Home work 4

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Q1.
First, we convert the given objective function from a minimization to a maximization function by multiplying coefficients with -1. Thus, our objective function becomes

$$maximize : -2x_1 - 7x_2 - x_3$$

Next, we check for variables with non-negativity constraints. In the given linear program, x_1 and x_3 do not follow the non-negative constraints. Therefore, we replace x_1 with $x_1^1 - x_1^{11}$. Since x_3 already has a constraint that $x_3 \leq 0$, we replace x_3 with $-x_3^1$. Thus, the linear program becomes

$$maximize : -2(x_1^1 - x_1^{11}) - 7x_2 + x_3^1$$

$$x_1^1 - x_1^{11} + x_3^1 = 7$$

$$3(x_1^1 - x_1^{11}) + x_2 \geq 24$$

$$x_1^1, x_1^{11}, x_2, x_3^1 \geq 0$$

Now, we replace all equality constraints with $\leq$ and $\geq$ constraints.

$$maximize : -2(x_1^1 - x_1^{11}) - 7x_2 + x_3^1$$

$$x_1^1 - x_1^{11} + x_3^1 \leq 7$$

$$x_1^1 - x_1^{11} + x_3^1 \geq 7$$

$$3(x_1^1 - x_1^{11}) + x_2 \geq 24$$

$$x_1^1, x_1^{11}, x_2, x_3^1 \geq 0$$

Finally, all the $\geq$ constraints are converted to $\leq$ constraints by multiplying them with -1 on both sides.

$$maximize : -2(x_1^1 - x_1^{11}) - 7x_2 + x_3^1$$
$$x_1^1 - x_1^{11} + x_3^1 \leq 7$$
$$-(x_1^1 - x_1^{11} + x_3^1) \leq -7$$
$$-(3(x_1^1 - x_1^{11}) + x_2) \leq -24$$
$$x_1^1, x_1^{11}, x_2, x_3^1 \geq 0$$

Thus, the standard form for the given linear program is as follows:

$$maximize : -2x_1^1 + 2x_1^{11} - 7x_2 + x_3^1$$
$$x_1^1 - x_1^{11} + x_3^1 \leq 7$$
$$-x_1^1 + x_1^{11} - x_3^1 \leq -7$$
$$-3x_1^1 + 3x_1^{11} - x_2 \leq -24$$
$$x_1^1, x_1^{11}, x_2, x_3^1 \geq 0$$

Q2.

To convert a given linear program into slack form, we first convert it into standard form. After following the four steps to convert a linear program into standard form, we get

$$maximize : 2x_1 - 6x_3$$
$$x_1 + x_2 - x_3 \leq 7$$
$$-3x_1 + x_2 \leq -8$$
$$x_1 - 2x_2 - 2x_3 \leq 0$$
$$x_1, x_2, x_3 \geq 0$$

To convert standard form into slack form, we introduce slack variables. form, we get

$$maximize : 2x_1 - 6x_3$$
$$x_4 = 7 - x_1 - x_2 + x_3$$
$$x_5 = -8 + 3x_1 - x_2$$
$$x_6 = -x_1 + 2x_2 + 2x_3$$

Here, x_4, x_5, x_6 are the **basic variables** and x_1, x_2, x_3 are the **non-basic variables**. Finally, we omit maximize and get the slack form.

$$z = 2x_1 - 6x_3$$

$$x_4 = 7 - x_1 - x_2 + x_3$$

$$x_5 = -8 + 3x_1 - x_2$$

$$x_6 = -x_1 + 2x_2 + 2x_3$$

Q3.

To solve using simplex, the given linear program should be in slack form. The slack form for the given linear program is as follows:

$$z = -x_1 - x_2 - x_3$$

$$x_4 = -10000 + 2x_1 + 7.5x_2 + 3x_3$$

$$x_5 = -30000 + 20x_1 + 5x_2 + 10x_3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

To find the initial feasible solution, we substitute 0 for all non-basic variables. But, by doing so, we get $x_4 = -10000$ and $x_5 = -30000$, which violates the constraint $x_4, x_5 \geq 0$. Therefore, the initial basic solution is not feasible. Thus, we need to find the solution using auxiliary linear program. The auxiliary linear program is as follows:

$$maximize - x_0$$

subject to

$$-2x_1 - 7.5x_2 - 3x_3 - x_0 \leq -10000$$

$$-20x_1 - 5x_2 - 10x_3 - x_0 \leq -30000$$

$$x_1, x_2, x_3, x_4, x_5, x_0 \geq 0$$

The slack form for the the above linear program is as follows

$$z = -x_0$$

$$x_4 = -10000 + 2x_1 + 7.5x_2 + 3x_3 + x_0$$

$$x_5 = -30000 + 20x_1 + 5x_2 + 10x_3 + x_0$$

We perform a pivot by making x_0 as the entering variable and x_5 as the leaving variable. This gives us

