Symmetry and Phaselocking in Chains of Weakly Coupled Oscillators

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Dedicated to the memory of C. Conley

Abstract

Weakly coupled chains of oscillators with nearest-neighbor interactions are analyzed for phase-locked solutions. It is shown that the symmetry properties of the coupling affect the qualitative form of the phaselocked solutions and the scaling behavior of the system as the number of oscillators grows without bound. It is also shown that qualitative behavior of these solutions depends on whether the coupling is "diffusive" or "synaptic", terms defined in the paper. The methods include the demonstration that the equations for phaselocked solutions can be approximated by a singularly perturbed two-point (continuum) boundary value problem that is easier to analyze; the issue of convergence of the phaselocked solutions to solutions of the continuum equation is closely related to questions involving numerical entropy in computation schemes for a conservation law. An application to the neurophysiology of motor behavior is discussed briefly.

1. Introduction

This is the second in a series of papers on chains of weakly coupled oscillators. The equations to be considered have the form

$$\dot{\mathbf{U}}_k = \mathbf{F}_k(\mathbf{U}_k) + \varepsilon[\mathbf{G}^+(\mathbf{U}_{k+1},\mathbf{U}_k) + \mathbf{G}^-(\mathbf{U}_{k-1},\mathbf{U}_k)] \tag{1.1}$$

with $\mathbf{U}_k \in R^m$, $k = 1,\cdots, N+1$, and

$$\mathbf{G}^+(\mathbf{U}_{N+2},\mathbf{U}_{N+1}) \equiv 0 \equiv \mathbf{G}^-(\mathbf{U}_0,\mathbf{U}_1). \tag{1.2}$$

In the absence of coupling, the k-th oscillator $\mathbf{U}_k$ satisfies

$$\dot{\mathbf{U}}_k = \mathbf{F}_k(\mathbf{U}_k) \tag{1.3}$$

which is assumed to have a stable limit cycle. Also, $\mathbf{F}_k \equiv \mathbf{F} + O(\varepsilon)$, $\varepsilon \ll 1$, so the

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oscillators are similar. The coupling is assumed to be small, to depend only on nearest neighbors, and to be independent of k to lowest order in ε. Otherwise it is completely general, at least at this stage; F, G^+, G^- are smooth.

In spite of the generality of the equations, the assumption of weak coupling implies powerful constraints on the solutions to (1.1). To see why, first note that the weak coupling, plus the existence of the limit cycles for (1.3), imply the existence of an asymptotically stable invariant torus of dimension $N + 1$ for (1.1). (At $\varepsilon = 0$, the product of the limit cycles is such a torus; for $\varepsilon > 0$, the result follows from the persistance of hyperbolic invariant manifolds; see [1].) Furthermore, on the torus, the equations have a very special form, as follows: Coordinates $\theta_k \in S^1$, $\mathbf{Y}_k \in R^{m-1}$ can be picked for each R^m in a neighborhood of the limit cycle so that θ_k measures phase around the limit cycle of (1.3) and $\mathbf{Y}_k = 0$ on the limit cycle. It was shown in [2] that if $\phi_k = \theta_{k+1} - \theta_k$, then, on the torus, (1.1) can be written as

$$\begin{aligned}\dot{\phi}_k = \varepsilon[\Delta_k + H^+(\phi_{k+1}) - H^+(\phi_k)\\ +H^-(-\phi_k) - H^-(-\phi_{k-1})] + O(\varepsilon^2),\end{aligned} \tag{1.4}$$

$$\dot{\theta}_1 = O(1). \tag{1.5}$$

Here $\varepsilon\Delta_k \equiv \omega_{k+1} - \omega_k$, where ω_k is the frequency of the oscillation for (1.3). H^+ (respectively H^-) is a 2π-periodic scalar-valued function which depends on $\mathbf{F}$ and $\mathbf{G}^+$ (respectively $\mathbf{G}^-$). $H^\pm$ can be explicitly computed using, for example, averaging techniques. (The result in [2] was stated for the special case $\mathbf{G}^\pm = D(\mathbf{U}_{k\pm1} - \mathbf{U}_k)$ or $D\mathbf{U}_{k\pm1}$, where D is a positive definite matrix. However, the techniques work more generally for (1.1) and give an equation of the same form (1.4), (1.5) on the invariant torus.) Conditions (1.2) translate to

$$H^+(\phi_{N+1}) \equiv 0 \equiv H^-(-\phi_0). \tag{1.6}$$

We shall be concerned with phaselocking, i.e., solutions to (1.1) that are periodic in time, so that all the oscillators (which may have different natural frequencies) move at the same frequency. For weakly coupled systems, phaselocked solutions correspond to critical points of the associated system

$$\phi_k' = \Delta_k + H^+(\phi_{k+1}) - H^+(\phi_k) + H^-(-\phi_k) - H^-(-\phi_{k-1}), \tag{1.7}$$

where $\phi_k' = d\phi_k/d\tau$, $\tau = \varepsilon t$. That is, each stable critical point of (1.7) gives rise to a stable periodic orbit of (1.4), (1.5), independent of the $O(\varepsilon^2)$ terms. (For details, see [3], p. 168.)

In [2], we analyzed equation (1.7) for the special case $H^+(\phi) = H^-(\phi) = \sin\phi$, and $\Delta_k \equiv \Delta < 0$. The choice of $\{\Delta_k\}$ corresponds to a (decreasing) linear gradient in natural frequencies, with gradient strength measured by Δ. In that case, it was possible to solve explicitly for the critical points of (1.7). It was shown that for $\Delta = O(1/N^2)$ sufficiently small, there are such critical points. The bulk of the paper was then devoted to analyzing the behavior of (1.7) just after phaselocking is lost; it was shown that, in the coupled system (1.1), there are "plateaus" of frequency.

In this paper, we discuss (1.7) for general H^+, H^-, but only in the regime in which there is phaselocking. For general H^+, H^- (indeed for any $H^{\pm}$ that are not odd functions of their arguments) the techniques of [2] do not hold, and there is no explicit way for determining the existence of critical points of (1.7). However, for N large, it is possible to approximate (1.7) by a partial differential equation (P.D.E.) which is, in some (mathematical, but not physical) sense its continuum limit. One then shows that the real system (1.7) behaves like the continuum limit. This P.D.E. predicts that phaselocking is not sustained for arbitrarily large frequency gradients, in contrast to continuum phase equations derived from reaction-diffusion equations; see [4]–[6].

The paper is organized as follows: In Section 2, we "derive" a continuum limit to (1.7). Critical points of (1.7) are associated with time-independent solutions to this P.D.E. The time-independent version of this P.D.E. is a nonlinear, singularly perturbed two-point boundary value problem which, under very general hypotheses on $H^{\pm}$, can be solved uniquely. The solution to this problem is discussed in some detail (but without proofs) since the methods to be used to analyze (1.7) (rigorously) closely mimic the procedures used in formally constructing solutions to singularly perturbed boundary value problems. We also state the main result of the paper, that (1.7), (1.6) does indeed behave like the continuum limit. This result requires an extra hypothesis on H, one which turns out to be related to issues of numerical stability of algorithms for discretizing stiff P.D.E.'s.

Before proving the main result, we pause in Section 3 to consider some of its implications. We show that the symmetry properties of the functions $H^{\pm}$ affect the scaling behavior of phaselocking as the frequency gradient is increased; in particular, if $H = H^+ = H^-$ is an odd function of its argument, a much smaller frequency gradient among oscillators will cause it to lose locking than if H also has a non-odd component. The frequency at which the oscillators lock is also dramatically affected by the symmetry properties.

Another distinction that turns out to be important for phaselocking is between diffusive and non-diffusive (or "synaptic") coupling. For ordinary diffusion (i.e., $\mathbf{G}^{\pm}(\mathbf{U}_k) = D(\mathbf{U}_{k\pm 1} - \mathbf{U}_k)$ with D a matrix), or any functions $\mathbf{G}^{\pm}$ which satisfy $\mathbf{G}^{\pm}(\mathbf{U},\mathbf{U}) \equiv 0$, the resulting functions $H^{\pm}$ have the property $H^{\pm}(0) = 0$. By extension, we shall label as "diffusive" any $\mathbf{G}^{\pm}$ for which $H^{\pm}$ satisfies $H^{\pm}(0) = 0$. By "synaptic" coupling, we shall mean any functions $H^{\pm}$ for which $H^{\pm}(0) \neq 0$. The word is taken from the interactions of biological cells *via* chemical synapses, for which the flux need not vanish when the two cells are in the same state. It follows from the analysis given in Section 2 that, for large numbers of oscillators, synaptic coupling leads to qualitatively different solutions than diffusive coupling.

Sections 4–6 contain the proof that, if the frequency gradient is not too large, there is a time-independent solution to (1.7) which behaves like the solution to the continuum boundary value problem. Section 4 analyzes the analogue of the "shock layer equation". The discrete version has features of the continuum shock layer equation, including the existence of a very useful integral. Section 5 investigates the analogue of the outer equation, a first-order difference equation which describes the solution on most of the interval. Section 6 discusses the "matching" of these partial solutions. The argument uses a combination of topological and implicit function theorem techniques.

Section 7 is devoted to stability and numerics. Using ideas closely related to monotone methods for analyzing P.D.E.'s, we prove that the time-independent solution of (1.7) constructed in Sections 4–6 is asymptotically stable. The key assumption needed for these methods turns out to be essentially the same as one that guarantees that some algorithms used to compute solutions to the Cauchy problem for quasilinear P.D.E.'s lead to the "right" weak solution. We also discuss numerical work in the absence of that hypothesis.

In Section 8, we discuss an application, to fish locomotion, which motivated much of the above work, in particular the distinction between diffusive and synaptic coupling. Finally, in the appendix, we reconsider the derivation of (1.4), (1.5) in order to show that it is valid for ε small, uniformly in N. We also give a plausibility argument for the assertion that the stable critical points of (1.7) correspond uniformly to phaselocked solutions of (1.6), (1.7).

2. The Model Continuum Equation

A. In this section, we derive and analyze a heuristic approximation to (1.7). The analysis motivates the more complex proofs of Sections 4–6. The continuum approximation follows from a decomposition of the functions $H^{\pm}$ into their symmetric and antisymmetric parts. Let $H^{\pm}(\phi) = H_0^{\pm}(\phi) + H_e^{\pm}(\phi)$, where $H_0^{\pm}(-\phi) = -H_0^{\pm}(\phi)$ and $H_e^{\pm}(-\phi) = H_e^{\pm}(\phi)$. (Any function can be so decomposed in a unique way.) Also, let $\beta_k/N \equiv \Delta_k$. (This scaling implies that the total frequency difference from one end of the chain to the other is $O(1)$ as $N \to \infty$.) Then the equation for the time-independent solution to (1.7) can be rewritten as

$$0 = \frac{\beta_k}{N} + [f(\phi_{k+1}) - f(\phi_{k-1})] + [g(\phi_{k+1}) - 2g(\phi_k) + g(\phi_{k-1})], \tag{2.1}$$

where

$$\begin{aligned} f &\equiv \tfrac{1}{2}(H_e^+ + H_e^-) + \tfrac{1}{2}(H_0^+ - H_0^-), \\ g &\equiv \tfrac{1}{2}(H_0^+ + H_0^-) + \tfrac{1}{2}(H_e^+ - H_e^-). \end{aligned} \tag{2.2}$$

Note that for the isotropic case $H^+ \equiv H \equiv H^-$, we have $f = H_e$, $g = H_0$. Equation (2.1) can be written in a way that is more suggestive:

$$\begin{aligned} 0 = \frac{1}{N}\bigg\{\beta_k &+ \frac{2[f(\phi_{k+1}) - f(\phi_{k-1})]}{2/N} \\ &+ \frac{1}{N}\frac{[g(\phi_{k+1}) - 2g(\phi_k) + g(\phi_{k-1})]}{1/N^2}\bigg\}. \end{aligned} \tag{2.3}$$

Suppose $\beta(x)$ is a smooth function of x, $0 \leqq x \leqq 1$, such that $\beta_k = \beta(k/(N+1))$, and we look for solutions $\phi_1, \cdots, \phi_N$ for which the graph of ϕ *vs.* k also lies approximately along a smooth curve $\phi(x)$, with $\phi_k \approx \phi(k/(N+1))$. Then, for N large, one might expect $\phi(x)$ to approximately satisfy the O.D.E.

$$0 = \beta(x) + 2f(\phi)_x + \frac{1}{N}g(\phi)_{xx}. \tag{2.4}$$

The boundary conditions (1.6) translate into the conditions

$$H^-(-\phi) = 0 \quad \text{at} \quad x = 0, \qquad H^+(\phi) = 0 \quad \text{at} \quad x = 1, \tag{2.5}$$

or, equivalently

$$f = g \quad \text{at} \quad x = 0, \qquad f = -g \quad \text{at} \quad x = 1. \tag{2.6}$$

The reader is encouraged to consider the special example $H^+ = H^- \equiv H \equiv A \sin\phi + B\cos\phi$, with $A > 0$. (Such an H occurs if $\mathbf{F}(\mathbf{U})$ is taken to be a simple "λ-ω" system, and the coupling is linear. The calculation is done in [2] with $\mathbf{G}^{\pm} = D(\mathbf{U}_{k\pm1} - \mathbf{U}_k)$, D a matrix, to yield $H = A\sin\phi + B(\cos\phi - 1)$; if instead, one uses $\mathbf{G}^{\pm} = D\mathbf{U}_{k\pm1}$, the result is $H = A\sin\phi + B\cos\phi$ for some A, B.)

Equations (2.4), (2.5) do not correspond to standard continuum limits associated with reaction-diffusion equations. At least in the case of isotropic (i.e., $H^+ = H^-$) linear diffusive coupling, (1.1) is related to the reaction-diffusion equations

$$\mathbf{U}_t = \mathbf{F}(\mathbf{U}, x) + \varepsilon D\nabla^2\mathbf{U}, \tag{2.7}$$

where D is a positive definite matrix and $x \approx k/(N+1)$, as above. If, for $0 \leqq x \leqq 1$, (2.7) is discretized into N steps, the resulting equations are

$$\dot{\mathbf{U}}_k = \mathbf{F}_k(\mathbf{U}_k) + N^2\varepsilon D(\mathbf{U}_{k+1} - 2\mathbf{U}_k + \mathbf{U}_{k-1}), \tag{2.8}$$

where $\mathbf{U}_k = \mathbf{U}(k/(N+1))$. For fixed ε, the coupling in (2.8) does not remain small as $N \to \infty$. Thus, the invariant manifold procedure discussed in the appendix does not work uniformly in N for fixed ε. The continuum limit of the model continuum equation (M.C.E.) maintains the weak coupling between adjacent oscillators. It does not correspond to a physical continuum, as does (2.7), but it provides the appropriate continuum "diagnostic equation" for (2.1).

B. Before analyzing (2.4), (2.6), we summarize our hypotheses on the functions f and g. The most crucial hypothesis is that

$$g'(\phi) > 0 \tag{2.9}$$

in a sufficiently large interval J around $\phi = 0$ which we shall specify later, and which will depend on the $\{\beta_k\}$. For $\{\beta_k\}$ too large in absolute value, the appropriate interval does not exist and, in fact, the phaselocking breaks down (see [2] and Section 3). Hypothesis (2.9) will be needed for (temporal) stability of the time-independent solution to (1.7) (see Section 7). Since the highest-order term in (2.3) is $g'(\phi) \cdot \phi_{xx}$, it can be guessed from the time-*dependent* version of (2.3) that, if (2.9) is violated, then (1.7) can behave, over part of its range, like the discrete analogue of a backward heat equation.

