Math Phys II HW 2

Vince Baker

January 25, 2015

Abstract

1 Problem 1

We seek solutions of the Kortweg-deVries equation:

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0 \tag{1.1}$$

We look for solutions $\psi(\xi)$, with $\xi = x - ct$. To write 1.1 in terms of ξ, we calculate the partial derivatives:

$$\frac{\partial \psi}{\partial t} = -c\frac{d\psi(\xi)}{d\xi}$$
$$\frac{\partial \psi}{\partial x} = \frac{d\psi(\xi)}{d\xi}$$
$$\frac{\partial^3 \psi}{\partial x^3} = \frac{d^3\psi(\xi)}{d\xi^3}$$

We can now write 1.1 in terms of ξ:

$$-c\frac{d\psi}{d\xi} + \psi\frac{d\psi}{d\xi} + \frac{d^3\psi}{d\xi^3} = 0 \tag{1.2}$$

This simplifies to:

$$(\psi - c)\frac{d\psi}{d\xi} + \frac{d^3\psi}{d\xi^3} = 0 \tag{1.3}$$

We can integrate 1.3 to find:

$$\frac{d^2\psi}{d\xi^2} = c\psi - \frac{\psi^2}{2} \tag{1.4}$$

We then integrate again and multiply by $\frac{d\psi}{d\xi}$:

$$\frac{d\psi}{d\xi} = \int (c\psi - \frac{\psi^2}{2}) \tag{1.5}$$

$$(\frac{d\psi}{d\xi})^2 = \frac{\psi^2}{2}(c - \frac{\psi^3}{3}) \tag{1.6}$$

$$\frac{d\psi}{d\xi} = \frac{\psi}{\sqrt{2}}(c - \frac{\psi^3}{3})^{\frac{1}{2}} \tag{1.7}$$

We can now integrate to find $\psi(\xi)$. Wolfram Alpha spits out some awful mess that can't possibly be an answer.

2 Problem 2

The general form of a second-order linear PDE is:

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x\partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} \tag{2.1}$$

The characteristic equation, with solutions $\xi(x,y)$ and $\eta(x,y)$, is:

$$A(\frac{dy}{dx})^2 + 2B(\frac{dy}{dx}) + C = 0 \tag{2.2}$$

We wish to write Eq. 1 in terms of ξ and η. We differentiate $\psi(\xi,\eta)$:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x} \tag{2.3}$$

Now we calculate the other partials with respect to η and ξ.

$$\frac{\partial}{\partial x}(\frac{\partial\psi}{\partial\xi}) = \frac{\partial^2\psi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x} \tag{2.4}$$

$$\frac{\partial}{\partial x}(\frac{\partial\psi}{\partial\eta}) = \frac{\partial^2\psi}{\partial\eta^2}\frac{\partial\eta}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x} \tag{2.5}$$

We use 3,4 and 5 to calculate $\frac{\partial^2\psi}{\partial x^2}$.

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} &= \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial\xi}{\partial x}(\frac{\partial^2\psi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x}) \\ &\quad + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial\eta}{\partial x}(\frac{\partial^2\psi}{\partial\eta^2}\frac{\partial\eta}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}) \\ &= \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2 + \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta} + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 + \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta}\end{aligned}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 + 2(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta}) \tag{2.6}$$

The calculation of $\frac{\partial^2\psi}{\partial y^2}$ is identical.

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\xi}{\partial y^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial y^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial y})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial y})^2 + 2(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2\psi}{\partial\xi\partial\eta}) \tag{2.7}$$

We now take $\frac{\partial}{\partial y}$ of equation 1:

$$\begin{aligned}\frac{\partial^2\psi}{\partial x\partial y} &= \frac{\partial^2\xi}{\partial x\partial y}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x\partial y}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}) \\ &\quad + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}) + \frac{\partial^2\psi}{\partial\xi\partial\eta}(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y})\end{aligned} \tag{2.8}$$

We can now write Eq. 1 in terms of ξ and η:

$$
\begin{aligned}
A\{\frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi}+\frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta}+\frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2+\frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2+2(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta})\} \\
+2B\{\frac{\partial^2\xi}{\partial x\partial y}\frac{\partial\psi}{\partial\xi}+\frac{\partial^2\eta}{\partial x\partial y}\frac{\partial\psi}{\partial\eta}+\frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y})+\frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y})+\frac{\partial^2\psi}{\partial\xi\partial\eta}(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y})\} \\
+C\{\frac{\partial^2\xi}{\partial y^2}\frac{\partial\psi}{\partial\xi}+\frac{\partial^2\eta}{\partial y^2}\frac{\partial\psi}{\partial\eta}+\frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial y})^2+\frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial y})^2+2(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2\psi}{\partial\xi\partial\eta})\} \quad (2.9)
\end{aligned}
$$

We now take a break to stop Eq. 9 from giving us a migraine brought on by eye strain.

We collect the coefficients of all the derivates of ψ:

$$
\begin{aligned}
\frac{\partial\psi}{\partial\xi}(A\frac{\partial^2\xi}{\partial x^2}+2B\frac{\partial^2\xi}{\partial x\partial y}+C\frac{\partial^2\xi}{\partial y^2}) \\
\frac{\partial\psi}{\partial\eta}(A\frac{\partial^2\eta}{\partial x^2}+2B\frac{\partial^2\eta}{\partial x\partial y}+C\frac{\partial^2\eta}{\partial y^2}) \\
\frac{\partial^2\psi}{\partial\xi^2}(A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2) \\
\frac{\partial^2\psi}{\partial\eta^2}(A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2)
\end{aligned}
$$

After the break, we recognize that since $\xi(x,y)$ and $\eta(x,y)$ are solutions to Eq. 1:

$$A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2=0 \quad (2.10)$$

$$A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2=0 \quad (2.11)$$

3 Problem 3

We are solving the characteristic equation for:

$$\frac{\partial^2\psi}{\partial t^2}-c(x)^2\frac{\partial^2\psi}{\partial x^2}=0$$

With A=1, B=0, and $c=-c(x)^2$, the characteristic equation is:

$$\frac{dx}{dt}^2-c(x)^2=0 \quad (3.1)$$

$$\frac{dx}{dt}=c(x) \quad (3.2)$$

$$dt=\frac{1}{c(x)}dx \quad (3.3)$$

With $c(x)=c_0(1+\frac{|x|}{a})$ the characteristic curve can be written:

$$t=\pm\frac{1}{c_0}(x+sgn(x)\frac{x^2}{2}) \quad (3.4)$$