Statmech II HW6

Vince Baker

November 29, 2015

1 Problem 8.15

a. We prove the relation:

$$X = \frac{2n\mu^{*2}}{(\frac{\partial \mu_0}{\partial x})|_{x=1/2}} = \frac{n\mu^{*2}}{kT}\frac{f_{1/2}(z)}{f_{5/2}(z)} \tag{1.1}$$

We start with the relation:

$$\mu_0(xN) = kT \ln(\frac{xN\lambda^3}{V}) = kT \ln z \tag{1.2}$$

$$\frac{\partial \mu_0}{\partial x} = kT\frac{\partial \ln z}{\partial x} \tag{1.3}$$

With $f_{3/2}(z) \simeq z$ we identify $f_{3/2}(z) = \frac{xN\lambda^3}{V}$ we find an expression for $\frac{\partial \ln z}{\partial x}$:

$$\frac{\partial f_{3/2}(z)}{\partial \ln z}\frac{\partial \ln z}{\partial x} = \frac{N\lambda^3}{V} \tag{1.4}$$

$$\frac{\partial \ln z}{\partial x} = \frac{f_{3/2}(z)}{xf_{1/2}(z)} \tag{1.5}$$

Using 1.3 and 1.6 we can now write 1.1 as:

$$X = \frac{2n\mu^{*2}}{kT/x}|_{x=1/2}\frac{f_{1/2}(z)}{f_{3/2}(z)} \tag{1.6}$$

$$X = \frac{n\mu^{*2}}{kT}\frac{f_{1/2}(z)}{f_{3/2}(z)} \tag{1.7}$$

At high temperatures $z \ll 1$ and (keeping terms to first order in z):

$$\frac{f_{1/2}(z)}{f_{3/2}(z)} = \frac{z - z^2 2^{-1/2} + ...}{z - z^2 2^{-1/2} + ...} \tag{1.8}$$

$$\frac{f_{1/2}(z)}{f_{3/2}(z)} \simeq \frac{1 - z2^{-1/2}}{1 - z2^{-3/2}} \tag{1.9}$$

$$\frac{f_{1/2}(z)}{f_{3/2}(z)} \simeq 1 - z2^{-3/2} \tag{1.10}$$

Where we have used the fact that $2^{-1/2} - 2^{-3/2} = 2^{-3/2}$. Using the high temperature expression $z = \frac{n\lambda^3}{2}$ we can write the susceptibility:

$$X = \frac{n\mu^{*2}}{kT}\left(1 - \frac{n\lambda^3}{2}2^{-3/2}\right) \tag{1.11}$$

$$X = \frac{n\mu^{*2}}{kT}\left(1 - \frac{n\lambda^3}{2^{5/2}}\right) \tag{1.12}$$

With $X_0 \equiv \frac{n\mu^{*2}}{kT}$ we have proved the provided relation.

At low temperatures we use the Sommerfeld expansions of the Fermi integrals:

$$\frac{f_{1/2}(z)}{f_{3/2}(z)} = \frac{3}{2}\frac{1}{\ln z}\left(1 - \frac{\pi^2}{6}(\ln z)^{-2} + ...\right) \tag{1.13}$$

Using the low-temperature approximation $\ln z = \frac{e_f}{kT}\left(1 - \frac{\pi^2}{12}(\frac{kT}{e_f})^2\right)$:

$$X = \frac{n\mu^{*2}}{kT}\frac{f_{1/2}(z)}{f_{3/2}(z)} \tag{1.14}$$

$$X = \frac{n\mu^{*2}}{kT}\frac{3}{2}\frac{kT}{e_f}\left(1 - \frac{\pi^2}{6}(\frac{kT}{e_f})^{-2} + ...\right)\left(1 + \frac{\pi^2}{12}(\frac{kT}{e_f})^2 + ...\right) \tag{1.15}$$

$$X = \frac{3n\mu^{*2}}{2e_f}\left(1 - \frac{\pi^2}{12}(\frac{kT}{e_f})^{-2} + ...\right) \tag{1.16}$$