Secondly, we assume that

$$f''(\phi) \neq 0 \quad \text{for} \quad \phi \in J. \tag{2.10}$$

This hypothesis says that equation (2.4) is "genuinely nonlinear". Note that it is satisfied for some interval J around $\phi = 0$ if $H^{\pm} = A^{\pm}\sin\phi + B^{\pm}\cos\phi$, provided $|A^{+} - A^{-}|$ is not too big.

The interval J must contain ϕ_L and ϕ_R, the solution to (2.6) which give the left-hand and right-hand boundary conditions for (2.4). Since f and g are periodic, there is, in general, more than one set of solutions. We assume that J is chosen so that there is exactly one set of solutions in J, i.e.,

(2.11)

$$\exists! \text{ solution } \phi_L \text{ (respectively } \phi_R) \text{ to } f = g \text{ (respectively } f = -g) \text{ for } \phi \in J.$$

(See Figure 2.1.) We shall then replace (2.6) by

$$\phi = \phi_L \quad \text{at} \quad x = 0, \qquad \phi = \phi_R \quad \text{at} \quad x = 1. \tag{2.12}$$

Finally, there is a hypothesis that always holds provided the nonisotropy is not too large. If $H^+ = H^-$ (isotropy), then $\phi = 0$ is a "turning point" of (2.4), i.e., $f'(0) = 0$. Furthermore, in that case, 0 lies between ϕ_L and $\phi_R = -\phi_L$. For $H_0^+ - H_0^-$ sufficiently small, there is a nearby turning point ϕ_T. Hence we assume

$$\begin{gathered}\exists \text{ a unique value } \phi = \phi_T \text{ in } J \text{ such that } f'(\phi_T) = 0,\\ \text{and } \phi_T \text{ lies between } \phi_L \text{ and } \phi_R.\end{gathered} \tag{2.13}$$

Note that if $f''(\phi) < 0$, then $\phi_R \leqq \phi_T \leqq \phi_L$ (see Figure 2.1); if $f''(\phi) > 0$ then $\phi_L \leqq \phi_T \leqq \phi_R$.

C. For N large, equations (2.4), (2.12) can be considered as a singularly perturbed, 2-point boundary value problem for $\phi(x)$. We now review how such problems may be solved, and summarize some of the properties of solutions to (2.4), (2.12). We shall not fill in the details of the existence proofs for (2.4), (2.12); the formal arguments motivate the closely related, but more intricate, proofs to be done in Sections 4–6.

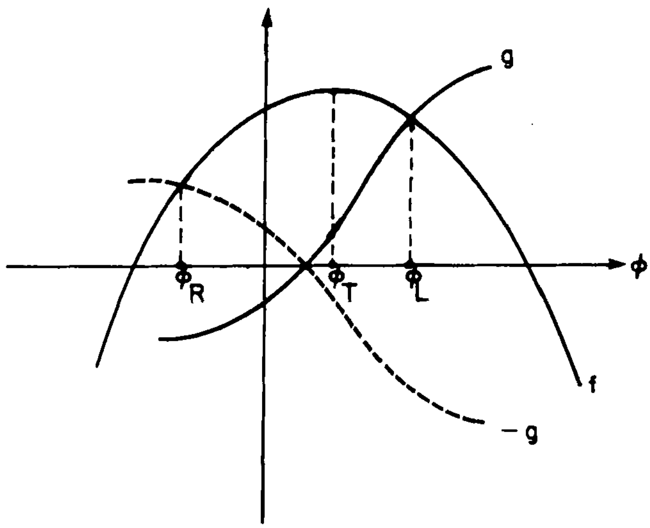

Figure 2.1. The functions $f(\phi)$ and $g(\phi)$. The dashed curve is the graph of $-g(\phi)$. ϕ_L satisfies $f(\phi) = g(\phi)$; ϕ_R satisfies $f(\phi) = -g(\phi)$. ϕ_T is the "turning point", i.e., the place at which $f'(\phi) = 0$. This picture corresponds to the case $f'' < 0$.

Recall that, for a singularly perturbed equation, the solution to the boundary value problem is expected to be like the solution to the "reduced problem"

$$0 = \beta(x) + 2f(\phi)_x. \tag{2.14}$$

Since (2.14) is of first order, and there are two boundary conditions (2.12), one expects at least one shock layer, which may be near one of the boundaries, or may be in the interior. In general, the latter is possible if (2.4) has a "turning point", as in (2.13).

The equation describing the shock layer is obtained from (2.4) by scaling the variable x around a point in the layer so that the term with the highest derivative becomes comparable in size to the others. A convenient point turns out to be the value x_0 (not yet known) at which $\phi = \phi_T$ (the turning point). Let $X = N(x - x_0)$. Then the shock layer equation is

$$0 = 2f(\phi)_X + g(\phi)_{XX}. \tag{2.15}$$

Note that (2.15) is independent of x_0.

The procedure for constructing (formally) a solution to (2.4), (2.12) is to find a solution $\Phi(x)$ to (2.14) satisfying one boundary condition (say the right) and a solution $\phi(X)$ to (2.15) satisfying the left-hand boundary condition, along with $\lim_{X \to \infty}\phi(X) = \Phi(0)$. The idea is that, since X is a "stretched" variable, ϕ goes to its asymptotic ($X \to \infty$) limit in an interval in x that goes to zero as $N \to \infty$.

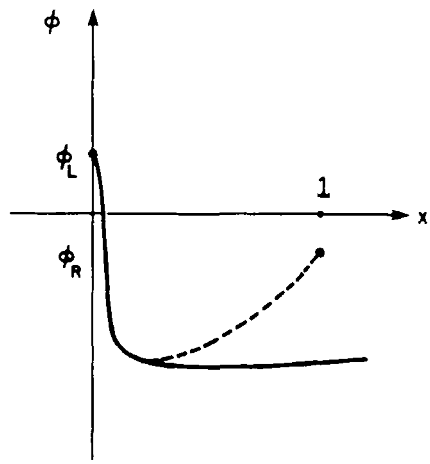

Figure 2.2. The shock layer and outer solutions for (2.7), $f'' < 0$. The dashed line is the solution $\Phi(x)$ to (2.14), $\Phi(1) = \phi_R$. The solid line is the solution $\phi(X)$ to (2.15), $\phi(0) = \phi_L$, $\lim_{X\to\infty}\phi(X) = \Phi(0)$.

(If the boundary layer is to be at the right-hand side, then $\Phi(x)$ must satisfy the left-hand boundary condition, $\phi(X)$ the right-hand one, and $\lim_{X\to-\infty}\phi(X) = \Phi(1)$.) The relevant solutions to the outer and shock layer equations, as proved in Proposition 2.1 below, are shown in Figure 2.2 for the case $f'' < 0$, $\beta(x) < 0$. For more details of this procedure, see [7].

The position of the boundary layer is determined by the sign of a quantity we shall now define. Let

$$\Omega = -\int_0^1 \beta(s)\,ds,$$

and

$$Q_c = f(\phi_L) - f(\phi_R) + \Omega.$$

PROPOSITION 2.1. *Suppose that J can be chosen so that* (2.9)–(2.11), (2.13) *hold. Assume, for definiteness, that $f'' < 0$ near $\phi = 0$. Suppose that $Q_c > 0$ (respectively $Q_c < 0$) and let $\Phi(x)$ be the solution to* (2.4) *with $\Phi(1) = \phi_R < \phi_T$ (respectively $\Phi(0) = \phi_L > \phi_T$). Assume that J can be chosen large enough to contain the range of Φ, for $0 \leqq x \leqq 1$. Also assume that $\Phi(x) \neq \phi_T$ for any x. Then for N sufficiently large, there is a unique (formal) solution to* (2.4), (2.12), *and this solution has a boundary layer on the left-hand (respectively right-hand) side. (If $f'' > 0$ and $Q_c < 0$ (respectively $Q_c > 0$) there is a similar result with $\Phi(1) = \phi_R$ (respectively $\Phi(0) = \phi_L$).)*

Remark 2.1. If $H^+ \equiv H^-$, then f is even, which implies that $f(\phi_L) = f(\phi_R)$, so $Q_c = \Omega$. In that case, with e.g. $\beta(x) < 0$ and $f'' < 0$, the boundary layer is on the left. The proposition implies that the position of the boundary layer is determined by a balance between the amount of the anisotropy $|H^+ - H^-|$ and the size of the frequency gradient $\beta(x)$. If $\beta(x) \equiv 0$, then the solutions to the outer equation are constant, $\Omega = 0$, and the position of the boundary layer is determined only by the anisotropy.

Proof of Proposition 2.1: We shall prove only the case $f'' < 0$ and $Q_c > 0$; the other cases are shown similarly. For this case, we let $x_0 = 0$. We integrate (2.15) subject to the matching condition $\lim_{X \to \infty} \phi(X) = \Phi(0)$, where Φ satisfies (2.14) and the boundary value condition $\Phi(1) = \phi_R$. This yields

$$f(\Phi(0)) - f(\phi(X)) = g'(\phi(X))\phi_X. \tag{2.16}$$

We will be finished if we show that there is a solution to (2.16) which satisfies $\phi(0) = \phi_L$.

Consider any solution to (2.16) with $\phi' < 0$ at $\phi = \Phi(0)$. Shooting backwards (i.e., $X \to -\infty$), it follows from (2.16) that, as long as ϕ satisfies $f(\Phi(0)) < f(\phi)$ (and $\phi \in J$), then $\phi_X < 0$ (i.e., ϕ increases as X gets more negative). Thus, either there is another value $\phi = \phi_{-\infty}$ for which $f(\Phi(0)) = f(\phi_{-\infty})$ (as in Figure 2.3), or else, for some X, $\phi = \phi_L$. If the latter is true, we are done, since all solutions to (2.16) are translates of one another; thus there is one for which $\phi = \phi_L$ at $X = 0$, i.e., $x = 0$. Suppose there is a $\phi_{-\infty}$ for which $f(\Phi(0)) = f(\phi_{-\infty})$. We claim that $\phi_{-\infty} > \phi_L$; hence, the solution passes $\phi = \phi_L$ and, as above, we are done. By the definition of Ω, $f(\phi_R) = f(\Phi(0)) + \Omega$. Hence, if $Q_c > 0$, then $f(\phi_L) - f(\phi_{-\infty}) > 0$. Since $\phi_{-\infty}, \phi_L > \phi_T$, $f' < 0$ between $\phi_{-\infty}$ and ϕ_L. Therefore, $\phi_L < \phi_{-\infty}$.

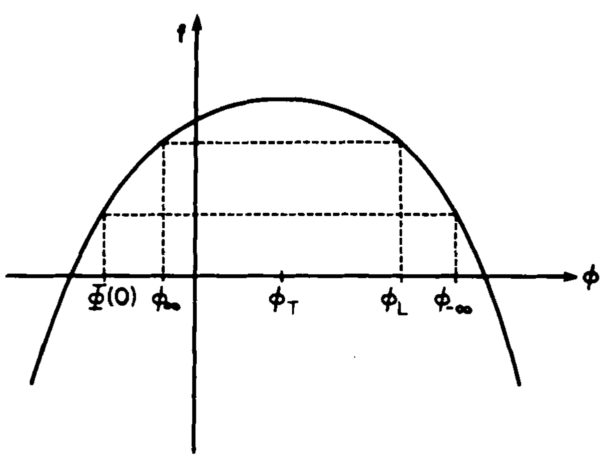

Figure 2.3. $\phi_{-\infty}$ satisfies $f(\phi_{-\infty}) = f(\Phi(0))$. For $f'' < 0$, $\Phi(0) < \phi_T < \phi_{-\infty}$. ϕ_∞ satisfies $f(\phi_\infty) = f(\phi_L)$.

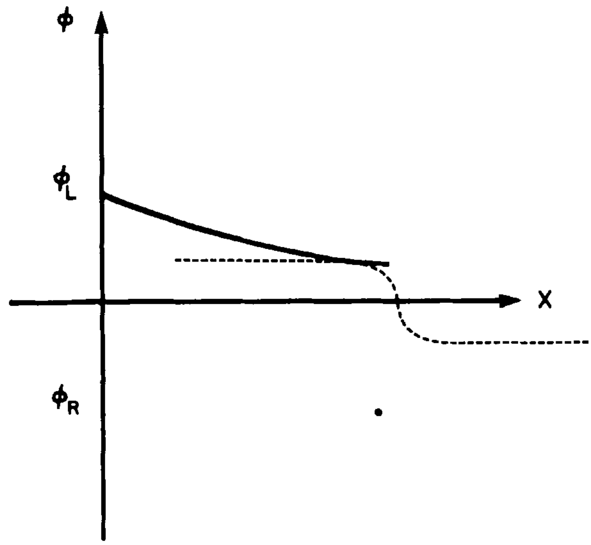

Figure 2.4. If $f'' < 0$, $Q > 0$, equations (2.4), (1.6) cannot be solved (formally) with a right-hand boundary layer. The solid curve is the solution $\Phi(x)$ to (2.14) with $\Phi(0) = \phi_L$. The dashed curve is the solution $\phi(X)$ to (2.15) which "matches" $\Phi(x)$, i.e., $\lim_{X \to -\infty} \phi(X) = \Phi(1)$. Note that $\lim_{X \to +\infty} \phi(X) > \phi_R$, so there is no X for which $\phi(X) = \phi_R$.

The uniqueness of the formal solution follows from further analysis of the shock layer equation. It is clear from (2.15) that (if $f'' < 0$) only monotonely decreasing solutions which pass $\phi = \phi_T$ are candidates for shock layer solutions; hence there is only one shock layer. To see that this layer cannot be near the right-hand endpoint, suppose that $x_0 = 1$ and $\beta(x)$ is sufficiently small so that the solution $\Phi(x)$ to (2.14) with $\Phi(0) = \phi_L$ stays above ϕ_T (see Figure 2.4). Then if a solution to (2.15) has the appropriate (matching) limit as $X \to -\infty$, its limit as $X \to +\infty$ lies above ϕ_R. (This follows from arguments using Q_c as above.) Hence, the right-hand boundary condition cannot be matched. The same argument rules out any interior layer.

Remark 2.2. If $Q_c = 0$, the above arguments show that for $0 < x_0 < 1$, there is a *formal* solution with a shock layer at x_0. That is, if Φ_L (respectively Φ_R) is the solution to (2.14) satisfying $\Phi(0) = \phi_L$ (respectively $\Phi(1) = \phi_R$), then one can match a boundary layer solution at *any* x_0 to Φ_R for $x > x_0$ and Φ_L for $x < x_0$ (see Figure 2.5). More sophisticated arguments show that the solution to (2.4), (2.12) is still unique. In some special cases, this is easy to see. For example, suppose that $\beta(x) \equiv 0$ (no frequency gradient), and $H^+ = H^-$ (isotropic coupling). Then f is even and g is odd. The solutions to (2.4) are then symmetric around the point x_0 at which $\phi = 0$. Also, $\phi_L = -\phi_R$. In this case, the only solution to (2.4), (2.12) has a boundary layer at $x_0 = \frac{1}{2}$.

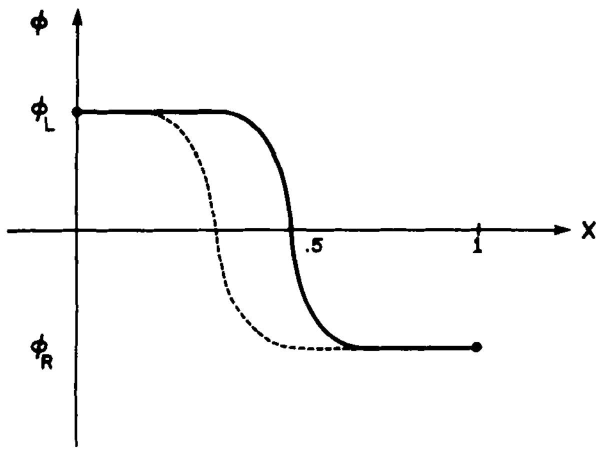

Figure 2.5. The solution to (2.4), (1.6) in the case $f'' < 0$, $\beta(x) \equiv 0$, $H^+ = H^-$. The boundary layer is then at $x_0 = \frac{1}{2}$. Formal arguments permit a shock layer at any x_0, as shown by the dashed curve.

