of order p with knot sequence \mathbf{z} is of the order $O(m^{-p})$; moreover, the points x_i are chosen such that

$$\int_{x_i}^{x_{i+1}} |g^{(p)}(x)|^{1/p} dx = \frac{1}{m} \int_a^b |g^{(p)}(x)|^{1/p} dx, \quad i = 0, 1, \dots, m-1. \quad (8.5.11)$$

Let $\{x_{jl}\}_{l=0}^m$ denote nonuniformly spaced breakpoints taken relative to $g(s) \equiv u_1(\xi_j, s)$ and let $[a, b] = [\xi_j, 1]$. Then a continuity vector $\mathbf{c}_j = \{c_{jl}\}_{l=0}^m$ is needed to form a knot sequence \mathbf{z}_j from $\{x_{jl}\}_{l=0}^m$ as described above. Now, set

$N_j = p + \sum_{l=1}^{m-1} (p - c_{jl})$, and let $\{B_{jl}\}_{l=1}^{N_j}$ be a collection of B -splines of order p associated with the knot sequence \mathbf{z}_j ; additional breakpoints are defined similarly at the points $\{x_{jl}\}_{l=-m}^{-1}$, and $\{B_{jl}\}_{l=-N_j}^{-1}$ will denote the B -splines associated with these points. Since B'_i in (8.5.8) is a spline of order $p-1$ with compact support $[z_i, z_{i+p}]$, we have

$$\int_0^1 u_1(\xi_j, s) B'_i(s) ds = \int_{z_i}^{z_{i+p}} u_1(\xi_j, s) B'_i(s) ds. \quad (8.5.12)$$

Here $u_1(\xi_j, s) B'_i(s)$ is approximated by an element from $\text{span}[B_{jl}]_{l=-N_j}^{N_j}$. Thus, if $\{B_{jl}\}$ is a collection of linear splines, then $u_1(\xi_j, s) B'_i(s)$ is approximated by $\sum_{l=-m}^m u_1(\xi_j, x_{il}) B'_i(x_{il}) B_{jl}(s)$. But if $\xi_j \notin [z_i, z_{i+p}]$, then the integral in (8.5.12) is approximated by a Newton–Cotes-type quadrature rule with equally

spaced points. On the other hand, if $\xi_j \in [z_i, z_{i+p}]$, then we choose $\{x_{il}\}$ as described above. We shall use the notation s_{jl} for points that are both uniformly (equally) or nonuniformly spaced, and define for any $g \in C[0, 1]$

$$(U_1^m g)(x) = \sum_{s_{il} \in [z_i, z_{i+p}]} W_{il} g(s_{il}), \quad (8.5.13)$$

where W_{il} are the weights in the quadrature or linear splines, with $x = \xi_j$, i.e.,

$$W_{il} = \int_{x_{i,p-1}}^{x_{i,p+1}} B_{jl}(s) ds.$$

Then Eq (8.5.8) in discrete form becomes

$$(I + P_N U_1^m D) \tilde{\phi}_N^m = P_N \hat{f}. \quad (8.5.14)$$

Now, let $I + A_N$ and $I + B_N^m$ denote the matrix form of the operator $I + P_N U D$ and $I + P_N U_1 D$, respectively. Then Eqs (8.5.9) and (8.5.14) reduce to the following system of linear equations:

$$\begin{aligned} (I + A_N) \tilde{\phi}_N &= P_N \hat{f}, \\ (I + B_N^m) \tilde{\phi}_N^m &= P_N \hat{f}, \end{aligned} \quad (8.5.15)$$

and the following result holds (Kaneko and Xu 1991): The inverse operators $(I + B_N^m)^{-1}$ exist and are uniformly bounded for all sufficiently large m and N . Moreover,

$$\|\tilde{\phi}_N - \tilde{\phi}_N^m\| \leq \text{const} \left\{ \| (A_N - B_N^m) \tilde{\phi} \| + \| A_N - B_N \| \|\tilde{\phi} - \tilde{\phi}_N\| \right\},$$

where $\tilde{\phi} = \phi(\xi_j)$ are the values of $\tilde{\phi}$ at the collocation points ξ_j . Finally,

$$\|\tilde{\phi} - \tilde{\phi}_N\| = O(\|\phi - P_N \phi\|).$$

Thus, $\|A_N - B_N^m\| \rightarrow 0$ as fast as $\|\phi - P_N \phi\| \rightarrow 0$. This result is similar to Atkinson's Theorem 2 (1985).

EXAMPLE 8.5.1. (Kaneko and Xu 1991) Consider

$$\phi(x) - \int_0^1 \frac{\phi(s) ds}{\sqrt{|x-s|}} = f(x), \quad 0 \leq x \leq 1,$$

with $f(x) = x - 2\sqrt{1-x} - \frac{4}{3} [x^{3/2} - (1-x)^{3/2}]$. The exact solution is $\phi(x) = x$. Eq (8.5.3) becomes

$$\phi(x) + \int_0^1 u_1(x, s) \phi'(s) ds = f(x) + 2(\sqrt{x} - \sqrt{1-x}),$$

where

$$u_1(x, s) = \begin{cases} 2(\sqrt{x} - \sqrt{x-s}) & \text{if } s \leq x, \\ 2(\sqrt{x} + \sqrt{s-x}) & \text{if } s > x. \end{cases}$$

Using B -splines with $m = 16$ breakpoints x_m , and replacing $\int_0^1 \frac{\phi(s) ds}{\sqrt{|x-s|}}$ by $\int_0^1 \frac{P_N(s) ds}{\sqrt{|x-s|}}$, where P_N is the interpolation projection of $[0, 1]$ onto the space of piecewise polynomials of degree N with a set of breakpoints $\{z_i\}_{i=0}^m$, for each i , let $z_1 < z_{i0} < z_{i1} < \dots < z_{in} \leq z_{i+1}$, and $(P_n \phi)(s) = \sum_{j=0}^n \phi(z_{jl}) \rho_{ij}(s)$ for $s \in [z_i, z_{i+1}]$, where ρ_{ij} is the unique polynomial of degree $\leq n$ such that $\rho_{ij}(z_{il}) = \delta_{jl}$. Then the approximating equation is

$$\phi_n(x) - \sum_{i=0}^{m-1} \int_{z_i}^{z_{i+1}} \frac{1}{\sqrt{|x-s|}} \sum_{j=0}^n \phi_n(z_{ij}) \rho_{ij}(s) ds = f(x). \quad (8.5.16)$$

Define the integral operator U_n ($n \geq 1$) by

$$(U_n \phi)(s) = \int_0^1 \frac{1}{\sqrt{|x-s|}} (P_n \phi)(x) dx.$$

Then Eq (8.5.15) can be written as

$$(I - U_n) \phi_n = f. \quad (8.5.17)$$

Set $x = z_{ij}$ ($i = 0, 1, \dots, m-1$; $j = 0, 1, \dots, n$) in (8.5.15). Then we get the system

$$\phi_n(z_{ij}) - \sum_{i=0}^{m-1} \int_{z_i}^{z_{i+1}} \frac{1}{\sqrt{|z_{ij}-s|}} \sum_{j=0}^n \phi_n(z_{ij}) \rho_{ij}(s) ds = f(z_{ij}), \quad (8.5.18)$$

where the matrix of the system is given by $a_{ij} = \int_0^1 \frac{1}{\sqrt{|x-s|}} \rho_{ij}(s) ds$, which is computed by evaluating the integrals $\int_0^1 \frac{1}{\sqrt{|z_{ij}-s|}} B_i(x) dx$ using the trapezoidal rule, where B_i is the linear B -spline with support $[z_i, z_{i+1}]$. The results

obtained by Kaneko and Xu (1991) are presented in Table 8.5.1 and are compared with results obtained by the collocation method is made.

Table 8.5.1

x	B -spline method	Collocation method	Exact
0.0	0.000285	0.000199	0.0
0.25	0.806078	0.806121	0.805927
0.5	0.999620	0.999529	1.0
0.75	0.805855	0.805845	0.805927
1.0	-0.000197	-0.000210	0.0 ■

EXAMPLE 8.5.2. (Vainikko and Ubas 1981, Koneko and Xu 1991) Consider

$$-0.2 \phi(x) - \int_{-1}^1 \frac{\phi(s) ds}{\sqrt{|s-x|}} = -0.2 x^2, \quad -1 \leq x \leq 1.$$

The solution $\phi(x)$ has an unbounded derivative at $x = 1$ and $0.306753 < \phi(1) < 0.306754$. Here

$$u_1(x, s) = \begin{cases} 2(\sqrt{x+1} - \sqrt{x-s}) & \text{if } s \leq x, \\ 2(\sqrt{x+1} + \sqrt{s-x}) & \text{if } s \geq x. \end{cases}$$

Using a linear B -spline over uniformly spaced knots $x_i = -1 + \frac{i}{2^{n-1}}$, $i = 0, 1, \dots, 2^n$, and the graded knots

$$x_0 = -1, \quad x_i = -1 + \left(\frac{i}{2^{n-1}}\right)^q, \quad i = 1, \dots, 2^{n-1},$$

$$x_{2^n-i} = -x_i, \quad i = 1, \dots, 2^{n-1}-1.$$

In the case of nonuniform knots $q = 4$ is taken. The results obtained by Kaneko and Xu (1991) are given in Table 8.5.2.

Table 8.5.2

n	Graded Knots	Uniform Knots
4	0.2986092	0.2333885
5	0.3022588	0.2534495
6	0.3043483	0.2696310
7	0.3061258	0.2807035 ■

8.5.3. From Log-Singular to Cauchy-Singular. Consider the equation

$$a \int_{-1}^x \psi(s) ds + \frac{b}{\pi} \int_{-1}^1 \psi(s) \log|x-s| ds = f(x), \quad |x| < 1, \quad (8.5.19)$$

where $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$, $\psi(x)$ is the unknown function, and $f(x) \in C^n[-1, 1]$ and $f \in H^\gamma[-1, 1]$ for $0 < \gamma \leq 1$. If $a = 0$, then Eq (8.5.19) becomes the well-known Carleman equation (Carleman 1922). If we introduce a new unknown function $\phi(x)$ defined by

$$\phi(x) = \int_{-1}^x \psi(s) ds - \phi_0 \frac{1+x}{2}, \quad (8.5.20)$$

where ϕ_0 is some constant, along with the boundary conditions $\phi(-1) = 0 = \phi(1)$, then Eq (8.5.19) becomes

$$a \phi(x) - \frac{b}{\pi} \int \frac{\phi(s)}{s-x} ds = F(x), \quad (8.5.21)$$

where

$$\begin{aligned} F(x) &= f(x) - a \phi_0 \frac{1+x}{2} - \frac{b}{\pi} \left[\left(\phi(s) + \phi_0 \frac{1+x}{2} \right) \log|x-s| \right]_{-1}^1 \\ &\quad + \frac{b \phi_0}{2\pi} \int_{-1}^1 \frac{1+s}{s-x} ds \\ &= f(x) - a \phi_0 \frac{1+x}{2} + \frac{b \phi_0}{\pi} - \frac{b \phi_0}{\pi} \log(1-x) \\ &\quad + \frac{b \phi_0}{2\pi} (1+x) \log \frac{1-x}{1+x} \\ &= f(x) - \phi_0 \left\{ a \frac{1+x}{2} + \frac{b}{\pi} - \frac{b}{2\pi} \left[(1-x) \log(1-x) \right. \right. \\ &\quad \left. \left. + (1+x) \log(1+x) \right] \right\}. \end{aligned} \quad (8.5.22)$$

Note that Eq (8.5.22) is a Cauchy singular equation of the second kind (CSK2) (see chapter 9). Now, we seek a solution $\phi(x)$ of Eq (8.5.21) of the form $\phi(x) = g(x)w(x)$, where $w(x) = (1-x)^\alpha(1+x)^\beta$ is the fundamental solution with index $\kappa = -(\alpha + \beta) = -1$ (see Section 9.3). Hence, Eq (8.5.21) is solvable for all $F \in L_2[-1, 1]$ such that the condition

$$\int_{-1}^1 w^{-1}(x) F(x) dx = 0 \quad (8.5.23)$$

is satisfied. We also have $g(x) = (\hat{A}F)(x)$, where

$$(\hat{A}F)(x) \equiv a w^{-1}(x) F(x) + \frac{b}{\pi} \int_{-1}^1 w^{-1}(s) \frac{F(s)}{s-x} ds. \quad (8.5.24)$$

We can use the condition (8.5.24) to determine the constant ϕ_0 in (8.5.20), which is given by (Capobianco 1995)

$$\phi_0 = \frac{1}{c} \int_{-1}^1 w^{-1}(x) f(x) dx, \quad (8.5.25)$$

where

$$c = \int_{-1}^1 w^{-1}(x) \left\{ a \frac{1+x}{2} - \frac{b}{\pi} - \frac{b}{2\pi} [(1-x) \log(1-x) + (1+x) \log(1+x)] \right\} dx.$$

Hence, the solution of Eq (8.5.19) is given by

$$\begin{aligned} \psi(x) &= \phi'(x) + \frac{\phi_0}{2} \\ &= a F'(x) + \frac{\phi_0}{2} + \frac{b}{\pi} \frac{d}{dx} \left[w(x) \int_{-1}^1 w^{-1}(s) \frac{F(s)}{s-x} ds \right]. \end{aligned} \quad (8.5.26)$$

For numerical solution of Eq (8.5.21), see Section 9.3.

8.5.4. A Class of Singular Integrals. In the quadrature method the following integrals occur in the case of log-singular and hypersingular kernels:

$$\begin{aligned} &\int \frac{(s-x)^j \ln |s-x| ds}{(1-s)^\alpha (1+s)^\beta}, \quad j = 0, 1, 2; \\ &\int \frac{(s-a)^j ds}{(1-s)^\alpha (1+s)^\beta}, \quad j = 0, 1, 2; \\ &\int \frac{ds}{(s-x)^j (1-s)^\alpha (1+s)^\beta}, \quad j = 1, 2. \end{aligned} \quad (8.5.27)$$

These integrals cover logarithmic-, Cauchy-, and Hadamard-type singularities. Kabir et al. (1999) consider the above integrals and study a piecewise quadratic polynomial method to numerically solve singular equations of the above types. We do not give the details here, but interested readers may consult their work. They

have solved the following three problems involving singular integral equations: (i) with logarithmic-type singularity, for example,

$$\frac{1}{\pi} \int_{-1}^1 \ln |s - x| \phi(s) ds = -\frac{T_n(x)}{n}, \quad n > 0, \quad (8.5.28)$$

with the additional condition $\int_{-1}^1 \phi(s) ds = 0$. The exact solution is $\phi(x) = (1 - x^2)^{-1/2} T_n(x)$. A numerical solution is computed for $n = 2$ using the fundamental function $w(x)$ with $\alpha = 1/2 = \beta$.

(ii) with Hadamard-type singularity, i.e.,

$$\phi(x) = \pi C(x) U_\infty A(x) - \frac{C(x)}{4} \int_{-b/2}^{b/2} \frac{\phi'(s) ds}{s - x}, \quad |x| < b/2. \quad (8.5.29)$$

This equation, known as the Prandtl integro-differential equation in aerodynamics, governs the circulation $\phi(x)$ around a finite wing span of arbitrary cross-section $C(x)$, with the Kutta condition $\phi(\pm b/2) = 0$. Integrating (8.5.29) by parts and using the Kutta conditions, we can reduce this equation to

$$\phi(x) - \frac{C(x)}{4} \int_{-b/2}^{b/2} \frac{\phi(s) ds}{(s - x)^2} = \pi U_\infty C(x) A(x), \quad |x| < b/2. \quad (8.5.30)$$

Applying a physical constraint corresponding to the finite lift L , which is given by

$$L = \rho U_\infty \int_{-b/2}^{b/2} \phi(s) ds = \frac{\pi \rho U_\infty \phi_0 b}{4},$$

we see that the exact solution of Eq (8.5.30) of an untwisted ($A = \text{const}$) wing of elliptic cross-section $C(x) = C_0 \left[1 - \left(\frac{2x}{b} \right)^2 \right]^{1/2}$ is given by

$$\phi(x) = \phi_0 \left[1 - \left(\frac{2x}{b} \right)^2 \right]^{1/2}.$$

(iii) with a strongly singular kernel, i.e.,

$$\int_a^b \frac{\phi(s) ds}{(s - x)^2} + \int_a^b k(x, s) \phi(s) ds = -\pi \left(\frac{1 + \kappa}{2\mu} \right) p(x), \quad (8.5.31)$$

with the additional condition same as in (i). This equations occurs in the crack problem in a semi-infinite plate where the crack lies orthogonal to the stress-free boundary and is subjected to surface tractions $p(x)$. The unknown function $\phi(x)$ is the crack-opening displacement (see Kaya and Erdogan 1987), and the kernel is defined by

$$k(x, s) = -\frac{1}{(s+x)^2} + \frac{12x}{(s+x)^3} - \frac{12x^2}{(s+x)^4}.$$

By using the substitutions

$$\begin{aligned} s &= \left(\frac{b-a}{2}\right)r + \frac{b+a}{2}, \quad x = \left(\frac{b-a}{2}\right)y + \frac{b+a}{2}, \\ \phi(s) &= \left(\frac{b-a}{2}\right)\psi(r), \end{aligned}$$

and assuming that tractions are uniform on the crack surface ($p(x) = p_0$), we reduce Eq (8.5.31) to

$$\int_{-1}^1 \frac{\psi(r)}{(r-y)^2} dr + \int_{-1}^1 k(y, r)\psi(r) dr = g(y), \quad |y| < 1, \quad (8.5.32)$$

where

$$\begin{aligned} g(y) &= -\pi \left(\frac{1+\kappa}{2\mu}\right) p_0, \\ k(y, r) &= -\frac{1}{(r+y+2m)^2} + \frac{12(y+m)}{(r+y+2m)^3} - \frac{12(y+m)^2}{(r+y+2m)^4}, \end{aligned}$$

and $m = (b+a)/(b-a)$.

8.6. Weakly Singular Volterra Equations

We consider an SVK2 of the form

$$\phi(x) - \int_0^x (x-s)^{-\alpha} \kappa(x, s) \phi(s) ds = f(x), \quad 0 < \alpha < 1, \quad x \in [0, X], \quad (8.6.1)$$

where $f \in C^m[0, X]$ and $\kappa \in C^m[T]$, where $T = \{(x, s) : 0 \leq s \leq x \leq X\}$ is the triangle domain in the (x, s) -plane. Equations of this form occur in many physical problems, generally with $\alpha = 1/2$ (see Brunner and van der Houwen 1986).

We assume the existence of a unique solution of Eq (8.6.1) in the interval $[0, x]$, although, in general, this solution cannot be found by analytical means. It is known that the exact solution is not smooth at $x = 0$ even if f and κ are smooth (Brunner and van der Houwen 1986). Thus, the numerical methods must have a high rate of convergence on the interval $[0, X]$. Lubich (1985) uses fractional linear multistep methods that require special construction of starting weights. Brunner (1983, 1984) uses collocation methods with either graded meshes instead of uniform ones or nonpolynomials instead of polynomials. We describe the polynomial spline collocation method with uniform meshes because it is easy to derive global convergence results and compare the results with others. This method leads to an implicit Runge–Kutta method. A block-by-block method can be obtained from the implicit Runge–Kutta method, as Linz does (1985, pp.114–116 and 136–137). These methods have the same rate of convergence as the collocation methods and show better stability. Other simpler methods, specially those by Kershaw (1982) and Lubich (1983), give good accuracy in the interval $[\varepsilon, X]$, $\varepsilon > 0$, which excludes the point $x = 0$.

The singular behavior of $\phi(x)$ at $x = 0$ can be resolved by writing $\phi(x) = \psi(x) + g(x)$, where $\psi \in C^m[0, X]$ and $g(x)$ is a known function, and then solving the equation

$$\psi(x) - \int_0^x (x-s)^{-\alpha} \kappa(x, s) \psi(s) ds = F(x), \quad (8.6.2)$$

where

$$F(x) = f(x) - g(x) + \int_0^x (x-s)^{-\alpha} \kappa(x, s) g(s) ds. \quad (8.6.3)$$

The success of this technique depends on a proper choice of the function $g(x)$ so that the approximate solution $\tilde{\psi}(x)$ of Eq (8.6.2) is smooth in the class $C^m[0, X]$ and it can be approximated by a collocation method. Then the required approximate solution $\tilde{\phi}(x)$ of Eq (8.6.1) can be obtained by adding $g(x)$ to $\tilde{\psi}(x)$.

First, we consider the singular part of the solution.

CASE 1: Let $\alpha = 1/2$. Then, according to Brunner and van der Houwen (1986, p.29), the function $\phi(x)$ in Eq (8.6.1) can be written as $\phi(x) = u(x) + \sqrt{x} v(x)$, where $u, v \in C^m[0, X]$. Let $v(x) = a + b x + c x^2 + G(x)$ for some $G(x) =$

$a_0x^3 + \dots$. Then

$$\phi(x) = u(x) + \sqrt{x}G(x) + \sqrt{x}(a + bx + cx^2) = \psi(x) + g(x),$$

where

$$\psi(x) = u(x) + \sqrt{x}G(x), \quad g(x) = \sqrt{x}(a + bx + cx^2).$$

Recall that $\psi(x)$ is unknown, $\psi(x) \in C^3[0, X]$, and $g(x)$ will be completely known once the coefficients a , b , and c are determined.

CASE 2: To consider the general case, let $0 < \alpha < 1$. The exact solution $\phi(x) \in C[0, X] \times C^m(0, X]$ and has the series representation

$$\phi(x) = f(x) + \sum_{j=1}^{\infty} \phi_j(x) x^{j(1-\alpha)}, \quad (8.6.4)$$

where $\phi_j(x) \in C^m[0, X]$. If we expand $\phi_j(x)$ by Taylor's series at $x = 0$, i.e.,

$$\phi_j(x) = \sum_{i=0}^i \frac{\phi_j^{(i)}(0)}{i!} x^i + R_j(x), \quad (8.6.5)$$

where the remainder $R_j(x) \rightarrow 0$ faster than x^m as $x \rightarrow 0$, then (8.6.4) becomes

$$\phi(x) = f(x) + \sum_{j=1}^{\infty} \sum_{i=1}^m \frac{\phi_j^{(i)}(0)}{i!} x^{i+j(1-\alpha)} + \sum_{j=1}^{\infty} R_j(x) x^{j(1-\alpha)}. \quad (8.6.6)$$

Since $f(x) \in C^m[0, X]$, we can write $\phi(x)$ as

$$\phi(x) = \psi_m(x) + g_{m,\alpha}(x), \quad (8.6.7)$$

where

$$\begin{aligned} \psi_m(x) &= \sum_{n=0}^m \frac{\psi_m^{(n)}(0)}{n!} x^n + R(x), \\ g_{m,\alpha}(x) &= \sum_{i,j}^* a_{i,j} x^{i+j(1-\alpha)}. \end{aligned} \quad (8.6.8)$$

Here ψ_m and $g_{m,\alpha}$ are the regular and singular parts of ϕ , respectively, $R(x)$ is the remainder such that $R(x) \in C^m[0, X]$ and $R(x) \rightarrow 0$ faster than x^m as $x \rightarrow 0$,

and $\sum_{i,j}^*$ denotes summation over all i, j such that $i + j(1 - \alpha)$ runs through all distinct positive, noninteger values less than m . Some examples of $g_{m,\alpha}$ are

$$\begin{aligned} g_{3,1/2}(x) &= \sum_{n=1,3,5} a_{0n} x^{n/2}, \\ g_{3,1/3}(x) &= \sum_{n=1,2,4} a_{0n} x^{2n/3} + \sum_{n=1,2} a_{1n} x^{1+2n/3}, \\ g_{3,2/3}(x) &= \sum_{n=1,3,5,7,8} a_{0n} x^{n/3}, \\ g_{3,0.9}(x) &= \sum_{i=0}^2 \sum_{j=1}^9 a_{ij} x^{i+j/10}. \end{aligned} \quad (8.6.9)$$

Now, to determine the constants a_{ij} in (8.6.8), we expand $f(x)$ and $\kappa(x, s)$ in terms of Taylor polynomials of degree m (with a remainder), and substitute these expansions into Eq(8.6.1), which yields the following system:

$$\begin{aligned} &\sum_{n=0}^m \frac{\psi_m^{(n)}(0)}{n!} x^n + \sum_{i,j}^* a_{ij} x^{i+j(1-\alpha)} \\ &= \sum_{n=0}^m \frac{f_m^{(n)}(0)}{n!} x^n + \sum_{n=0}^m \sum_{i=0}^m \sum_{j=0}^i \binom{i}{j} \frac{\psi_m^{(n)}(0)}{n! i!} \frac{\partial^i \kappa(0, 0)}{\partial x^j \partial s^{i-j}} \\ &\quad \times x^{i+n+j(1-\alpha)} B(i - j + n + 1, 1 - \alpha) \\ &+ \sum_{i,j}^* \sum_{p=0}^m \sum_{q=0}^p \frac{\binom{p}{q}}{p!} a_{ij} \frac{\partial^p \kappa(0, 0)}{\partial x^q \partial s^{p-q}} \\ &\quad \times x^{p+i+j(1-\alpha)+q(1-\alpha)} B(p - q + i + j(1 - \alpha) + 1, 1 - \alpha) \\ &+ \{\text{remainder terms}\}. \end{aligned} \quad (8.6.10)$$

By collecting the coefficients of different powers of x on both sides of (8.6.10), we obtain a system of algebraic equations which when solved, either analytically or numerically, yields the coefficients a_{ij} and, hence, $\psi_m^{(n)}(0)$ for $i = 0, 1, \dots, m$.

In the following two examples (Abdalkhani 1993) we use the notation

$$\begin{aligned} \kappa_1 &= \kappa(0, 0), \quad X_1 = \frac{\partial \kappa(0, 0)}{\partial x}, \quad S_1 = \frac{\partial \kappa(0, 0)}{\partial s}, \\ X_2 &= \frac{\partial^2 \kappa(0, 0)}{\partial x^2}, \quad S_2 = \frac{\partial^2 \kappa(0, 0)}{\partial s^2}, \\ M_1 &= \frac{\partial^2 \kappa(0, 0)}{\partial x \partial s}, \quad \psi_{m,0}^{(n)} = \psi_m^{(n)}(0). \end{aligned}$$

These examples show the complexity of formula (8.6.10), which increases as m increases and α takes different values.

EXAMPLE 8.6.1. Let $m = 3$ and $\alpha = 1/2$; we have already computed $g_{3,1/2}(x)$ in (8.6.9). Now, from (8.6.10) we get

$$\psi_{3,0} = f(0),$$

$$a_{01} = \psi_{3,0} \kappa_1 B(1/2, 1) = 2f(0) \kappa_1,$$

$$\psi'_{3,0} = f'(0) + a_{01} \kappa_1 B(1/2, 3/2) = f'(0) + \pi f(0) \kappa_1^2,$$

$$a_{03} = \psi_{3,0} [X_1 B(1/2, 1) + S_1 B(1/2, 2)] + \psi'_{3,0} \kappa_1 B(1/2, 2),$$

$$\psi''_{3,0} = f''(0) + 2a_{01} [X_1 B(1/2, 5/2) + S_1 B(1/2, 3/2)] + 2a_{03} \kappa_1 B(1/2, 5/2),$$

$$a_{05} = \frac{1}{2} \psi_{3,0} [X_2 B(1/2, 1) + S_2 B(1/2, 3) + 2M_1 B(1/2, 2)]$$

$$+ \psi'_{3,0} [X_1 B(1/2, 2) + S_1 B(1/2, 3)] + \frac{1}{2} \psi''_{3,0} \kappa_1 B(1/2, 3). \blacksquare$$

EXAMPLE 8.6.2. Let $m = 3$ and $\alpha = 2/3$; we have already computed $g_{3,2/3}(x)$ in (8.6.9). Now, from (8.6.10) we get

$$\psi_{3,0} = f(0),$$

$$a_{01} = \psi_{3,0} \kappa_1 B(1, 1/3) = 3f(0) \kappa_1,$$

$$a_{02} = a_{01} \kappa_1 B(4/3, 1/3),$$

$$\psi'_{3,0} = f'(0) + a_{02} \kappa_1 B(5/3, 1/3),$$

$$a_{04} = \psi_{3,0} [X_1 B(1, 1/3) + S_1 B(2, 1/3) + \psi'_{3,0} \kappa_1 B(2, 1/3)],$$

$$a_{05} = a_{01} [X_1 B(4/3, 1/3) + S_1 B(5/3, 1/3)] + a_{04} \kappa_1 B(7/3, 1/3),$$

$$\psi''_{3,0} = f''(0) + 2a_{02} [X_1 B(5/3, 1/3) + S_1 B(18/3, 1/3)] + 2a_{05} \kappa_1 B(8/3, 1/3),$$

$$a_{07} = 2\psi_{3,0} [X_2 B(1, 1/3) + S_2 B(3, 1/3) + 2M_1 B(2, 1/3)]$$

$$+ \psi'_{3,0} [X_1 B(2, 1/3) + S_1 B(3, 1/3)] + \frac{1}{2} \psi''_{3,0} \kappa_1 B(3, 1/3),$$

$$a_{08} = 2 [X_2 B(4/3, 1/3) + S_2 B(10/3, 1/3) + 2M_1 B(7/3, 1/3)]$$

$$+ a_{04} [X_1 B(7/3, 1/3) + S_1 B(10/3, 1/3)] + a_{07} \kappa_1 B(10/3, 1/3). \blacksquare$$

Now, we discuss some methods to numerically solve Eq (8.6.1) (see Abdalkhani 1993).

(a) COLLOCATION METHOD. This method is discussed in detail in Brunner and van der Houwen (1986), Ch. 5, pp. 347–398) and works as follows: Partition the interval $[0, X]$ at the points (knots) $x_j = jh$, $j = 0, 1, \dots, N$, so that the step size $h = X/N$, $N \geq 1$. Let $z_n = \{x_0, x_1, \dots, x_N\}$, and denote the set of all piecewise polynomials of degree m at the set $\{z_n\}$ by $P_m(z_n)$. Note that these polynomials may have finite discontinuities at the knots x_j . Now, to find the approximate solution $\tilde{\psi}(x)$ of Eq (8.6.2), define the collocation set $X(N)$ by

$$X(N) = \bigcup_{i=0}^{n-1} X_i, \\ X_i = \{x_n + c_i h : 0 \leq c_1 < c_2 < \dots < c_{m+1} \leq 1\}, \\ i = 1, 2, \dots, m+1; \quad n = 0, 1, \dots, N-1. \quad (8.6.11)$$

Let $\psi \in P_m(z_n)$ satisfy Eq (8.6.2) on the set $X(N)$ in the interval $[x_n, x_{n+1}]$, where $x = x_n + c_i h$. Then

$$\begin{aligned} \psi_n(x_n + c_i h) &= \tilde{F}(x_n + c_i h) \\ &+ h^{1-\alpha} \left[\sum_{p=0}^{n-1} \int_0^1 \frac{\kappa(x_n + c_i h, x_p + sh) \psi_p(x_p + sh)}{(n-p+c_i-s)^\alpha} ds \right. \\ &\quad \left. + \int_0^{c_i} \frac{\kappa(x_n + c_i h, x_n + sh) \psi_n(x_n + sh)}{(c_i - s)^\alpha} ds \right], \end{aligned} \quad (8.6.12)$$

where $\tilde{F}(x_n + c_i h)$ is the approximation of $F(x)$, computed from (8.6.3) by using a quadrature rule and by evaluating $g(x)$ analytically or numerically. The integrals (moments) in (8.6.12) are computed by quadrature, thus yielding the approximate solution

$$\begin{aligned} \tilde{\psi}_n(x_n + c_i h) &= \tilde{F}(x_n + c_i h) \\ &+ h^{1-\alpha} \left[\sum_{p=0}^{n-1} \sum_{j=1}^{m+1} w_{i,j}^{n,p} \kappa(x_n + c_i h, x_p + c_j h) \tilde{\psi}_p(x_p + sh) \right. \\ &\quad \left. + \sum_{j=1}^q b_{ij} \kappa(x_n + c_i h, x_n + c_j h) \tilde{\psi}_n(x_n + c_j h) \right], \end{aligned} \quad (8.6.13)$$

where $q = (i-1)$ or i , and the weights are given by

$$\begin{aligned} w_{i,j}^{n,p} &= \int_0^1 (n-p+c_i-s)^{-\alpha} l_j(s) ds, \\ b_{ij} &= \int_0^{c_i} (c_i-s)^{-\alpha} l_j^*(s) ds, \end{aligned} \quad (8.6.14)$$

where $l_j(s)$ and $l_j^*(s)$ are the corresponding Lagrange interpolation polynomials ($l_j^*(s)$ depends on i, j , and q). Note that $q = (i - 1)$ gives an explicit method, whereas $q = i$ gives an implicit one. Abdalkhani (1993) gives the error estimate as

$$|\psi(x) - \tilde{\psi}(x)| \leq C h^m \quad \text{for all } x \in [0, X] \text{ as } h \rightarrow 0^+,$$

where C is a constant independent of h and N ; this estimate holds for any choice of c_i , $i = 1, 2, \dots, m + 1$, which are defined as in (8.6.11).

(b) IMPLICIT RUNGE-KUTTA METHOD. (IRK) Take $m = 3$, $q = i = 1, 2, 3$, and $c_1 = 0$, $c_2 = 1/2$, $c_3 = 1$ in (8.6.13). Also, define

$$\begin{aligned} \psi_p^j &= \tilde{\psi}_p(x_p + c_i h), \quad p = 0, 1, \dots, n - 1, \\ \psi_n^i &= \tilde{\psi}_p(x_p + c_i h), \quad n = 0, 1, \dots, N - 1, \\ \psi_n^1 &= \psi_{n-1}^3, \quad n = 1, 2, \dots, N. \end{aligned}$$

Then for these choices of m , c_i , and $\alpha = 1/2$, the coefficients b_{ij} , defined in (8.6.14), are given by

$$\begin{aligned} b_{11} &= b_{12} = b_{13} = b_{23} = 0, \quad b_{21} = \sqrt{2}/3, \quad b_{22} = 2\sqrt{2}/3, \\ b_{31} &= 2/15, \quad b_{32} = 16/15, \quad b_{33} = 4/5. \end{aligned}$$

(c) A BLOCK-BY-BLOCK method is obtained using the IRK; details can be found in Linz (1985).

The methods in (b) and (c) have the same rate of convergence as that in (a).

EXAMPLE 8.6.3. (Abdalkhani 1993) Consider the SVK2

$$\phi(x) - \lambda \int_0^x e^{\gamma(x-s)} (x-s)^{-1/2} \phi(s) ds = (1 - \lambda^2 \pi x) e^{\gamma x}, \quad (8.6.15)$$

which has the exact solution $\phi(x) = (1 - 2\lambda \sqrt{x}) e^{\gamma x}$. The singular part of $\phi(x)$ for $m = 3$ is given by

$$g(x) = \sqrt{x} (2\lambda + 2\lambda \gamma x + \lambda \gamma^2 x^2). \quad (8.6.16)$$

Eq (8.6.15) is solved numerically for different values of λ and γ , and $g(x)$ given by (8.6.16); it is also solved for $g(x) = 0$, which is the case that includes singularity.

The integrals of the form $\int_0^x e^{\gamma(x-s)} (x-s)^{-1/2} ds$ are computed by quadrature

rules given in Krylov, Lugin and Yanavich (1963). The results for maximum relative errors are given below in Tables 8.6.3, where
 column (a) uses the IRK method, with $g = 0$, $h = 0.01$, $0 \leq x \leq 1$;
 column (b) uses the IRK method, with g as in (8.6.16), $h = 0.01$, $0 \leq x \leq 2$; and
 column (c) uses the block-by-block method, with g as in (8.6.16), $h = 0.01$,
 $0 \leq x \leq 2$.

Table 8.6.3. Maximum Relative Errors

	(a)	(b)	(c)
$\lambda = 1, \gamma = 0$	1.38 at $x = 0.9$	2.71(-6) at $x = 2$	1.97(-6) at $x = 2$
$\lambda = 1, \gamma = -1$	0.19 at $x = 1$	0.02 at $x = 2$	1.28(-6) at $x = 2$
$\lambda = 1, \gamma = 1$	1.41 at $x = 1$	0.205 at $x = 2$	9.96(-5) at $x = 2$
$\lambda = -1, \gamma = 0$	2.22(-3) at $x = 0.1$	5.58(-9) at $x = 1.9$	1.16(-8) at $x = 0.2$
$\lambda = -1, \gamma = -1$	2.009(-3) at $x = 0.1$	7.75(-4) at $x = 2$	3.16(-8) at $x = 2$
$\lambda = -1, \gamma = 1$	2.45(-3) at $x = 0.1$	2.62(-4) at $x = 2$	3.72(-8) at $x = 1.9$

Columns (b) and (c) show that the removal of singularity improves the accuracy of approximate solution. Also note that for some values of α , i.e., $\alpha = 0.9$, the computation of $g(x)$ becomes difficult even in the case of “good” equations because of the value of $g_{3,0.9}$ in (8.6.9). ■

REFERENCES USED: Abdalkhani (1993), Atkinson (1966, 1967, 1972, 1976, 1985), Auer and Gardner (1955), Baker (1978), de Boor (1973), Brunner (1983, 1984), Brunner and van der Houwen (1986), Capobianco (1995), Carleman (1922), de Hoog and Weiss (1973a,b), de Klerk, Eyre and Venter (1995), Delves and Mohamed (1985), Kabir, Madenci and Ortega (1999), Kaneko and Xu (1991), Kaya and Erdogan (1987), Kershaw (1982), Kondo (1991), Kussmaul and Werner (1968), Krylov, Lugin and Yanavich (1963), Laurie and Venter (1993), Linz (1969, 1985), Lubich (1983, 1985), Lyness and Ninham (1967), Phillips (1969), Piessens, de Donker-Kapenga, Überhuber and Kahaner (1983), Schneider (1979, 1981), Vainikko and Ubas (1981).

9

Cauchy Singular Equations

We first define singular integral equations of both the first and second kinds (SK1 and SK2) with the Cauchy kernel (also called Cauchy singular equations) and then present some useful numerical methods to solve them. These equations are encountered in many applications in aerodynamics, elasticity, and other areas.

9.1. Cauchy Singular Equations of the First Kind

A Cauchy singular equation of the first kind (CSK1) is defined in the complex plane C by

$$\frac{1}{i\pi} \int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta = f(z), \quad (9.1.1)$$

where Γ is a closed or nonclosed contour in C with $z = x + iy$, and ζ and z are (complex) points on Γ , $\phi(z)$ is the unknown function, $\frac{1}{\zeta - z}$ is the Cauchy kernel, and $f(z)$ is the free term. The integral on the left side of Eq (9.1.1) exists only as the Cauchy p.v. (see Appendix D). If Γ is a closed contour, special cases of Eq (9.1.1) are the following:

- (i) A CSK1 on the real axis has the form

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(s)}{s - x} ds = f(x), \quad -\infty < x < \infty. \quad (9.1.2)$$

(ii) A CSK1 on a finite interval $[a, b]$ has the form

$$\frac{1}{\pi} \int_a^b \frac{\phi(s)}{s-x} ds = f(x), \quad a \leq x \leq b < \infty. \quad (9.1.3)$$

In Eqs (9.1.1)–(9.1.3) it is assumed that all functions satisfy the Hölder condition. A general form of a CSK1 in C is

$$\frac{1}{i\pi} \oint_{\Gamma} \frac{g(z, \zeta)}{\zeta - z} d\zeta = f(z), \quad (9.1.4)$$

where $g(z, \zeta)$ is a given function that satisfies the Hölder condition with respect to both variables, and all other functions satisfy the Hölder condition as in Eqs (9.1.1)–(9.1.3).

If in Eq (9.1.1) we replace the variable z by ζ_1 , multiply by $\frac{1}{i\pi} \frac{d\zeta_1}{\zeta_1 - z}$, integrate along Γ , and change the order of integration by using the Poincaré–Betrand formula (E.2.13) with $g(z, z) = \phi(z)$ in this case, then we obtain

$$\frac{1}{i\pi} \int_{\Gamma} \frac{f(\zeta_1) d\zeta_1}{\zeta_1 - z} = \phi(z) + \frac{1}{i\pi} \int_{\Gamma} \phi(\zeta) d\zeta \cdot \frac{1}{i\pi} \oint_{\Gamma} \frac{d\zeta_1}{(\zeta_1 - z)(\zeta - \zeta_1)},$$

where the second integral on the right side for a closed contour Γ is

$$\oint_{\Gamma} \frac{d\zeta_1}{(\zeta_1 - z)(\zeta - \zeta_1)} = \frac{1}{\zeta - z} \left(\int_{\Gamma} \frac{d\zeta_1}{\zeta_1 - \zeta} - \int_{\Gamma} \frac{d\zeta_1}{\zeta_1 - z} \right) = \frac{1}{\zeta - z} (i\pi - i\pi) = 0.$$

Hence, the solution of Eq (9.1.1) for a closed contour Γ is

$$\phi(z) = \frac{1}{i\pi} \oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Some examples of CSK1 include:

(i) The solution of the CSK1 (9.1.2) is given by

$$\phi(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{s-x} ds.$$

The integral equation and its solution form a Hilbert transform pair in the symmetric form.

(ii) The integral equation (9.1.3) represents the problem of flow of an ideal inviscid fluid around a thin profile ($a \leq x \leq b$) under the assumption that $|a| + |b| < \infty$. Its solutions are as follows:

(a) The solution bounded at both endpoints is given by

$$\phi(x) = -\frac{\sqrt{(x-a)(b-x)}}{\pi} \int_a^b \frac{f(s)}{\sqrt{(s-a)(b-s)}} \frac{ds}{s-x},$$

provided that $\int_a^b \frac{f(s) ds}{\sqrt{(s-a)(b-s)}} = 0$.

(b) The solution bounded at the endpoint $x = a$ and unbounded at the other endpoint $x = b$ is given by

$$\phi(x) = -\frac{1}{\pi} \sqrt{\frac{x-a}{b-x}} \int_a^b \sqrt{\frac{b-s}{s-a}} \frac{f(s)}{s-x} ds.$$

(c) The solution unbounded at both endpoints is given by

$$\phi(x) = -\frac{1}{\pi \sqrt{(x-a)(b-x)}} \left[\int_a^b \frac{\sqrt{(s-a)(b-s)}}{s-x} f(s) ds + C \right],$$

where C is an arbitrary constant. Note that $\int_a^b \phi(s) ds = C/\pi$.

9.2. Approximations by Trigonometric Polynomials

Consider a CSK1 on the interval $[-1, 1]$ defined by

$$\frac{1}{\pi} \int_{-1}^1 \frac{\phi(s)}{s-x} ds + \frac{1}{\pi} \int_{-1}^1 k(x, s)\phi(s) ds = f(x). \quad (9.2.1)$$

This equation is encountered in problems of aerodynamics and plane elasticity (see Kalantyra 1973). We consider the following three cases.

CASE 1. SOLUTION UNBOUNDED AT EACH ENDPOINT. According to Mushkhelishvili (1992), this solution is of the form

$$\phi(x) = \frac{g(x)}{\sqrt{1-x^2}}, \quad (9.2.2)$$

where $g(x)$ is a bounded function on $[-1, 1]$. Substituting (9.2.2) into Eq (9.2.1) and setting $x = \cos \theta$, $s = \cos \sigma$, $0 \leq \theta, \sigma \leq \pi$, we get

$$\frac{1}{\pi} \int_{-1}^1 \frac{g(\cos \sigma)}{\cos \sigma - \cos \theta} d\sigma + \frac{1}{\pi} \int_{-1}^1 k(\cos \theta, \cos \sigma) g(\cos \sigma) d\sigma = f(\cos \theta). \quad (9.2.3)$$

We replace the function $g(x)$ by the Lagrange interpolation polynomials l_n with Chebyshev nodes $x_m = \cos \theta_m$, $\theta_m = \frac{2m-1}{2n} \pi$, $m = 1, \dots, n$, where

$$l_n(g; \cos \theta) = \frac{1}{n} \sum_{j=1}^n (-1)^{j+1} g(\cos \theta_j) \frac{\cos n\sigma \sin \theta_j}{\cos \theta - \cos \theta_j}. \quad (9.2.4)$$

For each j the fraction on the right side of (9.2.4) is an even trigonometric polynomial of degree $\leq (n-1)$, and

$$\frac{1}{\pi} \int_0^\pi \frac{\cos n\sigma}{\cos \sigma - \cos \theta} d\sigma = \frac{\sin n\theta}{\sin \theta}, \quad n = 0, 1, 2, \dots. \quad (9.2.5)$$

We write (9.2.4) as

$$l_n(g; \cos \theta) = \frac{2}{n} \sum_{j=1}^n g(\cos \theta_j) \sum_{m=0}^{n-1} \cos \theta_j \cos m\theta - \frac{1}{n} \sum_{j=1}^n g(\cos \theta_j).$$

Then we obtain the following quadrature rule for the singular part of Eq (9.2.1):

$$\frac{1}{\pi} \int_{-1}^1 \frac{\phi(s)}{s - x} ds = \frac{2}{n \sin \theta} \sum_{j=1}^n g(\cos \theta_j) \sum_{m=1}^{n-1} \cos m\theta_m \sin m\theta. \quad (9.2.6)$$

This formula is exact only in the case when $g(s)$ is a polynomial of degree $\leq (n-1)$.

For the second integral (regular part) on the left side of Eq (9.2.1) we use the relation

$$\frac{1}{\pi} \int_{-1}^1 \frac{P(x) dx}{\sqrt{1-x^2}} = \frac{1}{n} \sum_{j=1}^n P(\cos \theta_j),$$

where $P(x)$ is a polynomial of degree $\leq (2n-1)$. Then

$$\frac{1}{\pi} \int_{-1}^1 k(x, s) \phi(s) ds = \frac{1}{n} \sum_{j=1}^n k(\cos \theta, \cos \theta_j) g(\cos \theta_j). \quad (9.2.7)$$

Thus, substituting (9.2.6) and (9.2.7) into Eq (9.2.1), we get

$$\begin{aligned} \frac{2}{n \sin \theta} \sum_{j=1}^n g(\cos \theta_j) \sum_{m=1}^{n-1} \cos m\theta_j \sin m\theta + \frac{1}{n} \sum_{j=1}^n k(\cos \theta, \cos \theta_j) g(\cos \theta_j) \\ = f(\cos \theta). \end{aligned} \quad (9.2.8)$$

If we set $\theta = \theta_i$, $i = 1, \dots, n$, and use the relation

$$\sum_{m=1}^{n-1} \cos m\theta_j \sin m\theta_i = \frac{1}{2} \cot \frac{\theta_i \pm \theta_j}{2}, \quad (9.2.9)$$

where the \pm sign is chosen according as $|i - j|$ is even or odd, we obtain from (3.2.8) the following system of linear algebraic equations:

$$\begin{aligned} \sum_{j=1}^n a_{ij} g_j &= f_i, \quad f_i = f(\cos \theta_i), \quad i = 1, \dots, n, \\ a_{ij} &= \frac{1}{n} \left[\csc \theta_i \cot \frac{\theta_i \pm \theta_j}{2} + k(\cos \theta_i, \cos \theta_j) \right], \end{aligned} \quad (9.2.10)$$

which determines the approximate values of $g_j = g(\cos \theta_j)$. This solution for g_j is then substituted into (9.2.4) and (9.2.2) to obtain the approximate solution $\tilde{\phi}$ of Eq (9.2.1) at the nodes x_j .

CASE 2. SOLUTION BOUNDED AT ONE ENDPOINT. Let the solution be bounded at the endpoint $x = -1$. Then we take

$$\phi(x) = \sqrt{\frac{1-x}{1+x}} h(x), \quad (9.2.11)$$

where $h(x)$ is bounded on $[-1, 1]$. We choose the same interpolation nodes as in Case 1, replace $h(x)$ by the trigonometric polynomial

$$l_n(h; \cos \theta) = \frac{1}{n} \sum_{j=1}^n (-1)^{j+1} h(\cos \theta_j) \frac{\cos n\theta \sin \theta_j}{\cos \theta - \cos \theta_j}, \quad (9.2.12)$$

and substitute the result into the singular part of Eq (9.2.1). This gives the following quadrature rule for the singular part of Eq (9.2.1):

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{\phi(s)}{s-x} ds &= 2 \frac{1-\cos \theta}{n \sin \theta} \sum_{j=1}^n h(\cos \theta_j) \sum_{m=1}^{n-1} \cos m\theta_j \sin m\theta \\ &\quad - \frac{1}{n} \sum_{j=1}^n h(\cos \theta_j), \end{aligned} \quad (9.2.13)$$

which is exact only if $h(s)$ is a polynomial of degree $\leq (n - 1)$. The quadrature rule for the regular part of Eq (9.2.1) is

$$\frac{1}{\pi} \int_{-1}^1 k(x, s) \phi(s) ds = \frac{1}{n} \sum_{j=1}^n (1 - \cos \theta_j) k(\cos \theta, \cos \theta_j) h(\cos \theta_j), \quad (9.2.14)$$

which is exact only if the integrand is a polynomial of degree $\leq (2n - 2)$. Now, after substituting (9.2.13) and (9.2.14) into Eq (9.2.1) and using $\theta = \theta_i$ ($i = 1, \dots, n$) and the relation (9.2.9), we obtain the following system of algebraic equations:

$$\begin{aligned} \sum_{j=1}^n b_{ij} h_j &= f_i, \quad f_i = f(\cos \theta_i), \quad i = 1, \dots, n, \\ b_{ij} &= \frac{1}{n} \left[\tan \frac{\theta_i}{2} \cot \frac{\theta_i \pm \theta_j}{2} - 1 + 2 \sin^2 \frac{\theta_j}{2} k(\cos \theta_i, \cos \theta_j) \right], \end{aligned} \quad (9.2.15)$$

which determines the approximate value $h_j = h(\cos \theta_j)$. This solution for h_j is then used in (9.2.12) and (9.2.11) to obtain the approximate solution $\tilde{\phi}$ of Eq (9.2.1) at the nodes x_j .

Note that if the solution is unbounded at the other endpoint $x = 1$, we take

$$\phi(x) = \sqrt{\frac{1+x}{1-x}} h(x).$$

CASE 3. SOLUTION BOUNDED AT BOTH ENDPOINTS. In this case the solution $\phi(x)$ must satisfy the boundary conditions $\phi(-1) = 0 = \phi(1)$. We approximate $\phi(x)$ by an even trigonometric polynomial in θ , which is constructed for the nodes $x_j = \cos \theta_j$ of the corresponding Chebyshev polynomial of the second kind, namely, by

$$M_n(\phi; \cos \theta) = \frac{2}{n+1} \sum_{j=1}^n \phi(\cos \theta_j) \sum_{m=1}^n \sin m\theta_j \sin m\theta. \quad (9.2.16)$$

This leads to the following quadrature rule for the singular part of Eq (9.2.1):

$$\frac{1}{\pi} \int_{-1}^1 \frac{\phi(s)}{s-x} ds = -\frac{2}{n+1} \sum_{j=1}^n \phi(\cos \theta_j) \sum_{m=1}^n \sin m\theta_j \cos m\theta, \quad (9.2.17)$$

which is exact for any odd trigonometric polynomial ϕ of degree $\leq n$. For the regular part of Eq (9.2.1) we use the relation

$$\int_{-1}^1 \sqrt{1-x^2} P(x) dx = \frac{\pi}{n+1} \sum_{j=1}^n \sin^2 \theta_j P(\cos \theta_j), \quad (9.2.18)$$

where $P(x)$ is a polynomial of degree $\leq (2n - 1)$, as in Case 1. Thus, in view of (9.2.18), the quadrature rule for the regular part of Eq (9.2.1) is

$$\frac{1}{\pi} \int_{-1}^1 k(x, s)\phi(s) ds = \frac{1}{n+1} \sum_{j=1}^n \sin \theta_j k(\cos \theta, \cos \theta_j) \phi(\cos \theta_j). \quad (9.2.19)$$

Now, after we substitute (9.2.17) and (9.2.19) into Eq (9.2.1) and set $\theta = \theta_i$, $i = 1, \dots, n$, we obtain the following system of algebraic equations:

$$\begin{aligned} \sum_{j=1}^n c_{ij} h_j &= f_i, \quad f_i = f(\cos \theta_i), \quad i = 1, \dots, n, \\ c_{ij} &= \frac{\sin \theta_j}{n+1} \left[\frac{2e_{ij}}{\cos \theta_j - \cos \theta_i} + k(\cos \theta_i, \cos \theta_j) \right], \\ e_{ij} &= \begin{cases} 0 & \text{for even } |i-j|, \\ 1 & \text{for odd } |i-j|. \end{cases} \end{aligned} \quad (9.2.20)$$

Once this system is solved for $h_j = h(\cos \theta_j)$, the approximate solution $\tilde{\phi}$ is computed from (9.2.16).

While using this method, it is important to have the approximate solution of the form

$$\tilde{\phi}(x) = (1-x)^\alpha (1+x)^\beta \Psi(x), \quad (9.2.21)$$

where $\alpha = \pm 1/2 = \beta$, and $\Psi(x)$ is a bounded function on the interval $[-1, 1]$ with well-defined values $\Psi(-1)$ and $\Psi(1)$. If the representation (9.2.21) holds, then this method can also be applied to Cauchy singular integral equations of the second kind (CSK2); see the next section.

9.3. Cauchy Singular Equations of the Second Kind

A singular equation of the second kind with a Cauchy kernel (CSK2) is generally of the form

$$a(x)\phi(x) + \frac{b}{i\pi} \int_{\Gamma} \frac{g(x, s)}{s-x} \phi(s) ds = f(x), \quad (9.3.1)$$

where Γ is a closed or nonclosed contour, and s and x are points on Γ . It is assumed that the functions $a(x)$, $\phi(x)$, and $f(x)$ satisfy the Hölder condition on

Γ and the function $g(x, s)$ also satisfies the Hölder condition on Γ with respect to both variables. The kernel of Eq (9.3.1) can be written as

$$\frac{g(x, s)}{s - x} = \frac{g(x, s) - g(x, x)}{s - x} + \frac{g(x, x)}{s - x} = f(x),$$

and if we set $g(x, x) = b(x)$ and $\frac{g(x, s) - g(x, x)}{s - x} = i\pi k(x, s)$, then Eq (9.3.1) takes the form

$$a(x)\phi(x) + \frac{b(x)}{i\pi} \int_{\Gamma} \frac{\phi(s)}{s - x} ds + \int_{\Gamma} k(x, s)\phi(s) ds = f(x), \quad (9.3.2)$$

where $b(x)$ satisfies the Hölder condition on Γ , and $k(x, s)$ satisfies the Hölder condition everywhere on Γ except at the points $s = x$, where we have $|k(x, s)| < A|s - x|^{-\alpha}$, A a positive constant, and $0 \leq \alpha < 1$. Eq (9.3.2) is taken as the standard form of a CSK2, where $a(x)$ and $b(x)$ are called the coefficients of the equation, $\frac{1}{s - x}$ the Cauchy kernel, $k(x, s)$ the kernel of the regular part, and $f(x)$ the free term. The first term on the left side of Eq (9.3.2) is also known as the *characteristic part* of the equation.

Eqs (9.3.1) and (9.3.2) can be written in the operator form as

$$(K\phi)(x) \equiv (K_s + K_r)(x) = f(x), \quad (9.3.3)$$

where $K = K_s$ is known as the *singular operator* such that the equation

$$(K_s\phi)(x) \equiv a(x)\phi(x) + \frac{b(x)}{i\pi} \int_{\Gamma} \frac{\phi(s)}{s - x} ds = f(x) \quad (9.3.4)$$

is called the *characteristic equation* for Eq (9.3.2), and the operator K_s is sometimes called the *characteristic operator*. For the regular part of Eq (9.3.2) we have

$$(K_r\phi)(x) \equiv \int_{\Gamma} k(x, s)\phi(s) ds, \quad (9.3.5)$$

where K_r is called the *regular (Fredholm) operator*. Thus, Eq (9.3.2) can be written as

$$(K\phi)(x) \equiv (K_s\phi)(x) + (K_r\phi)(x) = f(x). \quad (9.3.6)$$

The equation

$$(K^T\phi)(x) \equiv a(x)\phi(x) - \frac{1}{i\pi} \int_{\Gamma} \frac{b(s)\phi(s)}{s - x} ds + \int_{\Gamma} k(s, x)\phi(s) ds = g(x) \quad (9.3.7)$$

is the *transposed* CSK2, where K^T is the transposed operator of the operator K . The equation

$$\left((K_s)^T \phi \right) (x) \equiv a(x)\phi(x) - \frac{1}{i\pi} \int_{\Gamma} \frac{b(s)\phi(s)}{s-x} ds = g(x) \quad (9.3.8)$$

is the *transposed characteristic equation* to Eq (9.3.4). Note that $(K_s)^T \neq (K^T)_s$, where

$$\left((K^T)_s \phi \right) (x) \equiv a(x)\phi(x) - \frac{b(x)}{i\pi} \int_{\Gamma} \frac{\phi(s)}{s-x} ds. \quad (9.3.9)$$

Some examples of CSK2 include the following:

(i) Consider the CSK2

$$a\phi(x) + \frac{b}{\pi} \int_{-1}^1 \frac{\phi(s)ds}{s-x} = f(x), \quad -1 < x < 1,$$

where, without loss of generality, we can take $a^2 + b^2 = 1$. We have three cases:

(a) The solution bounded at both endpoints:

$$\phi(x) = a f(x) - \frac{b}{\pi} \int_{-1}^1 \frac{g(x)}{g(s)} \frac{f(s)ds}{s-x}, \quad (9.3.10)$$

where $g(x) = (1+x)^\alpha(1-x)^{1-\alpha}$, and α is a root of the equation

$$a + b \cot(\pi\alpha) = 0 \quad (9.3.11)$$

on the interval $0 < \alpha < 1$. The above solution exists iff $\int_{-1}^1 \frac{f(s)}{g(s)} ds = 0$.

(b) The solution bounded at the endpoint $x = 1$ and unbounded at $x = -1$ is given by (9.3.10) with $g(x) = (1+x)^\alpha(1-x)^{-\alpha}$, where in this case α is the root of Eq (9.3.11) on the interval $-1 < \alpha < 0$.

(c) The solution unbounded at both endpoints:

$$\phi(x) = a f(x) - \frac{b}{\pi} \int_{-1}^1 \frac{g(x)}{g(s)} \frac{f(s)ds}{s-x} + C g(x),$$

where $g(x) = (1+x)^\alpha(1-x)^{-1-\alpha}$, C is an arbitrary constant, and α is a root of Eq (9.3.11) on the interval $0 < x < 1$.

(ii) The Tricomi equation

$$\phi(x) - \lambda \int_0^1 \left(\frac{1}{s-x} - \frac{1}{x+s-2xs} \right) \phi(s) ds = f(x), \quad 0 < x < 1,$$

has the solution

$$\phi(x) = \frac{1}{1+\lambda^2\pi^2} \left[f(x) + \int_0^1 \left(\frac{1}{s-x} - \frac{1}{x+s-2xs} \right) f(s) ds \right] + \frac{C(1-x)^\beta}{x^{1+\beta}},$$

where C is an arbitrary constant, and

$$\begin{aligned} \alpha &= \frac{2}{\pi} \tan^{-1}(\lambda\pi), \quad -1 < \alpha < 1, \\ \beta &= \frac{2}{\pi} \tan^{-1}(\lambda\pi), \quad -2 < \beta < 0. \end{aligned}$$

Eq (9.3.1) can be solved in a closed form if $a(x) = a$ and $b(x) = b$ are constants and the kernel $k(x, s)$ is analytic in each variable in the domain $\Omega^+ = \text{Int}(\Gamma)$, where Γ is the (closed) boundary of this domain. Then Eq (9.3.1) becomes

$$a \phi(x) + \frac{1}{i\pi} \int_{\Gamma} \frac{g(x, s)}{s-x} \phi(s) ds = f(x), \quad (9.3.12)$$

where $g(x, s) = b - i\pi k(x, s)$; hence, $g(x, x) = b$. Let $b \neq 0$ (otherwise Eq (9.3.1) is no longer singular), and write

$$\psi(x) = \frac{1}{i\pi b} \int_{\Gamma} \frac{g(x, s)}{s-x} \phi(s) ds. \quad (9.3.13)$$

Then Eq (9.3.12) can be written as

$$a \phi(x) + b \psi(x) = f(x). \quad (9.3.14)$$

Apply the operation (9.3.13) on Eq (9.3.14). This yields

$$a \psi(x) + b \phi(x) = w(x), \quad (9.3.15)$$

where

$$w(x) = \frac{1}{i\pi b} \int_{\Gamma} \frac{g(x, s)}{s-x} f(s) ds.$$

Hence, by solving Eqs (9.3.14) and (9.3.15) for ϕ , we get

$$\phi(x) = \frac{1}{a^2 - b^2} \left[f(x) - \frac{1}{i\pi} \int_{\Gamma} \frac{g(x, s)}{s - x} f(s) ds \right], \quad a \neq \pm b. \quad (9.3.16)$$

9.4. From CSK2 to FK2

This is known as the Carleman–Vekua reduction method (see Mushkhelishvili 1992, p.155). Consider a CSK2 of the form

$$\phi - \lambda K \circ \phi = f, \quad (9.4.1)$$

with type 2 convolution (see Section 1.2), where the kernel $k(x, s) = \frac{g(x, s)}{x - s}$. Let two distinct Cauchy singular kernels be defined as

$$k_1(x, s) = \frac{g_1(x)}{x - s}, \quad k_2(x, s) = \frac{g_2(x)}{x - s}, \quad (9.4.2)$$

where $g_1, g_2 \in C[a, b]$. We extend the type 2 convolution to

$$k_1 \circ k_2 = \int \frac{g_1(x)}{x - t} \frac{g_2(t)}{t - s} dt. \quad (9.4.3)$$

In general, we have $k_1 \circ (k_2 \circ f) \neq (k_1 \circ k_2) \circ f$. Let

$$R \equiv (k_1 \circ k_2) \circ f - k_1 \circ (k_2 \circ f) \quad (9.4.4)$$

denote the remainder. Since, in view of the Poincaré–Bertrand formula (D.2.14), we have

$$\int_a^b \frac{ds}{x - s} \int_a^b \frac{F(x, t)}{s - t} dt = \int_a^b dt \int_a^b \frac{F(x, t)}{(x - s)(s - t)} dt - \pi^2 F(x, x), \quad (9.4.5)$$

where $F \equiv g_1 g_2 f$, we apply (9.4.5) to $\frac{g_1(x)}{x - s}$ and $\frac{g_2(x)}{x - s}$, and get

$$R = \pi^2 g_1(x) g_2(x) f(x). \quad (9.4.6)$$

Now, let $g_1 = g_2 = g$, and multiply both sides of Eq (9.4.1) by λK from the left; this gives

$$\lambda K \circ \phi - \lambda^2 K \circ (K \circ \phi) = \lambda K \circ f, \quad (9.4.7)$$

which in view of (9.4.6) reduces to

$$K \circ (K \circ \phi) = K^{\circ 2} \circ \phi + \pi^2 g^2 \phi.$$

This relation together with Eq (9.4.1) is then applied to (9.4.7), which becomes

$$(I + \lambda^2 \pi^2 K^{\circ 2}) \circ \phi - \lambda^2 K^{\circ 2} \circ \phi = f + \lambda K \circ f,$$

or

$$[I + \lambda^2 \pi^2 K^{\circ 2}(x)] \phi(x) - \lambda^2 \int_a^b k^{\circ 2}(x, s) \phi(s) ds = f(x) + \lambda \int_a^b k(x, s) f(s) ds. \quad (9.4.8)$$

Since $k^{\circ 2} = k \circ k$ is bounded and, thus, no longer has a singular point, the integral on the left side of (9.4.8) is an ordinary definite integral. Hence, the CSK2 (9.4.1) is reduced to an FK2 (9.4.8) provided $1 + \lambda^2 \pi^2 g^2(x)$ does not vanish at any point on $[a, b]$.

EXAMPLE 9.4.1. (Villat 1916) If $k(x, s) = \frac{1}{2\pi} \cot \frac{x-s}{2}$, $0 \leq x \leq 2\pi$, we have

$$k^{\circ 2} = k \circ k = \frac{1}{4\pi^2} \int_0^{2\pi} \cot \frac{x-t}{2} \cot \frac{t-s}{2} dt = \frac{1}{4\pi^2} \cdot 2\pi = \frac{1}{2\pi}.$$

Then from (9.4.8) we find that

$$(1 + \lambda^2) \phi(x) - \frac{\lambda^2}{2\pi} \int_0^{2\pi} \phi(s) ds = f(x) + \lambda \int_0^{2\pi} k(x, s) f(s) ds \equiv G(x). \quad (9.4.9)$$

Since $G(x)$ is known, the solution of the CSK2 with the above kernel is

$$\phi(x) = \frac{G(x)}{1 + \lambda^2}, \quad (9.4.10)$$

where C is a constant which is determined by substituting this solution into (9.4.9) as

$$C = \frac{\lambda^2}{2\pi} \int_0^{2\pi} G(s) ds.$$

Also see the Hilbert kernel in Section 7.9. ■

9.5. Gauss–Jacobi Quadrature

Erdogan and Gupta (1972), Erdogan, Gupta and Cook (1973), and Krenk (1975) show that by a proper choice of collocation points both CSK1 and CSK2 can be solved using the Gauss–Jacobi quadrature rule. Consider a CSK2 in its general form

$$a_i \phi_i(x) + \frac{b_i}{\pi} \int_{-1}^1 \phi_i(s) \frac{ds}{s-x} + \lambda \sum_{i=1}^N \int_{-1}^1 k_{ij}(x, s) \phi(s) ds = f_i(x), \quad |x| < 1, \quad (9.5.1)$$

for $j = 1, \dots, N$, where a_i and b_i are real constants, the kernels $k_{ij}(x, s) \in H([-1, 1])$, and the free terms $f_i(x)$ are known functions. The unknown functions $\phi_i(x)$ or their first derivatives have integrable singularities at the endpoints $x = \pm 1$. Since a general closed-form solution of Eq (9.5.1) is not known, methods for numerical solution based on Gauss–Jacobi quadrature are given in Erdogan, Gupta and Cook (1973) who, in order to characterize the singular behavior of $\phi_i(x)$, have found a group of fundamental functions defined by

$$w_i(x) = (1-x)^{\alpha_j} (1+x)^{\beta_j}, \quad (9.5.2)$$

where

$$\begin{aligned} \alpha_j &= \frac{1}{2i\pi} \log \frac{a_j - ib_j}{a_j + ib_j} + N_j, \\ \beta_j &= -\frac{1}{2i\pi} \log \frac{a_j - ib_j}{a_j + ib_j} + M_j, \end{aligned} \quad (9.5.3)$$

N_j and M_j being integers for $j = 1, \dots, N$, and for each of the N equations in (9.5.1) the *index* κ_j of the integral operators K_{ij} is defined by

$$\kappa_j = -(\alpha_j + \beta_j) = -(N_j + M_j), \quad j = 1, \dots, N. \quad (9.5.4)$$

Since we have assumed that ϕ_i or their first derivatives have integrable singularities at the endpoints, the index must be $-1, 0, 1$ (see Mushkhelishvili 1992). The numerical solution of Eq (9.5.1) is given by

$$\phi_i(x) = g_i(x) w_i(x), \quad (9.5.5)$$

where

$$g_i(x) = \sum_{i=1}^{\infty} c_{ij} P_j^{(\alpha, \beta)}(x). \quad (9.5.6)$$

Here $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials of degree n with indices α and β , and c_{ij} are constants to be determined. The general scheme is to truncate the series (9.5.6) for $i = 1, \dots, n$, and determine methods to compute the unknown coefficients c_{ij} . We discuss this problem below for a simple case of Eq (9.5.1), but the method can be easily extended to Eq (9.5.1).

