# PHYS 502: Mathematical Physics II

*Winter 2014, Homework #3*

(Due February 7, 2014)

1. Find the three lowest-frequency modes of oscillation of acoustic waves in a hollow sphere of radius $R$. Assume a boundary condition $\partial u/\partial r = 0$ at $r = R$, where $u$ obeys the differential equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\,.$$

2. The neutron density $n$ inside a spherical sample of fissionable material obeys the equation

$$\nabla^2 n + \lambda n = \frac{1}{\kappa}\frac{\partial n}{\partial t}\,,$$

where $\lambda > 0$, $\kappa > 0$, and $n = 0$ on the surface of the sample.

(a) Suppose the sample is spherical, of radius $R$. By seeking spherically symmetric modes with time dependence $e^{\alpha t}$, find the critical radius $R_0$ such that $n$ is unstable and *increases* exponentially with time for $R > R_0$.

(b) Now suppose the sample is a hemisphere, again of radius $R$. Repeat part (a), for axially symmetric modes.

(c) Two hemispheres of the material, each just barely stable as in part (b), are brought together to form a sphere. This sphere is unstable, with

$$n \sim e^{t/\tau}\,.$$

Find the time constant $\tau$ of the resulting explosion.

3. The curved surface of a long cylinder of radius $b$ is kept at a constant temperature $T = 0$. Initially the cylinder is at a uniform temperature $T_0 > 0$. Derive an expression for the temperature at the center of the cylinder at any time $t > 0$, and write down a simplified solution (not $T = 0$!) valid in the limit $t \gg b^2/\kappa$, where $\kappa$ is the heat diffusion coefficient of the cylinder.

4. (a) Consider the *homogeneous* two-dimensional Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

where $u(r,\theta)$ is finite inside the circle $r = R$ and satisfies the *inhomogeneous* boundary condition

$$u(R,\theta) = f(\theta)\,,$$

where $f$ is some given function. By writing down the general separable solution to the homogeneous equation, show that the solution may be written in the form

$$u(r,\theta) = \int_0^{2\pi} K(r,\theta,\theta')\,f(\theta')\,d\theta'\,.$$

and determine the function $K$.

(b) Solve the above equation for $f(\theta) = \cos^2\theta$.

5. Show explicitly from the series solutions that

$$\begin{aligned} J_{1/2}(x) &= A\,x^{-1/2}\,\sin x \\ J_{-1/2}(x) &= B\,x^{-1/2}\,\cos x\,. \end{aligned}$$

Hence, taking $A = B = 1$ and using the recurrence relations, write down expressions for $J_{3/2}(x)$ and $J_{5/2}(x)$.