Remark 2.3. The hypothesis that $g'(\Phi(x)) > 0$ for all Φ in the outer solution is essential; if $|\beta(x)|$ is too large, this hypothesis can fail, and there will be no solution to (2.4), (2.12). In terms of the original problem (1.1), this means that phaselocking breaks down when the frequency gradient is too large. For discussion of the scaling properties of the breakdown point, see Section 3.

D. For the discrete problem (2.1), (1.6), hypotheses (2.9)–(2.11), (2.13) play the same role as in the continuum analogue. However, in order that the solution to (2.1), (1.6) mimic that of (2.4), (2.12), there is an extra hypothesis, one related to issues of numerical stability of discretized problems. In (2.4), $(1/N)g'(\phi)\phi_{xx}$ is a "viscosity term" which smooths the shocks that can occur in its absence. The ratio $g'(\phi)/f'(\phi)$ governs the width of the shock layer. (The larger the ratio, the wider the layer.) This width is $O(1/N)$, the order of magnitude of the mesh size of the discrete problem (2.1). Instabilities occur if the shock layer is too small relative to the mesh size. The hypothesis needed in our problem to rule this out turns out to be easy to state in terms of f and g: We assume that $|g'(\phi)| > |f'(\phi)|$ or, equivalently, since $g' > 0$,

$$g' + f' > 0, \quad g' - f' > 0 \quad \text{for all} \quad \phi \in J. \tag{2.17}$$

We note that

$$\begin{aligned} (f+g)(\phi) &= H^+(\phi), \\ (f-g)(\phi) &= H^-(-\phi). \end{aligned} \tag{2.18}$$

Also note that (2.17) implies (2.9).

We can now state the main result of this paper. Let

$$Q = f(\phi_L) - f(\phi_R) - \left(\sum_{j=1}^{N} \beta_j \right) / N.$$

THEOREM 2.1. *Assume the hypotheses of Proposition* 2.1, *and also* (2.17). *Then if* $Q \neq 0$, *there exists a temporally stable, time-independent solution to* (1.7), (1.6). *This solution converges* (*nonuniformly near the boundary layer*) *to the solution of* (2.4), (2.12) *as* $N \to \infty$.

3. Some Implications of the Analysis of the M.C.E.

As will be shown in Sections 4–6, the discrete equations (2.1), (1.6) behave like the boundary value problem (2.4), (2.12). Furthermore, as will be shown in Section 7, the associated solution is temporally stable (i.e., stable as a solution to (1.7)). Thus, it is easy to read the qualitative behavior of (2.1), (1.6) from the solution to (2.4), (2.12). We mention here some of the more striking conclusions.

1. The distinction between "diffusive" ($H^{\pm}(0) = 0$) and "synaptic" ($H^{+}(0) \neq 0$) coupling, as described in the introduction, is an important one, since the two different kinds of coupling lead to qualitatively very different solutions. Assume, for simplicity that the coupling is isotropic ($H^+ = H^-$). If $H(0) = 0$, then $\phi_L = 0 = \phi_R$, and $\phi \equiv 0$ is a solution. By contrast, the equations (2.4) are the same for synaptic coupling, but if $H(0) \neq 0$, then $\phi_L \neq 0 \neq \phi_R$. Hence $\phi \equiv 0$ does not satisfy (2.12).

Recall that $\phi(x) \sim \phi(k/(N+1)) = \phi_k$, where ϕ_k is the phase difference between oscillators k and $k+1$ ($\phi_k = \theta_{k+1} - \theta_k$). Thus, $\phi \equiv 0$ corresponds to phaselocking in synchrony; $\phi(x) \not\equiv 0$ means that there is a spatial pattern of phases. In an x-interval in which $\phi(x) > 0$, $\theta_{k+1} > \theta_k$, and so the wave $\theta =$ constant travels from the higher numbered oscillators to the lower ones; the opposite holds if $\phi(x) < 0$.

2. If $\beta(x) \not\equiv 0$, in either the diffusive or synaptic case, the solution $\phi(x) \not\equiv 0$, and the resulting wave travels at a nonconstant speed. To see this, note that, in this "continuum limit", $\phi(x)$ can be interpreted as $\partial\theta/\partial x$. Differentiating $\theta(x, t) =$ constant with respect to t, one gets

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial x}\frac{\partial x}{\partial t} = 0. \tag{3.1}$$

Since the solution represents a phaselocked situation in which $\partial\theta/\partial t$ is independent of x and t, (3.1) implies that the speed dx/dt along the wave is inversely proportional to the (absolute value) of ϕ at the point. (Thus, in Figure 2.2, except for the boundary layer, the wave speeds up as x increases.) The wave has nonzero constant speed in an interval in which $\phi(x)$ is constant and not zero. This holds (essentially) in the synaptic case when $\beta(x) \equiv 0$.

If $\beta(x) \equiv 0$, we note that synaptic coupling gives a mechanism for the production of stable, constant speed waves in the absence of any "pacemaker". The mechanism depends crucially on the fact that the chain is finite (however long).

3. If $\beta(x) \equiv 0$, the wave can go in either direction, depending on the anisotropy. They can even go in both directions. In Figure 2.5, the solution to (2.4), (2.12) with $\beta \equiv 0$, $H^+ = H^-$, $f'' < 0$ corresponds to a pair of waves going outward from the center. The mirror image of Figure 2.5 about $x_0 = \frac{1}{2}$ is the graph of the solution with $\beta \equiv 0$, $H^+ = H^-$, $f'' > 0$, which corresponds to waves going inward toward the center. The boundary layer position is very sensitive; almost any change in anisotropy or $\beta(x)$ will (in the continuum limit) move the shock layer to $x = 0$ or $x = 1$.

4. From equation (2.4) it is apparent why the symmetry properties of H^+ are important. If, for example, $H^+ = H^-$ is a purely odd function of ϕ, then the singularly perturbed equation (2.4) is missing its most important term $f(\phi)_x$. The purely odd (and isotropic) case turns out to be easy to solve explicitly (at least when there is phaselocking), so one can check directly the effects of omitting the even term. In [2], it was shown that, for a purely odd H, the largest linear frequency gradient that still allows phaselocking has a total frequency difference (from the first to the last oscillator) of $O(1/N)$. (This was done in [2] only for $H = \sin\phi$, but the analysis is the same for other odd functions.) Thus, for *any* fixed β, if the frequency difference between successive oscillators is β/N, locking is impossible for sufficiently large N. By contrast, Theorem 2.1 implies that if H is not purely odd, then phaselocking can be maintained with an $O(1)$ total frequency difference. (Accordingly to Theorem 2.1, there *are* stable phaselocked solutions if $\{\beta_k\}$ is not too large; the total frequency difference is $\Sigma_k(\beta_k/N)$. In both the odd and the non-odd cases, these frequency differences have been scaled with respect to ε, so the real frequency differences, for equations (1.3), are $O(\varepsilon)$. As shown in the appendix, the behavior with respect to N is uniform in ε.)

5. Proposition 2.1 and Theorem 2.1 also give information about the frequency at which the oscillators phaselock. If $H^+ = H^- \equiv H$, the frequency gradient is linear, and H is an odd function, then it can be shown directly from symmetry arguments that the frequency of the locked solution is the average of the natural frequencies. By contrast, with synaptic coupling (possible only with H not odd), the frequency of the locked system may even be above the highest or below the lowest natural frequency. For example, consider the case of a linear frequency gradient, and $f'' > 0$, so the boundary layer of (2.4), (2.12) is on the right. For a phaselocked solution, the frequency $\dot\theta_k$ is independent of k, and can be determined by the equation for $\dot\theta_1$:

$$\dot\theta_1 = \omega_1 + \varepsilon H(\phi_1) = \omega_1 + \varepsilon H(\phi_L) + O(\varepsilon^2). \tag{3.2}$$

In case $f'' > 0$, we have $H(\phi_L) < 0$, so $\dot\theta_k < \omega_1$; if $\varepsilon|H(\phi_L)| > |\omega_1 - \omega_N| = O(\varepsilon)$, then the locked frequency is lower than the lowest natural frequency ω_N. Similarly if $f'' < 0$, the locked frequency may be higher than the highest natural frequency.

4. Analysis of the Shock Layer Equation

A. In this section, we return to the discrete equations (2.1) and analyze the special case $\beta_k \equiv 0$ (no frequency gradient). As will be seen in Section 6, the resulting equations play the same role for the full equations (2.1) that the shock layer equation (2.15) does for the continuum problem (2.4), (2.12).

For the shock layer, the relevant problem is not a two-point boundary value problem (B.V.P.), but the limiting behavior as $k \to \infty$ (or $-\infty$) for an initial value problem. However, the results obtained about the initial value problem will enable us to solve the B.V.P. for $\beta_k \equiv 0$ with little further work.

The following lemma says that (2.1) can be treated as an initial value problem, shooting from the left, and, within some ranges of initial conditions, $\{\phi_k\}$ has a finite limit as $k \to \infty$. There is an analogous result for shooting backwards.

LEMMA 4.1. *Suppose $f'' < 0$ (respectively $f'' > 0$) and assume there exists $\phi_\infty < \phi_T$ (respectively $\phi_\infty > \phi_T$) such that $f(\phi_L) = f(\phi_\infty)$, and an open interval J containing $[\phi_\infty, \phi_L]$ (respectively $[\phi_L, \phi_\infty]$) in which hypotheses (2.9)–(2.11), (2.13) and (2.17) are valid (see Figure 2.3). Then there exists $\tilde{\phi}_1$ such that, for $\tilde{\phi}_1 < \phi_1 < \phi_L$ (respectively $\phi_L < \phi_1 < \tilde{\phi}_1$), equation (2.1), with $\beta_k \equiv 0$ and $\phi_0 = \phi_L$, ϕ_1 given, can be solved successively for $\phi_2, \phi_3, \cdots$ with $\phi_k < \phi_{k-1}$ (respectively $\phi_k > \phi_{k-1}$), and $\lim_{k\to\infty}\phi_k$ exists. This limit can be explicitly computed, given ϕ_0 and ϕ_1. If two solutions $\{\phi_k\}, \{\bar{\phi}_k\}$ satisfy $\phi_0 = \bar{\phi}_0$ and $\bar{\phi}_1 < \phi_1$, then $\bar{\phi}_k < \phi_k$ for all k.*

Proof: We shall consider only the case $f'' < 0$; the other case is similar. If $\beta_k \equiv 0$, (2.1) can be rewritten as

$$(4.1) \qquad H^+(\phi_{k+1}) - H^+(\phi_k) = -[H^-(-\phi_k) - H^-(-\phi_{k-1})].$$

(This can be seen more easily from (1.7), with ϕ'_k, Δ_k set equal to zero.) Assume, by induction, that $\phi_2, \cdots, \phi_k$ have been found, that $\phi_k < \phi_{k-1} < \cdots < \phi_0$, and that $\phi_k > \phi_R$. From hypothesis (2.17), $(d/d\phi)[-H^-(-\phi)] > 0$. Since $\phi_k < \phi_{k-1}$, the right-hand side of (4.1) is less than 0. Also by (2.17), $(d/d\phi)[H^+(\phi)] > 0$. This implies that if (4.1) can be solved for $\phi_{k+1} \in J$, then $\phi_{k+1} < \phi_k$; as long as the ϕ_k stay in a region in which $(d/d\phi)H^+ > 0$, the equations *can* be solved.

We now show that for $\phi_1 < \phi_0 \equiv \phi_L$ sufficiently close to ϕ_0, the $\{\phi_k\}$ form a Cauchy sequence which remains in J. This shows simultaneously that the process can be continued forever, and that the $\{\phi_k\}$ converge to a finite limit. To see this, we rewrite (4.1) (equivalently (2.1)) again, to get

$$\begin{aligned}(4.2)\qquad &[H^+(\phi_k) - H^+(\phi_{k+1})]\\ &= [H^+(\phi_{k-1}) - H^+(\phi_k)] - 2[f(\phi_{k-1}) - f(\phi_k)]\\ &\equiv \mathscr{C}_k[H^+(\phi_{k-1}) - H^+(\phi_k)],\end{aligned}$$

where

$$\mathscr{C}_k \equiv 1 - \frac{2(f(\phi_{k-1}) - f(\phi_k))}{H^+(\phi_{k-1}) - H^+(\phi_k)}. \tag{4.3}$$

From (2.17) and the monotonicity of the $\{\phi_k\}$, the denominator of the right-hand side of (4.3) is greater than 0. Suppose, for some k, that $\phi_1, \cdots, \phi_k > \phi_T$; in view of (2.13) and the fact that $f'' < 0$, it follows that $f' < 0$ for $\phi_k < \phi < \phi_{k-1}$, so $\mathscr{C}_k > 1$. Using (4.2), this shows that, for any $\phi_1 < \phi_0$, the solution must pass $\phi = \phi_T$ in a finite number of steps, the number (crudely) bounded above by $(\phi_0 - \phi_T)/(\phi_0 - \phi_1)$.

Let k_0 be the first value of k for which $\phi_k \leqq \phi_T$. Then, for $k > k_0$, $\phi_k < \phi_T$ is bounded away from ϕ_T. It follows that, for $k > k_0$, $\mathscr{C}_k < 1$ and bounded away from 1. From (4.2),

$$H^+(\phi_k) - H^+(\phi_{k+1}) = \prod_{j=k_0+1}^{k} \mathscr{C}_j[H^+(\phi_j) - H^+(\phi_{j+1})]. \tag{4.4}$$

This shows that, as long as the $\{\phi_k\}$ are defined, once they pass ϕ_T, they begin to converge geometrically.

The $\{\phi_k\}$ can cease to be defined only if the sequence exits from the region in which H^+ has positive derivative. We now show that there are initial conditions $\tilde{\phi}_1 < \phi_1 < \phi_0 = \phi_L$ for which this is guaranteed not to happen. As in the case of the analogous continuum equation (2.15), if $\beta_k \equiv 0$, equation (2.1) has an "integral", i.e., a quantity dependent on any pair ϕ_k and ϕ_{k+1} in a manner independent of k. To see this, we once again rewrite (2.1), with $\beta_k \equiv 0$:

$$\begin{aligned}(4.5)\qquad P(k, k+1) &\equiv f(\phi_{k+1}) + f(\phi_k) + g(\phi_{k+1}) - g(\phi_k)\\ &= f(\phi_k) + f(\phi_{k-1}) + g(\phi_k) - g(\phi_{k-1}).\end{aligned}$$

Thus, $P(k, k+1)$ is independent of k, and can be evaluated at $k = 0$:

$$P = f(\phi_1) + f(\phi_0) + g(\phi_1) - g(\phi_0). \tag{4.6}$$

Note that P is closely related to the integral of the continuum equation (2.15). Using P, we can compute $\bar{\phi}_\infty \equiv \lim_{k\to\infty}\phi_k$ in terms of ϕ_0 and ϕ_1 (providing that

ϕ_k is defined for all k). Replacing ϕ_k, ϕ_{k+1} by $\bar{\phi}_\infty$ in P, we get

$$P = 2f(\bar{\phi}_\infty). \tag{4.7}$$

(Compare (4.7) with the continuum case, in which $\bar{\phi}_\infty \equiv \lim_{X\to\infty}\phi(X)$ is calculated by $2f(\bar{\phi}_\infty) = f(\phi) + g'(\phi)\phi_X$, with ϕ, ϕ_X given initial conditions.) If $\phi_1 \approx \phi_0$, then from (4.6), $f(\phi_0) \approx f(\bar{\phi}_\infty)$, i.e., the limiting value of $\bar{\phi}_\infty$, as $\phi_1 \to \phi_0 = \phi_L$ is ϕ_∞. From this, it can be seen that there is a parameter interval for ϕ_1 such that the resulting trajectory $\{\phi_k\}$ stays in the region in which H^+ has a positive derivative, and hence is defined for all k.