9.5.1. Solution by Jacobi Polynomials. For the sake of simplicity we consider a special case of Eq (9.5.1) of the form

$$a\phi(x) + \frac{b}{\pi} \int_{-1}^1 \phi(s) \frac{ds}{s-x} + \lambda \int_{-1}^1 k(x,s)\phi(s) ds = f(x), \quad |x| < 1, \quad (9.5.7)$$

for which, by using the orthogonality properties of the Jacobi polynomials, Erdogan, Gupta and Cook (1973) derive an infinite system of linear algebraic equations

$$\phi(x) = \sum_{n=0}^{\infty} c_n w(x) P_n^{(\alpha,\beta)}(x), \quad (9.5.8)$$

where $w(x) = (1-x)^\alpha(1+x)^\beta$, and c_n , $n = 0, 1, \dots$, are constants to be determined. Before substituting (9.5.8) into Eq (9.5.7), note that for the index $\kappa = (-1, 0, 1)$ we have (see Tricomi 1957, Szegö 1939)

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 w(s) P_n^{(\alpha,\beta)}(s) \frac{ds}{s-x} &= \cot(\pi\alpha) w(x) P_n^{(\alpha,\beta)}(x) \\ &- \frac{2^{\alpha+\beta} \Gamma(\alpha)\Gamma(n+\beta+1)}{\pi\Gamma(n+\alpha+\beta+1)} F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-x}{2}\right), \\ &-1 < x < 1, \quad \Re\{\alpha\} > -1, \quad \Re\{\alpha\} \neq 0, 1, \dots, \quad \Re\{\beta\} > -1, \\ \cot(\pi\alpha) &= \cot\pi\left[\frac{1}{2i\pi} \log\left(\frac{a-ib}{a+ib}\right) + N\right] = -\frac{a}{b}, \quad \alpha + \beta = -\kappa, \end{aligned} \quad (9.5.9)$$

and

$$P_{n-\kappa}^{(-\alpha,-\beta)}(x) = \frac{\Gamma(n-\kappa-\alpha+1)}{\Gamma(1-\alpha)\Gamma(n-\kappa+1)} F\left(n+1, -n+\kappa; 1-\alpha; \frac{1-x}{2}\right). \quad (9.5.10)$$

Hence, combining (9.5.9) and (9.5.10), we get

$$\begin{aligned} a w(x) P_n^{(\alpha,\beta)}(x) + \frac{b}{\pi} \int_{-1}^1 w(s) P_n^{(\alpha,\beta)}(s) \frac{ds}{s-x} \\ = -2^{-\kappa} b \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\pi} P_{n-\kappa}^{(-\alpha,-\beta)}(x), \quad |x| < 1. \end{aligned} \quad (9.5.11)$$

Then, substituting (9.5.8) into Eq (9.5.7) and using (9.5.11), we obtain an infinite system of algebraic equations

$$\sum_{n=0}^{\infty} c_n \left[-\frac{2^{-\kappa} b}{\sin(\pi\alpha)} P_{n-\kappa}^{(-\alpha, -\beta)}(x) + h_n(x) \right] = f(x), \quad (9.5.12)$$

where

$$h_n(x) = \lambda \int_{-1}^1 w(s) P_n^{(\alpha, \beta)}(s) k(x, s) ds, \quad |x| < 1. \quad (9.5.13)$$

Now, we use the orthogonality relations

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(s) P_m^{(\alpha, \beta)}(s) w(s) ds = \begin{cases} 0 & \text{if } n \neq m, \\ \theta_m^{(\alpha, \beta)} & \text{if } n = m, \end{cases} \quad (9.5.14)$$

for $m = 0, 1, 2, \dots$, where (see Gradshteyn and Ryzik 1965)

$$\theta_m^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{2m+\alpha+\beta+1} \frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!\Gamma(m+\alpha+\beta+1)}, \quad m = 0, 1, 2, \dots, \quad (9.5.15)$$

and

$$\theta_0^{(\alpha, \beta)} = \int_{-1}^1 w(s) ds = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \quad (9.5.16)$$

Then we truncate the series (9.5.12) to obtain

$$-\frac{2^{-\kappa} b}{\sin(\pi\alpha)} \theta_m(-\alpha, -\beta) c_{m+\kappa} + \sum_{m=0}^N d_{nm} c_n = F_m, \quad m = 0, 1, \dots, N, \quad (9.5.17)$$

where

$$\begin{aligned} d_{nm} &= \int_{-1}^1 P_m^{(-\alpha, -\beta)}(x) w(-\alpha, -\beta, x) h_n(x) dx, \\ F_m &= \int_{-1}^1 P_m^{(-\alpha, -\beta)}(x) w(-\alpha, -\beta, x) f(x) dx, \\ w(-\alpha, -\beta, x) &= (1-x)^{-\alpha}(1+x)^{-\beta} = w^{-1}(x). \end{aligned} \quad (9.5.18)$$

There are three cases to consider.

CASE 1. $\kappa = -1$: Note that the first term of the series (9.5.12) is equal to a constant multiplied by $c_0 P_1^{(-\alpha, -\beta)}(x)$. Hence, in solving Eq (9.5.7) we can take

$c_{-1} = 0$. Also, since $P_0^{(-\alpha, -\beta)}(x) = 1$, it can be seen from Eqs (9.5.17) and (9.5.18) that the first equation obtained from (9.5.17) for $m = 0$ is equivalent to the consistency condition

$$\int_{-1}^1 \left[f(x) - \int_{-1}^1 k(x, s)\phi(s) ds \right] \frac{ds}{w(s)} = 0. \quad (9.5.19)$$

Thus, Eqs (9.5.17) give $(N+1)$ linear equations to compute the unknown constants c_0, \dots, c_N .

CASE 2. $\kappa = 0$: This case does not need any additional conditions, and Eqs (9.5.17) give the unique solution for c_0, \dots, c_N .

CASE 3. $\kappa = 1$: In this case there are $(N+2)$ unknown constants c_0, \dots, c_{N+1} but only $(N+1)$ equations given by (9.5.17). Thus, we need one more equation, which is provided by the equilibrium or compatibility condition $\int_{-1}^1 \phi(s) ds = A$, which, after its substitution into (9.5.8) and using the orthogonality condition, reduces to

$$c_0 \theta_0(\alpha, \beta) = A. \quad (9.5.20)$$

Then Eqs (9.5.17) together with (9.5.20) are solved to compute the $(N+2)$ constants c_0, \dots, c_{N+1} .

Erdogan et al. (1973) solve the following elasticity problems: (i) layered materials with an interface crack; and (ii) a punch with constant friction on an elastic half-plane. Details of the solution of these problems are not given here because they are not only lengthy but also pertain to specialized interests.

9.5.2. Collocation Method for CSK2. To develop this method we start with (9.5.11), and note that although the two terms on the left side in (9.5.11) are both singular, their sum is regular. Now, we shall form another kind of polynomials $Q_{n,m}^{(\alpha,\beta)}(x)$ from the Jacobi polynomials by defining

$$Q_{n,m}^{(\alpha,\beta)}(x) = P_{n-\kappa}^{(-\alpha, -\beta)}(x) P_m^{(\alpha, \beta)}(x) - P_n^{(\alpha, \beta)}(x) P_{m-\kappa}^{(-\alpha, -\beta)}(x). \quad (9.5.21)$$

If the degree of the polynomials $Q_{n,m}^{(\alpha,\beta)}(x)$ is equal to or less than n , then we have

$$\frac{P_{n-\kappa}^{(-\alpha, -\beta)}(x) P_m^{(\alpha, \beta)}(x) - P_n^{(\alpha, \beta)}(x) P_{m-\kappa}^{(-\alpha, -\beta)}(x)}{P_n^{(\alpha, \beta)}(x)} = - \sum_{j=1}^n \frac{q_j}{s_j - x}, \quad (9.5.22)$$

where the points s_j are roots of the equation

$$P_n^{(\alpha, \beta)}(s_j) = 0, \quad j = 1, 2, \dots, n, \quad (9.5.23)$$

and the coefficients q_j are yet to be determined. Krenk (1975) proves that the polynomials $Q_{n,m}^{(\alpha, \beta)}(x)$ are of degree $n - m - 1$ for $\kappa = -1, 0, 1$, with $n \geq m$, and the coefficients q_j are computed from

$$q_j = \frac{P_{n-\kappa}^{(-\alpha, -\beta)}(s_j) P_m^{(\alpha, \beta)}(s_j)}{P_n^{(\alpha, \beta)'}(s_j)}. \quad (9.5.24)$$

If we choose the collocation points x_i such that

$$P_{n-\kappa}^{(-\alpha, -\beta)}(x_i) = 0 \quad \text{for } i = 1, \dots, n - \kappa, \quad (9.5.25)$$

then we find from (9.5.22) and (9.5.24) that

$$P_{m-\kappa}^{(-\alpha, -\beta)}(x_i) = \sum_{j=1}^n \frac{P_{n-\kappa}^{(-\alpha, -\beta)}(s_j)}{P_n^{(\alpha, \beta)'}(s_j)} \frac{P_m^{(\alpha, \beta)}(s_j)}{s_j - x_i}. \quad (9.5.26)$$

Now, we use (9.5.26) to compute

$$\frac{a}{b} \phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{s - x} \equiv S(x), \quad (9.5.27)$$

which consists of the first two terms on the left side of Eq (9.5.7), where $\phi(x) = g(x)w(x)$. Let $\tilde{g}(x)$ denote the approximation of $g(x)$ by a truncated series of the form (9.5.6), i.e.,

$$\tilde{g}(x) = \sum_{m=0}^p c_m P_m^{(\alpha, \beta)}(x). \quad (9.5.28)$$

Then, an approximation of $S(x)$ at the points x_i is given by

$$\begin{aligned} \tilde{S}(x_i) &= -2^{-\kappa} \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\pi} \sum_{m=0}^p c_m P_{m-\kappa}^{(-\alpha, -\beta)}(x_i) \\ &= -2^{-\kappa} \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\pi} \sum_{m=0}^p \sum_{j=1}^n c_m \frac{P_{n-\kappa}^{(-\alpha, -\beta)}(s_j)}{P_n^{(\alpha, \beta)'}(s_j)} \frac{P_m^{(\alpha, \beta)}(s_j)}{s_j - x_i} \\ &= -2^{-\kappa} \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\pi} \sum_{j=1}^n \frac{P_{n-\kappa}^{(-\alpha, -\beta)}(s_j)}{P_n^{(\alpha, \beta)'}(s_j)} \frac{\tilde{g}(s_j)}{s_j - x_i}. \end{aligned} \quad (9.5.29)$$

First, note that in (9.5.29) the truncation of the series for $\tilde{g}(x)$, given by (9.5.28), is analogous to a quadrature of an integral; also, we must have $p \leq n$ which is equivalent to using $m \leq n$ for all m . Next, note that formula (9.5.29) to compute the first two terms of Eq (9.5.7) is, in fact, a quadrature rule of the form

$$\tilde{S}(x_i) = \sum_{j=1}^n W_j^n \frac{g(s_j)}{s_j - x_i}, \quad i = 1, 2, \dots, n - \kappa, \quad (9.5.30)$$

where the weights W_j^n are given by

$$W_j^n = -2^{-\kappa} \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\pi} \frac{P_{n-\kappa}^{(-\alpha, -\beta)}(s_j)}{P_n^{(\alpha, \beta)'}(s_j)}$$

$$= \begin{cases} -\frac{2^{\alpha+\beta}}{\pi} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n+1)!\Gamma(n+\alpha+\beta+2)} \frac{2n+\alpha+\beta+2}{P_n^{(\alpha, \beta)'}(s_j) P_{n+1}^{(\alpha, \beta)}(s_j)} & \text{for } j = n+1, \\ \frac{2^{\alpha+\beta}}{\pi} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{n!\Gamma(n+\alpha+\beta+1)} \frac{2n+\alpha+\beta}{P_n^{(\alpha, \beta)'}(s_j) P_{n-1}^{(\alpha, \beta)}(s_j)} & \text{for } j = n-1, \end{cases} \quad (9.5.31)$$

where the definition of $Q_{n,j}^{(\alpha, \beta)}(x)$, given by (9.5.21), is used (for details, see Krenk 1978). Note that the weights W_j^n , with the exception of the factor $1/\pi$, are the Gauss–Jacobi weights for the numerical integration of bounded, continuous functions (see Stroud and Secrest 1966, Abramowitz and Stegun 1968).

For $a = 0$ (i.e., for CSK1) we have $\alpha = \pm 1/2 = \beta$. Then the quadrature rule to be used is obtained by trigonometric formulas and explicit expressions for x_i and s_j . These specific formulas are given in Section 9.6.

EXAMPLE 9.5.1. Consider Eq (9.5.7) with $\phi(x) = g(x)(1-x)^\alpha(1+x)^\beta$. Then we have the following three cases.

CASE 1. $\kappa = 1$: In this case $\alpha < 0$ and $\beta < 0$. To solve Eq (9.5.7), we need the additional condition of the form $\frac{1}{\pi} \int_{-1}^1 \phi(s) ds = A$. Then by using (9.5.30) and the standard Gauss–Jacobi quadrature rule (see Stroud and Secrest 1966, Abramowitz and Stegun 1968) we obtain the following system of n linear equations to compute $\tilde{g}(s_j)$, $j = 1, 2, \dots, n$:

$$\sum_{j=1}^n W_j^n \tilde{g}(s_j) \left[\frac{b}{s_j - x_i} + \pi \lambda k(x_i, s_j) \right] = f(x_i), \quad i = 1, 2, \dots, n-1,$$

$$\begin{aligned} \sum_{j=1}^n W_j^n \tilde{g}(s_j) &= \frac{A}{\pi}, \\ P_n^{(\alpha, \beta)}(s_j) &= 0, \quad j = 1, 2, \dots, n, \\ P_{n-1}^{(-\alpha, -\beta)}(x_i) &= 0, \quad i = 1, 2, \dots, n-1. \end{aligned} \tag{9.5.32}$$

CASE 2. $\kappa = 0$: In this case $\alpha = -\beta$, and (9.5.30) yields the following system of n linear equations to compute $\tilde{g}(s_j)$, $j = 1, 2, \dots, n$:

$$\begin{aligned} \sum_{j=1}^n W_j^n \tilde{g}(s_j) \left[\frac{b}{s_j - x_i} + \pi \lambda k(x_i, s_j) \right] &= f(x_i), \quad i = 1, 2, \dots, n, \\ P_n^{(\alpha, \beta)}(s_j) &= 0, \quad j = 1, 2, \dots, n, \\ P_n^{(-\alpha, -\beta)}(x_i) &= 0, \quad i = 1, 2, \dots, n. \end{aligned} \tag{9.5.33}$$

CASE 3. $\kappa = -1$: In this case $\alpha > 0$ and $\beta > 0$. The solution needs to satisfy not only Eq (9.5.7) but also a consistency condition which is

$$\int_{-1}^1 \frac{S(x) dx}{(1-x)^\alpha (1+x)^\beta} = 0,$$

where $S(x)$ is defined in (9.5.27). By using (9.5.11) we can write the above consistency condition as

$$\sum_{m=0}^p c_m \int_{-1}^1 P_{m+1}^{(-\alpha, -\beta)}(x) (1-x)^{-\alpha} (1+x)^{-\beta} dx = 0,$$

which is an identity. By using (9.5.30) we obtain the following system of $(n+1)$ equations:

$$\begin{aligned} \sum_{j=1}^n W_j^n \tilde{g}(s_j) \left[\frac{b}{s_j - x_i} + \pi \lambda k(x_i, s_j) \right] &= f(x_i), \quad i = 1, 2, \dots, n+1, \\ P_n^{(\alpha, \beta)}(s_j) &= 0, \quad j = 1, 2, \dots, n, \\ P_{n+1}^{(-\alpha, -\beta)}(x_i) &= 0, \quad i = 1, 2, \dots, n+1. \end{aligned} \tag{9.5.34}$$

Then to solve this system it is sufficient to choose only n out of the $(n+1)$ collocation points x_i , since the additional equation in this case is used only to normalize the interval of integration. ■

9.5.3. A Special Case. The methods just described have two common features: (i) the singular behavior of the solution at the endpoints is built into the solution because $\phi(x)$ is taken equal to $g(x) w(x)$, where $w(x) = (1-x)^\alpha (1+x)^\beta$; and (ii) if we use a Gaussian quadrature rule the integrals are replaced by sums and the unknown values of $g(x)$ are computed at certain preassigned quadrature points by the collocation method, which yields a system of linear algebraic equations. Another important feature to recall is that unless λ is an eigenvalue of Eq (9.5.7), the preceding system of linear algebraic equations obtained through a Gaussian rule and collocation method is nonsingular.

We would like to discuss the special case $\kappa = 1$. In this case the solution of Eq (9.5.7) is *not unique* (Krenk 1975). But since in physical problems we seek a unique solution, we must impose an additional condition, generally of the form given in Example 9.5.1. The weights in this case are given by

$$W_j^n = -\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{2\pi} \frac{P_{n-1}^{(-\alpha,-\beta)}(s_j)}{P_n^{(\alpha,\beta)'}(s_j)}, \quad (9.5.35)$$

where

$$P_{m-1}^{(-\alpha,-\beta)}(s_j) = \frac{2\pi}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{j=1}^n W_j^n \frac{P_n^{(\alpha,\beta)'}(s_j)}{s_j - x_i}, \quad (9.5.36)$$

and s_j and x_i are defined in (9.5.32). For $m = 0$, note that $P_{-1}^{(-\alpha,-\beta)'}(s_j) = 0$ (by convention, see Szegő 1939, p.43). Then the above additional condition reduces to

$$\sum_{j=1}^n \frac{W_j^n}{s_j - x_i} = 0, \quad (9.5.37)$$

and the Gauss–Jacobi formula (9.5.32) yields the following system of linear algebraic equations to determine $g(s_j)$:

$$\begin{aligned} b \sum_{j=1}^n W_j^n \frac{g(s_j)}{s_j - x_i} + \pi\lambda \sum_{j=1}^n W_j^n k(x_i, s_j) g(s_j) &= f(x_i), \\ \sum_{j=1}^n W_j^n g(s_j) &= A, \quad i = 1, \dots, n-1. \end{aligned} \quad (9.5.38)$$

For $\lambda \neq 0$ the system (9.5.38) can be written as

$$(\mathbf{A} + \lambda \mathbf{C}) \mathbf{g} = \mathbf{f}, \quad (9.5.39)$$

where $\mathbf{A} = (a_{i,j})$ and $\mathbf{C} = (c_{i,j})$ are defined by

$$\begin{aligned} a_{i,j} &= \frac{b W_j^n}{s_j - x_i}, \quad i = 1, \dots, n-1; j = 1, \dots, n, \\ a_{n,j} &= W_j^n, \quad j = 1, \dots, n, \end{aligned} \quad (9.5.40)$$

$\mathbf{g} = [g(s_1), \dots, g(s_n)]^T$, and $\mathbf{f} = [f(x_1), \dots, f(x_{n-1}), A]^T$. Gerasoulis and Srivastav (1982) show that $\det|\mathbf{A}| \neq 0$ for $n \geq 2$ and that the magnitude of this determinant is bounded away from zero such that

$$[\det|\mathbf{A}|]^2 > \frac{1}{|b|} \left[\frac{\Gamma(\alpha+1)\Gamma(\alpha+2)}{4\pi} \right]^2 \quad \text{for } n \geq 2.$$

If $\alpha = \beta = -1/2$, then $[\det|\mathbf{A}|]^2 = 4^{n-1}/n^3$.

EXAMPLE 9.5.2. (Krenk 1975, Erdogan et al. 1973) Consider the problem

$$a p(x) - \frac{1}{\pi} \int_{-1}^1 \frac{p(s) ds}{s-x} = 0, \quad |x| < 1,$$

with the compatibility condition $\int_{-1}^1 p(s) ds = P$, where P is a constant. This represents a plane elasticity problem consisting of a plane rigid punch with sharp corners at $x = \pm 1$ sliding slowly in the negative x -direction on the surface of an elastic half-space with a constant coefficient of friction η , where $p(x)$ represents the normal pressure, and

$$a = \begin{cases} \frac{1-2\nu}{2(1-\nu)} \eta & \text{for plane strain,} \\ \frac{2-\nu}{4-\nu} \eta & \text{for plane stress,} \end{cases}$$

and ν is Poisson's ratio. The contact stress has integrable singularities at $x = -1$, and α and β are computed from

$$\alpha = -1 + \frac{1}{\pi} \tan^{-1} \left(\frac{1}{a} \right), \quad \beta = -\frac{1}{\pi} \tan^{-1} \left(\frac{1}{a} \right).$$

The exact solution is given by

$$p(x) = -\frac{P}{\pi} \sin(\pi\alpha) (1-x)^\alpha (1+x)^\beta.$$

For computational purposes take $\alpha = -0.34$, $\beta = -0.66$ so that $\kappa = -(\alpha + \beta) = 1$, and $P = -\pi/\sin(\pi\alpha)$. Then $a = \cot((\alpha + 1)\pi) = \cot(0.66\pi) = -0.54975465$; and $b = -1$ is given. Note that since the regular part is missing from the given equation, we use formula 9.5.40) with $\lambda = 0$. The results are given in Table 9.5.2.

Table 9.5.2

s_j	x_i	$g(s_j)$
0.99570161	0.98941625	111.306
0.96919398	0.95469739	21.5961
0.91882847	0.89647074	9.62524
0.84583484	0.81616593	5.65333
0.75200941	0.71575974	3.82474
0.63966222	0.59772435	2.82451
0.51155956	0.46496613	2.21646
0.37085573	0.32075402	1.8202
0.22101530	0.16863897	1.54969
0.06572782	0.01236655	1.35956
-0.09118299	-0.14421527	1.22417
-0.24585350	-0.29725094	1.1284
-0.39447517	-0.44297222	1.06321
-0.53338846	-0.57779094	1.02356
-0.65917283	-0.69838740	1.00744
-0.76873102	-0.80179206	1.01604
-0.85936525	-0.88545862	1.05532
-0.92884347	-0.94732641	1.14203
-0.97545332	-0.98586873	1.33095
-0.99802548	—	2.04284

The exact value of $g(x) = 1$. Our values differ from those of Krenk (1975) who gets each value of $g(s_j) = 1.0$. For computational details, see `krenk1.nb` and `krenk2.nb`. ■

9.5.4. Generalized Cauchy Kernel. Consider the CSK2 with generalized Cauchy kernel

$$\begin{aligned} a(x) \phi(x) + \frac{b(x)}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{s - x} + \frac{1}{\pi} \int_{-1}^1 k_{\pm}(x, s) \phi(s) ds \\ + \frac{1}{\pi} \int_{-1}^1 k(x, s) \phi(s) ds = f(x), \quad |x| < 1, \end{aligned} \tag{9.5.41}$$

where $\phi(x)$ is the unknown function, $f(x) \in C[-1, 1]$, $a(x)$ and $b(x)$ are known functions that can be real or complex and satisfy the Hölder condition, $k(x, s)$ is the regular kernel continuous in both x and s , and $k_{\pm}(x, s)$ is the generalized Cauchy kernel that has stationary singularities at the endpoints $x = \pm 1$ and is of the form

$$k_{\pm}(x, s) = \sum_{i=0}^N c_i(x) (1+x)^i \frac{d^i}{dx^i} \left(\frac{1}{s - z_1} \right) + \sum_{j=0}^M d_j(x) (1-x)^j \frac{d^j}{dx^j} \left(\frac{1}{s - z_2} \right), \quad (9.5.42)$$

where $c_i(x)$ and $d_j(x)$ are known Hölder-continuous functions, and

$$\begin{aligned} z_1 &= -1 + (1+x) e^{i\theta_1}, \quad 0 < \theta_1 < 2\pi, \\ z_2 &= 1 + (1-x) e^{i\theta_2}, \quad -\pi < \theta_2 < \pi. \end{aligned}$$

A unique solution of Eq (9.5.41) exists iff $\int_{-1}^1 \phi(x) dx = A$, where A is a known constant. Savruk et al. (1999) use the method discussed above based on the fundamental function $w(x)$ such that $\phi(x) = w(x) g(x)$ is assumed, where $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $-1 < \Re\{\alpha\}, \Re\{\beta\} < 0$, and the Gauss–Chebyshev quadrature is used. Expressing this function as

$$w(x) = 2^{\alpha+\beta} [X(x) + R(x)] + w_0(x), \quad (9.5.43)$$

where

$$\begin{aligned} X(x) &= \left(\frac{1-x}{1+x} \right)^{\alpha}, \quad R(x) = \left(\frac{1+x}{1-x} \right)^{\beta}, \quad 1 < \Re\{\alpha\}, \Re\{\beta\} < 0, \\ w_0(x) &= w_1(x) \psi_1(x), \\ w_1(x) &= (1-x)^{-\beta}(1+x)^{-\alpha}, \\ \psi_1(x) &= (1-x^2)^{\alpha+\beta} - 2^{\alpha+\beta} \left[(1+x)^{\alpha+\beta} + (1-x)^{\alpha+\beta} \right], \end{aligned}$$

and using the Gauss–Chebyshev quadrature, Savruk et al. (1999) solve the following problems numerically:

(i) The problem of a periodic array of collinear cracks along the interface of two isotropic semi-infinite elastic plates is governed by the equation (Theocaris and Ioakimidis 1977)

$$-i\gamma \phi(x) + \frac{\lambda}{2} \int_{-1}^1 \phi(s) \cot \left(\frac{\pi\lambda(s-x)}{2} \right) ds = 1, \quad |x| < 1, \quad (9.5.44)$$

subject to the condition $\int_{-1}^1 \phi(s) ds = 0$, where λ and γ are known parameters. If we take $\phi(x) = w(x) g(x)$, Eq (9.5.44) reduces to

$$-i\gamma g(x)w(x) + \frac{1}{\pi} \int_{-1}^1 g(s) w(s) \frac{ds}{s-x} + \frac{1}{\pi} \int_{-1}^1 g(s) w(s) k(x,s) ds = 1, \quad (9.5.45)$$

for $|x| < 1$, where the regular kernel $k(x,s)$ is given by

$$k(x,s) = \frac{\pi\lambda}{2} \cot\left(\frac{\pi\lambda(s-x)}{2}\right) - \frac{1}{s-x},$$

and $\alpha = -1/2 - i\varpi$, $\beta = -1/2 + i\varpi$, with $\varpi = \frac{1}{2\pi} \ln \frac{1+\gamma}{1-\gamma}$. Since the index $\kappa = -(\alpha + \beta) = 1$, the function $w_0(x)$ in Eq (9.5.43) vanishes, and the problem is solved using Gauss-Chebyshev quadrature with $\varpi = 1$, $\gamma = (e^{2\pi} - 1) / (e^{2\pi} + 1)$, and λ ranging from 0 to 0.9. Details are available in Savruk et al. (1999).

(ii) Consider the CSK2 with variable coefficients in the form

$$\sqrt{1-x^2} \phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{s-x} = 1+x, \quad |x| < 1, \quad (9.5.46)$$

subject to the same condition as in (i). With parameters $\alpha = -1/2 = \beta$, the function $w_0(x)$ also vanishes in this case. Using Gauss-Chebyshev quadrature, the following results are obtained: $g(1) = 0.99999999$ for $n = 2$ and $g(1) = 1.0$ for $n = 4$. The exact solution of Eq (9.5.46) is

$$\phi(x) = \frac{g(x)}{\sqrt{1-x^2}} \quad \text{with } g(x) = x.$$

(iii) Consider the CSK2 with variable coefficients associated with the singular kernel:

$$\phi(x) + \frac{x}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{s-x} = x, \quad |x| < 1, \quad (9.5.47)$$

with the index $\kappa = 0$. The exponents are $\alpha = -1/4 = \beta$, and the function $w_0(x)$ is given by

$$w_0(x) = -\sqrt{2} \frac{\sqrt[4]{1-x^2}}{\sqrt{1+x} + \sqrt{1-x} + \sqrt{2}}.$$

The function $g(x)$ is computed by using Gauss–Chebyshev quadrature ; for results see Savruk et al. (1999).

(iv) Consider the CSK2 (Erdogan and Gupta 1971)

$$\phi(x) + \frac{1}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{s - x} + \frac{1}{\pi} \int_{-1}^1 k_{\pm}(x, s) \phi(s) ds = 1, \quad |x| < 1, \quad (9.5.48)$$

subject to the same condition as above, where

$$k_{\pm}(x, s) = \frac{0.167}{s + x + 2} - \frac{3.493(1+x)}{(s + x + 2)^2} + \frac{2.329(1+x)^2}{(s + x + 2)^3},$$

which is the generalized Cauchy kernel. Taking $\alpha = -1/4$ and $\beta = -0.5649$ and using the Gauss–Chebyshev quadrature, Savruk et al. (1999) present a graphical solution.

9.6. Collocation Method for CSK1

This method is developed on the same lines as in Section 9.5.2. By substituting $\phi(x) = g(x)w(x)$ into the CSK1 of the form

$$\frac{1}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{s - x} + \lambda \int_{-1}^1 k(x, s) \phi(s) ds = f(x), \quad |x| < 1, \quad (9.6.1)$$

we get the equation

$$\frac{1}{\pi} \int_{-1}^1 \frac{g(s)w(s) ds}{s - x} + \lambda \int_{-1}^1 k(x, s) g(s)w(s) ds = f(x), \quad |x| < 1. \quad (9.6.2)$$

This equation is reduced to a system of linear equations in $g(s_j)$, $j = 1, \dots, n$, of the form

$$\frac{1}{\pi} \sum_{j=1}^n W_j^n g(s_j) \left[\frac{1}{s_j - x_i} + \pi \lambda k(x_i, s_j) \right] = f(x_i), \quad i = 1, \dots, n - \kappa, \quad (9.6.3)$$

where $w(x) = (1-x)^\alpha(1+x)^\beta$, $-1 < \alpha, \beta < 1$, $|x| < 1$, and the weights W_j^n are given in terms of Jacobi polynomials for $\kappa = 0$ but in terms of Chebyshev

polynomials $T_n(x)$ and $U_n(x)$ of the first and second kind for $\kappa = 1$ and $\kappa = -1$, respectively. The points s_j and x_i are determined from

$$\begin{aligned} P_n^{(\alpha, \beta)}(s_j) &= 0, \\ P_{n-\kappa}^{(\alpha+\kappa, \beta+\kappa)}(x_i) &= 0 \quad \text{for } \kappa = \mp 1, \\ P_n^{(\alpha+1, \beta-1)}(x_i) &= 0 \quad \text{for } \kappa = 0, -1 < \alpha < 0, 0 < \beta < 1, \\ P_n^{(\alpha-1, \beta+1)}(x_i) &= 0 \quad \text{for } \kappa = 0, 0 < \alpha < 1, -1 < \beta < 0. \end{aligned} \quad (9.6.4)$$

As in (9.5.27), we shall again define $S(x) = \frac{1}{\pi} \int_{-1}^1 \frac{g(s)w(s) ds}{s-x}$, where $g(x)$ are bounded in the interval $[-1, 1]$ and defined by (9.5.28). As before, there are three cases to consider:

CASE 1. $\kappa = 0$: This case has two parts:

(i) If $\alpha = -1/2, \beta = 1/2$, then $w(x) = (1-x)^{-1/2}(1+x)^{1/2}$. The function $S(x)$ is approximated by $\tilde{S}(x)$, which is given by

$$\tilde{S}(x) = \sum_{m=0}^p \sum_{j=1}^n c_m P_m^{(-1/2, 1/2)}(s_j) \frac{2(1+s_j)}{2n+1} \frac{1}{s_j - x_i}. \quad (9.6.5)$$

Since

$$\frac{1}{\pi} \int_{-1}^1 \left(\frac{1+s}{1-s}\right)^{1/2} \frac{g(s)}{s-x_i} = \sum_{j=1}^n \frac{2(1+s_j)}{2n+1} \frac{g(s_j)}{s_j - x_i}, \quad (9.6.6)$$

where the points s_j and x_i are given by

$$\begin{aligned} P_n^{(-1/2, 1/2)}(s_j) &= 0 \implies s_j = \cos\left(\frac{2j-1}{2n+1}\pi\right), \quad j = 1, 2, \dots, n, \\ P_n^{(1/2, -1/2)}(x_i) &= 0 \implies x_i = \cos\left(\frac{2i\pi}{2n+1}\right), \quad i = 1, 2, \dots, n, \end{aligned} \quad (9.6.7)$$

and since the Gauss–Jacobi quadrature for a bounded continuous function $g(s)$ is given by (see Abramowitz and Stegun 1968)

$$-\frac{1}{\pi} \int_{-1}^1 \left(\frac{1+s}{1-s}\right)^{1/2} g(s) ds \approx \sum_{j=1}^n \frac{2(1+s_j)}{2n+1} g(s_j), \quad (9.6.8)$$

where s_j is given in (9.6.7), we can formally regard (9.6.6) as a Gauss–Jacobi quadrature rule for the singular part of Eq (9.6.2) such that this rule is valid only at

finitely many collocation points $x_i, i = 1, \dots, n$. Thus, using (9.6.6) and (9.6.8) we reduce Eq (9.6.2) to the following system of n linear algebraic equations in $g(s_j)$:

$$\sum_{j=1}^n \frac{2(1+s_j)}{2n+1} \left[\frac{1}{s_j - x_i} + \pi \lambda k(x_i, s_j) \right] g(s_j) = f(x_i), \quad i = 1, \dots, n, \quad (9.6.9)$$

where x_i and s_j are given in (9.6.7).

(ii) If $\alpha = 1/2, \beta = -1/2$, then $w(x) = (1-x)^{1/2}(1+x)^{-1/2}$, and

$$\frac{1}{\pi} \int_{-1}^1 \left(\frac{1-s}{1+s} \right)^{1/2} \frac{g(s)}{s - x_i} ds = \sum_{j=1}^n \frac{2(1-s_j)}{2n+1} \frac{g(s_j)}{s_j - x_i}, \quad (9.6.10)$$

where the points s_j and x_i are given by

$$\begin{aligned} P_n^{(1/2, -1/2)}(s_j) &= 0 \implies s_j = \cos\left(\frac{2j\pi}{2n+1}\right), \quad j = 1, 2, \dots, n, \\ P_n^{(-1/2, 1/2)}(x_i) &= 0 \implies x_i = \cos\left(\frac{2i-1}{2n+1}\pi\right), \quad i = 1, 2, \dots, n, \end{aligned} \quad (9.6.11)$$

and Eq (9.6.2) reduces to the following system of n linear equations:

$$\sum_{j=1}^n \frac{2(1-s_j)}{2n+1} \left[\frac{1}{s_j - x_i} + \pi \lambda k(x_i, s_j) \right] g(s_j) = f(x_i), \quad i = 1, \dots, n, \quad (9.6.12)$$

where x_i and s_j are given in (9.6.11).

CASE 2. $\kappa = 1$: If the solution $\phi(x)$ has integrable singularities at the endpoints, then in this case $\alpha = -1/2 = \beta$; thus, $w(x) = (1-x^2)^{-1/2}$, and the polynomials $P_n^{(\alpha, \beta)}(x)$ reduce to Chebyshev polynomials $T_n(x)$ of the first kind. Then using $\phi(x) = w(x)g(x)$, $|x| < 1$, Eq (9.6.2) together with the condition $\int_{-1}^1 \phi(s) ds = A$ reduce to the following system of n algebraic equations:

$$\begin{aligned} \sum_{j=1}^n \frac{1}{n} \left[\frac{1}{s_j - x_i} + \pi \lambda k(x_i, s_j) \right] g(s_j) &= f(x_i), \\ \sum_{j=1}^n \frac{\pi}{n} g(s_j) &= A, \quad i = 1, \dots, n-1, \end{aligned} \quad (9.6.13)$$

where A is a constant, and x_i and s_j are given by

$$\begin{aligned} T_n(s_j) = 0 \implies s_j &= \cos\left(\frac{2j-1}{2n}\pi\right), \quad j = 1, 2, \dots, n, \\ U_{n-1}(x_i) = 0 \implies x_i &= \cos\left(\frac{i\pi}{n}\right), \quad i = 1, 2, \dots, n-1. \end{aligned} \quad (9.6.14)$$

The details of the derivation of Eq (9.6.13) can be found in Erdogan and Gupta (1972).

CASE 3. $\kappa = -1$: If the function $\phi(x)$ is bounded at both ends, then in this case $\alpha = 1/2 = \beta$; thus, $w(x) = (1 - x^2)^{1/2}$, and the polynomials $P_n^{(\alpha, \beta)}(x)$ reduce to Chebyshev polynomials $U_n(x)$ of the second kind. This leads to a Gauss–Chebyshev quadrature for the singular part of Eq (9.6.2). If we assume that $\tilde{g}(x)$

is defined, as in (9.5.28), by $\tilde{g}(x) = \sum_{m=0}^p c_m U_m(s)$, and use the relations

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 U_m(s) (1 - s^2)^{1/2} \frac{ds}{s - x} &= -T_{m+1}(x), \quad |x| < 1, \\ \frac{1}{\pi} \int_{-1}^1 T_j(s) T_m(s) (1 - s^2)^{-1/2} ds &= \begin{cases} 0 & \text{if } j \neq m, \\ 1 & \text{if } j = m = 0, \\ 1/2 & \text{if } j = m > 0, \end{cases} \end{aligned}$$

then Eq (9.6.2) gives

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \left[g(x) - \lambda \int_{-1}^1 k(x, s) \phi(s) ds \right] \frac{dx}{(1 - x^2)^{1/2}} \\ = - \sum_{m=1}^p c_m \int_{-1}^1 T_{j+1}(x) \frac{dx}{(1 - x^2)^{1/2}} = 0. \end{aligned} \quad (9.6.15)$$

Thus, the consistency condition is automatically satisfied, and Eq (9.6.2) reduces to the following system of algebraic equations:

$$\sum_{j=1}^n \frac{1 - s_j^2}{n + 1} \left[\frac{1}{s_j - x_i} + \pi k(x_i, s_j) \right] g(s_j) = f(x_i), \quad i = 1, \dots, n, \quad (9.6.16)$$

where the points x_i and s_j are given by

$$\begin{aligned} U_n(s_j) = 0 \implies s_j &= \cos\left(\frac{j\pi}{n+1}\right), \quad j = 1, 2, \dots, n, \\ T_{n+1}(x_i) = 0 \implies x_i &= \cos\left(\frac{2i+1}{2(n+1)}\pi\right), \quad i = 1, 2, \dots, n+1. \end{aligned} \quad (9.6.17)$$

The details of the derivation of Eq (9.6.16) can be found in Erdogan and Gupta (1972).

Erdogan et al.(1973) have solved the following problems: (i) a finite crack perpendicular to the interface of two bounded half-planes, and (ii) a half-plane containing a crack parallel to the boundary. Details of the solution of these problems are not given here because they are not only lengthy but also pertain to specialized interests.

9.6.1. Canonical Equation. If $\lambda = 0$ in (9.6.1), we obtain the so-called canonical equation

$$\frac{1}{\pi} \int_{-1}^1 \frac{\phi(s) ds}{s - x} = f(x), \quad |x| < 1, \quad (9.6.18)$$

which has a bounded solution iff

$$\frac{1}{\pi} \int_{-1}^1 \frac{f(x) dx}{\sqrt{1 - x^2}} = 0. \quad (9.6.19)$$

If Eq (9.6.18) has bounded solution, then

$$\int_{-1}^1 \frac{1}{\sqrt{1 - x^2}} \left\{ f(x) - \lambda \int_{-1}^1 k(x, s) \phi(s) ds \right\} dx = 0. \quad (9.6.20)$$

Since (9.6.20) contains $\phi(s)$, it cannot be used to check whether the necessity of the condition (9.6.19) is met unless $\int_{-1}^1 \frac{k(x, s)}{\sqrt{1 - x^2}} dx = 0$ for almost all s . But this is obviously the case when $k(x, s)$ is an odd function of x .

The collocation method given above to compute a bounded solution of Eq (9.6.18) consists of the following steps:

1. Set $\phi(x) = \sqrt{1 - x^2} g(x)$;
2. Replace the integrals by the Gaussian quadrature using the zeros of $U_n(x)$;
3. Generate the system of linear algebraic equations at the zeros of $T_{n+1}(x)$; and
4. Solve this system, which contains $(n + 1)$ equations in n unknowns; thus ignore one equation.

Jen and Srivastav (1983) observe that “in practice the most harmless point to neglect would be the one closest to $x = 0$.” Note that steps 2 and 3 use the zeros given in (9.6.14), and the quadrature rule is given by

$$\int_{-1}^1 \frac{\sqrt{1 - s^2} g(s) ds}{s_j - x_i} = \frac{1}{2n + 2} \sum_{j=1}^{2n+1} \frac{(1 - s_j^2) g(s_j)}{s_j - x_i}, \quad (9.6.21)$$

where $s_j, j = 1, \dots, 2n + 1$, are the quadrature points and $x_i, i = 1, \dots, 2n + 2$, are the Nyström points. Also, by a change of variable it is easy to verify that in the equation

$$\int_{-1}^1 \frac{\sqrt{1 - s^2} g(s) ds}{s_j - x_i} = f(x), \quad |x| < 1, \quad (9.6.22)$$

$g(s)$ is an odd function of s if $f(x)$ is even, and $g(s)$ is an even function of s if $f(x)$ is odd. Hence, to solve Eq (9.6.1), first we split Eq (9.6.18) into its ‘odd’ and ‘even’ parts, and then combine the regular part of Eq (9.6.1) with $f(x)$. Now, we consider the following two cases:

CASE 1. If $f(x)$ is odd and $g(s)$ is even, then a bounded solution always exists because the condition (9.6.19) is automatically satisfied. By using Gauss–Chebyshev rule, Eq (9.6.22) reduces to

$$\frac{1}{2n+1} \sum_{j=1}^n \frac{(1 - s_j^2) g(s_j) ds}{s_j - x_i} = f(x_i), \quad i = 1, \dots, 2n + 1,$$

which, by using symmetry, can be written as

$$\frac{2}{2n+1} x_i^2 \sum_{j=1}^n \frac{(1 - s_j^2) g(s_j) ds}{s_j^2 - x_i^2} = x_i f(x_i), \quad i = 1, \dots, n + 1. \quad (9.6.23)$$

The equation corresponding to $i = n + 1$ is automatically satisfied since $x_{n+1} = 0$. Hence, there are n equations in n unknowns $g(s_j)$, and the system is normal.

CASE 2. If $g(s)$ is odd, $f(x)$ is even and the condition (9.6.19) is satisfied, where the discrete form of this condition is $\sum_{i=1}^{2n+2} f(x_i) = 0$, which, since $f(x)$ is even and the distribution of the points x_i are symmetric, reduces to

$$\sum_{i=1}^{n+1} f(x_i) = 0, \quad (9.6.24)$$

then formula (9.6.21) yields the following system of linear algebraic equations:

$$\frac{1}{2n+2} \sum_{j=1}^n \frac{(1 - s_j^2) g(s_j) ds}{s_j - x_i} = f(x_i), \quad i = 1, \dots, 2n + 2. \quad (9.6.25)$$

This system is overdetermined but consistent (see Jen and Srivastav 1983, p.629). Notice that $s_{n+1} = 0$, and for continuous g we have $g(s_{n+1}) = 0$. Hence, the system (9.6.25) reduces to

$$\frac{1}{n+1} \sum_{j=1}^n \frac{s_j (1 - s_j^2) g(s_j) ds}{s_j^2 - x_i^2} = f(x_i), \quad i = 1, \dots, n+1, \quad (9.6.26)$$

which has $n+1$ equations and n unknowns $g(s_j)$. This system can be written in matrix form as

$$\mathbf{A} \mathbf{g} = \mathbf{f}, \quad (9.6.27)$$

where $\mathbf{A} = (a_{i,j})$ is an $(n+1) \times n$ matrix with its (i,j) th element given by

$$a_{i,j} = \frac{1}{n+1} \frac{s_j^2 (1 - s_j^2)}{s_j^2 - x_i^2}, \quad (9.6.28)$$

$\mathbf{g} = \{g_j\}$ is an $n \times 1$ vector with the j th element $\frac{g(s_j)}{s_j}$, and $\mathbf{f} = \{f_i\}$ is an $(n+1) \times 1$ vector with the i th element $f(x_i)$.

Now, we present a LEAST SQUARES SOLUTION of system (9.6.27): We introduce a ‘phantom’ variable $g(s_{n+1})$ that satisfies

$$\frac{1}{n+1} \sum_{j=1}^n \frac{s_j^2 (1 - s_j^2) g(s_j) / s_j ds}{s_j^2 - x_i^2} + g(s_{n+1}) = f(x_i), \quad i = 1, \dots, n+1, \quad (9.6.29)$$

and consider the system

$$(\mathbf{A} \mathbf{e}) \tilde{\mathbf{g}} = \mathbf{f}, \quad (9.6.30)$$

where $\mathbf{e} = [1, 1, \dots, 1]^T$, and $\tilde{\mathbf{g}}$ is the vector \mathbf{g} augmented by the element $g(s_{n+1})$; thus, $\tilde{\mathbf{g}}$ is an $(n+1) \times 1$ vector. By using the identity $\sum_{j=1}^n (s_j^2 - x_i^2)^{-1} = 0$, for $i = 1, \dots, n-1$, where s_j are the zeros of the polynomial $T_{2n}(x)$ and x_i the zeros of the polynomial $U_{2n-1}(x)$, the columns of the coefficient matrix in (9.6.30) form an orthogonal basis in \mathbb{R}^{n+1} . The condition (9.6.24) implies that the vector \mathbf{f} is orthogonal to \mathbf{e} , i.e., $g(s_{n+1}) = 0$. Assuming that f satisfies the condition (9.6.19) but not (9.6.24), we find that the least-squares solution $\mathbf{g}^* = \{g_j^*\}$, $j = 1, \dots, n$, of the system (9.6.27) satisfies the system

$$\mathbf{A}^T \mathbf{A} \mathbf{g}^* = \mathbf{A}^T \mathbf{f}, \quad (9.6.31)$$

which is a normal system. Since the columns of \mathbf{A} are mutually orthogonal, we find that

$$g_j^* = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{f(x_i)}{s_j^2 - x_i^2}, \quad j = 1, \dots, n. \quad (9.6.32)$$

Note that the least-squares solution \mathbf{g}^* coincides with the phantom solution $\tilde{\mathbf{g}}$ in the sense that $\tilde{g}_j = g_j^*$ for $j = 1, \dots, n$, and $\tilde{g}_{n+1} = \frac{1}{n+1} \sum_{i=1}^{n+1} f(x_i)$. The quantity

$$d = \frac{1}{n+1} \sum_{i=1}^{n+1} f(x_i), \quad (9.6.33)$$

which defines the accuracy of the Gaussian quadrature, can be used as a *measure of inconsistency*. We consider the case where d differs significantly from zero for moderately large n , and solve the overdetermined system (9.6.27) by deleting one row from the matrix \mathbf{A} . The question is, which row should be deleted. Suppose that $\mathbf{g}(p)$ is the solution vector of the system

$$\mathbf{A}^{(p)} \mathbf{g}^{(p)} = \mathbf{f}^{(p)}, \quad (9.6.34)$$

where $\mathbf{A}^{(p)}$ and $\mathbf{f}^{(p)}$ are obtained from the original system (9.6.27) by deleting the p th row. Define an $n \times 1$ vector $\boldsymbol{\epsilon}^{(p)}$ by

$$\boldsymbol{\epsilon}^{(p)} = \mathbf{g}^* - \mathbf{g}^{(p)} \quad (9.6.35)$$

and an $n \times 1$ vector \mathbf{d} by $d_j, j = 1, \dots, n$. Then $\boldsymbol{\epsilon}^{(p)}$ satisfies the system

$$\mathbf{A}^{(p)} \boldsymbol{\epsilon}^{(p)} = \mathbf{d}^{(p)}. \quad (9.6.36)$$

Instead of solving (9.6.36) directly, we use the properties of the matrix \mathbf{A} to compute $\boldsymbol{\epsilon}^{(p)}$ as follows: Define an $(n+1) \times 1$ vector \mathbf{d}^* by $d_j^* = d, j = 1, \dots, n+1, j \neq p$, and $d_p^* = -nd$, and consider the augmented system

$$(\mathbf{A} \ \mathbf{e}) \ \boldsymbol{\epsilon} = \mathbf{d}^*, \quad (9.6.37)$$

where \mathbf{e} is augmented by a ‘phantom’ element e_{n+1} . Since the right side of vector \mathbf{d}^* satisfies the consistency condition (9.6.24), we find by using the orthogonality properties of the coefficient matrix that

$$\epsilon_j^{(p)} = -d \frac{s_j}{s_j^2 - x_i^2}, \quad j = 1, \dots, n; p = 1, \dots, n+1, \quad (9.6.38)$$

determines the discrepancy in the j th element of the solution vector caused by deleting the p th row of the system. The error in $g(s)$ is, therefore, determined by

$$\epsilon_j^{*(p)} = -d \frac{s_j \sqrt{1 - s_j^2}}{s_j^2 - x_i^2}. \quad (9.6.39)$$

Jen and Srivastav (1983) compute $\epsilon_j^{*(p)}$ for $n = 20$, $d = 1$, and $p = 1, 7, 14, 18, 21$, and observe that the error is maximum near the deleted row. They advise that the choice of the row to be deleted must depend on the quantity of interest in the problem; e.g., in crack problems if the stress near the endpoints needs be computed, then choose the deleted value of s_j which is closest to zero. But if we seek the entire crack profile and maximum crack displacement, then choose the deleted value s_j which is farthest from zero.

EXAMPLE 9.6.1. (Jen and Srivastav 1983) Consider the canonical equation

$$\int_{-1}^1 \frac{g(s) \sqrt{1 - s^2}}{s - x} ds = \frac{1 - a^2}{1 - 2ax + a^2} - 1, \quad |x| < 1,$$

with $a^2 = 0.95$. Let $f_0(x)$ denote the even part of the right side $f(x)$. Then $\int_{-1}^1 \frac{f_0(x)}{\sqrt{1 - x^2}} dx = 0$, and the compatibility condition is satisfied. Thus, the exact solution is given by the odd part (case 2), i.e., $g(x) = -\frac{2a}{1 - 2ax + a^2}$. Tables 9.6.1(a) and 9.6.1(b) give the numerical solutions obtained by Jen and Srivastav (1983) with $n = 20$ and $n = 60$, respectively, and show the deleted row of the overdetermined system. Note that the discrepancy between the exact and computed values depends only on the value of d chosen, and not on the solution itself. Columns 2 and 3 give the exact values and the values obtained by the least-squares method, respectively, and the last three columns give the values of $\epsilon^*(s)$ for different values of p .

Table 9.6.1(a). For $n = 20$, $d = -0.2079$

s	$g(s)$	$\tilde{g}(s)$	$p = 21$	$p = 10$	$p = 1$
0.9972	-11.9377	-9.4564	-0.0156	-0.0369	3.7
0.9749	-4.3238	-3.4251	-0.0475	-0.1200	0.9372
0.9309	-2.5355	-2.0084	-0.0817	-0.2421	0.5352
0.8660	-1.7275	-1.3684	-0.1202	-0.5129	0.3620
0.7818	-1.2518	-0.9916	-0.1661	-2.7579	0.2247

Table 9.6.1(a). For $n = 20$, $d = -0.2079$, Continued

s	$g(s)$	$\tilde{g}(s)$	$p = 21$	$p = 10$	$p = 1$
0.6801	-0.9267	-0.7341	-0.2247	0.9263	0.1661
0.5633	-0.6811	-0.5296	-0.3062	0.3762	0.1202
0.4339	-0.4812	-0.3812	-0.4349	0.2104	0.0817
0.2948	-0.3082	-0.2442	-0.6849	0.1201	0.0475
0.1490	-0.1506	-0.1193	-1.4717	0.0555	0.0156
0.0	0.0	0.0	—	—	—

Table 9.6.1(b). For $n = 60$, $d = -0.0038$

s	$g(s)$	$\tilde{g}(s)$	$p = 61$	$p = 30$	$p = 1$
0.9937	-19.4869	-19.4121	-0.0001	-0.0002	0.1984
0.9838	-5.3774	-5.3568	-0.0007	-0.0015	0.0211
0.9449	-2.8574	-2.8464	-0.0013	-0.0032	0.0110
0.8827	-1.8726	-1.8654	-0.0020	-0.0063	0.0072
0.7998	-1.3302	-1.3251	-0.0029	-0.0161	0.0051
0.6979	-0.9733	-0.9696	-0.0039	0.0496	0.0037
0.5794	-0.7102	-0.7075	-0.0054	0.0095	0.0017
0.4471	-0.4994	-0.4975	-0.0077	0.0047	0.0019
0.3041	-0.3190	-0.3178	-0.0120	0.0025	0.0012
0.1539	-0.1556	-0.1550	-0.0248	0.0012	0.0006
0.0	0.0	-0.0038	—	—	— ■

REFERENCES USED: Abramowitz and Stegun (1968), Erdogan and Gupta (1971, 1972), Erdogan, Gupta and Cook (1973), Gerasoulis and Srivastav (1982), Gradshteyn and Ryzik (1965), Jen and Srivastav (1983), Kalandiya (1973), Kondo (1991), Krenk (1975, 1978), Kythe (1998), Mushkhelishvili (1992), Polyanin and Manzhirov (1998), Savruk, Madenci and Shkarayev (1999), Stroud and Secrest (1966), Szegö (1939), Theocaris and Ioakimidis (1977), Tricomi (1957), Villat (1916).

10

Sinc-Galerkin Methods

In this chapter we consider an FK2 of the form (1.2.2): $\phi(x) - \lambda \int_a^b k(x, s)\phi(s) ds = f(s)$, where $a \leq x, s \leq b$, and the kernel $k(x, s)$ has a weak singularity at an endpoint. In numerical approximations, whether in quadrature, finite differences, finite elements, and the like, the computational methods generally use polynomials as basis functions to obtain approximate solutions that are sufficiently accurate in a region where the function to be approximated is ‘smooth’ (or analytic). However, such methods fail significantly in a neighborhood of singularities of the function. An analytic function ϕ has a singularity at a point at which ϕ' does not exist and at the endpoints of an interval or a contour. On the other hand, the numerical approximation obtained by using Whittaker’s cardinal function $C(\phi, h, x)$ yields much better results than those obtained by polynomial methods in the case when singularities are present at an endpoint of an interval. This method, however, may or may not yield better results in the absence of singularities. The function $C(\phi, h, x)$ is called the *cardinal interpolant* to $\phi(x)$.

10.1. Sinc Function Approximations

The function $\text{sinc}(x)$ is defined on the real line by

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \quad (10.1.1)$$

Fig. 10.1.1 shows the graph of $\text{sinc}(x)$. The Whittaker’s cardinal function $C(\phi, h)$ of an arbitrary function ϕ is defined by

$$C(\phi, h, x) = \sum_{j=-\infty}^{\infty} \phi(jh) S(j, h)(x), \quad (10.1.2)$$

where this series converges for any step size $h > 0$, and

$$S(j, h)(x) = \text{sinc}\left(\frac{x - jh}{h}\right) = \frac{\sin \pi(x - jh)/h}{\pi(x - jh)/h}. \quad (10.1.3)$$

The approximation of ϕ using a finite number of terms of (10.1.2) has been extensively studied by Stenger (1981, 1993), which contain a review of the properties of $C(\phi, h, x)$ as well as new approximations derived by using $C(\phi, h, x)$ for interpolating, integrating, and approximating the Fourier series over $(-\infty, \infty)$, and Hilbert transforms over $(-\infty, \infty)$, $(0, \infty)$, and $(-1, 1)$. A detailed catalog of these properties is given in Appendix C.

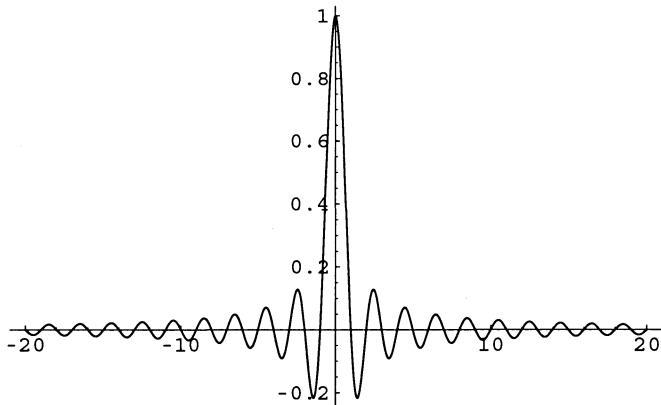


Fig. 10.1.1. Plot of $\text{sinc}(x)$.

We present the numerical approximation method obtained by using $S(j, h) \circ F$ as basis functions, where F denotes a suitable mapping (usually conformal) of an interval onto \mathbb{R} . Let $B(h)$ denote the family of all functions ϕ analytic in \mathbb{C} , such that $\phi \in L_2(\mathbb{R})$ and

$$|\phi(z)| \leq C e^{\pi|z|/h}, \quad z = x + i y \in \mathbb{C}, \quad (10.1.4)$$

for some constant C . Then $\phi(z) = C(\phi, h, z)$ for all $z \in \mathbb{C}$ (see McNamee, Stenger and Whitney 1971), and the results (C.1.7)–(C.1.9) hold if f in these results is replaced by ϕ . Thus, the sequence $\left\{ h^{-1/2} S(j, h) \right\}_{j=-\infty}^{\infty}$ is a complete orthonormal sequence in $B(h)$. Also, if $\phi \in B(h)$, then (Stenger 1976)

$$(i) \lim_{n \rightarrow \infty} \left[\int_{-nh}^{nh} \phi(s) ds - h \sum_{j=-n}^n \phi(jh) \right] = 0, \text{ and}$$

$$(ii) \int_{\mathbb{R}} |\phi(s)|^2 ds = h \sum_{j=-\infty}^{\infty} |\phi(jh)|^2.$$

Let $d > 0$, and let $B_p(\mathcal{D}_d)$, $p \geq 1$, denote the family of all functions ϕ that are analytic in the domain

$$\mathcal{D}_d = \{z = x + iy : |y| < d\}, \quad (10.1.5)$$

such that $\int_{-d}^d |\phi(x + iy)| dy \rightarrow 0$ as $x \rightarrow \pm\infty$, and

$$N(\phi, p, \mathcal{D}_d) = \lim_{y \rightarrow d^-} \left\{ \left[\int_{\mathbb{R}} |\phi(x+iy)|^p dx \right]^{1/p} + \left[\int_{\mathbb{R}} |\phi(x-iy)|^p dy \right]^{1/p} \right\} < \infty. \quad (10.1.6)$$

THEOREM 10.1.1. (Stenger 1976) Let h and d be positive, let $\phi \in B_1(\mathcal{D}_d)$, $d > 0$, and let the error $E(\phi)$ be defined by

$$E(\phi)(x) = \phi(x) - C(\phi, h, x), \quad x \in \mathbb{R}. \quad (10.1.7)$$

Then

$$\begin{aligned} \left| \int_{\mathbb{R}} \phi(x) dx - h \sum_{j=-\infty}^{\infty} \phi(jh) \right| &\leq \frac{e^{-\pi d/h} N(\phi, 1, \mathcal{D}_d)}{2 \sinh(\pi d/h)} \\ &\leq C_1 e^{-2\pi d/h} \quad \text{for some } C_1 > 0, \end{aligned} \quad (10.1.8)$$

and

$$|E(\phi)(x)| \leq \frac{N(\phi, 1, \mathcal{D}_d)}{2\pi d \sinh(\pi d/h)} \leq C_2 e^{-\pi d/h} \quad \text{for all } x \in \mathbb{R} \text{ and some } C_2 > 0. \quad (10.1.9)$$

10.2. Conformal Maps and Interpolation

Let Ω_d be a simply connected domain in the complex plane \mathbb{C} , let F be a conformal map of Ω_d onto \mathcal{D}_d , and let $G = F^{-1}$ denote the inverse map. Let $a = G(-\infty)$ and $b = G(\infty) \neq a$ be boundary points of Ω_d , and let

$$\Gamma = \{w \in D : w = G(x), -\infty \leq x \leq \infty\}, \quad w = u + iv, \quad (10.2.1)$$

be a closed contour in Ω_d . Given F and G , let z_j and μ_j denote the points $z_j = G(jh)$ and $\mu_j = G'(jh)$, respectively, $j = 0, \pm 1, \pm 2, \dots$. Also, let

$$N(\phi, \Omega_d) = \lim_{\Gamma \rightarrow \partial \Omega_d} \int_{\Gamma} |\phi(z) dz|, \quad (10.2.2)$$

where Γ is a closed contour in Ω_d . Define a one-dimensional sinc-interpolant I_{1N} on $\phi(z)$ by

$$I_{1N}\phi(z) = \sum_{j=-N}^N \phi(z_j) \operatorname{sinc}\left(\frac{F(z) - jh}{h}\right). \quad (10.2.3)$$

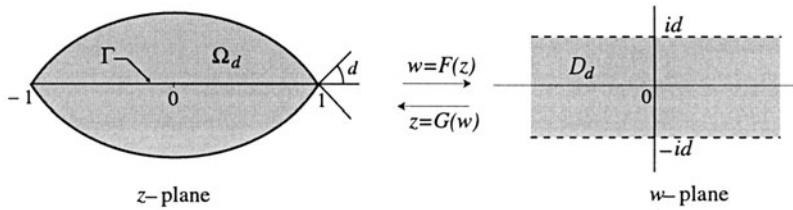
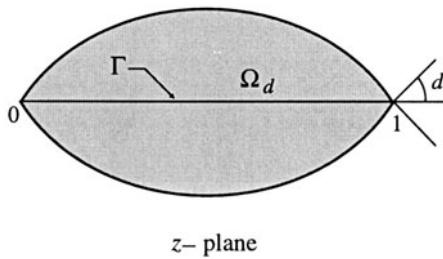
Then, by using conformal mapping we can convert the above error bounds to domains other than \mathcal{D}_d . We present four types of conformal maps F and the associated inverse map G that correspond to the intervals $\Gamma = [0, 1]$, $[-1, 1]$, and $[0, \infty)$, respectively.

TYPE 1. $\Gamma = [-1, 1]$: In this case

$$\Omega_d = \left\{ z : \left| \arg \left\{ \frac{1+z}{1-z} \right\} \right| < d \right\},$$

as shown in Fig. 10.2.1. The functions F and G at the points z_j are given by

$$\begin{aligned} w = F(z) &= \log \left(\frac{1+z}{1-z} \right), \quad z = G(w) = \tanh \frac{w}{2} = \frac{e^w - 1}{e^w + 1}, \\ z_j &= \tanh \frac{jh}{2}, \quad j = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (10.2.4)$$

Fig. 10.2.1. Type 1 domain Ω_d .Fig. 10.2.2. Type 2 domain Ω_d .

TYPE 2. $\Gamma = [0, 1]$: In this case

$$\Omega_d = \left\{ z : \left| \arg \left\{ \frac{z}{1-z} \right\} \right| < d \right\}.$$

The boundary points of D consist of two circular arcs that intersect at an angle $2d$ (radians) at $z = 0$ and 1 (see Fig. 10.2.2). The functions F and G at the points z_j are given by

$$w = F(z) = \log \left(\frac{z}{1-z} \right), \quad z = G(w) = \frac{e^w}{1+e^w}, \quad (10.2.5)$$

$$z_j = \frac{e^{j\pi}}{1+e^{j\pi}}, \quad j = 0, \pm 1, \pm 2, \dots$$

Note that in Fig. 10.2.2 as well as in Figs. 10.2.3 and 10.2.4, only the domain Ω_d is shown because the domain D_d in the w -plane remains the same as in Fig. 10.2.1.

TYPE 3. $\Gamma = [0, \infty]$: In this case

$$\Omega_d = \left\{ z : |\arg \{z\}| < d \right\},$$

as shown in Fig. 10.2.3.

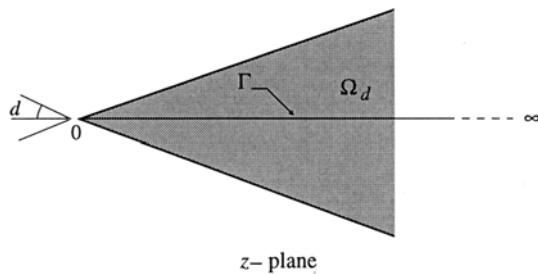


Fig. 10.2.3. Type 3 domain Ω_d .

The functions F and G at the points z_j are given by

$$\begin{aligned} w &= F(z) = \log z, \quad z = G(w) = e^w, \\ z_j &= e^{jh}, \quad j = 0, \pm 1, \pm 2, \dots \end{aligned} \tag{10.2.6}$$

TYPE 4. $\Gamma = [0, \infty)$: Another domain in this case is

$$\Omega_d = \left\{ z : |\arg \{\sinh z\}| < d \right\}, \quad 0 < d \leq \pi/2,$$

as shown in Fig. 10.2.4.

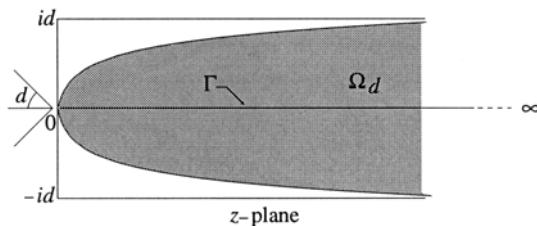


Fig. 10.2.4. Type 4 domain Ω_d .

The functions F and G at the points z_j are given by

$$\begin{aligned} w &= F(z) = \log \{ \sinh z \}, \quad z = G(w) = \log \left\{ e^w + \sqrt{1 + e^{2w}} \right\}, \\ z_j &= \log \left\{ e^{jh} + \sqrt{1 + e^{2jh}} \right\} = \sinh^{-1} (e^{jh}), \quad j = 0, \pm 1, \pm 2, \dots \end{aligned} \tag{10.2.7}$$

Note that the expression for z_j given in (10.2.7), although mathematically exact, gives inaccurate results for small e^{jh} . Therefore, we shall write (10.2.7) as

$$\begin{aligned} z_j &= \int_0^{4^{jh}} (1+t^2)^{-1/2} dt \\ &= e^{jh} - \frac{1}{6} e^{3jh} + \frac{3}{40} e^{5jh} - \frac{5}{112} e^{7jh} + \dots, \quad \text{where } e^{jh} < 0.1, \end{aligned} \quad (10.2.8)$$

and use the last expression in (10.2.8) for computational purposes if $e^{jh} < 0.1$, since an accurate computation of $\phi(z_j)$ is important, especially near a singularity.

The most effective use of the sinc-Galerkin method occurs when $\phi \in B(\Omega_d)$ and when

$$|\phi(x)| \leq C e^{-\alpha|F(x)|} \quad \text{for } x \in \Gamma, \quad (10.2.9)$$

where C and α are positive constants.

For QUADRATURE OVER Γ we have the following result (Stenger 1981): If $\phi \in B_1(\Omega_d)$, then

$$\left| \int_{\Gamma} \phi(x) dx - h \sum_{j=-\infty}^{\infty} \frac{\phi(z_j)}{F'(z_j)} \right| \leq \frac{e^{-\pi d/h}}{2 \sinh(\pi d/h)} N(\phi, \Omega_d). \quad (10.2.10)$$

Now, we shall study ERROR ESTIMATES for the Type 1 case in detail. The results for other cases are obtained similarly. If $N(\phi, \mathcal{D}_d) < \infty$, then the error bounds for the case of Type 1 domain, defined by

$$|E(\phi)| = \left| \int_{-1}^1 \phi(z) dz - h \sum_{j=-\infty}^{\infty} \mu_j \phi(jh) \right|$$

are given by (10.1.8). Further, if we set $g(z) = \frac{2\phi(z)}{1-z^2}$, and if $N(g, \Omega_d) < \infty$, then for some $C_2 > 0$ and all $z \in [-1, 1]$ (Stenger 1976)

$$\left| \phi(z) - \sum_{j=-\infty}^{\infty} \phi(z_j) \operatorname{sinc}\left(\frac{F(z) - jh}{h}\right) \right| \leq \frac{N(g, \Omega_d)}{2\pi d \sinh(\pi d/h)} \leq C_2 e^{-\pi d/h}. \quad (10.2.11)$$

These results are extended to include approximation by finite sum, as follows.