$$x_0 = 30000 - 20x_1 - 5x_2 - 10x_3 + x_5$$

Substituting this value in the above slack form, we get

$$z = -30000 + 20x_1 + 5x_2 + 10x_3 - x_5$$

$$x_4 = 20000 - 18x_1 + 2.5x_2 - 7x_3 + x_5$$

For the above auxiliary linear program, the initial basic solution is feasible. Hence, we repeatedly perform a pivot to get an optimal solution. Let x_2 be the entering variable and x_0 be the leaving variable, by using the tightest constraint. We get the following

$$z = -x_0$$

$$x_2 = 6000 - 4x_1 - 2x_3 + \frac{x_5}{5} - \frac{x_0}{5}$$

$$x_4 = 35000 - 28x_1 - 12x_3 + \left(\frac{3}{2}\right) x_5 - \frac{x_0}{2}$$

This is the optimal solution for the above auxiliary program. Updating the objective function and setting x_0 to 0, we get

$$z = -6000 + 3x_1 + x_3 - \frac{x_5}{5}$$

$$x_2 = 6000 - 4x_1 - 2x_3 + \frac{x_5}{5}$$

$$x_4 = 35000 - 28x_1 - 12x_3 + \left(\frac{3}{2}\right) x_5$$

Performing pivot by choosing x_1 as entering variable and x_2 as leaving variable, we get

$$z = -2250 - \left(\frac{2}{7}\right) x_3 - \left(\frac{3}{28}\right) x_4 - \left(\frac{11}{280}\right) x_5$$

$$x_1 = 1250 - \left(\frac{3}{7}\right) x_3 - \left(\frac{1}{28}\right) x_4 + \left(\frac{3}{56}\right) x_5$$

$$x_2 = 1000 - \left(\frac{2}{7}\right) x_3 + \left(\frac{1}{7}\right) x_4 - \left(\frac{1}{70}\right) x_5$$

Since all the co-efficients are negative, the basic solution is the optimal solution. Therefore $(x_1, x_2, x_3) = (1250, 1000, 0)$

Q4.

To solve using simplex, the given linear program should be in slack form. The slack form for the given linear program is as follows:

$$z = x_1 + 3x_2$$

$$x_3 = 8 - x_1 + x_2$$

$$x_4 = -3 + x_1 + x_2$$

$$x_5 = 2 + x_1 - 4x_2$$

To find the initial feasible solution, we substitute 0 for all non-basic variables. But, by doing so, we get $x_4 = -3$, which violates the constraint $x_4 \geq 0$. Therefore, the initial basic solution is not feasible. Thus, we need to find the solution using auxiliary linear program. The auxiliary linear program is as follows:

$$maximize - x_0$$

subject to

$$x_1 - x_2 - x_0 \leq 8$$

$$-x_1 - x_2 - x_0 \leq -3$$

$$-x_1 + 4x_2 - x_0 \leq 2$$

$$x_1, x_2, x_0 \geq 0$$

The slack form for the the above linear program is as follows

$$z = -x_0$$

$$x_3 = 8 - x_1 + x_2 + x_0$$

$$x_4 = -3 + x_1 + x_2 + x_0$$

$$x_5 = 2 + x_1 - 4x_2 + x_0$$

We perform a pivot by making x_0 as the entering variable and x_4 as the leaving variable. This gives us

$$x_0 = 3 - x_1 - x_2 + x_4$$

Substituting this value in the above slack form, we get

$$z = -3 + x_1 + x_2 - x_4$$

$$x_3 = 11 - 2x_1 + x_4$$

$$x_5 = 5 - 5x_2 + x_4$$

For the above auxiliary linear program, the initial basic solution is feasible. Hence, we repeatedly perform a pivot to get an optimal solution. Let x_1 be the entering variable and x_0 be the leaving variable. We get the following

$$z = -x_0$$

$$x_1 = 3 - x_0 - x_2 + x_4$$

$$x_3 = 5 + 2x_0 + 2x_2 - x_4$$

$$x_5 = 5 - 5x_2 + x_4$$

This is the optimal solution for the above auxiliary program. Updating the objective function and setting x_0 to 0, we get