Finally, we show that if $\bar{\phi}_1 < \phi_1$, then $\bar{\phi}_k \le \phi_k$ for all k. Let $P, \bar{P}$ be values of the integral (4.5) for the solutions $\{\phi_k\}, \{\bar{\phi}_k\}$. By hypothesis (2.17), $(f+g)' > 0$, so $\bar{\phi}_1 < \phi_1$ implies $\bar{P} < P$. Suppose that $\bar{\phi}_j < \phi_j$ for all $j \leqq k$, by induction. Then

$$P - \bar{P} = [(f+g)(\phi_{k+1}) - (f+g)(\bar{\phi}_{k+1})] + [(f-g)(\phi_k) - (f-g)(\bar{\phi}_k)] > 0.$$

Since $\bar{\phi}_k < \phi_k$ and $(f-g)' < 0$, the second quantity in brackets is negative. Thus, the first quantity in brackets is positive. Using again $(f+g)' > 0$, we conclude that $\bar{\phi}_{k+1} < \phi_{k+1}$.

Remark 4.1. The existence of ϕ_∞ is not an extra hypothesis for Theorem 2.1. If Q is such that the boundary layer is to be on the left, then (e.g. if $f'' < 0$) $f(\Phi(0)) < f(\phi_L)$, so if J is large enough to satisfy the hypotheses of Proposition 2.1, then there is automatically such a ϕ_∞. (See Proposition 2.1 for the definition of $\Phi(0)$.)

For the continuum equation (2.4), the solution to the shock layer equation (2.15) can be shown to be a good approximation to the real solution near the layer, e.g. for $x \leqq O(1/\sqrt{N})$. The next lemma says that this is also true for the discrete case: the solution to the shock layer equations stays close to that of the full equation (2.1) for $k = O(\sqrt{N})$, or equivalently, $k/N = O(1/\sqrt{N})$.

LEMMA 4.2. *Assume the hypotheses of the previous lemma. Let $\{\psi_k\}$ denote a solution to* (2.1) *with $\beta_k \equiv 0$, $\psi_0 = \phi_L$ and ψ_1 in the interval proved in the previous lemma* (*i.e., $\psi_1 > \bar{\phi}_1$ if $f'' < 0$, $\psi_1 < \bar{\phi}_1$ if $f'' > 0$*). *Let $\{\phi_k\}$ be the solution to the full equations* (2.1), *i.e., $\beta_k \not\equiv 0$, with $\phi_0 = \psi_0$, $\phi_1 = \psi_1$. Then, given δ, there exists a κ such that $|\phi_k - \psi_k| < \delta/\sqrt{N}$ for $k < \kappa\sqrt{N}$.*

Proof: As usual, we write out only the case $f'' < 0$. Let $\rho_k = \phi_k - \psi_k$. Then $\{\rho_k\}$ satisfy

$$\begin{aligned} 0 = \frac{\beta_k}{N} &+ [(f'+g')(\psi_{k+1})]\rho_{k+1} - 2g'(\psi_k)\rho_k \\ &+ [(-f'+g')(\psi_{k-1})]\rho_{k-1} + O(\rho_{k-1}^2, \rho_k^2, \rho_{k+1}^2) \end{aligned} \tag{4.8}$$

with $\rho_0 = 0 = \rho_1$. We wish to show that, for some κ, $|\rho_k| < \delta/\sqrt{N}$ for $k < \kappa\sqrt{N}$. Equation (4.8) can be rewritten as

$$\rho_{k+1} - \rho_k = A_k\rho_k - B_k\rho_{k-1} - \frac{\bar{\beta}_k}{N} + O\left(\rho_{k-1}^2, \rho_k^2, \rho_{k-1}^2\right), \tag{4.9}$$

where

$$A_k = \frac{(g' - f')(\psi_k)}{(g' + f')(\psi_{k+1})} + \frac{(g' + f')(\psi_k) - (g' + f')(\psi_{k+1})}{(g' + f')(\psi_{k+1})}, \tag{4.10}$$

$$B_k = \frac{(g' - f')(\psi_{k-1})}{(g' + f')(\psi_{k+1})}, \tag{4.11}$$

and $\bar{\beta}_k = \beta_k((g' + f')(\psi_{k+1})$. We may assume that (4.8) has been solved for ρ_{k+1}, so we may omit the term $O(\rho_{k+1}^2)$ from the right-hand side of (4.9). It follows from (4.9) that

$$|\rho_{k+1} - \rho_k| < B_k|\rho_k - \rho_{k-1}| + a_k\rho_k + \frac{\bar{\beta}_k}{N} + O\left(\rho_{k-1}^2, \rho_k^2\right), \tag{4.12}$$

where $a_k \equiv |A_k - B_k|$. Using the fact that the $\{\psi_k\}$ are known to converge exponentially for $k > k_0$ it can be seen from (4.10), (4.11) that the $\{a_k\}$ converge exponentially to zero. In particular, $\Sigma_{k=1}^{\infty} a_k$ is convergent. We shall use this and (4.12) to estimate the solution to (4.8). To handle the $O(\rho_i^2)$ terms, we assume, by induction, that $|\rho_i| < \delta/\sqrt{N}$, $i \leqq k$. Let $u_k \equiv \rho_k - \rho_{k-1}$, $u_0 = 0$. Then (4.12) implies

$$|u_{k+1}| < B_k|u_k| + a_k \sum_{i=1}^{k} |u_i| + \frac{\bar{\beta}_k}{N} + \frac{e_1\delta^2}{N}, \tag{4.13}$$

where $e_1\delta^2/N$ is a bound for $O(\rho_k^2, \rho_{k-1}^2)$ if $|\rho_i| < \delta/\sqrt{N}$.

Suppose, as in the previous lemma, that $\phi_1, \cdots, \phi_{k_0-1} > \phi_T$ and $\phi_{k_0} \leqq \phi_T$. Then, since $f'' < 0$, we have $f'(\phi_k) < 0$ for $k > k_0$, so $B_k \leqq 1$ and is bounded away from 1 for $k > k_0$. Let B be such a bound: $B_k \leqq B < 1$ for $k > k_0$. By picking k_0 still larger, we may assume that $\Sigma_{j=k_0+1}^{\infty} a_j < 1 - B$. We now sum (4.13) to get

$$\sum_{j=1}^{k} |u_{j+1}| < B \sum_{j=1}^{k} |u_j| + \sum_{j=1}^{k} a_j \sum_{i=1}^{j} |u_i| + \frac{1}{N} \sum_{j=1}^{k} \left(|\bar{\beta}_j| + e\delta\right). \tag{4.14}$$

Equivalently,

$$\begin{aligned} |u_{k+1}| + \alpha \sum_{j=k_0+1}^{k} |u_j| &< (1 - B) \sum_{j=1}^{k_0} |u_j| \\ &\quad + \sum_{j=1}^{k_0} a_j \sum_{i=1}^{j} |u_i| + \frac{1}{N} \sum_{j=1}^{k} \left(|\bar{\beta}_j| + e_1\delta^2\right), \end{aligned} \tag{4.15}$$

where $\alpha \equiv 1 - B - \Sigma_{j=k_0+1}^{\infty} a_j$. The first two sums on the right-hand side of (4.15) have a finite number of terms, the number independent of N. Furthermore, each u_k is $O(1/N)$, so the first two sums are bounded by e_2/N for some e_2. Finally,

$$|\rho_{k+1}| \leqq \sum_{j=1}^{k_0} |u_j| + \sum_{k_0+1}^{k+1} |u_j|. \tag{4.16}$$

Dividing (4.15) by $\alpha < 1$, we get from (4.16)

$$\frac{1}{N}\left\{e_3 + \frac{e_2}{\alpha} + \frac{k}{\alpha}\left(\beta + e_1\delta^2\right)\right\}, \tag{4.17}$$

where e_3/N is a bound for the first sum on the right-hand side of (4.16) and β is an upper bound for $|\bar{\beta}_j|$. Let $e = (\beta + e_1, \delta^2 + 1)/\alpha$. If $k \leqq \delta\alpha\sqrt{N}/e$, then (4.17) implies that, for N large, $|\rho_{k+1}| < \delta/\sqrt{N}$. Let $\kappa = \delta\alpha/e$, and we are done.

Remark. A similar result holds for backward shooting; in that case, the relevant solutions stay close for $N - O(\sqrt{N}) \leqq k \leqq N$.

There is one more result about the shock layer equation that we shall want as a lemma to solve the full equation (2.1), (1.6). The next lemma is used in Section 6 to match a solution to the shock layer equation to one of the outer equations (to be discussed in Section 5).

LEMMA 4.3. *Assume the hypotheses of Lemma* 4.1.

(a) *Let* $w_k \equiv \partial\psi_k/\partial\psi_1$, *where* $\{\psi_k\}$ *is the solution to* (2.1) *with* $\beta_k \equiv 0$, ψ_0 *given*, ψ_1 *a parameter as in Lemma* 4.2. *Then* $\lim_{k\to\infty} w_{k+1}/w_k = 1$.

(b) *Let* $\bar{w}_k = \partial\phi_k/\partial\phi_1$, *where* ϕ_k *is a solution to* (2.1) *with* $\phi_0 = \psi_0$, $\phi_1 = \psi_1$ *as in the previous lemma. Then for* $k = \kappa\sqrt{N}$, $\bar{w}_{k+1}/\bar{w}_k$ *is arbitrarily close to* 1 *for* N *large.*

Proof: As before, we consider only the case $f'' < 0$.

(a) The $\{w_k\}$ satisfy a linear variational equation which is very similar to, but simpler than, (4.8). That is,

$$0 = \left[(f' + g')(\psi_{k+1})\right] w_{k+1} - 2g'(\psi_k)w_k + \left[(g' - f')(\psi_{k-1})\right] w_{k-1} \tag{4.18}$$

with $w_0 = 0$, $w_1 = 1$. Equations (4.18) can be rewritten as

$$w_{k+1} - w_k = A_k w_k - B_k w_{k-1}, \tag{4.19}$$

where A_k and B_k are as in (4.10) and (4.11). Hence, as in Lemma 4.2,

$$|w_{k+1} - w_k| < B_k|w_k - w_{k-1}| + a_k w_k,$$

where the $\{a_k\}$ converge geometrically to zero. The argument of the previous lemma shows that $\lim_{k\to\infty} w_k$ exists and is finite. To conclude that $w_{k+1}/w_k \to 1$ as $k \to \infty$, we need show only that $\lim_{k\to\infty} w_k$ is non-zero. This follows from the last statement of Lemma 4.1, which implies that $w_k > w_1 = 1$ for all k.

(b) $\{\bar{w}_k\}$ satisfies the same equation as $\{w_k\}$, with $\{\psi_k\}$ replaced by $\{\phi_k\}$. Since, by Lemma 4.2, the $\{\phi_k\}$ stay arbitrarily close to the $\{\psi_k\}$ for $k \leq \kappa\sqrt{N}$, $\{\bar{w}_k\}$ stays arbitrarily close to $\{w_k\}$ for $k \leqq \kappa\sqrt{N}$. In particular, $k = \kappa\sqrt{N}$ can be made arbitrarily large by choosing N large, so $\bar{w}_{k+1}/\bar{w}_k$ can be made arbitrarily close to 1.

B. As noted earlier, the shock layer equation is the full equation for the special case $\beta_k \equiv 0$ for all k. Also, the previous lemmas provide all the information necessary to solve the B.V.P. (2.1), (1.6) in this case. Since this case is of special interest for one of the applications we have in mind (Section 8), we write the result here.

THEOREM 4.1. *Assume the hypotheses of Theorem* 2.1 *for the case* $\beta_k \equiv 0$ *for all* k. *Then there exists a solution to* (2.1), (1.6). *The solution converges (nonuniformly near the boundary layer) to the solution to* (2.4), (2.12) *as* $N \to \infty$.

Proof: As above, we write only the case $f'' < 0$. If $Q > 0$, the methods of Lemma 4.1 show there exist a ϕ_1 such that the solution to (2.1) with $\phi_0 = \phi_L$, ϕ_1 given, satisfies $\lim_{k\to\infty}\phi_k = \phi_R$, with a boundary layer near $k = 0$. Since $\partial\phi_N/\partial\phi_1 \neq 0$, as in Lemma 4.3, one can find a nearby ϕ_1 such that the associated solution satisfies $\phi_{N+1} = \phi_R$. Similarly, if $Q < 0$, there is a ϕ_N such that the solution $\{\phi_k\}$ with $\phi_{N+1} = \phi_R$, ϕ_N given, satisfies $\lim_{k\to-\infty}\phi_k = \phi_L$, and has a boundary layer near $k = N + 1$; again, for a nearby ϕ_N, the associated solution $\{\phi_k\}$ satisfies $\phi_0 = \phi_L$. In both cases, the solution is essentially constant away from the boundary layers; this constant is the same as in the solution to (2.4), (2.12).

5. Analysis of the Outer Equation

As in the continuum analogue, there is an "outer equation" for (2.1) which turns out to govern the behavior of (2.1), (1.6) except in some thin shock layer. For the discrete problem (2.1), (1.6), one cannot get the outer equation merely by omitting the terms whose continuum analogues are formally small; such a difference equation would be of second order, unlike the continuum outer equation (2.14), which is of first order. Instead, we work with a first-order equation that is closely related:

$$(5.1)_{N,k} \qquad 0 = \frac{\beta_k}{2N} + f(\phi_{k+1}) - f(\phi_k).$$

For definiteness, we assume a decreasing gradient, i.e., $\beta_k \leq 0$. (Recall that $\beta_k \equiv \beta < 0$ represents a decreasing linear frequency gradient.)

The first lemma says that (5.1) behaves like (2.14) for the appropriate $\beta(x)$.

LEMMA 5.1. *Assume there is a smooth function $\beta(x)$ such that $\{\beta_k\}$ of $(5.1)_{N,k}$ satisfy $\beta(k/(N+1)) - \beta_k = O(1/N^2)$. Let $\Phi(x) \equiv \Phi_L(x)$ (respectively $\Phi_R(x)$) be the solution to* (2.14) *with $\Phi(0) = \phi_L$ (respectively $\Phi(1) = \phi_R$), and assume that $|\beta(x)|$ is sufficiently small so that*

$$f'(\Phi(x)) \neq 0 \tag{5.2}$$

for all $0 \leq x \leq 1$, i.e., the outer solution does not hit any turning point of (2.4). *Let $\{\Phi_k\}$ denote the solution to* (5.1) *with $\Phi_0 = \phi_L$. Then $\{\Phi_k\}$ converges, as $N \to \infty$, to $\Phi_L(x)$. More specifically, for each $0 \leq x \leq 1$, and any sequence k_N such that $k_N/N \to x$ as $N \to \infty$, we have $\Phi_{k_N} \to \Phi(x)$ uniformly in x. The solution $\{\Phi_k\}$ is monotone decreasing in k. A similar result holds if $\Phi_{N+1} = \phi_R$ (instead of $\Phi_0 = \phi_L$); this yields a solution monotone increasing in k.*

Proof: We write out the case $\Phi(x) = \Phi_L(x)$. It is easier to work with $Z_k = f(\Phi_k)$ rather than the Φ_k. By (5.1), $Z_{k+1} - Z_k = -\beta_k/2N$. It follows that

$$Z_k = Z_0 + \frac{1}{2N}\sum_{j=1}^{k-1} \beta_j, \tag{5.3}$$

where $Z_0 = f(\Phi_0)$. Let $Z(x) = f(\Phi(x))$. By (2.14), $Z_x = -\frac{1}{2}\beta(x)$, so that $Z = -\frac{1}{2}\int_0^x \beta(s)\,ds + Z_0$. Since $f' \neq 0$, (5.2) implies that Z_k and $Z(x)$ can be solved, respectively, for Φ_k and $\Phi(x)$. It is clear that if $k_N/N \to x$, then $Z_{k_N} \to Z(x)$; hence, the same is true of Φ_k and $\Phi(x)$. The monotonicity of the $\{\Phi_k\}$ also follows from (5.2). A similar argument holds if the right-hand boundary condition Φ_{N+1} is given instead of Φ_0.