THEOREM 10.2.1. (Stenger 1976, 1981) Let ϕ be analytic in Ω_d . Then

(i) If $N(\phi, \Omega_d) < \infty$, $|\phi(z)| \leq C_1 (1 - z^2)^{\alpha-1}$ for $C_1, \alpha > 0$ and $z \in (-1, 1)$, and if $h = \sqrt{2\pi d}/(\alpha N)$, then there exists a $C_1 > 0$ such that

$$\left| \int_{-1}^1 \phi(z) dz - h \sum_{j=-N}^N \mu_j \phi(z_j) \right| \leq C_1 e^{-\sqrt{2\pi d \alpha N}}. \quad (10.2.12)$$

(ii) Let $g(z) = \frac{2\phi(z)}{1 - z^2}$. If $N(g, \Omega_d) < \infty$, then $|\phi(z)| < C_2 (1 - z^2)^\alpha$ for $C_2, \alpha > 0$ and $z \in (-1, 1)$, and if $h = \sqrt{\pi d}/(\alpha N)$, then there exists a $C_2 > 0$ such that for all $z \in [-1, 1]$

$$|\phi(z) - I_{1N}\phi(z)| \leq C_2 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}. \quad (10.2.13)$$

PROOF. (i) From (10.2.11) we have

$$\begin{aligned} |E| &\equiv \left| \int_{-1}^1 \phi(z) dz - h \sum_{j=-N}^N \mu_j \phi(z_j) \right| \\ &\leq \tilde{C}_1 e^{-2\pi d/h} + h \left\{ \sum_{j=N+1}^{\infty} \mu_j |\phi(z_j)| + \sum_{j=-\infty}^{-N-1} \mu_j |\phi(z_j)| \right\} \\ &\leq \tilde{C}_1 e^{-2\pi d/h} + 2C_1 h \sum_{j=N+1}^{\infty} \frac{2^{2\alpha-1}}{(e^{jh} + 2 + e^{-jh})^\alpha} \\ &\leq \tilde{C}_1 e^{-2\pi d/h} + 2^{2\alpha} C_1 h \sum_{j=N+1}^{\infty} e^{-\alpha j h} \\ &\leq \tilde{C}_1 e^{-2\pi d/h} + \frac{2^{2\alpha} C_1 h e^{-\alpha(N+1)h}}{1 - e^{-\alpha h}} \\ &\leq \tilde{C}_1 e^{-2\pi d/h} + \frac{2^{2\alpha}}{\alpha} C_1 e^{-\alpha Nh}, \quad \text{since } \alpha h \leq e^{\alpha h} - 1, \\ &\leq (\tilde{C}_1 + 2^{2\alpha} C_1 / \alpha) e^{-\sqrt{2\pi d \alpha N}} \quad (\text{by definition of } h). \end{aligned}$$

(ii) The result follows by a similar argument, leading to

$$|\phi(z) - I_{1N}\phi(z)| \leq \left(\tilde{C}_2 + \frac{C_2}{2^{1-2\alpha}} \sqrt{\frac{N}{2\pi d}} \right) e^{-\sqrt{\pi d \alpha N}}.$$

For INTERPOLATION OVER Γ we note that when $\phi \in B_1(\Omega_d)$ but $F'\phi \notin B_1(\Omega_d)$, then $\lim_{x \rightarrow a} \phi(x) = \phi(a)$ and $\lim_{x \rightarrow b} \phi(x) = \phi(b)$ exist and are bounded, where the limits are taken along Γ . Then in this case

$$\gamma(z) = \phi(x) - \frac{\phi(a)}{1 + e^{F(x)}} - \frac{\phi(b)}{1 + e^{-F(x)}}. \quad (10.2.14)$$

EXAMPLE 10.2.1. Let $\Gamma = [-1, 1]$, and let ϕ be analytic and bounded on Ω_d of Fig. 10.2.1. On this Γ the condition (10.2.9) yields

$$|\phi(x)| \leq C(1+x)^{-\alpha}(1-x)^\alpha, \quad \alpha > 0, C > 0. \quad (10.2.15)$$

With $h = \sqrt{\pi d/(\alpha N)}$, we find from (10.2.13) that

$$\left| \phi(x) - \frac{h}{\pi} \sin \left(\frac{\pi}{h} \log \frac{1+x}{1-x} \right) \sum_{j=-N}^N \frac{(-1)^j \phi(z_j)}{\log \frac{1+x}{1-x} - jh} \right| \leq C_1 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}, \quad (10.2.16)$$

where z_j are given in (10.2.4), and C depends only on ϕ , d , and α . If $\phi(-1) \neq 0$ or $\phi(1) \neq 0$, then the function γ defined in (10.2.14) becomes

$$\gamma(x) = \phi(x) - \frac{(1-x)\phi(0)}{2} - \frac{(1+x)\phi(1)}{2}. \quad (10.2.17)$$

The approximation in (10.2.16) is useful in interpolating functions like $\phi(x) = x^{1/3}(1-x)^{-1/2} \log x$ or $\phi(x) = \sin \pi x \log(1-x)$. ■

EXAMPLE 10.2.2. Let $\Gamma = [0, 1]$. This case is similar to that of Example 10.2.1. Let ϕ be analytic and bounded on Ω_d of Fig. 10.2.2. On this Γ the condition (10.2.9) yields

$$|\phi(x)| \leq C x^{-\alpha}(1-x)^\alpha, \quad \alpha > 0, C > 0, \quad (10.2.18)$$

and with $h = \sqrt{\pi d/(\alpha N)}$, we find from (10.2.13) that (10.2.16) holds if we replace the factor $(1+x)$ by x everywhere in it, and the points z_j are taken as those defined in (10.2.5). If $\phi(0) \neq 0$ or $\phi(1) \neq 0$, then the function γ defined in (10.2.14) becomes

$$\gamma(x) = \phi(x) - (1-x)\phi(0) - (1+x)\phi(1). \blacksquare \quad (10.2.19)$$

EXAMPLE 10.2.3. Let $\Gamma = [0, \infty)$, and Ω_d is of Type 3 as in Fig. 10.2.3. On this Γ the condition (10.2.9) yields

$$|\phi(x)| \leq \begin{cases} C x^\alpha & \text{if } 0 \leq x \leq 1, \\ C x^{-\alpha} & \text{if } x \geq 1, \end{cases} \quad (10.2.20)$$

where $\alpha > 0$ and $C > 0$. Then if we take $h = \sqrt{\pi d / (\alpha N)}$, the estimate (10.2.13) yields

$$\left| \phi(x) - \frac{h}{\pi} \sin \left(\frac{\pi}{h} \log x \right) \sum_{j=-N}^N \frac{(-1)^j \phi(z_j)}{\log x - jh} \right| \leq C_1 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}, \quad (10.2.21)$$

for all $x \in [0, \infty)$, where C_1 depends only on ϕ, d , and α , and z_j are defined in (10.2.6). If $\phi(0) \neq 0$ or $\phi(\infty) \neq 0$, then the function γ defined in (10.2.14) becomes

$$\gamma(x) = \phi(x) - \frac{\phi(0)}{1+x} - \frac{x \phi(1)}{1+x}. \quad (10.2.22)$$

The formula (10.2.21) accurately interpolates functions ϕ of the form $\phi(x) = \frac{x^{2/3} \log x}{1+x}$, or $\phi(x) = x^{5/2} e^{-x} \sin \frac{x}{2}$. ■

EXAMPLE 10.2.4. Let $\Gamma = [0, \infty)$ of Fig. 10.2.4. On this Γ the condition (10.2.9) becomes

$$|\phi(x)| \leq \begin{cases} C x^\alpha & \text{if } 0 \leq x \leq 1, \\ C e^{-\alpha x} & \text{if } x \geq 1, \end{cases} \quad (10.2.23)$$

where $\alpha > 0$ and $C > 0$, and under the same choice of h as in Example 10.2.3 the estimate (10.2.13) yields

$$\left| \phi(x) - \frac{h}{\pi} \sin \left(\frac{\pi}{h} \log(\sinh x) \right) \sum_{j=-N}^N \frac{(-1)^j \phi(z_j)}{\log(\sinh x) - jh} \right| \leq C_1 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}, \quad (10.2.24)$$

for all $x \in [0, \infty)$, where C_1 depends only on ϕ, d , and α , and z_j are defined by (10.2.8). If $\phi(0) \neq 0$ or $\phi(\infty) \neq 0$, then the function γ defined in (10.2.14) becomes

$$\gamma(x) = \phi(x) - \frac{\phi(0)}{1 + \sinh x} - \frac{x \sinh x \phi(1)}{1 + \sinh x}. \quad (10.2.25)$$

The formula (10.2.24) performs very well at interpolating functions on $[0, \infty)$ which may show oscillatory behavior on $(0, \infty)$ and have a singularity at $x = 0$,

e.g., for functions ϕ of the form $\phi(x) = x^\alpha \log \left[1 - \left(\frac{\sin x}{x} \right)^2 \right] e^{-\alpha x}$, or $\phi(x) = x^{4/5} e^{-x}$. ■

10.3. Approximation Theory

Let ϕ be defined on $[-1, 1]$, and set $\ell\phi(x)$ as the linear interpolant of ϕ at ± 1 , i.e., $\ell\phi(x) = \frac{1-x}{2}\phi(-1) + \frac{1+x}{2}\phi(1)$. Define

$$u(x) = \phi(x) - \ell\phi(x), \quad g(z) = \frac{2u(z)}{1-z^2}.$$

LEMMA 10.3.1. Let ϕ be analytic in Ω_d and satisfy the Lipschitz condition on $\overline{\Omega}_d$ for some $\alpha > 0$. Then, for u and g defined above we have

- (i) $N(g, \Omega_d) < \infty$, and
- (ii) there exists a $C > 0$ such that for $z \in [-1, 1]$

$$|u(z)| < C (1-z^2)^\alpha. \quad (10.3.1)$$

PROOF. (i) Recall that by (10.2.2)

$$N(g, \Omega_d) \equiv \lim_{\Gamma \rightarrow \partial \Omega_d} \int_{\Gamma} |g(z)| dz = \lim_{\Gamma \rightarrow \partial \Omega_d} 2 \int_{\Gamma} \frac{|u(z)|}{|1-z^2|} dz.$$

Now, since ϕ satisfies the Lipschitz condition on $\overline{\Omega}_d$ for some $\alpha > 0$, there is a constant $C_1 > 0$ such that

$$|\phi(z) - \phi(z')| \leq C_1 |z - z'|^\alpha.$$

Thus, u satisfies the Lipschitz condition on $\overline{\Omega}_d$ for some $\alpha > 0$, since

$$\begin{aligned} |u(z) - u(z')| &\leq |\phi(z) - \phi(z')| + |\ell\phi(z) - \ell\phi(z')| \\ &\leq C_1 |z - z'|^\alpha + \frac{1}{2} |\phi(-1)| |z - z'| + \frac{1}{2} |\phi(1)| |z - z'| \\ &\leq C_2 |z - z'|^\alpha \quad \text{for some } C_2 > 0. \end{aligned}$$

Also, since for some z near $+1$, we have $|u(z)| = |u(z) - u(1)| \leq C_2 (1-z)^\alpha$, and $\frac{1}{|1-z^2|} \leq \frac{C_3}{|1-z|}$ for some $C_3 > 0$, we have for z near $+1$

$$\frac{|u(z)|}{|1-z^2|} \leq C_2 C_3 |1-z|^{\alpha-1},$$

which is integrable near ± 1 , say inside a compact set. A similar argument holds for z near -1 . Finally, the boundedness of $g(z)$ elsewhere implies that $N(g, \Omega_d) < \infty$.

(ii) is obvious from the assumption that ϕ satisfies the Lipschitz condition on $\overline{\Omega}_d$ for some $\alpha > 0$ and the above argument. ■

THEOREM 10.3.2. Let ϕ be defined as in Lemma 10.3.1, and set $h = \sqrt{\pi d/(\alpha N)}$. Then there is a constant $C > 0$ such that for all $z \in [-1, 1]$

$$|\phi(z) - [I_{1N}u(z) + \ell\phi(z)]| \leq C \sqrt{N} e^{-\sqrt{\pi d\alpha N}}. \quad (10.3.2)$$

PROOF. From Lemma 10.3.1 we know that Theorem 10.2.1 can be applied to $u(z)$, but

$$\begin{aligned} |\phi(z) - [I_{1N}u(z) + \ell\phi(z)]| &= |\phi(z) - \ell\phi(z) - I_{1N}u(z)| \\ &= |u(z) - I_{1N}u(z)|, \end{aligned}$$

which yields the desired result. ■

10.4. Convergence

For convergence for $z \in [-1, 1]$ we have the following results.

THEOREM 10.4.1. Let $\phi \in B_1(\mathcal{D}_d)$, and choose a d^* such that $d > d^* > 0$; then for all $z \in \mathcal{D}_{d^*}$

$$|\phi(z) - C(\phi, h, x)| \leq \frac{N(\phi, 1, \mathcal{D}_d)}{2(d - d^*)} e^{-\pi(d-d^*)/h}. \quad (10.4.1)$$

PROOF. Set $\Gamma_n = \{(s + it) : |s| \leq (n + 1/2)h, t = \pm d, \text{ or } s = \pm(n + 1/2)h, |t| \leq d\}$ for a positive integer n . By Cauchy's theorem

$$\left| \phi(z) - \sum_{j=-n}^n \phi(jh) \operatorname{sinc}\left(\frac{z-jh}{h}\right) \right| = \left| \lim_{c \rightarrow 1^-} \frac{\sin(\pi z/h)}{2i\pi} \int_{\Gamma_n} \frac{\phi(cw) dw}{(w-z) \sin(\pi w/h)} \right|.$$

Note that $|\sin(u+iv)| = \sqrt{\sin^2 u + \sinh^2 v}$. Consider a vertical segment of Γ_n ; then for the integral representation of the error E ,

$$\begin{aligned} &\left| \frac{\sin(\pi z/h)}{2i\pi} \int_{-d}^d \frac{\phi((n+1/2)h+it) dt}{[(n+1/2)h-x]^2 + (t-y)^2} \cdot \sin \pi ((n+1/2)h+it)/h \right| \\ &\leq \frac{\sin(\pi z/h)}{2\pi} \frac{1}{C} \int_{-d}^d |\phi((n+1/2)h+it)| dt, \end{aligned}$$

since $\left| \frac{\sin \pi ((n+1/2)h + it)}{h} \right| \leq 1$, and we have assumed n large enough to make $[(n+1/2)h - x]^2 + (t-y)^2 > C$, which can be done for any $z \in \mathcal{D}_{d^*}$. Hence, for all $z \in \mathcal{D}_{d^*}$

$$\lim_{n \rightarrow \infty} \frac{\sin(\pi z/h)}{2i\pi} \times \int_{-d}^d \frac{\phi((n+1/2)h + it) dt}{[(n+1/2)h - x]^2 + (t-y)^2} = 0.$$

Now, we are left only with horizontal segments of Γ_n for which

$$\begin{aligned} & \left| \frac{\sin(\pi z/h)}{2i\pi} \int_{\mathbb{R}} \frac{\phi(s + id^-) ds}{(s + id^- - z) \sin \pi(s + id^-)/h} \right| \\ & \leq \frac{|\sin(\pi z/h)|}{2(d - d^*) \sinh(\pi d/h)} \int_{\mathbb{R}} |\phi(s + id^-)| ds. \end{aligned}$$

Since $|s + id - z| = \sqrt{(s-x)^2 + (d+y)^2} > d - d^*$, and $|\sin \pi(s + id)/h| \geq \sinh(\pi d/h)$, we find that

$$\begin{aligned} |E| \equiv |\phi(z) - C(\phi, h, z)| & \leq \frac{N(\phi, 1, \mathcal{D}_d)}{2\pi(d - d^*)} \frac{|\sin(\pi z/h)|}{\sin(\pi d/h)} \\ & \leq \frac{N(\phi, 1, \mathcal{D}_d)}{2\pi(d - d^*)} \frac{\cosh(\pi d^*/h)}{\sin(\pi d/h)}, \end{aligned}$$

which, when written with exponentials, yields the result. ■

The change of variables behaves exactly as in (10.2.11). Thus, if $N(g, \Omega_d) < \infty$, then

$$|\phi(z) - C(\phi, h, F(z))| \leq \tilde{C}_2 e^{-\pi(d-d^*)/h}$$

for all $z \in \Omega_{d^*}$ and some $\tilde{C}_2 > 0$. Also, since $|\text{sinc}(z/h)| \leq C e^{\pi d^*}/h$ for all $z \in \mathcal{D}_{d^*}$, we have the following result.

THEOREM 10.4.2. Let ϕ be analytic in Ω_d , $g(z) = \frac{z\phi(z)}{1-z^2}$, $N(g, \Omega_d) < \infty$, and $|\phi(z)| \leq C(1-z^2)^\alpha$ for some $C, \alpha > 0$ and all $z \in (-1, 1)$. Then there exists a constant $A > 0$ such that when $h = A/\sqrt{N}$ we have $|\phi(z) - I_{1N}\phi(z)| = O(e^{-\gamma\sqrt{N}})$ for some $\gamma > 0$ and all $z \in \Omega_{d^*}$.

PROOF. In view of the proof of Theorem 10.2.1 we need only consider

$$\begin{aligned} \sum_{j=N+1}^{\infty} \left| \phi(x_j) \operatorname{sinc}\left(\frac{F(z) - jh}{h}\right) \right| &\leq C e^{\pi d^*/h} \sum_{j=N+1}^{\infty} |\phi(x_j)| \\ &\leq \frac{C}{\alpha h} e^{\pi d^*/h} e^{-\alpha Nh} \quad (\text{as in Theorem 10.2.1}) \\ &\leq \frac{C\sqrt{N}}{A\alpha} e^{(A\alpha - \pi d^*/A)\sqrt{N}}, \quad \text{where } h = A/\sqrt{N}. \end{aligned}$$

This result holds if A is taken sufficiently large. ■

We consider an extension to the case when ϕ is analytic in $(\Omega_d) \cap C(\overline{\Omega}_d)$ but does not satisfy the Lipschitz condition on $\overline{\Omega}_d$ for $\alpha > 0$. In this case pointwise convergence of $I_{1N}\phi$ to ϕ is proved in the following theorem where we have assumed without loss of generality that $\phi(-1) = 0 = \phi(1)$. The theorem is proved in \mathcal{D}_d with $u(w) = \phi(G(w))$, where, in view of continuity of ϕ , we have $u \rightarrow 0$ in \mathcal{D}_d as $\Re\{w\}$ goes along lines of constant $\Im\{w\}$; in fact, this limiting process is uniform for $\Im\{w\} \in [-d, d]$.

THEOREM 10.4.3. If $u(w)$ is described as above and $h = 1/\sqrt{N}$, then $I_{1N}u(z) \rightarrow u(z)$ as $N \rightarrow \infty$ for all $z \in \mathbb{R}$.

PROOF. Fix $z \in \mathbb{R}$ and assume that $|z| < \sqrt{N}$ and $h = 1/\sqrt{N} < d$. Set Γ_n as in the proof of Theorem 10.4.1. Then, in view of Theorem 10.4.1,

$$|u(z) - I_{1N}u(z)| = \left| \frac{\sin(\pi z/h)}{2i\pi} \int_{\Gamma_n} \frac{\phi(w) dw}{(w-z) \sin(\pi w/h)} \right|.$$

Consider a vertical segment of Γ_n , on which

$$\begin{aligned} |E| &\equiv \left| \frac{\sin(\pi z/h)}{2i\pi} \int_{-h}^h \frac{u((N+1/2)h+it) dt}{((N+1/2)h+it-z) \sin \pi ((N+1/2)h+it)/h} \right| \\ &\leq \frac{\|u\|_{\Omega_d}}{2\pi} \int_{-\sqrt{h}}^{\sqrt{h}} \frac{dt}{\sqrt{\sigma^2 + t^2}}, \end{aligned}$$

where $|\sigma| = |(N+1/2)h - z| > 0$ by assumption,

$$\leq \frac{\|u\|_{\Omega_d}}{2\pi} \log \sqrt{\frac{(N+1/2)h - z)^2 + h + \sqrt{h}}{(N+1/2)h - z)^2 + h - \sqrt{h}}} \rightarrow 0 \quad \text{for fixed } z \text{ as } N \rightarrow \infty.$$

Also, on a horizontal segment of Γ_n we have

$$\begin{aligned}
& \left| \frac{\sin(\pi d/h)}{2i\pi} \int_{-(N+1/2)h}^{(N+1/2)h} \frac{u(s + i\sqrt{h})}{\sqrt{(s-z)^2 + h} \cdot \sin \pi(s + i\sqrt{h})/h} ds \right| \\
& \leq \frac{\|u\|_{\Omega_d}}{2\pi \sinh(\pi/\sqrt{h})} \int_{-3Nh}^{3Nh} \frac{ds}{\sqrt{s^2 + h}} \\
& = \frac{\|u\|_{\Omega_d}}{2\pi} e^{-\pi N^{1/4}} \left\{ \log \left(\sqrt{(3Nh)^2 + h} + 3Nh \right) \right. \\
& \quad \left. - \log \left(\sqrt{(3Nh)^2 + h} - 3Nh \right) \right\} \\
& \leq \frac{5\|u\|_{\Omega_d}}{\pi} \sqrt{N} e^{-\pi N^{1/4}} \quad \text{for sufficiently large } N \\
& \rightarrow 0 \quad \text{for fixed } z \text{ as } N \rightarrow \infty.
\end{aligned}$$

Note that the best choice of h is to take it proportional to $1/\sqrt{N}$; then all previous results hold with this choice.

10.5. Sinc-Galerkin Scheme

The first step in applying this method is to determine a priori the regions where the solution of an FK2 (VK2) is analytic. This method becomes more useful in the case of nonlinear integral equations since it is very difficult to determine the nature of singularity for such equations. However, we do not deal with this situation in this book. Unlike the approximation technique of the Galerkin method, the results of Sections 10.2 and 10.3 are used to replace an integral equation by a system of algebraic equations without carrying out any numerical integration. Thus, we consider an FK2 of the form (1.2.18) with $\lambda = 1$, i.e.,

$$\phi(x) = (K\phi)(x) + f(x), \quad (10.5.1)$$

where $(K\phi)(x) = \int_a^b k(x, s)\phi(s) ds$, or $(K\phi)(x) = \int_a^x k(x, s)\phi(s) ds$, under the assumption that ϕ as well as $K\phi$ are analytic in Ω_d and satisfy a Lipschitz condition on $\bar{\Omega}_d$ for $0 < \alpha \leq 1$. Then for an FK2 we can approximate ϕ and f on

$\Gamma = [a, b]$ by

$$\begin{aligned}\phi(x) &\approx \phi_N(x) = \sum_{j=-N-1}^{N+1} c_j G_j(x), \\ f(x) &\approx f_N(x) = \sum_{j=-N-1}^{N+1} d_j G_j(x),\end{aligned}\tag{10.5.2}$$

where

$$\begin{aligned}G_{-N-1}(x) &= \frac{b-x}{b-a}, \\ G_j(x) &= S(j, h) \circ F(x), \quad j = -N, -N+1, \dots, N, \\ G_{N+1} &= \frac{x-a}{b-a}.\end{aligned}\tag{10.5.3}$$

Then set

$$\mu_i(x) = (KG_i)(x), \quad i = -N-1, -N, \dots, N+1,\tag{10.5.4}$$

where μ_i are approximated on Γ by

$$\mu_i(x) = \mu_{iN}(x) = \sum_{j=-N-1}^{N+1} e_{ij} G_j(x).\tag{10.5.5}$$

Substituting these approximations in (10.5.1), we obtain the following system of algebraic equations:

$$c_j - \sum_{i=-N-1}^{N+1} e_{ij} c_i = d_j, \quad j = -N-1, -N, \dots, N+1,\tag{10.5.6}$$

which determines the unknown coefficients c_j . This is known as the SINC-GALERKIN SCHEME.

An accurate computation of μ_i in (10.5.5) is important in the preceding scheme. The approximations of previous sections become very useful in this computation, especially in the case of singular integral equations. If we assume that the step size $h = \sqrt{\pi d}/(\alpha N)$, the function f in (10.5.2) and μ_i in (10.5.5) can be approximated, according to Stenger (1981), to within an error of $O(\sqrt{N}e^{-\pi d \alpha N})$, which yields an approximation of ϕ by ϕ_N accurate to within an error of $O(N^{3/2}e^{-\pi d \alpha N})$.

EXAMPLE 10.5.1. Consider an VK2

$$\phi(x) = \int_0^x [k(s)\phi(s) + f(s)] ds + r(x), \quad x \in [0, 1], \quad (10.5.7)$$

where k, r and $f \in B_1(\Omega_d)$, and f satisfies a Lipschitz condition on $[0, 1]$ for $\alpha > 0$. For $x = 1$ this equation becomes an FK2. The approximations on $[0, 1]$ are

$$\begin{aligned} \phi(x) &\approx \phi_N(x) = \sum_{j=-N-1}^{N+1} c_j G_j(x), \\ r(x) &\approx r_N(x) = \sum_{j=-N-1}^{N+1} \rho_j G_j(x), \\ G_{-N-1}(x) &= 1 - x, \quad G_{N+1} = x, \\ G_j(x) &= S(j, h) \circ \log \frac{x}{1-x}, \quad j = -N, -N+1, \dots, N, \\ \rho_j &= -(1 - z_j) \rho_{-N-1} - z_j \rho_{N+1} + r(z_j), \\ z_j &= \frac{e^{jh}}{1 + e^{jh}}, \quad j = -N, -N+1, \dots, N. \end{aligned} \quad (10.5.8)$$

Substituting these values in (10.5.7) and using (C.2.2) as well as the approximation

$$\int_0^1 f(x) dx \approx h \sum_{j=-N}^N z_j (1 - z_j) f(z_j),$$

and equating the coefficients of G_j on both sides, we obtain the system

$$\begin{aligned} c_{-N-1} &= \rho_{-N-1}, \\ c_j &= h \sum_{m=-N}^N \sigma_{j-m} z_m (1 - z_m) \left\{ k_m [(1 - z_m) c_{-N-1} + c_m + z_m c_{N+1}] \right. \\ &\quad \left. + f_m - c_{N+1} + \rho_{N+1} \right\}, \quad j = -N, -N+1, \dots, N, \\ c_{N+1} &= h \sum_{m=-N}^N z_m (1 - z_m) \left\{ k_m [c_m + (1 - z_m) c_{-N-1} + z_m c_{N+1}] + f_m \right\} \\ &\quad + \rho_{N+1}, \end{aligned} \quad (10.5.9)$$

where $z_m = \frac{1}{2} \left(1 + \tanh \frac{mh}{2} \right)$. Take $h = \sqrt{\pi s / (\alpha N)}$ and solve the system for c_j . This yields the approximate value of ϕ_N , which is accurate to within an error of order $O(N e^{-\sqrt{\pi d \alpha N}})$. ■

NOTE: For an extension of this method to two-dimensional potential problems, see Section 10.8.

10.6. Computation Guidelines

Consider the approximation

$$I(\phi) = \int_{\Gamma} \phi(x) dx \approx T_h(\phi) \equiv h \sum_{j=-\infty}^{\infty} \frac{\phi(z_j(h))}{F'(z_j(h))}, \quad (10.6.1)$$

where $\phi \in B_1(\Omega_d)$. Also, let

$$M_h(\phi) = h \sum_{j=-\infty}^{\infty} \frac{\phi(z_j(h/2))}{F'(z_j(h/2))}, \quad (10.6.2)$$

Then

$$T_{(h/2)}(\phi) = \frac{1}{2} [T_h(\phi) + M_h(\phi)]. \quad (10.6.3)$$

In view of (10.2.10) we find that when h is replaced by $h/2$, the precision in significant digits in the computation of (10.6.1) doubles, i.e., we achieve twice the original precision obtained with h alone. For example, let $h = 1$, say; then compute $T_1(\phi)$ and $M_1(\phi)$, in that order. If the difference $|T_1(\phi) - M_1(\phi)| = \varepsilon$, then

$$|I(\phi) - T_{(1/2)}(\phi)| = \left| I(\phi) - \frac{1}{2} [T_1(\phi) - M_1(\phi)] \right| = O(\varepsilon^2) < \varepsilon. \quad (10.6.4)$$

Another important aspect in computing the approximation of $I(\phi)$ is the fact that we can never compute the entire infinite sum in (10.6.1) and (10.6.2). However, if we assume that $\frac{\phi(x)}{F'(x)} = O(e^{-\alpha|F(x)|})$, as in (10.2.9), then we have a useful stopping rule for computing the infinite sums. Suppose, we stop the summation in (10.6.1) at some $N > 0$ when $\frac{|\phi(z_N)|}{F'(z_N)} = O(e^{-\alpha Nh}) < \varepsilon/3$. Then we may expect that

$$\begin{aligned} \left| h \sum_{j=N+1}^{\infty} \frac{\phi(z_j(h))}{F'(z_j(h))} \right| &\leq O\left(h \sum_{j=N+1}^{\infty} e^{-\alpha j h}\right) \\ &= O\left(\frac{h e^{-\alpha(N+1)h}}{1 - e^{-\alpha h}}\right) = O(e^{-\alpha Nh}) = O(\varepsilon). \end{aligned} \quad (10.6.5)$$

This means that the remainder after the term $j = N$ should be of the same order of magnitude as the N th term. However, in practice we should avoid the stopping rule at or near a zero of ϕ . For this purpose a more reliable test is (Stenger 1981)

$$\frac{|\phi(z_N)|}{F'(z_N)} + \frac{|\phi(z_{N+1})|}{F'(z_{N+1})} + \frac{|\phi(z_{N+2})|}{F'(z_{N+2})} < \frac{\varepsilon}{3}. \quad (10.6.6)$$

For negative j we should carry out a similar stopping rule since the function $\frac{\phi(x)}{F'(x)}$ may converge to zero at different rates as $x \rightarrow a$ or as $x \rightarrow b$ along Γ .

10.7. Sinc-Collocation Method

We shall develop a collocation method using the sinc function as the basis functions. The approximate solution will be shown to have an error of the form $C\sqrt{m} \log m e^{-\gamma m}$, where C and γ are positive constants and m denotes the number of basis functions used. A useful feature of this method is that it provides a natural grading of collocation points near the singularity of the exact solution of a VK2 and that of the nodes in the quadrature rule to be used to compute the integrals.

Consider a VK2 of the form (1.2.18):

$$\phi(x) = (K^\alpha \phi)(x) + f(x), \quad 0 \leq x \leq a, \quad (10.7.1)$$

where $K^\alpha : C[0, a] \mapsto C[0, a]$ denotes the weakly singular linear Volterra operator defined by

$$(K^\alpha \phi)(x) = \int_0^x \frac{k(x, s)}{(x - s)^\alpha} \phi(s) ds, \quad 0 \leq x \leq a, \quad 0 < \alpha < 1, \quad (10.7.2)$$

and the free term f and the kernel k are smooth on the interval $[0, a]$ and $\{(x, s) : 0 \leq s \leq x \leq a\}$, respectively. According to Brunner and van der Houwen (1986), the VK2 (10.7.1) has a unique solution $\phi \in C[0, a]$, although ϕ has an unbounded derivative at $x = 0$; in fact, $\phi'(x) = O(x^{-\alpha})$ as $x \rightarrow 0$.

Using the sinc function approximation on a finite interval $(0, X)$ we see that the conformal map $w = F(z) \equiv F_X(z) = \log \frac{z}{X - z}$ of type 2, Section 10.2, carries the eye-shaped domain Ω_d onto the infinite strip \mathcal{D}_d . Note that in the present case

the boundary of Ω_d consists of two circular arcs that intersect symmetrically with respect to the real axis at an angle $2d$ at the endpoints 0 and X . Thus, the basis functions are given by

$$S(j, h, X)(x) = \text{sinc}\left(\frac{F_X(x) - jh}{h}\right), \quad (10.7.3)$$

which behave like the Kronecker delta at the nodes $x_m \in (0, X)$ defined by (see (10.2.5))

$$x_m = G_X(mh) = X \frac{e^{mh}}{1 + e^{mh}}. \quad (10.7.4)$$

We denote these nodes by

$$x_m = X \mu_m, \quad \text{where } \mu_m = \frac{e^{mh}}{1 + e^{mh}}. \quad (10.7.5)$$

Note the range of summation, which, unlike previous sections, is not from $j = -N$ to $j = N$.

As we know from previous sections, if a function g is defined on $(0, X)$, then the function

$$C(g, h, x) = \sum_{j=-M}^N g(X\mu_j) S(j, h, X)(x) \quad (10.7.6)$$

agrees with g at each node $x_j = X\mu_j$, $j = -M, -M+1, \dots, N$ (see (10.2.3)). Moreover, if we integrate $C(g, h, x)$ over the interval $(0, X)$, we obtain the sinc-quadrature rule

$$\int_0^X g(x) dx \approx Xh \sum_{j=-M}^N g(X\mu_j) \mu_j (1 - \mu_j) \quad (10.7.7)$$

(see Example 10.5.1). The error estimate (10.2.13) yields the following result.

THEOREM 10.7.1. Let a function g be analytic and bounded on Ω_d and be such that

$$|g(x)| \leq \begin{cases} C_1 x^\beta & \text{if } 0 < x \leq X/2, \\ C_1 (X - x)^\gamma & \text{if } X/2 < x < X, \end{cases}$$

where C_1 , β , and γ are positive constants. Then taking $h = \sqrt{\pi d / (\beta M)}$ and $N = [\beta M / \gamma] + 1$, we have the sinc-quadrature rule

$$|g(x) - C(g, h, x)| \leq C_3 \sqrt{M} e^{-\sqrt{\pi d \beta M}}, \quad (10.7.8)$$

where the constant C_2 depends on g, d, β , and γ . The error bound for the rule (10.7.7) is given by the following result (Stenger 1981; Riley 1992).

THEOREM 10.7.2. Let a function g be analytic on the domain Ω_d such that

$$\int_{G_X(x+L)} |g(z) dz| \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty,$$

where $L = \{ib : |b| < d\}$, and

$$\lim_{C \rightarrow \partial\Omega_d} \inf_{C \subseteq \Omega_d} \int_C |g(z) dz| < \infty.$$

Further, suppose there are positive constants C_3, β , and γ such that

$$|g(x)| \leq \begin{cases} C_3 x^{\beta-1}, & \text{if } 0 < x \leq X/2, \\ C_3 (X - x)^{\gamma-1}, & \text{if } X/2 < x < X. \end{cases}$$

Then, if we take $h = \sqrt{2\pi d/(\gamma M)}$ and $M = [\gamma N/\beta] + 1$, we have

$$\left| \int_0^X g(x) dx - Xh \sum_{j=-M}^N g(X\mu_j) \mu_j (1 - \mu_j) \right| \leq C_4 e^{-\sqrt{2\pi d \gamma N}}, \quad (10.7.9)$$

where C_4 depends only on g, d, β , and γ . Note that these theorems pertain to any function g satisfying the hypotheses, but for our purpose g shall be replaced by the solution ϕ of the VK2 under consideration.

The SINC-COLLOCATION SCHEME to compute an approximate nonsmooth solution $\tilde{\phi}$ of Eqs (10.7.1) and (10.7.2) over the interval $(0, X)$, $x \leq a$, is as follows: Assuming that ϕ is analytic and bounded on Ω_d , the translated function

$$\Phi(x) = \phi(x) - \left[\phi(0) + \frac{\phi(X) - \phi(0)}{X} x \right]$$

satisfies the hypotheses of Theorem 10.7.1 with $\beta = 1 - \alpha$ and $\gamma = 1$. The function

$$C(\phi, h, x) = \phi(0) + \sum_{j=-M}^N \Phi(X\mu_j) S(j, h, X)(x) + \frac{\phi(X) - \phi(0)}{X} x \quad (10.7.10)$$

interpolates ϕ at the points $0, X$, and each $X\mu_j$, $j = -M, -M + 1, \dots, N$, and with $h = \sqrt{\pi d / ((1 - \alpha)M)}$ and $N = [(1 - \alpha)M] + 1$, we find from (10.7.7) that

$$|\phi(x) - C(\phi, h, x)| \leq C_5 \sqrt{M} e^{-\sqrt{\pi d (1 - \alpha) M}}. \quad (10.7.11)$$

Thus, in view of the sinc-interpolant (10.7.10), a trial solution of the form

$$\tilde{\phi} \equiv \phi_{M,N}(x) = c_{-M-1} + \sum_{j=-M}^N c_j S(j, h, X)(x) + \frac{c_{N+1}}{X} x \quad (10.7.12)$$

will be used to approximate the exact solution ϕ of Eq (10.7.1). (For extending the summation to include the boundary points in a two-dimensional case, see Section 10.2.) The first coefficient $c_{-M-1} = \phi_{M,N}(0) = f(0)$ is easy to compute. The remaining coefficients c_j of $\phi_{M,N}$ are determined by using the condition that $\phi_{M,N}$ satisfy (10.7.1) at suitably chosen collocation points (see below how these collocation points are chosen), by using an appropriate quadrature rule to compute integrals involving the operator K^α defined in (10.7.2). Thus, we use Theorem 10.7.2 to obtain an approximation for K^α as

$$(K_{M,N}^\alpha \phi)(x) = x^{1-\alpha} \hat{h} \sum_{m=-M}^N k(X, X\mu_m) \phi(X\mu_m) \mu_m (1 - \mu_m)^{1-\alpha}, \quad (10.7.13)$$

where μ_m are defined by (10.7.5), and $\hat{h} = \sqrt{2\pi d / ((1-\alpha)M)}$. In view of (10.7.9) the approximation (10.7.13) holds if the product $k(x, \cdot)\phi$ is analytic and bounded with \hat{h} as defined above. Therefore, we assume that

$$\left| (K^\alpha \phi)(x) - (K_{M,N}^\alpha \phi)(x) \right| \leq C_6 e^{-\sqrt{2\pi d(1-\alpha)M}} \quad (10.7.14)$$

for all $x \in [0, a]$. This assumption is valid if the functions $k(x, \cdot)\phi$ are analytic on Ω_d and are uniformly bounded there for all $x \in [0, a]$. Also,

$$\left| \int_0^1 \frac{ds}{(1-s)^\alpha} - \hat{h} \sum_{j=-M}^N u_m (1 - u_m)^{1-\alpha} \right| \leq C_7 e^{-\sqrt{2\pi d(1-\alpha)M}}. \quad (10.7.15)$$

We use the Kronecker delta behavior of sinc functions to choose the collocation points as $x_i = X\mu_i$ for $i = -M, -M+1, \dots, N$, and $x_{N+1} = X$. Thus, the remaining $(M+N+2)$ unknown coefficients c_j 's in (10.7.12) are computed by requiring that

$$\phi_{M,N}(x) = f(x_i) + (K_{M,N}^\alpha \phi_{M,N})(x_i), \quad i = -M, -M+1, \dots, N+1, \quad (10.7.16)$$

or by writing it as the linear system $\mathbf{A}\mathbf{c} = \mathbf{f}$, i.e.,

$$(A_{M,N} - E_{M,N}) \{\mathbf{c}\} = \left(\begin{bmatrix} & & x_{-M}/X \\ I & & \\ \vdots & & \\ 0 & \dots & 0 & 1 \end{bmatrix} - E_{M,N} \right) \{\mathbf{c}\} = \{\mathbf{f}\}, \quad (10.7.17)$$

where I is the identity matrix of order $M + N + 1$, and the matrix $E_{M,N}$ has the elements

$$\begin{aligned} e_{ij} &= (K_{M,N}^\alpha S(j, h, X))(x_i), \quad i = -M, -M + 1, \dots, N + 1, \\ &\quad j = -M, -M + 1, \dots, N, \\ e_{i,N+1} &= (K_{M,N}^\alpha p)(x_i), \quad i = -M, -M + 1, \dots, N + 1; p(x) = \frac{x}{X}; \end{aligned}$$

$\{\mathbf{c}\}$ denotes the vector of unknown coefficients of $\phi_{M,N}$, and the vector $\{\mathbf{f}\}$ has components

$$\begin{aligned} f_i &= f(x_i) + f(0) (K_{M,N}^\alpha f_0)(x_i), \quad i = -M, -M + 1, \dots, N + 1, \\ f_0(x) &= 1. \end{aligned}$$

The matrix $A_{M,N}$ in (10.7.17) is a well-behaved nonsingular matrix, and the system $\mathbf{A}\mathbf{c} = \mathbf{f}$ determines a unique vector \mathbf{c} provided the norm $\|E_{M,N}\|$ is small; in fact, the bound on this norm, as given by Riley (1992), is

$$\|E_{M,N}\| \leq C_8 X^{1-\alpha} \log M, \quad (10.7.18)$$

which implies that for all M we can choose X such that $\|A_{M,N}^{-1} E_{M,N}\| < 1$, and hence, for sufficiently small X , the system $\mathbf{A}\mathbf{c} = \mathbf{f}$ has a unique solution and, according to Riley (1992),

$$\left\| \left(A_{M,N}^{-1} - E_{M,N} \right) \right\| \leq 4 \quad \text{for all } M. \quad (10.7.19)$$

At each collocation point x_i we find from (10.7.10) and (10.7.16) that

$$\begin{aligned} C_{M,N}(\phi, h, x_i) - \phi_{M,N}(x_i) &= \phi(x_i) - \phi_{M,N}(x_i) \\ &= (K^\alpha \phi)(x_i) - (K_{M,N}^\alpha \phi_{M,N})(x_i) \\ &= (K^\alpha \phi)(x_i) - K_{M,N}^\alpha C_{M,N}(\phi, h, x_i) \\ &\quad + (K_{M,N}^\alpha [C_{M,N}(\phi, h) - \phi_{M,N}])(x_i), \end{aligned}$$

which can be written as the linear system

$$(A_{M,N} - E_{M,n}) (\mathbf{b} - \mathbf{c}) = (\mathbf{K}^\alpha \phi) - \mathbf{K}_{M,N}^\alpha C_{M,N}(\phi, h), \quad (10.7.20)$$

where the matrix $(A_{M,N} - E_{M,n})$ is the same as in (10.7.17), and the vectors \mathbf{b} and \mathbf{c} contain the coefficients of $C_{M,N}(\phi, h)$ and $\phi_{M,N}$, respectively. The vectors $(\mathbf{K}^\alpha \phi)$ and $(\mathbf{K}_{M,N}^\alpha C_{M,N})(\phi, h)$ contain the elements $(K^\alpha \phi)(x_i)$ and

$(K_{M,N}^\alpha C_{M,N}(\phi, h))(x_i)$, respectively. Now, we use (10.7.19), (10.7.11), and the triangle inequality to obtain

$$\begin{aligned}\|\mathbf{b} - \mathbf{c}\| &\leq 4 (\|\mathbf{K}^\alpha \phi - \mathbf{K}_{M,N}^\alpha \phi\| + \|\mathbf{K}_{M,N}^\alpha (\phi - C_{M,N}(\phi, h))\|) \\ &\leq C_9 \sqrt{M} e^{-\sqrt{\pi d(1-\alpha)M}}.\end{aligned}\quad (10.7.21)$$

This inequality provides a bound for $\|\phi - \phi_{M,N}\|$ at each collocation point, which is valid for sufficiently small X . This means that the approximate solution $\phi_{M,N}$ may be valid on a short interval $[0, X]$ as compared to the original interval $[0, a]$. This restriction on X is a consequence of the bound (10.7.18), although in numerical computation this restriction in many cases may not be a serious handicap. The choice of M is also important, since an increase in M leads to a crowding of collocation points near the endpoint 0 with $\lim_{M \rightarrow \infty} x_{-M} = 0$. This situation will increase the rounding errors, thus affecting the accuracy of computed results. Also, for large M the coefficient matrix in the linear system (10.7.17) is full, contrary to the typical triangular structure. But this method works very well for the specified class of analytic functions, provided M is kept relatively small.

EXAMPLE 10.7.1. Consider the VK2

$$\phi(x) = \sqrt{x} + \frac{\pi x}{2} - \int_0^x \frac{\phi(s)}{\sqrt{x-s}} ds, \quad 0 \leq x \leq a,$$

with $k(x, s) = 1$. The exact solution of this equation is $\phi(x) = \sqrt{x}$. Theorem 10.7.1 holds on the eye-shaped domain $\Omega_{\pi/2}$, and we choose M, N, h , and \hat{h} as in that theorem with $\alpha = 1/2$ and $d = \pi/2$. We shall apply the sinc-collocation scheme on the interval $[0, 1]$, with $h = 0.01$ and $M = 2(2)10, 20(20)100$. The error estimate is given by $\sqrt{M} \log M e^{-\sqrt{\pi d(1-\alpha)M}} = \sqrt{M} \log M e^{-\pi \sqrt{M/4}}$, which is presented in Table 10.7.1.

Table 10.7.1. Error Estimates

M	Error estimate	\parallel	M	Error estimate
2	0.106312	\parallel	20	0.0207106
4	0.119814	\parallel	40	0.00173513
6	0.093614	\parallel	60	1.64836(-4)
8	0.069178	\parallel	80	3.10113(-5)
10	0.050693	\parallel	100	6.94007(-6)

Computational details are available in `riley1.nb`. ■

10.8. Single-Layer Potential

An extension of the sinc-Galerkin method to two-dimensional potential problems is investigated in Schwing (1976) to compute the numerical solution of the FK2

$$\phi(P) = \lambda \iint_S k(P, Q) \phi(Q) dA_Q = f(P), \quad (10.8.1)$$

where P and Q are points on a two-dimensional surface S embedded in \mathbb{R}^3 , λ is a complex number that may or may not be an eigenvalue of the kernel (P, Q) which is possibly singular, and $f(P)$ is a known function on S . It is assumed that the kernel may have a singularity of the form $k(P, Q) = \frac{\hat{f}(P, Q)}{d(P, Q)}$, where $d(P, Q)$ represents the distance from point P to Q , and \hat{f} is bounded. Thus, all singularities of the kernel are included in $\frac{1}{d(P, Q)}$. Equations of this type arise in potential theory of electricity and magnetism, for instance, the solution of the Neumann problem over a volume V , bounded by S , defined by $\nabla^2 u = 0$ in V , $\frac{\partial u}{\partial n} = f$ on S , which leads to a single-layer potential

$$u(P) = \iint_S \frac{\rho_Q}{d(P, Q)} dA_Q, \quad P \in V,$$

where the unknown density function ρ satisfies the FK2 (see Mikhlin 1967)

$$\rho(P) + \frac{1}{2\pi} \iint_S \frac{\partial}{\partial n_P} \left(\frac{1}{d(P, Q)} \right) \rho(Q) dA_Q = \frac{f(P)}{2\pi}, \quad P \in S. \quad (10.8.2)$$

The sinc-Galerkin method is used to solve (10.8.1). Initially we consider the case when the surface S is the square $S = [-1, 1] \times [-1, 1]$, and the kernel is of the type

$$k(x, y; s, t) = \frac{\kappa(x, y; s, t)}{\sqrt{(x - s)^2 + (y - t)^2}}, \quad (x, y), (s, t) \in S.$$

The following notation is used:

- (i) $\Omega = \{(x, y) \in C^2 : \text{if } x \in [-1, 1] \Rightarrow y \in \Omega_d, \text{ or if } x \in [-1, 1] \Rightarrow x \in \Omega_d\}$,
 $\Omega^* = \{(x, y) \in C^2 : \text{if } x \in [-1, 1] \Rightarrow y \in \Omega_{d^*}, \text{ or if } x \in [-1, 1] \Rightarrow x \in \Omega_{d^*}\}$;
- (ii) $w_{ij} = \mu_i \mu_j$;

(iii) If $F(x, y)$ is defined on Ω , then $F^x(y)$ represents $F(x, y)$ for x fixed in $[-1, 1]$; similarly, $F^y(x)$ represents $F(x, y)$ for y fixed in $[-1, 1]$.

$$(iv) g^x(y) = \frac{2F^x(y)}{1 - y^2}, \text{ and } g^y(x) = \frac{2F^y(x)}{1 - x^2}.$$

$$(v) I_{2N}\phi(x, y) = \sum_{i,j=1}^N \phi(x_i, y_j) \operatorname{sinc}((F(x) - ih)/h) \operatorname{sinc}((F(y) - jh)/h).$$

$$(vi) d(P, Q) \equiv d(x, y; s, t) = \sqrt{(x-s)^2 + (y-t)^2}.$$

(vii) $K\phi(x, y) = \iint_S k(x, y; s, t)\phi(s, t) ds dt$. Then Eq (10.8.1) is written as

$$(I - \lambda K)\phi = f. \quad (10.8.3)$$

It is assumed that ϕ satisfies CONDITION A1 if for all fixed $x \in [-1, 1]$ the function ϕ^x is analytic in $\Omega_d \cap C(\overline{\Omega}_{d^*})$, and for all fixed $y \in [-1, 1]$ the function ϕ^y is analytic in $\Omega_d \cap C(\overline{\Omega}_{d^*})$, where d, d^* are fixed numbers such that $d > d^* > 0$. We denote by \mathcal{X} the Banach space of functions ϕ defined on Ω_d such that ϕ satisfies condition A1. Similarly, ϕ is said to satisfy CONDITION A2 if for some α and for all $x \in [-1, 1]$ the function ϕ^x satisfies the Lipschitz condition in $\overline{\Omega}_d$ and for all $y \in [-1, 1]$ the function ϕ^y satisfies the Lipschitz condition in $\overline{\Omega}_d$. We denote by X^α the space of functions ϕ defined on Ω_d such that ϕ satisfies condition A2. Also, ϕ is said to satisfy the CONDITION A3 if for some $\alpha > 0$ the inequality $|\phi(x, y)| \leq C|1-x^2|^{\alpha-1}|1-y^2|^{\alpha-1}$ for $x, y \in \overline{\Omega}$, while both ϕ^x and ϕ^y are analytic in Ω_d .

Some properties of the operator K defined in (10.8.3) are as follows:

- (i) If $\phi \in C(S)$, then $K\phi$ satisfies a Hölder condition of order $\alpha = 1/3$ on S ;
- (ii) There is a constant $C > 0$ that does not depend on ϕ such that $\|K\phi\|_S \leq C\|\phi\|_S$.

(iii) There is a positive constant L_k such that $\iint_S |k(x, y; s, t)| ds dt \leq L_k$.

(iv) K is a compact operator from $C(S)$ to $C(S)$.

(v) If $\phi \in C(S)$, then $K\phi \in \mathcal{X}$.

(vi) $K\phi$ is analytic at a point $x_0 \in \Omega_d \setminus \{-1, 1\}$.

(vii) If $x = p + iq$, then $\lim_{q \rightarrow 0} K\phi(x, y) = K\phi(p, y)$.

For proofs of these results, see Schwing (1976, Ch. 3).

Let X_N denote the approximation space defined as follows: Set

$$\nu_{ij}(x, y) = \operatorname{sinc}\left(\frac{F(x) - ih}{h}\right) \operatorname{sinc}\left(\frac{F(y) - jh}{h}\right); \quad (10.8.4)$$

then

$$X_N = \text{span} \left\{ \nu_{ij} \right\}_{i,j=-N}^N. \quad (10.8.5)$$

The two-dimensional sinc-interpolant I_{2N} is a projection operator from X_N to \mathcal{X} :

$$I_{2N}\phi(x, y) = \sum_{i,j=-N}^N \phi(x_i, y_j) \nu_{ij}(x, y). \quad (10.8.6)$$

We have the following results on interpolation.

THEOREM 10.8.1. Let ϕ satisfy condition A3 for some $\alpha > 0$. If $h = \sqrt{2\pi d}/(\alpha N)$, then there exists a constant $C_1 > 0$ such that

$$\left| \iint_S \phi(x, y) dx dy - h^2 \sum_{i,j=-N}^N w_{ij} \phi(x_i, y_j) \right| \leq C_1 e^{-\sqrt{2\pi d \alpha n}}. \quad (10.8.7)$$

THEOREM 10.8.2. Let F be defined on Ω such that F satisfies conditions A1 and A2 for some $\alpha > 0$, and let $\phi \equiv 0$ on ∂S . Then, if $h = \sqrt{2\pi d}/(\alpha N)$, then there exists a constant $C_2 > 0$ such that

$$|\phi(x, y) - I_{2N}\phi(x, y)| \leq C_2 N^{3/2} e^{-\sqrt{\pi d \alpha N}} \quad \text{for all } (x, y) \in S. \quad (10.8.8)$$

THEOREM 10.8.3. Let ϕ be defined on Ω and satisfy the condition A2 for some $\alpha > 0$, and let $\phi \equiv 0$ on ∂S . Then there exist constants C and γ such that for $h = C/\sqrt{N}$

$$|\phi(x, y) - I_{2N}\phi(x, y)| = O\left(e^{-\gamma\sqrt{N}}\right) \quad \text{as } N \rightarrow \infty \text{ for all } (x, y) \in \Omega^*. \quad (10.8.9)$$

These results can be compared with (10.2.13). The proofs of these theorems are available in Schwing (1976, Theorem 2.18).

The Bubnov–Galerkin method applied to (10.8.3) leads to the problem of finding the approximate solution $\phi_N \in X_N$ such that

$$\phi_N - \lambda I_{2N} K \phi_N = I_{2N} f, \quad (10.8.10)$$

or

$$\begin{aligned} & \sum_{i,j} \phi_{ij} \nu_{ij}(x, y) - \lambda \left\{ \sum_{i,j} \iint_S k(x_i, y_j; s, t) \right. \\ & \times \left. \left[\sum_{m,n} \phi_{mn} \nu_{mn}(s, t) \right] ds dt \right\} \nu_{ij}(x, y) = \sum_{i,j} f_{ij} \nu_{ij}(x, y), \end{aligned} \quad (10.8.11)$$

where $\phi_{ij} = \phi_N(x_i, y_j)$ and $f_{ij} = f(x_i, y_j)$, respectively. By using the linear ordering on interpolation points $\{P_m\}_{m=1}^{(2N+1)^2} = \{(x_i, y_j)\}_{i,j=-N}^N$, Eq.(10.8.11) can be rewritten as a linear system of algebraic equations

$$A \bar{\phi} = \bar{f}, \quad (10.8.12)$$

where $A = I - \lambda B$, $B = [b_{mn}]$, $\bar{\phi} = [\phi_N(P_m)]$, $\bar{f} = [f(P_m)]$, and

$$b_{mn} = \iint_S k(P_m, Q) \nu_{mn}(Q) dA_Q. \quad (10.8.13)$$

Note that the sinc-interpolant I_{2N} provides a good approximation only if ϕ or f is zero on the boundary of S . But to deal with nonzero boundary conditions we must subtract a function from ϕ so that their difference is zero on the boundary. This can be achieved in \mathbb{R}^2 by using the bilinearly bounded Boolean sum defined by

$$P_1 \oplus P_2 \phi = P_1 \phi + P_2 \phi - P_1 P_2 \phi, \quad (10.8.14)$$

where

$$\begin{aligned} P_1 \phi(x, y) &= \frac{1}{2} [(1-x)\phi(-1, y) + (1+x)\phi(1, y)], \\ P_2 \phi(x, y) &= \frac{1}{2} [(1-y)\phi(x, -1) + (1+y)\phi(x, 1)]. \end{aligned}$$

Let

$$\begin{aligned} b\phi(x, y) &= \frac{1}{4} [(1-x)(1-y)\phi(-1, -1) + (1-x)(1+y)\phi(-1, 1) \\ &\quad + (1+x)(1-y)\phi(1, -1) + (1+x)(1+y)\phi(1, 1)] = P_1 P_2 \phi. \end{aligned}$$

Here the projection P_1 interpolates ϕ for $x = \pm 1$, P_2 interpolates ϕ for $y = \pm 1$, and the bilinear function $b\phi$ interpolates ϕ at the vertices of S ; also, $P_1 \oplus P_2 \phi$ is an interpolant to ϕ on the boundary of S . Thus, we can apply I_{2N} to $\phi - P_1 \oplus P_2 \phi$, although we have a problem here since points like $(-1, y)$ are not known in advance. But we can remedy this situation by constructing $b\phi$ by adding interpolation points at the vertices of S . In order to add additional points to the boundary of S so that $P_1 \oplus P_2 \phi$ can be completely computed, we define $\hat{\phi}(x, y) = \phi(x, y) - b\phi(x, y)$, and approximate terms like $\hat{\phi}(-1, y)$ instead by taking, e.g., $\hat{\phi}(-1, y) = I_{1N} \hat{\phi}(-1, y)$, and similarly for $\hat{\phi}(1, y)$, $\hat{\phi}(x, -1)$ and $\hat{\phi}(x, 1)$, where I_{1N} is defined by (10.2.3). Using the notation

$$\begin{aligned} B_N \phi(x, y) &= \frac{1}{2} [(1-x)\hat{\phi}(-1, y) + (1+x)\hat{\phi}(1, y) + (1-y)\hat{\phi}(x, -1) \\ &\quad + (1+y)\hat{\phi}(x, 1)] + b\phi(x, y), \end{aligned} \quad (10.8.15)$$

it can be proved that $p_1 \oplus P_2\phi(x_i, y_j) = B_N\phi(x_i, y_j)$ for all $i, j = -N, -N + 1, \dots, N$, where (x_i, y_j) are the interpolation points of I_{2N} (see Schwing 1976, Lemma 4.4).

- Now, we define a new set of basis functions $\{\nu_{ij}\}_{i,j=-N-1}^{N+1}$ as follows:
- (a) $\nu_{ij} = \text{sinc}\left(\frac{F(x) - ih}{h}\right) \text{sinc}\left(\frac{F(y) - jh}{h}\right)$, $-N \leq i, j \leq N$, for interpolation at (x_i, y_j) ;
 - (b) $\nu_{-N-1,j} = \left[\frac{1-x}{2} - I_{1N}\left(\frac{1-x}{2}\right)\right] \text{sinc}\left(\frac{F(y) - jh}{h}\right)$ for interpolation at $(-1, y_j)$;
 - $\nu_{N+1,j} = \left[\frac{1+x}{2} - I_{1N}\left(\frac{1+x}{2}\right)\right] \text{sinc}\left(\frac{F(y) - jh}{h}\right)$ for interpolation at $(1, y_j)$;
 - $\nu_{i,-N-1} = \left[\frac{1-y}{2} - I_{1N}\left(\frac{1-y}{2}\right)\right] \text{sinc}\left(\frac{F(x) - ih}{h}\right)$ for interpolation at $(x_i, -1)$;
 - $\nu_{i,N+1} = \left[\frac{1+y}{2} - I_{1N}\left(\frac{1+y}{2}\right)\right] \text{sinc}\left(\frac{F(x) - ih}{h}\right)$ for interpolation at $(x_i, 1)$;
 - (c) $\nu_{-N-1,-N-1} = \left[\frac{1-x}{2} - I_{1N}\left(\frac{1-x}{2}\right)\right] \left[\frac{1-y}{2} - I_{1N}\left(\frac{1-y}{2}\right)\right]$ for interpolation at $(-1, -1)$;
 - $\nu_{-N-1,N+1} = \left[\frac{1-x}{2} - I_{1N}\left(\frac{1-x}{2}\right)\right] \left[\frac{1+y}{2} - I_{1N}\left(\frac{1+y}{2}\right)\right]$ for interpolation at $(-1, 1)$;
 - $\nu_{N+1,-N-1} = \left[\frac{1+x}{2} - I_{1N}\left(\frac{1+x}{2}\right)\right] \left[\frac{1-y}{2} - I_{1N}\left(\frac{1-y}{2}\right)\right]$ for interpolation at $(1, -1)$;
 - $\nu_{N+1,N+1} = \left[\frac{1+x}{2} - I_{1N}\left(\frac{1+x}{2}\right)\right] \left[\frac{1+y}{2} - I_{1N}\left(\frac{1+y}{2}\right)\right]$ for interpolation at $(1, 1)$.

This introduces a new projection operator $P_N : \mathcal{X} \mapsto X_N$ such that $P_N\phi = I_{2N}(\phi - B_N\phi) + B_N\phi$, and, thus, enlarges the definition of X_N to $X_N = \text{span}\{\nu_{ij}\}_{i,j=-N-1}^{N+1}$; this admits boundary interpolation as shown in Fig. 10.8.1.

Note that now $P_N\phi(x, y) = \sum_{i,j=-N-1}^{N+1} \phi_{ij} \nu_{ij}(x, y)$; so, if $\phi \in X^\alpha$ for some $\alpha > 0$, then $P_1 \oplus P_2\phi \in X^\alpha$; then $\phi - P_1 \oplus P_2\phi$ satisfies the hypotheses of Theorems 10.8.2 and 10.8.3, and if $h = \sqrt{\pi d/(\alpha N)}$, there exists a positive constant C such that for all $(x, y) \in S$

$$|\phi(x, y) - P_N \phi(x, y)| \leq C N^{3/2} e^{-\sqrt{\pi d \alpha N}}, \quad (10.8.16)$$

and there exist positive constants C_1 and γ such that for $h = C_1/\sqrt{N}$

$$|\phi(x, y) - P_N \phi(x, y)| = O(e^{-\gamma \sqrt{N}}) \quad \text{as } N \rightarrow \infty \text{ for all } (x, y) \in \Omega^*. \quad (10.8.17)$$

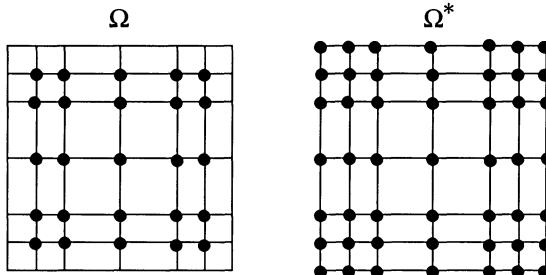


Fig. 10.8.1. Interpolation Points from $N \times N$ to $(N + 1) \times (N + 1)$ Mesh.

This introduces a new projection operator $P_N : \mathcal{X} \mapsto X_N$ such that $\phi = I_{2N}(\phi - B_N \phi) + B_N \phi$, and, thus, it enlarges the definition of X_N to $X_{N+1} = \text{span} \left\{ \nu_{ij} \right\}_{i,j=-N-1}^{N+1}$, as shown in Fig. 10.8.2. Note that now $P_N \phi(x, y) = \sum_{i,j=-N-1}^{N+1} \phi_{ij} \nu_{ij}(x, y)$; also, if $\phi \in X^\alpha$ for some $\alpha > 0$, then $P_1 \oplus P_2 \phi \in X^\alpha$; thus, $\phi - P_1 \oplus P_2 \phi$ satisfies the hypotheses of Theorems 10.8.2 and 10.8.3, and if $h = \sqrt{\pi d}/(\alpha N)$, there exists a positive constant C such that for all $(x, y) \in S$

$$|\phi(x, y) - P_N \phi(x, y)| \leq C N^{3/2} e^{-\sqrt{\pi d \alpha N}},$$

and there exist positive constants C_1 and γ such that for $h = C_1/\sqrt{N}$ the order of convergence is given by

$$|\phi(x, y) - P_N \phi(x, y)| = O(e^{-\gamma \sqrt{N}}) \quad \text{as } N \rightarrow \infty \text{ for all } (x, y) \in \Omega^*.$$

It implies that $\|K - K_N\| = O(e^{-\gamma \sqrt{N}})$ as $N \rightarrow \infty$.

In the case when we cannot explicitly integrate

$$b_{mn} = \iint_S k(x, y; s, t) \nu_{mn}(s, t) ds dt,$$

as defined in (10.8.13), where (x_i, y_j) are interpolation points P_N and ν_{mn} the associated basis functions, we employ a QUADRATURE RULE Q_L ($L > 0$) defined as follows: Since the kernel $k(x, y; s, t)$ has a singularity of the type $\frac{1}{|d(x, y; s, t)|}$ at the point $(x, y) = (s, t)$, we partition S into subrectangles R_l , $l = 1, 2, 3, 4$, based on the location of the singularity. Next, we use the affine transformation $T_l : S \mapsto R_l$, $l = 1, 2, 3, 4$ (see Fig. 10.8.2). For example, for $(\sigma, \tau) \neq (s, t) \in S$

$$T_1(\sigma, \tau) = \begin{pmatrix} s_1(\sigma, \tau) \\ t_1(\sigma, \tau) \end{pmatrix} = \begin{pmatrix} 0.5[(1-\sigma)x + (1+\sigma)] \\ 0.5[(1-\tau)y + (1+\tau)] \end{pmatrix}.$$

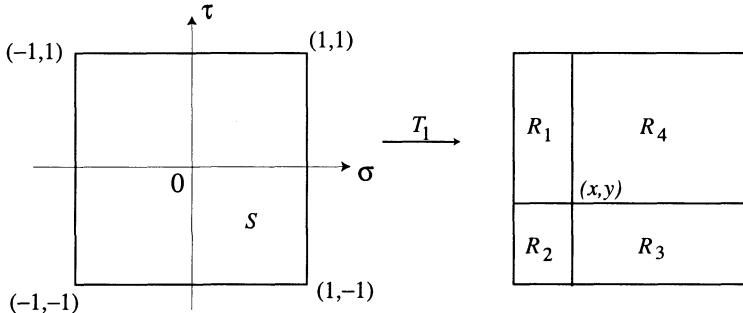


Fig. 10.8.2. Transformation T_1 .

Thus,

$$\begin{aligned} \iint_S k(x, y; s, t) \nu(s, t) ds dt &= \sum_{l=1}^4 \iint_{R_l} k(x, y; s, t) \nu(s, t) ds dt \\ &= \sum_{l=1}^4 J_{xy}^l \iint_S k^l(x, y; \sigma, \tau) \nu^l(\sigma, \tau) d\sigma d\tau, \end{aligned}$$

where $k^l(x, y; \sigma, \tau) = k(x, y; s_l(\sigma, \tau), t_l(\sigma, \tau))$, the quantities ν^l are defined similarly as before, and J_{xy}^l is the Jacobian of the transformation T_l ; for example, $J_{xy}^1 = 0.25(1-x)(1-y)$. The above change of variable (x, y) to (σ, τ) provides

a perturbed quadrature of the original integrals b_{mn} which places any singularity at a vertex of S . For example, for $l = 1$ the singularity occurs at $(-1, -1)$. Thus, the quadrature rule Q_L transforms the system (10.8.13) into a perturbed system

$$\begin{aligned} \hat{A}\hat{\phi} &= \hat{f}, \\ \hat{A} = I - \lambda \hat{B}_L &= I - \lambda (B + E_L), \quad \hat{B}_L = [\hat{b}_{mn}], \\ E_L &= [e_{mn}], \quad e_{mn} = b_{mn} - \hat{b}_{mn}, \\ \hat{b}_{mn} &= \sum_{l=1}^4 J_m^l h^2 \sum_{i,j=-L}^L w_{ij} k^l(P_m, Q_{ij}) \nu_n^l(Q_{ij}). \end{aligned} \quad (10.8.18)$$

So far we have used the Bubnov–Galerkin method to compute an approximate solution of Eq (10.8.3), by choosing an N and setting up the interpolation points $\{(x_i, y_j)\}$ and the associated basis functions ν_{ij} . For the square S recall from (10.2.4) that $(x_i, y_j) = \left(\frac{e^{ih} - 1}{e^{ih} + 1}, \frac{e^{jh} - 1}{e^{jh} + 1}\right)$ for $-N \leq i, j \leq N$, and $x_{-N-1} = y_{-N-1} = -1$, $x_{N+1} = y_{N+1} = 1$; so these points are easily generated. These points are placed in an order which is used to set up the linear algebraic system (10.8.12) if the integrals b_{mn} can be computed explicitly; otherwise we use the perturbed system (10.8.18), which employs the quadrature Q_L with a prescribed tolerance. This yields $\phi - \hat{\phi}$ as the approximate solution of Eq (10.8.1), where the total number of computation points is $M = (2N + 1)^2$ and the rate of convergence is of the order $O(e^{-\delta M^{1/4}})$.

EXAMPLE 10.8.1. Eq (10.8.2) for the unknown single-layer potential density $\rho(P)$ can be written as

$$(I - \lambda K)\rho = f. \quad (10.8.19)$$

The adjoint operator $K^*\rho$ is known to have an eigenvalue $\lambda = -\frac{1}{2\pi}$ (since $\iint_S \frac{\partial}{\partial n_q} \left(\frac{1}{d(P, Q)} \right) dA_Q = -\iint_S \frac{\cos \alpha}{d^2(P, Q)} dA_Q = -2\pi$). Hence, $\rho \equiv 1$ is an eigenfunction for the homogeneous adjoint problem $(I_\lambda K^*)\rho = 0$. Thus, a solution of Eq (10.8.19) exists only if f is orthogonal (in the L_2 sense) to $\rho = 1$, namely, if

$$\iint_S f dA = 0. \quad (10.8.20)$$

For example, if V is the unit ball with boundary S (unit sphere), then the outward normal to S at a point P is given by P itself, and the kernel can be represented as

$$k(P, Q) = \frac{\partial}{\partial n_P} \left(\frac{1}{d(P, Q)} \right) = \frac{(x - s)x - (y - t)y - (z - v)z}{d^3(P, Q)},$$

where $P = (x, y, z)$ and $Q = (s, t, v)$. Let the boundary values be given by $g(x, y, z) = a(1 - 3z^2)$ for some constant a . Then since

$$a \iint_S (1 - 3z^2) dA = a \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} (1 - 3\sin^2 \theta) \cos \theta d\phi d\theta = 0,$$

we set $f(P) = \frac{g(P)}{2\pi}$, and the condition (10.8.20) is satisfied. For the boundary values $g = 2(1 - 3z^2)$ the solution of the Neumann problem ($\nabla^2 u = 0$ in V , $\frac{\partial u}{\partial n} = f$ on S mentioned in the beginning of this section) is known to be $u = x^2 + y^2 - 2z^2 + c$, where c is an arbitrary constant. Thus, Eq (10.8.19) is an eigenvalue problem, and the solution ρ in the spherical coordinate system becomes

$$\begin{aligned} \rho &= x^2 + y^2 - 2z^2 + c = r^2 (\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi - 2 \sin^2 \theta) + c \\ &= r^2 (\cos^2 \theta - 2 \sin^2 \theta) + c = r^2 (1 - 3 \sin^2 \theta) + c, \end{aligned}$$

which yields

$$\frac{\partial \rho}{\partial n} = \frac{\partial \rho}{\partial r} = 2r (1 - 3 \sin^2 \theta) = 2 \left(r - \frac{3z^2}{r} \right) \Big|_{r=1} = 2(1 - 3z^2).$$

Also, the operator K is self-adjoint on S since

$$\begin{aligned} k(P, Q) &= \frac{\partial}{\partial n_P} \left(\frac{1}{d(P, Q)} \right) = \frac{(x - s)x - (y - t)y - (z - v)z}{d^3(P, Q)} \\ &= \frac{[1 - (xs + yt_z v)]}{2\sqrt{2} [1 - (xs + yt + zv)]^{3/2}} = \frac{-1}{2d(P, Q)} \\ &= \frac{\partial}{\partial n_Q} \left(\frac{1}{d(P, Q)} \right) = k^*(P, Q). \end{aligned} \tag{10.8.21}$$

Hence, we find that $\rho \equiv 1$ is a basis for the eigenfunctions of Eq (10.8.19) associated with the eigenvalue $\lambda = -\frac{1}{2\pi}$ on the unit sphere S . Now, we compute the solution of Eq (10.8.19) which in view of (10.8.21) reduces to $\rho(P) + \frac{1}{2\pi} \iint_S k(P, Q) \rho(Q) dA_Q = f(P)$, or $\rho(P) - \frac{1}{4\pi} \iint_S \frac{\rho(P)}{d(P, Q)} dA_Q = \frac{g(P)}{2\pi}$, or $\rho(P) = \frac{u(P) + 2g(P)}{4\pi}$, where $u(P)$ is the solution of the Neumann problem mentioned earlier.

