Statmech II HW5

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1 Problem 8.1

Using the linear approximation for the Fermi distribution at low temperatures, we have the following approximate Fermi integral:

$$f_v(z) = \frac{1}{\Gamma(z)} \left(\int_0^{\xi-2} z^{v-1} + \int_{\xi-2}^{\xi+2} \left(\frac{\xi+2-z}{4}) \right) z^{v-1} \right) \tag{1.1}$$

Solving for $f_{5/2}$ and $f_{3/2}$:

$$f_{5/2}(z) = \frac{4}{3\sqrt{\pi}} \left(\int_0^{\xi-2} z^{3/2} dz + \frac{\xi+2}{4} \int_{\xi-2}^{\xi+2} z^{3/2} dz + \frac{1}{4} \int_{\xi-2}^{\xi+2} z^{5/2} dz \right) \tag{1.2}$$

$$\begin{aligned} f_{5/2}(z) = &\frac{4}{3\sqrt{\pi}} (\frac{2}{5}(\xi-2)^{5/2} + \frac{1}{10}(\xi+2)^{7/2} - \frac{1}{10}(\xi+2)(\xi-2)^{5/2} \\ &+ \frac{1}{14}(\xi+2)^{7/2} - (\xi-2)^{7/2}) \end{aligned} \tag{1.3}$$

$$\begin{aligned} f_{5/2}(z) = &\frac{4}{3\sqrt{\pi}} \{ \frac{2}{5}(\xi-2)^{3/2} - \frac{1}{10}(\xi+2)(\xi-2)^{5/2} \\ &+ \frac{12}{70}(\xi+2)^{7/2} - \frac{1}{14}(\xi-2)^{7/2} \} \end{aligned} \tag{1.4}$$

$$f_{3/2}(z) = \frac{2}{\sqrt{\pi}} \left(\int_0^{\xi-2} z^{1/2} dz + \frac{\xi+2}{4} \int_{\xi-2}^{\xi+2} z^{1/2} dz + \frac{1}{4} \int_{\xi-2}^{\xi+2} z^{3/2} dz \right) \tag{1.5}$$

$$\begin{aligned} f_{3/2}(z) = &\frac{2}{\sqrt{\pi}} (\frac{2}{3}(\xi-2)^{3/2} + \frac{1}{6}(\xi+2)^{5/2} - \frac{1}{6}(\xi+2)(\xi-2)^{3/2} \\ &+ \frac{1}{10}(\xi+2)^{5/2} - (\xi-2)^{5/2}) \end{aligned} \tag{1.6}$$

$$\begin{aligned} f_{3/2}(z) = &\frac{2}{\sqrt{\pi}} \{ \frac{2}{3}(\xi-2)^{3/2} - \frac{1}{6}(\xi+2)(\xi-2)^{3/2} \\ &+ \frac{4}{15}(\xi+2)^{5/2} - \frac{1}{10}(\xi-2)^{5/2} \} \end{aligned} \tag{1.7}$$

We substitute $\frac{e_f}{kT}$ for ξ. At low temperatures z is large, so we keep only the highest powers of z in the two expressions.

$$\frac{f_{5/2}(z)}{f_{3/2}(z)} = \frac{2}{3}\left(\frac{\frac{12}{70}(\frac{e_f}{kT}+2)^{7/2} - \frac{1}{14}(\frac{e_f}{kT}-2)^{7/2}}{\frac{4}{15}(\frac{e_f}{kT}+2)^{5/2} - \frac{1}{10}(\frac{e_f}{kT}-2)^{5/2}}\right) \tag{1.8}$$