The next lemma, which is considerably less obvious, says that (5.1) is indeed an outer equation for (2.1). That is, there are solutions to (2.1) (those without boundary layers) which are uniformly close to solutions to (5.1). As in Section 4, we write out the argument for only one of the cases ($f'' < 0$ or $f'' > 0$; $Q < 0$ or $Q > 0$). In Section 4, we picked a choice for which the boundary layer is on the left, because it is notationally easier to shoot forwards than backwards. In this section, for similar reasons, we choose a case in which $\Phi(0) = \phi_L$ rather than $\Phi(1) = \phi_R$. These cases belong to incompatible choices; in Section 6, in which the matching is done, we shall work with an outer solution satisfying $\Phi(1) = \phi_R$.

LEMMA 5.2. *Assume the hypotheses of Proposition* 2.1 *and the hypothesis on $\{\beta_k\}$ of Lemma* 5.1. *Then, for N sufficiently large, there is a γ and an interval I in the parameter ϕ_1 (respectively ϕ_N) such that the solution to* (2.1) *with $\phi_0 = \Phi_L(0)$ (respectively $\phi_{N+1} = \Phi_R(1)$) satisfies $|\phi_k - \Phi_k| < \gamma/N$ for all $1 \leq k \leq N$.*

Proof: We write out the case $f'' > 0$, $Q > 0$, so $\Phi(x) = \Phi_L(x)$; the argument for the other cases are similar. In this case, the boundary condition ϕ_L satisfies $\phi_L < \phi_T$.

Let $\eta_k \equiv \phi_k - \Phi_k$. We are looking for an interval I in the parameter η_1 such that, if $\phi_0 = \Phi_0$, $\phi_1 = \Phi_1 + \eta_1$, $\eta_1 \in I$, then $|\eta_k| < \gamma/N$ for all $k \leqq N$. Let $v_k \equiv \eta_k - \eta_{k-1}$ with $\eta_0 = 0$ (so $v_1 = \eta_1$). As in Section 4, given ϕ_0 and ϕ_1, (2.1) defines a sequence $\{\phi_k\}$ by shooting, a sequence that continues to exist as long as $(H^+)' \equiv (f+g)' > 0$. Since the estimates below will show that $\{\phi_k\}$ stay close to $\{\Phi_k\}$, for which $(f+g)' > 0$ by hypothesis, the sequence is well defined. Thus, we have a sequence of maps M_k: $R^1 \to R^1$ given by $M_k(v_1) = v_k$. Let $\bar{J}$ be the interval $[-\gamma/N^2, \gamma/N^2]$, where γ is to be determined, and $I_k = M_k^{-1}(\bar{J})$. We shall show that there is a value of γ such that $I_1 \supset I_2 \supset \cdots \supset I_N$, and $v_1 \in I_k$ implies $|v_k| < \gamma/N^2$. Since $|\eta_k| \leqq \Sigma_{j=1}^{k}|v_j| < \gamma/N$, we may then let $I = I_n$.

From (5.1), we get

$$\frac{\beta_k}{N} + f(\Phi_{k+1}) - f(\Phi_{k-1}) = (\beta_k + \beta_{k-1})/2N. \tag{5.4}$$

Thus, $\{\eta_k\}$ satisfy

$$\begin{aligned} 0 = {} & [(f'+g')(\Phi_{k+1}) + O(\eta_{k+1})]\eta_{k+1} + [-2g'(\Phi_k) + O(\eta_k)]\eta_k \\ & + [(-f'+g')(\Phi_{k-1}) + O(\eta_{k-1})]\eta_{k-1} + (\beta_{k-1} - \beta_k)/2N \\ & + [g(\Phi_{k+1}) - 2g(\Phi_k) + g(\Phi_{k-1})]. \end{aligned} \tag{5.5}$$

We first assume $\phi_T \neq \phi_L$. From Lemma 5.1, since $f' \neq 0$, it follows that $f(\Phi_{k-1}) - f(\Phi_k) = O(1/N)$ uniformly in k; given the smoothness of f and g, and the fact that f' is bounded away from zero on the Φ_k, this implies that $[g(\Phi_{k+1}) - 2g(\Phi_k) + g(\Phi_{k-1})]$ is $O(1/N^2)$. Moreover, by hypothesis, $(\beta_{k-1} - \beta_k)/N = O(1/N^2)$. We abbreviate all these terms by $O(1/N^2)$. Equation (5.5) may be rewritten as

$$\eta_{k+1} - \eta_k = E_k\eta_k - D_k\eta_{k-1} + O(1/N^2), \tag{5.6}$$

where

$$E_k = \frac{2g'(\Phi_k) + O(\eta_k)}{(f'+g')(\Phi_{k+1}) + O(\eta_{k+1})} - 1,$$

$$D_k = \frac{(-f'+g')(\Phi_{k-1}) + O(\eta_{k-1})}{(f'+g')(\Phi_{k+1}) + O(\eta_{k+1})}.$$

Let $d_k \equiv E_k - D_k$. Then

$$d_k = \frac{[-f'(\Phi_{k+1}) + f'(\Phi_{k-1})] + [-g'(\Phi_{k+1}) + 2g'(\Phi_k) - g'(\Phi_{k-1})]}{(f' + g')(\Phi_{k+1}) + O(\eta_{k+1})} \tag{5.7}$$
$$+ \text{h.o.t.}$$

The terms labelled h.o.t. come from the $O(\eta_j)$ terms in E_k, D_k, $j = k, k \pm 1$:

$$\text{h.o.t.} = [-f''(\Phi_{k+1})\eta_{k+1} + f''(\Phi_{k-1})\eta_{k-1}]$$
$$+ [-g''(\Phi_{k+1})\eta_{k+1} - 2g''(\Phi_k)\eta_k + g''(\Phi_{k-1})\eta_{k-1}] + O(\eta_j^2).$$

Using the fact that $\Phi_{j+1} - \Phi_j = O(1/N)$ for all j, and the induction hypothesis $|\eta_j| < \gamma/N$ for $j \leqq k$, we see that h.o.t. $= O(1/N^2)$. Also, the second term in brackets in (5.7) is $O(1/N^2)$. Thus

$$d_k = \frac{[-f'(\Phi_{k+1}) + f'(\Phi_{k-1})]}{(f' + g')(\Phi_{k+1}) + O(\eta_{k+1})} + O(1/N^2). \tag{5.8}$$

For the case $f'' > 0$, $Q > 0$, we have by Lemma 5.1 that $\Phi_T > \Phi_1 > \cdots > \Phi_N$. Hence $f'(\Phi_{k+1}) < f'(\Phi_{k-1})$. Also, by (2.17), the denominator of d_k is greater than 0. Thus, for sufficiently large N, $d_k > 0$. It follows that if $\eta_k, \eta_k - \eta_{k-1} > 0$ or $\eta_k, \eta_k - \eta_{k-1} < 0$, then (5.6) implies

$$|v_{k+1}| > D_k|v_k| - C_1/N^2 \tag{5.9}$$

for some C_1.

For the case $f'' > 0$, we have $f'(\Phi_j) < 0$ since $\Phi_j < \Phi_T$ by Lemma 5.1. Thus $D_k > 1$, and is bounded away from 1 uniformly in k. (As we shall see, this will imply *exponential expansion* in η_k as k increases, in contrast to the exponential *convergence* in Section 4. This is not an artifact of the choice of sign of f''; if we worked with $f'' < 0$ and $\Phi_{N+1} = \phi_R$, then the shooting would be done backwards, and there would be exponential expansion of the η_k as *k decreases*.) Let $D > 1$ be a lower bound for $\{D_k\}$.

We assume by induction that $I_1 \supset \cdots \supset I_k$. Let $v_1 = \bar{v}_1^k$ be on the boundary of I_k, and let $\bar{v}_k = M_k(\bar{v}_1^k)$. By definition, $|\bar{v}_k| = \gamma/N^2$. Thus, from (5.9), $\bar{v}_{k+1} \equiv M_{k+1}(\bar{v}_1^k)$ satisfies

$$|\bar{v}_{k+1}| > \frac{D\gamma}{n^2} - \frac{C_1}{N^2}. \tag{5.10}$$

Choose $\gamma > C_1/(D-1)$. Then (5.10) implies that $|\bar{v}_{k+1}| > \gamma/N^2$. Consequently, $M_{k+1}(I_k) \supset \bar{J}$, or $I_{k+1} \equiv M_{k+1}^{-1}(\bar{J}) \subset M_k^{-1}(\bar{J}) \equiv I_k$.

This finishes the proof under the assumption that $\phi_L \neq \phi_T$. Although this is a generic assumption, it fails for the particularly important case of ordinary diffusive coupling, for which $\phi_T = 0$, $H^{\pm}(0) = 0$. Hence, we extend the argument here. Let $H_\alpha^{\pm}$ be a 1-parameter family of 2π-periodic functions H, satisfying $H_\alpha^{\pm} \to H^{\pm}$ as $\alpha \to 0$. Let f_α, g_α be the associated functions f, g satisfying (2.9)–(2.11), (2.13), (2.17), and $\phi_L(\alpha), \phi_T(\alpha)$ the associated boundary conditions and turning points. We continue to analyze only the case $f'' > 0$, $Q > 0$. Assume that, for $\alpha > 0$, $\phi_L(\alpha) < \phi_T(\alpha)$, but $\phi_L = \phi_T$ for $\alpha = 0$. (It is easy to construct such families.)

By the previous arguments, for each α there is an interval of solutions to the full equations (2.1), (1.6) which stay close to the relevant solution to (5.1). By Lemma 5.1, these solutions are close to the relevant solution to the continuum outer equation (2.14), which varies continuously as $\alpha \to 0$. Hence, the collection of these solutions to (2.1), (1.6), for α near 0, is uniformly bounded. It follows that there is a subsequence such that the solution is arbitrarily close to solving (2.1), (1.6) for $\alpha = 0$. To prove that these "approximate solutions" converge (to something which then automatically satisfies (2.1), (1.6) for $\alpha = 0$), it suffices to establish a certain transversality condition: Let $\mathscr{F}_k(\phi_{k-1}, \phi_k, \phi_{k+1})$ denote the right-hand side of equation (2.1). The transversality condition is

$$\left| \frac{\partial(\mathscr{F}_1, \cdots, \mathscr{F}_N)}{\partial(\phi_1, \cdots, \phi_N)} \right| \neq 0. \tag{5.11}$$

The Jacobian matrix in (5.11) is tri-diagonal. The non-zero entries in the k-th column are $(-f' + g')(\phi_k)$, $-2g'(\phi_k)$, $(f' + g')(\phi_k)$. Thus, the matrix is diagonally dominant, and, by the circle theorem (see [8], p. 304), the eigenvalues are in the closure of the left half-plane. The proof of this theorem shows further that, for a matrix with this structure, diagonally dominant implies no zero eigenvalues. Thus we are done.

Let $\{\phi_k\}$ denote an "outer solution" to (2.1), i.e., a solution to the full equations which stays uniformly close to a solution Φ_k to the outer equation. The final lemma of this section will be useful for matching. It says that, in the case analyzed above, divergences from such an outer solution increase exponentially as k increases. In the cases in which the outer solution satisfies $\Phi_{n+1} = \phi_R$, the divergences increase exponentially as k *decreases* from $k = N + 1$.

LEMMA 5.3. *Assume the hypotheses of Lemma* 5.2, *and let* $\{\phi_k\}$ *be the solution to* (2.1) *constructed in that lemma. Let* $y_k = \partial\phi_k/\partial\phi_1$. *If* $Qf'' > 0$ (*respectively* $Qf'' < 0$), *then* $y_{k+1}/y_k > 1$ (*respectively* $y_{k+1}/y_k < 1$), *and is bounded away from* 1 *uniformly in* k.

Proof: This follows almost immediately from the estimates in Lemma 5.2. For definiteness, assume $f'' > 0$, $Q > 0$, as above. The $\{y_k\}$ satisfy an equation similar to (but simpler than) (5.5):

$$(5.12)\quad 0 = [(f' + g')(\phi_{k+1})]\, y_{k+1} - 2g'(\phi_k) y_k + [(-f' + g')(\phi_{k-1})]\, y_{k-1}$$

with $y_0 = 0$, $y_1 = 1$. As in Lemma 5.2,

$$(5.13)\qquad (y_{k+1} - y_k) > D(y_k - y_{k-1}) - \bar{d}_k y_k,$$

where $D > 1$ and $\bar{d}_k = O(1/N)$. Assume by induction that $y_k/y_{k-1} > \bar{D}$, where $D > \bar{D} > 1$. From (5.13)

$$(5.14)\qquad \frac{y_{k+1}}{y_k} - 1 > D\left(1 - \frac{1}{\bar{D}}\right) - \bar{d}_k.$$

For N sufficiently large, (5.14) implies that $y_{k+1}/y_k > \bar{D}$.

6. Matching

In the previous two sections, we constructed solutions to the outer equation (5.1) and the shock layer equation (4.1). We also saw that there are solutions to the full (time-independent) equations (2.1) which can be made arbitrarily close, for all k, to a given solution of (5.1), or for $k \leqq O(\sqrt{N})$ to a given solution to (4.1). It remains to show that such solutions can be pieced together. For definiteness, we shall assume that $f'' < 0$ and $Q > 0$; the other cases are similar. By the analysis of Section 2, we then expect a solution qualitatively like Figure 2.2, with a boundary layer on the left.

Using the integral P given in (4.5), it is easy to match formally, i.e., to find a solution $\{\Phi_k\}$ to (5.1) with $\Phi_{N+1} = \phi_R$ and $\{\psi_k\}$ to (4.1) with $\psi_0 = \phi_L$, $\lim_{k\to\infty}\psi_k = \Phi_0$. (This is the analogue of formal matching for the continuum equation (2.4).) As seen in Section 4, ψ_k gets very close to its limit in a finite number of steps, independent of N. A formal approximation to a solution is then $\phi_k = \psi_k$ for $k \leqq \bar{\kappa}\sqrt{N}$, $\phi_k = \Phi_k$ for $k > \bar{\kappa}\sqrt{N}$ for any $\bar{\kappa} > 0$.

We shall now show that, for N large and $\bar{\kappa} = \kappa$ as defined in Lemma 4.2, this is indeed a good approximation, i.e., that under the hypotheses of Theorem 2.1, there is a solution to (2.1), (1.6), and it is close to the $\{\phi_k\}$ constructed above. We shall shoot from the right and from the left. Let $\{\psi_k\}$ and $\{\Phi_k\}$ denote the above solutions to (4.1) and (5.1), respectively, which match to "solve" (2.1), (1.6) as above. Let $\{\phi_k\}$ denote solutions to (2.1) which stay close to ψ_k for $k \leqq \kappa\sqrt{N}$, as in Lemma 4.2. Let $\{\bar{\phi}_k\}$ denote solutions to (2.1) which stay close to Φ_k for all k, as in Lemma 5.2. These solutions are parameterized, respectively, by ϕ_1 and $\bar{\phi}_N$ ($\phi_0 = \phi_L$ and $\bar{\phi}_{N+1} = \phi_R$ are fixed). We shall show that ϕ_1 and $\bar{\phi}_N$ can be chosen so that the resulting solutions agree at two consecutive indices k, $k + 1$, $k = O(\sqrt{N})$. Since (2.1) is a second-order equation, this will imply that the solutions agree for all k. (The continuum analogue is to require that the two solutions agree, at some x, on both ϕ and ϕ'.) Let $\bar{\phi}_N^0 \equiv \Phi_N$, $\phi_1^0 = \psi_1$.

