DEPARTMENT OF PHYSICS

PhD Qualifying Exam — **Modern Physics**

Friday, September 17, 1999 — 1 pm - 4 pm

PRINT YOUR NAME________________________

EXAM CODE__________________

1. PUT YOUR EXAM CODE, **NOT YOUR NAME,** ON EACH PIECE OF PAPER YOU HAND IN. (This allows us to grade each student only on the work presented.)

2. Do each problem or question on a separate sheet of paper.
   (This allows us to grade them simultaneously.)

Answer 5 of 7 short answer questions in part I and 3 of 5 longer problems in part II, with at least one problem from Group A and one from Group B. *Circle* the numbers below to indicate which questions you have answered—write nothing on the lines.

Short questions	*Long Problems*
1. _____	A1. _____
2. _____	A2. _____
3. _____	A3. _____
4. _____	B1. _____
5. _____	B2. _____
6. _____	
7. _____	

MODERN PHYSICS

PART I: Short answers (25%)

ANSWER 5 OF 7 QUESTIONS

1. In a muonic atom, a μ^- particle (m= 106 MeV/c^2) replaces an electron. A free μ^- particle decays ($N(t) = N_o e^{-t/\tau}$) with a lifetime τ of 2.2 μs. For a particular element, 42% of the muonic atoms with the muon in the 1s state have the muon captured by the nucleus, and therefore there is no muon decay. After 5 μs, how many muons in such a muonic atom remain in the 1s state?

2. In the preceding problem, would you expect 2s and 2p captures to be more or less likely, and why?

3 The interaction of an electron with a classical (i.e. non-quantized) plane wave electromagnetic field is described by the Hamiltonian

$$H = \frac{1}{2m}\left[\vec{P} - q\vec{A}\left(\vec{R},t\right)\right]^2 + V_C(R) - \frac{q}{m}\vec{S}\ \ \vec{B}\left(\vec{R},t\right),$$

where q and m denote the electron's charge and mass, $V_C(R)$ is the Coulomb potential energy due to the nucleus, and the last term on the right hand side is the Pauli correction.

After expanding the first term on the right hand side, organize all the contributions of the Hamiltonian in order of decreasing importance and discuss the relative sizes of each.

4. For the first three levels of the hydrogen atom (n=1,2,3) briefly discuss the consequences of the Pauli correction to the Schroedinger description of the system

5. Write the partition function you would need to begin a computation of the density of the atmosphere as a function of height above the earth.

6. A simple method to determine the ratio C_p/C_v is the following: A wire is used to heat a certain amount of gas with initial temperature T_o, pressure P_o and volume V_o. First the experiment is done at constant volume with the pressure changing from P_o to P_1, then it is repeated (after the system cools back to T_o) at constant pressure, with the volume changing from V_o to V_1. The time for heating, t, is the same in both experiments. Find the ratio C_p/C_v in terms of the initial and final volumes and pressures (assume the gas to be ideal.)

7. If you are given the canonical partition function for a system, show how to obtain the specific heat from this information.

PART II: Long problems (75%)

ANSWER 3 OF 5 QUESTIONS, WITH AT LEAST ONE FROM GROUP A AND ONE FROM GROUP B.

A1. Two for the price of one!

Consider a single mode of the electromagnetic field with frequency ω and wave vector $\vec{k}$. This observable is dynamically equivalent to a simple 1-D harmonic oscillator. This field undergoes a two-photon interaction process, e. g. parametric amplification in which a wave of frequency 2ω generates two quanta of frequency ω. This process is described by the Hamiltonian

$$H = \hbar\omega\, a^\dagger a + i\hbar\Lambda\left(a^{\dagger 2} e^{-2i\omega t} - h.a.\right)$$

where $a^\dagger$ and a are the creation and annihilation operators for the mode and Λ is a real quantity characterizing the strength of the interaction. Our final objective is to calculate the average number of photons created in the vacuum by the parametric process. For this purpose, here is a convenient road map.

a) Write the Heisenberg equations of motion for $a(t)$. Let

$$a(t) = b(t)e^{-i\omega t}$$

and now also write the equations of motion for $b(t)$ and $b^\dagger(t)$.

b) Define the auxiliary operators

$$b_P(t) = \frac{b(t) + b^\dagger(t)}{2}$$

$$b_Q(t) = \frac{b(t) - b^\dagger(t)}{2i}$$

and construct their Heisenberg equations. Solve these equations assuming that $b_P(0)$ and $b_Q(0)$ are known.

c) Assume that at t=0 the electromagnetic field is in the vacuum state. Calculate the mean number of photons $\langle N\rangle_t$ at time t.