2 Problem 8.19

We begin from the relation from problem 8.13, expression 8.1.21 and 8.4.7:

$$C_V = \frac{\pi^2}{3}k^2Ta(e_F) \tag{2.1}$$

$$a(e) = \frac{gV}{h^3}4\pi p^2\frac{dp}{de} \tag{2.2}$$

$$\frac{de}{dp} = \frac{p/m}{\left(1 + (\frac{p}{mc})^2\right)^{1/2}} \tag{2.3}$$

Using these expressions with $g = 2$ we find:

$$a(e) = \frac{8\pi mV}{h^3}p\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2} \tag{2.4}$$

$$\frac{C_V}{k} = \frac{8\pi^3 mV}{3}kTp\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2} \tag{2.5}$$

Using the value of N from equation 8.4.4 we show:

$$\frac{C_V}{Nk} = \pi^2 mkT \frac{\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2}}{p_f^2} \tag{2.6}$$

Making the substitution $x = \frac{p_f}{mc}$ we have proved the desired result.

$$\frac{C_V}{Nk} = \pi^2 \frac{kT}{mc^2} \frac{\left(x^2 + 1\right)^{1/2}}{x^2} \tag{2.7}$$

For the nonrelativistic case $p_f << mc, x << 1$ we have:

$$\frac{C_V}{Nk} \simeq \pi^2 \frac{mkT}{p_f^2} \tag{2.8}$$

With $e_f = \frac{p_f^2}{2m}$ we recover the nonrelativistic result $\frac{C_V}{Nk} = \frac{\pi^2}{2}\frac{kT}{e_f}$.
For the extreme relativistic case $p_f >> mc, x >> 1$ we obtain:

$$\frac{C_V}{Nk} \simeq \frac{\pi^2 kT}{p_f c} \tag{2.9}$$

We do not have any particular result for the specific heat of an ultrarelativistic Fermi gas to compare this to. However, we note that using ultrarelativistic energy $e = pc$ we recover the standard form of the specific heat without the factor of $\frac{1}{2}$.

3 Problem 7.3

Using the relation from note 6 in chapter 7 of Pathria:

$$\frac{g_{3/2}(z)}{g_{3/2}(1)} = \left(\frac{T_c}{T}\right)^{3/2} \tag{3.1}$$

Truncating Pathria D.9 two two terms and instering into 1:

$$g_{3/2}(e^{-a}) = \frac{\Gamma(-1/2)}{a^{-1/2}} + \xi(\frac{3}{2}) + ... \tag{3.2}$$

$$g_{3/2}(e^{-a}) = \xi(\frac{3}{2}) - 2\sqrt{\pi}a^{1/2} \tag{3.3}$$

$$1 - \frac{2\sqrt{\pi}a^{1/2}}{\xi(\frac{3}{2})} = \left(\frac{T}{T_c}\right)^{3/2} \tag{3.4}$$

$$a^{1/2} = \frac{\xi(\frac{3}{2})}{2\sqrt{\pi}} \left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right) \tag{3.5}$$

Now make a Taylor expansion of $1 - \left(\frac{T}{T_c}\right)^{3/2}$.

$$1 - \left(\frac{T}{T_c}\right)^{3/2} \simeq (1-1) - \frac{3}{2}\frac{1}{T_c^{3/2}} T_c^{1/2}(T - T_c) \tag{3.6}$$

$$1 - \left(\frac{T}{T_c}\right)^{3/2} \simeq -\frac{3}{2}\frac{T - T_c}{T_c} \tag{3.7}$$

Inserting the approximation into 5 and squaring both sides:

$$a \simeq \frac{1}{\pi}\left(\frac{3\xi(3/2)}{4}\right)^2 \left(\frac{T - T_c}{T_c}\right)^2 \tag{3.8}$$