Let j be the integer $j = [\kappa\sqrt{N} - 1]$, where $[\cdot]$ denotes "largest integer in $\cdot$". Let S: $R^2 \to R^2$ map $(\phi_1, \bar{\phi}_N)$ into $(\bar{\phi}_j - \phi_j, \bar{\phi}_{j+1} - \phi_{j+1})$. If $S(\phi_1, \bar{\phi}_N) = (0,0)$, then $\phi_k \equiv \bar{\phi}_k$ for all k. We first show that, for N sufficiently large, there is a rectangle $\mathscr{R}$ in $\phi_1, \bar{\phi}_N$-space, containing $(\phi_1^0, \bar{\phi}_N^0)$, such that the image of $\partial\mathscr{R}$ under S winds once around $(0,0)$. We demonstrate further that, for all $(\phi_1, \bar{\phi}_N) \in \mathscr{R}$,

$$\frac{\partial\left(\bar{\phi}_j - \phi_j, \bar{\phi}_{j+1} - \phi_{j+1}\right)}{\partial\left(\phi_1, \bar{\phi}_N\right)} \neq 0. \tag{6.1}$$

This will prove that there is a point $(\phi_1, \bar{\phi}_N)$ that gets mapped to $(0,0)$, and that the point is locally unique. We note that (6.1) alone does not suffice, because we have no solution to (2.1), (1.6) to "perturb off", only a "singular solution" patched together from simpler limiting equations. This is common to singularly perturbed equations.

From Lemma 5.2, $\nu_\pm$ can be chosen so that $|\eta_j| \equiv |\bar{\phi}_j - \Phi_j| = \gamma/N$ for $\bar{\phi}_N = \bar{\phi}_N^0 + \nu_+$, $\bar{\phi}_N = \bar{\phi}_N^0 - \nu_-$. Also, given b, γ can be chosen so that

$$\begin{aligned} \bar{\phi}_j - \bar{\phi}_{j+1} &> \frac{b}{N} \quad \text{if} \quad \bar{\phi}_N = \bar{\phi}_N^0 + \nu_+, \\ \bar{\phi}_{j+1} - \bar{\phi}_j &> \frac{b}{N} \quad \text{if} \quad \bar{\phi}_N = \phi_N^0 - \nu_-. \end{aligned} \tag{6.2}$$

This is true since $\Phi_{j+1} - \Phi_j = O(1/N)$ and the η_j diverge geometrically as j decreases (see Figure 6.1). Similarly, the estimates of Lemma 4.2 imply that $\mu_\pm$ can be chosen so that $|\rho_j| \equiv |\phi_j - \psi_j| = \delta/\sqrt{N}$ for $\phi_1 = \phi_1^0 + \mu_+$, $\phi_1 = \phi_1^0 - \mu_-$. δ is chosen large enough so that $\bar{\phi}_i - \phi_i > 0$ (respectively < 0) for $i = j, j+1$ if $\phi_1 = \phi_1^0 - \mu_-$ (respectively $\phi_1^0 + \mu_+$). That this is possible again follows from previous statements. That is, for $\psi_1 = \phi_1 = \phi_1^0$, $\Phi_N = \bar{\phi}_N = \bar{\phi}_N^0$, we have, by definition, $\lim_{k\to\infty}\psi_k = \Phi_0$. Since $j = O(\sqrt{N})$, ψ_j is exponentially close (in N) to $\lim_{k\to\infty}\psi_k$. Also, $|\Phi_0 - \Phi_j| = O(1/\sqrt{N})$; hence $|\Phi_j - \psi_j| = O(1/\sqrt{N})$. Since $|\phi_j - \psi_j| = O(1/\sqrt{N})$ and $|\bar{\phi}_j - \Phi_j| = O(1/N)$, the former term dominates the estimate of $\bar{\phi}_j - \phi_j$.

Let $\mathscr{R} = [\phi_1^0 - \mu_-, \phi_1^0 + \mu_+] \times [\phi_N^0 - \nu_-, \phi_N^0 + \nu_+]$. We can now demonstrate that if $\partial\mathscr{R}$ is traversed clockwise, its image under S winds clockwise once around $(0,0)$. To do this, we show that

(i) the image of the side $\phi_1 = \phi_1^0 + \mu_+$ (respectively $\phi_1^0 - \mu_-$) is contained in the 3-rd quadrant (respectively 1-st quadrant);

(ii) along the side $\bar{\phi}_N = \bar{\phi}_N^0 + \nu_+$, as ϕ_1 increases, the image of the line goes from the 1-st to the 3-rd quadrant, passing through the 4-th (but not the 2-nd) quadrant (see Figure 6.2). Similarly, along $\bar{\phi}_n = \bar{\phi}_N^0 - \nu_-$, as ϕ_1 increases, the image curve passes through the 2-nd, but not the 4-th, quadrant.

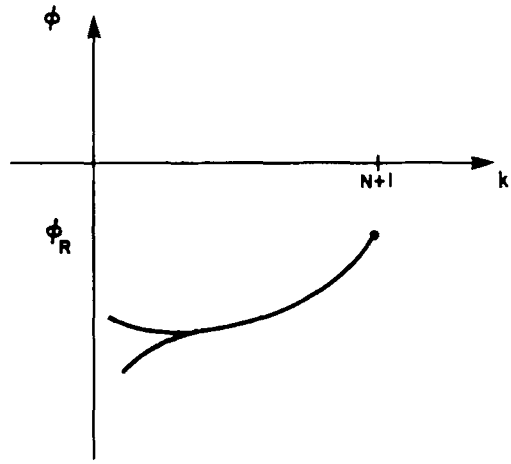

Figure 6.1. Two solutions $\{\phi_k\}$ to (2.1) close to the solution Φ_k to (5.1) satisfying $\Phi(1) = \phi_R$; the solutions both satisfy $\phi_{N+1} = \phi_R$, but have different ϕ_N.

Statement (i) follows directly from the construction of $\mathscr{R}$ since, for $\phi_1 = \phi_1^0 + \mu_+$, $\bar{\phi}_i - \phi_i < 0$, $i = j, j+1$. To establish (ii), we note that the edges of the image are in the 1-st and 3-rd quadrants (as claimed) by continuity. To get from the 1-st to the 3-rd quadrant, the image must pass through the horizontal axis, where $\bar{\phi}_{j+1} = \phi_{j+1}$ (see Figure 6.3). By definition, along the image of $\bar{\phi}_N = \bar{\phi}_N^0 + \nu_+$, we have $\phi_j > \phi_{j+1}$. Also, it follows from the methods of Lemma 4.1 that, for

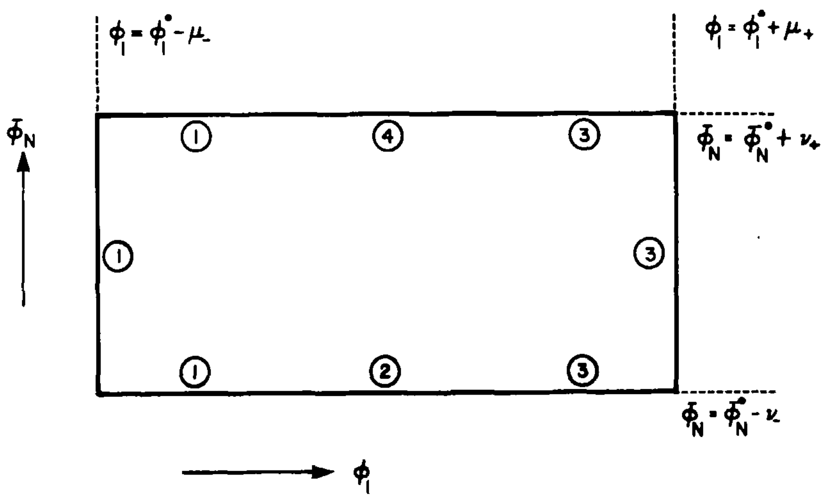

Figure 6.2. The rectangle $\mathscr{R}$. The numbers in circles represent the quadrant of the image of that portion of $\partial\mathscr{R}$ under the map S.

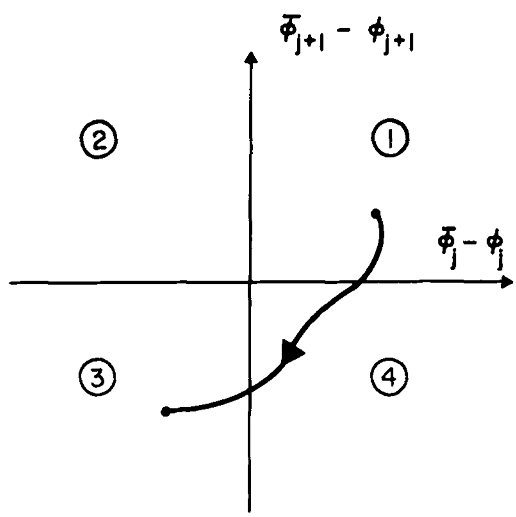

Figure 6.3. The image of $\bar{\phi}_N = \phi_N^0 + \nu_+$ under S. The curve goes through the axis $\bar{\phi}_{j+1} = \phi_{j+1}$ at a place where $\bar{\phi}_j > \phi_j$, i.e., between the first and fourth quadrants.

any solution $\{\phi_k\}$ along this arc, $|\phi_{j+1} - \phi_j| \leqq b/N$ for some b. Hence, if γ is chosen so that (6.2) holds for this b, then at the crossing point (i.e., where $\bar{\phi}_{j+1} = \phi_{j+1}$) $\bar{\phi}_j - \phi_{j+1} > b/N$, so that $\bar{\phi}_j > \phi_j$; thus, the crossing occurs between the 1-st and 4-th quadrants. This argument also rules out a crossing between the 2-nd and 3-rd quadrants, and a similar argument works for $\bar{\phi}_N = \bar{\phi}_N^0 - \nu_-$.

To prove (6.1) we must compute dS. Since ϕ_i (respectively $\bar{\phi}_i$), $i = j, j+1$, depends only on ϕ_1 (respectively $\bar{\phi}_N$), the left-hand side of (6.1) is

$$dS = \begin{pmatrix} -\dfrac{\partial \phi_j}{\partial \phi_1} & \dfrac{\partial \bar{\phi}_j}{\partial \bar{\phi}_N} \\ -\dfrac{\partial \phi_{j+1}}{\partial \phi_1} & \dfrac{\partial \bar{\phi}_{j+1}}{\partial \bar{\phi}_N} \end{pmatrix}.$$

By Lemma 4.3,

$$\frac{\partial \phi_{j+1}}{\partial \phi_1} \Big/ \frac{\partial \phi_j}{\partial \phi_1} \to 1 \quad \text{as} \quad N \to \infty, \quad j = \kappa\sqrt{N}. \tag{6.3}$$

By Lemma 5.3,

$$\frac{\partial \bar{\phi}_{j+1}}{\partial \bar{\phi}_N} \Big/ \frac{\partial \bar{\phi}_j}{\partial \bar{\phi}_N} < 1 \quad \text{and bounded away from 1.} \tag{6.4}$$

(The proof was done for $f'' > 0$, shooting forwards, and showed that $(\partial\bar{\phi}_{j+1}/\partial\bar{\phi}_N)/(\partial\bar{\phi}_j/\partial\bar{\phi}_N) > 1$; the same argument establishes (6.4) if $f'' < 0$ and the shooting is backwards.) Estimates (6.3) and (6.4) together imply (6.1).

The above matching completes the proof of Theorem 2.1.

7. Asymptotic and Numerical Stability of Phaselocked Solutions

A. Theorem 2.1 provides the *existence* of a phaselocked solution to (1.7), (1.6). We shall now prove that this solution is asymptotically stable. We shall use ideas closely related to monotone methods for parabolic P.D.E.'s. We assume the hypotheses of Theorem 2.1.

DEFINITION 7.1. An upper (respectively lower) solution to (1.7), (1.6) is a solution $\{\phi_k(\tau)\}$ such that, for all $1 \leqq k \leqq N$, $d\phi_k/d\tau < 0$ (respectively > 0) at $\tau = 0$.

We shall show below (Corollary 7.1) that an upper (respectively lower) solution decreases (respectively increases) for all τ. We first prove

LEMMA 7.1. *Suppose that $\phi_k^1(\tau)$ and $\phi_k^2(\tau)$ are two solutions to* (1.7), (1.6) *such that $\phi_k^1(0) < \phi_k^2(0)$. Then $\phi_k^1(\tau) \leqq \phi_k^2(\tau)$ for all $\tau > 0$.*

Proof: From (1.7),

$$\begin{aligned}\frac{d}{d\tau}(\phi_k^2 - \phi_k^1) = {} & \left[(f+g)(\phi_{k+1}^2) - (f+g)(\phi_{k+1}^1)\right] \\ & +\left[(g-f)(\phi_{k-1}^2) - (g-f)(\phi_{k-1}^1)\right] \\ & +2\left[-g(\phi_k^2) + g(\phi_k^1)\right].\end{aligned} \tag{7.1}$$

By hypothesis, at $\tau = 0$, $(\phi_k^2 - \phi_k^1) > 0$ for all $1 \leqq k \leqq N$. Suppose, for some k_1 and τ_1, that $\phi_{k_1}^1(\tau_1) = \phi_{k_1}^2(\tau_1)$. Then $g(\phi_{k_1}^1(\tau_1)) = g(\phi_{k_1}^2(\tau_1))$. Also, since $\phi_k^1(\tau_1) \leqq \phi_k^2(\tau_1)$ for all k, it follows from hypothesis (2.17) that the quantities in the first two brackets are non-negative. This shows that $(d/d\tau)(\phi_k^2 - \phi_k^1) \geqq 0$, and so ϕ_k^1 cannot cross ϕ_k^2 at $\tau = \tau_1$ for any τ_1.

COROLLARY 7.1. *If $\{\phi_k^u\}$ (respectively $\{\phi_k^l\}$) is an upper (respectively lower) solution to* (1.7), (1.6), *then $(d/d\tau)\phi_k^u \leqq 0$ for all τ (respectively $(d/d\tau)\phi_k^l \geqq 0$ for all τ).*

Proof: Consider an upper solution $\{\phi_k^u\}$. Since, at $\tau = 0$, $(d/d\tau)\phi_k^u < 0$, we have $\phi_k^u(\tau + h) < \phi_k^u(\tau)$ for $\tau = 0$ and all h sufficiently small. By Lemma 7.1, it follows that $\phi_k^u(\tau + h) < \phi_k^u(\tau)$ for all $\tau > 0$, which implies that $(d/d\tau)(\phi_k^u) \leqq 0$ for all $0 \leqq k \leqq N + 1$ and $\tau > 0$. A similar argument holds for $\{\phi_k^l\}$.

The next lemma says that Theorem 2.1 can be used to construct, not only time independent solutions to (1.7), (1.6), but also upper and lower solutions which bracket the time-independent one.