Schwing (1976) provides a Fortran77 code to approximate the singular integrals over the square $[-1, 1] \times [-1, 1]$ which occur in the computation of $I - \lambda K$. It

computes the density function ρ . With $u = x^2 + y^2 - 2z^2$, and the expected value $\hat{\rho} = \frac{u + 2g}{4\pi}$, the results are given in Table 10.8.1 for $N = 1$, and in Table 10.8.2 for $N = 2$. The last column in each table gives the approximate solution, which matches with the exact solution $\rho = 1$. The eigenvalues λ_i at the points P_i , $i = 1, \dots, 7$, are

$$\{27.027, 1.287, 1.248, 1.070, 1.129, 3.436, 4.926\}.$$

Table 10.8.1. $N = 1$

P_i	ρ	$\hat{\rho}$	$\rho - \hat{\rho}$
$P_1 = (0, -0.6638, 0.7479)$	0.722	-0.270	0.992
$P_2 = (0.6638, -0.4965, 0.5594)$	1.057	0.024	1.033
$P_3 = (0, 0, 1)$	0.196	-0.795	0.991
$P_4 = (0.6638, 0, 0.7479)$	0.721	-0.270	0.991
$P_5 = (0, -1, 0)$	1.388	0.398	0.990
$P_6 = (0.6638, -0.7479, 0)$	1.333	0.398	0.935
$P_7 = (1, 0, 0)$	1.394	0.398	0.996

Table 10.8.2. $N = 2$

P_i	ρ	$\hat{\rho}$	$\rho - \hat{\rho}$
$P_1 = (0, -0.8171, 0.5765)$	0.9995	0.0012	0.9983
$P_2 = (0.5084, -0.4154, 0.4964)$	1.1050	0.1037	1.0013
$P_3 = (0.8171, -0.4710, 0.3323)$	1.3556	0.3539	1.0017
$P_4 = (0, -0.5084, 0.9611, 0.7479)$	0.5117	-0.4873	0.9990
$P_5 = (0.5084, -0.4378, 0.7415)$	0.7436	-0.2585	1.0021
$P_6 = (0.8171, -0.2931, 0.4964)$	1.1049	0.1037	1.0012
$P_7 = (0, 0, 1)$	0.2027	-0.7958	0.9985
$P_8 = (0.5084, 0, 0.8611)$	0.5114	-0.4873	0.9987
$P_9 = (0.8171, 0, 0.5765)$	0.9994	0.0012	0.9982
$P_{10} = (0, -1, 0)$	1.3972	0.3979	0.9993
$P_{11} = (0.5084, -0.8611, 0)$	1.3856	0.3979	0.9877
$P_{12} = (0.8171, 0.5765, 0)$	1.3903	0.3979	0.9924
$P_{13} = (0, 0, 1)$	1.3976	0.3979	0.9997 ■

10.9. Double-Layer Problem

Another example where the sinc-quadrature method is used for the FK2 with a Mellin-type singularity defined by

$$\phi(x) - \int_0^1 k(x, s) [\phi(s) - \phi(0)] ds + \gamma(x) \phi(0) = f(x), \quad 0 \leq x \leq 1, \quad (10.9.1)$$

is discussed by Kress, Sloan, and Stenger (1998). Here the functions γ and the free term $f(x)$ are both assumed to be 1-periodic and continuous, i.e., $\gamma(0) = \gamma(1)$, and $f(0) = f(1)$. This problem arises from the double-layer integral equation formulation of the Dirichlet problem of potential theory in plane domains with corners. The kernel $k(x, s)$ is assumed to be periodic with period 1 with respect to x and be continuous in both x and s , $0 \leq x, s \leq 1$, except for the four corners of the square $[0, 1] \times [0, 1]$, at which corners the kernel has a Mellin-type singularity. The solution sought is also a 1-periodic function $\phi(x)$, such that $\phi(0) = \phi(1)$. A Nyström method with sinc quadrature is used to solve this problem. For details see the work by Kress et al. (1998).

REFERENCES USED: Brunner (1985), Brunner and van der Houwen (1986), Higgins (1985), Hoog and Weiss (1974), Kress, Sloan, and Stenger (1998), Linz (1969), McNamee, Stenger, and Whitney (1971), Mikhlin (1967), Riley (1992), Schwinger (1976), Stenger (1972, 1976, 1981, 1993), Whittaker and Watson (1973).

11

Equations of the First Kind

We have discussed Fredholm equations of the first kind off and on in previous chapters. These equations occupy a special place since they possess an inherent ill-posedness. This property makes their numerical evaluation difficult; different techniques are needed to compute such solutions. We shall discuss some of the well-known methods in this chapter.

11.1. Inherent Ill-Posedness

A Fredholm integral equation of the first kind (FK1) has the form

$$\int_a^b k(x, s)\phi(s) ds = f(x), \quad a \leq x \leq b. \quad (11.1.1)$$

These equations are inherently ill-posed problems (see Groetsch 1984, Kress 1989, Hansen 1992, and Golberg and Chen 1997), meaning that the solution is generally unstable. This ill-posedness makes numerical solutions very difficult, as a small error can lead to an unbounded error. This is justified as follows: It is known that when $k(x, s)$ is continuous, symmetric, and nondegenerate, the eigenvalue problem

$$\int_a^b k(x, s)\phi(s) ds = \lambda \phi(x) \quad (11.1.2)$$

has an infinite number of real eigenvalues λ_i such that $\lim_{i \rightarrow \infty} \lambda_i = 0$, with associated eigenfunctions $\phi_i(x)$ (Hansen 1992). The function $\phi_i(x)$ can be normalized to

$\psi_i(x)$ such that $\langle \psi_i(x), \psi_i(x) \rangle = 1$. Thus,

$$\int_a^b k(x, s) \psi_i(s) ds = \lambda_i \psi_i(x). \quad (11.1.3)$$

Both $f(x)$ and $\phi(x)$ can be expanded into a series of $\psi_i(x)$:

$$f(x) = \sum_{i=1}^{\infty} f_i \psi_i(x), \quad f_i = \langle f, \psi_i \rangle, \quad \phi(x) = \sum_{i=1}^{\infty} \phi_i \psi_i(x). \quad (11.1.4)$$

Substituting (11.1.3) and (11.1.4) into Eq (11.1.1) and comparing coefficients of $\psi_i(x)$ on both sides, we get

$$f_i = \lambda_i \phi_i. \quad (11.1.5)$$

Hence, $\phi(x)$ can be expressed as

$$\phi(x) = \sum_{i=1}^{\infty} \frac{f_i}{\lambda_i} \psi_i(x). \quad (11.1.6)$$

In practice, we use an approximation of the series (11.1.6) by truncating it to n terms; thus,

$$[\phi(x)]_n \approx \sum_{i=1}^n \frac{f_i}{\lambda_i} \psi_i(x). \quad (11.1.7)$$

Suppose that a small error ϵ_i is introduced in the evaluation of f_i , resulting in an error $\delta[\phi(x)]_n$ in the values of $[\phi(x)]_n$. Then

$$[\phi(x)]_n + \delta[\phi(x)]_n \approx \sum_{i=1}^{\infty} \frac{(f_i + \epsilon_i)}{\lambda_i} \psi_i(x), \quad \delta[\phi(x)]_n \approx \sum_{i=1}^n \frac{\epsilon_i}{\lambda_i} \psi_i(x), \quad (11.1.8)$$

which in view of the orthonormality of $\psi_i(x)$ gives

$$\left\langle \delta[\phi(x)]_n, \delta[\phi(x)]_n \right\rangle = \sum_{i=1}^n \frac{\epsilon_i^2}{\lambda_i^2}. \quad (11.1.9)$$

Thus, no matter how small the errors ϵ_i are, the squared error in $\phi_i(x)$ will grow.

Another source of ill-posedness comes from the Riemann-Lebesgue lemma, which states that if any function $f(x)$ is square-integrable, then

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \sin \frac{n\pi(x-a)}{b} dx = 0. \quad (11.1.10)$$

This result implies that if $\phi(x)$ is a solution of Eq (11.1.1) and $k(x, s)$ is square-integrable, then $\phi(x) + C \sin \frac{n\pi(x-a)}{b}$ for $n \rightarrow \infty$ will satisfy Eq (11.1.1) for any value of C . Another aspect of the problem is that if $\int_a^b k(x, s)g(s) ds = 0$ for any $g \in C[a, b]$ and $\phi(x)$ is any solution of Eq (11.1.1), then $\phi(x) + g(x)$ is also a solution of Eq (11.1.1). Moreover, if both $\phi(x)$ and $k(x, s)$ are continuous but $f(x)$ is not continuous, then Eq (11.1.1) is not solvable.

Another criterion for the existence of the solution to FK1 was stated by Fox and Goodwin (1953):

THEOREM 11.1.1. If $k(x, s)$ satisfies a linear differential equation $L[k(x, s)] = 0$, where $L \equiv D^n + p_1(x)D^{n-1} + \dots + p_n(x)$, $D \equiv \frac{\partial}{\partial x}$, then $f(x)$ must satisfy the same differential equation.

The proof is obvious; Assume that we can interchange the order of integration with respect to s and differentiation with respect to x ; then applying the operator L to both sides of Eq (11.1.1), we have

$$L\{f(x)\} = L \left\{ \int_a^b k(x, s)\phi(s) ds \right\} = \int_a^b L\{k(x, s)\}\phi(s) ds = 0.$$

11.2. Separable Kernels

In the case when the kernel $k(x, s)$ is separable, i.e.,

$$k(x, s) = \sum_{i=1}^n a_i(x) \bar{b}_i(s) = \sum_{i=1}^n a_i \otimes b_i, \quad (11.2.1)$$

substituting (11.2.1) into Eq (11.1.1), we get

$$f(x) = \sum_{i=1}^n \int_a^b a_i(x) \bar{b}_i(s) \phi(s) ds = \sum_{i=1}^n \alpha_i a_i(x), \quad (11.2.2)$$

where the coefficients α_i , defined by

$$\alpha_i = \int_a^b \bar{b}_i(s) \phi(s) ds, \quad (11.2.3)$$

are known since the functions $a_i(x)$ and $f(x)$ are known. Thus, the problem reduces to solving Eqs (11.2.3) with both α_i and $\bar{b}_i(s)$ known and $\phi(s)$ unknown. It is reasonable to assume that

$$\phi(s) = \sum_{i=1}^n \beta_j \bar{b}_j(s). \quad (11.2.4)$$

Substituting (11.2.4) into (11.2.3), we find that

$$\alpha_i = \sum_{j=1}^n \beta_j \langle b_i, b_j \rangle, \quad i = 1, 2, \dots, n, \quad (11.2.5)$$

where (11.2.5) represents a system of n linear equations in n unknowns β_j . Since the functions $b_i(s)$ are assumed to be linearly independent, the matrix for the system (11.2.5) will be nonsingular and a unique solution $\{\beta_1, \beta_2, \dots, \beta_n\}$ exists. However, if in (11.2.3) we replace $\phi(s)$ by $\phi(s) + g(s)$, where $\langle g(s), b_i(s) \rangle = 0$, then there are an infinite number of solutions for $\phi(x)$ from Eq (11.2.3).

Consider the homogeneous equation

$$\int_a^b k^T(s, x) \psi(s) ds = 0, \quad (11.2.6)$$

with transposed kernel

$$k^T(s, x) = \sum_{i=1}^n \bar{a}_i(s) b_i(x). \quad (11.2.7)$$

Then Eq (11.2.6) reduces to

$$b_i(x) \langle \psi, a_i \rangle = 0. \quad (11.2.8)$$

Since $b_i(x)$ are linearly independent, we have

$$\langle \psi, a_i \rangle = 0. \quad (11.2.9)$$

Obviously, an infinite number of functions $\psi(x)$ satisfy Eq (11.2.9). We find from (11.2.2) that all such functions $\psi(x)$ must be orthogonal to $f(x)$. We can similarly show that

$$\int_a^b k(s, x)\psi(x) ds = 0 \quad (11.2.10)$$

has an infinite number of solutions. This analysis proves the following theorem:

THEOREM 11.2.1. A necessary and sufficient condition for an FK1 of the form (11.1.1) with a separable kernel to have an L_2 -solution is that the given functions $f(x)$ be of the form (11.2.2) and that it be orthogonal to all solutions of Eq (11.2.6).

11.3. Some Theorems

Some useful theorems for integral equations of the first kind are given below; their proofs are available in well-known texts, such as Pogorzelski (1966) and Kondo (1991). We state a few theorems for symmetric kernels.

THEOREM 11.3.1. If $k(x, s)$ is a symmetric, positive-definite kernel and an FK1 is solvable, then the following iteration scheme yields a solution: Choose an arbitrary L_2 -function $\phi_0(x)$ and a number λ such that $0 < \lambda < 2\lambda_1$, where λ_1 is the smallest eigenvalue of the kernel $k(x, s)$, then $\phi_n(x)$, defined by

$$\phi_{n+1}(x) = \phi_n(x) + \lambda[f(x) - K\phi_n(x)], \quad (11.3.1)$$

converges to the L_2 -solution $\phi(x)$ of FK1.

THEOREM 11.3.2. If $k(x, s)$ is a closed symmetric kernel, then an FK1 has a unique solution in $[a, b]$ if and only if the series $\sum_{n=1}^{\infty} |\lambda_n f_n|^2$ converges, where λ_n are the eigenvalues of the kernel $k(x, s)$ and $f_n = \langle f, \phi_n \rangle$, where ϕ_n are the normalized orthogonal eigenfunctions of the kernel $k(x, s)$ corresponding to the eigenvalue λ_n .

EXAMPLE 11.3.1. Consider the FK1 (Kondo 1991)

$$\int_0^1 (1 + xs)\phi(s) ds = 2x + 3.$$

Since $k(x, s) = 1 + xs$ is not a closed kernel, because $\int_0^1 (1 + xs)(-1 + 6s - 6s^2) ds = 0$, the function $\phi(x) = 6x + \alpha(-1 + 6x - 6x^2)$ is a solution of the integral equation for all α and, therefore, is not unique. However, it does satisfy the conditions of Theorem 11.2.1, and hence, a solution exists. ■

If the L_2 -kernel $k(x, s)$ is not symmetric, then the following results hold:

Let $\phi(x)$ and $\psi(x)$ satisfy the following integral equations:

$$\begin{aligned}\phi(x) &= \lambda \int_a^b k(s, x)\psi(s) ds, \\ \psi(x) &= \lambda \int_a^b k(x, s)\phi(s) ds.\end{aligned}\tag{11.3.2}$$

Then it can be shown that both $\phi(x)$ and $\psi(x)$ satisfy the following integral equations with a symmetric kernel:

$$\begin{aligned}\phi(x) &= \lambda^2 \int_a^b h_1(x, s)\phi(x) ds, \\ \psi(x) &= \lambda^2 \int_a^b h_2(x, s)\psi(s) ds,\end{aligned}\tag{11.3.3}$$

where

$$h_1(x, s) = \int_a^b k(t, x)k(t, s) dt, \quad h_2(x, s) = \int_a^b k(x, t)k(s, t) dt.\tag{11.3.4}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ denote the eigenvalues of Eqs (11.3.3) and the associated normalized eigenfunctions be denoted by $\phi_1, \phi_2, \dots, \phi_n \dots$ and $\psi_1, \psi_2, \dots, \psi_n \dots$, respectively; then the following theorem holds.

SCHMIDT'S THEOREM. Every function $f(x)$ having one of the two forms

$$f(x) = \int_a^b k(x, s)g(s) ds \quad \text{or} \quad f(x) = \int_a^b k(s, x)g(s) ds,\tag{11.3.5}$$

where both $k(x, s)$ and $f(x)$ are L_2 -functions, is the sum of its Fourier series with respect to the system $\psi_n(x)$ in the case of the first form and with respect to the system $\phi_n(x)$ in the case of the second form.

LEMMA 11.3.1. If the series $\sum_i \frac{\phi_i(x)\psi_i(x)}{\lambda_i}$ is uniformly convergent, then the kernel $k(x, s)$ has the series representation

$$k(x, s) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\psi_i(x)}{\lambda_i}.$$

Picard's theorem plays an important role in the solution of an FK1 with non-symmetric kernels.

PICARD'S THEOREM. If the kernel h_2 defined in (11.3.4) is closed, then an FK1 possesses an L_2 -solution iff the series

$$\sum_{n=1}^{\infty} |\lambda_n f_n|^2 \quad (11.3.6)$$

converges, where $\{\lambda_n^2\}$ is the sequence of eigenvalues of the kernel $h_2(x, s)$ and the constants f_n are the Fourier coefficients of $f(x)$ with respect to the system of the eigenfunctions $\psi_n(x)$ of the kernel $h_2(x, s)$.

11.4. Numerical Methods

Some numerical methods to solve an FK1 are discussed below.

11.4.1. Quadrature Method. A simple numerical scheme to solve an FK1 is as follows:

Divide $[a, b]$ into n equal parts of length $h = (b - a)/n$; then Eq (11.1.1) can be approximated as

$$f(x) \approx \sum_{j=1}^n k(x, s_j) \phi(s_j) h. \quad (11.4.1)$$

Set $x = x_1, x_2, \dots, x_n$ in (11.4.1); this gives

$$f_i = \sum_{j=1}^n k_{ij} \phi_j, \quad i = 1, 2, \dots, n, \quad (11.4.2)$$

where $f_i = f(x_i)$, $k_{ij} = k(x_i, s_j) h$, and $\phi_j = \phi(s_j)$. The set of equations (11.4.2) forms a system of n equations in n unknowns ϕ_i and can be solved if the determinant of the system

$$\begin{vmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{vmatrix} \quad (11.4.3)$$

is nonsingular. However, the solution obtained in this manner depends on the choice of the values of x , and experience has shown that different choices generate different solutions. We have not seen any work that optimizes the choice of the values of x that will yield the best solution. While it is customary to choose the quadrature points s_j the same as the Nyström points x_i , it is, however, not always possible to do so, e.g., in the case of a kernel that is not defined at $x = s$ or becomes unbounded there. The advantage of choosing x_i the same as s_j is that the determinant (11.4.3) is symmetric if the kernel is symmetric.

It is observed that the quadrature method for FK1 is very unsatisfactory. The method basically consists of replacing the integral in Eq (11.1.1) by a quadrature rule with weights w_j and quadrature points s_j , $j = 0, 1, \dots, n$. This leads to the system of equations

$$\sum_{j=0}^n w_j k(x, s_j) \phi(s_j) = f(x). \quad (11.4.4)$$

If we write

$$\sum_{j=0}^n w_j k(x, s_j) \tilde{\phi}(s_j) = f(x), \quad (11.4.5)$$

and attempt to compute the values of $\tilde{\phi}(s_j)$ that satisfy Eq (11.4.5), we shall in general fail to find a solution, because, in view of (11.4.4), a solution exists only if $f(x)$ is a linear combination of the functions $k(x, s_j)$, $j = 0, 1, \dots, n$. Even if the FK1 has an analytical solution, the above method does not provide it. But if we choose the points x_0, x_1, \dots, x_m in $[a, b]$, we may satisfy the equations

$$\sum_{j=0}^n w_j k(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad i = 0, 1, \dots, m. \quad (11.4.6)$$

Let us assume that $m = n$, and let Eqs (11.4.6) determine $\tilde{\phi}(s_0), \dots, \tilde{\phi}(s_n)$ under the condition that $\det[k(x_i, s_j)] \neq 0$. Then the system becomes the Nyström system (Section 1.6), where the points x_i need not be the same as the points s_j ; but, as mentioned above, sometimes it works to choose them as the same points.

EXAMPLE 11.4.1. Consider the FK1 with

$$k(x, s) = \begin{cases} x(1-s) & \text{if } x \leq s, \\ s(1-x) & \text{if } x \geq s, \end{cases}$$

defined on $[0, 1]$. We apply the repeated trapezoidal rule, and choose $x_i = s_i = ih$, $h = 1/n$. Now, since

$$k(0, s) = k(1, s) = k(x, 0) = k(x, 1) = 0$$

for $0 \leq x \leq s \leq 1$, the system (11.4.6) reduces to

$$\sum_{j=1}^n h k(ih, jh) \tilde{\phi}(jh) = f(ih), \quad i = 0, 1, \dots, n, \quad (11.4.7)$$

where $f(ih) = f(0) = 0$ for $i = 0$ and $f(ih) = f(1) = 0$ for $i = n$. Thus, the system (11.4.7) fails to compute the values $\tilde{\phi}(0)$ and $\tilde{\phi}(1)$. Suppose that $f(0) = f(1) = 0$; otherwise this method yields no solution to the given equation. Note that

$$h k(ih, jh) = \begin{cases} ih^2(1 - jh) & \text{if } i \leq j, \\ jh^2(1 - ih) & \text{if } i \geq j, \end{cases}$$

and therefore, the system (11.4.6) reduces to

$$\begin{aligned} \tilde{\Phi} &\equiv \left\{ \begin{array}{c} \tilde{\phi}(h) \\ \tilde{\phi}(2h) \\ \tilde{\phi}(3h) \\ \vdots \\ \tilde{\phi}(1-2h) \\ \tilde{\phi}(1-h) \end{array} \right\} \\ &= -\frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \left\{ \begin{array}{c} f(h) \\ f(2h) \\ f(3h) \\ \vdots \\ f(1-2h) \\ f(1-h) \end{array} \right\}, \end{aligned} \quad (11.4.8)$$

and

$$\begin{aligned} \tilde{\phi}(ih) &= -\frac{1}{h^2} [f((i-1)h) - 2f(ih) + f((i+1)h)], \\ f(0) &= 0, \quad f(1) = 0. \end{aligned}$$

Obviously, $\tilde{\phi}(ih) \approx -f''(ih)$, where it is assumed that $f \in C^2[0, 1]$. Thus, the trapezoidal rule is satisfactory only if we introduce $f(0)$ and $f(1)$ as zero values. It is also clear from (11.4.8) that the computation of the derivative $f''(ih)$ is unstable due to roundoff errors in $f(x)$.

If we use the repeated Simpson's rule instead of the trapezoidal, we obtain the system of equations similar to (11.4.8) except that the left-side vector is

$$\tilde{\Phi} \equiv \left[\frac{4}{3} \tilde{\phi}(h), \frac{2}{3} \tilde{\phi}(2h), \frac{4}{3} \tilde{\phi}(3h), \dots, \frac{2}{3} \tilde{\phi}(1-2h), \frac{4}{3} \tilde{\phi}(1-h) \right]^T,$$

with $h = 1/n$ and $n = 2m$. In this case

$$\tilde{\phi}(ih) \rightarrow \begin{cases} -\frac{3}{4} f''(ih) & \text{if } i \text{ is odd,} \\ -\frac{3}{2} f''(ih) & \text{if } i \text{ is even.} \end{cases}$$

This situation tends to yield inaccurate results for $\tilde{\Phi}$. There is another problem that may arise in such examples. Suppose if $k(x_i, s_j) = 0$ for some i and all s_j , or if $k(x_i, s_j) = 0$ for some j and all x_i ; then $\det(\mathbf{k}) = 0$. Moreover, by making an "unsuitable" choice of the Nyström points x_i or the quadrature points s_j , we can find ourselves in a situation where the matrix \mathbf{k} becomes almost singular or ill-conditioned. This then leads to unacceptable solutions. ■

An alternative scheme is to use the Gaussian quadrature; then for each point x we can write

$$f(x) = \sum_{j=1}^n w_j k(x, s_j) \phi(s_j), \quad (11.4.9)$$

where w_j are weights connected to the particular quadrature rule used. As before, we can generate a system of n equations in n unknowns $\phi(s_j)$. If the determinant of the system is nonsingular, we can find the approximate values $\tilde{\phi}(x)$ of $\phi(x)$ at n quadrature points in $[a, b]$. In general, it is convenient to change the interval of integration to $[0, 1]$. This can be accomplished by a linear transformation, provided a and b are finite. Unfortunately, in a majority of cases the matrix of the system is ill-conditioned and the method is not very successful. The remarks concerning the quadrature method are also valid for Gaussian quadrature. Baker (1978, p.635) remarks that, in practical terms, accurate results are difficult to obtain. He also points out that the smoother the kernel is, the more ill-conditioned the equation becomes.

EXAMPLE 11.4.2. Consider the FK1 of Example 11.3.1. The minimum norm solution of the integral equation is $\phi(x) = 6x$. Using a four-point Gauss–Chebychev quadrature, Kondo’s and our values (using Mathematica) of $\tilde{\phi}(x)$ are given in Table 11.4.2.

Table 11.4.2

x	$\tilde{\phi}(x)$ (Kondo)	$\tilde{\phi}(x)$ (Mathematica)
0.102673	202.685303	6.94159
0.406204	-182.9258184	-4.59716
0.593769	73.620687	-1.61587
0.897327	19.482761	11.2714

Clearly, neither set of values is acceptable. For computational details, see Kondo . n.b.

■ However, quadrature methods should not be totally discarded. Sometimes they do yield acceptable results. Consider the following example from Baker (1978) and Caldwell (1994).

EXAMPLE 11.4.3. Let the FK1 be

$$\int_0^1 \ln|x-s|\phi(s)ds = 0.5 [x^2 \ln x + (1-x^2) \ln(1-x) - x - 0.5].$$

The exact solution is given by $\phi(x) = x$. Baker uses the midpoint rule for $n = 3, 9$, and 27; however, when we used the midpoint rule with $n = 3$, we could not duplicate Baker’s results. He does not give the values of x used to generate the system of Eqs (11.4.2) for this problem. Using Mathematica, with different values of x_1, x_2, x_3 , such as (i) 0.22, 0.555, 0.89, and (b) 0.29, 0.59, 0.91, we noticed a considerable fluctuation. For our best results we used the values of the set (ii) for $n = 3$. The results for $\tilde{\phi}(x)$ are given in Table 11.4.3(a), in which the first two columns represent the quadrature points x and the exact value of $\phi(x)$, respectively; the next three columns represent the values of $\tilde{\phi}(x)$ obtained by Baker (1978) for $n = 3, 9, 27$; and the last two columns are based on Mathematica for $n = 3$ for the values of x_1, x_2, x_3 from the sets (i) and (ii), respectively, used for the corresponding values of $\tilde{\phi}(x)$. The results indicate the dependence of the values of $\tilde{\phi}(x)$ on the choice of x_1, x_2, x_3 .

Table 11.4.3(a)

x	Exact	Baker $n = 3$	Baker $n = 9$	Baker $n = 27$	Set (i) $n = 3$	Set (ii) $n = 3$
1/6	0.6666	0.1579	0.16637	0.1664	0.167652	0.183435
1/2	0.5000	0.5000	0.50000	0.5000	0.509197	0.565688
5/6	0.8333	0.8421	0.83360	0.8333	0.828419	0.893304

For computational details, see `baker1.nb`. ■

We could not duplicate Baker's results by using the midpoint rule with 9 quadrature points. Our results for the 9-point rule are given in Table 11.4.3(b), where the first column corresponds to the quadrature points at which $\tilde{\phi}(x)$ is evaluated; the second column to the values of the solution obtained from the system of Eqs (11.4.1) by using the set (a) $x = \{0.1, 0.2, 0.3, 0.4, 0.51, 0.6, 0.7, 0.8, 0.9\}$; and the third column to values of the set (b) $x = \{0.11, 0.21, 0.31, 0.37, 0.48, 0.59, 0.7, 0.81, 0.92\}$. Values of $\tilde{\phi}(x)$ in the third column are monotonically increasing, as they should be.

Table 11.4.3(b)

x	$\tilde{\phi}(x)$ for set (a)	$\tilde{\phi}(x)$ for set (b)
1/18	0.0744906	0.0395499
1/6*	0.126046	0.180864
5/18	0.436291	0.323793
7/18	0.325066	0.391586
1/2*	0.426527	0.469519
11/18	0.496846	0.608349
13/18	0.738199	0.72643
5/6*	0.907256	0.849342
17/18	1.10364	0.97774

The mark * indicates the points at which Baker evaluated the solution. For computational details, see `baker1.nb`. ■

11.4.2. Method of Moments. Divide the interval $[a, b]$ into N equal subintervals of length $h = (b - a)/N$. Let $\phi(s) = C_i$ in the subinterval $\xi_{i-1} = a + (i - 1)h \leq s < a + ih = \xi_i$, where C_i is a constant equal to the average value of $\phi(s)$ in this subinterval. Then Eq (11.1.1) can be written in the form of

the system

$$\sum_{i=1}^N C_i m_i(x) = f(x), \quad \text{where } m_i(x) = \int_{\xi_{i-1}}^{\xi_i} k(x, s) ds, \quad (11.4.10)$$

which can be used to yield a system of N algebraic equations by letting $x = x_i$, $a + (i - 1)h \leq x_i \leq a + ih$, $i = 1, 2, \dots, N$. This system can then be solved to yield the average value of $\phi(x)$ in any subinterval. This value can be regarded as the value of $\tilde{\phi}(x)$ at the middle point of the subinterval and will be reasonably accurate unless the interval contains a turning point of the function.

11.4.3. Regularization Methods. Phillips (1962) develops a method for solving FK1 by using the quadrature rule together with a regularization technique; later Twomey (1963) improved this method. We present these two techniques.

(i) In Eq (11.1.1) if $f(x)$ is not known accurately, then an error $\epsilon(x)$ may occur in the evaluation of $f(x)$. Thus, we consider a slightly modified equation

$$\int_a^b k(x, s)\phi(s) ds = f(x) + \epsilon(x), \quad (11.4.11)$$

where $\epsilon(x)$ is an arbitrary function except for a condition on its size. Because $\epsilon(x)$ is arbitrary, we get a family \mathcal{F} of solutions of Eq (11.4.11). The problem then reduces to that of determining the true solution. To do this, we need more information about the solution. If the functional form of ϕ were known, we could use the least-squares method, but, in general, that is not the case. Phillips, therefore, assumes that ϕ is a reasonably smooth function and suggests choosing the solution that function from the family \mathcal{F} that is the smoothest in some sense. For this purpose he uses the condition that the function should have a piecewise continuous second derivative, that is,

$$\int_a^b \phi_s''(x) dx = \min_{\phi \in \mathcal{F}} \int_a^b \phi''(x) dx. \quad (11.4.12)$$

In order to apply the quadrature rule to solve Eqs (11.4.11) and (11.4.12), divide the interval $[a, b]$ into n equal (for convenience) parts at the points $a = s_0 < s_1 < s_2 < \dots < s_n = b$, $n(s_i - s_{i-1}) = b - a$, and replace Eq (11.4.11) by the linear system

$$\sum_{j=0}^n \omega_j k_{ij} \phi_j = f_i + \epsilon_i, \quad i = 0, 1, \dots, n, \quad (11.4.13)$$

where $\phi_j = \phi(s_j)$, $f_i = f(x_i)$, $\epsilon_i = \epsilon(x_i)$, $k_{ij} = k(x_i, s_j)$, and ω_j are the weights depending on the quadrature rule used. The condition to be imposed on the magnitude of $\epsilon(x)$ is

$$\sum_{i=0}^n \epsilon_i^2 = e^2, \quad (11.4.14)$$

where e is a constant. Note that an alternate condition $\sum_{i=0}^n p_i \epsilon_i^2 = e^2$, $p_i \geq 0$, can also be used.

Eq (11.4.12) can also be expressed in terms of the second difference as

$$\sum_{i=0}^n (\phi_{i+1}^s - 2\phi_i^s + \phi_{i-1}^s)^2 = \min_{\phi \in \mathcal{F}^*} \sum_{i=0}^n (\phi_{i+1} - 2\phi_i + \phi_{i-1})^2, \quad (11.4.15)$$

where $\Phi^s = [\phi_0^s, \phi_1^s, \dots, \phi_n^s]^T$ and \mathcal{F}^* is a set of vectors satisfying Eqs (11.4.13) and (11.4.14). In Eq (11.4.15) it is assumed that ϕ vanishes outside the interval $[a, b]$, and thus, $\phi_{-1} = \phi_{n+1} = 0$. Introducing the matrices

$$\mathbf{A} = (\omega_j k_{ij}), \quad \mathbf{A}^{-1} = (\alpha_{ij}),$$

Eq (11.4.13) can be written as $\mathbf{A} \Phi = \mathbf{f} + \boldsymbol{\epsilon}$, which implies that

$$\Phi = \mathbf{A}^{-1}(\mathbf{f} + \boldsymbol{\epsilon}) = \mathbf{A}^{-1}\mathbf{f} + \mathbf{A}^{-1}\boldsymbol{\epsilon}. \quad (11.4.16)$$

Thus, we find from Eq (11.4.16) that ϕ_j are linear functions of ϵ_i and

$$\frac{\partial \phi_i}{\partial \epsilon_j} = \alpha_{ij}, \quad i, j = 0, 1, \dots, n. \quad (11.4.17)$$

Using Eqs (11.4.14) and (11.4.17) and the constraint (11.4.14), the conditions which ϕ_j must satisfy can be expressed as

$$\sum_{j=0}^n (\phi_{j+1}^s - 2\phi_j^s + \phi_{j-1}^s) (\alpha_{i+1,j} - 2\alpha_{ij} + \alpha_{i-1,j}) + \gamma^{-1} \epsilon_i = 0, \quad (11.4.18)$$

for $i = 0, 1, \dots, n$, along with the constraint (11.4.14), where γ^{-1} is the Lagrange multiplier. In a standard problem of this type ϵ_i are known and γ has to be determined. In this case, of course, both ϵ_i and γ are unknown. Therefore, we consider the problem as that of finding the vector \mathbf{f}^s satisfying (11.4.13) so as to minimize the expression

$$\gamma \sum_{j=0}^n (\phi_{j+1}^s - 2\phi_j^s + \phi_{j-1}^s)^2 + \sum_{i=0}^n \epsilon_i^2, \quad (11.4.19)$$

for a given constant value of γ . From Eq. (11.4.18) we note that $\gamma \geq 0$. For $\gamma = 0$ the problem reduces to that of solving $\mathbf{A} \Phi = \mathbf{f}$. For $\gamma > 0$ Eq (11.4.18) can be reduced to

$$\begin{aligned} & \sum_{j=0}^n (\phi_{j+1}^s - 2\phi_j^s + \phi_{j-1}^s) (\alpha_{i+1,j} - 2\alpha_{ij} + \alpha_{i-1,j}) + \gamma^{-1} \epsilon_i \\ &= \sum_{m=0}^n (\alpha_{m-2,j} - 4\alpha_{m-1,j} + \alpha_{m,j}) - 4\alpha_{m+1,j} + \alpha_{m+2,j}) \phi_m + \gamma^{-1} \epsilon_i = 0, \end{aligned}$$

or

$$\gamma \sum_{m=0}^n \beta_{mj} \phi_m + \epsilon_i = 0,$$

which in matrix form is

$$\gamma \mathbf{B} \Phi^s = \boldsymbol{\epsilon}, \quad (11.4.20)$$

where

$$\begin{aligned} \beta_{mj} &= \alpha_{m-2,j} - 4\alpha_{m-1,j} + \alpha_{m,j} - 4\alpha_{m+1,j} + \alpha_{m+2,j}, \\ \alpha_{-2,j} &= -\alpha_{0,j}, \quad \alpha_{-1,j} = 0 = \alpha_{n+1,j}, \\ \alpha_{n+2,j} &= \alpha_{n,j}, \quad j = 0, 1, \dots, n. \end{aligned} \quad (11.4.21)$$

These values are obtained by equating the coefficients of $\phi_0, \phi_1, \phi_{n-1}$, and ϕ_n from Eqs (11.4.18) and (11.4.20). Substituting (11.4.20) into Eq (11.4.16), we find that

$$\Phi^s = (\mathbf{A} + \gamma \mathbf{B})^{-1} \mathbf{f}, \quad \boldsymbol{\epsilon} = -\gamma \mathbf{B} \Phi^s. \quad (11.4.22)$$

The only question left to answer is, how to choose γ ? We note that increasing γ improves the smoothness. Therefore, for the smoothest Φ , one might want to choose a very large γ , but for large γ , the error e will be large, *so we choose the largest γ that keeps e below its preassigned value*. In any case, the solution for a few values of γ will be adequate to determine the best solution.

EXAMPLE 11.4.4. Phillips (1962) and later Baker et al. (1964) consider the following integral equation

$$\int_{-6}^6 k(x-s) \phi(s) ds = f(x),$$

where

$$k(z) = \begin{cases} 1 + \cos \frac{\pi z}{3} & \text{if } |z| \leq 3, \\ 0 & \text{if } |z| > 3. \end{cases}$$

$$f(x) = (6 - |x|) \left(1 + \frac{1}{2} \cos \frac{\pi x}{3} \right) + \frac{9}{\pi} \sin \frac{\pi |x|}{3}.$$

The exact solution of this equation is $\phi(x) = k(x)$. Table 11.4.4 gives the values of $\phi(x)$ obtained by using trapezoidal and Simpson's rules.

Table 11.4.4

x	0.0	± 1.0	± 2.0	± 3.0	± 4.0	± 5.0	± 6.0
$\tilde{\phi}(x)$ (Trapezoidal)	1.962	1.538	0.462	0.019	0.0	0.0	0.0
$\tilde{\phi}(x)$ (Simpson)	2.943	1.154	0.693	0.014	0.0	0.0	0.0
$\phi(x)$ (Exact)	2.000	1.500	0.500	0.0	0.0	0.0	0.0
$\tilde{\phi}(x)$, $\gamma = 0.0011$	2.355	1.404	0.385	0.078	-0.033	0.003	-0.007
$\tilde{\phi}(x)$, $\gamma = 0.011$	2.111	1.492	0.46	-0.009	0.047	0.02	-0.097
$\tilde{\phi}(x)$, $\gamma = 0.03$	2.054	1.534	0.506	-0.036	0.033	0.033	-0.097
$\tilde{\phi}(x)$, $\gamma = 0.1$	1.985	1.574	0.588	-0.050	-0.035	0.046	-0.027
$\tilde{\phi}(x)$, $\gamma = 0.5$	1.852	1.492	0.653	0.007	-0.095	0.019	0.038
$\tilde{\phi}(x)$, $\gamma = 1.0$	1.775	1.414	0.654	0.053	-0.087	-0.007	0.024

Our results obtained using Mathematica are somewhat different from those obtained by Phillips. However, our results bear out the main conclusions by Phillips that well-designed regularization improves the accuracy of the solution. The unregularized results using trapezoidal rule were quite accurate. For computational details, see `phillips.nb`. ■

(ii) Soon after Phillips published his work, Twomey (1963) suggested a simplification of the process as follows: Instead of minimizing the expression (11.4.19) with respect to ϵ_i , minimize it with respect to ϕ_j . Differentiation of (11.4.19) with respect to ϕ_j yields

$$\sum_{j=0}^n \epsilon_i a_{ij} + \gamma (\phi_{j-2} - 4\phi_{j-1} + 6\phi_j - 4\phi_{j+1} + \phi_{j+2}) = 0, \quad i = 0, 1, \dots, n, \quad (11.4.23)$$

where a_{ij} is an element of \mathbf{A} . Then Eq (11.4.23) in matrix form is

$$\mathbf{A}^T \boldsymbol{\epsilon} + \gamma \mathbf{H} \Phi = 0, \quad (11.4.24)$$

where \mathbf{A}^T is the transpose of \mathbf{A} and \mathbf{H} is the matrix

Eliminating ϵ from (11.4.16) and (11.4.24), we obtain

$$\Phi = (\mathbf{A}^T \mathbf{A} + \gamma \mathbf{H})^{-1} \mathbf{A}^T \mathbf{f}, \quad (11.4.26)$$

which can be written as

$$\mathbf{A} \Phi + \gamma (\mathbf{A}^{-1})^T \mathbf{H} \Phi = \mathbf{f}$$

or

$$\mathbf{A} \Phi + \gamma \mathbf{B} \Phi = \mathbf{f}, \quad (11.4.27)$$

where $(\mathbf{A}^{-1})^T \mathbf{H} = \mathbf{B}$. Note that Eq (11.4.27) has the same solution as given by Phillips (1962, in his Eq (P.12)). Twomey (1963) points out that one can use higher-order regularization by minimizing the third difference rather than the second difference. He also gives an alternate form of the matrix \mathbf{H} . These techniques will work only if the matrix \mathbf{A} is invertible. In fact, depending on the order of smoothness that we expect from the solution, we can minimize $\|L\phi\|$ in any norm, where L is a linear differential operator.

EXAMPLE 11.4.5. We shall solve Example 11.4.4 by using Twomey's regularization scheme. The results are given in Table 11.4.5 for the values of $x \in [-3, 3]$; the values of $x = \pm 4, \pm 5, \pm 6$ are not given since they are found to be close to zero and are, therefore, not significant. Also, notice that the values for different values of γ are not symmetric about $x = 0$. For computational details, see `phillips.nb`.

Table 11.4.5

x	$\gamma = 0.0011$	$\gamma = 0.011$	$\gamma = 0.03$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 1.0$
-3	0.0773	-0.0031	-0.0172	-0.0134	0.1889	0.0449
-2	0.3829	0.4639	0.5051	1.0562	0.6513	0.6824
-1	1.4068	1.4798	1.4897	1.4879	1.4694	1.4549
0	2.3562	2.0791	2.0101	1.9402	1.8414	1.8007
1	1.3996	1.4764	1.4881	1.4941	1.4913	1.4832
2	0.3950	0.4722	0.5157	0.5719	0.6557	0.6816
3	0.0802	-0.0030	-0.0207	-0.0266	-0.0163	-0.0055 ■

11.4.4. Other Regularizations. Some other regularizations are given below.

(i) Lavrentiev (1967) proposes the following: If in Eq (11.1.1) both $f(x), \phi(x) \in L_2$, and the kernel $k(x, s)$ is square-integrable, symmetric, and positive-definite,

then for all $\phi(x) \in L_2$ we have

$$\int_a^b \int_a^b k(x, s)\phi(s)\phi(x) dx ds \geq 0,$$

where equality holds only for $\phi(x) \equiv 0$. Under these conditions we modify Eq (11.1.1) and consider

$$\int_a^b k(x, s)\phi(s) ds + \gamma\phi(x) = f(x), \quad (11.4.28)$$

where γ is known as the regularization parameter and Eq (11.4.28) is an FK2 whose solution, denoted by $\phi_\gamma(x)$, can be found. Substituting $\phi_\gamma(x)$ for $\phi(x)$ in Eq (11.1.1), we get

$$\int_a^b k(x, s)\phi_\gamma(s) ds = f_\gamma(x).$$

If $\|f(x) - f_\gamma(x)\| \leq \delta$, where δ is a preassigned quantity representing the tolerance of error, then the function $\phi_\gamma(x)$ is considered an acceptable approximate solution to Eq (11.1.1).

(ii) Tikhonov (1963), under the same restrictions as imposed by Lavrentiev, considered the following FK2:

$$\int_a^b k^*(x, s)\phi(s) ds + \gamma\phi(x) = f^*(x),$$

where $k^*(x, s) = k^*(s, x) = \int_a^b k(t, s)k(t, x)dt$, $f^*(x) = \int_a^b k(t, x)f(t) dt$, and γ is the above-mentioned regularization parameter. Again, following the same procedure as in the Lavrentiev case, we can find an approximate solution.

(iii) Essah and Delves (1989) develop a cross-validation scheme to automatically set the two regularization parameters (constraints) introduced by them in the solution of an FK1 of the form (11.1.1) by using a Chebyshev series method. It is assumed that the solution $\phi \in L_2[a, b]$. The interval is mapped onto $[-1, 1]$, and the solution is expressed as

$$\phi(x) \approx \phi_N(x) = \sum_{i=0}^N a_i T_i(x), \quad -1 \leq x \leq 1, \quad (11.4.29)$$

where $T_i(x)$ are the Chebyshev polynomials of the first kind. Following Babolian and Delves (1979), the Galerkin equations in the matrix form for the coefficients a_i are

$$\mathbf{B} \mathbf{a} = \mathbf{f}, \quad (11.4.30)$$

where $\mathbf{a} = \{a_0, a_1, \dots, a_N\}$, and $\mathbf{B} = (B_{ij})$ and $\mathbf{f} = \{f_i\}$ are given by

$$\begin{aligned} B_{ij} &= \int_{-1}^1 \int_{-1}^1 \frac{k(x, s) T_i(s) T_j(x)}{\sqrt{1-x^2}} dx ds, \\ f_i &= \int_{-1}^1 \frac{f(x) T_i(x)}{\sqrt{1-x^2}} dx, \quad i = 0, 1, \dots, N. \end{aligned} \quad (11.4.31)$$

The values of the elements B_{ij} and f_i can be approximated by FFT techniques (see Delves 1977, Delves and Abd-Elal 1977). Since the FK1 is ill-posed, we run into computational instability in solving the system (11.4.30), but this can be offset by imposing a set of constraints on the vector \mathbf{a} . These constraints arise from the hypothesis that the solution $\phi(x)$ of the FK1 (11.1.1) is represented as an infinite series

$$\phi(x) = \sum_{i=0}^N b_i T_i(x), \quad -1 \leq x \leq 1. \quad (11.4.32)$$

Thus, we require that $\phi(x)$ be in $L_2[-1, 1]$ and the series (11.4.32) be convergent.

This means that $\sum_{i=0}^N |b_i|^2 < \infty$. In accordance with Babolian and Delves (1979) we shall, therefore, assume that $|b_i| = O(i^{-n})$, $n > 1/2$, i.e., there exist constants C'_ϕ and $\rho (> 1/2)$, depending on analytic properties of ϕ , such that

$$|b_i| \leq C'_\phi \hat{i}^{-\rho}, \quad \hat{i} = \max \{1, i\}.$$

Let C_ϕ and r be the approximate values of $C'\phi$ and ρ . The constants C_ϕ (or λ) and r are called *regularization parameters*. Now, the problem (11.4.30) reduces to that of minimizing $\|\mathbf{Ba} - \mathbf{f}\|$ subject to the condition $|a_j| \leq C_\phi \hat{j}^{-r} \equiv \delta_j$, $j = 0, 1, \dots, N$. This problem can be regarded as a standard linear programming (LP) problem by using the L_1 - (or L_∞ -) norm. In the L_1 -norm case this problem can be stated as

$$\begin{aligned} \text{Minimize } w &= \sum_{i=0}^M \left| \sum_{j=0}^N B_{ij} a_j - f_i \right| \\ \text{subject to } |a_j| &\leq \delta_j, \quad j = 0, 1, \dots, N. \end{aligned}$$

Babolian and Delves (1979) discuss three heuristic strategies to compute suitable values of C_ϕ and r , but as Belward (1985) shows, these strategies are often satisfactory but not always. The remedy lies in the use of the cross-validation technique (Wahba 1977) along with the Nyström method, which works as follows: Compute the solution ϕ_N at a set of points x_i , $i = 1, \dots, N$; next, introduce a sequence of regularized solutions $\tilde{\phi}_i(x)$, computed by omitting the i th Nyström equation associated with the point x_i , and then for the error estimate in $\tilde{\phi}_i(x)$ use the residual r_i in the omitted i th equation. An overall measure of the residual can be formed from all r_i , $i = 1, \dots, N$. This measure depends on C_ϕ and r , which can be set to minimize the overall measure. Thus, by successively omitting the i th Galerkin equation, the cross-validation estimate is developed as follows: Let \mathbf{a}^m denote the computed solution vector obtained by minimizing $\|\mathbf{B}^m \mathbf{a}^m - \mathbf{f}^m\|$ subject to $|a_j^m| \leq \delta_j$ for $j = 0, 1, \dots, N$ and $m = 1, 1, \dots, M$, where $\|\mathbf{B}^m\|$ represents the matrix \mathbf{B} with the m th row removed, and $\|\mathbf{a}^m\|$ and $\|\mathbf{f}^m\|$ each represents the vector \mathbf{a} and \mathbf{f} with the m th element removed. Take $M = N + 1$. Note that for a good choice of the parameters C_ϕ (or λ) and r the residual of the removed equation must be relatively very small for all m . The L_1 -norm of the vector \mathbf{r} of residuals in the omitted Galerkin equation is given by

$$V(\lambda, r) = \sum_{m=0}^M \left| \sum_{j=0}^N B_{mj} a_j^m - f_m \right|. \quad (11.4.33)$$

The values of λ and r are chosen so that the function $V(\lambda, r)$ is a minimum. Varah (1983) shows that the function $V(\lambda, r)$ is almost flat over the relevant ranges of λ and r , and this is another source of difficulty. The algorithm to compute the values of the regularization parameters consists of using the bounds

$$m_1 T - m_2 e_N \leq V(\lambda, r) \leq M_1 T + M_2 e_N,$$

where m_1, m_2, M_1, M_2 are positive constants, $T = T_N(\lambda, r) = \|\mathbf{b}_N - \mathbf{a}_N\|$, and

$$e_N = \frac{C_b C'_\phi (2p - 1) N^{-(q+\rho)+1}}{(p-1)(q+\rho-1)}, \quad p > 1, q + \rho > 1, C_b > 0(\text{const}).$$

Note that $T_N(\lambda, r)$ remains “close” to $V(\lambda, r)$.

As Essah and Delves (1989) point out, there are two major difficulties in the implementation of this technique. Firstly, it is time-consuming in terms of processing since each $V(\lambda, r)$ requires $M + 1$ LP problems to be solved for \mathbf{a}^m , $m = 0, 1, \dots, M$; and secondly, the function $V(\lambda, r)$, being flat, is difficult to minimize. Essah and Delves present two procedures:

PROCEDURE 1. Assume that the solution \mathbf{a} is a good approximation to \mathbf{a}^m in the sense that $\mathbf{a}^m = \mathbf{a} + \mathbf{z}^m$ for $m = 0, 1, \dots, M$, where $|\mathbf{z}^m|$ is expected to be small. Then solve the LP problem:

$$\begin{aligned} & \text{Minimize } e^T (\mathbf{U} + \mathbf{v}), \\ & \text{subject to } \mathbf{B}^m \mathbf{z}^m + \mathbf{u} - \mathbf{v} = \mathbf{f}^m - \mathbf{B}^m \mathbf{a}, \\ & \mathbf{z}^m + \mathbf{s}' = \boldsymbol{\delta} + \mathbf{a}, \\ & -\mathbf{z}^m + \mathbf{s}'' = \boldsymbol{\delta} + \mathbf{a}, \end{aligned}$$

and compute \mathbf{z}^m and, hence, \mathbf{a}^m for $m = 0, 1, \dots, M$. In most cases \mathbf{z}^m will have many zero elements, thus reducing computing time.

PROCEDURE 2. Reduce the number of LPs needed in computing $V(\lambda, r)$, since in many cases \mathbf{a}^m is identical with the solution \mathbf{a} . Essah and Delves (1989) remark that “if we can tell in advance that this will be so, we need not compute \mathbf{a}^m . In fact, we can tell this by looking at the final tableau of the LP problem for \mathbf{a} .” The guiding factors to foresee if the solution $\mathbf{a}^m = \mathbf{a}$ are: (i) No basic variables for \mathbf{a} appear in the m th equation (removed row); and (ii) No “slack” variable appears in the m th equation. Essah and Delves (1989) have done extensive research in determining the time savings achieved with these procedures. They also consider the following examples.

EXAMPLE 11.4.6. (Babolian and Delves 1979) Consider the FKI with

(a) $k(x, s) = s \exp\{x^\beta\}$, $f(x) = \frac{\exp\{x^\beta + 1\} - 1}{x^\beta + 1}$, $0 \leq x, s \leq 1$, and $\beta = 1, 2$, which has the exact solution $\phi(s) = e^\beta$. The functions k , f , and ϕ are analytic in x and s , and $\phi \in C^\infty[0, 1]$, and thus the regularization parameter r can take any value.

(b) $k(x, s) = \begin{cases} x(s-1) & \text{if } x \leq s, \\ s(x-1) & \text{if } x > s, \end{cases}$, for $0 \leq x, s \leq 1$,

which has exact solution $\phi(s) = f''(s)$ provided $f''(x)$ exists on $[0, 1]$ and $f(0) = f(1) = 0$. The free term is

$$f(x) = \begin{cases} e^x + (1-e)x - 1 & \text{if } x \leq s, \\ (x^3 - x)/6 & \text{if } x > s. \end{cases}.$$

For these two cases the kernel k has a discontinuous first-order partial derivative on the line $x = s$, which can be handled by the FFT technique mentioned earlier; the functions ϕ and f are analytic. Thus, λ and r can take any value.

(c) $k(x, s) = \sqrt{x^2 + s^2}$, $f(x) = [(1+x^2)^{3/2} - x^3]/3$, $0 \leq x, s \leq 1$, with solution $\phi(s) = s$. The first partial derivatives of k have a discontinuity at $(0, 0)$, but f and ϕ are analytic. ■

EXAMPLE 11.4.7. (Belward 1985) Consider the FK1 with

$$(a) k(x, s) = e^{xs}, f(x) = \frac{2 + x - 2e^{x/2}}{2x^2}, 0 \leq x, s \leq 1, \text{ with solution}$$

$$\phi(s) = \begin{cases} s - 1/2 & \text{if } 0 \leq x \leq 1/2, \\ 0 & \text{if } 1/2 < x \leq 1. \end{cases}$$

The functions k, f are analytic, ϕ is continuous on $[0, 1]$, but ϕ' has a finite jump discontinuity; thus, $\phi \in C^1[0, 1]$, and $r = 2$.

$$(b) k(x, s) = \begin{cases} s(x-1) & \text{if } s \leq x, \\ x(s-1) & \text{if } x < s, \end{cases}, 0 \leq x, s \leq 1, \text{ and}$$

$$f(x) = \begin{cases} (4x^2 - 3x)/8 & \text{if } 0 \leq x \leq 1/2, \\ (x-1)/8 & \text{if } 1/2 < x \leq 1, \end{cases}$$

with solution $\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2, \\ 0 & \text{if } 1/2 < x \leq 1. \end{cases}$ Thus, $\phi \in C^0[0, 1]$, and so $r = 1$. ■

EXAMPLE 11.4.8. (te Riele 1984) Consider the FK1 with $k(x, s) = (x + s)^{-1}$, $f(x) = x^{-1} \ln \{(1+x)(1+x/5)^{-1}\}$, $1 \leq x, s \leq 5$, with solution $\phi(s) = 1/s$. The functions k, f , and ϕ are analytic in $[1, 5]$. ■

Extensive computational results for λ and r on these examples are available in Essah and Delves (1989).

11.4.5. Comparison. Caldwell (1994) compares different methods for the numerical solution of FK1. He considers the usual quadrature methods and finds that classical quadrature methods do not give good results in general but can be useful for special problems. He also proposes a modification of the quadrature method by replacing the FK1 (11.1.1) by

$$\phi(s_i) \int_a^b k(s_i, s) ds + \sum_{j=0, j \neq i}^n \omega_j k(s_i, s_j) [f(s_j) - f(s_i)] = f(s_i). \quad (11.4.34)$$

Clearly, to use this method we should be able to compute $\phi(s_i) \int_a^b k(s_i, s) ds$ to a high degree of accuracy. However, Caldwell comments that this approach suffers from the same defect as quadrature methods but might be better in some cases. He also considers the boundary integral method. In this approach, instead of approximating $k(x, s)\phi(s)$, we approximate only $\phi(s)$. In order to avoid having to calculate the weights for each x , he proposes using the midpoint rule. This

method is based on the assumption $\phi(x)$ is slowly varying and in each subinterval s_j, s_{j+1} can be assumed to be constant equal to $\phi(s_{jm})$, where s_{jm} is the middle point of the subinterval. The method results in the following equation:

$$\sum_{j=0}^{n-1} \phi(s_{jm}) \int_{s_j}^{s_{j+1}} k(s_i, s) ds = f(s_i). \quad (11.4.35)$$

This method is particularly useful when $k(x, s)$ is badly behaved and $\phi(x)$ is not.

Caldwell also considers the simplest form of regularization method, which is as follows: Consider an FK2 of the form

$$\int_a^b k(x, s) \phi_\epsilon(s) ds + \epsilon \phi_\epsilon(x) = f(x), \quad (11.4.36)$$

where ϵ is a small, positive parameter. The sequence of solutions $\phi_\epsilon(x)$ is obtained for some suitable values of ϵ and the results are then extrapolated to $\epsilon \rightarrow 0$. Caldwell points out that we could establish that in the L_2 -norm

$$\lim_{\epsilon \rightarrow 0} \|\phi_\epsilon(x) - \phi(x)\| = 0, \quad (11.4.37)$$

but this convergence is not uniform. His last method is the iteration method for a symmetric kernel, which has the form

$$\Phi_{(n+1)} = \Phi_{(n)} + \alpha (\mathbf{f} - \mathbf{k} \Phi_{(n)}), \quad \Phi_{(n)} = 0, \quad (11.4.38)$$

where α is less than twice the minimum eigenvalue of the homogenous equation. It can be shown that this iterative scheme converges to the true solution or to the solution with the minimum norm in the case of nonuniqueness. This method can also be considered a regularization procedure.

EXAMPLE 11.4.9. Caldwell (1994) considers the following five equations:

$$(i) \int_0^1 e^{xs} \phi(s) ds = \frac{e^{(x+1)} - 1}{x+1}, \quad \phi(x) = e^x, \quad (11.4.39)$$

$$(ii) \int_0^1 \sin xs \phi(s) ds = \frac{\sin x - x \cos x}{x^2}, \quad \phi(x) = x, \quad (11.4.40)$$

$$(iii) \int_0^1 \frac{\phi(s)}{x+s+\epsilon} ds \\ = 1 + (x+\epsilon) \ln \left(\frac{x+\epsilon}{1+x+\epsilon} \right), \quad \phi(x) = x, \quad (11.4.41)$$

$$\begin{aligned}
 \text{(iv)} \quad & \int_0^1 \frac{\phi(s)}{\sqrt{x+s+\epsilon}} ds \\
 &= \frac{1}{3} [\{2 - 4(x+\epsilon)\}\sqrt{1+x+\epsilon} + 4(x+\epsilon)\sqrt{x+\epsilon}], \quad \phi(x) = x,
 \end{aligned} \tag{11.4.42}$$

$$\begin{aligned}
 \text{(v)} \quad & \int_0^1 \ln|x-s|\phi(s) ds \\
 &= \frac{1}{2} [x^2 \ln x + (1-x^2) \ln(1-x) - x - \frac{1}{2}], \quad \phi(x) = x.
 \end{aligned} \tag{11.4.43}$$

Table 11.4.9 gives the results obtained by Caldwell:

Table 11.4.9

Method	Rule	(i)	(ii)	(iii)	(iv)	(v)
Quadrature	Boundary-integral	+	+	+	+	10^{-2}
	Gauss-Legendre	+	+	+	+	10^{-2}
	Gauss-Chebychev	+	+	+	+	10^{-2}
Regularized Quadrature	Boundary-integral	10^{-2}	10^{-3}	10^0	10^{-1}	10^{-2}
	Gauss-Legendre	10^{-4}	10^{-3}	10^0	10^{-2}	10^{-2}
Modified Quadrature	Gauss-Chebychev	10^{-2}	10^{-2}	10^0	10^{-2}	10^{-2}
	Gauss-Legendre	+	+	+	10^0	10^{-3}
	Gauss-Chebychev	10^{-1}	10^0	+	10^0	10^{-2}
Iteration	Mid-point rule	10^0	10^0	+	10^{-2}	10^{-3}
	Gauss-Legendre	10^{-1}	10^{-1}	10^0	10^0	10^{-2}
	Gauss-Chebychev	10^{-1}	10^{-1}	10^0	10^0	10^{-2}
	Midpoint rule	10^{-2}	10^{-1}	10^0	10^0	10^{-3}

In this table the last 5 columns represent the order of error for the corresponding method and the FK1, and the “+” sign indicates that the error is greater than 10^0 . It is clear from this table that, in general, the direct application of a quadrature rule does not give good results. The modified quadrature is only a little better, the regularized quadrature gives the best results, and the iteration the next best results. For the FK1 in Eq (11.4.43) all methods give acceptable results. In his computations, the maximum value of n Caldwell uses is 40. ■

11.4.6. From FK1 to FK2. In certain cases an FK1 can be changed into an FK2. Thus, if the kernel $k(x, s)$ is discontinuous along a curve $s = g(x)$ and the discontinuity is finite, then an FK1 can be changed to an FK2, as follows:

Starting with

$$f(x) = \int_a^b k(x, s)\phi(s) ds = \int_a^{g(x)} k(x, s)\phi(s) ds + \int_{g(x)}^b k(x, s)\phi(s) ds,$$

and differentiating both sides with respect to x , we get

$$\begin{aligned} f'(x) &= \int_a^b \frac{\partial k(x, s)}{\partial x} \phi(s) ds + k(x, g(x)_-) \phi(g(x)) g'(x) \\ &\quad - k(x, g(x)_+) \phi(g(x)) g'(x) \\ &= \int_a^b \frac{\partial k(x, s)}{\partial x} \phi(s) ds + S(x) \phi(g(x)) g'(x), \end{aligned}$$

where $k(x, g(x)_-)$ and $k(x, g(x)_+)$ are defined everywhere in (a, b) , and the difference $S(x) = k(x, g(x)_-) - k(x, g(x)_+)$. On dividing the above equation by $S(x) g'(x)$ and replacing x by $x = g^{-1}(y)$, we obtain an FK2. Now, the following theorem can be stated.

THEOREM 11.4.1. If (i) $k(x, s)$ is bounded in the domain $\Omega = [a, b] \times [a, b]$ and continuous except on the curve $s = g(x)$, where $g(x)$ has a nonzero continuous derivative in $[a, b]$, with $g(a) = a$, and $g(b) = b$, (ii) $S(x) \in C[a, b]$, (iii) $\frac{\partial k(x, s)}{\partial x} = k_x(x, s)$ is real and exists in Ω , and (iv) $f(x)$ and $f'(x)$ are continuous in $[a, b]$, then, if the quantity

$$\left| \frac{k_x(x, s)}{S(x) g'(x)} \right|_{x=g^{-1}(y)}$$

does not vanish in Ω , then an FK1 can be changed into the following FK2:

$$\int_a^b \left| \frac{k_x(x, s)}{S(x) g'(x)} \right|_{x=g^{-1}(y)} \phi(s) ds + \phi(y) = \left| \frac{f'(x)}{S(x) g'(x)} \right|_{x=g^{-1}(y)}. \quad (11.4.44)$$

Note that $\int_a^b k(x, s)\phi(s) ds = f(x) + C$ will also reduce to the above FK2. Thus, the solution to the FK2 will match the solution of $\int_a^b k(x, s)\phi(s) ds = f(x) + C$ for some specific value of C .

If $k(x, s)$ is continuous in Ω , but if $\frac{\partial^n k(x, s)}{\partial x^n}$ for some n has a finite discontinuity at $s = g(x)$, then the theorem can be generalized.

11.4.7. Equations over a Contour. Linear equations FK1 are not always in the form (11.1.1). Sometimes the integral in the equation appears as a contour integral where the free variable lies on the contour C . However, such equations can be transformed to the form (11.1.1) by parametrizing the contour C by the arc length measured from a fixed point on C . If C is a closed contour, the integrand $k(x, s)\phi(s)$ becomes periodic with a period equal to the length of C . In such cases the trapezoidal rule yields high-order local truncation errors if $k(x, s)$ and $\phi(x)$ are smooth.

EXAMPLE 11.4.10. Consider an FK1 with a weakly singular kernel

$$\int_C \ln |z - \zeta| \phi(\zeta) |d\zeta| = f(z), \quad z \in C,$$

where z and ζ are complex and the contour C is a circle with center at $z = 0$ and radius r . Such equations arise in conformal mappings (see Chapter 14; also Kythe 1998). Take a point P on the contour C as $\zeta = re^{i\theta}$, where θ is the angle that the radius vector OP makes with the real axis. Thus, $|d\zeta| = r|d\theta| = r d\theta$. Let $z = re^{i\delta}$, where r is fixed and δ is the free variable, $0 \leq \delta \leq 2\pi$, be a point on C . Then

$$\int_C \ln |z - \zeta| \phi(\zeta) |d\zeta| = r \int_0^{2\pi} \ln |r(e^{i\delta} - e^{i\theta})| \phi(re^{i\theta}) d\theta = f(re^{i\theta}).$$

This example, though an FK1, illustrates the formulation on integral equations, which generally involves contour integration. ■

An equation of this form is SYMM'S EQUATION, which is discussed in the next example.

EXAMPLE 11.4.11. Consider the case of integral equations that involve integration on a contour C (closed or unclosed), which has the form

$$\phi(z) - \lambda \int_C k(z, \sigma) \phi(\sigma) d\sigma = f(z). \quad (11.4.45)$$

A method for solving this equation consists of choosing points s_0, s_1, \dots, s_n on C (note that $s_n = s_0$ if C is closed) and denoting the arc joining the points s_j, s_{j+1} by C_j , with length $h_j = |s_{j+1} - s_j|$. Then Eq (11.4.45) in the case of a closed contour C can be written as

$$\phi(z) - \lambda \sum_{j=0}^{n-1} \int_{C_j} k(z, \sigma) \phi(\sigma) d\sigma = f(z). \quad (14.4.46)$$

Let z_j be a point on C_j . Then we will seek to determine the values $\tilde{\phi}(z_m)$, $m = 0, 1, \dots, n$, such that

$$\tilde{\phi}(z_m) - \lambda \sum_{j=0}^{n-1} \tilde{\phi}(z_j) \int_{C_j} k(z_m, \sigma) d\sigma = f(z_m). \quad (11.4.47)$$

But it may be difficult to compute $\int_{C_j} k(z_m, \sigma) d\sigma$, even when C is a line segment $[a, b]$. Therefore, if $m \neq j$, we shall apply the quadrature

$$\int_{C_j} k(z_m, \sigma) d\sigma = \sum_{i=0}^{n_j} w_{ij} k(z_m, \sigma_{ij}) \quad (11.4.48)$$

to compute the integral on the left side over those points of C_j that do not contain the point z_m (in this case the integrand remains bounded). For the case when $m = j$ we need some method to avoid unbounded quantities (singularities). One such method to solve SYMM'S EQUATION (Symm (1966) involves an approximate solution for the integral $\int_{C_j} \log |z_m - \sigma| d\sigma$ by using Simpson's rule:

$$\begin{aligned} & \int_{C_j} \log |z_m - \sigma| |d\sigma| \\ &= \begin{cases} \frac{h_j}{6} \left\{ \log |z_m - s_j| + 4 \log |z_m - z_j| + \log |z_m - s_{j+1}| \right\} & \text{if } m \neq j, \\ |z_j - s_j| \left\{ \log |z_j - s_j| - 1 \right\} \\ \quad + \log |z_j - z_{j+1}| \left\{ \log |z_j - s_{j+1}| - 1 \right\} & \text{if } m = j, \end{cases} \end{aligned} \quad (11.4.49)$$

where z_j is the middle point of C_j . For more information on Symm's equation, and other equations of conformal mapping including Lichtenstein's, Gershgorin's, Carrier's, Bannin's, Waeschawski-Stiefel's, Theodorsen's, Arbenz', and Mikhlin's equation, see Trefethen (1986) and Kythe (1998). ■

11.5. Volterra Equations of the First Kind

Under certain circumstances a VK1 of the form

$$\int_a^x k(x, s) \phi(s) ds = f(x), \quad 1 \leq x \leq b, \quad (11.5.1)$$

can be transformed into a VK2. We have the following result for this purpose.

