Math Phys II HW5

Vince Baker

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1 Problem 1

We investigate the solution of the 3D Helmholtz equation with the boundary that the solution represents a traveling wave as $r \to \infty$. The general solution is:

$$U(r,\theta,\phi) = \sum_{\ell mn} (a_m j_m(kr) + b_m n_m(kr)) Y_\ell^m(\theta,\phi) \tag{1.1}$$

We look at the solution near $\vec{x}\,' = \vec{x}$, where we require $u(r) = -\frac{1}{4\pi r}$. We see that $-\frac{1}{4\pi} j_0(x) = (-\frac{1}{4\pi})\frac{\cos x}{x}$ will satisfy this. Looking at the requirement that the exterior solution becomes a traveling wave as $r \to \infty$, we see that we can add $n_0(x) = \frac{\sin x}{x}$ to the solution to get an exponential form (spherical Henkel function $h_0^{(1)}$). With spherical symmetry our final Green's function is:

$$G(\vec{x},\vec{x}\,') = -\frac{e^{ikr}}{4\pi r} \tag{1.2}$$

2 Problem 2

Our trial Green's function places an image point of $\mathbf{x}$ at $\mathbf{x}_1 = \alpha\mathbf{x}$. Without loss of generality we examine the solution for $\mathbf{x}$ lying on the x-axis in Cartesian coordinates. The intersection points with the sphere, where $G(\mathbf{x},\mathbf{x}^{'}) = 0$, are at $\pm a$. We then have:

$$\beta = \frac{|\mathbf{x}^{'} - \alpha\mathbf{x}|}{|\mathbf{x}^{'} - \mathbf{x}|} \tag{2.1}$$

$$|\mathbf{x}^{'}| = \pm a \tag{2.2}$$

$$\frac{|a - \alpha x|}{|a - x|} = \frac{|a + \alpha x|}{|a + x|} \tag{2.3}$$

$$\alpha = (\frac{a}{x})^2 \tag{2.4}$$

$$\beta = \frac{a + \frac{a^2}{x}}{a + x} \tag{2.5}$$

$$\beta = \frac{1 + \frac{a}{x}}{1 + \frac{x}{a}} = \frac{a}{x} \tag{2.6}$$

For a Dirichlet boundary condition $f(\theta^{'}, \phi^{'})$ defined at $r^{'} = a$ we need to compute only the surface integral contribution to the solution. We find the normal derivative of the Green's function using the geometric identity $|\mathbf{x}^{'} - \frac{a^2}{r^2}\mathbf{x}| = \frac{a}{r}|\mathbf{x}^{'} - \mathbf{x}|$:

$$G(\mathbf{x}^{'}, \mathbf{x}) = \frac{1}{4\pi}(-\frac{a}{r|\mathbf{x}^{'} - \frac{a^2}{r^2}\mathbf{x}|} + \frac{1}{|\mathbf{x}^{'} - \mathbf{x}|}) \tag{2.7}$$

$$\nabla G = \frac{1}{4\pi}\frac{(1 - \frac{a^2}{r^2})\mathbf{x}^{'}}{|\mathbf{x}^{'} - \mathbf{x}|^3} \tag{2.8}$$

$$\frac{\mathbf{x}^{'}}{|\mathbf{x}^{'}|} \cdot \nabla G = \frac{1}{4\pi a}\frac{a^2 - r^2}{c^3} \tag{2.9}$$

We can now set up the solution as the integral on the surface of sphere $r^{'} = a$.

$$u(r, \theta, \phi) = \frac{1}{4\pi a}\int_{\Omega^{'}} \frac{a^2 - r^2}{|\mathbf{x}^{'} - \mathbf{x}|^3} f(\theta^{'}, \phi^{'})\ d\Omega^{'} \tag{2.10}$$

We compare this to the series solution of Laplace's equation subject to an inhomogenous boundary condition on the surface of the sphere. The regular series solution for Laplace's equation inside a sphere is:

$$u(r, \theta, \phi) = \sum_{\ell, m} A_{\ell m} r^{\ell} Y_{\ell}^{m}(\theta, \phi) \tag{2.11}$$

Applying the surface boundary condition $u(a, \theta, \phi) = f(\theta, \phi)$ we get an expansion of $f(\theta, \phi)$ in spherical harmonics:

$$\sum_{\ell,m} A_{\ell m} a^\ell Y_\ell^m(\theta, \phi) = f(\theta, \phi) \tag{2.12}$$

$$\alpha_{\ell m} = A_{\ell m} a^\ell \tag{2.13}$$

$$\alpha_{\ell m} = \int_\Omega Y_\ell^m(\theta, \phi) f(\theta, \phi)\ d\Omega \tag{2.14}$$

We can see a similarity between the two solutions if we expand $|\mathbf{x}^{'} - \mathbf{x}|$ in spherical harmonics.

3 Problem 3

We examine the inhomogenous wave equation with a point source moving on a trajectory. The retarded potential solution is:

$$\phi(\mathbf{x}, t) = -\frac{c}{4\pi} \int d^3\mathbf{x}'\ dt'\ f(\mathbf{x}', t') \frac{\delta\left[|\mathbf{x} - \mathbf{x}'| - c(t - t')\right]}{|\mathbf{x} - \mathbf{x}'|} \tag{3.1}$$

With $f(\mathbf{x}', t^{'}) = \delta\left[\mathbf{x}' - \xi(t^{'})\right]$, then $\mathbf{x}' = \xi(t^{'})$. We then have:

$$\phi(\mathbf{x}, t) = -\frac{c}{4\pi} \int dt'\ \frac{\delta\left[|\mathbf{x} - \xi(t')| - c(t - t')\right]}{|\mathbf{x} - \xi(t')|} \frac{d\xi}{dt} \tag{3.2}$$

$$|\mathbf{x} - \xi(t')| = c(t - t') \tag{3.3}$$

$$\phi(\mathbf{x}, t) = -\frac{c}{4\pi} \int dt'\ \frac{\delta\left[|\mathbf{x} - \xi(t')| - c(t - t')\right]}{t - t'} \frac{d\xi}{dt} \tag{3.4}$$