LEMMA 7.2. *Let* $\{\phi_k^\mu\}$ *be time-independent solutions to* (1.7), (1.6) *with* $\beta_k \leqq 0$ *replaced by* $\beta_k + \mu$. *(For* μ *sufficiently small, these exist by continuity.) Then for* $\mu > 0$ *(respectively* $\mu < 0$*), the solution* $\phi_k^\mu(\tau)$ *to* (1.7), (1.6) *with initial conditions* ϕ_k^μ *is an upper (respectively lower) solution for* (1.7), (1.6) *with the original* $\{\beta_k\}$. *Furthermore, for* $\mu^+ > 0$, $\mu^- < 0$, $\phi_k^{\mu^-} < \phi_k < \phi_k^{\mu^+}$, *where* $\phi_k \equiv \phi_k^0$.

Proof: Inserting $\{\phi_k^\mu(\tau)\}$ into (1.7) at $\tau = 0$, we get $(d/d\tau)\phi_k^\mu(\tau) = -\mu$, so $\phi_k^\mu(\tau)$ is indeed an upper (respectively lower) solution for $\mu > 0$ (respectively $\mu < 0$). The other statement follows from the detailed construction of the time-independent solution. It is easy to see that for $\mu > 0$ (respectively $\mu < 0$), the outer solution of $\phi_k^\mu(\tau)$ lies above (respectively below) the outer solution to $\phi_k(\tau)$. For the boundary layer, recall that the shock layer equation is independent of β_k (and hence of μ); the shock layer solutions for $\{\phi_k^\mu\}$ and $\{\phi_k\}$ solve the same equation, with the same (right-hand or left-hand) boundary condition, but with different asymptotes. (They "match" the appropriate outer solution.) It then follows from the last statement of Lemma 4.1 that $\phi_k^\mu < \phi_k$ (respectively $\phi_k^\mu > \phi_k$) for $\mu < 0$ (respectively $\mu > 0$).

LEMMA 7.3. *Let* $\{\phi_k^u(\tau)\}$ *and* $\{\phi_k^l(\tau)\}$ *be any upper and lower solutions, respectively, constructed as above. Then* $\lim_{\tau\to\infty}\phi_k^u(\tau) = \phi_k = \lim_{\tau\to\infty}\phi_k^l(\tau)$.

Proof: We have seen that $\phi_k^u(\tau)$ decreases as τ increases, and is bounded below by ϕ_k. Hence $\phi_k^u(\tau)$ approaches some limit $\bar{\phi}_k$ for each k. We claim that $\phi_k = \bar{\phi}_k$ for all k. Let $\{\phi_k^\mu(\tau)\}$ be the family of upper solutions constructed in the previous lemma, and let μ_1 be the largest value of μ for which $\phi_{k_1}^\mu = \bar{\phi}_{k_1}$, $\phi_k^\mu \geqq \bar{\phi}_k$ for all k and some k_1. By construction, the solution $\bar{\phi}_k(\tau)$ to (1.7), (1.6) with initial conditions $\bar{\phi}_k$ is time-independent; unless $\mu_1 = 0$, $\phi_k^{\mu_1}(\tau)$ decreases. Since $\bar{\phi}_k(\tau) \leqq \phi_k^{\mu_1}(\tau)$ at $\tau = 0$, this is true for all τ by Lemma 7.1. Thus, we must have $\mu_1 = 0$ and $\bar{\phi}_k = \phi_k$ for all k. A similar argument shows that ϕ_k^l converges to ϕ_k for all k.

Finally, we may prove asymptotic stability:

THEOREM 7.2. *Let* $\phi_k^u(\tau)$ *and* $\phi_k^l(\tau)$ *be any upper and lower solution constructed as above. Let* $\{\hat{\phi}_k\}$ *satisfy* $\phi_k^l(0) \leqq \hat{\phi}_k \leqq \phi_k^u(0)$. *Then the solution* $\hat{\phi}_k(\tau)$ *to* (1.7), (1.6) *with initial conditions* $\hat{\phi}_k$ *converges as* $\tau \to \infty$ *to the time-independent solution* ϕ_k *constructed in Theorem* 2.1.

Proof: By Lemma 7.1, $\phi_k^l(\tau) \leqq \hat{\phi}_k(\tau) \leqq \phi_k^u(\tau)$. Since $\phi_k^l(\tau) \to \phi_k$ and $\phi_k^u(\tau) \to \phi_k$, we are done.

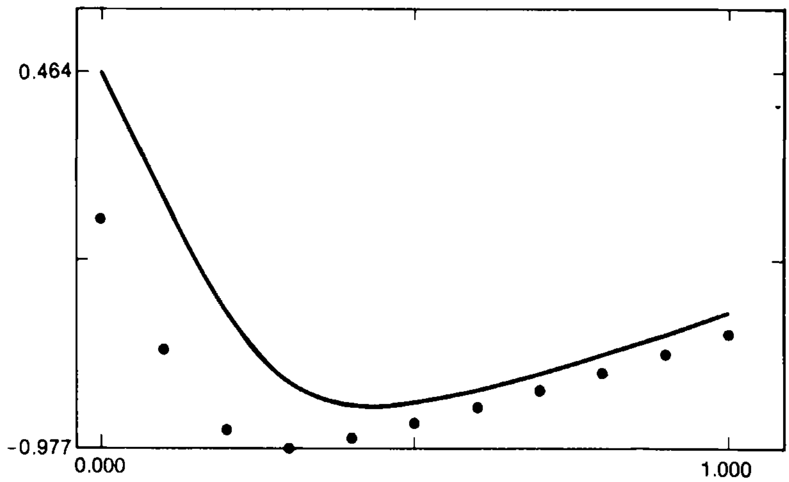

Figure 7.1. Comparison of the discrete equations, (2.1), (1.6) with numerically computed solutions of the continuum model, (2.4), Here, and in subsequent figures, $f(u) = -a\cos u$, $g(u) = b\sin u$, and $\beta(x) \equiv -\beta$, a constant. Solid lines are continuum and small circles are the discrete solutions. $N = 11$, $a = -0.5$, $b = 1.0$, $\beta = 0.5$.

Remark 7.1. Monotone methods are often used to prove the existence as well as the stability of solutions. The difficulty with such an approach in the present case is that there is no obvious way to construct lower solutions (if $\beta_k < 0$) *until* Theorem 2.1 has been proved. Indeed, for $\{\beta_k\}$ too large in absolute value, there is no phaselocked solution.

B. Theorems 2.1 and 7.1 concern the behavior of (2.1) (1.6) only in the limit of N large. Therefore, it is of interest to have simulations for N not so large. Figures 7.1–7.3 compare the solution of (2.1), (1.6) to that of (2.4), (2.12) for $N = 11, 21, 41$. (The solution to (2.1), (1.6) was computed, using a stiff O.D.E. solver, by integrating (1.7) from initial conditions until a steady state was reached. The solution to (2.4), (2.12) was computed using PDECOL, a finite element methods for parabolic equations, by integrating the time dependent version of (2.4) until a steady state was reached.) Figure 7.4 shows the solution to (1.7), (1.6) for $N = 41$ and the solution to the outer equation (2.14) satisfying the right-hand boundary condition.

In Figure 7.5, we present a simulation for f, g which satisfy (2.9)–(2.11), (2.13), but violate (2.17). As is seen, there is still an asymptotically stable, time-independent solution, but the solution now has spatial oscillations. Thus, hypothesis (2.17), which was the major assumption needed in order to use the monotone methods, appears also to be a crucial hypothesis for guaranteeing convergence of the solution of (2.1), (1.6) to that of (2.4), (2.12) as $N \to \infty$.

C. A hypothesis closely related to (2.17) occurs in an investigation (see [10]) of stable numerical algorithms for computing weak solutions to quasilinear

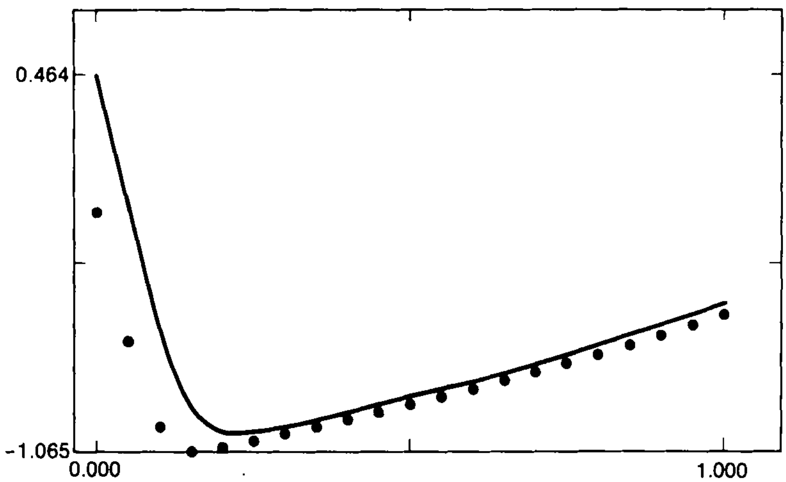

Figure 7.2. Same as Figure 7.1, $N = 21$.

P.D.E.'s of the form

$$u_t + F(u)_x = 0, \qquad u \in R^1, \quad x \in R^1, \tag{7.2}$$

with initial conditions $u(x,0) = u_0(x)$ given. (Beware: the u and F are unrelated to previous uses of these letters in this paper.) As is well known, weak solutions to (7.2) are not unique, and some condition, called an entropy condition, is

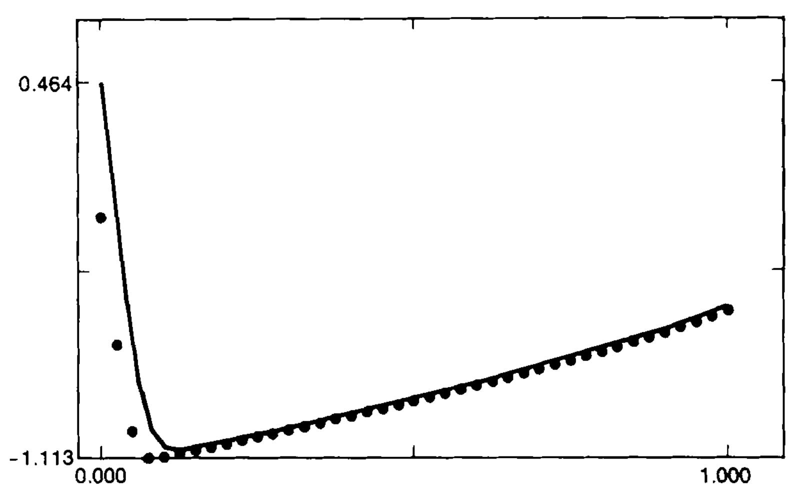

Figure 7.3. Same as Figure 7.2, $N = 41$.

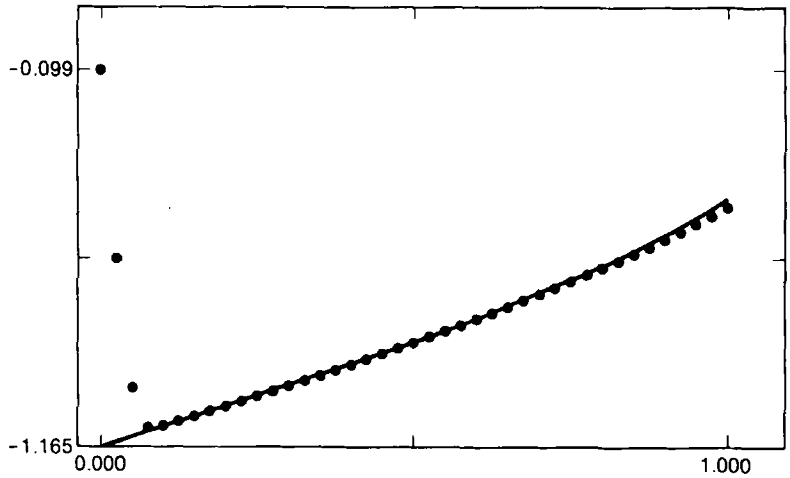

Figure 7.4. Comparison of the solution to the outer equation, (2.14) (solid lines) with the solution to the discrete equations (2.1), (1.6) (circles). Parameters are the same as in Figure 7.3.

required to pick out the physically correct one, which is the limit, as $\varepsilon \to 0$, of solutions to the associated equations

$$u_t + F(u)_x = \varepsilon u_{xx}, \qquad u(x,0) = u_0(x). \tag{7.3}$$

One such hypothesis is that the algorithm is a monotone finite difference scheme.

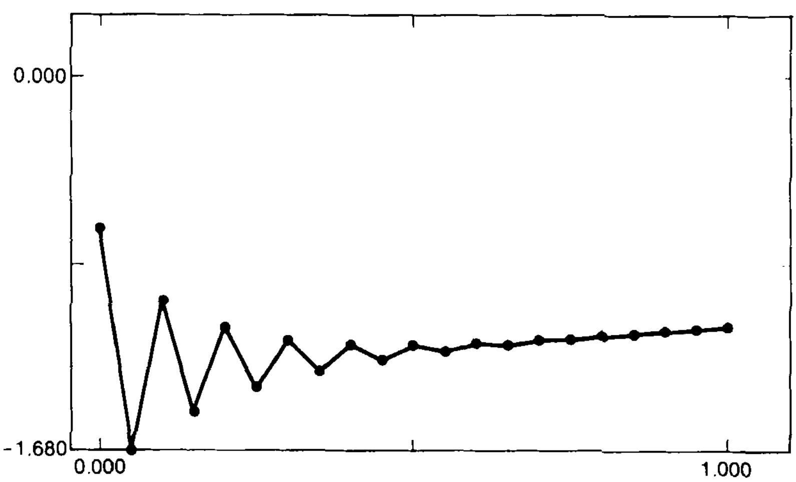

Figure 7.5. Violation of hypothesis (2.17). Solutions to (2.1), (1.6) when $N = 21$, $a = -2.0$, $b = 1.0$, $\beta = 0.8$.

A finite difference scheme

$$u_k^{n+1} = \mathscr{S}\left(u_{k-j}^n, u_{k-j+1}^n, \cdots, u_{k+j}^n\right)$$

is said to be monotone if $\mathscr{S}$ is a monotone increasing function of each of its arguments. As shown in [9], monotone schemes for a single conservation law, when convergent, always converge to the physically relevant solution (as the mesh size goes to zero). If $\beta(x) \equiv 0$, solutions to (2.1) can be considered fixed points of the discrete algorithm

$$(7.4) \qquad \phi_k^{n+1} = \phi_k^n + q\{(g+f)(\phi_{k+1}^n) - 2g(\phi_k^n) + (g-f)(\phi_{k-1}^n)\}$$

for any $q \in R'$. For $q > 0$ and not too large, hypothesis (2.17) implies that (7.4) is a monotone scheme.

Harten et al. [9] do not prove that such schemes converge. Le Roux [10] has investigated a class of numerical algorithms for (7.2) and shown, under conditions closely related to monotonicity, that they converge to the right weak solution. In spite of the similarity of the hypotheses, the techniques of [10] do not suffice to establish Theorem 2.1 for $\beta(x) \not\equiv 0$. The difficulty is related to the problem of using monotone methods: the methods of [10] require *a priori* bounds on the solution. This is not known in the absence of Theorem 2.1, and not true for $\{|\beta_k|\}$ too large.

A better way to discretize (2.4), (2.12) is to use one-sided difference schemes (see [11]) which have been applied to singular perturbation problems (see [12]). However, the real equations (1.7), (1.6) correspond to a central difference scheme for (2.4), (2.12).

8. An Application: Locomotor Central Pattern Generators in Fish

A. Many stereotypic rhythmic motions (walking, running, swimming) are thought to be governed by programs in the central nervous system that can operate in the absence of feedback or descending control from structures above the spinal cord (see [16]); these programs are known as central pattern generators (C.P.G's). This paper was partially motivated by attempts to understand such C.P.G's.