THEOREM 11.5.1. If the function $f(x)$ and the kernel $k(x, s)$ satisfy the following conditions:

- (a) $f(x)$ is continuously differentiable in $(a, a + h)$ and $f(a)$ is zero; and
- (b) $k(x, s)$ and its derivative with respect to x are continuous on $(a, a + h)$ and, in addition, $k(x, x) \neq 0$; then Eq (11.5.1) can be reduced to a VK2.

To see this we differentiate Eq (11.5.1) with respect to x :

$$\int_a^x \frac{\partial k(x, s)}{\partial x} \phi(s) ds + k(x, x)\phi(x) = f'(x). \quad (11.5.2)$$

Then a VK2 is obtained on dividing Eq (11.5.2) by $k(x, x)$. If $k(x, x) = 0$, then we impose condition (b) on $\frac{\partial k(x, s)}{\partial x}$ and differentiate once more. Thus, as long as $\frac{\partial^n k(x, s)}{\partial x^n}$ exists and is not zero at $s = x$ for some n , a VK1 can be reduced to a VK2.

Now, we present some methods for numerically solving a VK1.

11.5.1. Quadrature Method. (Mikhlin and Smolitskiy 1967) It is obvious that $f(a) = 0$ for Eq (11.5.1), and we assume that $k(x, x) \neq 0$. Consider

$$\int_a^{x_n} k(x_n, s) \phi(s) ds = f(x_n). \quad (11.5.3)$$

Then using any quadrature rule, we can write Eq (11.5.1) in the form

$$\sum_{i=0}^n A_i^{(n)} k(x_n, x_i) \phi(x_i) = f(x_n), \quad (11.5.4)$$

which is a system of $(n+1)$ equations in $(n+1)$ unknowns $\phi(x_i)$, where the first equation (for $n = 0$) is, however, identically zero, i.e., $A_0^0 k(a, a)\phi(a) = f(a) = 0$. Since $k(a, a) \neq 0$ and $\phi(a)$ is not specified and depends on $f(x)$, it cannot be arbitrarily zero. Hence $A_0^0 = 0$. Therefore, the system has only n useful equations, and we have to evaluate $\phi(a)$ by other means. Let $x = a + h$, $h > 0$ in Eq (11.5.3); then we get

$$\int_a^{a+h} k(a+h, s) \phi(s) ds = f(a+h). \quad (11.5.5)$$

Applying the mean-value theorem to Eq (11.5.5), we have

$$k(a+h, a+bh)\phi(a+bh) = \frac{1}{h}[f(a+bh) - f(a)], \quad 0 < b < 1, \quad (11.5.6)$$

which, on taking the limit as $h \rightarrow 0$, gives

$$k(a, a)\phi(a) = f'(a). \quad (11.5.7)$$

The system of equations (11.5.4) can now be solved for $\phi(x_i)$.

11.5.2. Method of Linz. Linz (1969) begins by making the the following assumptions:

- (a) $f(0) = 0$;
- (b) $k(x, x) \neq 0$ for all x on the interval of integration; and
- (c) $k(x, t)$ and $f(x)$ are bounded and sufficiently smooth so that all derivatives exist.

Under the above conditions a unique and smooth solution will exist. In order to solve Eq (11.5.1), where a is henceforth assumed to be zero, divide the interval $[0, b]$ into N subintervals of size $h = b/N$. The endpoints of these intervals are $x_0 = 0$, $x_i = ih$, $x_N = Nh = b$. The approximate solution will be denoted by $\tilde{\phi}_i$. The following definitions are stated for convenience.

DEFINITION 11.5.1. Let $\tilde{\phi}_0(h), \tilde{\phi}_1(h), \dots$ denote approximate values of $\phi(x)$ obtained by any method with step size h such that $Nh = a$. Then the method is convergent iff

$$\lim_{h \rightarrow 0, N \rightarrow \infty} \left[\max_{0 \leq i \leq N} |\tilde{\phi}_i(h) - \phi_i(x_i)| \right] = 0.$$

DEFINITION 11.5.2. A method is said to be *of order p* if p is the largest number for which a finite constant C exists such that

$$\max_{0 \leq i \leq N} |\tilde{\phi}_i(h) - \phi_i(x_i)| \leq Ch^p \quad \text{for all } 0 < h < a.$$

The following lemma is also needed for the development of the methods given below.

LEMMA 11.5.1. If

$$|\xi_n| \leq A \sum_{i=0}^{n-1} |\xi_i| + B \quad \text{for } n = 1, 2, \dots; A > 0, B > 0, \quad (11.5.8)$$

then

$$|\xi_n| \leq (B + A |\xi_0|) (1 + A)^{n-1}. \quad (11.5.9)$$

If $A = hM$ and $nh = x$, then

$$|\xi_n| \leq (B + hM |\xi_0|) e^{Mx}.$$

(i) Rectangular rule: Applying the rectangular rule to Eq (11.5.1) on the interval $[0, x_n]$, we have

$$h \sum_{i=0}^{n-1} k(x_n, x_i) \tilde{\phi}_i = f(x_n), \quad n = 1, 2, \dots, \quad (11.5.10)$$

or

$$\tilde{\phi}_{n-1} = \frac{f(x_n)}{h k(x_n, x_{n-1})} - \sum_{i=0}^{n-2} \frac{k(x_n, x_i)}{k(x_n, x_{n-1})} \tilde{\phi}_i. \quad (11.5.11)$$

THEOREM 11.5.2. The approximation (11.5.11) is at least of order 1.

PROOF. Let $\epsilon_r = \phi(x_r) - \tilde{\phi}_r$. Then Eqs (11.5.1) and (11.5.10) yield

$$h \sum_{i=0}^{n-1} k(x_n, x_i) \epsilon_i = h \sum_{i=0}^{n-1} k(x_n, x_i) \phi(x_i) - \int_0^{x_n} k(x_n, s) \phi(s) ds.$$

Subtracting the above equation for n from the corresponding equation for $(n+1)$, we get

$$\begin{aligned} & k(x_{n+1}, x_n) \epsilon_n + \sum_{i=0}^{n-1} [k(x_{n+1}, x_i) - k(x_n, x_i)] \epsilon_i \\ &= k(x_{n+1}, x_n) \phi(x_n) - \frac{1}{h} \int_{x_n}^{x_{n+1}} k(x_{n+1}, s) \phi(s) ds \\ &+ \sum_{i=0}^{n-1} [k(x_{n+1}, x_i) - k(x_n, x_i)] \phi(x_i) \\ &- \frac{1}{h} \int_0^{x_n} [k(x_{n+1}, s) - k(x_n, x_i)] \phi(s) ds, \quad n = 1, 2, \dots. \end{aligned} \quad (11.5.12)$$

Assuming that all quantities in Eq (11.5.12) possess adequate smoothness and noting that $k(x, x) \neq 0$, we therefore conclude that for sufficiently small h , and $k(x_{n-1}, x_n) \neq 0$, there exist constants M_1, M_2, M_3 such that

$$|\epsilon_n| \leq h M_1 \sum_{i=0}^{n-1} |\epsilon_i| + h M_2,$$

with $|\epsilon_0| \leq h M_3$. Then, in view of (11.5.9),

$$|\epsilon_n| \leq (h M_2 + h^2 M_1 M_3) e^{M_1 x_n}.$$

This establishes the theorem.

(ii) Midpoint rule: Using the midpoint instead of the left endpoints of subintervals, the rectangular rule for Eq (11.5.1) yields

$$h \sum_{i=0}^{n-1} k(x_n, \xi_i) \tilde{\phi}_i = f(x_n), \quad n = 1, 2, \dots, \quad (11.5.13)$$

which gives

$$\tilde{\phi}_{n-1} = \frac{f(x_n)}{h k(x_n, \xi_n)} - \sum_{i=0}^{n-2} \frac{k(x_n, \xi_i)}{k(x_n, \xi_n)} \tilde{\phi}_i, \quad (11.5.14)$$

where $\xi_i = \frac{1}{2}(x_i + x_{i+1})$, and $\tilde{\phi}_i$ denotes the approximation to $\phi\left(\frac{x_i + x_{i+1}}{2}\right)$.

(iii) Trapezoidal rule: Using the trapezoidal rule for Eq (11.5.1), we obtain

$$\begin{aligned} & \frac{1}{2} h k(x_n, x_0) \tilde{\phi}_0 + h \sum_{i=1}^{n-1} k(x_n, x_i) \tilde{\phi}_i \\ & + \frac{1}{2} h k(x_n, x_n) \tilde{\phi}_n = f(x_n). \end{aligned} \quad (11.5.15)$$

From Eq (11.5.15), we can solve for $\tilde{\phi}_n$; thus,

$$\tilde{\phi}_n = \frac{2f(x_n)}{h k(x_n, x_n)} - \frac{k(x_n, x_0)}{k(x_n, x_n)} \tilde{\phi}_0 - 2 \sum_{i=0}^{n-2} \frac{k(x_n, x_i)}{k(x_n, x_n)} \tilde{\phi}_i, \quad n = 1, 2, \dots, \quad (11.5.16)$$

where the value of $\phi(0)$ is given by (11.5.7).

It can be shown, as before, that both the midpoint and trapezoidal rules yield second-order approximations. Linz (1969) points out that one has to be careful about selecting the quadrature rules. He demonstrates that the error may become large if Simpson's or Gregory's formulas are used. His estimates show that for the rectangular rule the error $\epsilon_n = h e(x_n) + O(h^2)$, where $e(x)$ is the solution of

$$\begin{aligned} e(x) = & - \int_0^x \frac{k_x(x, s)}{k(x, x)} e(s) ds - \frac{1}{2k(x, x)} \left[k_s(x, x)\phi(x) + k(x, x)\phi'(x) \right. \\ & \left. + \int_0^x \frac{\partial}{\partial \xi} \{ k_x(x, \xi) \phi(\xi) \} d\xi \right]. \end{aligned}$$

For the midpoint rule the error is $\epsilon_n = h^2 e(\xi_n) + O(h^3)$, where $e(x)$ is the solution of

$$\begin{aligned} e(x) = & - \int_0^x \frac{k_x(x, s)}{k(x, x)} e(s) ds - \frac{1}{24k(x, x)} \left[k_{ss}(x, x)\phi(x) + 2k_s(x, x)\phi'(x) \right. \\ & + k(x, x)\phi''(x) + k_{xs}(x, x)\phi(x) + k_x(x, x))\phi'(x) \\ & \left. - k_{xs}(x, 0)\phi(0) - k_x(x, 0))\phi'(0) \right]. \end{aligned}$$

For the trapezoidal rule Linz introduces the average error $\eta_r = (\epsilon_r + \epsilon_{r-1})/2$ which is given by

$$\eta_r = h^2 \eta(x_r) + O(h^3),$$

where $\eta(x)$ is the solution of

$$\begin{aligned} \eta(x) = & - \int_0^x \frac{k_x(x, s)}{k(x, x)} \eta(s) ds + \frac{1}{12k(x, x)} \left[k_{ss}(x, x)\phi(x) + 2k_s(x, x)\phi'(x) \right. \\ & + k(x, x)\phi''(x) + k_{xs}(x, x)\phi(x) + k_x(x, x)\phi'(x) \\ & \left. - k_{xs}(x, 0)\phi(0) - k_x(x, 0))\phi'(0) \right]. \end{aligned}$$

(iv) Improvement of accuracy: By using different step sizes, say h and ht , for the rectangular rule, we can write

$$\begin{aligned} \tilde{\phi}_n(h) &= \phi(x_n) + h \epsilon(x_n) + O(h^2), \\ \tilde{\phi}_n(ht) &= \phi(x_n) + ht \epsilon(x_n) + O(h^2). \end{aligned}$$

Eliminating the error term of $O(h)$, we have the improved estimate for $\tilde{\phi}_n^c$:

$$\tilde{\phi}_n^c = \frac{1}{t-1} \left(t \tilde{\phi}_n(h) - \tilde{\phi}_n(ht) \right) = \phi(x_n) + O(h^2).$$

This technique is known as *Richardson's extrapolation*. One should take care to make an appropriate choice of t in order to ensure that the mesh points of evaluation coincide in the two choices of step sizes.

EXAMPLE 11.5.1. (Linz 1969) Consider the VK1

$$\int_0^x \cos(x-s)\phi(s) ds = \sin x,$$

with exact solution $\phi(x) = 1$. The results, using the rectangular rule, are given in Table 11.5.1.

Table 11.5.1

x	$\tilde{\phi}_n$	$\tilde{\phi}_n^c$
0.15	1.00459	1.00188
0.30	1.00837	1.00561
0.45	1.01216	1.00842
0.60	1.01596	1.01122
1.05	1.02746	1.01961
1.50	1.03909	1.02800

Obviously, the extrapolated values are more accurate. We now discuss the same problem using the midpoint rule. Using Mathematica, the midpoint rule gives a constant value at all points. This constant value is 0.999895836588493108 for $h = 0.05$ and 0.999067636365641 for $h = 0.15$. If we apply Richardson's extrapolation, the value actually becomes 1. Using the trapezoidal rule, we find that $\tilde{\phi}_{2n} = 1$ and $\tilde{\phi}_{2n+1} = 1.00042$. We did not experiment with extrapolation in this case. For computational details, see `linz.nb`. ■

EXAMPLE 11.5.2. (Linz 1969) Consider

$$\int_0^x \cos(x-s)\phi(s) ds = 1 - \cos x,$$

whose exact solution is $\phi(x) = x$. The accuracy with the rectangular rule was not particularly good. The trapezoidal rule yielded the results given in Table 11.5.2(a).

Table 11.5.2(a)

x	$\tilde{\phi}(x), h = 0.3$	$\tilde{\phi}(x), h = 0.1$	Extrapolation
0.30	0.29975	0.297756	0.29999
0.60	0.59950	0.59551	0.59999
0.90	0.89925	0.89327	0.89999
1.20	1.19900	1.19103	1.2

Using Mathematica and the midpoint rule our values are exactly the same as those obtained by Linz (1969), as listed in Table 11.5.2(b).

Table 11.5.2(b)

x	$\tilde{\phi}(x), h = 0.3$	$\tilde{\phi}(x), h = 0.1$	Extrapolation
0.15	0.15057	0.15006	0.14999
0.45	0.45171	0.45019	0.45
0.75	0.75285	0.75031	0.749993
1.05	1.05398	1.05044	1.05
1.35	1.35512	1.35056	1.34999
1.65	1.65626	1.65069	1.64999

Obviously, both the trapezoidal and midpoint rules along with Richardson's extrapolation method give very good results for these two examples. For computational details, see `linz.nb`. ■

11.5.3. Product-Integration Method. Linz (1971) also develops a method for solving a VK1 by product-integration method, as follows: To solve Eq (11.5.1) on the interval $[0, x_N]$, divide this interval into n equal subintervals $[x_{i-1}, x_i]$ of length h , where $x_i = ih$, $x_0 = 0$. A constant $\tilde{\phi}_i$ approximates the value of $\phi(x)$ in each subinterval $[x_{i-1}, x_i]$. Using this in Eq (11.5.1) and letting $x = x_m$ for different values of m , we have

$$\sum_{i=1}^m w_i(x_m) \tilde{\phi}_i = f(x_m), \quad m = 1, 2, \dots, N, \quad (11.5.17)$$

where

$$w_i(x) = \int_{x_{i-1}}^{x_i} k(x, s) ds \quad (11.5.18)$$

are the moment integrals. Eq (11.5.17) can be solved successively for $\tilde{\phi}_m$ by

$$\tilde{\phi}_m = \frac{f(x_m)}{w_m(x_m)} - \sum_{i=1}^{m-1} \frac{w_i(x_m)}{w_m(x_m)} \tilde{\phi}_1, \quad m = 1, 2, \dots, N. \quad (11.5.19)$$

It should be noted that for a sufficiently small h , the solution $\tilde{\phi}_i$ will approximate $\phi\left(\frac{x_i + x_{i-1}}{2}\right)$ fairly accurately except when the interval $[x_{i-1}, x_i]$ contains a turning point of $\phi(x)$. Linz (1971) gives error estimates for this method and shows that $\tilde{\phi}_i$ is a second-order approximation for $\phi(x_i - h/2)$. This method can be used even if the moment integrals are inexact, but error estimates are then different (see Linz 1971).

EXAMPLE 11.5.3. (Linz 1971) Consider

$$\begin{aligned} \int_0^x (4 + \cos s)\phi(s) ds &= 1.999x + \frac{x^2}{10} + \frac{x \cos 10x}{1000} \\ &\quad + \left(\frac{x}{10} + \frac{x^2}{100} \right). \end{aligned} \quad (11.5.20)$$

Its exact solution is $1 + x/10$. The numerical results obtained by Mathematica are given in Table 11.5.3. The first column is the value of x , the second column represents the exact value of $\phi(x)$, the third column the approximate values $\tilde{\phi}(x)$ for $h = 1/30$, the fourth column the approximate values $\tilde{\phi}(x)$ for $h = 1/10$ and the last column the approximate values $\tilde{\phi}(x)$ obtained by extrapolation.

Table 11.5.3

x	$\phi(x)$	$\tilde{\phi}(x), h = 1/30$	$\tilde{\phi}(x), h = 1/10$	Extrapolation
0.05	1.005	1.004998	1.004981	1.005
0.15	1.015	1.014989	1.014901	1.015
0.25	1.025	1.024983	1.024851	1.025
0.35	1.035	1.034995	1.034961	1.035
0.45	1.045	1.045015	1.045129	1.045
0.55	1.055	1.055013	1.055113	1.055
0.65	1.065	1.064994	1.064946	1.065
0.75	1.075	1.074968	1.074716	1.075
0.85	1.085	1.084947	1.084538	1.085
0.95	1.095	1.094984	1.094875	1.095

Clearly, a smaller value of h gives more accurate results, but after using Richardson's extrapolation the results are exact for practical purposes. For computational details, see `linz1.nb` ■

EXAMPLE 11.5.4. Consider the integral equations

$$\int_0^x (4 + x \cos s) \phi(s) ds = \begin{cases} f_1(x) \\ f_2(x) \end{cases}, \quad (11.5.21)$$

where

$$f_1(x) = x + 6x^2 + 3x \cos x + (x + 3x^2) \sin x,$$

$$f_2(x) = \frac{1}{8}(-32 + 31x + 2x^3 + 32 \cos x + x \cos 2x + 40x \sin x + 2x^2 \sin 2x).$$

Their exact solutions are $\phi_1(x) = 1 + 3x$ and $\phi_2(x) = 1 + x \cos x$, respectively. Table 11.5.4 gives the exact values (obtained from the function directly using Mathematica) and the values obtained from the algorithm (11.5.9) for $h = 1/10$. The results show fairly good accuracy even without using the extrapolation. For computational details, see `linz.nb`.

Table 11.5.4

x	$\phi_1(x)$	$\tilde{\phi}_1(x)$	$\tilde{\phi}_2(x)$	Extrapolation
0.05	1.15	1.15	1.04987	1.04994
0.15	1.45	1.44998	1.14812	1.14832
0.25	1.75	1.74995	1.24190	1.24223
0.35	2.05	2.04990	1.32833	1.32878
0.45	2.35	2.34984	1.40463	1.40520
0.55	2.65	2.64977	1.46821	1.46889
0.65	2.95	2.94968	1.51668	1.51745
0.75	3.25	3.24959	1.54792	1.54877
0.85	3.55	3.54949	1.56010	1.56099
0.95	3.85	3.84938	1.55169	1.55260 ■

11.6. Abel's Equation

For an introduction to Abel's equation, see Section 8.5(c). Piessens and Verbaeten (1973) developed a numerical solution for the Abel's equation

$$\int_0^x (x-s)^{-\alpha} \phi(s) ds = f(x), \quad 0 \leq \alpha \leq 1. \quad (11.6.1)$$

Tricomi (1957) gives the following explicit solution for this equation:

$$\phi(x) = \frac{\sin(\pi\alpha)}{\pi} \left[\frac{f(0)}{x^{(1-\alpha)}} + \int_0^x \frac{f'(s)}{(x-s)^{(1-\alpha)}} ds \right]. \quad (11.6.2)$$

This formula loses its usefulness if either the algebraic form of $f(x)$ is not explicitly known or its derivative is difficult to evaluate. The method of Piessens and Verbaeten (1973) is for the special case when $f(x)$ can be approximated by a finite series in terms of the Chebyshev polynomials $T_m(x)$ of the first kind. Let

$$f(x) \approx x^\beta \sum_{m=0}^N c_m T_m(1-2x), \quad (11.6.3)$$

where $\beta > \alpha$ is a free parameter. While Piessens and Verbaeten (1973) assume that $0 \leq x \leq 1$, they emphasize that it is not a necessary restriction. First, consider the integral equation

$$\int_0^x (x-s)^{-\alpha} g_n(s) ds = x^\beta T_m(1-2x).$$

Its solution can be easily derived by applying Laplace transform; thus,

$$\mathcal{L}\{x^\beta T_m(1-2x)\} = \frac{\Gamma(\beta+1)}{p^{\beta+1}} {}_3F_1\left(\begin{matrix} -n, n, \beta+1 \\ 1/2 \end{matrix}; \frac{1}{p}\right),$$

where

$${}_pF_q\left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p \\ \beta_1, \beta_2, \dots, \beta_q \end{matrix}; Z\right) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^p (\alpha_i)_n}{\prod_{i=1}^q (\beta_i)_n} \cdot \frac{Z^n}{n!}, \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

Hence,

$$\mathcal{L}\{g_n(x)\} = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha)} \frac{1}{p^{\alpha+\beta}} {}_3F_1\left(\begin{matrix} -n, n, \beta+1 \\ 1/2 \end{matrix}; \frac{1}{p}\right),$$

and, therefore,

$$g_n(x) = \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha)} {}_3F_2\left(\begin{matrix} -n, n, \beta+1 \\ 1/2, \alpha+\beta \end{matrix}; x\right). \quad (11.6.4)$$

If the approximate value of $f(x)$ is given by Eq (11.6.3), then $\phi(x)$ is approximated by

$$\tilde{\phi}(x) \simeq \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha)} \sum_{m=0}^{\infty} c_m f_m(x), \quad (11.6.5)$$

where

$$f_m(x) = {}_3F_2\left(\begin{matrix} -m, m, \beta+1 \\ 1/2, \alpha+\beta \end{matrix}; x\right). \quad (11.6.6)$$

Piessens uses Fasenmyer's method as given in Raineville (1960) to develop the following recurrence formula for the evaluation of $f_n(x)$:

$$f_n(x) + (A_n + B_n x) f_{n-1}(x) + (C_n + D_n x) f_{n-2}(x) + E_n f_{n-3}(x) = 0, \quad (11.6.7)$$

where

$$\begin{aligned} A_n &= -\frac{1}{n-2} \left[n-3 + \frac{(n-1)(2n-3)}{n+\alpha+\beta-1} \right], \\ B_n &= \frac{4(n+\beta)}{n+\alpha+\beta-1}, \\ C_n &= \frac{1}{n-2} \left[-1 + \frac{(n-1)}{n+\alpha+\beta-1} \right], \\ D_n &= -\frac{4(n-\beta-3)(n-1)}{(n+\alpha+\beta-1)(n-2)}, \\ E_n &= -\frac{(n-\alpha-\beta-2)(n-1)}{(n+\alpha+\beta-1)(n-2)}. \end{aligned}$$

Starting values for $f_n(x)$ are

$$\begin{aligned} f_0(x) &= 1, & f_1(x) &= 1 - \frac{2(\beta+1)}{\alpha+\beta} x, \\ f_2(x) &= 1 - \frac{8(\beta+1)}{\alpha+\beta} x + \frac{8(\beta+1)(\beta+2)}{(\alpha+\beta)(\alpha+\beta+1)} x^2. \end{aligned}$$

For large n , the recurrence relation (11.6.7) reduces to

$$f_n + (4x - 3)f_{n-1} - (4x - 3)f_{n-2} - f_{n-3} = 0.$$

The first numerical example considered in Piessens and Verbaeten (1973) is given below.

EXAMPLE 11.6.1. Consider the equation

$$\int_0^x (x-s)^{-0.5} \phi(s) ds = e^x - 1.$$

Its exact solution is $\phi(x) = \frac{1}{\pi} e^x \operatorname{erf}(\sqrt{x})$. Using $\beta = 1$ and $N = 12$, the exact values of $\phi(x)$ and its approximate values from formula (11.6.5) using Mathematica are given in Table 11.6.1. According to Piessens and Verbaeten (1973), the maximum absolute error that they found is of the order 10^{-14} . The results we obtained using Mathematica give the maximum absolute error of the order 10^{-9} . For computational details, see `piessens.nb`.

Table 11.6.1

x	$\phi(x)$	$\tilde{\phi}(x)$	$ \phi(x) - \tilde{\phi}(x) $
0.1	0.215291	0.215291	1.71681(-9)
0.2	0.325884	0.325884	2.18754(-9)
0.3	0.427560	0.427560	2.37261(-9)
0.4	0.529333	0.529333	2.38010(-9)
0.5	0.635032	0.635032	2.26239(-9)
0.6	0.747040	0.747040	2.05575(-9)
0.7	0.867188	0.867188	1.79052(-9)
0.8	0.997089	0.997089	1.49493(-9)
0.9	1.138300	1.138300	1.19658(-9)
1.0	1.292390	1.292390	9.23337(-10)

Fettis (1964) also gives a numerical form of the solution to Abel's integral equation. He uses Gauss-Jacobi quadrature for a form of Tricomi's solution. Minerbo and Levy (1960) uses orthogonal polynomials to invert Abel's equation. Wing (1991) has written a useful primer on the integral equations of the first kind. ■

An equation of the form

$$\int_0^x \frac{\kappa(x,s)}{(x^p - s^p)^\alpha} \phi(s) ds = f(x), \quad 0 \leq x \leq X, \quad (11.6.8)$$

where $p = 1$ or 2 , and $0 < \alpha < 1$, is called a GENERALIZED ABEL'S EQUATION.

EXAMPLE 11.6.2. (Atkinson 1971) Consider Abel's equation

$$\int_0^x \frac{\phi(s) ds}{\sqrt{x^2 - s^2}} = f(x), \quad 0 \leq x \leq X. \quad (11.6.9)$$

Although the integral operator K with kernel

$$k(x, s) = \begin{cases} (x^2 - s^2)^{-1/2} & \text{if } s \leq x, \\ 0 & \text{if } s > x, \end{cases}$$

is not compact on $[0, X]$, the solution of this equation is given by

$$\begin{aligned} \phi(x) &= \frac{2}{\pi} \frac{d}{dx} \int_0^x \frac{s f(s)}{\sqrt{x^2 - s^2}} ds, \quad 0 \leq x \leq X, \\ \phi(0) &= \frac{2}{\pi} f(0), \end{aligned} \quad (11.6.10)$$

assuming that the integral exists. We use the trapezoidal rule to approximate integrals of the form $\int_0^x (x^2 - s^2)^{-1/2} \phi(s) ds$ appearing in the given Eq (11.6.9), and obtain the following system of equations:

$$\begin{aligned} \sum_{j=0}^m v_j(mh) \tilde{\phi}(jh) &= f(mh), \quad m = 1, 2, \dots, n, \\ \tilde{\phi}(0) &= \frac{2}{\pi} f(0), \end{aligned} \quad (11.6.11)$$

where $nh = X$, h is the step size, and as in Section A.3,

$$\begin{aligned} v_0(mh) &= \frac{1}{h} \int_0^h \frac{h-s}{\sqrt{(mh)^2 - s^2}} ds, \\ v_j(mh) &= \frac{1}{h} \int_{(j-1)h}^{(j+1)h} \frac{\delta_i(s)}{\sqrt{(mh)^2 - s^2}} ds, \quad 1 \leq j \leq m-1, \\ v_m(mh) &= \frac{1}{h} \int_{(m-1)h}^{mh} \frac{s - (m-1)h}{\sqrt{(mh)^2 - s^2}} ds, \\ \delta_i(s) &= \begin{cases} \frac{1}{h} [s - (j-1)h] & \text{if } (j-1)h \leq s \leq jh, \\ \frac{1}{h} [(j+1)h - s] & \text{if } jh \leq s \leq (j+1)h. \end{cases} \end{aligned} \quad (11.6.12)$$

Two interesting cases are

- (i) If $f(x) = x^2$ in (11.6.9), then the exact solution is $\phi(x) = \frac{\pi}{4}x^2$. In this case, for $X = 5$ the error found by Atkinson (1971) for a different choice of h is as follows: $1.6(-3)$ for $h = 1/10$; $4.1(-4)$ for $h = 1/20$; $1.0(-4)$ for $h = 1/40$; and $2.6(-5)$ for $h = 1/80$.
- (ii) If $f(x) = \frac{\pi}{2}J_0(x)$ in (11.6.9), then the exact solution is $\phi(x) = \cos x$. In this case, the errors at $x = 2(2)20$ given in Table 11.6.2 are from Atkinson (1971). ■

Table 11.6.2

x	$h = 1/10$	$h = 1/20$
2.0	2.9(-4)	7.7(-5)
4.0	5.6(-4)	1.4(-4)
6.0	-7.5(-4)	-1.9(-4)
8.0	7.1(-5)	2.2(-5)
10.0	16.9(-4)	1.7(-4)
12.0	-6.5(-4)	-1.7(-4)
14.0	-1.5(-4)	3.5(-4)
16.0	7.7(-4)	2.0(-4)
18.0	-4.9(-4)	-1.3(-4)
20.0	-3.6(-4)	-8.9(-5)

11.7. Iterative Schemes

We now present two iteration methods, the first for an FK1 and the second for an FK1/VK1.

11.7.1. Hanna and Brown's Scheme. Hanna and Brown (1991) propose an iteration scheme that appears to be quite successful. Consider the FK1 (11.1.1), and assume that an approximate solution $\tilde{\phi}(x)$ has been found or guessed. Introduce a CORRECTOR FUNCTION $C(x)$ such that $\phi(x) = C(x)\tilde{\phi}(x)$; then Eq (11.1.1) becomes

$$\int_a^b k(x, s) C(s) \tilde{\phi}(s) ds = f(x). \quad (11.7.1)$$

Now the problem reduces to determining the unknown function $C(x)$, where the kernel of the new FK1 is $k(x, s)C(s)$. At this stage, we use one-point Gauss quadrature (see Section A.2, Rule Q_{12}) and obtain

$$C(s_1) \int_a^b k(x, s) \tilde{\phi}(s) ds = f(x), \quad (11.7.2)$$

where $s_1(x)$ is defined by

$$s_1(x) = \frac{\int_a^b s k(x, s) \tilde{\phi}(s) ds}{\int_a^b k(x, s) \tilde{\phi}(s) ds}. \quad (11.7.3)$$

For a given $x \in [a, b]$, Eq (11.7.3) determines the point $s_1(x)$ at which $C(s)$ is determined by Eq (11.7.2). Once $C(s)$ is determined at a point, we have a better approximation $\tilde{\phi}(s)$ for $\phi(s)$ and we can continuously repeat this procedure until $C(s) \approx 1$. By choosing several values of s , we can determine ϕ at different values of $s(x)$. Since the Gauss one-point quadrature is quite accurate for linear functions, it is reasonable to expect that the solution to an FK1 will also be a good approximation.

The numerical implementation of this scheme is straightforward. In the absence of any clue for a first approximation, we can use the procedure outlined above by assuming the zeroth approximation as follows:

Applying one-point Gauss quadrature to Eq (11.1.1), we obtain

$$s_0(x) = \frac{\int_a^b s k(x, s) ds}{\int_a^b k(x, s) ds} \quad (11.7.4)$$

and

$$\phi(s_0) \int_a^b k(x, s) ds = f(x). \quad (11.7.5)$$

From these two equations we have

$$s_0(x) = \frac{\int_a^b s k(x, s) ds}{\int_a^b k(x, s) ds}$$

and

$$\phi(s_0) = \frac{f(x)}{\int_a^b k(x, s) ds}, \quad (11.7.6)$$

where $s_0 = s_0(x)$. Now continuing with this approximation and applying the process explained above, we will obtain in succession

$$s_1(x) = \frac{\int_a^b s k(x, s) \phi_0(s) ds}{\int_a^b k(x, s) \phi_0(s) ds},$$

$$C_1(s_1) = \frac{f(x)}{\int_a^b k(x, s) \phi_0(s) ds},$$

where s_1 depends on x , and thus, the first approximation is $\phi_1(s) = C_1(s) \phi_0(s)$.

Repeating the foregoing steps, we obtain the second approximation as

$$s_2(x) = \frac{\int_a^b s k(x, s) \phi_1(s) ds}{\int_a^b k(x, s) \phi_1(s) ds},$$

$$C_2(s_2) = \frac{f(x)}{\int_a^b k(x, s) \phi_1(s) ds},$$

where $s_2 = s_2(x)$ and $\phi_2(s) = C_2(s) \phi_1(s)$. Continuing in this manner we can obtain the desired number of iterations. A good approximation is reached when $C_n(s) \approx 1$.

The error analysis in this scheme is as follows: For any approximation $\phi_n(s)$, we have $\phi(s) = C(s) \phi_n(s)$ and the error $E(s)$ is given by $\phi(s) = \phi_n(s) + E(s)$. From these two relations, we get

$$\frac{E(s)}{\phi(s)} = \frac{C(s) - 1}{C(s)}. \quad (11.7.7)$$

The process outlined here yields an accurate solution when $C_n(s) \approx 1$. Even when this is not the case, we have a fair estimate of the error.

EXAMPLE 11.7.1. (Hanna and Brown 1991) Consider the FK1

$$\int_0^\infty e^{-sx} \phi(s) ds = \frac{p!}{x^{p+1}}.$$

The exact solution is $\phi(x) = x^p$. We first determine the initial (zeroth) approximation:

$$s_0(x) = \frac{\int_0^\infty s e^{-xs} ds}{\int_0^\infty e^{-xs} ds} = \frac{1}{x}, \quad (11.7.8)$$

and

$$C_0(s_0) \left(\frac{1}{x} \right) = f(x) = \frac{p!}{x^{p+1}} = p!(s_0)^{p+1}. \quad (11.7.9)$$

Thus, the first approximation is

$$\phi_1(s) = C_0(s) \phi_0(s) = p! s^p. \quad (11.7.10)$$

Repeating the process, we find the second approximation to be

$$s_1(x) = \frac{\int_0^\infty p! s^{p+1} e^{-xs} ds}{\int_0^\infty p! s^p e^{-xs} ds} = \frac{(p+1)}{x}, \quad (11.7.11)$$

and

$$\frac{C_1(s_1) (p!)^2}{x^{p+1}} = \frac{p!}{x^{p+1}}; \quad C_1(s_1) = \frac{1}{p!}. \quad (11.7.12)$$

The new approximation is then $\phi_2(s) = p! s^p \frac{1}{p!} = s^p$. We can now anticipate that the next iteration will simply yield $C_2 = 1$, which is true, as can be easily verified. ■

EXAMPLE 11.7.2. (Babolian and Delves 1979, Delves and Mohamed 1985, Hanna and Brown 1991) Consider the FK1

$$\int_0^1 e^{sx} \phi(s) ds = f(x) = \frac{e^{x+1} - 1}{x + 1}, \quad (11.7.13)$$

which has the exact solution $\phi(x) = e^x$. We present the details of the numerical solution carried out by Hanna and Brown (1991). They use the trapezoidal rule with two Richardson extrapolations. The minimum number of quadrature points (mq) used is 30. Assuming that the first approximation is $\phi_0(s) = 1$, the corrector function $C_1(x)$ and the point $s_1(x)$ are given by

$$C_1(s_1) = \frac{x(e^{x+1} - 1)}{(1+x)(e^x - 1)}, \quad s_1(x) = \frac{1 + (x-1)e^x}{x(e^x - 1)}. \quad (11.7.14)$$

Since $\phi_1(s) = 1$, the next approximation is given by

$$\phi_2(s_1) = C_1(s_1) = \frac{x(e^{x+1} - 1)}{(1+x)(e^x - 1)}. \quad (11.7.15)$$

At this stage, if it were simple to evaluate x in terms of s_1 , we could find $\phi_1(s_1)$ as an explicit function of s and continue the process analytically. However, even

if we could complete this second approximation, the inversion process in the third approximation is likely to be even more complex. Therefore, in order to carry out the next approximations, we have to evaluate $\phi_1(s)$ at a discrete set of points. The general procedure is as follows: The first step is to evaluate the integrals $\int_a^b s k(x, s) ds$ and $\int_a^b k(x, s) ds$ by any sound quadrature method. Mathematica gives very accurate results, and quite often exact evaluations can be obtained. For the next step we do need to use a reliable quadrature rule, like the trapezoidal or midpoint rule with Richardson extrapolation or Gaussian quadrature with error monitoring. First, we choose the points x_j and evaluate the corresponding s_j . For any given iteration it is helpful to use a fixed set of quadrature points s_j . Then evaluate $k(x_i, s_j)$ at all required quadrature points and store them. Now, $C(s_j)$, and consequently $\tilde{\phi}(s_j)$, can be evaluated from the following integrals:

$$s_j = \frac{\int_a^b s k(x_j, s) \tilde{\phi}(s) ds}{\int_a^b k(x_j, s) \tilde{\phi}(s) ds}, \quad C_j(s_j) = \frac{f(x_j)}{\int_a^b k(x_j, s) \tilde{\phi}(s) ds}. \quad (11.7.16)$$

For the next iteration, the values of s_j will be different from the previous ones. However, to use the same set of values s_j , we have to use interpolation or extrapolation or the latest approximation for $C(s)$. After the new values are obtained, the necessity for further iteration can be determined from the error estimate in Eq (11.7.7). Hanna and Brown (1991) use $\phi_0(s) = 1, 1 + 0.1 s^2$ for the solution without noise and consider an additional third function $1 - 0.1 s^2$ as the approximation for $\phi_0(s)$ for noisy data. They use linear least squares ($l = 1$), quadratic least-squares ($l = 2$), and cubic least squares ($l = 3$) for approximating the values of $C(s)$ and $\phi(s)$. They present their results in the form of Tables 11.7.1 and 11.7.2, which show the evaluations without any noise, whereas Tables 11.7.3 and 11.7.4 represent the results after introducing pseudo-random noise to the data $g(x_j)$ with standard deviation 0.005. Note that in these tables “iter” means number of iterations.

Table 11.7.1. Effects of different degrees of extrapolation, with $\phi_0(s) = 1$

s	Exact ϕ	$\tilde{\phi}, l = 3, \text{iter}=9$	$\tilde{\phi}, l = 2, \text{iter}=6$	$\tilde{\phi}, l = 1, \text{iter}=3$
0.0	1.00	0.991	0.990	0.844
0.2	1.22	1.23	1.22	1.119
0.4	1.49	1.50	1.50	1.54
0.6	1.82	1.82	1.82	1.89
0.8	2.23	2.22	2.22	2.24
1.0	2.72	2.73	2.72	2.59

These results indicate that while there is considerable improvement from $l = 1$ and two iterations to $l = 2$ and six iterations, there is no improvement from $l = 2$ and six iterations to $l = 3$ and nine iterations. The results for a different initial guess ($\phi_0(s) = 1 + 0.1s^2$) in Table 11.7.2 are exactly the same for $l = 2$ and six iterations, but the results for $l = 1$ and two iterations have improved but not significantly. However, the results are very accurate in all cases for $l \geq 2$.

Table 11.7.2. Effects of different degrees of extrapolation, with $\phi_0(s) = 1 + 0.1s^2$

s	Exact ϕ	$\tilde{\phi}, l = 2, \text{iter}=6$	$\tilde{\phi}, l = 1, \text{iter}=3$
0.0	1.0	0.99	0.90
0.2	1.22	1.22	1.12
0.4	1.49	1.50	1.52
0.6	1.82	1.82	1.86
0.8	2.23	2.22	2.23
1.0	2.72	2.72	2.64

In Tables 11.7.3 and 11.7.4 below, “sd” means standard deviation of pseudo-random noise quadrature points for evaluation of integrals.

Table 11.7.3. Effects of noise, with $\phi_0(s) = 1 + 0.1s^2$ and sd = 0.05

s	Exact ϕ	$\tilde{\phi}, l = 2, \text{iter}=14$	$\tilde{\phi}, l = 1, \text{iter}=3$
0.0	1.0	1.87	0.828
0.2	1.22	1.36	1.117
0.4	1.49	1.22	1.52
0.6	1.82	1.40	1.86
0.8	2.23	2.03	2.21
1.0	2.72	3.61	2.55

Table 11.7.4. Effects of noise, with sd = 0.05, and iter=3 for different $\phi_0(s)$

s	Exact ϕ	$\tilde{\phi}, \phi_0 = 1 + 0.1s^2, l = 2$	$\tilde{\phi}, \phi_0 = 1 - 0.1s^2, l = 1$
0.0	1.0	0.883	0.761
0.2	1.22	1.18	1.17
0.4	1.49	1.50	1.54

Table 11.7.4 Continued

s	Exact ϕ	$\tilde{\phi}, \phi_0 = 1 + 0.1s^2, l = 2$	$\tilde{\phi}, \phi_0 = 1 - 0.1s^2, l = 1$
0.6	1.82	1.84	1.90
0.8	2.23	2.20	2.22
1.0	2.72	2.60	2.49

It is obvious from these tables that the procedure developed by Hanna and Brown yields quite good results. With noisy data, the best results are obtained from the initial choice of $\phi_0(s) = 1 + 0.1s^2$ and the worst from the choice $\phi_0(s) = 1$ and quadratic least-squares approximation. ■

11.7.2. van den Berg's Scheme. We consider an FK1/VK1 of the type (1.2.15) over a domain D and present iterative schemes that provide a least-squares error correction criterion and yield a system of linear algebraic equations. This error also provides a measure of the approximate solution. We start with an initial guess and a set of arbitrarily chosen correction functions and develop a convergent iteration scheme. We also discuss some suitable choices for the correction functions and determine the rate of convergence. The scheme with two minimization steps shall then be applied to the problem of scattering and diffraction of waves by an obstacle (see Example 11.7.3). Consider the FK1/VK1 $f = K\phi$ defined on a domain D , i.e.,

$$f(x) = \int_{s \in D} k(x, s) \phi(s) ds, \quad x \in D. \quad (11.7.17)$$

Let the root mean-square error E in the approximate solution $\tilde{\phi}$ be defined by

$$E = \left\langle \overline{(f - K\tilde{\phi})}, (f - K\tilde{\phi}) \right\rangle^{1/2} = \|f - K\tilde{\phi}\| \geq 0, \quad (11.7.18)$$

where a bar over a quantity denotes its complex conjugate, and equality in (11.7.18) occurs only if $\phi = \tilde{\phi}$. We minimize E by an iterative scheme and aim at its convergence to zero. The normalized error is defined by

$$\hat{E} = \frac{\|f - K\tilde{\phi}\|}{\|f\|}. \quad (11.7.19)$$

An outline of the first minimization of E which leads to the solution of Eq (11.7.17) is as follows: Assume that there exists an iterative scheme in which n steps have been carried out which results in the approximate value $\phi_{(n)}$. The root mean-square error $E_{(n)}$ after n steps of iterations is

$$E_{(n)} = \left\langle \overline{r_{(n)}}, r_{(n)} \right\rangle^{1/2} = \|r_{(n)}\|, \quad (11.7.20)$$

where $r_{(n)} \equiv r_{(n)}(x) = f - K\phi_{(n)}$ is the residual (error). In going from the $(n-1)$ st step to the n th step, we take

$$\phi_{(n)} = \phi_{(n-1)} + \delta_{(n)} F_{(n)}, \quad (11.7.21)$$

where $\delta_{(n)}$ is a variational parameter and $F_{(n)} = F_{(n)}(x)$ is a suitably chosen variational function. Thus, using (11.7.21) we find the residual is

$$r_{(n)} = r_{(n-1)} - \delta_{(n)} KF_{(n)}, \quad (11.7.22)$$

and

$$[E_{(n)}]^2 = [E_{(n-1)}]^2 - 2 \Re \left\{ \delta_{(n)} A_{(n)} \right\} + |\delta_{(n)}|^2 B_{(n)}, \quad (11.7.23)$$

where

$$A_{(n)} = \langle \overline{r_{(n-1)}}, KF_{(n)} \rangle, \quad (11.7.24)$$

$$B_{(n)} = \langle \overline{KF_{(n)}}, KF_{(n)} \rangle = \|KF_{(n)}\|^2. \quad (11.7.25)$$

Hence, from (11.7.23) we get

$$[E_{(n)}]^2 = [E_{(n-1)}]^2 - \frac{|A_{(n)}|}{B_{(n)}} + B_{(n)} \left| \delta_{(n)} - \frac{\overline{A_{(n)}}}{B_{(n)}} \right|^2. \quad (11.7.26)$$

The right side of (11.7.26), taken as a function of $\delta_{(n)}$, has a minimum at $\delta_{(n)} = \frac{\overline{A_{(n)}}}{B_{(n)}}$. Substituting it in (11.7.26) gives

$$[E_{(n)}]^2 = [E_{(n-1)}]^2 - \frac{|A_{(n)}|^2}{B_{(n)}}, \quad (11.7.27)$$

which implies that if we impose the restriction $A_{(n)} \neq 0$, we get an improvement in the solution of the FK1. But this restriction limits our choice of $F_{(n)}$. If we substitute the above minimum value of $\delta_{(n)}$ into (11.7.22) and use (11.7.24) and (11.7.25), the minimization procedure leads to

$$\langle \overline{r_{(n)}}, KF_{(n)} \rangle = 0, \quad (11.7.28)$$

which is an orthogonality property on D . The dependence of $A_{(n)}$ on $F_{(n)}$ is obtained by interchanging the integration in the inner product and operator definitions, which gives

$$A_{(n)} = \langle K^T \overline{r_{(n-1)}}, F_{(n)} \rangle, \quad (11.7.29)$$

where the transposed integral operator K^T is defined by

$$(K^T \phi)(x) = \int_{s \in D} k(s, x) \phi(s) ds.$$

In the case when K is a matrix, we take the transposed matrix form of K under the integral sign. Thus, property (11.7.28) can be written as

$$\langle K^T \overline{r(n)}, F_{(n)} \rangle = 0. \quad (11.7.30)$$

For the second minimization step we decrease the right side of (11.7.27) by minimizing $B_{(n)}$ while keeping $A_{(n)}$ fixed. In view of (11.7.30) if we replace n by $n - 1$, the value of $A_{(n)}$ remains unchanged if the function $F_{(n)}$ on the right side of (11.7.29) is replaced by $F_{(n)} - G_{(n)} F_{(n-1)}$, where $G_{(n)}$ is a second variational parameter. This replacement yields a new value $\widehat{B}_{(n)}$ of $B_{(n)}$. Thus, from (11.7.25)

$$\begin{aligned} \widehat{B}_{(n)} &= \left\langle \overline{K(F_{(n)} - G_{(n)} F_{(n-1)})}, K(F_{(n)} - G_{(n)} F_{(n-1)}) \right\rangle \\ &= \widehat{B}_{(n)} - 2\Re \{G_{(n)} C_{(n)}\} + |G_{(n)}|^2 B_{(n-1)} \\ &= B_{(n)} - \frac{|C_{(n)}|^2}{B_{(n-1)}} + B_{(n-1)} \left| G_{(n)} - \frac{\overline{C_{(n)}}}{B_{(n-1)}} \right|^2, \end{aligned} \quad (11.7.31)$$

where

$$C_{(n)} = \langle \overline{KF_{(n)}}, KF_{(n-1)} \rangle.$$

The right side of (11.7.31), as a function of $G_{(n)}$, has a minimum at $G_{(n)} = \frac{\overline{C_{(n)}}}{B_{(n-1)}}$. Substituting it in (11.7.31) gives

$$\widehat{B}_{(n)} = B_{(n)} - \frac{|C_{(n)}|^2}{B_{(n-1)}}, \quad (11.7.32)$$

which implies that $\widehat{B}_{(n)} < B_{(n)}$ if $C_{(n)} \neq 0$. Also, if we substitute this minimum into

$$KF_{(n)} - G_{(n)} KF_{(n-1)} \equiv K\widehat{F}_{(n)}, \quad (11.7.33)$$

we find that

$$\left\langle \overline{K\widehat{F}_{(n)}}, KF_{(n-1)} \right\rangle = 0. \quad (11.7.34)$$

Note that the improvement in the iterative scheme depends on the choice of $F_{(n)}$. Thus, if the original $F_{(n)}$ is chosen such that $C_{(n)} = 0$, then this second minimization step will be useless and will automatically stop after being executed once. But if this minimization continues, the resulting error decreases and is given by

$$[E_{(n)}]^2 = [E_{(n)}]^2 - \frac{|A_{(n)}|^2}{B_{(n)} - \frac{|C_{(n)}|^2}{B_{(n-1)}}}. \quad (11.7.35)$$

Once $C_{(n)}$ is determined, we replace $F_{(n)}$ by $\widehat{F}_{(n)}$, where

$$\widehat{F}_{(n)} = F_{(n)} - C_{(n)} F_{(n-1)}. \quad (11.7.36)$$

Note that the second minimization step can be executed for $n \geq 2$ since $F_{(0)}$ is not defined. For a given set of functions $F_{(n)}$ each of the two minimization steps is presented as ALGORITHM 11.7.1 and ALGORITHM 11.7.2.

Once $\widehat{F}_{(n)}$ is defined as in (11.7.35), the orthogonality property (11.7.28) becomes

$$\langle \overline{\widehat{r}_{(n)}}, K\widehat{F}_{(n)} \rangle = 0, \quad (11.7.37)$$

where

$$\widehat{r}_{(n)} = r_{(n-1)} - \frac{\overline{A_{(n)}}}{\overline{B_{(n)}}} K\widehat{F}_{(n)}. \quad (11.7.38)$$

If we multiply the complex conjugate of Eq (11.7.38) by $KF_{(n-1)}$ and integrate over D , using the orthogonality property (11.7.28) where n is replaced by $n-1$, we obtain

$$\langle \overline{\widehat{r}_{(n)}}, KF_{(n-1)} \rangle = 0. \quad (11.7.39)$$

Then, after substituting (11.7.33) into (11.7.37) and using (11.7.39), we find that $\langle \overline{r_{(n)}}, KF_{(n)} \rangle = 0$, or

$$\langle K^T \overline{\widehat{r}_{(n)}}, F_{(n)} \rangle = 0, \quad (11.7.40)$$

which is an improved form of the orthogonality property (11.7.30).

Since we require that $A_{(n)} \neq 0$ in every iteration, we find from (11.7.29) that

$$F_{(n)} = K^T \overline{\widehat{r}_{(n-1)}}, \quad (11.7.41)$$

which is known as the gradient method (GM). But if we apply the method of steepest descent to the iterative solution of the linear system of algebraic equations

(see Sarkar, Starkiewicz, and Stratton 1981), we substitute (11.7.41) into (11.7.29) and obtain

$$A_{(n)} = \langle \overline{F_{(n)}}, F_{(n)} \rangle. \quad (11.7.42)$$

Note that after executing the second minimization step, the variational function $F_{(n)}$ in the following iteration is given by

$$\overline{F_{(n+1)}} = K^T \widehat{r}_{(n)}, \quad (11.7.43)$$

where $\widehat{r}_{(n)}$ is defined in (11.7.38). Since, in view of (11.7.40) and (11.7.43)

$$\langle \overline{F_{(n+1)}}, F_{(n)} \rangle = 0, \quad (11.7.44)$$

which defines the orthogonality property between two subsequent iterations, we find from (11.7.41) that

$$\langle \overline{KF_{(n+1)}}, r_{(n-1)} \rangle = 0. \quad (11.7.45)$$

This property is used to simplify $G_{(n)}$ as follows: In the expression $C_{(n+1)} = \langle \overline{KF_{(n+1)}}, K\widehat{F}_{(n)} \rangle$ substitute the value of $K\widehat{F}_{(n)}$ from (11.7.38) and use (11.7.45) and $A_{(n+1)} = \langle \overline{\widehat{r}_{(n)}}, KF_{(n+1)} \rangle$, which yields

$$\begin{aligned} C_{(n+1)} &= -\widehat{B}_{(n)} \frac{\overline{A_{(n+1)}}}{\overline{A_{(n)}}}, \\ G_{(n+1)} &= -\frac{A_{(n+1)}}{A_{(n)}} = \frac{\overline{C_{(n+1)}}}{\overline{\widehat{B}_{(n)}}}. \end{aligned} \quad (11.7.46)$$

Thus, $G_{(n+1)}$ can be computed in advance in the $(n+1)$ st integration provided the value of $A_{(n)}$ in the n th iteration is known. This leads to the conjugate gradient method (CGM), which generates the variational functions as

$$\begin{aligned} F_{(1)} &= \overline{K^T} r_{(0)}, \\ F_{(n)} &= \overline{K^T} r_{(n-1)} + \frac{A_{(n)}}{A_{(n-1)}} F_{(n-1)}, \quad n \geq 2. \end{aligned} \quad (11.7.47)$$

The final iteration scheme is given by (11.7.47) and (11.7.42) and is presented as ALGORITHM 11.7.3.

ALGORITHM 11.7.1

 $n = 0$ Initial guess: $\phi_{(0)}$ Residual error: $r_{(0)} = f - K\phi_{(0)}$ Error: $E_{(0)} = \|r_{(0)}\|$ do for $n = n + 1$ Variational function: $F_{(n)}$ $A_{(n)} = \langle \overline{r_{(n-1)}}, KF_{(n)} \rangle$ $B_{(n)} = \|KF_{(n)}\|^2$

$$\delta_{(n)} = \frac{\overline{A_{(n)}}}{B_{(n)}} .$$

Iterated estimate: $\phi_{(n)} = \phi_{(n-1)} + \delta_{(n)} F_{(n)}$ Residual error: $r_{(n)} = r_{(n-1)} - \delta_{(n)} KF_{(n)}$ Error: $E_{(n)} = \|r_{(n)}\|$

enddo

Storage: $\phi_{(n)}, r_{(n)}$

ALGORITHM 11.7.2

 $n = 0$ Initial guess: $\phi_{(0)}$ Residual error: $r_{(0)} = f - K\phi_{(0)}$ Error: $E_{(0)} = \|r_{(0)}\|$ do for $n = n + 1, n \geq 2$

Second minimization:

$C_{(n)} = \langle \overline{KF_{(n)}}, KF_{(n-1)} \rangle$

$G_{(n)} = \frac{\overline{C_{(n)}}}{B_{(n-1)}}$

Set $B_{(n)} := B_{(n)} - G_{(n)} C_{(n)}$ $F_{(n)} := F_{(n)} - C_{(n)} F_{(n-1)}$ $KF_{(n)} := KF_{(n)} - C_{(n)} KF_{(n-1)}$ Iterated estimate: $\phi_{(n)} = \phi_{(n-1)} + \delta_{(n)} F_{(n)}$ Residual error: $r_{(n)} = r_{(n-1)} - \delta_{(n)} KF_{(n)}$ Error: $E_{(n)} = \|r_{(n)}\|$

enddo

Storage: $\phi_{(n)}, r_{(n)}, F_{(n)}, B_{(n)}, [KF_{(n)}]$

ALGORITHM 11.7.3

$n = 0$

Initial guess: $\phi_{(0)}$

Residual error: $r_{(0)} = f - K\phi_{(0)}$

Error: $E_{(0)} = \|r_{(0)}\|$

do for $n = n + 1$, $n \geq 2$

Variational function: $F_{(n)}$

$$A_{(n)} = \|K^T r_{(n-1)}\|^2$$

Gradient Method (GM):

$$F_{(n)} = \overline{K^T} r_{(n-1)}$$

Conjugate Gradient Method (CGM):

$$F_{(1)} = \overline{K^T} r_{(0)} \text{ if } n = 1$$

$$F_{(n)} = \overline{K^T} r_{(n-1)} + \frac{A_{(n)}}{A_{(n-1)}} F_{(n-1)} \text{ if } n \geq 2$$

$$B_{(n)} = \langle \overline{K F_{(n)}}, K F_{(n)} \rangle = \|K F_{(n)}\|^2$$

Iterated estimate: $\phi_{(n)} = \phi_{(n-1)} + \delta_{(n)} F_{(n)}$

Residual error: $r_{(n)} = r_{(n-1)} - \delta_{(n)} K F_{(n)}$

Error: $E_{(n)} = \|r_{(n)}\|$

enddo

Storage for GM: $\phi_{(n)}, r_{(n)}$

Storage for CGM: $\phi_{(n)}, r_{(n)}, A_{(n)}, F_{(n)}$

EXAMPLE 11.7.3. The following specialized applications are presented in van den Berg (1980):

(i) Herman (1981) has studied the problem of scattering of transient acoustic waves in three-dimensional inhomogeneous penetrable finite objects of arbitrary shape formulated in terms of a volume-integral equation over the domain of the scatterer (for a definition of the scatterer, see Kythe 1995). The problem considered by Van den Berg (1980) is about scattering of a Gaussian pulsed plane wave by a cube, where the incident wave propagates with speed c and pulse width $\tau = 3a/c$, and solves Eq (11.7.17) by using Algorithm 11.7.3 (GM). The cube is divided into 96 identical tetrahedra with time step $\Delta t = 0.12 \tau$, and the initial guess as the incident field. The results are presented graphically (Eq (11.7.23) vs. number of iterations up to 12). The CGM is not used because of a huge storage problem.

(ii) Van Onsellen (1980) has formulated the problem of elastic wave motion in a semi-infinite solid, excited by a vibrating disk on its stress-free surface ("vibroseis problem") in terms of Eq (11.7.17). Van den Berg (1981) considers this problem for a unit rectangular plate that vibrates with a frequency of 100 Hz, where the plate is divided into 11×11 subsquares. The solution is obtained by using both GM and CGM, and the results are presented graphically (Eq (11.7.23) vs. number of iterations up to 8). It is found that CGM converges faster than GM.

(iii) Van den Berg (1981) considers the problem of heat generated in a distribution of biological tissue (hyperthermia problem) where the incident field is produced by a 27-MHz cooled rigid waveguide applicator, and the two-dimensional (cross-section) model of the human body is represented by a plane domain-integral equation of the form (11.7.17). As a particular case the cross-section of the human pelvis is divided into 0.005×0.005 m subsquares so as to achieve a total of about 2500 cells. The solution is obtained by using both GM and CGM, and the results are presented graphically (Eq (11.7.23) vs. number of iterations up to 3). ■

REFERENCES USED: Atkinson (1971), Babolian and Delves (1979), Baker (1978), Belward (1985), Caldwell (1994), Delves (1977), Delves and Abd-Elal (1977), Essah and Delves (1989), Fettis (1964), Fox and Goodwin (1953), Golberg and Chen (1997), Groetsch (1984), Hanna and Brown (1991), Hansen (1992), Herman (1981), Kanwal (1997), Kondo (1991), Kress (1989), Kythe (1998), Lavrentiev (1967), Linz (1969, 1971, 1973), Mikhlin and Smolitskiy (1967), Minerbo and Levy (1960), Phillips (1962), Piessens and Verbaeten (1973), Pogorzeleski (1966), Rainville (1960), Sarkar, Starkiewicz, and Stratton (1981), Symm (1969), te Riele (1984), Tikhonov (1963), Trefethen (1986), Tricomi (1957), Twomey (1963), van der Berg (1980), Van Onsellen (1980), Varah (1983), Wahba (1977), Wing (1991).

12

Inversion of Laplace Transforms

The problem of the inversion of Laplace transforms is a special case of integral equations of the first kind on the infinite interval $(0, \infty)$. In these equations the free term $F(s)$ is the Laplace transform of an unknown function $f(t)$, $0 < t < \infty$, where s is the variable of the transform. In this chapter we present different numerical methods for computing the function $f(t)$ since it is known that this problem is ill-posed.

12.1. Laplace Transforms

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \int_0^\infty e^{-st} f(t) dt, \quad (12.1.1)$$

provided the function $F(s)$ exists and $\Re\{s\} > a$, where all the singularities of $F(s)$ lie to the left of $\Re\{s\} > a$. The sufficient conditions for the existence of $F(s)$ are that

- (1) $f(t)$ is piecewise continuous on the interval $0 \leq t \leq T$ for any $T > 0$, and
- (2) $|f(t)| \leq A e^{a_0 t}$ for $T \geq T_0$, where $A > 0$, $a_0 \leq a$, and $T_0 > 0$ are real constants.

Under these conditions $F(s)$ exists for $s > a$.

The problem of obtaining $f(t)$ from a known $F(s)$ is called the inversion of the Laplace transform, which is equivalent to solving the FK1 (12.1.1) for $f(t)$.

However, as is well known, this problem is ill-posed. In general, the problem does not have a solution if $F(s)$ is not analytic for $s > a$.

The analytical solution of Eq (12.1.1) is

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)e^{st} ds, \quad (12.1.2)$$

The proof of (12.1.2) is well known (see Sneddon 1978 or Churchill 1972). Several tables of Laplace transforms, namely, by Erdyli (1954), Oberhettinger and Badii (1973), and Roberts and Kaufman (1966) are available, but these tables often do not contain the needed inverse. Frequently, in real problems $F(s)$ is very complicated and it is not easy to obtain its analytical inverse; sometimes even when the inverse can be found, it is hard to interpret. In such cases the methods of numerical solution of Eq (12.1.1) become very useful.

Some PROPERTIES OF LAPLACE TRANSFORMS are as follows: If $\mathcal{L}f(t) = F(s)$, then

- (a) $\mathcal{L}t^k f(t) = (-1)^k \frac{d^k F(s)}{ds^k}$, and $\mathcal{L}^{-1}s^k F(s) = \frac{d^k f(t)}{dt^k}$.
- (b) $\mathcal{L}e^{-kt} f(t) = F(s+k)$, and $\mathcal{L}^{-1}F(s+k) = e^{-kt} f(t)$.
- (c) $\mathcal{L}kf(kt) = F(s/k)$, and $\mathcal{L}^{-1}F(s/k) = kf(kt)$.
- (d) $\mathcal{L}\frac{d^n f(t)}{dt^n} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)(0)}$.
- (e) $\mathcal{L}H(t-a)f(t) = e^{-as} F(s)$, and $\mathcal{L}^{-1}e^{-as} F(s) = H(t-a)f(t)$, where $H(t-a)$ is the Heaviside step function.
- (f) If $\mathcal{L}g(t) = G(s)$, then $\mathcal{L}^{-1}G(s) F(s) = f \star g = g \star f = \int_0^t f(t-\xi) g(\xi) d\xi$.
- (g) $\lim_{s \rightarrow 0} s F(s) = f(0)$.

If $\Re\{s\} > 0$ in Eq (12.1.1) and all the singularities of $F(s)$ lie to the left of $\Re\{s\} = a$, and if $f(t)$ is of exponential order, then $\lim_{\Re\{s\} \rightarrow \infty} F(s) \rightarrow 0$. This implies that

- (h) $F(s) = s^{-\alpha} \phi(s)$, where $\phi(s)$ is regular and bounded in the half-plane $\Re\{s\} \geq a$, and $\phi(s)$ can be also expanded in polynomials in $1/s$.

There are several basic techniques for a numerical solution of Eq (12.1.2). Of these the earliest are based on the use of orthogonal polynomials and corresponding quadrature formulas (see, e.g., Widder 1934, 1935, Lanczos 1956, Papoulis 1956, Bellman et al. 1966). Some use a Fourier series representation, discretizing the

Bromwich integral (see, e.g., Dubner and Abate 1968, Durbin 1974, Crump 1976, Tzou et al. 1994, Tzou 1997), yet another numerical inversion method is based on representing the indicial function as a sum of exponentials (see, e.g., Papoulis 1956) and on rational fractions or continued fractions (see, e.g., Schlessinger 1968, Schüttler and Scalpino 1986). A bibliography of numerical methods for the inversion of the Laplace transforms was published by Piessens (1975) and Piessens and Dang (1976), and a survey and comparison of inversion techniques was published by Davies and Martin (1979). An excellent presentation of introductory material on numerical inversion of the Laplace transform is available in Krylov and Skoblya (1969).

12.2. General Interpolating Scheme

In view of the fact that $\phi(s)$ in property (h) is regular and bounded in the half-plane $\Re s > 0$, it can be interpolated by a linear combination of a complete system of functions. It is convenient to choose interpolating functions with argument s^{-1} and the interpolating points along the real axis. Thus if $V_k(s^{-1})$ are $n+1$ polynomials of degree n in s^{-1} and we choose tabular points $s = s_0, s_1, \dots, s_n$ to the right of a , then

$$F(s) = s^{-\alpha} \phi(s), \quad \phi(s) = \sum_{k=0}^n \frac{V_k(s^{-1})}{V_k(s_k^{-1})} \phi(s_k) + r_n(s), \quad (12.2.1)$$

where

$$V_k(s^{-1}) = \frac{\prod_{i=0}^n (s^{-1} - s_i^{-1})}{(s^{-1} - s_k^{-1})}. \quad (12.2.2)$$

The polynomials $V_k(s^{-1})$ so defined are the interpolating polynomials for $\phi(s)$ at the tabular points. Using Eqs (12.2.1) and (12.2.2) in Eq (12.1.2), we have

$$f(t) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \left[s^{-\alpha} e^{st} \sum_{k=0}^n \frac{V_k(s^{-1})}{V_k(s_k^{-1})} \phi(s_k) + r_n(s) \right] ds = A_k(t) \phi(s_k) + R_n, \quad (12.2.3)$$

where

$$\begin{aligned} A_k(t) &= \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \left[s^{-\alpha} \frac{V_k(s^{-1})}{V_k(s_k^{-1})} \phi(s_k) e^{st} \right] ds, \\ R_n &= \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} s^{-\alpha} e^{st} r_n(s) ds. \end{aligned} \quad (12.2.4)$$

To evaluate $A_k(t)$, we proceed as follows:

$$\frac{V_k(s^{-1})}{V_k(s_k^{-1})} \phi(s_k) = \sum_{i=0}^k \beta_{ki} s^{-i},$$

where β_{ki} are known constants; thus,

$$A_k(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} [s^{-\alpha} e^{st} \sum_{i=0}^k \beta_{ki} s^{-i}] ds = \sum_0^k \frac{\beta_{ki}}{\Gamma(\alpha + i)} t^{\alpha+i-1}. \quad (12.2.5)$$

We can simplify the calculations by choosing s_i to be equidistant such that $s_i - s_{i-1} = 1$ and by making a horizontal translation along the real axis, it is possible always to choose $s_0 = 1$.

