PHYS 502: Mathematical Physics II

Winter 2015

Solutions to Homework #4

1. (a) Bessel's equation is

$$x^2y + xy + (x^2 - m^2)y = 0\,.$$

Seeking a series solution of the form $y(x) = x^\alpha \sum_{n=0}^{\infty} c_n x^n$ and substituting into the equation we find, as usual

$$\begin{aligned} \alpha &= \pm m\,, \\ c_1 &= 0\,, \\ c_n &= \frac{-c_{n-2}}{(n+\alpha)^2 - m^2}\,. \end{aligned}$$

(i) For $m = \frac{1}{2}, \alpha = \frac{1}{2}$, we have $(n+\alpha)^2 - m^2 = (n+1)n$, so

$$\begin{aligned} c_{2k} &= \frac{(-1)^k c_0}{(2k+1)!}\,, \\ J_{\frac{1}{2}}(x) &= c_0 x^{\frac{1}{2}}(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots) \\ &= c_0 x^{-\frac{1}{2}} \sin x\,. \end{aligned}$$

(ii) For $m = \frac{1}{2}, \alpha = -\frac{1}{2}$, we have $(n+\alpha)^2 - m^2 = n(n-1)$, so

$$\begin{aligned} c_{2k} &= \frac{(-1)^k c_0}{2k!}\,, \\ J_{\frac{1}{2}}(x) &= c_0 x^{-\frac{1}{2}}(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots) \\ &= c_0 x^{-\frac{1}{2}} \cos x\,. \end{aligned}$$

Using the recurrence relation $J_{m+1}(x) = (2m/x)J_m(x) - J_{m-1}(x)$ (and setting each $c_0 = 1$ for simplicity), we find

$$\begin{aligned} J_{\frac{3}{2}}(x) &= x^{-1} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) &= x^{-\frac{3}{2}}(\sin x - x\cos x) \\ J_{\frac{5}{2}}(x) &= 3x^{-1} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) &= x^{-\frac{5}{2}}(3\sin x - 3x\cos x - x^2 \sin x)\,. \end{aligned}$$

(b) Writing $f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn}x)$, we have

$$\int_0^1 [f(x)]^2 x\,dx = \int_0^1 \left[\sum_{n=1}^{\infty} a_n J_m(\alpha_{mn}x)\right] \left[\sum_{k=1}^{\infty} a_k J_m(\alpha_{mk}x)\right] x\,dx\,.$$

Because of the orthogonality condition

$$\int_0^1 J_m(\alpha_{mn}x) J_m(\alpha_{mk}x) x\,dx = \tfrac{1}{2}[J_{m+1}(\alpha_{mn})]^2$$

(see H&R §9.5.3) only the terms with $k = n$ survive, so

$$\int_0^1 [f(x)]^2 \, x \, dx = \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 \, [J_{m+1}(\alpha_{mn})]^2 .$$

2. The temperature satisfies the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \, \nabla^2 T.$$

Separating out the time dependence $T(\mathbf{x}, t) = \chi(\mathbf{x}) e^{-\kappa k^2 t}$, we have

$$\nabla^2 \chi + k^2 \chi = 0,$$

with χ regular as $r = |\mathbf{x}| \to 0$ and $\chi = 0$ at $r = b$. The general solution is a sum of terms of the form

$$\chi(r, \phi) = J_m(kr) \, e^{im\phi},$$

where we have assumed that the solution is independent of z. The axisymmetric initial and boundary conditions imply that only the $m = 0$ term contributes, and the boundary condition at $r = b$ implies $J_0(kb) = 0$, so $k = k_n = \alpha_{0n}/b$, where α_{mn} is the n-th root of J_m. Thus the solution is

$$T(r, t) = \sum_n a_n \, J_0(k_n r) \, e^{-\kappa k_n^2 t}.$$

We determine the a_n by satisfying the initial conditions:

$$u(r, 0) = T_0 = \sum_n a_n \, J_0 \left(\frac{\alpha_{0n} r}{b} \right) .$$

Inverting this Bessel series gives

$$a_n = \frac{2T_0}{b^2 J_1^2(\alpha_{0n})} \int_0^b J_0 \left(\frac{\alpha_{0n} r}{b} \right) r dr.$$