A2. A raisin' in the field

The simplest Hamiltonian describing the interaction of a single mode of the electromagnetic field (formally equivalent to a simple harmonic oscillator) and a two-level system, e.g. a spin-1/2 particle, is given by

$$H = H_0 + H_1,$$

where

$$H_0 = \hbar\omega\, a^\dagger a + \frac{\hbar\omega_0}{2}\sigma_z$$

$$H_1 = \hbar\kappa\left(a^\dagger\sigma^- + a\sigma^+\right), \qquad\qquad \kappa = \text{real and} >0.$$

H_0 is the unperturbed Hamiltonian, i.e. just the sum of the energy operators of the two subsystems, and H_1 describes their interaction. As usual,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are the Pauli spin operators.

(a) Write down the eigenstates and eigenvalues of the operators $\hbar\omega\, a^\dagger a$, $\frac{\hbar\omega_0}{2}\sigma_z$ and H_0

(b) Provide a physical interpretation of the action of the operators $a^\dagger\sigma^-$ and $a\sigma^+$ when they act on the eigenstates of H_0.

(c) For simplicity, now assume $\omega = \omega_0$ (resonance) in the calculations that follow. We are interested in finding the eigenstates and eigenvalues of the total Hamiltonian H. As a first step, verify that

$$\left[H_0, H_1\right] = 0\,.$$

What is the consequence of this fact?

(d) Verify that the state $|n=0\rangle|-1\rangle \equiv |0,-1\rangle$ is one of the required eigenstates. Calculate the corresponding eigenvalue.

(e)For the remaining eigenstates, let

$$|\varphi_n\rangle = A|n+1,-1\rangle + B|n,+1\rangle, \qquad (n\geq 0)$$

where A, B are coefficients to be determined. Using this ansatz as a convenient expansion, find the eigenstates and eigenvalues of the total Hamiltonian.

A3. **The state I'm in.**

Consider a physical system whose three dimensional state space is spanned by the orthonormal basis u_1 u_2 u_3. In this basis, the Hamiltonian of the system is given by

$$H = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and an observable G is given by

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

At time t = 0, the physical system is the state $\psi(0)$ written as

$$\psi(0) = (1/\sqrt{2})u_1 + (1/2)\, u_2 + (1/2)\, u_3$$

(a) At t = 0 the energy of the system is measured. What values can be found and with what probabilities?

(b) Calculate the mean value of H and the root-mean-square deviation of H for the system in $\psi(0)$.

(c) At t = 0, one measures G instead of H. What results can be found and with what probabilities?

B1. **Spin Power**

The nuclei in a certain crystalline solid have spin one, such that each nucleus can be in any of the three quantum states labelled by the quantum number m, where m = 1, 0 or –1. Because of the ellipsoidal nature of the charge distribution, a nucleus has the same energy $E = \varepsilon$ in the state m =1 and m=–1, compared with an energy E = 0 in the state m = 0.

(a) What is the probability of finding a nucleus in each of the three states at absolute temperature T.

(b) Find an expression, as a function of T, of the nuclear contribution to the molar internal energy of the solid.

(c) Find an expression, as a function of T, of the nuclear contribution to the molar heat capacity. What are the limiting values of the molar heat capacity at very low and very high temperature? Plot the molar heat capacity as a function of temperature.

B2. **Around the cycle**

Consider an ideal gas undergoing the following thermodynamic cycle:

$A \rightarrow B$, Gas compresses adiabatically from volume V_A to V_B.
$B \rightarrow C$, Gas heated at constant volume
$C \rightarrow D$, Gas expanded adiabatically from volume V_B to V_A.
$D \rightarrow A$, Gas cooled at constant volume.

The entropy at state A is S_A and the entropy at state D is S_D.

(a) Draw a diagram of this process in the S-V plane.

(b) Derive an expression for the work *W* done by the engine during one cycle by determining the work done at each of the four steps above. Express your answer in terms of the (temperature independent) heat capacity, the number of particles *N*, Boltzman's constant *k*, and the volumes and entropies V_A andV_B and S_A and S_B.

(c) Derive an expression for the heat energy *Q absorbed* by the engine in terms of the heat capacity, the number of particles *N*, Boltzman's constant *k*, and the volumes and entropies V_A andV_B and S_A and S_D.

(d) If the efficiency is defined as W / Q, show that the efficiency is given by

$$\varepsilon = 1 - (V_A / V_B)^{-2/3}$$

for a monatomic ideal gas.