4 Problem 7.5

a) We prove the following relations for the isothermal compressibility and adiabatic compressibility of an ideal Bose gas:

$$\kappa_T = \frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \tag{4.1}$$

$$\kappa_S = \frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S \tag{4.2}$$

Using the expressions for P and n and taking the derivatives with T held constant and with z held constant:

$$P = kT\frac{1}{\lambda^3} g_{5/2}(z) \tag{4.3}$$

$$\frac{\partial P}{\partial z} = \frac{kT}{\lambda^3}\frac{1}{z} g_{3/2}(z) \tag{4.4}$$

$$\frac{\partial P}{\partial T} = \frac{5}{2}\frac{(2\pi m)^{3/2}}{h^3} k^{5/2} T^{3/2} g_{5/2}(z) \tag{4.5}$$

$$n = \frac{1}{\lambda^3} g_{3/2}(z) \tag{4.6}$$

$$\frac{\partial n}{\partial z} = \frac{1}{\lambda^3}\frac{1}{z} g_{1/2}(z) \tag{4.7}$$

$$\frac{\partial n}{\partial T} = \frac{3}{2}\frac{(2\pi m k)^{3/2}}{h^3} T^{1/2} g_{3/2}(z) \tag{4.8}$$

Writing the compressibility expressions as functions of n and using the appropriate derivatives, we show:

$$\kappa_T = \frac{V}{N}\left(\frac{\partial(\frac{N}{V})}{\partial P}\right)_T = \frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_T \tag{4.9}$$

$$\kappa_T = \frac{1}{nkT}\frac{g_{1/2}(z)}{g_{3/2}(z)} \tag{4.10}$$

$$\kappa_S = \frac{V}{N}\left(\frac{\partial(\frac{N}{V})}{\partial P}\right)_S = \frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_S \tag{4.11}$$

$$\kappa_S = \frac{3}{5nkT}\frac{g_{3/2}(z)}{g_{5/2}(z)} \tag{4.12}$$

b) We now derive the relations:

$$\gamma = \frac{C_p}{C_v} = 1 + \frac{4}{9}\frac{C_v}{Nk}\frac{g_{1/2}(z)}{g_{3/2}(z)} \tag{4.13}$$

$$= \frac{5}{3}\frac{g_{5/2}(z)g_{1/2}(z)}{\left(g_{3/2}(z)\right)^2} \tag{4.14}$$

We first note that $\frac{C_p - C_v}{C_v} = \gamma - 1$, so $\gamma = 1 + \frac{C_p - C_v}{C_v}$. We calculate $\frac{\partial P}{\partial T}|_V$:

$$P = \frac{2U}{3V} \tag{4.15}$$

$$\left(\frac{\partial P}{\partial T}\right)|_V = \frac{2}{3V}\left(\frac{\partial U}{\partial T}\right)_V \tag{4.16}$$

$$\left(\frac{\partial P}{\partial T}\right)|_V = \frac{2}{3V}C_V \tag{4.17}$$

Using the provided relation for $C_P - C_V$ and plugging in the expression for $\left(\frac{\partial P}{\partial T}\right)|_V$:

$$\frac{C_P - C_V}{C_V} = \frac{4T}{9V}C_V\frac{1}{nkT}\frac{g_{1/2}(z)}{g_{3/2}(z)} \tag{4.18}$$

$$\frac{C_P - C_V}{C_V} = \frac{4}{9}\frac{C_V}{Nk}\frac{g_{1/2}(z)}{g_{3/2}(z)} \tag{4.19}$$

$$\gamma = 1 + \frac{4}{9}\frac{C_V}{Nk}\frac{g_{1/2}(z)}{g_{3/2}(z)} \tag{4.20}$$

Using the relation $\frac{C_P}{C_V} = \frac{\kappa_T}{\kappa_S}$ and substituting our previous values for the compressibilities:

$$\gamma = \frac{C_P}{C_V} = \frac{5}{3}\frac{g_{5/2}(z)g_{1/2}(z)}{\left(g_{3/2}(z)\right)^2} \tag{4.21}$$