One well-studied example is the locomotor C.P.G in fish, which are primitive vertebrates. For technical reasons, much of the work has been done in dogfish and lamprey, but some results have been corroborated in other species (see [17]). Fish of many kinds propel themselves through water by rhythmic undulations of their bodies, undulations which are caused by waves of contractions that pass down the axial muscles. The contractions are induced by activity in the motor neurons which emerge from the cord periodically along the backbone, corresponding to the segmentation of the backbone. Measurements at these positions (" ventral roots"), or from the muscle segments, show temporally periodic bursts of activity at each such point, with a uniform frequency, and a phase lag between two points that is proportional to the distances between the points. That is, the neural activity is a constant speed travelling wave.

The propagation speed of the neural wave is proportional to the swimming speed of the fish, which can vary greatly (see [17], [18]). Furthermore, it is also proportional to the frequency of oscillation at each position, so the wave length of the wave is independent of the frequency. (This contrasts with the dispersion function for propagation down an axon in Hodgkin-Huxley models; see [19].) The ratio of the wave length to the size of the fish is species dependent. For dogfish, the wave length is about twice the fish; for thinner animals, such as lamprey, the ratio is smaller.

Dogfish can swim backwards in response to appropriate stimulation. When this happens, the wave of neural activity passes from the tail to the head (i.e., "caudal" to "rostral") instead of vice versa; see [18]. In forward swimming, the "leading segment" (the segment at which the phase is most advanced) need not be at the head of the fish, but can be anywhere in the most rostral third; one then observes neural waves travelling outward in both directions from the leading segment; see [18]. The position of the leading segment is easily changed by experimental alterations; see [18]. Finally, there is nothing special about the ends of the spinal cord; small pieces of the cord, functionally isolated from the rest, have the ability to "swim" forward or backward with appropriate phase relationships; see [18]. (These experiments were done on "spinal" dogfish, in which a lesion was made to remove the influence of upper brain structures, and with the head fixed in place. The resulting neural activity is believed to correspond to neural activity in intact animals; see [20].)

The results of this paper lead to an understanding of how the travelling waves of electrical activity may be generated, and why they have the properties described above. It also leads to some physiological conclusions, including a suggested mechanism for changing the speed of the fish while maintaining the wave length. The major assumption is that the cord can be viewed as a discrete collection of coupled oscillators, presumably, but not necessarily, with one pair of oscillators per segment. (These "segmental oscillators" have not yet been found, but their existence is a common working hypothesis; see [20].) It is also assumed that the coupling is "weak". (This is unknown.) Finally, the model uses nearest-neighbor interactions; however, as will be shown in a subsequent paper, the conclusions are expected to hold for a much wider class of interactions, including non-weak interactions built up out of multiple-neighbor connections.

The model of the locomotor C.P.G is then given by equations (1.1), where U_k is the k-th segmental oscillator. The crucial hypothesis is that of "synaptic coupling"; this generic hypothesis is much more physiological than diffusive coupling for collections of cells which communicate *via* synapses. From the analysis of Section 2, and from Theorem 2.1, we see that there are phaselocked solutions to (1.1), (1.2) with the property that the position of the most phase-advanced point is not necessarily at the ends, provided that the coupling is synaptic. (This requires $f'' < 0$.) If there is no significant frequency gradient, waves travelling outward from this point have constant speed. (For a model of the lamprey C.P.G using frequency gradients, see [21]. This model involves diffusive coupling, and gives rise to waves with nonconstant speed.) Solutions with waves travelling simultaneously in both directions occur only if the coupling is almost isotropic. (Otherwise, the most phase-advanced point is at an edge.) In this case, as observed in dogfish, the leading point is very sensitive to parameter changes. If the coupling is indeed close to being isotropic, a small change in the relative strengths of the forward and backward coupling moves the leading point to the caudal edge; this is a simple mechanism for the production of backward swimming. (With $f'' < 0$, an increase in the descending coupling moves the leading segment in the caudal direction.)

Forward swimming at different speeds, keeping the wave length constant, can also be achieved in a simple way, by uniformly increasing the frequencies of the oscillators. (There are control or initiation systems with global effects on the spinal cord; see [20].) The wave length is determined by the phase lag between successive oscillators, which depends on the boundary value ϕ_L or ϕ_R, and so is unaffected by such frequency changes if the coupling $H^{\pm}$ (or even just the solution to (1.6)) is unchanged. Alterations in the size of the cord (e.g. by transection) likewise leave the wave length unchanged if the remaining oscillators are numerous enough.

The mathematics also has implications for neural development in fish. For more details, and for applications to fin movement in electric fish, see [22], [23]. Simpler but related mathematics, with relevance to C.P.G's, is discussed in [22].

B. The "synaptic coupling" mechanism for producing constant speed waves may be compared with a mechanism for reaction-diffusion equations (2.7) which produce such waves. If $\mathbf{F}(\mathbf{U}, x)$ is independent of x, and $\mathbf{U}_t = \mathbf{F}(\mathbf{U})$ has a stable limit cycle, there is a 1-parameter family of (constant speed) travelling waves, with small wave number; see [24]. The plane waves have, in general, a nontrivial dispersion function; the wave length is not independent of frequency; see [24]. One does not expect to see any of the inhomogeneous solutions unless there is a mechanism for producing the initial conditions. Such a mechanism is provided by a local inhomogeneity ("pacemaker"), which acts to increase the local frequency (see [25], [26]), and the resulting plane wave depends on the frequency of the pacemaker. This contrasts with the synaptic mechanism, for which a pacemaker is not needed.

Appendix

Equation (1.4) was derived from (1.1) in [2] for a finite number $(N + 1)$ of oscillators. There are two steps in the derivation. The first is the assertion that there is an attracting invariant torus of dimension $N + 1$ for (1.1), for ε fixed and N arbitrarily large. The second is that coordinates $\theta_1, \phi_1, \cdots, \phi_N$ can be chosen on the torus so that the equations have the form (1.4).

The proof of the second step is uniform in N as $N \to \infty$, for fixed ε. The first step is also true, uniformly in N, but this is not obvious from the proof written in

[2], which uses a perturbation theorem of Fenichel [1] to get an invariant torus T^{N+1} for $\varepsilon \leqq \varepsilon_0(N)$ sufficiently small. To see that ε can be chosen independently of N requires estimates on the size of the permissible ε. Such estimates are given explicitly in a proof by Chow and Hale [27], pp. 425–429, for equations of the form

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} + \varepsilon\boldsymbol{\Theta}(\boldsymbol{\theta}, \boldsymbol{\rho}, \varepsilon),$$

$$\mathbf{R} = \mathbf{A}\boldsymbol{\rho} + \varepsilon\mathbf{R}(\boldsymbol{\theta}, \boldsymbol{\rho}, \varepsilon),$$

where $\boldsymbol{\theta} \in R^{N+1}$, $\boldsymbol{\rho} \in R^M$, $\boldsymbol{\Theta}$ is a 2π-periodic function of each component θ_j of $\boldsymbol{\theta}$, and $\mathbf{A}$ is an $(M+1) \times (M+1)$ matrix whose eigenvalues have negative real parts. Under the hypothesis that (1.3) has a stable limit cycle, (1.1) can be put in this form, with $M = (N+1)(m-1)$ and a $\boldsymbol{\Theta}$ in which the k-th component Θ_k depends only on θ_j, ρ_j, $j = k, k \pm 1$. Here, the $\{\theta_k\}$ are the angular variables around the limit cycles of (1.3), and $\rho_j = \varepsilon r_j$, where $r_j \in R^{m-1}$ is a normal variable to θ_j in a neighborhood of the limit cycle to (1.3), and $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_{N+1})$. It can be seen in a straightforward way from estimate (5.12) of [27] and this form of Θ that ε_0 can be chosen independently of N.

In addition to the proof that the derivation of (1.4) is uniform in N, there is another issue to be addressed in order to show that phaselocking exists, uniformly in N for fixed ε small and other parameters held fixed. Standard averaging theory (see [3]) states that if (1.7) has a stable critical point, then (1.4), (1.5) have a stable limit cycle for $\varepsilon \leqq \varepsilon_0$ sufficiently small. The size of ε_0 depends in general on the stability of the critical point: the closer to zero the least negative eigenvalue, the smaller the perturbation required to lose stability. Thus, a necessary condition that ε_0 can be chosen independent of N is that the eigenvalues of the linearization of the critical point of (1.7) be bounded away from zero uniformly in N. Because of the boundary layer in the solution to (1.7), (1.6), this is not straightforward to show. We present below a plausibility argument, and leave to others the task of formulating and proving a rigorous result.

The linearization of the system (1.7), (1.6) around the critical point is given by a tri-diagonal matrix L whose non-zero entries in the k-th row are $((-f' + g')(\phi_{k-1}), -2g'(\phi_k), (f' + g')(\phi_{k+1}))$, where $(\phi_1, \cdots, \phi_N)$ denotes the solution to (1.7), (1.6). This matrix can be symmetrized by a change of coordinates a diagonal matrix. Let L_s denote the negative of the resulting matrix. Then L_s is tri-diagonal with the non-zero elements of the k-th row given by $(-B_{k,k-1}, 2A_k, -B_{k,k+1})$, where

$$A_k = g'(\phi_k),$$

$$B_{k,k-1} = ((g' + f')(\phi_k) \cdot (g' - f')(\phi_{k-1}))^{1/2},$$

$$B_{k,k+1} = ((g' + f')(\phi_{k+1}) \cdot (g' - f')(\phi_k))^{1/2}.$$

Note that $B_{k,k-1} = B_{k-1,k}$. By construction, the arguments of the square roots are positive, and the positive square root is taken for $B_{k,k\pm 1}$.

If $A_k \equiv A$ and $B_{k,k\pm 1} \equiv B$ were independent of k, with $A - B > 0$, the result would now follow easily from the Gershgorin circle theorem (see [8]), yielding a lower bound on the eigenvalues of $2(A - B)$, independent of N. Indeed, if (1.7) is linearized around any constant function $\phi_k \equiv \phi$, with $f'(\phi) \neq 0$, then $A - B > 0$ does hold, so the eigenvalues are bounded below.

In the full nonconstant coefficient case, the Gershgorin theorem is not easy to apply. To conclude that the eigenvalues are bounded away from zero, this theorem requires that the sum of the entries of each row be so bounded. Outside of the boundary layer, this is true, since $\phi_{k+1} = \phi_k + o(1)$, so $2A_k - B_{k,k-1} - B_{k,k+1}$ is bounded below by $2\min\{g'(\phi) - ([g'(\phi)]^2 - [f'(\phi)]^2)^{1/2}\} - o(1)$, for ϕ in the range of the outer solution. (In this region, f' is bounded away from zero.) The reasoning fails only for a finite number of rows, the number independent of N. It remains to find out if the conclusion still holds in the generality of the hypotheses of this paper.

Remark A. It is crucial for this argument that $f' \not\equiv 0$. In the case $f' \equiv 0$, the conclusion fails: even for a linearization around a constant function $\phi_k \equiv \phi$, the matrix L is a multiple of the tri-diagonal matrix $(1, -2, 1)$, whose smallest eigenvalues tend to zero as $N \to \infty$. This suggests that chains of oscillators coupled by functions H that are odd (so $f \equiv 0$) may lose phaselocking for N large due to instabilities, as well as for the reason discussed in Section 3. (The critical point itself disappears as $N \to \infty$ for any fixed total frequency difference.)

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Bibliography

[1] Fenichel, N., *Persistence and smoothness of invariant manifolds for flows*, Indiana Univ. Math. J. 21, 1971, pp. 193–226.

[2] Ermentrout, G. B., and Kopell, N., *Frequency plateaus in a chain of weakly coupled oscillators, I*, SIAM J. Math. Anal. 15, 1984, pp. 215–237.

[3] Guckenheimer, J., and Holmes, P., *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer, New York, 1983.

[4] Howard, L. N., and Kopell, N., *Slowly varying waves and shock structures in reaction-diffusion equations*, Studies in Appl. Math. 56, 1977, pp. 95–145.

[5] Hagen, P., *Target patterns in reaction-diffusion systems*, Adv. in Appl. Math. 2, 1981, pp. 400–416.

[6] Neu, J., *Chemical waves and the diffusive coupling of limit cycle oscillators*, SIAM J. Appl. Math. 36, 1979, pp. 509–515.
[7] Lin, C. C., and Segel, L. A., *Mathematics Applied to Deterministic Problems in the Natural Sciences*, MacMillan, New York, 1974.
[8] Strang, G., *Linear Algebra and its Applications*, Second Edition, Academic Press, New York, 1980.
[9] Harten, A., Hyman, J. M., and Lax, P. D., *On finite difference approximations and entropy conditions for shocks*, Comm. Pure and Appl. Math. 29, 1976.
[10] Le Roux, A. Y., *A numerical conception of entropy for quasi-linear equations*, Math. of Comp. 31, 1977, pp. 848–872.
[11] Engquist, B., and Osher, S., *One-sided difference approximations for nonlinear conservation laws*, Math. of Comp. 36, 1981, pp. 321–351.
[12] Osher, S., *Nonlinear singular perturbation problems and one-sided difference schemes*, SIAM J. Numer. Anal. 18, 1981, pp. 129–144.
[13] Traub, R. D., *Simulation of intrinsic bursting in CA3 hippocampal neurons*, Neuroscience 7, 1982, pp. 1233–1242.
[14] Wilson, H. R., and Cowen, J. D., *A mathematical theory of the functional dynamics of cortical and thalamic tissue*, Kybernetik 13, 1973, pp. 55–80.
[15] Kopell, N., *Time periodic but spatially irregular solution to a model reaction-diffusion equation*, Ann. N.Y. Acad. Sci. 357, 1980, pp. 397–409.
[16] Grillner, S., *Locomotion in vertebrates: central mechanisms and reflex interaction*, Physiol. Rev. 55, 1975, pp. 247–304.
[17] Grillner, S., and Kashin, S., *On the generation and performance of swimming in fish*, in *Neural Control of Locomotion*, Herman, R. M., Grillner, S., Stein, P. S. G., and Stuart, D. G., eds., Plenum, New York, 1976.
[18] Grillner, S., *On the generation of locomotion in the spinal dogfish*, Exp. Brain Res. 20, 1974, pp. 459–470.
[19] Rinzel, J., *Models in neurobiology*, in *Nonlinear Phenomena in Physics and Biology*, Enns, R. H., Jones, B. L., Miura, R. M., and Rangnekar, S. S., eds., Plenum, New York, 1981.
[20] Grillner, S., *Neurobiological bases of rhythmic motor acts in vertebrates*, Science 228, 1985, pp. 143–149.
[21] Cohen, A. H., Holmes, P. J., and Rand, R. H., *The nature of the coupling between segmental oscillators of the lamprey spinal generator for locomotion: a mathematical model*, J. Math. Biol. 13, 1982, pp. 345–369.
[22] Kopell, N., *Toward a theory of modelling central pattern generators*, to appear in *Neural Control of Rhythmic Movements*, Cohen, A. H., Rossignol, S., and Grillner, S., eds., J. Wiley, New York.
[23] Kopell, N., *Coupled oscillators and locomotion by fish*, to appear in *Oscillators in Chemistry and Biology*, 1985, Othmer, H., ed., Lectures in Biomath., 66, Springer-Verlag, New York, 1986.
[24] Kopell, N., and Howard, L. N., *Plane wave solutions to reaction-diffusion equations*, Studies in Appl. Math. 52, 1973, pp. 291–328.
[25] Kopell, N., *Target pattern solutions to reaction-diffusion equations in the presence of impurities*, Adv. in Appl. Math. 2, 1981, pp. 389–399.
[26] Hagan, P., *Target patterns in reaction-diffusion systems*, Adv. in Appl. Math. 2, 1981, pp. 400–416.
[27] Chow, S. N., and Hale, J., *Methods of Bifurcation Theory*, Springer-Verlag, New York, 1982.

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