We can evaluate the integral using the recurrence relation $x J_0(x) = [x J_1(x)]'$, to find

$$\begin{aligned}
\int_0^b J_0 \left(\frac{\alpha_{0n} r}{b} \right) r dr &= \frac{b^2}{\alpha_{0n}^2} \int_0^{\alpha_{0n}} s J_0(s) \, ds \\
&= \frac{b^2}{\alpha_{0n}^2} \int_0^{\alpha_{0n}} [s J_1(s)]' \, ds \\
&= \frac{b^2}{\alpha_{0n}} \, J_1(\alpha_{0n}),
\end{aligned}$$

resulting in

$$a_n = \frac{2T_0}{\alpha_{0n} J_1(\alpha_{0n})} .$$

The central temperature is

$$\begin{aligned}
T(0, t) &= \sum_n a_n \, e^{-\kappa k_n^2 t} \\
&\approx a_1 \, e^{-\kappa k_1^2 t} = \frac{2T_0}{\alpha_{01} J_1(\alpha_{01})} \, e^{-\kappa \alpha_{01}^2 t / b^2} ,
\end{aligned}$$

where the leading term dominates the sum if

$$\kappa t(\alpha_{02}^2 - \alpha_{01}^2)/b^2 \gg 1,$$

or (since $\alpha_{01} = 2.40$, $\alpha_{02} = 5.52$)

$$t \gg \frac{b^2}{24.7\kappa}.$$

3. (a) Inside the spherical cavity formed by the two hemispheres, the general solution of Laplace's equation is

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(b_{lm} r^{-l-1} + c_{lm} r^l \right) P_l^m(\cos\theta)\, e^{im\phi},$$

where r, θ, ϕ are spherical polar coordinates and the line of contact between the hemispheres is at $\theta = \frac{\pi}{2}$. For ϕ to be regular at $r = 0$ we must have $b_{lm} = 0$; axial symmetry implies that $c_{lm} = 0$ for $m \neq 0$. Thus the solution is of the form

$$\phi(r, \theta) = \sum_{l=0}^{\infty} c_l\, r^l\, P_l(\cos\theta).$$

The boundary condition at $r = a$ is

$$\phi(a, \theta) = \sum_{l=0}^{\infty} c_l\, a^l\, P_l(\cos\theta) = \begin{cases} +V_0, & 0 \leq \theta < \frac{\pi}{2}, \\ -V_0, & \frac{\pi}{2} < \theta \leq \pi. \end{cases}$$

Inverting this Legendre series, we find

$$\begin{aligned} c_l\, a^l \left(\frac{2}{2l+1} \right) &= \int_0^{\pi} \phi(a, \theta) P_l(\cos\theta) d(\cos\theta) \\ &= V_0 \left[\int_{-1}^{0} -P_l(\mu) d\mu + \int_0^1 P_l(\mu) d\mu \right] \\ &= \begin{cases} 0 & (l \text{ even}) \\ 2V_0 \int_0^1 P_l(\mu) d\mu & (l \text{ odd}) \end{cases} \end{aligned}$$

Hence, for $l = 2n+1$, we have

$$c_{2n+1} = \tfrac{1}{2}(4n+3) a^{-2n-1} \cdot 2V_0 \int_0^1 P_{2n+1}(\mu) d\mu = V_0 \frac{4n+3}{2(n+1)} a^{-2n-1} P_{2n}(0) \quad \text{(see H\&R §9.1.2)},$$

so

$$\phi(r, \theta) = V_0 \sum_{n=0}^{\infty} \frac{4n+3}{2(n+1)} P_{2n}(0) \left(\frac{r}{a} \right)^{2n+1} P_{2n+1}(\cos\theta)\, .$$

(b) Now we are seeking a solution to the wave equation

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0,$$

with boundary conditions specified on the same hemispheres as in part (a), except that now the boundary values are variable,

$$\Phi(t, r = a) = \pm V_0 e^{-i\omega t},$$

and we want the solution for $r > a$.