12.2.1 Inverse in Terms of Legendre Polynomials. Papoulis (1956) introduced the logarithmic time scale x defined by

$$e^{-\sigma t} = x, \quad \sigma > 0. \quad (12.2.6)$$

The time interval $[0, \infty)$ is then mapped onto the interval $(0, 1]$ and

$$f(t) = f\left(-\frac{\ln x}{\sigma}\right) = \phi(x). \quad (12.2.7)$$

Then Eq (12.1.1) becomes

$$\sigma F(s) = \int_0^1 x^{(s/\sigma)-1} \phi(x) dx, \quad (12.2.8)$$

from which we obtain

$$\sigma F[(2k+1)\sigma] = \int_0^1 x^{2k} \phi(x) dx. \quad (12.2.9)$$

Using the fact that Legendre polynomials form a complete orthogonal set in the interval $(-1, 1)$, we can express $\phi(x)$ as a series of Legendre polynomials. To do that we extend the definition of $\phi(x)$ to the interval $(-1, 1)$. This is done by defining $\phi(-x) = \phi(x)$. Thus, $\phi(x)$ becomes an even function and, therefore, can be expanded in terms of a series of even Legendre polynomials, i.e.,

$$\phi(x) = \sum_{k=0}^{\infty} C_k P_{2k}(x),$$

hence,

$$f(t) = \sum_{k=0}^{\infty} C_k P_{2k}(e^{-\sigma t}). \quad (12.2.10)$$

To determine C_k , we take the transform of $\psi_{2k}(t) = P_{2k}(e^{-\sigma t})$ and get

$$\Psi_{2k}(s) = \frac{N(s)}{s(s+2\sigma)\dots(s+2k\sigma)}, \quad (12.2.11)$$

where $N(s)$ is polynomial of degree less than $2k$. One the properties of Legendre polynomials is that

$$\int_0^1 x^{2n} P_{2k}(x) dx = 0 \quad \text{for } n < k. \quad (12.2.12)$$

Now, Eqs (12.2.9) and (12.2.12) imply

$$\Psi_{2k}[(2n+1)\sigma] = 0, \quad n = 0, 1, \dots, k-1;$$

therefore, the roots of $N(s)$ are

$$(2n+1)\sigma, \quad n = 0, 1, \dots, k-1,$$

and $\Psi_{2k}(s)$ can be expressed as

$$\Psi_{2k}(s) = \frac{(s-\sigma)(s-3\sigma)\dots[s-(2k-1)\sigma]}{s(s+2\sigma)\dots(s+2k\sigma)} A, \quad (12.2.13)$$

where A is a constant to be determined from the initial-value theorem, which in this case means that

$$\lim_{s \rightarrow \infty} s \Psi_{2k}(s) = A = P_{2k}(1) = 1.$$

Hence, the Laplace transform of $P_{2k}(e^{-\sigma t})$ is

$$\Psi_{2k}(s) = \frac{(s-\sigma)(s-3\sigma)\dots[s-(2k-1)\sigma]}{s(s+2\sigma)\dots(s+2k\sigma)}. \quad (12.2.14)$$

In view of Eq (12.2.14), Eq (12.2.10) yields

$$F(s) = \frac{C_0}{s} + \sum_{k=0}^{\infty} \frac{(s-\sigma)(s-3\sigma)\dots[s-(2k-1)\sigma]}{s(s+2\sigma)\dots(s+2k\sigma)} C_k. \quad (12.2.15)$$

Replacing s by odd multiples of σ in Eq (12.2.15), we obtain equations of the form

$$\sigma F[(2k+1)\sigma] = \sum_{m=0}^k \frac{2^m k!}{(2k+1)(2k+3)\dots(2k+2m+1)(k-m)!} C_m. \quad (12.2.16)$$

We can now solve for any number of coefficients C_k .

12.2.2 Inverse in Terms of Shifted Legendre Polynomials. (Lanczos 1956) Since

$$F(s) = \mathcal{L}f(t) = \int_0^\infty f(t)e^{-st} dt, \quad \xi = e^{-t} \Rightarrow f(t) = \int_0^1 f(-\ln \xi) \xi^{s-1} d\xi,$$

we have

$$F(k+1) = \int_0^1 f(-\ln \xi) \xi^k d\xi = \int_0^1 \phi(\xi) \xi^k d\xi = y_k,$$

where $\phi(s) = f(-\ln s)$. Consider any polynomial

$$p_n(\xi) = \sum_{i=0}^n \alpha_n^i \xi^i = \alpha_n^0 + \alpha_n^1 \xi + \alpha_n^2 \xi^2 + \cdots + \alpha_n^n \xi^n,$$

and note that

$$\int_0^1 \phi(\xi) p_n(\xi) d\xi = \sum_{i=0}^n \alpha_n^i \int_0^1 \phi(\xi) \xi^i d\xi = \sum_{i=0}^n \alpha_n^i y_i = c_n,$$

where c_n are known because α_n^i , the coefficients of the polynomial, and $y_i = F(i+1)$ are known. Now, consider the shifted Legendre polynomials $P_n^*(x)$ on the interval $[0,1]$; recall that $P_n^*(x)$ are obtained from $P_n(\eta)$ by the substitution $x = \frac{\eta+1}{2}$, $\eta = 2x+1$, and $P_n^*(x)$ has the additional advantage that its coefficients are all integers and satisfy the normality condition

$$\int_0^1 [P_n^*(x)]^2 dx = \frac{1}{2n+1}. \quad (12.2.17)$$

Because of the orthogonality of $P_n^*(x)$, any well-behaved function can be expanded into a series of shifted Legendre polynomials. Thus, we can write

$$\phi(x) = \sum_{n=0}^{\infty} A_n P_n^*(x), \quad A_n = (2n+1) \int_0^1 \phi(x) P_n^*(x) dx = (2n+1)c_n. \quad (12.2.18)$$

We will write the inverse of $F(s)$ in terms of shifted Legendre polynomials, as follows: Since

$$\phi(x) = f(-\ln x) = \sum_{n=0}^{\infty} (2n+1)c_n P_n^*(x),$$

we have

$$f(t) = \sum_{n=0}^{\infty} (2n+1)c_n P_n^*(e^{-t}), \quad (12.2.19)$$

where $c_n = \sum_{i=0}^n \alpha_n^i y_i$, and α_n^i are the integer coefficients of $P_n^*(x)$.

The drawbacks to this form of inverse are that the terms $2k+1$ and α_n^i are large and that even a small variation in c_n results in a large variation in the value of $f(t)$. For example, the coefficient of x^7 in the expression for $P_n^*(x)$ is -2333760 . In general, we would be required to evaluate c_n to well over 10 places beyond the decimal, and even then it would be difficult to consider more than about 10 terms in the series in Eq (12.2.19).

12.2.3. Method of Bellman, Kalaba, and Lockett. We consider an application of Gaussian quadrature in the numerical inversion of Laplace transforms, as developed by Bellman, Kalaba, and Lockett (1966). Since we are required to evaluate $f(t)$ for a given value of t and known $F(s)$ from the formula (12.1.1), the substitution

$$\tau = e^{-t}, t = -\ln \tau, \quad dt = -\frac{1}{\tau} d\tau \quad (12.2.20)$$

into the formula (12.1.1) gives

$$\int_0^1 \tau^{s-1} f(-\log \tau) d\tau = F(s).$$

Let

$$g(\tau) = f(-\log \tau).$$

Then applying (12.2.20), we have

$$F(s) = \int_0^1 \tau^{s-1} g(\tau) d\tau = \sum_{k=1}^N w'_k \tau_k^{s-1} g(\tau_k). \quad (12.2.21)$$

We define $c_k = g(\tau_k)$, and (12.2.21) reduces to

$$F(s) = \sum_{k=1}^N w'_k \tau_k^{s-1} c_k. \quad (12.2.22)$$

Since w'_k , τ_k^{s-1} and $F(s)$ are known, Eq (12.2.22) can be reduced to a system of N equations by choosing N values for s . Let $s = n + 1$, $n = 0, 1, 2, \dots, N - 1$; then

$$F(n + 1) = \sum_{k=1}^N w'_k \tau_k^n c_k, \quad (12.2.23)$$

where by solving the system (12.2.23) for c_k we obtain the values of $g(\tau)$ for N points. From this data $g(\tau)$ can be approximated by a polynomial of degree $N - 1$. The difficulty arises in the inversion of the matrix $\{w'_k \tau_k^n\}$, which is often ill-conditioned.

12.2.4. Solution of the System. The system (12.2.23) has a solution of the type

$$c_k = g(\tau_k) = \sum_{n=0}^{N-1} \alpha_{kn} F(n + 1).$$

To solve the system of equations (12.2.23) we proceed as follows: Let $y_k = w'_k g(\tau_k) = w'_k c_k$; then the system (12.2.23) reduces to

$$\sum_{k=1}^N \tau_k^{i-1} y_k = a_i = F(i). \quad (12.2.24)$$

Let the matrix of the system have the elements $m_{ik} = \tau_k^{i-1}$, $i = 1, 2, \dots, N$. Multiply both sides of the system (12.2.24) by q_i and add them to obtain

$$\sum_{k=1}^N y_k \left(\sum_{i=1}^N m_{ik} q_i \right) = \sum_{i=1}^N a_i q_i, \quad (12.2.25)$$

where q_i are yet to be determined. If we define $l(\tau) = \sum_{k=1}^N q_k \tau^{k-1}$, the system (12.2.25) becomes

$$\sum_{k=1}^N y_k l(\tau_k) = \sum_{i=1}^N a_i q_i. \quad (12.2.26)$$

The idea is to choose q_i in such a way that only one of the unknowns y_k remains in Eq (12.2.26) and the coefficients of the other unknowns become zero. To carry out this computation, we need to choose N values of the set q_i . Let these values be denoted by q_{ij} . Then, we define

$$l_j(\tau) = \sum_{k=1}^N q_{kj} \tau^k,$$

and choose q_{kj} such that

$$l_j(\tau_k) = \begin{cases} 0 & \text{if } k \neq j, \\ 1 & \text{if } k = j. \end{cases}$$

It is possible to determine such q_{kj} , because $l_j(\tau)$ is a polynomial of degree $N - 1$ and therefore, satisfies N conditions. Having determined q_{kj} , we find from

$$(12.2.26) \text{ that } y_i = \sum_{k=1}^N a_k q_{ki}, \text{ or}$$

$$g(\tau_i) = \sum_{k=1}^N \frac{1}{w'_i} a_k q_{ki}.$$

It turns out that $l_j(\tau)$ is a Lagrange interpolating polynomial given by

$$l_j(\tau) = \frac{P_N^*(\tau)}{(\tau - \tau_j)P'_N(\tau_j)}.$$

For numerical implementation we will use the ninth-order Legendre polynomial and two simple functions $\sin t$ and e^{-t} with Laplace transforms $\frac{1}{s^2 + 1}$ and $\frac{1}{s + 1}$. Zeros x_i for the shifted polynomial are 0.01592, 0.0819845, 0.193314, 0.337874, 0.5, 0.6621265, 0.806686, 0.9180155, 0.98408, and the modified weights $w_i = w_{10-i}$ are 0.0406376, 0.0903246, 0.1303056, 0.1561712, 0.1651198.

EXAMPLE 12.2.1. $F(s) = \frac{1}{s^2 + 1}$, $f(t) = \sin t$.
In this case the set of equations (12.2.23) is given by

$$\begin{aligned} F(n+1) &= (0.0406376)(0.01592)^n c_1 + (0.0903246)(0.0819845)^n c_2 \\ &\quad + (0.1303056)(0.193314)^n c_3 + (0.1561712)(0.337874)^n c_4 \\ &\quad + (0.1651198)(0.5)^n c_5 + (0.1561712)(0.6621265)^n c_6 \\ &\quad + (0.1303056)(0.806686)^n c_7 + (0.0903246)(0.9180155)^n c_8 \\ &\quad + (0.0406376)(0.98408)^n c_9 \quad \text{for } n = 0, 1, \dots, 8. \end{aligned}$$

The solution for c_i is

$$\{-0.555427, 0.531849, 1.0262, 0.867557, 0.650197, 0.392641, 0.219162, 0.0810746, 0.018805\}.$$

Thus, the function $f(t) = f(-\ln x) = g(x)$, $g(x_1) = c_i$ is known at the nine points x_1 . Further, if we assume that

$$g(x) = \sum_{k=1}^9 B_k x^{k-1},$$

then we can obtain nine equations by letting $x = x_i$ for $i = 1, 2, \dots, 9$, and solve for B_k . We find that B_k are given by

$$\{ -0.5554273121669501, 0.5318493617139866, 1.0262029177215746, \\ 0.8675569112458106, 0.6501966073073655, 0.39264073646123254, \\ 0.21916193172192688, 0.08107463934338578, 0.018805014457441416 \}.$$

Now, $f(t)$ can be expressed as

$$\begin{aligned} f(t) = & -0.953120649827127 + 26.85593411635183e^{-t} \\ & - 120.02973389488155e^{-2t} + 133.11599313846204e^{-3t} \\ & + 511.05194566693484e^{-4t} - 1962.8192889883146e^{-5t} \quad (12.2.27) \\ & + 2790.690732420141e^{-6t} - 1864.0777367913424e^{-7t} \\ & + 486.19656491268586e^{-8t}. \end{aligned}$$

Table 12.2.1 gives the values of $\sin t$ and the approximate values obtained from (12.2.27). For computational details see `bellman.nb`.

Table 12.2.1

t	$f(t)$	$\tilde{f}(t)$
0.0	0.0	0.0312
0.5	0.479426	0.48193
1.0	0.841471	0.825406
1.5	0.997495	1.01933
2.0	0.909297	0.910953
10.0	-0.544021	-0.951902

Clearly, the values are not very accurate in comparison with other methods that we discuss in the sequel. However, for small t the error is between 1% to 3%, but for large t the method failed. The next example discusses a much simpler function.

EXAMPLE 12.2.2. $F(s) = \frac{1}{s+1}$, $f(t) = e^{-t}$.

Following the the same procedure as in Example 12.2.1, the approximation for e^{-t} corresponding to (12.2.45) is

$$\begin{aligned} f(t) = & -(5.400809667158862)10^{-6} + 1.000508184397117 e^{-t} \\ & - 0.011280122194696447 e^{-2t} + 0.09671775139664975 e^{-3t} \\ & - 0.399801991834753 e^{-4t} + 0.8833083132385336 e^{-5t} \\ & - 1.067983373238557 e^{-6t} + 0.6661710928983806 e^{-7t} \\ & - 0.1676435041570735 e^{-8t}. \end{aligned} \tag{12.2.28}$$

Table 12.2.2 gives the values of e^{-t} and the approximate values obtained from (12.2.28). For computational details, see `bellman.nb`.

Table 12.2.2

t	$f(t)$	$\tilde{f}(t)$
0	1.0	0.999991
1	0.367879	0.367883
2	0.135335	0.135332
3	0.0497871	0.0497887
4	0.0183156	0.0183163

The results for this example are reasonably accurate. This method has been improved upon by several authors, e.g., Felts and Cook (1969). First, they use the transformation $t = -\ln(x/T)$ to reduce the range of the integral for $F(s)$ to $(0, T)$. Next, they divide this range into two parts: $(0, T/2)$ and $(T/2, T)$. Then they again transform the variable of integration to reduce the range of each integral to $(0, 1)$. Finally, they apply Gaussian quadrature using the shifted Legendre polynomials. Piessens (1969) also improves upon the method of Bellman et al. (1966). He uses a different set of orthogonal polynomials, namely $\phi_k^{(N)}(x)$, for his quadrature formula, where $\phi_k^{(N)}(x)$ are given by

$$\phi_k^{(N)}(x) = (-1)^{N+1} \frac{(2N+1)!}{((N_1)!)^2} P_{N-1}^*(x).$$

12.2.5. Inverse in Terms of Laguerre Polynomials. The work of several authors is presented in this section.

(a) LANCZOS (1956): First, normalize both $f(t)$ and $F(s)$ as follows: Shift the complex variable s by the substitution $s = z + a$ and then multiply both sides of Eq (12.1.2) by e^{-at} ; it becomes

$$\phi(t) = f(t)e^{-at} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} F(z+a)e^{zt} dz = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \Phi(z)e^{zt} dz, \quad (12.2.29)$$

and

$$\Phi(z) = \int_0^\infty \phi(t)e^{-zt} dt. \quad (12.2.30)$$

We assume that $\phi(0) = 0$; if it is not, we replace it by $\phi^*(t) = \phi(t) - \alpha e^{-t}$, where $\alpha = \phi(0) = \lim_{z \rightarrow \infty} z F(z)$. The asterisk shall be suppressed in the sequel. Now, $\Phi(z)$ is analytic in the right half-plane including the imaginary axis. We map the entire right half-plane into the inside of a unit circle by the transformation

$$z = \frac{1-v}{1+v} = -1 + \frac{2}{1+v}, \quad dz = -\frac{2dv}{(1+v)^2} = -\frac{(z+1)^2}{2} dv,$$

and the imaginary axis becomes the unit circle. Then Eq (12.2.29) becomes

$$\phi(t) = \frac{1}{2i\pi} \int_{-1}^1 \Phi\left(\frac{1-v}{1+v}\right) e^{(1-v)t/(1+v)} \frac{2}{(1+v)^2} dv.$$

We know that $\frac{2}{(1+v)^2} \Phi\left(\frac{1-v}{1+v}\right)$ is analytic inside and on the unit circle and therefore has Maclaurin's expansion; thus,

$$\Phi\left(\frac{1-v}{1+v}\right) \frac{2}{(1+v)^2} = \sum_{k=0}^{\infty} C_k v^k,$$

or

$$\frac{(z+1)^2}{2} \Phi(z) = \sum_{k=0}^{\infty} C_k \left(\frac{1-z}{1+z}\right)^k,$$

or

$$\Phi(z) = \sum_{k=0}^{\infty} 2C_k \frac{(1-z)^k}{(1+z)^{k+2}}.$$

Let $z = 1 + z_1$; then

$$\Phi(1 + z_1) = \Psi(z_1) = \sum_{k=0}^{\infty} 2C_k \frac{(-1)^k z_1^k}{(2 + z_1)^{k+2}}.$$

Note that

$$\Phi(1 + z_1) = \int_0^\infty \phi(t)e^{-(1+z_1)t} dt = \int_0^\infty \phi(t)e^{-t}e^{-z_1 t} dt = \Psi(z_1).$$

Thus, the problem reduces to finding the Laplace inverse of $\Psi(z_1)$. Having done that, we can find the Laplace inverse of $\Phi(z)$. Now, consider the term $2\frac{(-1)^k z_1^k}{(2+z_1)^{k+2}}$. Using properties (a) and (b) of the Laplace transforms, we find that

the Laplace inverse of $\frac{1}{(2+z_1)^{k+2}}$ is $\frac{t^{k+1}e^{-2t}}{(k+1)!}$; therefore, the Laplace inverse of $2\frac{(-1)^k z_1^k}{(2+z_1)^{k+2}}$ is $\frac{d^k(\frac{t^{k+1}e^{-2t}}{(k+1)!})}{dt^k}$. The Laguerre polynomials are defined as

$$L_n(t) = e^t \frac{d^n(t^n e^{-t})}{dt^n}; \quad (12.2.31)$$

hence,

$$L_n(2t) = e^{2t} \frac{d^n(t^n e^{-2t})}{dt^n}.$$

Thus, $\mathcal{L}\frac{e^{-2t} L_n(2t)}{n!} = \frac{z_1^n}{(2+z_1)^{n+1}}$, and if $g(t) = e^{-2t} \left[\frac{L_k(2t)}{k!} - \frac{L_{k+1}(2t)}{(k+1)!} \right]$, then $G(z_1) = \frac{z_1^k}{(2+z_1)^{k+1}} - \frac{z_1^{k+1}}{(2+z_1)^{k+2}} = \frac{2z_1^k}{(2+z_1)^{k+2}}$. Going back to the original variable z , we have

$$\mathcal{L}^{-1} \left[\frac{2(1-z)^k}{(1+z)^{k+2}} \right] = (-1)^k e^{-t} \left[\frac{L_k(2t)}{k!} - \frac{L_{k+1}(2t)}{(k+1)!} \right]. \quad (12.2.32)$$

Introducing the function

$$\phi_k(t) = \frac{e^{-t/2} L_k(t)}{k!},$$

we can now write Eq (12.2.32) as

$$\mathcal{L}^{-1} \left[\frac{2(1-z)^k}{(1+z)^{k+2}} \right] = (-1)^k [\phi_k(2t) - \phi_{k+1}(2t)].$$

Hence, we have recovered the function $\phi(t)$, and it is given by

$$\phi(t) = \sum_0^\infty (-1)^k C_k [\phi_k(2t) - \phi_{k+1}(2t)]. \quad (12.2.33)$$

The original $\phi(t)$ can now be obtained by adding αe^{-t} to the right side of Eq (12.2.33).

(b) PAPOULIS (1956): The generating formula for Laguerre polynomials is

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^{-n} e^{-t}).$$

If we define

$$\phi_n(t) = e^{-t} L_n(t),$$

then the Laplace transform $\Phi_n(s)$ of $\phi_n(t)$ is given by

$$\Phi_n(s) \equiv \int_0^\infty \phi_n(t) e^{-st} dt = \frac{s^n}{(s+1)^{n+1}}. \quad (12.2.34)$$

It is clear from Eq (12.2.34) that the derivatives of $\Phi_n(s)$ of order less than n vanish at $s = 0$; so we must have

$$\begin{aligned} \left. \frac{d^k \Phi_n(s)}{ds^k} \right|_{s=0} &= (-1)^k \int_0^\infty t^k \phi_n(t) e^{-st} dt \Big|_{s=0} \\ &= (-1)^k \int_0^\infty t^k \phi_n(t) dt = 0, \quad k \leq n-1. \end{aligned}$$

If we express $f(t)$ as

$$f(t) = \sum_{k=0}^{\infty} C_k \phi_k(t) \quad (12.2.35)$$

and take the Laplace transform of both sides of Eq (12.2.35), we have

$$F(s) = \sum_{k=0}^{\infty} C_k \frac{s^k}{(s+1)^{k+1}}. \quad (12.2.36)$$

Expanding the right side of Eq (12.2.36) into a power series in s , we get

$$F(s) = \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} (-1)^n \binom{n+k}{k} s^{k+n} \right] C_k. \quad (12.2.37)$$

Expanding $F(s)$ into a power series at $s = 0$, we have

$$F(s) = \sum_{k=0}^{\infty} a_k s^k. \quad (12.2.38)$$

Comparing Eqs (12.2.37) and (12.2.38) and equating equal powers of s on the right sides, we find that

$$a_k = \sum_{n=0}^k (-1)^{k-n} \binom{k}{k-n} C_n. \quad (12.2.39)$$

The system of Eqs (12.2.39) can be easily solved to give

$$C_k = \sum_{j=0}^k \binom{k}{j} a_{k-j}.$$

Since a_k are known, one can find any finite number of the coefficients C_k , thereby determining $f(t)$. If $f(t)$ is approximated by $f_N(t) = \sum_{k=0}^N C_k \phi_k(t)$, then the first N derivatives of $f(t)$ and $f_N(t)$ are equal at $t = 0$.

(c) WEEKS (1966): He assumes that the indicial function $f(t)$ satisfies the following integrability conditions:

$$\int_0^\infty e^{-ct} |f(t)| dt < \infty, \quad (12.2.40a)$$

$$\int_0^\infty e^{-2ct} |f(t)|^2 dt < \infty \quad (12.2.40b)$$

whenever $c \geq c_0$. Under these conditions, the integral in Eq (12.1.1) defines a function $F(s)$ which is analytic for $\Re\{s\} > c_0$, and moreover, $f(t)$ and $F(s)$ are related by the Parseval theorem

$$\int_0^\infty e^{-2ct} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{i\infty} |F(c + i\omega)|^2 d\omega. \quad (12.2.41)$$

Now, we define a set of orthonormal polynomials

$$\Phi_n(x) = e^{-x/2} L_n(x), \quad n = 0, 1, 2, \dots,$$

where $L_n(x)$ are Laguerre polynomials defined in Eq (12.2.31). These polynomials satisfy the orthonormality condition

$$\int_0^\infty \Phi_m(x) \Phi_n(x) dx = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n, \end{cases}$$

and $\Phi(0) = 1$. Any function $f(t)$ can be expressed as an infinite series of the form

$$f(t) = e^{ct} \sum_{n=0}^{\infty} a_n \Phi_n \left(\frac{t}{T} \right), \quad a_n = \frac{1}{T} \int_0^{\infty} e^{-ct} f(t) \Phi_n \left(\frac{t}{T} \right) dt. \quad (12.2.42)$$

Denote the partial sum of the first N terms in the series for $f(t)$ by $f_N(t)$. Then $f_N(t)$ approximates $f(t)$ by taking N to be sufficiently large. The decay factor c and scale factor $T > 0$ are chosen to accelerate the convergence. If we define

$$F_n(s) = \int_0^{\infty} f_N(t) e^{-st} dt,$$

then Weeks (1966) has shown that $F_n(s)$ converges to $F(s)$. Note that

$$\begin{aligned} \mathcal{L}e^{ct}\Phi_n\left(\frac{t}{T}\right) &= \mathcal{L}e^{ct-t/(2T)}L_n\left(\frac{t}{T}\right) \\ &= \mathcal{L}e^{ct+t/(2T)}\frac{d^n}{dt^n}[e^{-t/T}t^n] = \frac{(s-c-t/(2T))^n}{(s-c+t/(2T))^{n+1}}. \end{aligned}$$

Hence,

$$\mathcal{L}f_N(t) = F_N(s) = \sum_{n=0}^N a_n \frac{(s-c-t/(2T))^n}{(s-c+t/(2T))^{n+1}}.$$

Let $s = c + i\omega$; then

$$F(c+i\omega) \approx \sum_{n=0}^N a_n \frac{\left(i\omega - \frac{t}{2T}\right)^n}{\left(i\omega + \frac{t}{2T}\right)^{n+1}}. \quad (12.2.43)$$

Changing the variable ω into θ by the relation

$$\omega = \frac{1}{2T} \cot \frac{\theta}{2},$$

Eq (12.2.43) yields

$$\left(\frac{1}{2T} + \frac{i}{2T} \cot \frac{\theta}{2} \right) F \left(c + \frac{i}{2T} \cot \frac{\theta}{2} \right) \approx \sum_{n=0}^N a_n e^{in\theta}. \quad (12.2.44)$$

Now, we introduce the real part $F_r(\sigma, \omega)$ and imaginary part $F_i(\sigma, \omega)$ of $F(\sigma+i\omega)$; then equating the real parts on both sides of Eq (12.2.44), we obtain

$$\begin{aligned} \frac{1}{2T} \left[F_r \left(c, \frac{1}{2T} \cot \frac{\theta}{2} \right) - \frac{1}{2T} \cot \frac{\theta}{2} F_i \left(c, \frac{1}{2T} \cot \frac{\theta}{2} \right) \right] &\approx \Re \left\{ \sum_{n=0}^N a_n e^{in\theta} \right\} \\ &= \sum_{n=0}^N a_n \cos n\theta = h(\theta). \end{aligned} \quad (12.2.45)$$

Using the trigonometric interpolation formulas (see Hildebrand 1956), the values of a_n are given by

$$\begin{aligned} a_0 &= \frac{1}{N+1} \sum_{j=0}^N h(\theta_j), & a_n &= \frac{2}{N+1} \sum_{j=0}^N h(\theta_j) \cos n\theta_j, & n &\neq 0, \\ \theta_j &= \left(\frac{2j+1}{N+1} \right) \frac{\pi}{2}. \end{aligned} \quad (12.2.46)$$

The interpolation formulas in Eq (12.2.46) are similar to the interpolation formulas using Chebyshev polynomials.

It now remains to discuss the scale factors T and c . Although Weeks (1966) establishes the convergence of the series in Eq (12.2.42) for all positive T and for $c > c_0$, it is reasonable to assume that the convergence might be faster for some specific values of T and c . Based on the behavior of Laguerre functions $L_n(x)$, he conjectures that $f_N(t)$ should give a good approximation for $f(t)$ for $0 < t < t_{\max}$, where

$$t_{\max} < 4NT.$$

He found that a satisfactory choice of T is $T = \frac{t_{\max}}{N}$.

Theoretically, the smallest value of c to ensure the convergence of the series in Eq (12.2.42) is $\Re\{s\} = c_0$ such that the rightmost singularity of $F(s)$ lies on the line $\Re\{s\} = c_0$. However, Weeks (1966) determines empirically that a satisfactory choice of c is

$$c = \left(c_0 + \frac{1}{t_{\max}} \right) u \left(c_0 + \frac{1}{t_{\max}} \right), \quad (12.2.47)$$

where $u(x)$ is the Heaviside unit step function. Once a_n are determined, we can approximate $f(t)$ by using Eq (12.2.42).

EXAMPLE 12.2.3. Define

$$\omega_j = \frac{1}{2T} \cot \frac{\theta_j}{2} = \frac{1}{2T} \left(\frac{1 + \cos \theta_j}{\sin \theta_j} \right),$$

and use Eqs (12.2.45) and (12.2.46) and the definition of $f_N(t)$. Table 12.2.3 contains the values for the function $\frac{2}{3}e^{-t/2} \sin \frac{3t}{2}$ obtained from Weeks' algorithm for $N = 30$ (see Weeks 1966) using the transformed function $1/(s^2 + s + 1)$ and the indicial function. The computation was carried out on Mathematica. For

computational details see `weeks.nb`.

Table 12.2.3

t	$f(t)$	$\tilde{f}(t), N = 30$	$\tilde{f}(t), N = 60$	Durbin
0.0	0.0	7.88(-8)	-7.5(-10)	0.002026
0.5	0.353907	0.377345	0.383849	0.377345
1	0.403341	0.533507	0.513614	0.533507
1.5	0.245024	0.525424	0.479157	0.525425
2	0.034610	0.419280	0.358920	0.419279
2.5	-0.109170	0.274110	0.214463	0.274111
3	-0.145411	0.133243	0.086115	0.133242
3.5	-0.099507	0.022128	-0.006185	0.022130
4	-0.025210	-0.049530	-0.058397	-0.049531
4.5	0.031623	-0.083449	-0.076518	-0.083450
5	0.051331	-0.087942	-0.071111	-0.087944
5.5	0.039320	-0.073722	-0.053192	-0.073720
6	0.013679	-0.050892	-0.031747	-0.050895
6.5	-0.008259	-0.027239	-0.012717	-0.027236
7	-0.017710	-0.007644	0.000957	-0.007647
7.5	-0.015174	0.005714	0.008682	0.005718
8	-0.006552	0.012715	0.011356	0.012711
8.5	0.001736	0.014511	0.010544	0.014516
9	0.005953	0.012805	0.007881	0.012780
9.5	0.005731	0.009302	0.004699	0.009308
10.0	0.002921	0.005385	0.001878	0.005380

When we took $c = 0$, our values in column 3 obtained using Weeks (1966) algorithm matched his values exactly; however, our values obtained using the indicial function $f(t)$ did not match his values. A simple test shows that our values are correct. The first negative value for $f(t)$ in our evaluations appears at $t = 2.5$. The sign of the function depends on the sign of $\sin(3t/2)$ where $(3t/2)|_{t=2.5} = 3.75$, which is in the third quadrant and, therefore, the function should be negative. It appears that the second column in Table 1 of Weeks' article was accidentally replaced by his third column, since the two columns are exactly alike. The third column in Table 12.2.3 corresponds to Weeks' algorithm with $N = 60$, and the values are generally closer to the actual values of the function, but not sufficiently accurate. We also note that if we use formula (12.2.47) to determine c , then its value for $t_{\max} = 10$ should be 0.05. If we take this value of c , the values obtained from the algorithm are worse than those obtained for $c = 0$. For

$c = 0$, $N = 30$, the time Mathematica took to produce the table values was about 210 seconds, and for $N = 60$, it was about 747 seconds. Durbin's algorithm (1974, see Section 12.3.4) with $N = 2000$ took about 64 seconds and produced results comparable to the Weeks' algorithm. We tried Durbin's algorithm for $N = 4000$; the evaluation time nearly doubled without any significant improvement in the accuracy. We note here that Durbin's algorithm produced excellent results for the examples that he used in his article.

Piessens' algorithm (1971) for numerical inversion of the Laplace transform using Laguerre functions is as follows:

$$F(s) = s^{-a} \sum_{k=0}^{\infty} c_k L_k(bs^{-1}),$$

Inverting term-by-term, we get

$$f(t) = \frac{t^{a-1}}{\Gamma(a)} \sum_{k=0}^{\infty} c_k \phi_k(bt),$$

where

$$\phi_k(x) = {}_1F_2\left(\begin{matrix} -k \\ 1, a \end{matrix}; x\right).$$

Some other authors who have used Laguerre polynomials are Shohat (1940), Lyness and Giunta (1986), Garbow, Giunta, Lyness, and Murli (1988a, 1988b), and Giunta, Laccetti, and Rizzardi (1988).

12.2.6. Inverse in Terms of Chebyshev Polynomials. The work of several authors is presented in this section.

(a) **Lanczos' Scheme** (Lanczos 1956) First normalize $f(t)$ such that $f(0) = 0$ as was done in Section 12.2.5 for the case of the inverse in terms of the Laguerre polynomials. In this section we denote the normalized function by $f(t)$. Next, we substitute $\xi = e^{-t}$ in Eq (12.1.1) and obtain

$$F(s) = \int_0^1 f(-\ln \xi) \xi^{s-1} d\xi = \int_0^1 \phi(\xi) \xi^{s-1} d\xi,$$

and substituting $\xi = (1 + \cos \theta)/2$, we get

$$f(-\ln \xi) = \phi(\xi) = \phi((1 + \cos \theta)/2) = \psi(\theta).$$

We note that because of the normalization $\psi(0) = 0$ and since $\theta = \pi$ corresponds to $t \rightarrow \infty$, we have $\psi(\pi) = 0$. Hence, $\psi(\theta)$ can be expanded in terms of $\sin k\theta$ as

$$\psi(\theta) = \frac{4}{\pi} \sum_{k=1}^{\infty} b_k \sin k\theta, \quad b_k = \frac{1}{2} \int_0^\pi \psi(\theta) \sin k\theta \, d\theta. \quad (12.2.48)$$

If we transform θ back to ξ , then b_k from Eq (12.2.48) can be expressed as

$$b_k = \frac{1}{2} \int_0^\pi \psi(\theta) \sin k\theta \, d\theta = \int_0^1 \phi(\xi) \frac{\sin k\theta}{\sin \theta} \, d\xi = \int_0^1 \phi(\xi) U_{k-1}(\xi) \, d\xi,$$

where $U_{k-1}(\xi)$ is the Chebyshev polynomial of the second kind and order $k - 1$. The coefficients of these polynomials are integers. If

$$U_k(\xi) = \sum_{i=0}^k \alpha_i \xi^i,$$

then

$$\begin{aligned} b_k &= \int_0^1 \phi(\xi) U_{k-1}(\xi) \, d\xi = \int_0^1 \phi(\xi) \sum_{i=0}^{k-1} \alpha_i \xi^i \, d\xi \\ &= \sum_{i=0}^{k-1} \alpha_i \int_0^1 \phi(\xi) \xi^i \, d\xi = \sum_{i=0}^{k-1} \alpha_i F(i+1). \end{aligned}$$

Thus, b_k can be determined since α_i and $F(i+1)$ are known.

(b) Papoulis' Scheme. Papoulis (1956) also uses the Chebyshev polynomials to evaluate the Laplace inverse numerically. To use his procedure, first normalize $f(t)$ such that $f(0) = 0$; then transform t in Eq (12.1.1) by the substitution

$$e^{-\sigma t} = \cos \theta, \quad t = -\frac{1}{\sigma} \ln \cos \theta, \quad \sigma > 0.$$

Then Eq (12.1.1) becomes

$$\begin{aligned} \sigma F(s) &= \int_0^{\pi/2} f\left(-\frac{1}{\sigma} \ln \cos \theta\right) (\cos \theta)^{(s-\sigma)/\sigma} \sin \theta \, d\theta \\ &= \int_0^{\pi/2} \phi(\theta) (\cos \theta)^{(s-\sigma)/\sigma} \sin \theta \, d\theta; \end{aligned}$$

if $s = (2k+1)\sigma$, $k = 0, 1, 2, \dots$, then

$$\sigma F[(2k+1)\sigma] = \int_0^{\pi/2} \phi(\theta) (\cos \theta)^{2k} \sin \theta \, d\theta. \quad (12.2.49)$$

Since $\phi(0) = 0$, the function $\phi(\theta)$ can be expanded in the interval $(0, \pi/2)$ into a series of odd sine functions:

$$\phi(\theta) = \sum_{k=0}^{\infty} C_k \sin(2k+1)\theta. \quad (12.2.50)$$

The coefficients C_k are determined by expressing $(\cos \theta)^{2n} \sin \theta$ in terms of a series of $\sin(2k+1)\theta$, $k = 0, 1, 2, \dots, n$, as follows:

$$\begin{aligned} (\cos \theta)^{2n} \sin \theta &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right)^{2n} \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right) \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} e^{i(2n-k)\theta} \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right) \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \left[\frac{e^{i(2n-k+1)\theta} - e^{i(2n-k-1)\theta}}{2i} \right] \\ &= \frac{1}{2^{2n}} \sum_{k=0}^n \frac{(2n-2k+1)(2n)!}{k!(2n-k+1)!} \sin(2n-2k+1)\theta. \end{aligned} \quad (12.2.51)$$

From Eq(12.2.49), we get

$$\begin{aligned} \sigma F[(2k+1)\sigma] &= \frac{1}{2^{2n}} \int_0^{\pi/2} \phi(\theta) \sum_{k=0}^n \frac{(2n-2k+1)(2n)!}{k!(2n-k+1)!} \sin(2n-2k+1)\theta d\theta \\ &= \frac{\pi}{4 \cdot 2^{2n}} \sum_{k=0}^n \frac{(2n-2k+1)(2n)!}{k!(2n-k+1)!} C_{n-k} \\ &= \frac{\pi}{4 \cdot 2^{2n}} \sum_{k=0}^n \frac{(2k+1)(2n)!}{(n-k)!(n+k+1)!} C_k, \end{aligned} \quad (12.2.52)$$

where

$$C_k = \frac{4}{\pi} \int_0^{\pi/2} \phi(\theta) \sin(2k+1)d\theta.$$

For a known $F(s)$, we can generate a system of equations for C_k which can be solved for any finite k . This leads to the determination of $\phi(\theta)$ and, hence, of $f(t)$. Using Eq (12.2.50), we have

$$f(t) = \phi(\cos^{-1}(e^{-\sigma t})) = \sum_{k=0}^{\infty} C_k \sin[(2k+1)\cos^{-1}(e^{-\sigma t})],$$

which can be expressed in terms of Chebyshev polynomials of the second kind. Note that

$$\frac{\sin(n+1)\theta}{\sin \theta} = U_n(\cos \theta),$$

where $U_n(x)$ are Chebyshev polynomials of order n and $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - e^{-2\sigma t}}$. Then from Eq (12.2.50), we get

$$f(t) = \sqrt{1 - e^{-2\sigma t}} \sum_{k=0}^{\infty} C_k U_{2k}(e^{-\sigma t}).$$

Since all C_k can be determined, $f(t)$ can be determined uniquely.

Many other authors have used orthogonal polynomials to numerically invert the Laplace transform. Shohat (1940) uses the Laguerre polynomials; Salzer (1955) uses the generalized Bessel polynomials discovered by Krall and Frink (1949); Piessen (1969b) improves upon Salzer's work and gives a new quadrature formula for the inversion of the Laplace transform; Piessen (1971) also develops another quadrature formula using the generalized Bessel polynomials. These investigations are for a class of functions $F(s)$ that can be approximated by polynomials in $1/s$.

Luke (1968) in his review of the book by Bellman et al (1966) suggests the use of the shifted Jacobi polynomials to carry out the numerical inversion of the Laplace transform. Piessens (1972), however, is the one to develop the details of such an approach, achieving better accuracy than the previous methods. The main aspects of his method are given below.

12.2.7. Inverse in Terms of Jacobi Polynomials. (Piessen 1972)
Assume that $F(s)$ can be represented by the following series:

$$F(s) = s^{-\alpha} \sum_{k=0}^{\infty} c_k P_k^{(\alpha, \beta)}(1 - bs^{-1}), \quad (12.2.53)$$

where $P_k^{(\alpha, \beta)}$ is the Jacobi polynomial of degree k and α, β , and b are free parameters. Their choice will be discussed later. The coefficients c_k are given by

$$c_k = \frac{1}{h_k} \int_{-1}^1 (1-u)^\alpha (1+u)^\beta P_k^{(\alpha, \beta)}(u) \Psi(u) du, \quad (12.2.54)$$

where

$$\Psi(u) = \left(\frac{b}{1-u} \right)^\alpha F \left(\frac{b}{1-u} \right)$$

and

$$h_k = \int_{-1}^1 (1-u)^\alpha (1+u)^\beta [P_k^{(\alpha,\beta)}(u)]^2 du.$$

Inverting the series (12.2.53) term-by-term, we find that

$$f(t) = \frac{t^{a-1}}{\Gamma(a)} \sum_{k=0}^{\infty} c_k \frac{(\alpha+1)_k}{k!} \phi_k(bt/2), \quad (12.2.55)$$

where $\phi_k(x)$ is a polynomial of degree k given by

$$\phi_k(x) = {}_2F_2\left(\begin{matrix} -k, k+\alpha+\beta+1 \\ \alpha+1, a \end{matrix}; x\right). \quad (12.2.56)$$

To evaluate numerical values of c_k , we note that if the series (12.2.53) is truncated after $M+1$ terms, we get a least-squares approximation of $F(s)$ over the interval $[b/2, \infty]$ with the weight function

$$s^{-\alpha-2}(2-bs^{-1}).$$

EVALUATION OF THE COEFFICIENTS c_k : Using the Gauss–Jacobi quadrature formula, we can evaluate the integral in Eq (12.2.54). Thus, the approximation for $f(t)$ is given by

$$f(t) \approx \frac{t^{a-1}}{\Gamma(a)} \sum_{k=0}^M \left[\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(k+\beta+1)} \phi_k\left(\frac{bt}{2}\right) \right. \\ \left. \times \sum_{j=1}^N V_j P_k^{(\alpha,\beta)}(u_j) \right],$$

where N is the order of the Gauss–Jacobi quadrature formula, u_j are the abscissas, $V_j = w_j \Psi(u_j)$, w_j are the weights of the Gauss–Jacobi quadrature formula, and M is the value of k at which the series in Eq (12.2.55) is truncated. In the special case of $\alpha = \beta = -0.5$, the computation becomes considerably simple. Eqs (12.2.55) and (12.2.56) reduce to

$$f(t) \approx \frac{t^{a-1}}{\Gamma(a)} \sum_{k=0}^{M'} c_k \phi_k\left(\frac{bt}{2}\right),$$

and

$$\phi_k(x) = {}_2F_2\left(\begin{matrix} -k, k \\ 1/2, a \end{matrix}; x\right).$$

Piessens uses the Clenshaw's method to compute c_k and obtains

$$c_k \approx \frac{2}{N} \sum_{m=0}^{N''} \Psi \left(\cos \frac{m\pi}{N} \right) \cos \left(\frac{mk\pi}{N} \right) \quad (12.2.57)$$

or

$$c_k \approx \frac{2}{N+1} \sum_{m=0}^N \Psi \cos \left(\frac{(2m+1)\pi}{2(N+1)} \right) \cos \left(\frac{(2m+1)k\pi}{2(N+1)} \right), \quad k \leq N. \quad (12.2.58)$$

In the formula (12.2.57) we need the value of $\lim_{s \rightarrow \infty} sF(s)$. If this limit is not known, we use formula (12.2.58). In general, it is convenient to take $\alpha = \beta = -0.5$.

CHOICE OF FREE PARAMETERS: The parameter a must be such that $F(s) \rightarrow \tilde{s}^{-a}$ as $s \rightarrow \infty$. However, there may be functions $F(s)$ for which such an a does not exist.

The choice $\alpha = \beta = -0.5$ simplifies the calculations considerably. These values of α and β are not suitable when the Laplace transform is known in a small interval on the real line. In that case formula (12.2.57) must be used and the value of N must be so low that it satisfies the condition

$$\cos \frac{\pi}{2(N+1)} \leq 1 - \frac{b}{A},$$

where $[0, A]$ is the interval in which $F(s)$ is known. This restricts the number of coefficients c_k that can be calculated. The problem can be avoided by taking α large and $\beta \approx 1$.

The value of b is related to the interval of convergence on the real line for the series in (12.2.53). The minimum value of $\Re\{s\}$ for which the series in (12.2.53) is convergent is $b/2$.

Piessens evaluates the polynomials $\phi_k(x)$ and gives a generating formula for these polynomials. Hypergeometric functions are available in Mathematica; see Piessens.nb for details. Finally, the error is given by

$$\epsilon(t) \approx \frac{t^{a-1}}{\Gamma(a)} c_{M+1} \phi_{M+1}(bt/2).$$

12.3. Inversion by Fourier Series

We discuss the methods of Dubner and Abate (1968) and Durbin (1974) in this section.

12.3.1. Dubner and Abate's Method. It has been shown by many researchers that a Fourier series representation of an unknown function $f(t)$ can be used effectively for numerical evaluation of the function $f(t)$. After substituting $s = a + i\omega$, Eq (12.1.1) becomes

$$\int_0^\infty e^{-(a+i\omega)t} f(t) dt = F(s), \quad (12.3.1)$$

or

$$\begin{aligned} \Re\{F(s)\} &= \int_0^\infty e^{-at} f(t) \cos \omega t dt, \\ \Im\{F(s)\} &= - \int_0^\infty e^{-at} f(t) \sin \omega t dt, \end{aligned} \quad (12.3.2)$$

which are the Fourier cosine and sine transform of $f(t)$, respectively. The Fourier inversion formula yields

$$f(t) = \frac{2e^{at}}{\pi} \int_0^\infty \Re\{F(s)\} \cos \omega t d\omega = -\frac{2e^{at}}{\pi} \int_0^\infty \Im\{F(s)\} \sin \omega t d\omega. \quad (12.3.3)$$

In all these relations $a > a_0$, where a_0 is a real constant such that all singularities of $F(s)$ lie to the left of $\Re\{s\} = a_0$.

Now, the problem is to numerically evaluate either of the integrals in (12.3.3). Consider a real function $h(t)$ such that $h(t) = 0$ for $t < 0$. Divide the interval $[0, \infty]$ into subintervals of length T . Then the n th subinterval is given by $(nT, (n+1)T)$. We cover the whole range by letting $n \rightarrow \infty$. Consider $h(t)$ in the interval $((n-1)T, (n+1)T)$ and define an even periodic function $g_n(t)$ of period $2T$ such that

$$g_n(t) = \begin{cases} h(t) & \text{if } nT \leq t \leq (n+1)T, \\ h(2nT - t) & \text{if } (n-1)T \leq t \leq nT. \end{cases} \quad (12.3.4)$$

After replacing t by $nT + t$ on the right side of (12.3.4), we have for even values of n

$$g_n(t) = \begin{cases} h(nT + t) & \text{if } 0 \leq t \leq T, \\ h(nT - t), & \text{if } -T \leq t \leq 0. \end{cases} \quad (12.3.5)$$

Similarly, for odd values of n

$$g_n(t) = \begin{cases} h((n+1)T - t) & \text{if } 0 \leq t \leq T, \\ h((n+1)T + t) & \text{if } -T \leq t \leq 0. \end{cases} \quad (12.3.6)$$

Therefore, the Fourier series for each $g_n(t)$ is given by

$$g_n(t) = \frac{A_{n,0}}{2} + \sum_{k=1}^{\infty} A_{n,k} \cos(\omega_k t), \quad (12.3.7)$$

where $\omega_k = k\pi/T$, and the coefficients $A_{n,k}$ are given by

$$A_{n,k} = \frac{2}{T} \int_0^T h(nT + x) \cos\left(\frac{k\pi x}{T}\right) dx, \quad n = 0, 2, 4, \dots, \quad (12.3.8)$$

and

$$A_{n,k} = \frac{2}{T} \int_0^T h((n+1)T - x) \cos\left(\frac{k\pi x}{T}\right) dx, \quad n = 1, 3, 5, \dots. \quad (12.3.9)$$

Eqs (12.3.8) and (12.3.9) can be combined into a single equation

$$A_{n,k} = \frac{2}{T} \int_{nT}^{(n+1)T} h(x) \cos\left(\frac{k\pi x}{T}\right) dx. \quad (12.3.10)$$

Adding all the $g_n(t)$, we have

$$\sum_{n=0}^{\infty} g_n(t) = \frac{2}{T} \left[\frac{A(\omega_0)}{2} + \sum_{k=1}^{\infty} A(\omega_k) \cos \omega_k t \right], \quad (12.3.11)$$

where

$$A(\omega_k) = \int_0^{\infty} h(t) \cos \omega_k t dt.$$

If we let $h(t) = e^{-at} f(t)$, then Eq (12.3.11) represents a Laplace transform of the real-valued function $f(t)$, where the transform parameter is $s = a + i\omega_k$. This implies that $A(\omega_k) = \Re\{F(s)\}$. After multiplying both sides of Eq (12.3.11) by e^{at} , we get

$$\begin{aligned} \sum_{n=0}^{\infty} e^{at} g_n(t) &= f(t) + \sum_{n=1}^{\infty} e^{at} g_n(t) \\ &= \frac{2e^{at}}{T} \left[\frac{1}{2} \Re\{F(a)\} + \sum_{k=1}^{\infty} \Re\{F(a + i\omega_k)\} \cos \omega_k t \right]. \end{aligned} \quad (12.3.12)$$

Since the first term in the sum on the left side is $e^{at}g_0(t) = f(t)$ in the interval $0 \leq t \leq T$, the rest of the terms $\sum_{n=1}^{\infty} e^{at} g_n(t)$ represent the error. Thus, for the interval $(0, T)$ we can write the error as

$$\begin{aligned} E_1 &= \sum_{n=1}^{\infty} e^{at} g_n(t) = e^{at} \sum_{n=1}^{\infty} [h(2n+t) + h(2n-t)] \\ &= \sum_{n=1}^{\infty} e^{-2anT} [f(2n+t) + e^{2at} f(2n-t)]. \end{aligned} \quad (12.3.13)$$

When the trapezoidal rule is applied to Eq (12.3.3) with step size π/T , we get

$$f(t) \approx \frac{2e^{at}}{T} \left[\frac{1}{2} \Re\{F(a)\} + \sum_1^{\infty} \Re\{F(a + i\omega_k)\} \cos \omega_k t \right], \quad (12.3.14)$$

which is the same as (12.3.12) if the error term is dropped.

The above analysis is, however, important because it yields an error estimate that depends not only on the step size but also on a , which can be chosen appropriately in order to minimize the error. It is shown in Section 12.3.2 that the error can be made reasonably small only for $t \leq T/2$. Thus, formulas (12.3.11) and (12.3.14) will yield the numerical value of the inverse Laplace transform to a desired accuracy in the interval $(0, T/2)$.

12.3.2. Error Analysis. Since a is chosen such that no singularities of $F(s)$ lie to the right of $\Re\{s\} = a$, we assume that no singularities of $F(s)$ lie to the right of the origin. This can always be done by a translation of the origin and an adjustment to $F(s)$. Thus, we assume that $f(t)$ is bounded by some function of the type Ct^m , where C is a constant and m is any nonnegative integer. First, let us assume that $m = 0$; then from Eq (12.3.13) we have

$$\text{error} = E_{11} \leq C e^{a(t-T)} \frac{\cosh at}{\sinh aT}.$$

Clearly, the error will become large as $t \rightarrow T$. It is therefore prudent to limit the use of formula (12.3.14) to the interval $(0, T/2)$. Then the error in this interval is limited to $C e^{-aT/2}$. By a procedure similar to the above we can show that if $f(t) \leq Ct^m$, then the error in the interval $(0, T/2)$ will be bounded by

$$E_{11} \leq C e^{-aT/2} \frac{\cosh aT/2}{\sinh aT} \approx C (1.5T)^m e^{-aT}.$$

12.3.3. Examples. The procedure for numerical evaluation is as follows: After determining the interval of t for which the evaluation is needed, say, $0 \leq t \leq t_1$, take $T = 2t_1$ and determine a from Eq (12.3.17) such that the error is within acceptable limits.

EXAMPLE 12.3.1. Consider $F(s) = \frac{1}{s^2 + s + 1}$ for which the function $f(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin \sqrt{3}t/2$. In Table 12.3.1 we give approximate values from formula (12.3.14) using $F(s)$ and exact values for $f(t)$, with $aT = 8$, $T = 20$, and take the first 2000 terms in formula (12.3.14). For computational details, see `dubabate.nb`.

Table 12.3.1

t	$f(t)$	$\tilde{f}(t)$
0.0	0.0	0.0020
0.5	0.3773	0.3773
1.0	0.5335	0.5335
1.5	0.5254	0.5254
2.0	0.4193	0.4193
10.0	0.0054	0.0054

EXAMPLE 12.3.2. Consider $F(s) = \frac{1}{s^2 + 1}$ for which $f(t) = \sin t$. In Table 12.3.2 we give approximate values from formula (12.3.14) using $F(s)$ and exact values for $f(t)$. If we use the same values of a and T , then the error according to formula (12.3.17) should appear in the fourth place after the decimal. In this table we give approximate values from formula (12.3.14) using $F(s)$ and taking the first 2000 terms and exact values for $f(t)$. For computational details, see `dubabate.nb`.

Table 12.3.2

t	$f(t)$	$\tilde{f}(t)$
0.0	0.0	0.0020
0.25	0.2474	0.2474
0.5	0.4794	0.4794
0.75	0.6816	0.6816
1.0	0.8414	0.8414
1.25	0.9489	0.9489
1.5	0.9974	0.9974
1.75	0.9839	0.9839

Table 12.3.2 Continued

t	$f(t)$	$\tilde{f}(t)$
2.0	0.9092	0.9092
4.0	-0.7568	-0.7568
9.0	0.4121	0.4120
10.0	-0.5440	-0.5443

Thus, we see that the error at the fourth place after the decimal appears near $T/2$. It should be noted that the function considered for this example converges very slowly.

12.3.4. Durbin's Improvement. Durbin (1974) improves upon the above method by constructing an infinite set of odd functions $k_n(t)$ with a period $2T$ defined as follows:

$$k_n(t) = \begin{cases} h(t) & \text{if } nT \leq t \leq (n+1)T, \\ -h(2nT - t) & \text{if } (n-1)T \leq t \leq nT, \end{cases} \quad (12.3.15)$$

for $n = 0, 1, 2, \dots$. Thus, $k_n(t)$ in the intervals $(-T, 0)$, $(0, T)$, and $(T, 2T)$ is given, respectively, by

$$k_n(t) = \begin{cases} -h(nT - t), & \text{if } -T \leq t \leq 0, \\ h(nT + t) & \text{if } 0 \leq t \leq T, \\ -h(nT - t) & \text{if } T \leq t \leq 2T, \end{cases} \quad (12.3.16)$$

for $n = 0, 2, 4, \dots$, and

$$k_n(t) = \begin{cases} h((n+1)T + t) & \text{if } -T \leq t \leq 0, \\ -h((n+1)T - t) & \text{if } 0 \leq t \leq T, \\ h((n-1)T + t) & \text{if } T \leq t \leq 2T, \end{cases} \quad (12.3.17)$$

for $n = 1, 3, 5, \dots$. The Fourier series representation for each $k_n(t)$ is

$$k_n(t) = \sum_{k=0}^{\infty} B_{n,k} \sin \omega_k t, \quad (12.3.18)$$

where

$$B_{n,k} = \int_{nT}^{(n+1)T} e^{-at} f(t) \sin \omega_k t dt. \quad (12.3.19)$$

Summing $B_{n,k}$ in (12.3.19) over n and using (12.3.2), we have

$$\sum_{n=0}^{\infty} B_{n,k} = \frac{2}{T} \int_0^{\infty} e^{-at} f(t) \sin \omega_k t dt = -\frac{2}{T} \Im\{F(a + i\omega_k)\}. \quad (12.3.20)$$

Multiplying both sides of (12.3.20) by $e^{at} \sin \omega_k t$ and using (12.3.18), we get

$$\sum_{n=0}^{\infty} e^{at} k_n(t) = \sum_{n=0}^{\infty} B_{n,k} e^{at} \sin \omega_k t = -\frac{2e^{at}}{T} [\Im\{F(a + i\omega_k)\} \sin \omega_k t]. \quad (12.3.21)$$

Note that (12.3.21) is similar to (12.3.12). As we did for (12.3.12), we let $h(t) = e^{-at} f(t)$ and use the last two representations in (12.3.16) and (12.3.17). Thus, we find that

$$\sum_{n=0}^{\infty} e^{at} k_n(t) = f(t) + \sum_{k=1}^{\infty} e^{-2akT} [f(2kT + t) - e^{2at} f(2kT - t)]. \quad (12.3.22)$$

Therefore, $f(t)$ can be represented as

$$f(t) + \text{Error}(E) = -\frac{2e^{at}}{T} [\Im\{F(a + i\omega_k)\} \sin \omega_k t]. \quad (12.3.23)$$

Adding (12.3.12) and (12.3.23) and dividing by 2, we get

$$\begin{aligned} f(t) &+ \frac{1}{2} (E_1 + E_2) \\ &= \frac{e^{at}}{T} \left[\frac{1}{2} \Re\{F(a)\} + \sum_{k=1}^{\infty} \{\Re\{F(a + i\omega_k)\} \cos \omega_k t \right. \\ &\quad \left. - \Im\{F(a + i\omega_k)\} \sin \omega_k t\} \right]. \end{aligned} \quad (12.3.24)$$

12.3.5. Error Analysis. In the formula (12.3.24) derived by Durbin (1974), the error is the average of the errors from Eqs (12.3.13) and Eq (12.3.23). We denote it by E_3 , and find that

$$E_3 = \frac{1}{2} (E_1 + E_2) = \sum_{k=1}^{\infty} e^{-2akT} f(2kT + t). \quad (12.3.25)$$

If $|f(t)| < C$, then

$$E_3 < \sum_{k=1}^{\infty} C e^{-2akT} = \frac{C e^{-2aT}}{1 - e^{-2aT}} = \frac{C}{e^{2aT} - 1}. \quad (12.3.26)$$

Obviously, the error estimate given by Eq (12.3.26) is superior in that it does not depend on t and actually decreases as aT increases. Moreover, it allows the representation of $f(t)$ by Eq (12.3.24) to be used for the interval $(0, T)$.

In the case when $|f(t)| \leq Ct^m$, Eq (12.3.25) yields

$$|E_3| \leq \sum_{k=1}^{\infty} C e^{-2akT} (t + 2kT)^m \leq C (2T)^m \sum_{k=1}^{\infty} e^{-2akT} (k+1)^m. \quad (12.3.27)$$

By comparing the series on the right of Eq (12.3.27) to the integral

$$\int_1^{\infty} e^{-2axT} (x+1)^m dx,$$

we can show that the error E_3 is bounded by (see Durbin 1974)

$$|E_3| \leq A (2T)^m e^{-2aT} \sum_{k=1}^{m+1} \frac{\alpha_k}{(2aT)^k},$$

where A and α_k are constants.

12.3.6. Examples. To carry out numerical implementation Durbin (1974) takes the average of the formulas (12.3.14) and (12.3.23), thus

$$f(t) \approx \frac{e^{at}}{T} \left[\frac{1}{2} \Re\{F(a)\} + \sum_1^{\infty} \Re\{F(a + i\omega_k)\} \cos \omega_k t - \{\Im\{F(a + i\omega_k)\} \sin \omega_k t\} \right]. \quad (12.3.28)$$

He suggests that T be replaced by $T/2$ in the formula (12.3.14) in order to extend its applicability to the interval $(0, 2T)$. In our view, T should be replaced by $2T$ instead of $T/2$ to cover the interval $(0, 2T)$. In fact, we get more accurate results by following the substitution suggested here. Replacing T by $T/2$ in (12.3.14) and (12.3.28), we get Durbin's Eqs (41) and (42) which are

$$f(t) \approx \frac{4e^{at}}{T} \left[\frac{1}{2} \Re\{F(a)\} + \sum_1^{\infty} \Re\{F(a + i\omega_k)\} \cos 2\omega_k t \right] \quad (12.3.29)$$

and

$$\begin{aligned} f(t) \approx & \frac{2e^{at}}{T} \left[\frac{1}{2} \Re\{F(a)\} \right. \\ & \left. + \sum_1^{\infty} \Re\{F(a + i\omega_k)\} \cos 2\omega_k t - \{\Im\{F(a + i\omega_k)\} \sin 2\omega_k t\} \right], \end{aligned} \quad (12.3.30)$$

respectively. Replacing T by $2T$ in (12.3.14) and (12.3.28), we get

$$f(t) \approx \frac{e^{at}}{T} \left[\frac{1}{2} \Re\{F(a)\} + \sum_1^{\infty} \Re\{F(a + i\omega_k)\} \cos \omega_k t / 2 \right] \quad (12.3.31)$$

and

$$\begin{aligned} f(t) \approx & \frac{e^{at}}{2T} \left[\frac{1}{2} \Re\{F(a)\} \right. \\ & \left. + \sum_1^{\infty} \Re\{F(a + i\omega_k)\} \cos \omega_k t / 2 - \{\Im\{F(a + i\omega_k)\} \sin \omega_k t / 2\} \right], \end{aligned} \quad (12.3.32)$$

respectively. Both Dubner and Abate (1968) and Durbin (1974) proposed the use of Fast Fourier transform (FLIT = fast Laplace inverse transform) for more efficient numerical evaluation. Here we give only Durbin's results for the FLIT. If $f(t)$ is needed at equidistant points, then set

$$t_j = j \Delta t, \quad \Delta t = \frac{jT}{N}, \quad j = 0, 1, 2, \dots, N-1,$$

and define

$$s_k = a + 2i\omega_k, \quad F(s_k) = \Re\{F(s_k)\} + i\Im\{F(s_k)\}. \quad (12.3.33)$$

Hence, we have Durbin's Eq (42) as

$$\begin{aligned} f(t) = E_3 + E_t + E_N = & \frac{e^{2aj\Delta t}}{2T} \left[\frac{1}{2} \Re\{F(a)\} \right. \\ & \left. + \sum_1^N \Re\{F(s_k)\} \cos \omega_k t_j / 2 - \{\Im\{F(s_k)\} \sin 2\omega_k t_j\} \right], \end{aligned}$$

where Eq (12.3.33) is derived by replacing T by $T/2$. However, we find that the correct replacement should be T by $2T$, which yields

$$\begin{aligned} f(t) = E_3 + E_t + E_n \\ = \frac{e^{4aj\Delta t}}{2T} \left[\frac{1}{2} \Re\{F(a)\} + \sum_1^N \Re\{F(s_k)\} \cos 2\omega_k t_j - \{\Im\{F(s_k)\} \sin 2\omega_k t_j\} \right], \end{aligned} \quad (12.3.34)$$

where E_t and E_r represent the truncation and roundoff errors, respectively. Durbin's examples are given below.

EXAMPLE 12.3.3. (Durbin 1974) Consider $F(s) = s(s^2 + 1)^{-2}$ for which $f(t) = 0.5t \sin t$. In Table 12.3.3 the second column corresponds to Dubner and Abate's method (replacing T by $T/2$, $\tilde{f}(t)_1$), the third column to Dubner and Abate's method (replacing T by $2T$, $\tilde{f}(t)_2$), the fourth column to Durbin's (replacing T by $T/2$, $\tilde{f}(t)_3$), the fifth column to Durbin's (replacing T by $2T$, $\tilde{f}(t)_4$), the sixth column to FLIT1 of Durbin (1974), and the seventh column to FLIT2 with T replaced by $2T$, and the last column to the exact solution $f(t)$. We take $T = 20$, $aT = 5$, and $N = 2000$, where N is determined from the error estimates. It will be clear from Table 12.3.3 that the results in all cases when T is replaced by $2T$ are much more accurate than those when T is replaced by $T/2$. We used Mathematica for our evaluations and some of our numerical values from Durbin's (1974, formula (42)), in particular for $t = 1$, are different from those obtained by Durbin. The results obtained using the FLIT correspond to the formula used for the FLIT. For computational details, see `durbin.nb`.

Table 12.3.3

t	$\tilde{f}(t)_1$	$\tilde{f}(t)_2$	$\tilde{f}(t)_3$	$\tilde{f}(t)_4$	FLIT1	FLIT2	Exact
0	0.1243	-1.2(-7)	0.0621	-6.1(-8)	0.0621	-6.1(-8)	0.0
1	0.4970	0.4207	0.4797	0.4207	0.4797	0.4207	0.4207
2	0.7846	0.9092	0.9077	0.9092	0.9077	0.9092	0.9092
3	-0.1038	0.2116	0.1452	0.2116	0.1452	0.2116	0.2116
4	-1.7074	-1.5136	-1.5868	-1.5136	-1.5868	-1.5136	-1.5136
5	-2.0114	-2.3973	-2.4075	-2.3973	-2.4075	-2.3973	-2.3973
6	0.1760	-0.8382	-0.7705	-0.8382	-0.7705	-0.8382	-0.8382
7	3.0208	2.2994	2.3865	2.2994	2.3865	2.2994	2.2994
8	2.8200	3.9574	3.9821	3.9574	3.9821	3.9574	3.9574
9	-1.5723	1.8545	1.7886	1.8545	1.7886	1.8545	1.8545
10	-5.6405	-2.7201	-2.8202	-2.7201	-2.8202	-2.7201	-2.7201
11	-2.5924	-5.4999	-5.5413	-5.4999	-5.5413	-5.4999	-5.4999
12	7.6657	-3.2194	-3.1588	-3.2194	-3.1588	-3.2194	-3.2194
13	13.5386	2.7310	2.8427	2.7310	2.8427	2.7310	2.7310
14	1.3008	6.9342	6.9941	6.9342	6.9941	6.9342	6.9342
15	-24.5049	4.8772	4.8254	4.8771	4.8254	4.8771	4.8771
16	-34.2959	-2.3030	-2.4241	-2.3032	-2.4241	-2.3032	-2.3032
17	-3.4398	-8.1718	-8.2515	-8.1718	-8.2515	-8.1718	-8.1718
18	42.8424	-6.7592	-6.7196	-6.7588	-6.7196	-6.7588	-6.7588
19	44.7389	1.4230	1.5513	1.4238	1.5513	1.4238	1.4238

We computed the results in this table not only for $N = 2000$ but also for

$N = 100, 500$, and 600 . The error appeared at the fourth digit for $N = 100$ and at the sixth place for both $N = 500$ and 600 .

EXAMPLE 12.3.4. (Durbin 1974) Consider $F(s) = \frac{e^{-10s}}{s}$ for which $f(t) = U(t - 10)$, where $U(x)$ is the Heaviside step function. Table 12.3.4 represents the computed data for the same values of a, T , and N as in Example 12.3.3; the last five columns (2–6) represent the results obtained from formulas corresponding to those for Table 12.3.3, and the last column corresponds to the exact values of $f(t)$. For computational details, see `durbin.nb`.

Table 12.3.4

t	$f(t)_1$	$f(t)_2$	$\tilde{f}(t)_3$	$\tilde{f}(t)_4$	$f(t)$
0	0.01356	0.00003	0.00678	0.00001	0.0
1	0.01796	0.00004	0.00678	0.00002	0.0
2	0.02522	0.00005	0.00678	0.00003	0.0
3	0.03718	0.00007	0.00679	0.00005	0.0
4	0.05690	0.00010	0.00679	0.00007	0.0
5	0.08942	0.00014	0.00680	0.00011	0.0
6	0.14303	0.00022	0.00682	0.00018	0.0
7	0.23142	0.00036	0.00685	0.00031	0.0
8	0.37715	0.00067	0.00693	0.00061	0.0
9	0.61742	0.00164	0.00717	0.00157	0.0
10	1.01331	0.49957	0.50665	0.49949	0.5
11	1.01797	0.99749	1.00614	0.99740	1.0
12	1.02522	0.99844	1.00638	0.99833	1.0
13	1.03719	0.99871	1.00645	0.99857	1.0
14	1.05691	0.99879	1.00649	0.99863	1.0
15	1.08943	0.99879	1.00651	0.99860	1.0
16	1.14304	0.99874	1.00652	0.99852	1.0
17	1.23143	0.99864	1.00655	0.99838	1.0
18	1.37716	0.99850	1.00659	0.99819	1.0
19	1.61743	0.99833	1.00666	0.99795	1.0

Table 12.3.4 confirms the remarks made after Table 12.3.3. Furthermore, Dubner and Abate's formula (15) does not give the average value at the point of discontinuity when T is replaced by $T/2$, but it does yield the average value at the point of discontinuity when T is replaced by $2T$. ■

EXAMPLE 12.3.5. (Durbin 1974) Consider $F(s) = \frac{2}{s(1 + e^{-2s})}$ for which $f(t) = 2 \sum_{k=0}^{\infty} (-1)^k U(t - 2k)$, where $U(x)$ is the Heaviside step function. In Table 12.3.5, we give the values obtained from formula (12.3.31) in the second column, values obtained by Durbin (1974) using the FLIT in the third column, and the values directly obtained from the function in the last column. For computational details, see `durbin.nb`.

Table 12.3.5

t	$\tilde{f}(t)$	FLIT	$f(t)$
0	0.99802	1.006615	1.0
1	1.99212	2.012790	2.0
2	0.99863	1.006321	1.0
3	0.00503	0.000511	0.0
4	0.99863	1.007322	1.0
5	1.99217	2.012833	2.0
6	0.99840	1.005719	1.0
7	0.00458	0.000549	0.0
8	0.99798	1.008010	1.0
9	1.99133	2.012913	2.0
10	0.99734	1.005268	1.0
11	0.00327	0.000820	0.0
12	0.99639	1.009086	1.0
13	1.98940	2.013551	2.0
14	0.99501	1.005602	1.0
15	0.00048	0.002814	0.0
16	0.99303	1.017248	1.0
17	1.98538	2.019597	2.0
18	0.99021	1.017248	1.0
19	-0.00525	0.039362	0.0

We note that the accuracy of the two methods is about the same.