$$z = 3 + 2x_2 + x_4$$

$$x_1 = 3 - x_2 + x_4$$

$$x_3 = 5 + 2x_2 - x_4$$

$$x_5 = 5 - 5x_2 + x_4$$

Performing pivot by choosing x_2 as entering variable and x_5 as leaving variable, using tightest constraint, we get

$$z = 5 + \left(\frac{7}{5}\right) x_4 - \left(\frac{2}{5}\right) x_5$$

$$x_2 = 1 + \left(\frac{1}{5}\right) x_4 - \left(\frac{1}{5}\right) x_5$$

$$x_1 = 2 + \left(\frac{4}{5}\right) x_4 + \left(\frac{1}{5}\right) x_5$$

$$x_3 = 7 - \left(\frac{3}{5}\right) x_4 - \left(\frac{2}{5}\right) x_5$$

Performing pivot by choosing x_4 as entering variable and x_3 as leaving variable, using tightest constraint, we get

$$z = \left(\frac{64}{3}\right) - \left(\frac{7}{3}\right) x_3 - \left(\frac{4}{3}\right) x_5$$

$$x_4 = \left(\frac{35}{3}\right) - \left(\frac{5}{3}\right) x_3 - \left(\frac{2}{3}\right) x_5$$

$$x_2 = \left(\frac{10}{3}\right) - \left(\frac{1}{3}\right) x_3 - \left(\frac{1}{3}\right) x_5$$

$$x_1 = \left(\frac{34}{3}\right) - \left(\frac{4}{3}\right) x_3 - \left(\frac{1}{3}\right) x_5$$

Since all the co-efficients are negative, the basic solution is the optimal solution. Therefore $(x_1, x_2) = \left(\frac{34}{3}, \frac{10}{3}\right)$

Q5.

Let n be the unknown number till which there are minimal number of number which are power of 2. The numbers which are not power of 2 are higher than compared to number which are power of 2.

Let t(i) be the cost of i^{th} operation. Then total cost of all n operations is $\sum_{i=1}^{n} t(i)$.

$$\sum_{i=1}^{n} t(i) = \sum_{i!=2^m} t(i) + \sum_{i=2^m} t(i)$$

As we know that there are higher number of numbers not of power 2. so,

$$\sum_{i=1}^{n} t(i) = \sum_{i!=2^m} t(i) + \sum_{i=2^m} t(i)$$

$$\sum_{i!=2^m} t(i) + \sum_{i=2^m} t(i) <= \sum_{i!=2^m} t(i) + n$$

$$= \sum_{i=1}^{\lceil \log n \rceil} 2^i + n$$

$$= 2^{1+\lceil \log n \rceil} + n$$

$$= 2^1 * 2^{\lceil \log n \rceil} + n$$

approximating it gives the below as...

$$= 2n + n$$

$$\sum_{i=1}^{n} t(i) = 3n$$

total cost of the operation is $\mathcal{O}(n)$

Amortized cost per operation can be obtained by averaging the total cost of operation, which is 3n/n = n.

Amortized cost of operation is $\mathcal{O}(1)$

Q6.

We use the results from the above problem. We got that Amortized cost for each operation is 3. Let t_i be the cost of operation and $\hat{t_i}$ be the amortized cost for operation.
t_i is i if i is the power of 2 and 1 if it's not power of 2. $\hat{t_i}$, for i^{th} operation is 3.
From accountig method, credit = Amortized cost - actual cost. Credit has to be positive.
for any sequence of operations,