As usual, we seek $e^{-i\omega t}$ time dependence, so the spatial part χ of the solution satisfies the Helmholtz equation

$$\nabla^2\chi + k^2\chi = 0,$$

with $k = \omega/c$. We require axisymmetry, so the general solution is

$$\Phi(r,\theta,t) = e^{-i\omega t}\sum_l \left[A_l j_l(kr) + B_l n_l(kr)\right]\, P_l(\cos\theta),$$

where we must retain both the j_l and the n_l solutions for $r > a$. Since the asymptotic forms are

$$j_l(x) \sim \frac{1}{x}\cos\left[x - \tfrac{\pi}{2}(l+1)\right], \quad n_l(x) \sim \frac{1}{x}\sin\left[x - \tfrac{\pi}{2}(l+1)\right],$$

as $x \to \infty$, the combination $h_l^{(1)} = j_l + i n_l$ clearly satisfies the "outgoing wave" condition as $r \to \infty$ and the solution takes the form

$$\Phi(r,\theta,t) = e^{-i\omega t}\sum_l C_l h_l^{(1)}(kr)\, P_l(\cos\theta).$$

At $r = a$,

$$\Phi(a,\theta,t) = e^{-i\omega t}\sum_l C_l h_l^{(1)}(ka)\, P_l(\cos\theta) = \pm V_0 e^{-i\omega t},$$

and the solution is essentially the same as in part (a):

$$c_{2n+1}\, h_{2n+1}^{(1)}(ka) = V_0\,\frac{4n+3}{2(n+1)} P_{2n}(0),$$

so

$$\Phi(r,\theta,t) = V_0\, e^{-i\omega t}\sum_n \frac{4n+3}{2(n+1)} P_{2n}(0)\,\frac{h_{2n+1}^{(1)}(kr)}{h_{2n+1}^{(1)}(ka)}\, P_{2n+1}(\cos\theta).$$

4. We wish to evaluate

$$\phi = \int d^3r_1 \int d^3r_2\ \psi^*(\mathbf{r}_1)\psi^*(\mathbf{r}_2)\,\frac{e^2}{r_{12}}\,\psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\,,$$

where $\psi(\mathbf{r}) = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}$, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$, and $Z = 2$ here. Expand the r_{12}^{-1} term in spherical harmonics:

$$\frac{1}{r_{12}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\ Y_l^m(\theta_1,\phi_1)^*\, Y_l^m(\theta_1,\phi_2)\ \frac{r_<^l}{r_>^{l+1}}\,,$$

where $r_< = \min(r_1, r_2)$ and $r_> = \max(r_1, r_2)$, with $r = |\mathbf{r}|$. Writing $d^3r = r^2 dr d\Omega$, the angular (Ω_1 and Ω_2) integrals give zero except for $l = m = 0$, and

$$\int d\Omega_1\, Y_l^m(\theta_1,\phi_1)^* = \int d\Omega_2\, Y_l^m(\theta_2,\phi_2) = \sqrt{4\pi}\,,$$

so

$$\phi = 16\pi^2 e^2\left(\frac{Z^3}{\pi a_0^3}\right)^2 \int r_1^2 dr_1 \int r_2^2 dr_2\ e^{-2Zr_1/a_0} e^{-2Zr_2/a_0}\ \frac{1}{\max(r_1,r_2)}\,.$$

Splitting the r_2 integral into two parts ($0 < r_2 < r_1$ and $r_1 < r_2 < \infty$), we have

$$\phi = \frac{16Z^6e^2}{a_0^6} \int_0^\infty dr_1\, r_1\, e^{-2Zr_1/a_0} \left[\int_0^{r_1} dr_2\, r_2^2 e^{-2Zr_2/a_0} + r_1 \int_{r_1}^\infty dr_2\, r_2 e^{-2Zr_2/a_0} \right] .$$

After some algebra (or application of Maple), this yields the desired result

$$\phi = \frac{5Ze^2}{8a_0} = \frac{5e^2}{4a_0} .$$