Several authors, e.g., Crump (1976), Albrecht and Honig (1977), De Hoog, Knight and Stokes (1982), Honig and Hirdes (1984), and D'Amore, Laccetti and Murli (1999a, 1999b), have used Durbin's formula (36), which is our (12.3.24), for numerical implementation, and they improve the summation of Fourier series by using various algorithms. Crump (1976) and Honig and Hirdes (1984) use Wynn's (1956) epsilon-algorithm (EPAL) to accelerate the convergence. However, we are not sure that such algorithms will improve the speed of computation using Mathe-

matica. We checked the algorithm for the series $\sum_{k=1}^N \frac{1}{n^2}$ for $N = 86$; Mathematica took 0.03 seconds to compute the sum, giving 1.63337 directly, and it took 1926.69 seconds to compute $\epsilon_{17}^{(1)} = 1.63334$. The algorithm is perhaps more suited for use in Fortran codes.

12.4. Inversion by the Riemann Sum

This method has been proposed by Tzou, Özisik and Chiffelle (1994) and Tzou (1997). The inverse $f(t)$ of the Laplace transform $F(s)$ is given by (12.1.2), i.e.,

$$f(t) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} F(s)e^{st} ds, \quad (12.4.1)$$

where a is taken to the right of all singularities of $F(s)$. Substituting $s = a + i\omega$, we can write

$$\frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} F(s)e^{st} ds = \frac{e^{at}}{2\pi} \int_{-\infty}^{\infty} F(a + i\omega)e^{i\omega t} d\omega. \quad (12.4.2)$$

Replacing ω by $\frac{n\pi}{\tau}$ and $\Delta\omega_n$ by π/τ , the integral on the right side of Eq (12.4.2) can be expressed as a Riemann sum:

$$\begin{aligned} \frac{e^{at}}{2\pi} \int_{-\infty}^{\infty} F(a + i\omega)e^{i\omega t} d\omega &= \frac{e^{at}}{2\pi} \sum_{n=-\infty}^{n=\infty} F(a + \frac{in\pi}{\tau}) e^{i(\frac{n\pi t}{\tau})} \\ &= \frac{e^{at}}{2\pi} \left[F(a) + \sum_{n=1}^{\infty} \left[F(a + \frac{in\pi}{\tau}) e^{i(\frac{n\pi t}{\tau})} + F(a - \frac{in\pi}{\tau}) e^{-i(\frac{n\pi t}{\tau})} \right] \right] \\ &= \frac{e^{at}}{\tau} \left[\frac{1}{2} F(a) + \Re \left\{ \sum_{n=1}^{\infty} \left(a + \frac{in\pi}{\tau} \right) e^{i(\frac{n\pi t}{\tau})} \right\} \right]. \end{aligned} \quad (12.4.3)$$

Set $\tau = t$; then

$$f(t) = \frac{e^{at}}{t} \left[\frac{1}{2} F(a) + \Re \left\{ \sum_{n=1}^{\infty} \left(a + \frac{in\pi}{t} \right) (-1)^n \right\} \right]. \quad (12.4.4)$$

Thus, integration has been reduced to an infinite series. The accuracy depends on the choice of a and N . Note that formula (12.4.3) is equivalent to formula

(12.3.28). Further, formula (12.4.4) was also given by Crump (1976). According to Tzou (1997) and Tzou et al. (1994), the best choice for a appears to be $a = 4.7/t$. The disadvantage, as Crump (1976) points out, is that one has to choose a different a for each t . This can be circumvented by replacing at by b and changing formula (12.4.4) to

$$f(t) = \frac{e^b}{t} \left[\frac{1}{2} F\left(\frac{b}{t}\right) + \Re \left\{ \sum_{n=1}^{\infty} \left(\frac{b}{t} + \frac{in\pi}{t} \right) (-1)^n \right\} \right]. \quad (12.4.5)$$

EXAMPLE 12.4.1. Consider $f(t) = 0.5t \sin t$, for which $F(s) = \frac{s}{s^2 + 1}$; and $f(t) = U(t - 10)$, for which $F(s) = \frac{e^{-10s}}{s}$. The values of $\tilde{f}(t)$ obtained from formula (12.4.5) are given in Table 12.4.1. There is no change in these values for $n = 1000$ or 10000 . Since formula (12.4.5) has singularity at $t = 0$, we have evaluated $\tilde{f}(t)$ at $t = 0.0001$. For computational details see `tzou.nb`.

Table 12.4.1

t	$0.5t \sin t$	$\tilde{f}(t)$	$U(t - 10)$	$\tilde{f}(t)$
0	0.0	5.00372(-9)	0.0	0.0
1	0.42073	0.42075	0.0	3.87(-21)
2	0.90929	0.90922	0.0	3.41(-9)
3	0.21168	0.21183	0.0	6.83(-9)
4	-1.51360	-1.51387	0.0	8.25(-5)
5	-2.39731	-2.39691	0.0	8.27(-5)
6	-0.83824	-0.83880	0.0	7.47(-5)
7	2.29945	2.30018	0.0	5.61(-5)
8	3.95743	3.95653	0.0	0.00020
9	1.85453	1.8556	0.0	0.00003
10	-2.72011	-2.72133	0.5	0.50003
11	-5.49995	-5.49858	1.0	1.00024
12	-3.21944	-3.22091	1.0	1.00018
13	2.73109	2.73264	1.0	0.99997
14	6.93425	6.93266	1.0	1.00017
15	4.87716	4.87874	1.0	1.00008
16	-2.30323	-2.30475	1.0	0.99994
17	-8.17188	-8.17046	1.0	0.99991
18	-6.75889	-6.76013	1.0	1.00018
19	1.42383	1.42486	1.0	1.00029

Clearly, the method gives very accurate results for the functions considered. This method was used to verify the analytical inverse of a rather complicated $F(s)$ by Jordan, Puri and Boros (2000); the results completely matched with the numerical values obtained from the exact inverse. We should, however, recall that formulas (12.4.5) and (12.3.32) are closely related.

12.5. Approximate Formulas

Some approximate formulas not covered in the previous sections are given below.

(1) Post (1930) and Widder (1934) derive the following well-known formula:

$$f(t) = \lim_{k \rightarrow 0} \sum \left[\frac{(-1)^k}{k!} \left(\frac{k}{t} \right)^{k+1} F^{(k)} \left(\frac{k}{t} \right) \right]. \quad (12.5.1)$$

(2) ter Haar (1951) proposes and Cost (1964) derives the following formula:

$$f(t) \approx t^{-1} F(t^{-1}). \quad (12.5.2)$$

Schapery (1962) modifies it to

$$f(t) \approx (2t)^{-1} F((2t^{-1})). \quad (12.5.3)$$

(3) An extrapolation formula for an approximate inverse of the Laplace transform was developed by Stehfest (1970a, 1970b):

$$f(t) \approx \frac{\ln 2}{t} \sum_{n=1}^N K_n F \left(\frac{n \ln 2}{t} \right), \quad (12.5.4)$$

where

$$K_n = (-1)^{n+N/2} \sum_{k=(n+1)/2}^{\min(n, N/2)} \frac{k^{N/2} (2k)!}{(N/2 - k)!(k-1)!(n-k)!(2k-n)!}$$

(4) Cost (1964) gives a summary of some of the approximate formulas based on Widder's general inversion formula, which is based on the sifting property of the Dirac delta function. In analytical form this property is

$$\int_0^\infty f(t) \delta(t - t_0) dt = f(t_0). \quad (12.5.5)$$

The following result is the basis of Widder's general inversion formula:

$$\frac{d^n F}{ds^n} = \int_0^\infty (-1)^n t^n f(t) e^{-st} dt. \quad (12.5.6)$$

Because of the sifting property, we have

$$\int_0^\infty \delta(t - t_0) dt = 1. \quad (12.5.7)$$

The formula $\int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}}$ implies that

$$\int_0^\infty s^{n+1} \frac{t^n}{n!} e^{-st} dt = 1. \quad (12.5.8)$$

Hence, we can reasonably assume that

$$s^{n+1} \frac{t^n}{n!} e^{-st} \approx \delta(t - t_0), \quad (12.5.9)$$

where t_0 is a point where $s^{n+1} \frac{t^n}{n!} e^{-st}$ has maximum value. Thus, in this case t_0 is n/s or $s = n/t$. Thus, we have

$$\delta(t - n/s) \approx s^{n+1} \frac{t^n}{n!} e^{-st}. \quad (12.5.10)$$

Cost points out that with the use of Sterling's formula, one can show that in the limit as $n \rightarrow \infty$, Eq (12.5.10) becomes exact. Using Eq (12.5.10) in Eq (12.5.6), we have

$$(-1)^n \frac{s^{n+1}}{n!} \frac{d^n F}{ds^n} = \int_0^\infty f(t) \delta(t - n/s) dt \approx f(t)|_{t=n/s}. \quad (12.5.11)$$

Eq (12.4.11) can be expressed in a somewhat different form as

$$f(t) = \lim_{n \rightarrow \infty} \left[(-1)^n \frac{s^{n+1}}{n!} \frac{d^n F}{ds^n} \right]_{s=n/t}. \quad (12.5.12)$$

Several approximations derived by others are special cases of formula (12.5.12).

(5) Alfrey's approximate formula: If we take $n = 1$ in Eq (12.5.12), it becomes

$$f(t) = \left[-s^2 \frac{dF}{ds} \right]_{p=1/t}. \quad (12.5.13)$$

Alfrey (1944) develops formula (12.5.13). Obviously, it is a special case of Widder's formula (12.5.11).

(6) ter Haar's approximate formula: Ter Haar (1951) proposed an approximation for $f(t)$ as

$$f(t) \approx [sF(s)]_{s=1/t}. \quad (12.5.14)$$

This approximation cannot be derived directly from Eq (12.5.11), but the ideas used in deriving Eq (12.5.14) are similar to those used in deriving Eq (12.5.11). Since $\int_0^\infty te^{-st} dt = 1/s^2$, we can consider $s^2 te^{-st}$ to be an approximation of $\delta(t - t_0)$ at an appropriate point. Hence, formula (12.5.5) can be expressed as

$$s^2 F(s) = \int_0^\infty \frac{f(t)}{t} s^2 te^{-st} dt \approx \int_0^\infty \frac{f(t)}{t} \delta(t - t_0) dt, \quad (12.5.15)$$

where t_0 can be taken to be the point at which $s^2 te^{-st}$ takes its maximum value. This value is clearly $st = 1$. Hence, Eq (12.5.15) becomes

$$s^2 F(s) \approx \left[\frac{f(t)}{t} \right]_{t=1/s}, \quad (12.5.16)$$

which is obviously equivalent to Eq (12.5.14).

(7) Schapery (1962), Cost (1964), and Rizzo and Shippy (1970) develop a Laplace inverse as a series with terms of the type e^{bt} plus a linear function of t . Let

$$f(t) = A + Bt + \sum_{i=0}^N A_k e^{-b_k t}. \quad (12.5.17)$$

Then the Laplace transform of both sides yields

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \sum_{i=0}^N \frac{A_k}{s + b_k}. \quad (12.5.18)$$

Now we can solve for b_k by setting $s = b_k$ and generating N equations, where A and B represent the values at $t = 0$ of $f(t)$ and $f'(t)$, respectively. If these values are not known, we can choose some convenient values for s to generate two more equations. This leaves the choices for b_k and N . Obviously, the choice is random, but we can make intelligent choices by considering the known properties and behaviour of $f(t)$ and the known expression of $F(s)$.

REFERENCES USED: Albrecht and Honig (1977), Alfrey (1944), Bellman (1966), Bellman, Kalaba, and Lockett (1966), Churchill (1972), Cost (1964), Crump (1976),

D'Amore, Laccetti, and Murli (1999a, 1999b), Davies and Martin (1979), de Hoog, Knight, and Stokes (1982), Dubner and Abate (1968), Durbin (1974), Erdyli (1954), Felts and Cook (1969), Garbow, Giunta, Lyness, and Murli (1988,1988a), Giunta, Laccetti, and Rizzardi (1988), Hildebrand (1956), Honig and Hirdes (1984), Jordan, Puri, and Boros (2000), Krall and Frink (1949), Krylov and Skoblya (1969), Lanczos (1956), Luke (1968), Lyness and Giunta (1986), Oberhettinger and Badii (1973), Papoulis (1956), Piessens (1969, 1971, 1972, 1975), Piessens and Dang (1976), Post (1930), Rizzo and Shippy (1970), Roberts and Kaufman (1966), Salzer (1955), Schapery (1962), Schlessinger (1968), Schüttler and Scalpino (1986), Shohat (1940), Sneddon (1978), Stehfest (1970a, 1970b), ter Haar (1951), Tzou (1994, 1997), Tzou, Özisik, and Chiffelle (1994), Weeks (1966), Widder (1934, 1935), Wynn (1956).

A

Quadrature Rules

A quadrature rule Q is defined as

$$\int_a^b F(s) ds = \sum_{j=1}^{n+1} w_j F(s_j) + E(F), \quad (\text{A.1.1})$$

where w_j are the weights (or coefficients) and s_j are the quadrature points (also called base points or nodes). The interval $[a, b]$ is a finite or infinite interval of integration. The error term $E(F) = 0$ iff $F(x)$ is a polynomial of degree $\leq n$.

A.1. Newton–Cotes Quadratures

Let the interval $[a, b]$ be finite and partitioned by n equally spaced points

$$s_1 = a, \quad s_2 = a + \frac{b-a}{n}, \quad \dots, \quad s_n = a + (n-1) \frac{b-a}{n} = b - \frac{b-a}{n}, \quad s_{n+1} = a.$$

Some frequently used quadrature rules are as follows:

$$Q_1: \text{Repeated rectangle rule: } \int_a^b F(s) ds = \frac{b-a}{n} \sum_{j=1}^{n+1} F(s_j).$$

$$Q_2: \text{Repeated trapezoidal rule: } \int_a^b F(s) ds = \frac{b-a}{2n} \sum_{j=1}^{n+1} \{2F(s_j)\},$$

where the double primes on the summation indicate that the first and the last terms are halved.

Q_3 : Repeated midpoint rule:

$$\int_a^b F(s) ds = h \sum_{j=0}^{n-1} F(a + (j + 1/2)h), \quad h = \frac{b-a}{n}.$$

Q_4 : Repeated Simpson's rule:

$$\begin{aligned} \int_a^b F(s) ds = & \frac{b-a}{3n} \left[F(s_1) + 4F(s_2) + 2F(s_3) + 4F(s_4) \right. \\ & \left. + 2F(s_5) + \cdots + 2F(s_{n-1}) + 4F(s_n) + F(s_{n+1}) \right]. \end{aligned}$$

Q_5 : Repeated Weddle's rule:

$$\begin{aligned} \int_a^b F(s) ds = & \frac{3(b-a)}{10n} \left[F(s_1) + 5F(s_2) + F(s_3) + 6F(s_4) + F(s_5) + 5F(s_6) \right. \\ & + 2F(s_7) + 5F(s_8) + F(s_9) + 6F(s_{10}) \\ & \left. + F(s_{11}) + \cdots + F(s_{n-1}) + 5F(s_n) + F(s_{n+1}) \right]. \end{aligned}$$

For $n = 7$ it reduces to the simple Weddle's rule.

Q_6 : Newton-Cotes (N-C) rule:

$$\int_a^b F(s) ds = (b-a) \sum_{j=1}^{n+1} w_j F(s_j),$$

where $\sum_{j=1}^{n+1} w_j = 1$. The weights w_j for $n = 1, 2, \dots, 6$ are given in Table A.1.1.

For $n = 1$ this formula gives the simplest rectangle rule; for $n = 2$ it gives Simpson's two-strip rule:

$$\int_a^b F(s) ds = \frac{b-a}{6} [F(s_1) + 4F(s_2) + F(s_3)];$$

for $n = 3$ it gives Simpson's 3/8-rule:

$$\int_a^b F(s) ds = \frac{b-a}{8} [F(s_1) + 3F(s_2) + 3F(s_3) + F(s_4)];$$

for $n = 4$ it gives the four-strip rule:

$$\int_a^b F(s) ds = \frac{b-a}{90} [7F(s_1) + 32F(s_2) + 12F(s_3) + 32F(s_4) + 7F(s_5)];$$

similarly, for $n = 5$ it gives the five-strip rule and for $n = 6$ the six-strip rule, and so on; these rules can be written from Table A.1.1. A simple guideline to use this rule is as follows: When n is even (i.e., when there is an even number of subintervals or an odd number of base points), this rule is exact when $F(x)$ is a polynomials of degree $n + 1$ or less; when n is odd, this rule is exact for polynomials of degree n or less; conversely, a polynomial of degree n is integrated exactly by choosing $n + 1$ base points.

Table A.1.1. Weights for Newton–Cotes rule

n	w_1	w_2	w_3	w_4	w_5	w_6	w_7
1	1/2	1/2					
2	1/6	4/6	1/6				
3	1/8	3/8	3/8	1/8			
4	7/90	32/90	12/90	32/90	7/90		
5	19/288	75/288	50/288	50/288	75/288	19/288	
6	41/840	216/840	27/840	272/840	27/840	216/840	41/840

Q_7 : Tangential rule:

$$\int_a^b F(s) ds = \frac{b-a}{n} \sum_{j=1}^{n+1} F(s_j),$$

where $s_1 = a$, $s_2 = a + \frac{b-a}{2n}$, $s_3 = a + 3\frac{b-a}{2n}, \dots, s_{n+1} = b$.

Notes: (i) N–C rules are Riemann sums; thus, if $n \rightarrow \infty$, they converge to the exact value of the integral. (ii) Trapezoidal rule Q_2 is sometimes modified to Romberg's scheme (see Davis and Rabinowitz 1967, p.434) or Gregory's scheme (see Section 4.4, and Baker 1978, p.120). These schemes provide modifications of this rule with an end correction.

If we use the step size $h = (b - a)/n$, then the approximation formula (A.1.1) becomes

$$\int_0^{mh} F(s) ds = \sum_{j=0}^m W_{mj} F(jh), \quad m = 1, 2, \dots, n. \quad (\text{A.1.2})$$

Weights W_{mj} for some useful quadrature rules are given in following tables.

Table A.1.2. Weights for a single trapezoidal rule and repeated Simpson's rule*

$m \setminus j$	0	1	2	3	4	5
1	$h/2$	$h/2$				
2	$h/3$	$4h/3$	$h/3$			
3	$h/2$	$5h/6$	$4h/3$	$h/3$		
4	$h/3$	$4h/3$	$2h/3$	$4h/3$	$h/3$	
5	$h/2$	$5h/6$	$4h/3$	$2h/3$	$4h/3$	$h/3$

Table A.1.3. Weights for repeated Simpson's rule and a single trapezoidal rule

$m \setminus j$	0	1	2	3	4	5
1	$h/2$	$h/2$				
2	$h/3$	$4h/3$	$h/3$			
3	$h/3$	$4h/3$	$5h/6$	$h/2$		
4	$h/3$	$4h/3$	$2h/3$	$4h/3$	$h/3$	
5	$h/3$	$4h/3$	$2h/3$	$4h/3$	$5h/6$	$h/2$

Table A.1.4. Weights for a single 3/8-rule and repeated Simpson's rule

$m \setminus j$	0	1	2	3	4	5	6
1	$h/2$	$h/2$					
2	$h/3$	$4h/3$	$h/3$				
3	$3h/8$	$9h/8$	$9h/8$	$3h/8$			
4	$h/3$	$4h/3$	$2h/3$	$4h/3$	$h/3$		
5	$3h/8$	$9h/8$	$9h/8$	$17h/24$	$4h/3$	$h/3$	
6	$h/3$	$4h/3$	$2h/3$	$4h/3$	$2h/3$	$4h/3$	$h/3$

*Trapezoidal rule is used for integral \int_0^h and Simpson's rule for the remainder of the integral.

Table A.1.5. Weights for repeated Simpson's rule and a single 3/8-rule

$m \setminus j$	0	1	2	3	4	5	6	7
1	$h/2$	$h/2$						
2	$h/3$	$4h/3$	$h/3$					
3	$3h/8$	$9h/8$	$9h/8$	$3h/8$				
4	$h/3$	$4h/3$	$2h/3$	$4h/3$	$h/3$			
5 *	$h/3$	$4h/3$	$17h/24$	$9h/8$	$9h/8$	$3h/8$		
6	$h/3$	$4h/3$	$2h/3$	$4h/3$	$2h/3$	$4h/3$	$h/3$	
7 *	$h/3$	$4h/3$	$2h/3$	$4h/3$	$17h/24$	$9h/8$	$9h/8$	$3h/8$

The asterisk (*) indicates values that differ from corresponding values in Table A.1.4. This table has practical advantages over the previous table.

A.2. Gaussian Quadrature

Q_8 : Gauss-Legendre rule:

$$\int_a^b F(s) ds = \sum_{j=1}^n w_j F(s_j),$$

where

$$s_j = \frac{a + b + (b - a)\xi_j}{2}, \quad w_j = \frac{b - a}{(1 - \xi_j)^2 [P'_n(\xi_j)]^2};$$

here ξ_j are the Gauss points for the interval $[-1, 1]$ which are the zeros of the Legendre polynomials $P_n(x)$ in the interval $(-1, 1)$. Although tables for the Gauss points ξ_j and weights w_j are readily available (see, e.g., Abramowitz and Stegun (1968)), they can be easily computed by Mathematica for any n (see, e.g., `gausspoints.nb` where they are also computed for the interval $(0, 1)$). These rules can always be transformed to the interval $(-1, 1)$ as follows:

$$\int_{-1}^1 F(s) ds = \sum_{j=1}^n w_j f(s_j), \tag{A.2.1}$$

where w_j are the weights and x_j the zeros of $P_n(x)$ such that

$$w_j = \frac{1}{P'_n(x_j)} \int_{-1}^1 \frac{P_n(x)}{(x - x_j)} dx. \tag{A.2.2}$$

A MODIFICATION OF GAUSS QUADRATURE is as follows: For its application to the numerical inversion of the Laplace transform, we need to modify the quadrature formula to the interval $(0, 1)$. In order to accomplish this, we make the substitution $x = 2r - 1$, $dx = 2dr$, which gives

$$\int_{-1}^1 f(x) dx = \int_0^1 f(2r - 1) 2 dr = \sum_{j=1}^N w_j f(2r_j - 1)$$

or

$$\int_0^1 f(2r - 1) dr = \sum_{j=1}^n \frac{w_j}{2} f(2r_j - 1). \quad (\text{A.2.3})$$

Now, if we define $f(2r - 1) = \phi(r)$, then the quadrature rule reduces to

$$\int_0^1 \phi(r) dr = \sum_{j=1}^n w'_j \phi(r_j), \quad (\text{A.2.4})$$

where $r_j = \frac{x_j + 1}{2}$, $w'_j = \frac{w_j}{2}$, and r_j are the zeros of the shifted Legendre polynomials $P_n^*(r) = P_n(2r - 1)$.

Q_9 : Chebyshev* rule:

$$\int_a^b F(s) ds = h \sum_{j=1}^n F(s_j),$$

where $h = \frac{b-a}{n}$, and $s_j = \frac{b-a}{2} + \frac{b-a}{2} \xi_j$; here ξ_j denote the Chebyshev points that are the zeros of the Chebyshev polynomials $T_n(x)$, i.e., $\xi_j = \cos \frac{(2j-1)\pi}{2n}$, $j = 1, 2, \dots$. An advantage in using this rule is that the weights are all equal to h . See `chebyshevpoints.nb` for computing ξ_j for the intervals $(-1, 1)$ and $(0, 1)$ since they are not readily available.

Q_{10} : Clenshaw–Curtis rule:

$$\int_{-1}^1 F(s) ds = \sum_{j=0}^n w_j F\left(\cos \frac{j\pi}{n}\right),$$

*‘Chebyshev’ is a transliteration from the Russian name Чебышев; another spelling, Tschebyscheff, is sometimes used.

where

$$w_j = \frac{4}{n} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{\cos(2jm\pi/n)}{1-4m^2}, \quad j \neq 0, n; \quad w_0 = w_n = \frac{1}{n^2-1}.$$

This rule requires $(n+1)^2$ multiplications and additions, and thus it takes longer to compute. However, the weights w_j can be regarded as discrete Fourier cosine transform and computed directly with $O(n \ln n)$ operations (see Clenshaw and Curtis 1960).

Q_{11} : Cubic spline rule:

$$\int_a^b F(s) ds = \frac{h}{2} \sum_{j=1}^{n+1}'' (2F_j) - \frac{h^3}{4!} \sum_{j=1}^{n+1}'' (2m_j),$$

where $h = \frac{b-a}{n}$, $F_j = F(s_j)$, and $m_j = F''(s_j)$ for $j = 1, \dots, n+1$ (see King 1984, p. 175).

Q_{12} : One-point Gauss quadrature:

$$\int_a^b f(s) g(s) ds = w_1 g(s_1), \quad (\text{A.2.5})$$

where the quantities w_1 and s_1 are determined in terms of $f(s)$ as follows: Require that Eq (A.2.5) be exact for $g(s) = s^m$, $m = 0, 1$. Thus,

$$\int_a^b f(s) ds = w_1, \quad \int_a^b s f(s) ds = w_1 t_1, \quad (\text{A.2.6})$$

which determine both w_1 and s_1 .

Note that the built-in Mathematica object `NIntegrate` uses an adaptive algorithm, which recursively divides the interval of integration as needed; the Gauss points specify the number of initial points to choose. If an explicit setting for maximum points is given, `NIntegrate` also uses quasi-Monte Carlo methods to get an estimate of the result, sampling at most the numbers specified (see Wolfram, *The Mathematica Book*, 1996, p.1145).

The error estimate $E_x(F)$ for the above quadrature rules Q_j of degree p is bounded by

$$|E_x(F)| = \left| \int_a^b F(s) ds - \sum_{j=1}^n w_j F(s_j) \right| \leq C(F) n^{-p}, \quad (\text{A.2.7})$$

where the constant $C(F)$ depends on the function $F(x)$. The degree p is called the *order* of convergence. The above inequality implies that $\lim_{n \rightarrow \infty} |E_x(F)| = 0$ for any sequence $\{Q_j\}$ of quadrature rules. This condition is met by any n -panel Newton–Cotes rules of degree p for any fixed p and increasing n , and also Gauss–Legendre and Chebyshev n -point rules for increasing n . But this condition is not met by n -panel Newton–Cotes rules for any fixed n and increasing p . We should avoid this situation. Thus, in practice repeated trapezoidal or Simpson’s rule and one-panel Gauss–Legendre or Chebyshev rules make the best choice since these rules guarantee not only convergence but also a fast rate of convergence.

In some cases the Gauss–Legendre quadrature formulas become unstable for large n , say $n > 20$. This arises because of the fact that the points and weights are rational numbers and, therefore, must be rounded off. In general, this quadrature rule is superior to most other rules with the same number of points. However, for some integrals it is not true that a Gauss rule is always the best one. For example, the n -point trapezoidal rule is much better than the n -point Gauss–Legendre rule for the integral

$$\int_0^1 \frac{2}{2 + \sin(10\pi x)} dx,$$

as the following results show: Exact value $= 2/\sqrt{3} \approx 1.1547$; 4-point trapezoidal rule gives 1.91667, while 4-point Gauss–Legendre rule gives 2.53883; 12-point trapezoidal rule gives 2.1594, while 12-point Gauss–Legendre rule gives 2.25809. For computational details, see `app1.nb`.

A.3. Integration of Products

We shall construct a quadrature rule for an integral with an integrand that is the product of any two functions $w(x)$ and $F(x)$ over an interval $[a, b]$. Such a rule is of the form

$$\int_a^b w(s)F(s) ds = \sum_{j=0}^n v_j F(s_j), \quad (\text{A.3.1})$$

where s_j are the quadrature points, $a \leq s_j \leq b$, and the weights v_j depend on a , b as well as the function $w(x)$. If the function $w \equiv 1$, the rule (A.3.1) reduces to the quadrature rules presented in Section A.1–A.2. We discuss two cases:

CASE 1. If the function $F(x)$ is “badly behaved”, e.g., $F(x)$ behaves like $(x - s)^{-1/2}$ near $x = s$ but is continuous everywhere else, we can find a function $g(x)$ such that $F(x) = g(x)(x - s)^{-1/2}$. In general, if we approximate $F(x)$ by using a polynomial interpolant

$$I(x) = \sum_{j=0}^n l_j(x) F(s_j),$$

where

$$l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^n \frac{x - s_i}{s_j - s_i},$$

which agrees with $F(x)$ at the quadrature points s_j , $j = 0, 1, \dots, n$, then

$$\int_a^b w(s) I(s) ds = \sum_{j=0}^n v_j F(s_j),$$

where

$$v_j = \int_a^b w(s) l_j(s) ds.$$

Since $F \in C[a, b]$, we can choose the quadrature points s_0, s_1, \dots, s_n such that $I(x)$ is a reasonable approximation of $F(x)$.

EXAMPLE A.3.1. Suppose $s_0 = a$, $s_1 = b$, and $F'' \in C[a, b]$. Then $l_0(x) = \frac{b-x}{b-a}$ and $l_1(x) = \frac{x-a}{b-a}$, which gives $I(x) = \frac{b-x}{b-a} F(a) + \frac{x-a}{b-a} F(b)$, and $\int_a^b w(s) F(s) ds = v_0 F(a) + v_1 F(b)$, where

$$v_0 = \int_a^b \frac{b-s}{b-a} w(s) ds, \quad v_1 = \int_a^b \frac{s-a}{b-a} w(s) ds. \blacksquare$$

Formulas of the form (A.3.1) that are exact when $F(x)$ is a polynomial of degree $(2n + 1)$ are known as Gauss-type quadrature rules. In particular, for $a = 0, b = \infty$, $w(x) = e^{-x}$, we have the Gauss–Laguerre rule; for $a = -\infty, b = \infty$, $w(x) = e^{-x^2}$, we have the Gauss–Hermite rule; and for $a = -1, b = 1$, $w(x) = (1 - x^2)^{-1/2}$, we have the Gauss–Chebyshev rule.

CASE 2. Since integral equations involve integrals of the form $\int_a^b k(x, s)\phi(s) ds$, we keep x fixed, write $w(s) = k(x, s)$, and have an integral of the form (A.3.1), where the function $w(x)$ may be well behaved, or it may be badly behaved as, e.g., in the case of weakly singular integral equations where $w(s)$ may be of the form $|x - s|^{-\alpha} g(x, s)$, $0 < \alpha < 1$. If $\phi \in C^{n+1}[a, b]$, and $a \leq s_i \leq b$, $i = 0, 1, \dots, n$, then there is a value ξ depending on a such that

$$\phi(x) - I(x) = \frac{\phi^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - s_i).$$

Hence, in the case when a and b are finite,

$$\int_a^b w(s)\phi(s) ds - \int_a^b w(s)I(s) ds = \int_a^b w(s) \frac{\phi^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (s - s_i) ds,$$

where $\xi = \xi(s)$. If in (A.3.1) we set

$$v_j = \int_a^b w(s) \prod_{i \neq j} \frac{s - s_i}{s_j - s_i} ds,$$

then the error in the above approximation is bounded by

$$\|\phi^{(n+1)}(x)\|_\infty \int_a^b \left| w(s) \prod_{i \neq j} \frac{s - s_i}{s_j - s_i} \right| ds, \quad (\text{A.3.2})$$

provided $\int_a^b |w(s)| ds < \infty$. This leads to the quadrature rule

$$\int_a^b k(x, s) \phi(s) ds = \sum_{j=0}^n v_j \phi(s_j), \quad (\text{A.3.3})$$

which is exact when $\phi(x)$ is a polynomial of degree n .

To obtain an improved approximation, we partition $[a, b]$ at the points s_j , $j = 0, 1, \dots, n$, by $a = s_0 < s_1 < \dots < s_n = b$, and write

$$\int_a^b w(s)F(s) ds = \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} w(s)F(s) ds = \sum_{j=0}^{n-1} \left\{ v_0^{(j)} F(s_j) + v_1^{(j)} F(s_{j+1}) \right\}, \quad (\text{A.3.4})$$

where

$$\begin{aligned} v_0^{(j)} &= \int_{s_j}^{s_{j+1}} \frac{s_{j+1} - s}{s_{j+1} - s_j} w(s) ds, \\ v_1^{(j)} &= \int_{s_j}^{s_{j+1}} \frac{s - s_j}{s_{j+1} - s_j} w(s) ds. \end{aligned} \quad (\text{A.3.5})$$

The approximation is still of the form (A.3.1), and the error is bounded by

$$\frac{1}{2} \|F''(x)\|_\infty \sum_{j=0}^n \int_{s_j}^{s_{j+1}} |w(s)(s - s_j)(s - s_{j+1})| ds,$$

which is of the order $O(h^2)$ for $s_j = a + j h$, where $h = (b - a)/n$ is the step size, provided $\int_a^b |w(s)| ds < \infty$. This quadrature is known as the GENERALIZED TRAPEZOIDAL RULE since it is an extension of the repeated trapezoidal rule. A GENERALIZED MIDPOINT RULE is obtained by taking n even and setting $s_j = a + j h$ as above, with $v_0 = v_2 = v_4 = \dots = 0$, and $v_{2m+1} = \int_{s_{2m}}^{s_{2m+2}} w(s)F(s) ds$ for $m = 0, 1, \dots$. This rule is exact if $F(x)$ is piecewise-constant, i.e., constant in each subinterval $[s_{2m}, s_{2m+2}]$.

An extension of Simpson's rule is obtained for even n by setting $s_j = a + j h$, as above. Then in the subinterval $[s_{2m}, s_{2m+2}]$ the function $F(x)$ is approximated by the interpolant

$$\begin{aligned} I(x) &= \frac{1}{h^2} \left[\frac{1}{2} (x - s_{2m+1})(x - s_{2m+2}) F(s_{2m}) + (x - s_{2m})(x - s_{2m+2}) \right. \\ &\quad \times F(s_{2m+1}) + \frac{1}{2} (x - s_{2m})(x - s_{2m+1}) F(s_{2m+2}) \left. \right]. \end{aligned}$$

This gives the GENERALIZED SIMPSON'S RULE:

$$\int_a^b w(s)F(s) ds = \sum_{m=0}^{n/2-1} \int_{s_{2m}}^{s_{2m+2}} w(s)I(s) ds = \sum_{j=0}^n v_j F(s_j), \quad (\text{A.3.6})$$

where

$$\begin{aligned}
 v_0 &= \frac{1}{2h^2} \int_a^{a+2h} (s - s_1)(s - s_2) w(s) ds, \\
 v_{2j+1} &= -\frac{1}{h^2} \int_{s_{2j}}^{s_{2j+2}} (s - s_{2j})(s - s_{2j+2}) w(s) ds, \\
 v_{2j} &= \frac{1}{2h^2} \left\{ \int_{s_{2j-2}}^{s_{2j}} (s - s_{2j-2})(s - s_{2j-1}) w(s) ds \right. \\
 &\quad \left. + \int_{s_{2j}}^{s_{2j+2}} (s - s_{2j+1})(s - s_{2j+2}) w(s) ds \right\}, \quad j \neq 0 \text{ or } m, \\
 v_n &= \frac{1}{2h^2} \int_{b-h}^b (s - s_{2m-2})(s - s_{2m-1}) w(s) ds.
 \end{aligned} \tag{A.3.7}$$

If $F''(x) \in C[a, b]$, then the error in this rule is of the order $O(h^3)$. Note that the error may not be of the order $O(h^4)$ even if $F^{(4)}(x) \in C[a, b]$ unless $w(x) \equiv 1$ (see Wang 1976).

If the function $w(x)$ is of the form $(x - s)^{-\alpha} g(x, s)$, then the generalized trapezoidal rule is obtained by taking $\alpha = 1/2$, $s_j = a + j h$, as above, $g(x, s) = 1$, and $x = s_m$, $m = 0, 1, \dots, n$. Then

$$\int_a^b (s_m - s)^{-1/2} F(s) ds = \sum_{j=0}^{n-1} \left[v_{0,m}^{(j)} F(s_j) + v_{1,m}^{(j)} F(s_{j+1}) \right], \tag{A.3.8}$$

where

$$\begin{aligned}
 v_{0,m}^{(j)} &= \frac{1}{h} \int_{s_j}^{s_{j+1}} (s_{j+1} - s)(s_m - s)^{-1/2} ds, \\
 v_{1,m}^{(j)} &= \frac{1}{h} \int_{s_j}^{s_{j+1}} (s - s_j)(s_m - s)^{-1/2} ds.
 \end{aligned} \tag{A.3.9}$$

Alternatively, if we set

$$\begin{aligned}
 \psi_0(r) &= \int_0^a (1 - s)|r - s|^{-1/2} ds, \\
 \psi_1(r) &= \int_0^a s|r - s|^{-1/2} ds,
 \end{aligned}$$

then

$$\begin{aligned}
 v_{0,m}^{(j)} &= \sqrt{h} \psi_0(m - j), \\
 v_{1,m}^{(j)} &= \sqrt{h} \psi_1(m - j),
 \end{aligned} \tag{A.3.10}$$

where

$$\begin{aligned}\psi_0(r) &= \psi_2(r) - \psi_1(r), \\ \psi_1(r) &= \frac{2}{3} \left[\frac{(r-1)^2}{|r-1|^{1/2}} - \frac{r^2}{|r|^{1/2}} \right] + r\psi_2(r), \\ \psi_2(r) &= 2 \left[\frac{r}{|r|^{1/2}} - \frac{r-1}{|r-1|^{1/2}} \right].\end{aligned}$$

Further, the quadrature rule for $w(s) = \ln |s_m - s|$ is given by

$$\int_a^b \ln |s_m - s| F(s) ds = \sum_{j=0}^{n-1} \left[\hat{v}_{0,m}^{(j)} F(s_j) + \hat{v}_{1,m}^{(j)} F(s_{j+1}) \right], \quad (\text{A.3.11})$$

where by setting

$$\begin{aligned}\hat{\psi}_0(r) &= \int_0^a (1-s) \ln |r-s| ds, \\ \hat{\psi}_1(r) &= \int_0^a s \ln |r-s| ds,\end{aligned}$$

we have

$$\begin{aligned}\hat{v}_{0,m}^{(j)} &= \frac{1}{2} h \ln h + h \hat{\psi}_0(m-j), \\ \hat{v}_{1,m}^{(j)} &= \frac{1}{2} h \ln h + h \hat{\psi}_1(m-i);\end{aligned} \quad (\text{A.3.12})$$

or, alternatively, $\hat{\psi}_0(r) = \hat{\psi}_2(r) - \hat{\psi}_1(r)$, where

$$\begin{aligned}\hat{\psi}_1(r) &= \frac{1}{2} \left\{ (r-1)^2 \ln |r-1| - r^2 \ln |r| + \frac{1}{4} [r^2 - (r-1)^2] \right\} + r\hat{\psi}_2(r), \\ \hat{\psi}_2(r) &= r \ln |r| = (r-1) \ln |r-1| - 1.\end{aligned}$$

A word of caution: While computing above formulas for $v_{0,m}^{(j)}$ and $\hat{v}_{1,m}^{(j)}$, to avoid cancellation error take the limit values as $r \rightarrow 0$ or $r \rightarrow 1$. Also, note the following GENERALIZATIONS OF THE MIDPOINT RULE for even n :

$$\int_a^b |s_m - s|^{-1/2} F(s) ds = \sum_{j=0}^{n/2-1} w_m^{2j+1} F(a + (2j+1)h) \quad (\text{A.3.13})$$

and

$$\int_a^b \ln |s_m - s| F(s) ds = \sum_{j=0}^{n/2-1} \hat{w}_m^{2j+1} F(a + (2j+1)h), \quad (\text{A.3.14})$$

where

$$\begin{aligned} w_m^{2j+1} &= v_{0,m}^{(2j)} + v_{1,m}^{(2j)} + v_{0,m}^{(2j+1)} + v_{1,m}^{(2j+1)}, \\ \hat{w}_m^{2j+1} &= \hat{v}_{0,m}^{(2j)} + \hat{v}_{1,m}^{(2j)} + \hat{v}_{0,m}^{(2j+1)} + \hat{v}_{1,m}^{(2j+1)}. \end{aligned}$$

Thus,

$$w_m^{(2j+1)} = 2\sqrt{h} \left[\frac{m-2j}{|m-2j|^{1/2}} - \frac{m-2j-2}{|m-2j-2|^{1/2}} \right],$$

whence by taking the limit values we get

$$w_{2j+2}^{(2j+1)} = w_{2j}^{(2j+1)} = 2\sqrt{2h}.$$

Another MODIFICATION OF THE QUADRATURE METHOD in the case when the kernel $k(x, s)$ is badly behaved at $x = s$, i.e., when it is discontinuous or when one of its derivatives is discontinuous at $x = s$, consists of writing the FK2 of the form (1.2.2) as

$$\phi(x) - \lambda \phi(x) \int_a^b k(x, s) [\phi(s) - \phi(x)] ds = f(x), \quad (\text{A.3.15})$$

and using a quadrature rule to obtain

$$\{1 - \lambda A(x)\} \tilde{\phi}(x) - \lambda \sum_{j=0}^n w_j k(x, s_j) \{\tilde{\phi}(s_j) - \tilde{\phi}(x)\} = f(x), \quad (\text{A.3.16})$$

where

$$A(x) = \int_a^b k(x, s) ds. \quad (\text{A.3.17})$$

If we use the notation

$$\Delta(x) = \sum_{j=0}^n w_j k(x, s_j) - A(x), \quad (\text{A.3.18})$$

we obtain from (A.3.16)

$$\{1 + \lambda \Delta(x)\} \tilde{\phi}(x) - \lambda \sum_{j=0}^n w_j k(x, s_j) \tilde{\phi}(s_j) = f(x). \quad (\text{A.3.19})$$

If we set $x = x_i$, $i = 0, 1, \dots, n$, in (A.3.19), we have the MODIFIED QUADRATURE RULE

$$\{1 + \lambda \Delta(x_i)\} \tilde{\phi}(x) - \lambda \sum_{j=0}^n w_j k(x_i, s_j) \tilde{\phi}(s_j) = f(x_i), \quad (\text{A.3.20})$$

which in matrix notation is

$$(\mathbf{I} + \lambda (\Delta - \mathbf{K}\mathbf{D})) \tilde{\Phi} = \mathbf{f}, \quad (\text{A.3.21})$$

where \mathbf{I} is the identity matrix, $\mathbf{k} = (k(x_i, s_j))_{ij}$, $\Delta = \text{diag}\{\Delta(x_0), \Delta(x_1), \dots, \Delta(x_n)\}$, and $\mathbf{D} = \text{diag}\{w_0, w_1, \dots, w_n\}$. Note that the integral $A(x)$ needs be computed accurately, either analytically or numerically, for this method.

Lastly, the generalized rules can be extended to the case when $F(x)$ is the polynomial interpolant. For example, let $h = (b - a)/n$, and $n = Nm$, where N and m are integers; set $s_j = a + j h$, $j = 1, \dots, n$. For $s_{lm} \leq x \leq s_{lm+m}$ approximate $F(x)$ by the interpolant $I(x) \equiv P_{lm, lm+1, \dots, lm+m}(x)$ of degree m such that $I(x)$ agrees with $F(x)$ at the points $s_{lm}, s_{lm+1}, \dots, s_{lm+m}$. Then

$$\int_a^b w(s) F(s) ds = \sum_{l=0}^{N-1} \int_{s_{lm}}^{s_{lm+m}} s(s) I(s) ds = \sum_{j=0}^n v_j F(s_j), \quad (\text{A.3.22})$$

where

$$v_j = \begin{cases} v_{lm+i} = \int_{s_{lm}}^{s_{lm+m}} \prod_{\substack{j=l m \\ j \neq l m+i}} \frac{s - s_j}{s_{lm+i} - s_j} w(s) ds & \text{if } j = lm + i, i \neq 0, \\ v_{lm} = \int_{s_{lm}}^{s_{lm+m}} \prod_{l m+1}^{l m+m} \frac{s - s_j}{s_{lm} - s_j} w(s) ds \\ \quad + \int_{s_{l m-m}}^{s_{l m}} \prod_{l m-m}^{l m-1} \frac{s - s_j}{s_{l m+i} - s_j} w(s) ds & \text{if } j = lm. \end{cases} \quad (\text{A.3.23})$$

A.4. Singular Integrals

Sometimes we shall encounter singular integrals of the form $\int_a^b w(x) F(x) dx$. In such integrals the integrand is not easy to handle because of the following features:

- (i) A singularity occurs in the integrand, or the integrand has a low-order derivative;
- (ii) The integrand is discontinuous, or has a low-order derivative; and
- (iii) The integrand oscillates rapidly.

Integrals of type (i) are usually found in the study of integral equations; those of type (ii) can be integrated piecewise over the subintervals where the integrand is continuous; and those of type (iii) are not discussed here.

To compute type (i) integrals there are four methods:

- (1). It is often possible to factor the singularity out of the integrand so that only $w(x)$ becomes singular and $F(x)$ becomes regular. In such cases we can then use a Gauss rule for the weight function $w(x)$ provided one is available (see Stroud and Secrest 1966).
- (2). A Chebyshev or Clenshaw–Curtis rule may be used, provided that, e.g., $F(x)$ has the Chebyshev expansion

$$F(x) = \sum_{i=0}^{\infty}' a_i T_i(x), \quad (\text{A.4.1})$$

where the prime over the summation indicates that the first term is halved. Then

$$\int_a^b w(x) F(x) dx = \sum_{i=0}^{\infty}' a_i \int_{-1}^1 w(x) T_i(x) dx = \sum_{i=0}^{\infty}' a_i m_i, \quad (\text{A.4.2})$$

provided that the moments m_i are available. Generally, we truncate the series (A.4.1) at $i = n$ and replace a_i by approximations in terms of the known values of $F(\cos \frac{j\pi}{n})$, $j = 0, 1, \dots, n$.

- (3). If the singularity in $F(x)$ can be subtracted out, i.e., if it is possible to write the integral at a singularity x_0 in the form

$$\begin{aligned} \int_a^b w(x) F(x) dx &= \int_a^b w(x) [F(x) - F(x_0)] dx + F(x_0) \int_a^b w(x) dx \\ &= \int_a^b w(x) g(x) dx + F(x_0) m_0, \end{aligned} \quad (\text{A.4.3})$$

the integral $\int_a^b w(x)g(x) dx$ can then be estimated provided that the moment $m_0 = \int_a^b w(x) dx$ is known. For example, let $w(x) = x^{-1/2}$, and consider $\int_0^1 x^{-1/2} F(x) dx$, which has singularity at $x_0 = 0$. Then the subtraction method gives

$$\begin{aligned} \int_0^1 x^{-1/2} F(x) dx &= \int_0^1 x^{-1/2} [F(x) - F(0)] dx + F(0) \int_0^1 x^{-1/2} dx \\ &= \int_0^1 x^{-1/2} [F(x) - F(0)] dx + 2F(0). \end{aligned}$$

Since in most cases $\lim_{x \rightarrow 0} [F(x) - F(x_0)] = 0$, the integrand $x^{-1/2} [F(x) - F(0)]$ will be finite (in fact, zero) at $x = 0$. Thus, this integral is easier to compute as an improper integral. In some cases this subtraction method can be repeated successively until the singularity is weakened (consider, e.g., $\int_0^1 x^{-5/2} F(x) dx$).

(4). If the singularity is weak, it can be ignored. In fact, integrands that are regular everywhere but have singular derivatives of some low order only make the convergence slower. In many cases, if the integrand is singular at x_0 , we can choose quadrature rules that avoid the point x_0 as an abscissa and thus they converge rapidly to ‘exact’ results. See Davis and Rabinowitz (1975, Section 2.12) for details on this method.

A.5. Infinite-Range Integrals

We shall encounter infinite-range integrals, which have the general forms

$$I_1 = \int_a^\infty w(x) F(x) dx, \quad I_2 = \int_{-\infty}^\infty w(x) F(x) dx, \quad (\text{A.5.1})$$

where the integrand usually vanishes near the infinite ends of the interval. If this is the case, then we take

$$\begin{aligned} I_1 &= \lim_{R \rightarrow \infty} \int_0^R w(x) F(x) dx = \lim_{R \rightarrow \infty} I_1(R), \\ I_2 &= \lim_{R \rightarrow \infty} \int_{-R}^R w(x) F(x) dx = \lim_{R \rightarrow \infty} I_2(R), \end{aligned} \quad (\text{A.5.2})$$

i.e., we solve the infinite-range integrals by truncating the range to some large $R > 0$; the slower the decay of the integrand, the large the value of R is chosen.

The infinite-range Gauss–Laguerre quadrature rule is

$$\int_0^\infty e^{-x} F(x) dx = \sum_{j=1}^n w_j F(\xi_j), \quad (\text{A.5.3})$$

where ξ_j are the zeros of the Laguerre polynomial $L_n(x)$ and

$$w_j = \frac{(n!)^2}{\xi_j [L'_n(\xi_j)]^2}, \quad j = 1, \dots, n.$$

The infinite-range Gauss–Hermite rule is

$$\int_{-\infty}^{\infty} e^{-x^2} F(x) dx = \sum_{j=1}^n w_j F(\xi_j), \quad (\text{A.5.4})$$

where ξ_j are the zeros of the Hermite polynomial $H_n(x)$ and

$$w_j = \frac{2^{n+1} n! \sqrt{\pi}}{[H'_n(\xi_j)]^2}, \quad j = 1, \dots, n.$$

A.6. Linear Transformation of Quadratures

Under the transformation

$$\begin{aligned} x &= \gamma u, \quad u = \frac{1}{\gamma} (x - \beta), \\ dx &= \gamma du, \quad \gamma = \frac{b-a}{d-c}, \quad \beta = \frac{ad-bc}{d-c}, \end{aligned} \quad (\text{A.6.1})$$

the interval $a \leq x \leq b$ is transformed into the interval $c \leq u \leq d$. Hence,

$$\int_a^b w(x) F(x) dx = \int_c^d w^*(u) G(u) du, \quad (\text{A.6.2})$$

where

$$G(u) \equiv F(\gamma u + \beta), \quad w^*(u) \equiv w(\gamma u + \beta). \quad (\text{A.6.3})$$

Then under the linear transformation (A.6.1) the quadrature formula

$$\int_a^b w(x) F(x) dx = \sum_{j=1}^n w_j F(x_j) + E(F) \quad (\text{A.6.4})$$

transforms into

$$\int_a^b w^*(u) G(u) dx = \sum_{j=1}^n w_j^* G(u_j) + E^*(G), \quad (\text{A.6.5})$$

where

$$\begin{aligned} u_k &= \frac{1}{\gamma} (x_k - \beta), \quad w_k^* = \frac{1}{\gamma} w_k \quad (k = 1, \dots, n), \\ E^*(G) &= \frac{1}{\gamma} E(F). \end{aligned} \tag{A.6.6}$$

EXAMPLE A.1. The 4-point N-C formula, known as Simpson's $\frac{3}{8}$ -rule, is

$$\int_a^b F(x) dx = \frac{3h}{8} F(a) + \frac{9h}{8} F(a+h) + \frac{9h}{8} F(a+2h) + \frac{3h}{8} F(b), \tag{A.6.7}$$

where $h = (b-a)/3$. To transform this onto the interval $[-1, 1]$, we have $c = -1$, $d = 1$, $\gamma = \frac{b-a}{2}$, and $\beta = \frac{a+b}{2}$. Also,

$$\begin{aligned} x = a &\longrightarrow u = -1, & x = a + h &\longrightarrow u = -\frac{1}{3}, \\ x = a + 2h &\longrightarrow u = \frac{1}{3}, & x = b &\longrightarrow u = 1. \end{aligned}$$

Thus, $F(a) = G(-1)$, $F(a+h) = G(-1/3)$, $F(a+2h) = G(1/3)$, $F(b) = G(1)$, and the weights become $w_1^* = \frac{1}{4} = w_4^*$, $w_2^* = \frac{3}{4} = w_3^*$. Hence the formula (A.6.7) is transformed into

$$\int_{-1}^1 G(u) du = \frac{1}{4} G(-1) + \frac{3}{4} G\left(-\frac{1}{3}\right) + \frac{3}{4} G\left(\frac{1}{3}\right) + \frac{1}{4} G(1),$$

where $G(u) = F\left(\frac{b-a}{2}u + \frac{a+b}{2}\right)$, and the error term becomes $E^*(G) = \frac{2E(F)}{b-a}$. ■

A.7. Trigonometric Polynomials

Let $F(x)$ be a trigonometric polynomial of degree m defined as a linear combination of the following functions: 1, $\cos x$, $\sin x$, $\cos^2 x$, $\cos x \sin x$, $\sin^2 x$, \dots , $\cos^m x$, $\cos^{m-1} x \sin x$, \dots , $\sin^m x$, or, equivalently, a linear combination of the functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos mx, \sin mx.$$

Then the approximation

$$\int_0^{2\pi} F(x) dx \approx \sum_{j=1}^{n+1} w_j F(x_j) \quad (\text{A.7.1})$$

cannot be exact for all trigonometric polynomials of degree m no matter how we choose w_k and x_k for $k = 1, \dots, n+1$. However, if all the points x_k in (A.7.1) are chosen as equally spaced and all w_k as equal, then we obtain a formula of degree n which is the highest degree possible. Thus, define the step size $h = \frac{2\pi}{n}$, and let β be any real number such that $0 \leq \beta < h$. Also define x_k and w_k by

$$x_m = \beta + (m-1)h, \quad w_m = h, \quad m = 1, \dots, n+1. \quad (\text{A.7.2})$$

Then the quadrature rule (A.7.1) is exact for all trigonometric polynomials of degree $\leq n$. This choice of x_m and w_m is known as the REPEATED MIDPOINT FORMULA. Thus, e.g., the $(n+1)$ -point repeated trapezoidal rule is

$$\int_0^{2\pi} F(x) dx \approx \frac{h}{2} F(0) + h F(h) + h F(2h) + \dots + h F(nh) + \frac{h}{2} F(2\pi). \quad (\text{A.7.3})$$

If $F(x)$ is a periodic function with period 2π , then $F(0) = F(2\pi)$, and the rule (A.7.3) becomes

$$\int_0^{2\pi} F(x) dx \approx h \sum_{j=1}^{n+1} F(jh). \quad (\text{A.7.4})$$

A.8. Condition Number

A quadrature rule reduces the problem of finding the solution of an integral equation, or that of an eigenvalue problem, to solving a system of algebraic equations of the form $\mathbf{A}\tilde{\Phi} = \mathbf{f}$, or

$$\sum_{j=0}^n A_{ij} \tilde{\phi}_j = f_i, \quad i = 0, 1, \dots, n, \quad (\text{A.8.1})$$

where $\det |(A_{ij})| \neq 0$. The solution of this system may be obtained by Gaussian elimination with partial pivoting or total condensation. In practice, we work with the augmented matrix $[A | f]$ and interchange rows. Also, if we interchange the columns of A and rearrange the elements $\{\tilde{\phi}_i\}$, we obtain complete pivoting. In either case, the basic idea of pivoting is to restrict the effects of roundoff error, because even though $\det |(A_{ij})|$ may be large, the solution may become sensitive to roundoff error. In such cases we say that the system (A.8.1) is ill conditioned, and the degree of ill-conditioning is measured by the *condition number* $\rho(A)$ of the matrix A , which is generally defined as $\rho(A) = \|A\| \|A\|^{-1}$, where A is a square nonsingular matrix and $\|\cdot\|$ denotes the matrix norm. Thus,

- (i) $\rho(A)$ is undefined if A is singular.
- (ii) The system (A.8.1) is said to be ill conditioned if $\rho(A)$ is “large” which depends on the choice of norm.

In fact, an error analysis shows that if $A\tilde{\Phi} = f$, and $(A + \delta A)(\tilde{\Phi} + \delta\tilde{\Phi}) = f + \delta f$, where A is nonsingular and the perturbation δA is relatively small, then $\frac{\|\delta\tilde{\Phi}\|}{\|\tilde{\Phi}\|}$ can be bounded in terms of $\frac{\|\delta\Phi\|}{\|\tilde{\Phi}\|}$ and $\frac{\|\delta f\|}{\|f\|}$ by a quantity that is large when $\rho(A)$ is large. Large condition numbers imply possible ill-conditioning.

For example, consider the FK2 $(K\phi)(x) = f(x)$, where

$$k(x, s) = \begin{cases} x(1-s) & \text{if } x \leq s, \\ s(1-x) & \text{if } x \geq s, \end{cases}$$

and $a = 0, b = 1$. There exists a solution of this equation if $f(0) = f(1) = 0$ and $f''(x)$ exists. This solution $\phi(x) = -f''(x)$ is then continuous if $f'' \in C[0, 1]$. If we solve this equation by the repeated trapezoidal rule, and take $s_i = ih$, $h = 1/n$, then $k(0, s) = k(1, s) = k(x, 0) = k(x, 1) = 0$ for $0 \leq x, s \leq 1$, and this equation reduces to the system

$$\sum_{j=1}^{n-1} h k(ih, jh) \tilde{\phi}(jh) = h(ih), \quad i = 0, 1, \dots, n, \quad (\text{A.8.2})$$

where $i = 0$ and $i = n$ correspond to $f(0)$ and $f(1)$, respectively. An explicit solution of the system (A.8.2) is

$$\left\{ \begin{array}{l} \tilde{\phi}(h) \\ \tilde{\phi}(2h) \\ \tilde{\phi}(3h) \\ \vdots \\ \tilde{\phi}(1-h) \end{array} \right\} = -\frac{1}{h^2} \left[\begin{array}{ccccc} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 \end{array} \right] \left\{ \begin{array}{l} f(h) \\ f(2h) \\ f(3h) \\ \vdots \\ f(1-h) \end{array} \right\}, \quad (\text{A.8.3})$$

where

$$\tilde{\phi}(ih) = -\frac{f((i-1)h) - 2f(ih) + f((i+1)h)}{h^2}, \quad f(0) = f(1) = 0.$$

If we proceed with Simpson's rule instead, we obtain the solution of the system (A.8.2) as

$$\left\{ \begin{array}{l} \frac{4}{3}\tilde{\phi}(h) \\ \frac{2}{3}\tilde{\phi}(2h) \\ \frac{3}{4}\tilde{\phi}(3h) \\ \vdots \\ \frac{4}{3}\tilde{\phi}(1-h) \end{array} \right\} = -\frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 \end{bmatrix} \left\{ \begin{array}{l} f(h) \\ f(2h) \\ f(3h) \\ \vdots \\ f(1-h) \end{array} \right\}. \quad (\text{A.8.4})$$

Note that both these quadrature rules produce a system of the form $h \mathbf{K} \tilde{\Phi} = \mathbf{f}$ with $\tilde{\Phi} = [\tilde{\phi}(h), \tilde{\phi}(2h), \dots, \tilde{\phi}(1-h)]^T$, $\mathbf{f} = [f(h), f(2h), \dots, f(1-h)]^T$, where $[h \mathbf{K}]^{-1}$ is a tridiagonal matrix $-h^{-2} \mathbf{T}$ with the diagonal elements as -2 and codiagonal elements as 1 . Thus, a condition number of $[h \mathbf{K}]$ is given by $\rho(h \mathbf{K}) = \|h \mathbf{K}\|_\infty \|h^{-2} \mathbf{T}\|_\infty$. But $\|h \mathbf{K}\|_\infty \rightarrow \|\mathbf{K}\|_\infty$ as $h \rightarrow 0$, whereas $\|h^{-2} \mathbf{T}\|_\infty = \frac{4}{h^2} \rightarrow \infty$ as $h \rightarrow 0$; hence, this condition number increases like h^{-2} as $h \rightarrow 0$.

However, in the case of an eigenvalue problem $(K\phi)(x) = \mu f$, the condition number

$$\rho \equiv \rho(\mu) = \frac{|\langle \phi, f \rangle|}{\|\phi\|_2 \|f\|_2} \quad (\text{A.8.5})$$

indicates the sensitivity of the characteristic value μ to a perturbation of the eigenvalue problem, and the smaller the value of μ is, the more difficult the computation of the approximate value $\tilde{\mu}$ is, since $0 < \rho \leq 1$. If μ is not a simple characteristic value, the ill-conditioning gets more complicated; in general, a multiple characteristic value of a non-Hermitian kernel is likely to be ill conditioned. The situation reverses for the eigenvalue $\lambda = 1/\mu$.

Geometrically, the definition (A.8.5) may be regarded as "the 'cosine' of the angle between corresponding left and right eigenfunctions" (Baker 1978, p.173). In practice, it is not possible to compute $\rho(\mu)$ although we can estimate this number if we compute the left and right eigenfunctions and find the cosine of the angle between them.

The condition number of a matrix A is sometimes defined in some linear algebra books as the ratio of the largest eigenvalue to the smallest of eigenvalues of the matrix, which is the product of A with its transpose A^T . While the two different definitions will give different condition numbers, they will both be large or small together.

The Mathematica Book (Wolfram 1996, p.851) defines the condition number as the ratio of the largest singular value of a matrix to the smallest one. This ratio determines, e.g., the accuracy of numerical matrix inverses. Note that very small singular values are usually numerically meaningless. The built-in Mathematica object `SingularValues` removes any singular values of a matrix that are smaller than a preassigned tolerance multiplied by its largest singular value. The option `Tolerance` specifies the preassigned tolerance to be used in such computations.

A.9. Quadrature Tables

Various tables are available for quadrature, and still many others can be made through computational techniques. If Mathematica is accessible, there is no need to collect published quadrature tables. Any required table can be produced using Mathematica.

However, if Mathematica is not accessible, there are some important published tables which contain certain quadrature formulas and tables. Of such literature the following description explains the contents and extent of such tables.

(1) *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* by Abramowitz and Stegun (1968) contains formulas and tables on numerical integration in Ch. 25. The details are as follows:

- §25.4.1: Trapezoidal rule
- §25.4.2: Extended trapezoidal rule
- §25.4.4: Modified trapezoidal rule
- §25.4.5: Simpson's rule
- §25.4.6: Extended Simpson's rule
- §25.4.7: Euler–Maclaurin summation formula
- §25.4.8: Lagrange formula
- §25.4.11: Equally spaced abscissas
- §25.4.13: Simpson's 3/8-rule
- §25.4.14: Bode's rule
- §25.4.13–20: Newton–Cotes formulas (closed type)

§25.4.21–26: Newton–Cotes formulas (open type)

§25.4.28: Chebyshev's equal weight integration formula

§25.4.29–46: Gaussian-type integration formula

§25.4.29: Gauss' formula on the interval $[-1, 1]$

§25.4.30: Gauss' formula on arbitrary interval

§25.4.31: Radau's formula on the interval $[-1, 1]$

§25.4.32: Lobatto's formula on the interval $[-1, 1]$

§25.4.47: Filon's formula

Table 25.4, pp.916–919: Abscissas and weights for Gaussian integration on the interval $[-1, 1]$ for $n = 2(1)10, 12(4)24(8)48(16)96$

Table 25.5, p.920: Abscissas for equal weight Chebyshev integration on the interval $[-1, 1]$ for $n = 2(1)7, 9$

Table 25.6, p.920: Abscissas and weights for Labato integration on the interval $[-1, 1]$ for $n = 3(1)7, 10$

Table 25.7, p.920: Abscissas and weights for Gaussian integration for integrands with a logarithmic singularity on the interval $[0, 1]$ for $n = 2(1)4$.

Table 25.8, pp.921–922: Abscissas and weights for Gaussian integration of moments on the interval $[0, 1]$ for $n = 1(1)8$ and $k = 1(1)5$

Table 25.9, p.923: Abscissas and weights for Laguerre integration on the interval $[0, \infty)$ for $n = 2(1)10, 12, 15$

Table 25.10, p.924: Abscissas and weights for Hermite integration on the interval $(-\infty, \infty)$ for $n = 2(1)10, 12(4)20$

Table 25.11, p.924: Coefficients for Filon's quadrature

(2) The *Index of Mathematical Tables* by Fletcher, Miller, Rosenhead and Comrie (1962) contains:

- §23.5 Numerical integration
- §23.51 Cotes formulae
- §23.512 Sard's formulae
- §23.515 Other formulae with equal intervals
- §23.518 Integration of linear sums of exponential functions
- §23.52 Part-range formulae and others
- §23.53 Formulae for forward integration
- §23.54 Steffensen's formulae
- §23.55 Maclaurin's formulae
- §23.56 Filon's formulae
- §23.57 Chebyshev's formulae
- §23.575 Other formulae with equal coefficients
- §23.58 Formulae with unequal intervals and coefficients
- §23.581 Gaussian quadrature formulae
- §23.5815 Radau's quadrature formulae

§23.582	Gauss–Laguerre quadrature formulae
§23.583	Generalized Gauss–Laguerre quadrature formulae
§23.584	Gauss–Hermite quadrature formulae
§23.585	Other quadrature formulae with unequal intervals and coefficients
§23.588	A formula for repeated integration
§23.59	Lagrangian integration polynomials
§23.6	Integration and summation, using differences
§23.61	Single integrals anywhere in the interval
§23.62	Repeated integrals anywhere in the interval
§23.67	Osculatory quadrature formulae
§23.68	$J = 1/(nw) \int_0^{nw} f(x) dx$ in terms of differences
§23.69	Lubbock coefficients
§23.695	Summation of slowly convergent series
§23.7	Double integrals

(3) The *NBS Handbook* (1964) contains:

- pp.886–887 Newton–Cotes formulas (closed type), $n = 2(1)11$
 p.887 Newton–Cotes formulas (open type), $n = 2(1)7$
 p.915 Lagrangian integration coefficients $\int_{x_m}^{x_{m+1}} f(x) dx \approx \frac{1}{h} \sum_i A_i(m) f(x_i)$,
 $n = 3(1)10$
 pp.916–919 Abscissas and weights for Gaussian integration, $n = 2(1)12(4)24(8)$
 $48(16)96$
 p.920 Abscissas for equal weight Chebyshev integration $\int_{-1}^1 f(x) dx \approx \frac{2}{n} \sum_{i=1}^n f(x_i)$, $n = 2(1)7, 9$
 p.920 Abscissas and weights for Lobatto integration, $n = 3(1)10$
 p.920 Abscissas and weights for Gaussian integration for integrands with logarithmic singularity, $n = 2(1)12$
 pp.921–922 Abscissas and weights for Gaussian integration of moments
 $\int_0^1 x^m f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$, $m = 0(1)5, n = 1(1)8$
 p.923 Abscissas and weights for Laguerre integration, $n = 2(1)10, 12, 15$
 p.924 Abscissas and weights for Hermite integration, $n = 2(1)10, 12, 15$
 p.924 Coefficients for Filon’s quadrature formula, $n = \text{number of quadrature points}$

(4) *Gaussian Quadrature Formulas* by Stroud and Secrest (1966) contains integration formulas to 30 significant figures:

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i) \text{ (Gauss)}, \quad n = 2(1)64(4)96(8)168, 256, 384, 512.$$

$$\begin{aligned}
\int_{-1}^1 (1-x^2)^\alpha f(x) dx &\approx \sum_{i=1}^n w_i f(x_i) \quad n = 2(1)20; \quad \alpha = -0.5, 0.5, 1, 1.5. \\
\int_{-1}^1 (1+x)^\beta f(x) dx &\approx \sum_{i=1}^n w_i f(x_i) \quad n = 2(1)30 \text{ for } \beta = 1; \quad n = 2(10)20 \text{ for } \\
&\beta = 2, 3, 4. \\
\int_{-1}^1 |x|^\alpha f(x) dx &\approx \sum_{i=1}^n w_i f(x_i) \quad n = 2(1)20; \quad \alpha = 1, 2, 3, 4. \\
\int_{-\infty}^{\infty} e^{-x^2} f(x) dx &\approx \sum_{i=1}^n w_i f(x_i) \text{ (Hermite)}, \quad n = 2(1)64(4)96(8)136. \\
\int_{-\infty}^{\infty} e^{-x} f(x) dx &\approx \sum_{i=1}^n w_i f(x_i) \text{ (Laguerre)}, \quad n = 2(1)32(4)68. \\
\int_{-\infty}^{\infty} |x|^\alpha e^{-x^2} f(x) dx &\approx \sum_{i=1}^n w_i f(x_i), \quad n = 2(1)20; \quad \alpha = 1, 2, 3. \\
\int_{-\infty}^{\infty} |x|^\alpha e^{-|x|} f(x) dx &\approx \sum_{i=1}^n w_i f(x_i), \quad n = 2(1)20; \quad \alpha = 1, 2, 3. \\
\int_0^1 \ln(1/x) f(x) dx &\approx \sum_{i=1}^n w_i f(x_i) \quad n = 2(1)16. \\
\frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} p^{-1} e^p F(p) dp &\approx \sum_{i=1}^n w_i F(p_i), \quad n = 2(1)24. \\
\int_{-1}^1 f(x) dx &\approx Af(-1) + \sum_{i=1}^n A_i f(x_i) + Af(1) \text{ (Lobatto)}, \quad n = 2(1)32(4)96. \\
\int_{-1}^1 f(x) dx &\approx Af(-1) + \sum_{i=1}^n A_i f(x_i) \text{ (Radau)}, \quad n = 2(1)19(4)47. \\
\int_{-1}^1 f(x) dx &\approx \sum_{i=1}^n A_i f(x_i) + \sum_{j=0}^m B_{2j} f^{2j}(0) \text{ (Lobatto)}, \quad n = 2(2)16; \quad m = \\
&1, 2, 3. \\
\int_{-1}^1 e^{-x^2} f(x) dx &\approx \sum_{i=1}^n A_i f(x_i) + \sum_{j=0}^m B_{2j} f^{2j}(0) \text{ (Lobatto)}, \quad n = 2(2)16; \quad m = \\
&1, 2, 3.
\end{aligned}$$

(5) Boujot and Maroni (1968) provide tables of Gauss formulas for $n = 2(1)12, 12$ for the following weights:

$$x^{\alpha-1} (1-x)^\beta, \quad \alpha, \beta > 0, \text{ over the interval } [0, 1].$$

$$x^{\alpha-1} (1 \log 1/x)^\beta, \quad \alpha, \beta > 0, \text{ over the interval } [0, 1].$$

$$e^{-x} x^{\alpha-1}, \quad \alpha > 0, \text{ over the interval } [0, \infty).$$

$w(x)/x$ and $w(x)/x(1-x)$ over the interval $[0, 1]$, where

$$w(x) = \frac{1}{\pi^2 + \log^2 \left(\frac{1-x}{x} \right)}.$$

(6) Krylov and Pal'tsev (1974) have tabulated Gauss formulas for $n = 1(1)10, 15S^*$ for the following weights:

$x^\alpha \log e/x$, $\alpha = -0.9(0.01)0(0.1)5$, over the interval $[0, 1]$.