$$\sum_{i=1}^{n} \hat{t_i} \geq \sum_{i=1}^{n} t_i$$

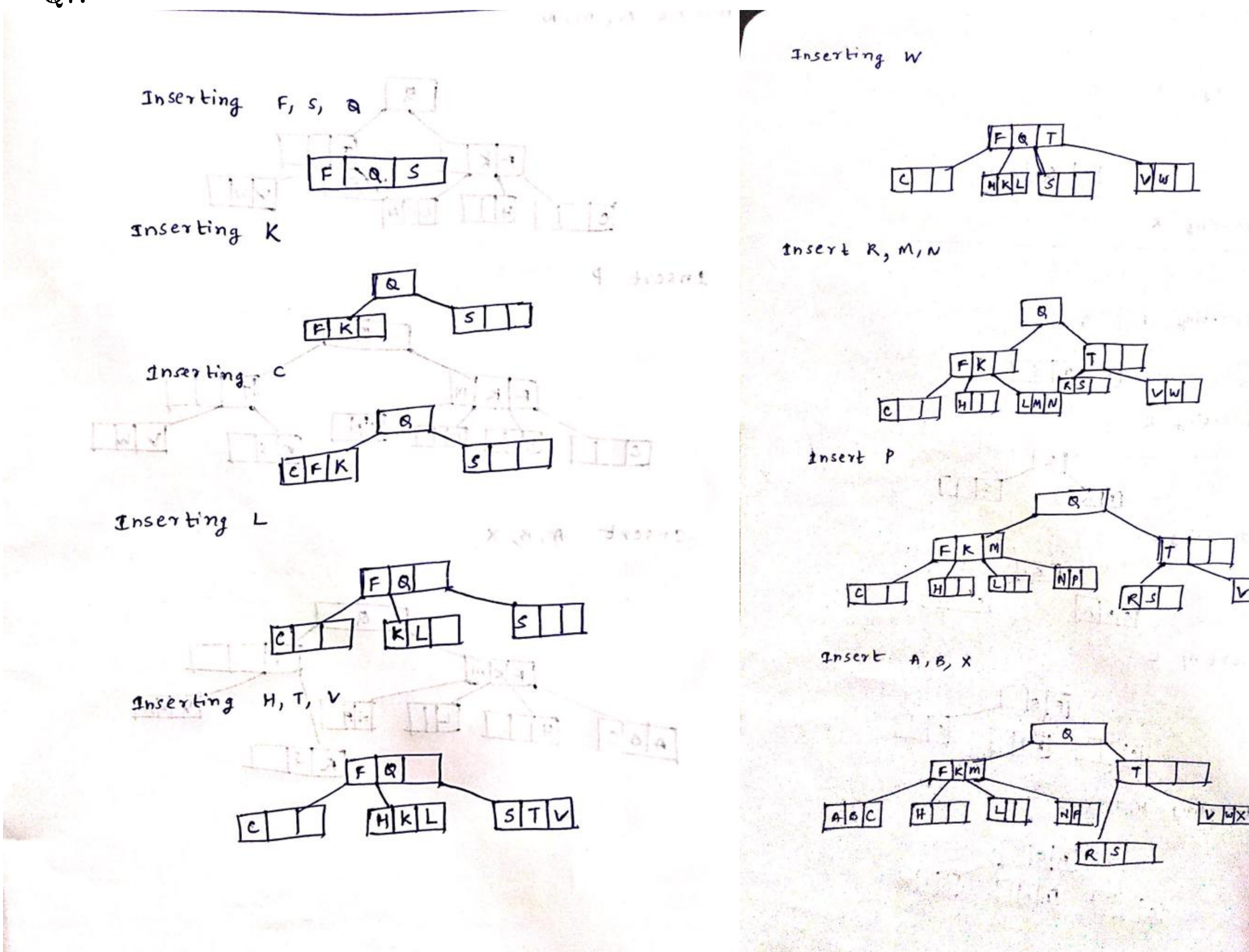
We know from the last problem, $\sum_{i=1}^{n} t_i \leq 3n$. From the above inequality equation we can say that $\sum_{i=1}^{n} \hat{t_i} = 3n$.
Now let's say we have a positive credit after performing $2^i th$ operation. Each of the 2^i - 1 operations

have the credit of 1. For each operation we pay cost of 3, creating a credit of 2 from each of them. Giving us the credit...

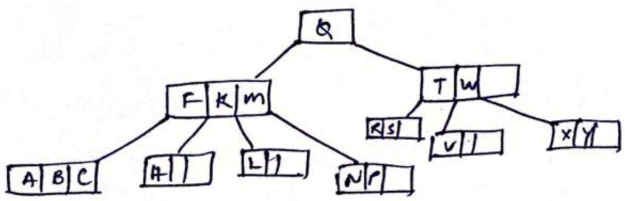
$$2(2^i - 1) = 2^{i+1} - 1$$

Then for the 2^{i+1}th operation, the 3 credits we pay gives us a total of $2^{i+1} + 1$ to use towards the actual cost of 2^{i+1}, leaving us with 1 credit. Since the amortized cost of each operation is $\mathcal{O}(1)$ and the total cost of n operations is $\mathcal{O}(n)$

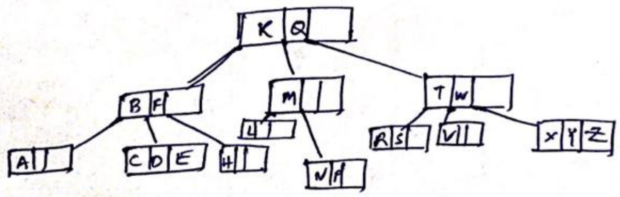
Q7.



Insert Y



Inserting D, Z, E



Final Configuration.