$x^\beta \log e/x \log(e/(1-x))$, $\beta = 0(1)5$, over the interval $[0, 1]$.

$\log(1/x)$ over the interval $[0, 1]$.

$x^\beta e^{-x} \log(1+1/x)$, $\beta = 0(1)5$, over the interval $[0, \infty)$.

(7) Berger (1969a,b) has tabulated zeros and weights for Gauss–Laguerre quadrature to 23–24S for $n = 100, 150, 200(100)900$, and for Gauss–Hermite quadrature to 26–27S for $n = 200(200)1000, 2000$.

(8) Piessens and Branders (1975) have tabulated Gauss formulas for selected values of n to $25S$ for the following weights:

$x^\alpha e^{-ax}$, $\alpha = -0.5, 0, 0.5$; $a = 1, 2, 5$, over the interval $[-1, 1]$.

e^{-ax^3} , $a = 1, 2, 5, 10$, over the interval $[-1, 1]$.

$x^{-\alpha} \log(1/x)$, $\alpha = -1/2, -1/3, 1/5, 1/4, 1/3, 1/2$, over the interval $[-1, 1]$.

$(1-x)^{-\alpha} x^{-\beta} \log(1/x)$, $\alpha, \beta = -1/2, -1/3, 1/5, 1/4, 1/3, 1/2$, over the interval $[0, 1]$.

$\cos x$, over the interval $[-\pi, \pi]$.

$\sin x$, over the interval $[-\pi, \pi]$.

e^{-ax^2} , $a = 1, 2, 5, 10$, over the interval $[-1, 1]$.

$(x+a)^{-\alpha}$, $a = 1.001, 1.01$; $\alpha = 1/2, 1, 2$, over the interval $[-1, 1]$.

$(x^2 + a^2)^{-\alpha}$, $a = 0.001, 0.01, 0.1, 1$; $\alpha = 1/2, 1, 2$, over the interval $[-1, 1]$.

(9) Miller (1960) in *Quadrature in terms of equally spaced Function values* has given almost every formula developed up to that date which expresses an integral as a weighted sum of equally spaced values of the integrand; both finite difference formulas and formulas in terms of equally spaced function values are available. A number of special formulas are also given. In many formulas the coefficients are given both as exact fractions and in decimal form. Error coefficients are given both for truncation and roundoff errors.

(10) Aizenshtat, Krylov and Metleskii (1962) in *Tables for calculating Laplace transforms and integrals of the form* $\int_0^\infty x^s e^{-x} f(x) dx$ give x_i, w_i , and $w_i e^{x_i}$

*Here, and in the sequel, S means “significant digits.”

for $s = -0.9(0.02)0, 0.55(0.05)3, -\frac{3}{4}, -\frac{1}{4}, m + j/3, \quad m = -1(1)2, \quad j = 1, 2, \quad n = 1(1)15, 8S.$

(11) Shao, Chen and Frank (1964) in *Tables of zeros and Gaussian weights of certain associated Laguerre polynomials and the related generalized Hermite polynomials* give x_i, w_i for generalized Laguerre integration for $\alpha = -0.5(0.5)10, n = 4, 8(8)32(16)64(32)128$, and for generalized Hermite integration for $\lambda = 0(1)10, n = 8(8)32(16)64(32)128(64)256$ to $25S$.

(12) Kronrod (1965) in *Nodes and Weights of Quadrature Formulas* gives tables in both decimal and octal forms for $n = 1(1)40$ to $16S$, arranging decimal forms on even-numbered and octal forms on odd-numbered pages.

REFERENCES USED: Abramowitz and Stegun (1968), Aizenshtat, Krylov and Metleskii (1962), Baker (1978), Berger (1969), Boujot and Maroni (1968), Clenshaw and Curtis (1960), Davis and Rabinowitz (1967, 1975), Fletcher, Miller, Rosenhead and Comrie (1962), Golub and Welsh (1967), Henrici (1964), Hornback (1975), Isaacson and Keller (1966), King (1984), Kronrod (1965), Krylov (1962), Krylov and Pal'tsev (1974), Kuo (1972), Miller (1960), NBS Handbook (1964), Piessens and Branders (1975), Shao, Chen, and Frank (1964), Squire (1970), Stroud (1974), Stroud and Secrest (1966), Wang (1976), Wolfram (1996).

B

Orthogonal Polynomials

A set of polynomials $\{f_i\}$ with degree i and such that $\langle f_i, f_j \rangle = 0$ for $i \neq j$ is called a set of orthogonal polynomials with respect to the inner product $\langle f_i, f_j \rangle$. Let $w(x)$ be an admissible weight function on a finite or infinite interval $[a, b]$. If we orthonormalize the powers $1, x, x^2, \dots$, we obtain a unique set of polynomials $p_n(x)$ of degree n and leading coefficient positive, such that

$$\int_a^b w(x) p_n(x) p_m(x) dx = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n, \end{cases} \quad (\text{B.1})$$

where δ_{mn} is known as the Kronecker delta. Table B.1 gives the classical polynomials corresponding to their specific weights and intervals.

Table B.1

Name	Symbol	Interval	$w(x)$
Chebyshev, 1st kind	$T_n(x)$	$[-1, 1]$	$(1 - x^2)^{-1/2}$
Chebyshev, 2nd kind	$U_n(x)$	$[-1, 1]$	$(1 - x^2)^{1/2}$
Gegenbauer (ultraspherical)	$C_n^\mu(x)$	$[-1, 1]$	$(1 - x^2)^{\mu-1/2}, \mu > -1/2$
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Jacobi	$P_n^{(\alpha, \beta)}(x)$	$[-1, 1]$	$(1 - x)^\alpha (1 + x)^\beta, \alpha, \beta > 1$
Laguerre	$L_n(x)$	$[0, \infty)$	e^{-x}
Generalized Laguerre	$L_n^{(\alpha)}$	$[0, \infty)$	$x^\alpha e^{-x}, \alpha > 1$
Legendre	$P_n(x)$	$[-1, 1]$	1

Orthogonal polynomials with respect to the above inner product satisfy another type of orthogonality, known as “discrete” orthogonality. Let \mathcal{P}_n denote the class of all polynomials $p_i(x)$, $i = 1, \dots, n+1$, such that

$$p_n(x_i) = \alpha_i, \quad i = 1, \dots, n+1, \quad (\text{B.2})$$

where $x_1 < x_2 < \dots < x_{n+1}$ are $(n+1)$ distinct points and $\alpha_1, \dots, \alpha_{n+1}$ arbitrary numbers.

Let $p_0(x), p_1(x), \dots, p_n(x), p_{n+1}(x)$ be orthonormal polynomials with the weight function $w(x)$ on $[a, b]$. Let x_1, \dots, x_{n+1} be $(n+1)$ zeros of $p_{n+1}(x)$ and let w_1, \dots, w_{n+1} be the respective Gaussian weights. Then, in view of Gaussian integration (see Q7, Appendix A)

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n+1} w_i f(x_i) \quad (\text{B.3})$$

for all $f \in \mathcal{P}_{2n+1}$. Now, since $p_j(x)p_k(x) \in \mathcal{P}_{2n+1}$ for $j, k \leq n$, we have

$$\sum_{i=1}^{n+1} w_i p_j(x_i) p_k(x_i) = \int_a^b w(x)p_j(x)p_k(x)dx = \delta_{jk}. \quad (\text{B.4})$$

Thus, p_0, p_1, \dots, p_n are orthonormal on the zeros of p_{n+1} with respect to the above inner product. This means that if we start with the monomials $1, x, x^2, \dots, x^n$ and orthonormalize them with respect to the discrete inner product $\langle f, g \rangle = \int_a^b w(x)f(x)g(x)dx$, we shall obtain orthonormal polynomials with respect to the continuous inner product defined above. Also, if $p_0(x), p_1(x), \dots$ are polynomials with $p_n(x) = c_n x^n + \dots, c_n > 0$, which are orthogonal with respect to the inner product $\langle f, g \rangle$, then we have the recurrence relation

$$p_{n+1}(x) = (\gamma_n x - a_{mn}) p_n(x) - a_{n,n-1} p_{n-1}(x) - \dots - a_{n0} p_0(x), \quad (\text{B.5})$$

for $n = 0, 1, \dots$, where

$$p_0(x) \equiv c_0, \quad \gamma_n = \frac{c_{n+1}}{c_n},$$

$$a_{nk} = \frac{\gamma_n \langle xp_n, p_k \rangle}{\langle p_k, p_k \rangle}, \quad k = 0, 1, \dots, n.$$

Moreover, if the above inner product satisfies the further condition $\langle xf, g \rangle = \langle f, xg \rangle$, then the recurrence relation (B.5) reduces to the three-term recurrence relation

$$p_{n+1}(x) = (\gamma_n x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x), \quad n = 0, 1, \dots, \quad (\text{B.6})$$

where we take $p_{-1}(x) = 0$, and

$$\alpha_n = \frac{\gamma_n \langle xp_n, p_n \rangle}{\langle p_n, p_n \rangle}, \quad n = 0, 1, \dots,$$

$$\beta_n = \frac{\gamma_n \langle xp_n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} = \frac{\gamma_n}{\gamma_{n-1}} \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle}, \quad n = 1, 2, \dots.$$

The three-term recurrence relation for $p_n(x)$ yields a “backward” recurrence for an efficient computation of a series expansion of the form

$$f(x) = \sum_{k=0}^N c_k p_k(x).$$

Simply set

$$B_k = \begin{cases} 0 & \text{for } k > n, \\ c_k + (\gamma_{k+1}x - \alpha_{k+1})B_{k+1} - \beta_{k+2}B_{k+2} & \text{for } 0 \leq k \leq n. \end{cases} \quad (\text{B.7})$$

Then

$$f(x) = \gamma_0 B_0. \quad (\text{B.8})$$

EXAMPLE B.1. To compute Chebyshev expansions $f(x) = \sum_{k=0}^N c_k T_k(x)$, choose $\alpha_k = 0$, $\beta_k = 1$, $\gamma_0 = \gamma_1 = 1$, and $\gamma_k = 2$ for $k \geq 2$.

Finally, orthogonal polynomials satisfy, among others, the Christoffel–Darboux identity

$$\sum_{k=0}^n \frac{p_k(x)p_k(y)}{h_k} = \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{\gamma_n h_n(x-y)}, \quad (\text{B.9})$$

where $h_k = \langle p_k, p_k \rangle$.

B.1. Zeros of Some Orthogonal Polynomials

Zeros of some orthogonal polynomials are used in quadrature rules. We consider some of the most frequently used polynomials. Information on others is available in the references cited ahead.

1. Chebyshev Polynomials of the First Kind $T_n(x)$ over the interval $[-1, 1]$, such that $T_n(1) = 1$. The m th zero $x_{n,m}$ of $T_n(x)$ is given by

$$x_{n,m} = \cos \frac{(2m-1)\pi}{2n}.$$

Other relevant data are as follows:

Norm: $\int_{-1}^1 (1-x^2)^{-1/2} [T_n(x)]^2 dx = \begin{cases} \frac{\pi}{2}, & n \neq 0, \\ \pi, & n = 0 \end{cases}$

Series form: $T_n(x) = \frac{n}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k} = \cos(n \arccos x)$

Indefinite and definite integrals: $\int T_0 dx = T_1, \int T_1 dx = \frac{T_2}{4},$

$$\int T_n dx = \frac{1}{2} \left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1} \right], \quad \int_{-1}^1 T_n dx = \begin{cases} \frac{2}{1-n^2}, & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$$

Inequality: $|T_n(x)| \leq 1, \quad -1 \leq x \leq 1$

Rodrigues' formula: $T_n(x) = \frac{(-1)^n (1-x^2)^{1/2} \sqrt{\pi}}{2^{n+1} \Gamma(n+1/2)} \frac{d^n}{dx^n} \{(1-x^2)^{n-1/2}\}.$

2. Chebyshev Polynomials of the Second Kind $U_n(x)$ over the interval $[-1, 1]$, such that $U_n(1) = n+1$. The m th zero $x_{n,m}$ of $U_n(x)$ is given by

$$x_{n,m} = \cos \frac{m\pi}{n+1}.$$

Other relevant data include:

Norm: $\int_{-1}^1 (1-x^2)^{1/2} [U_n(x)]^2 dx = \frac{\pi}{2}$

Series form: $U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k)!}{k!(n-2k)!} (2x)^{n-2k} = \frac{T'_{n+1}(x)}{n+1},$

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$$

Definite integral: $\int_{-1}^1 U_n dx = \begin{cases} \frac{2}{n+1}, & n = 2m, \\ 0, & n = 2m+1 \end{cases}$

Inequality: $|U_n(x)| \leq n+1, \quad -1 \leq x \leq 1,$

Rodrigues' formula: $U_n(x) = \frac{(-1)^n (n+1) \sqrt{\pi}}{(1-x^2)^{1/2} 2^{n+1} \Gamma(n+3/2)} \frac{d^n}{dx^n} \{(1-x^2)^{n+1/2}\}.$

3. Gegenbauer (or Ultraspherical) Polynomials $C_n^\mu(x)$ over the interval $[-1, 1]$ such that $C_n^\mu(1) = \binom{n+2\mu-1}{n}$. Other relevant data are the following:

$$\text{Norm: } \int_{-1}^1 (1-x^2)^{\mu-1/2} [C_n^\mu(x)]^2 dx = \frac{\pi 2^{1-2\mu} \Gamma(n+2\mu)}{n! (n+\mu) [\Gamma(\mu)]^2}$$

$$\text{Series form: } C_n^\mu(x) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(\mu+n-k)!}{k!(n-2k)!} (2x)^{n-2k}$$

$$\text{Inequality: } \max_{-1 \leq x \leq 1} |C_n^\mu(x)| = \begin{cases} \binom{n+2\mu-1}{n}, & \text{if } \mu > 0, \\ |C_n^\mu(x')|, & \text{if } -1/2 < \mu < 0 \end{cases}$$

where $x' = 0$ if $n = 2k$; $x' = \text{maximum point nearest zero}$ if $n = 2k + 1$.

$$\begin{aligned} \text{Rodrigues' formula: } C_n^\mu(x) &= \frac{(-1)^n 2^n n! \Gamma(\mu+n+1/2)}{\Gamma(\mu+1/2) \Gamma(n+2\mu) (1-x^2)^{\mu-1/2}} \times \\ &\quad \times \frac{d^n}{dx^n} \{(1-x^2)^{n+\mu-1/2}\}. \end{aligned}$$

4. Hermite Polynomials $H_n^{(\alpha)}(x)$ over the interval $(-\infty, \infty)$, such that

$$\text{Norm: } \int_{-\infty}^{\infty} e^{-x^2} [H_n^{(\alpha)}(x)]^2 dx = \sqrt{\pi} 2^n n!,$$

$$\text{Inequality: } |H_{2n}(x)| \leq e^{x^2/2} 2^{2n} n! \left[2 - \frac{1}{2^{2n}} \binom{2n}{n} \right],$$

$$|H_{2n+1}(x)| \leq |x| e^{x^2/2} \frac{(2n+2)!}{(n+1)!}$$

$$\text{Rodrigues' formula: } H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

5. Jacobi Polynomials $P_n^{\alpha,\beta}(x)$ over the interval $[-1, 1]$, such that $P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n}$. Other relevant data are:

$$\begin{aligned} \text{Norm: } \int_{-1}^1 (1-x)^\alpha (1+x)^\beta [P_n^{\alpha,\beta}(x)]^2 dx &= \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta)n! \Gamma(n+\alpha+\beta+1)} \end{aligned}$$

$$\text{Series form: } P_n^{\alpha,\beta}(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k} (x+1)^k,$$

$$\text{Inequality: } \max_{-1 \leq x \leq 1} |P_n^{\alpha, \beta}(x)| = \begin{cases} \binom{n+q}{n} \sim n^q & \text{if } q = \max(\alpha, \beta) \geq -1/2, \\ |P_n^{\alpha, \beta}|(x') \sim n^{-1/2} & \text{if } q < -1/2 \end{cases}$$

where x' is one of the two maximum points nearest $(\beta - \alpha)/(\alpha + \beta + 1)$,,

$$\begin{aligned} \text{Rodrigues' formula: } P_n^{\alpha, \beta}(x) &= \frac{(-1)^n}{2^n n! (1-x)^\alpha (1+x)^\beta} \\ &\times \frac{d^n}{dx^n} \{(1-x)^{n+\alpha} (1+x)^{n+\beta}\}. \end{aligned}$$

6. Laguerre Polynomials $L_n(x)$ over the interval $[0, \infty)$, such that $L_n(0) = n!$ and

$$\int_0^\infty e^{-x} L_n(x) L_m(x) dx = \begin{cases} 0 & \text{if } n \neq m, \\ (n!)^2 & \text{if } n = m. \end{cases}$$

Its m th zero $x_{n,m}$ is given by

$$x_{n,m} = \frac{j_m^2}{4k_n} \left(1 + \frac{j_m^2 - 2}{48k_n^2} \right) + O(n^{-5}),$$

where $k_n = n + 1/2$ and j_m is the m -th positive zero of the Bessel function $J_n(x)$. Other relevant data are

$$\text{Norm: } \int_0^\infty e^{-x} [L_n(x)]^2 dx = 1,$$

$$\text{Series form: } L_n(x) = \sum_{k=0}^n (-1)^k \binom{n}{n-k} \frac{1}{k!} x^k,$$

$$\text{Inequality: } |L_n(x)| = \begin{cases} e^{x/2}, & \text{if } x \geq 0, \\ \left[2 - \frac{1}{n!} \right] e^{x/2}, & \text{if } x \geq 0, \end{cases}$$

$$\text{Rodrigues' formula: } L_n(x) = \frac{1}{n! e^{-x}} \frac{d^n}{dx^n} \{x^n e^{-x}\}.$$

7. Generalized Laguerre Polynomials $L_n^{(\alpha)}(x)$ over the interval $[0, \infty)$, such that its m th zero $x_{n,m}$ is given by

$$x_{n,m} = \frac{j_{\alpha, \beta}^2}{4k_n} \left(1 + \frac{2(\alpha^2 - 1) + j_{\alpha, m}^2}{48k_n^2} \right) + O(n^{-5}),$$

where $k_n = n + (\alpha + 1)/2$, $\alpha > -1$, and $j_{\alpha, m}$ is the m -th positive zero of the

Bessel function $J_n(x)$. Other relevant data are

$$\text{Norm: } \int_0^\infty x^\alpha e^{-x} [L_n^{(\alpha)}(x)]^2 dx = \frac{\Gamma(n + \alpha + 1)}{n!}$$

$$\text{Series form: } L_n^{(\alpha)}(x) = \sum_{k=0}^n (-1)^k \binom{n + \alpha}{n - k} \frac{1}{k!} x^k$$

$$\text{Inequality: } |L_n^{(\alpha)}(x)| = \begin{cases} \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} e^{x/2}, & \text{if } x \geq 0, \alpha \geq 0, \\ \left[2 - \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(n + 1)} \right] e^{x/2}, & \text{if } x \geq 0, -1 < \alpha < 0, \end{cases}$$

$$\text{Rodrigues' formula: } L_n^{(\alpha)}(x) = \frac{1}{n! x^\alpha e^{-x}} \frac{d^n}{dx^n} \{x^{n+\alpha} e^{-x}\}.$$

8. Legendre Polynomials $P_n(x)$ over the interval $[-1, 1]$, such that $P_n(1) = 1$. If $x_{n,m}$ denotes the m th zero of $P_n(x)$, where $x_{n,1} > x_{n,2} > \dots > x_{n,n}$, then

$$x_{n,m} = \left(1 - \frac{1}{8n^2} + \frac{1}{8n^3}\right) \cos \frac{(4m-1)\pi}{4n+2} + O(n^{-4}).$$

Other relevant data are:

$$\text{Norm: } \int_{-1}^1 [P_n(x)]^2 dx = \frac{2}{2n+1},$$

$$\text{Series form: } P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k},$$

$$\text{Indefinite Integral: } \int P_n(x) dx = \frac{1}{2n+1} [P_{n+1}(x) - P_{n-1}(x)],$$

$$\text{Inequality: } |P_n(x)| \leq 1, \quad -1 \leq x \leq 1,$$

$$\text{Rodrigues' formula: } P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \{(1-x^2)^n\}.$$

Zeros of the Legendre polynomial of degree 9 are

$$x_1 = -0.96816, x_2 = -0.836031, x_3 = -0.613372, x_4 = -0.324253, \\ x_5 = 0, x_6 = 0.324253, x_7 = 0.613372, x_8 = 0.836031, x_9 = 0.96816$$

Zeros of the shifted Legendre polynomial of degree 9 are

$$y_1 = 0.01592, y_2 = 0.081984, y_3 = 0.193314, y_4 = 0.337874, y_5 = 0.5, \\ y_6 = 0.662127, y_7 = 0.806686, y_8 = 0.918016, y_9 = 0.98408.$$

Some of the weights w'_j corresponding to the shifted Legendre polynomials are $w'_1 = w'_9 = 0.040637$, $w'_2 = w'_8 = 0.090325$, $w'_3 = w'_7 = 0.13030$, $w'_4 = w'_6 = 0.15617$, $w'_5 = 0.16512$.

REFERENCES USED: Abramowitz and Stegun (1964), Davis and Rabinowitz (1967).

C

Whittaker's Cardinal Function

C.1. Basic Results

Let $h > 0$ be the step size, and let $\mathcal{B}(h)$ denote the family of functions $f \in L_2(\mathbb{R})$ that are analytic on the entire complex plane \mathbb{C} such that

$$|f(z)| \leq C e^{\pi|z|/h}, \quad (\text{C.1.1})$$

where C is a positive constant. Let m denote an integer, and set

$$S(m, h)(z) = \frac{\sin \frac{\pi}{h}(z - mh)}{\frac{\pi}{h}(z - mh)}. \quad (\text{C.1.2})$$

Then for f defined in \mathbb{R} the Whittaker's cardinal function for f with step size h is defined by

$$C(f, h)(z) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} f(mh) S(m, h)(z). \quad (\text{C.1.3})$$

This function was discovered by E. T. Whittaker (1915), who studied its mathematical properties and used it as an alternate expression for entire functions. He posed the interpolation problem of finding a function that passes through the points $(a + nw, f_n)$, where a, w are complex, and n is an integer. He called the class

of all such functions the *cotabular set* associated with $\{f_n\}$ and showed that the sum of the cardinal series, as it is called now, picks out a special member that we call now the “cardinal” function of this cotabular set. Whittaker found that this function is special because it is entire and has no “violent oscillations,” i.e., it is “band-limited.” The Whittaker series for the cardinal function f , as given in his (1915) article, has the form $\sum f(a_n w) \frac{\sin \pi(x - a - nw)/w}{\pi(x - a - nw)/w}$. The name “cardinal series” was assigned to it by his second son, J. M. Whittaker, around 1920. The younger Whittaker extended this study in his 1927 and 1935 publications. The function $C(f, h)$ has played an important role in engineering applications for approximating f in the transformation of information (Hartley 1928, Nyquist 1928, and Shannon 1948). Since then engineers have called $C(f, h)$ the “sinc function” or the “band-limited” expansion of f , as mentioned above. An interesting account about the philosophy and historical analysis of this function can be found in Higgins (1985).

We now discuss some properties of the function $C(f, h)$. Let

$$\delta_{jm}^{(n)} = S^{(n)}(j, 1)(m) = \left(\frac{d}{dx} \right)^n S(j, 1)(x) \Big|_{x=m}. \quad (\text{C.1.4})$$

In particular,

$$\begin{aligned} \delta_{jm}^{(0)} &= \begin{cases} 1 & \text{if } j = m, \\ 0 & \text{if } j \neq m, \end{cases} \\ \delta_{jm}^{(1)} &= \begin{cases} 0 & \text{if } j = m, \\ \frac{(-1)^{m-j}}{m-j} & \text{if } j \neq m, \end{cases} \\ \delta_{jm}^{(2)} &= \begin{cases} -\frac{\pi^2}{3} & \text{if } j = m, \\ -\frac{2(-1)^{m-j}}{(m-j)^2} & \text{if } j \neq m. \end{cases} \end{aligned} \quad (\text{C.1.5})$$

The following results present explicit relations between $C(f, h)$ and $S(m, h)$. Let $f \in \mathcal{B}(h)$. Then

$$f(z) = C(f, h)(z) \quad \text{for all } z \in \mathbb{C}; \quad (\text{C.1.6})$$

$$\int_{\mathbb{R}} f(x) dx = h \sum_{m=-\infty}^{\infty} f(mh); \quad (\text{C.1.7})$$

$$\int_{\mathbb{R}} |f(x)|^2 dx = h \sum_{m=-\infty}^{\infty} |f(mh)|^2; \quad (\text{C.1.8})$$

thus, the set $\{h^{-1/2} S(m, h)\}_{m=-\infty}^{\infty}$ is a complete orthonormal sequence in $\mathcal{B}(h)$.

There exists a unique function $g \in L_2(-\pi/h, \pi/h)$ such that

$$f(z) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-izs} g(s) ds, \quad (\text{C.1.9})$$

where $g(x)$ is defined as the complex Fourier transform

$$g(x) = \int_{\mathbb{R}} e^{ixs} f(s) ds = \begin{cases} h \sum_{m=-\infty}^{\infty} f(mh) e^{-imhx} & \text{if } -\frac{\pi}{h} < x < \frac{\pi}{h}, \\ 0 & \text{if } |x| > \frac{\pi}{h}. \end{cases} \quad (\text{C.1.10})$$

Thus,

$$f(z) = \frac{1}{h} \int_{\mathbb{R}} f(s) \frac{\sin \frac{\pi}{h}(z-s)}{\frac{\pi}{h}(z-s)} ds; \quad (\text{C.1.11})$$

$$f' \in \mathcal{B}(h), ; \quad (\text{C.1.12})$$

$$f^{(n)}(mh) = h^{-n} \sum_{j=-\infty}^{\infty} \delta_{jm}^{(n)} f(jh), \quad (\text{C.1.13})$$

where $\delta_{jm}^{(n)}$ is defined by (C.1.4), and thus, in view of (C.1.12) and (C.1.13), we have

$$f^{(n)}(x) = h^{-n} \sum_{j=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} \delta_{jm}^{(n)} f(jh) \right] S(m, h)(x). \quad (\text{C.1.14})$$

If g is defined as in (C.1.9), then

$$\int_0^x f(s) ds = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} g(s) \frac{1 - e^{-isx}}{is} ds, \quad (\text{C.1.15})$$

and, in particular,

$$\int_0^x S(m, h)(s) ds = h \left[\sigma_m + \frac{1}{\pi} \int_0^\pi \frac{\sin \left(\frac{x}{h} - m \right) s}{s} ds \right], \quad (\text{C.1.16})$$

where

$$\sigma_n = \frac{1}{\pi} \int_0^\pi \frac{\sin ns}{s} ds = \int_0^n \frac{\sin \pi x}{\pi x} dx \quad \text{by taking } ns = \pi x. \quad (\text{C.1.17})$$

Moreover, if $\int_{\mathbb{R}} f(s) ds = 0$, and if $\int_{-\infty}^x f(s) ds$ is in $\mathcal{B}(h)$, then

$$\int_{-\infty}^x f(s) ds = h \sum_{m=-\infty}^{\infty} \left[\sum_{j=-\infty}^{\infty} \sigma_{m-j} f(jh) \right] S(m, h)(x). \quad (\text{C.1.18})$$

Note that σ_n in (C.1.16) defines the integral of the sinc function for $n = 0, 1, 2, \dots$; these values are computed in `sinc1.nb`, and their distribution is presented in Fig.C.1.1.

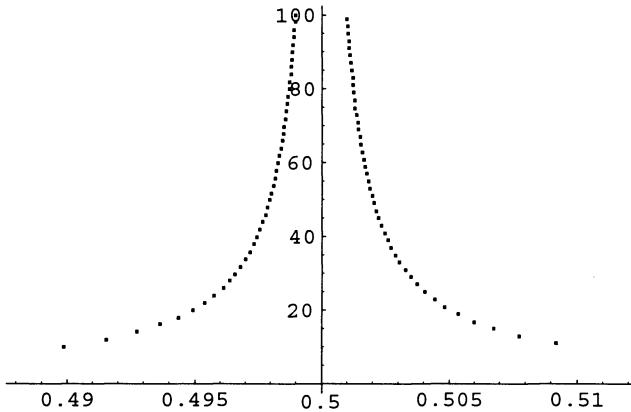


Fig. C.1.1. Distribution of σ_n for $n = 0, 1, \dots, 100$.

Let $0 < \alpha < 1$, and let g be defined as in (C.1.10). Then

$$\int_{\mathbb{R}} |x-s|^{\alpha-1} f(s) ds = \frac{\Gamma(\alpha) \cos(\pi\alpha/2)}{\pi} \int_{-\pi/h}^{\pi/h} |s|^{-\alpha} g(s) e^{-ixs} ds, \quad (\text{C.1.19})$$

and, in particular,

$$\begin{aligned} & \int_{\mathbb{R}} |x-s|^{\alpha-1} S(m, h)(s) ds \\ &= \frac{2h^\alpha \Gamma(\alpha) \cos(\pi\alpha/2)}{\pi} \int_0^\pi s^{-\alpha} \cos((x-mh)s/h) ds, \end{aligned} \quad (\text{C.1.20})$$

so that

$$\begin{aligned} & \int_{\mathbb{R}} |x-s|^{\alpha-1} f(s) ds \\ &= \frac{2h^{\alpha}\Gamma(\alpha) \cos(\pi\alpha/2)}{\pi} \sum_{m=-\infty}^{\infty} f(mh) \int_0^{\pi} s^{-\alpha} \cos((x-mh)s/h) ds. \end{aligned} \quad (\text{C.1.21})$$

Finally, let g be defined as in (C.1.10). Then

$$\int_{\mathbb{R}} \log|x-s| f(s) ds = \frac{1}{4} \int_{-\pi/h}^{\pi/h} \frac{g(0) - e^{-ixs} g(s)}{|s|} ds - \frac{1}{2} \left[\gamma + \log\left(\frac{\pi}{h}\right) \right] g(0), \quad (\text{C.1.22})$$

and, in particular,

$$\begin{aligned} & \int_{\mathbb{R}} \log|x-s| S(m, h)(s) d \\ &= -\frac{h}{2} \left\{ \gamma + \log\left(\frac{\pi}{h}\right) - \int_0^{\pi} \frac{1 - \cos((x-mh)s/h)}{s} ds \right\}, \end{aligned} \quad (\text{C.1.23})$$

so that

$$\begin{aligned} & \int_{\mathbb{R}} \log|x-s| f(s) ds \\ &= -\frac{h}{2} \sum_{m=-\infty}^{\infty} f(mh) \left\{ \gamma + \log\left(\frac{\pi}{h}\right) - \int_0^{\pi} \frac{1 - \cos((x-mh)s/h)}{s} ds \right\}. \end{aligned} \quad (\text{C.1.24})$$

The cardinal series

$$\sum_{n=1}^{\infty} c_n \frac{\sin \pi(t-n)}{\pi(t-n)} = \frac{\sin \pi t}{\pi} \sum_{n=1}^{\infty} c_n \frac{(-1)^n}{t-n}, \quad (\text{C.1.25})$$

can be obtained by some simple methods, two of which are as follows:

(i) Consider a function $f(z)$ at points $z = -m, \dots, 9, \dots, m$. The Lagrange interpolation formula that interpolates $f(z)$ at these points is given by

$$H_m(z) \left\{ \frac{f(0)}{z} + \sum_{n=1}^{\infty} \frac{f(n)}{H'_m(n)(z-n)} + \frac{f(-n)}{H'_m(-m)(z+n)} \right\}, \quad (\text{C.1.26})$$

where

$$H_m(z) = z \prod_{n=1}^m \left(1 - \frac{z^2}{n^2}\right),$$

and the cardinal series (C.1.25) is obtained by letting $m \rightarrow \infty$. This method is due to Brown (1915–1916) and Ferrar (1925, p.270).

(ii) A formal approach is to regard the cardinal series as a special case of Cauchy's partial-fractions expansion for a suitably restricted meromorphic function $F(z)$ with poles at the points t_n , i.e.,

$$F(z) = \sum \text{Res of } \frac{F(w)}{z-w} \text{ at the points } w = t_n;$$

then apply this to $F(z) = f(z) \csc \pi z$, where f is entire (see Ferrar 1925, p. 281).

C.2. Approximation of an Integral

The approximation of an integral on an interval $\Gamma = [a, x]$ of the type

$$I(x) = \int_a^x g(t) dt, \quad x \in \Gamma, \quad (\text{C.2.1})$$

is given by the following result (Stenger 1981).

THEOREM C.2.1. Let $g \in B_1(\Omega_d)$ and let $\left| \frac{g(x)}{F'(x)} \right| \leq C e^{-\alpha' |F(x)|}$ along Γ , where C and α' are positive constants. Let $0 < \beta < \pi/d$, and let

$$q(x) = \int_a^x g(t) dt - \frac{e^{\beta F(x)/2}}{e^{\beta F(x)/2} e^{-\beta F(x)/2}} \int_a^b g(t) dt \in B_1(\Omega_d), \quad (\text{C.2.2})$$

where Ω_d is defined in §10.2. If $\alpha = \min(\alpha', \beta)$, then for $h = \sqrt{\pi d / (\alpha N)}$ and all $x \in \Gamma$

$$\begin{aligned} \left| q(x) - h \sum_{j=-N}^N \left\{ \sum_{m=-N}^N \sigma_{j-m} \left[\frac{g(z_m)}{F'(z_m)} - \frac{\beta}{(e^{\beta m h/2} - e^{-\beta m h/2})^2} \int_\Gamma g(t) dt \right] \right\} \right. \\ \times S(j, h) \circ F(x) \Big| \leq C_1 N e^{-\sqrt{\pi d \alpha N}}, \end{aligned} \quad (\text{C.2.3})$$

where σ_j is defined by (C.1.17), and C_1 depends only on g, d and α .

The formula (C.2.3) is useful in approximating integrals like $\int_0^x t^{-2/3}(1-t)^{-5/4} \log t dt$, $x \in [0, 1]$. However, in practice, the integral (C.2.1) is generally approximated by the formula

$$\left| \int_{\Gamma} g(t) dt - h \sum_{j=-N}^N \frac{g(z_j)}{F'(z_j)} \right| \leq C_1 e^{-\sqrt{2\pi d \alpha N}}, \quad (\text{C.2.4})$$

where C_1 depends only on g, d , and α .

REFERENCES USED: Brown (1915–1916), Carslaw (1921), Ferrar (1925), Hartley (1928), Higgins (1985), McNamee, Stenger and Whitney (1971), Nyquist (1928), Shannon (1948), Stenger (1981), Walker (1991), Whittaker (1915), Whittaker and Watson (1973).

D

Singular Integrals

D.1. Cauchy Principal-Value Integrals

Consider the improper integral $\int_a^b \frac{dx}{x-s}$, $a < s < b$, which is evaluated as follows:

$$\begin{aligned}\int_a^b \frac{dx}{x-s} &= \lim_{\substack{\varepsilon_1 \rightarrow 0 \\ \varepsilon_2 \rightarrow 0}} \left(- \int_a^{s-\varepsilon_1} \frac{dx}{s-x} + \int_{s+\varepsilon_2}^b \frac{dx}{x-s} \right) \\ &= \ln \frac{b-s}{s-a} + \lim_{\substack{\varepsilon_1 \rightarrow 0 \\ \varepsilon_2 \rightarrow 0}} \ln \frac{\varepsilon_1}{\varepsilon_2}.\end{aligned}\tag{D.1.1}$$

The limit on the right side depends on how ε_1 and ε_2 go to zero. This integral is known as a *singular integral*. But this integral can become meaningful if we assume that a relationship exists between ε_1 and ε_2 . For example, if $\varepsilon_1 = \varepsilon_2 = \varepsilon$, i.e., if the deleted interval is symmetric about the point s , then it leads to the concept of the Cauchy principal value (p.v.) of the singular integral, which, denoted by a bar across the integral sign and defined as

$$\bar{\int}_a^b \frac{dx}{x-s}, \quad a < s < b,\tag{D.1.2}$$

is the number $\lim_{\varepsilon \rightarrow 0} \left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b \right) \frac{dx}{x-s}$ which, in view of (D.1.1), has the value

$$\int_a^b \frac{dx}{x-s} = \ln \frac{b-s}{s-a}. \quad (\text{D.1.3})$$

The notation used for the Cauchy p.v. integrals is, however, not necessary, because if an integral of the type $\int_a^b \frac{\phi(x)}{x-s} dx$ exists as a proper or improper integral, then it exists only in the sense of the Cauchy p.v., and their values coincide. But if we regard a singular integral in the sense of the Cauchy p.v. integral, the above notation becomes very useful.

For a generalization, consider a singular integral over the interval $[a, b]$ such that the integrand has a singularity of the type $\frac{1}{x-s}$ at an interior point s , $a < s < b$, and the regular part of the integrand is a function $\phi(x) \in C^0[a, b]$ which satisfies the Hölder condition

$$|\phi(x) - \phi(s)| \leq A |x-s|^\alpha,$$

where $0 < \alpha \leq 1$, and $|A| < \infty$. Then the Cauchy p.v. integral is defined by

$$\int_a^b \frac{\phi(x)}{x-s} dx = \lim_{\varepsilon \rightarrow 0} \left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b \right) \frac{\phi(x)}{x-s} dx, \quad a < s < b, \quad (\text{D.1.4})$$

where

$$\begin{aligned} \int_a^s \frac{\phi(x)}{x-s} dx &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_a^{s-\varepsilon} \frac{\phi(x)}{x-s} dx - \phi(s) \ln \varepsilon \right\}, \\ \int_s^b \frac{\phi(x)}{x-s} dx &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{s+\varepsilon}^b \frac{\phi(x)}{x-s} dx + \phi(s) \ln \varepsilon \right\}, \end{aligned} \quad (\text{D.1.5})$$

provided these limiting processes are taken together, since each of them does not exist independently. In fact, using the identity

$$\int_a^b \frac{\phi(x)}{x-s} dx = \int_a^b \frac{\phi(x) - \phi(s)}{x-s} dx + \phi(s) \int_a^b \frac{dx}{x-s}$$

and (D.1.3), we find that

$$\int_a^b \frac{\phi(x)}{x-s} dx = \int_a^b \frac{\phi(x) - \phi(s)}{x-s} dx + \phi(s) \ln \frac{b-s}{s-a}, \quad (\text{D.1.6})$$

since $\phi(x)$ satisfies the Hölder condition on $[a, b]$.

D.2. P.V. of a Singular Integral on a Contour

Consider the singular integral

$$\int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta \quad \text{for } z, \zeta \in \Gamma, \quad (\text{D.2.1})$$

on a contour Γ with endpoints a and b ($a = b$ if Γ is closed). Draw a circle of radius ρ with center at a point $z \in \Gamma$, and let z_1 and z_2 be the points of intersection of this circle with Γ . We take ρ small enough so that the circle has no other points of intersection with Γ . Let γ denote the part of the contour Γ cut out by this circle. Then the integral on the remaining contour $\Gamma - \gamma$ is $\int_{\Gamma - \gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta$, and the principal value of the singular integral (D.2.1) is given by $\lim_{\rho \rightarrow 0} \int_{\Gamma - \gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta$.

Since

$$\int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta = \int_{\Gamma} \frac{\phi(\zeta) - \phi(z)}{\zeta - z} d\zeta + \phi(z) \int_{\Gamma} \frac{d\zeta}{\zeta - z},$$

we find in the same way as in §D.1 that the singular integral (D.2.1) exists as a Cauchy p.v. integral for any function $\phi(\zeta)$ that satisfies the Hölder condition, and

$$\int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta = \int_{\Gamma} \frac{\phi(\zeta) - \phi(z)}{\zeta - z} d\zeta + \phi(z) \ln \frac{b - z}{z - a}, \quad (\text{D.2.2a})$$

or

$$\int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta = \int_{\Gamma} \frac{\phi(\zeta) - \phi(z)}{\zeta - z} d\zeta + \phi(z) \left(\ln \frac{b - z}{z - a} + i\pi \right). \quad (\text{D.2.2b})$$

Moreover, if the contour Γ is closed, then we let $a = b$ and obtain

$$\int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta = \int_{\Gamma} \frac{\phi(\zeta) - \phi(z)}{\zeta - z} d\zeta + i\pi \phi(z). \quad (\text{D.2.3})$$

Now consider the Cauchy-type integral

$$\Phi(z) = \frac{1}{2i\pi} \int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta, \quad (\text{D.2.4})$$

where Γ is a closed or nonclosed contour and $\phi(\zeta)$ satisfies the Hölder condition on Γ . This integral has limit values $\Phi^+(z)$ and $\Phi^-(z)$ at any point $z \in \Gamma$ (z not

equal to a or b) as $z \rightarrow \zeta$ from the left or from the right, respectively, along any path. These two limit values can be expressed in terms of the function $\phi(\zeta)$, which is known as the *density* of the integral, by the PLEMELJ FORMULAS

$$\begin{aligned}\Phi^+(z) &= \frac{1}{2} \phi(z) + \frac{1}{2i\pi} \int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta, \\ \Phi^-(z) &= -\frac{1}{2} \phi(z) + \frac{1}{2i\pi} \int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta,\end{aligned}\tag{D.2.5}$$

such that

$$\begin{aligned}\Phi^+(z) + \Phi^-(z) &= \frac{1}{i\pi} \int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta, \\ \Phi^+(z) - \Phi^-(z) &= \phi(z).\end{aligned}\tag{D.2.6}$$

If Γ is the real axis, then the Plemelj formulas become

$$\begin{aligned}\Phi^+(x) &= \frac{1}{2} \phi(x) + \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\phi(s)}{s - x} ds, \\ \Phi^-(x) &= -\frac{1}{2} \phi(x) + \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\phi(s)}{s - x} ds,\end{aligned}\tag{D.2.7}$$

and

$$\Phi^+(\infty) = \frac{1}{2} \phi(\infty), \quad \Phi^-(\infty) = -\frac{1}{2} \phi(\infty).\tag{D.2.8}$$

Hence, from (D.2.7) and (D.2.8) we have

$$\Phi^+(\infty) + \Phi^-(\infty) = 0, \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\phi(s)}{s - x} ds = 0.\tag{D.2.9}$$

We use the first property in (D.2.9) to represent a piecewise analytic function in the upper and the lower half-plane by an integral over the real axis. Consider a Cauchy-type integral over the real axis:

$$\Phi(z) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\phi(x)}{x - z} dx,\tag{D.2.10}$$

where z is complex and $\Phi(z)$ is a complex-valued function of a real variable x , which satisfies the Hölder condition on the real axis. If a function $\phi(x)$ is analytic on the upper half-plane D^+ , is continuous on the closed upper half-plane, and satisfies the Hölder condition on the real axis, then

$$\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\phi(x)}{x - z} dx = \begin{cases} \phi(z) - \frac{1}{2} \phi(\infty) & \text{for } \Im\{z\} > 0, \\ -\frac{1}{2} \phi(\infty) & \text{for } \Im\{z\} < 0; \end{cases}\tag{D.2.11}$$

also,

$$\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(\infty)}{x - z} dx = \begin{cases} \frac{1}{2}\phi(\infty) & \text{for } \Im\{z\} > 0, \\ -\phi(z) + \frac{1}{2}\phi(\infty) & \text{for } \Im\{z\} < 0, \end{cases} \quad (\text{D.2.12})$$

provided $\phi(z)$ is analytic on the lower half-plane D^- , is continuous on the closed lower half-plane, and satisfies the Hölder condition on the real axis.

EXAMPLE D.2.1. If $\Gamma = \{|z| = 1\}$, and $\phi(\zeta) = \frac{2}{\zeta(\zeta - 2)} = \frac{1}{\zeta - 2} - \frac{1}{\zeta}$, and $\phi(\infty) = 0$, then

$$\begin{aligned} \Phi(z) &= \frac{1}{2i\pi} \int_{\Gamma} \frac{1}{\zeta - 2} \frac{d\zeta}{\zeta - z} - \frac{1}{2i\pi} \int_{\Gamma} \frac{1}{\zeta} \frac{d\zeta}{\zeta - z} \\ &= \begin{cases} \frac{1}{z - 2} - \frac{1}{z} & \text{for } \Im\{z\} > 0, \\ \frac{1}{z} & \text{for } \Im\{z\} < 0, \end{cases} \end{aligned}$$

by (D.2.11) and (D.2.12). Hence,

$$\Phi^+(z) = \frac{1}{z - 2} \quad \text{and} \quad \Phi^-(z) = \frac{1}{z}. \blacksquare$$

Another useful formula is the POINCARÉ–BERTRAND FORMULA:

$$\frac{1}{i\pi} \int_{\Gamma} \frac{d\zeta}{\zeta - z} \cdot \frac{1}{i\pi} \int_{\Gamma} \frac{g(\zeta, \zeta_1)}{\zeta - \zeta_1} d\zeta_1 = g(z, z) + \frac{1}{i\pi} \int_{\Gamma} d\zeta_1 \cdot \frac{1}{i\pi} \int_{\Gamma} \frac{g(\zeta, \zeta_1)}{(\zeta - z)(\zeta - \zeta_1)} d\zeta, \quad (\text{D.2.13})$$

or, alternatively,

$$\int_{\Gamma} \frac{d\zeta}{\zeta - z} \cdot \int_{\Gamma} \frac{g(\zeta, \zeta_1)}{\zeta - \zeta_1} d\zeta_1 = \pi^2 g(z, z) + \int_{\Gamma} d\zeta_1 \cdot \int_{\Gamma} \frac{g(\zeta, \zeta_1)}{(\zeta - z)(\zeta - \zeta_1)} d\zeta, \quad (\text{D.2.14})$$

where the function $g(\zeta, \zeta_1)$ satisfies the Hölder condition with respect to both variables. This formula is useful when changing the order of integration in singular integrals. Thus, the pair of iterated singular integrals

$$\begin{aligned} F(\zeta) &= \frac{1}{i\pi} \int_{\Gamma} \frac{d\zeta}{\zeta - z} \cdot \frac{1}{i\pi} \int_{\Gamma} \frac{g(\zeta, \zeta_1)}{\zeta - \zeta_1} d\zeta_1, \\ G(\zeta) &= \frac{1}{i\pi} \int_{\Gamma} d\zeta_1 \cdot \frac{1}{i\pi} \int_{\Gamma} \frac{g(\zeta, \zeta_1)}{(\zeta - z)(\zeta - \zeta_1)} d\zeta, \end{aligned}$$

are not equal, although they differ in the order of integration.

D.3. Hadamard's Finite-Part Integrals

If $\phi(x) \in C^1[a, b]$, the Cauchy p.v. integral (D.1) becomes the first-order two-sided Hadamard's (or finite-part) integral, and the integrals (D.2) are then equivalent to first-order one-sided finite-part integrals.

Consider an improper integral on the interval $[a, b]$ such that (i) the integrand has a singularity of the type $\frac{1}{(x-s)^2}$ at an interior point s , $a < s < b$, and (ii) the regular part of the integrand is a function $\phi(x)$, $a \leq x \leq b$, which satisfies a Hölder continuous first-derivative condition

$$|\phi(x) - \phi(s) - (x-s)\phi'(s)| \leq A|x-s|^{\alpha+1},$$

where $0 < \alpha \leq 1$ and $|A| < \infty$, as before. Then a Hadamard's (finite-part) integral is defined by

$$\oint_a^b \frac{\phi(x)}{(x-s)^2} dx = \lim_{\varepsilon \rightarrow 0} \left\{ \left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b \right) \frac{\phi(x)}{(x-s)^2} dx - \frac{2\phi(s)}{\varepsilon} \right\}, \quad (\text{D.3.1})$$

where the neighborhood ε is symmetric about the singular point s . This integral can also be evaluated on both sides of the singular point as first-order one-sided integrals

$$\begin{aligned} \oint_a^s \frac{\phi(x)}{(x-s)^2} dx &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_a^{s-\varepsilon} \frac{\phi(x)}{(x-s)^2} dx - \frac{\phi(s)}{\varepsilon} - \phi'(s) \ln s \right\}, \\ \oint_s^b \frac{\phi(x)}{(x-s)^2} dx &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{s+\varepsilon}^b \frac{\phi(x)}{x-s} dx - \frac{\phi(s)}{\varepsilon} + \phi'(s) \ln s \right\}, \end{aligned} \quad (\text{D.3.2})$$

both of which must be taken together.

Differentiation of the Cauchy p.v. integrals with respect to s is carried out by

using the Leibniz rule* which yields

$$\begin{aligned} \frac{d}{ds} \int_a^b \frac{\phi(x)}{x-s} dx &= \lim_{\varepsilon \rightarrow 0} \frac{d}{ds} \left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b \right) \frac{\phi(x)}{x-s} dx \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b \right) \frac{\phi(x)}{(x-s)^2} dx - \frac{\phi(s-\varepsilon) + \phi(s+\varepsilon)}{\varepsilon} \right\}. \end{aligned} \quad (\text{D.3.3})$$

Since ϕ is continuous on $a \leq x \leq b$, we have $\phi(s-\varepsilon) = \phi(s+\varepsilon) = \phi(s)$ as $\varepsilon \rightarrow 0$. Then formula (D.3.3) becomes

$$\frac{d}{ds} \int_a^b \frac{\phi(x)}{x-s} dx = \lim_{\varepsilon \rightarrow 0} \left\{ \left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b \right) \frac{\phi(x)}{x-s} dx - \frac{2\phi(s)}{\varepsilon} \right\}, \quad (\text{D.3.4})$$

which is the same as (D.3.1). Hence, we obtain the formula

$$\frac{d}{ds} \int_a^b \frac{\phi(x)}{x-s} dx = \int_a^b \frac{\phi(x)}{(x-s)^2} dx, \quad (\text{D.3.5})$$

which is very useful in evaluating Hadamard's finite-part integrals.

D.4. Two-Sided Finite-Part Integrals

If we take $\phi(x) = \phi(x) - \phi(s) + \phi(s)$ in the Cauchy p.v. integral (D.1), then

$$\int_a^b \frac{\phi(x)}{x-s} dx = \int_a^b \frac{\phi(x) - \phi(s)}{x-s} dx + \phi(s) \int_a^b \frac{dx}{x-s}, \quad (\text{D.4.1})$$

where

$$\int_a^b \frac{dx}{x-s} = [\ln|x-s|]_a^b = \ln \left| \frac{b-s}{s-a} \right|. \quad (\text{D.4.2})$$

*The Leibniz rule for differentiating an integral with respect to a parameter is

$$\frac{\partial}{\partial y} \int_a^b g(t, y) dt = \int_a^b g_y(t, y) dt,$$

$$\frac{\partial}{\partial y} \int_{h_1(y)}^{h_2(y)} g(t, y) dt = \int_{h_1(y)}^{h_2(y)} g_y(t, y) dt + h'_2(y) g(h_2(y), y) - h'_1(y) g(h_1(y), y).$$

For details, see Williamson, Crowell, and Trotter (1962, p.316).

If $\phi \in C^1[a, b]$, the continuity requirement on $\phi'(x)$ is a necessary condition for a Taylor's series expansion of the function $\phi(x)$. But if $\phi(x) \notin C^1$, i.e., it does not have a high degree of continuity so as not to allow a Taylor's series expansion, a sufficient condition for the existence of finite-part integral is that $\phi(x)$ satisfy a Hölder or Lipschitz condition, i.e., $|\phi(x) - \phi(s)| \leq A|x - s|^\alpha$, $0 < \alpha \leq 1$, $|A| < \infty$. This condition also guarantees that the first integral on the right side of (D.4.1) is, at most, weakly singular. Now, integrating (D.1.4) by parts we get

$$\int_a^b \frac{\phi(x)}{x-s} dx = \phi(b) \ln|b-s| - \phi(a) \ln|a-s| - \int_a^b \phi'(x) \ln|x-s| dx, \quad (\text{D.4.3})$$

where $\phi(x) \in C^1[a, b]$. Then differentiation of (D.4.3) with respect to s yields

$$\frac{d}{ds} \int_a^b \frac{\phi(x)}{x-s} dx = \frac{\phi(a)}{a-s} - \frac{\phi(b)}{b-s} + \int_a^b \frac{\phi'(x)}{x-s} dx, \quad (\text{D.4.4})$$

where $\phi'(x)$ satisfies the Hölder condition $|\phi'(x_1) - \phi'(x_2)| \leq A|x_1 - x_2|^\alpha$.

Similarly, the finite-part integral of $\phi(x) \in C^2[a, b]$ is defined by

$$\begin{aligned} \int_a^b \frac{\phi(x)}{(x-s)^2} dx &= \int_a^b \frac{\phi(x) - \phi(s) - (x-s)\phi'(s)}{(x-s)^2} dx \\ &\quad + \phi(s) \int_a^b \frac{dx}{(x-s)^2} + \phi'(s) \int_a^b \frac{dx}{x-s}, \end{aligned} \quad (\text{D.4.5})$$

where on the right side the last integral is evaluated in (D.4.2), while the second integral is given by

$$\int_a^b \frac{dx}{(x-s)^2} = \left[-\frac{1}{x-s} \right]_a^b = \frac{1}{a-s} - \frac{1}{b-s}. \quad (\text{D.4.6})$$

In the case when $\phi(x) \notin C^2$, i.e., it does not have a high degree of continuity to allow a Taylor's series expansion, a sufficient condition for the existence of the finite-part integral in (D.4.5) is that $\phi(x)$ satisfy a Hölder-continuous first derivative condition as in §D.2, which also guarantees that the first integral on the right side of (D.4.5) is, at most, weakly singular.

Integrating a finite-part integral by parts we get

$$\int_a^b \frac{\phi(x)}{(x-s)^2} dx = \frac{\phi(a)}{a-s} - \frac{\phi(b)}{b-s} + \int_a^b \frac{\phi'(x)}{x-s} dx, \quad (\text{D.4.7})$$

where $\phi(x) \in C^1[a, b]$ satisfies the Hölder condition. If we compare (D.4.7) and (D.4.4), we obtain

$$\frac{d}{ds} \int_a^b \frac{\phi(x)}{x-s} dx = \int_a^b \frac{\phi(x)}{(x-s)^2} dx, \quad (\text{D.4.8})$$

which shows that differentiation can be carried out under the integral sign.

D.5. One-Sided Finite-Part Integrals

Consider the case when the integrand has a singularity at an endpoint of the interval $[a, b]$, that is, either $a \equiv s$ or $b \equiv s$. Then the finite-part of a first-order function $\phi(x) \in C^1[a, s]$ or $\phi(x) \in C^1[s, b]$ is defined, respectively, by

$$\int_a^s \frac{\phi(x)}{x-s} dx = \int_a^s \frac{\phi(x) - \phi(s)}{x-s} dx + \phi(s) \int_a^s \frac{dx}{x-s} \quad (\text{D.5.1})$$

or

$$\int_s^b \frac{\phi(x)}{x-s} dx = \int_s^b \frac{\phi(x) - \phi(s)}{x-s} dx + \phi(s) \int_s^b \frac{dx}{x-s}, \quad (\text{D.5.2})$$

where

$$\int_a^s \frac{dx}{x-s} = \text{finite part of } [\ln|x-s|]_a^s = -\ln|a-s| \quad (\text{D.5.3})$$

or

$$\int_s^b \frac{dx}{x-s} = \text{finite part of } [\ln|x-s|]_s^b = \ln|b-s|, \quad (\text{D.5.4})$$

If $\phi(x) \notin C^1$, a sufficient condition for the existence of the finite-part integrals in (D.5.1) and (D.5.2) is that $\phi(x) \in C^0$ and satisfies a Hölder condition. If we use (D.5.1) and (D.5.2) to derive (D.4.1), we find that finite-part integrals exist iff the function $\phi(x)$ is continuous at the singular point s .

Similarly, the second-order one-sided finite-part integral of a function $\phi(x) \in C^2$ is given by

$$\begin{aligned} \int_a^s \frac{\phi(x)}{(x-s)^2} dx &= \int_a^s \frac{\phi(x) - \phi(s) - (x-s)\phi'(s)}{(x-s)^2} dx \\ &\quad + \phi(s) \int_a^s \frac{dx}{(x-s)^2} + \phi'(s) \int_a^s \frac{dx}{x-s}, \end{aligned} \quad (\text{D.5.5})$$

or

$$\begin{aligned} \int_s^b \frac{\phi(x)}{(x-s)^2} dx &= \int_s^b \frac{\phi(x) - \phi(s) - (x-s)\phi'(s)}{(x-s)^2} dx \\ &\quad + \phi(s) \int_s^b \frac{dx}{(x-s)^2} + \phi'(s) \int_s^b \frac{dx}{x-s}, \end{aligned} \quad (\text{D.5.6})$$

according as the singularity is at the endpoint b or the endpoint a of the interval $[a, b]$. In these cases then

$$\int_a^s \frac{dx}{(x-s)^2} = \text{finite part of } \left[-\frac{1}{x-s} \right]_a^s = \frac{1}{a-s}, \quad (\text{D.5.7})$$

or

$$\int_s^b \frac{dx}{(x-s)^2} = \text{finite part of } \left[-\frac{1}{x-s} \right]_s^b = \frac{1}{s-b}, \quad (\text{D.5.8})$$

respectively. If $\phi(x) \notin C^2$, a sufficient condition for the existence of the finite-part integrals in (D.5.5) and (D.5.6) is that $f(x) \in C^1$ and satisfy a Hölder condition. If we use (D.5.1) and (D.5.2) to derive (D.4.1), we find that finite-part integrals exist iff the function $\phi'(x)$ is continuous at the singular point s .

D.6. Examples of Cauchy P.V. Integrals

EXAMPLE D.1. Let a Cauchy p.v. integral, denoted by $C_\nu(s)$, be defined by

$$C_\nu(s) = \int_0^1 \frac{x^\nu}{x-s} dx, \quad 0 < s < 1, \quad \nu > -1. \quad (\text{D.6.1})$$

Then the following results hold for rational ν :

- (i) $C_{\nu+1}(s) = s C_\nu(s) + \frac{1}{\nu+1};$
- (ii) $C_\nu(s) = -\pi s^\nu \cot \pi \nu - \sum_{n=0}^{\infty} \frac{s^{n-\nu}}{n-\nu} \quad -1 < \nu < 0, \nu \text{ not integer}; \quad (\text{D.6.2})$
- (iii) $C_n(s) = s^n \ln \left(\frac{1-s}{s} \right) + \sum_{k=1}^n \frac{s^{n-k}}{k} \quad \text{for integer } n.$

PROOF. (i) Take $x = x - s + s$; then

$$C_{\nu+1}(s) = \int_0^1 \frac{x^{\nu+1}}{x-s} dx = \int_0^1 x^\nu dx + s \int_0^1 \frac{x^\nu}{x-s} dx,$$

which proves the result.

(ii) For $-1 < \nu < 0$, set $x = su$; then

$$\begin{aligned} C_\nu(s) &= \int_0^1 \frac{x^\nu}{x-s} dx = s^\nu \int_0^1 \frac{x^\nu}{x-1} dx \\ &= s^\nu \left[\int_0^\infty \frac{x^\nu}{x-1} dx - \int_{1/s}^\infty \frac{x^\nu}{x-1} dx \right] \\ &= -\pi s^\nu \cot \pi \nu - s^\nu \int_0^s \frac{z^{-\nu-1}}{n-\nu} dz, \quad \text{where } x = 1/z, \\ &= -\pi s^\nu \cot \pi \nu - \sum_{n=0}^{\infty} \frac{s^n}{n-\nu}, \quad -1 < \nu < 0, \nu \neq n, \end{aligned}$$

where we have used $\frac{1}{1-s} = \sum_{n=0}^{\infty} s^n$.

(iii) For $\nu = n$ (integer) we use recursion and get

$$\begin{aligned} C_0(s) &= \int_0^1 \frac{1}{x-s} dx = \ln \left(\frac{1-s}{s} \right), \\ C_1(s) &= \int_0^1 \frac{x}{x-s} dx = 1 - s \ln \left(\frac{1-s}{s} \right), \\ C_2(s) &= \int_0^1 \frac{x^2}{x-s} dx = \frac{1}{2} + s + s^2 \ln \left(\frac{1-s}{s} \right), \\ C_3(s) &= \int_0^1 \frac{x^3}{x-s} dx = \frac{1}{3} + \frac{s}{2} + s^2 + s^3 \ln \left(\frac{1-s}{s} \right), \end{aligned}$$

and so on. Thus, by induction we have

$$\begin{aligned} C_n(s) &= s^n \ln \left(\frac{1-s}{s} \right) + s^{n-1} + \frac{s^{n-2}}{2} + \cdots + \frac{1}{n} \\ &= s^n \ln \left(\frac{1-s}{s} \right) + \sum_{k=1}^n \frac{s^{n-k}}{k}. \blacksquare \end{aligned}$$

EXAMPLE D.2. If we denote the Cauchy p.v. integral by

$$I_{1,\nu} \equiv C_\nu(s) = \int_0^1 \frac{x^\nu}{x-s} dx, \quad 0 < s < 1,$$

then by repeated differentiation we obtain the integral $I_{\alpha,\nu}$ for positive integer α as

$$I_{\alpha,\nu} = \int_0^1 \frac{x^\nu}{(x-s)^\alpha} dx, \quad 0 < s < 1. \quad (\text{D.6.3})$$

PROOF. We use formula (D.4.8), which, in view of (D.6.2(ii)), for non-integral ν yields

$$\begin{aligned} I_{1,\nu} &= \int_0^1 \frac{x^\nu}{x-s} dx = -\pi s^\nu \cot \pi \nu - \sum_{n=0}^{\infty} \frac{s^{n-\nu}}{n-\nu}, \\ I_{2,\nu} &= \frac{d}{ds} (I_{1,\nu}) = -\pi \cot \pi \nu \cdot \nu s^{\nu-1} - \sum_{n=1}^{\infty} \frac{n s^{n-1}}{n-\nu}, \\ I_{3,\nu} &= \frac{d}{ds} (I_{2,\nu}) = -\pi \cot \pi \nu \cdot \nu(\nu-1) s^{\nu-1} - \sum_{n=2}^{\infty} \frac{n(n-1) s^{n-2}}{n-\nu}, \end{aligned}$$

and so on. Thus, by induction

$$I_{\alpha,\nu} = -\frac{\pi \Gamma(\nu+1) s^{\nu-\alpha+1} \cot \pi \nu}{\Gamma(\nu-\alpha+2) (\alpha-1)!} - \sum_{n=\alpha-1}^{\infty} \frac{n! s^{n-\alpha+1}}{(n-\alpha+1)! (\alpha-1)! (n-\nu)}. \quad (\text{D.6.4})$$

For integer $\nu = n$ we have

$$\begin{aligned} I_{\alpha,n} &= \sum_{j=0}^{\alpha-2} \binom{n}{j} \frac{s^{n-j} [(1-s)^{j-\alpha+1} - (-s)^{j-\alpha+1}]}{j-\alpha+1} \\ &\quad + i n \binom{n}{\alpha-1} s^{n-\alpha+1} \ln \left(\frac{1-s}{s} \right) \\ &\quad + \sum_{j=\alpha}^n \binom{n}{j} \frac{s^{n-j} [(1-s)^{j-\alpha+1} - (-s)^{j-\alpha+1}]}{j-\alpha+1}. \end{aligned} \quad (\text{D.6.5})$$

For details, see Bertram and Ruehr (1992). ■

D.7. Examples of Hadamard's Finite-Part Integrals

First, we define the Hadamard transform $H_2[\phi]$ for a function $\phi(x)$ as

$$H_2[\phi] = \int_0^1 \frac{\phi(x)}{(x-s)^2} dx, \quad 0 < s < 1. \quad (\text{D.7.1})$$

Notice that this transform is the integral $I_{2,\nu}$. The results, given in Table D.1, hold for various choices of the function ϕ .

Table D.1.

$\phi(s)$	$H_2[\phi]$
s^ν (ν rational)	$-\pi\nu s^{\nu-1} \cot \pi\nu - \frac{1}{1-s} - \nu \sum_{n=0}^{\infty} \frac{s^n}{n-\nu+1}$
s^m (m integer)	$\frac{s^{m-1}}{1-s} + ms^{m-1} \ln \frac{1-s}{s} + \sum_{j=2}^m \frac{\binom{m}{j} s^{m-j} [(1-s)^{j-1} - (-s)^{j-1}]}{j-1}$
$[s(1-s)]^{3/2}$	$\frac{\pi}{2} \left[\frac{3}{4} - 6s(1-s) \right]$
$\frac{1}{2} + \frac{s-c}{2 s-c }$	$\frac{1}{s-1} + \frac{1}{c-s}$
$\frac{1}{2} [c + (1-2c)s + s-c]$	$(c-1) \ln s - c \ln 1-s + \ln s-c $
$\frac{1}{2} + \frac{1}{4\varepsilon} [s-c+\varepsilon + s-c-\varepsilon]$	$\frac{1}{s-1} + \frac{1}{2\varepsilon} \ln \left \frac{s-c-\varepsilon}{s-c+\varepsilon} \right $
$\frac{3}{4} (s-c)^2 - \frac{1}{4} (s-c) s-c + \frac{1}{2} (s-c) (c^2 + 2c - 1) + \frac{1}{2} c (c^2 - 1)$	$c+1 + \left(s + \frac{c^2}{2} - \frac{1}{2} \right) \ln 1-s + (s-c) \ln s-c - \left(2s + \frac{c^2}{2} - c - \frac{1}{2} \right) \ln s $
$\sin \pi s$	$-\pi \sin \pi s [\text{Si}(\pi(1-s)) + \text{Si}(\pi s)] + \pi \cos \pi s [\text{Ci}(\pi s) - \text{Ci}(\pi(1-s))]$

Note that if the function $\phi(x) \in C^2[a, b]$ satisfies a Hölder-continuous second-derivative condition, then Hadamard's second-order two-sided finite-part integrals contain the singular part $\frac{1}{(x-s)^3}$ and can be defined as above (see Ioakimidas 1988, for details). This concept can be generalized further, but we shall not go into this generalization.

NOTES: Kutt (1975) contains weights w_i to $30S$ for the equally spaced rule

$$\int_0^1 \frac{f(x)}{x^4} dx \approx \sum_{i=1}^n w_i f\left(\frac{i}{n}\right), \quad n = 3(1)20,$$

and the nodes x_i and weights w_i to $30S$ for the Gauss-type quadrature rule

$$\int_0^1 \frac{f(x)}{x^4} dx \approx \sum_{i=1}^n w_i f(x_i), \quad n = 2(1)20,$$

for $\lambda = 1, 4/3, 3/2, 5/3, 2, 3, 4, 5$. This work also contains weights for computation of the finite-part integrals of the form

$$\int_0^1 \frac{f(x)}{(x-s)^4} dx, \quad \lambda = 2, 3, 4, 5$$

for the equally spaced case.

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