We now plug this approximations into the expression for energy and differentiate with respect to T:

$$U = \frac{3}{2}NkT\frac{f_{5/2}(z)}{f_{3/2}(z)} \tag{1.9}$$

$$U = \frac{3}{2}NkT\frac{2}{3}\left(\frac{\frac{12}{70}(\frac{e_f}{kT}+2)^{7/2} - \frac{1}{14}(\frac{e_f}{kT}-2)^{7/2}}{\frac{4}{15}(\frac{e_f}{kT}+2)^{5/2} - \frac{1}{10}(\frac{e_f}{kT}-2)^{5/2}}\right) \tag{1.10}$$

$$\frac{C_v}{Nk} = Nk\frac{2}{3}\left(\frac{\frac{12}{70}(\frac{e_f}{kT}+2)^{7/2} - \frac{1}{14}(\frac{e_f}{kT}-2)^{7/2}}{\frac{4}{15}(\frac{e_f}{kT}+2)^{5/2} - \frac{1}{10}(\frac{e_f}{kT}-2)^{5/2}}\right) \tag{1.11}$$

2 Problem 8.2

We find a general expression for the temperature:

$$\frac{1}{v} = \frac{g}{\lambda^3}f_{3/2}(z) \tag{2.1}$$

$$\lambda = \frac{h}{\sqrt{2\pi mkT}} \tag{2.2}$$

$$T = \frac{h^2}{2\pi mk}\left(gvf_{3/2}(z)\right)^{-2/3} \tag{2.3}$$

At T_0, $z = 1$, and we find (from E.14 in Pathria):

$$f_{3/2}(1) = (1 - \frac{1}{\sqrt{2}})\zeta(\frac{3}{2}) = 0.7650 \tag{2.4}$$

We set $t_f = \frac{e_f}{k}$ and use Pathria equation 8.1.24 for e_f.

$$T_f = \left(\frac{3}{4\pi gv}\right)^{2/3}\frac{h^2}{2mk} \tag{2.5}$$

$$\frac{T_f}{T_0} = \frac{1}{\pi}\left(\frac{4\pi}{3f_{3/2}(1)}\right)^{2/3} = 0.9889 \tag{2.6}$$

So the temperature at which $\mu = 0$ is approximately the Fermi temperature.

3 Problem 8.3

Starting from the expression for pressure and using the recurrence relation for the Fermi integrals:

$$P = kT\frac{g}{\lambda^3}f_{5/2}(z) \tag{3.1}$$

$$P = kg\left(\frac{h}{\sqrt{2\pi mk}}\right)^3 T^{5/2}f_{5/2}(z) \tag{3.2}$$

$$\left(\frac{\partial P}{\partial T}\right)_P = kg\left(\frac{h}{\sqrt{2\pi mk}}\right)^3 \frac{5}{2}T^{\frac{3}{2}}f_{5/2}(z) + kg\left(\frac{h}{\sqrt{2\pi mk}}\right)^3 T^{5/2}\frac{1}{z}f_{3/2}(z)\left(\frac{\partial z}{\partial T}\right)_P \tag{3.3}$$

$$\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_P = -\frac{5}{2T}\frac{f_{5/2}(z)}{f_{3/2}(z)} \tag{3.4}$$

From equation 8.1.9 in Pathria we can prove the desired relation γ:

$$\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_v = -\frac{3}{2T}\frac{f_{3/2}(z)}{f_{1/2}(z)} \tag{3.5}$$

$$\gamma = \frac{C_P}{C_v} = \frac{5}{3}\frac{f_{5/2}(z)f_{1/2}(z)}{(f_{3/2}(z))^2} \tag{3.6}$$

Using eqns 8.1.30-8.1.32 of Pathria:

$$\gamma = \frac{5}{3}\frac{\frac{8}{15\sqrt{\pi}}(\ln z)^{5/2}(1+\frac{5\pi^2}{8}(\ln z)^{-2}+...)\frac{2}{\sqrt{\pi}}(\ln z)^{1/2}(1-\frac{\pi^2}{24}(\ln z)^{-2}+...)}{\left(\frac{4}{3\sqrt{\pi}}(1+\frac{\pi^2}{8}(\ln z)^{-2}+...)\right)^2} \tag{3.7}$$

Keeping up to second-order terms:

$$\gamma = \frac{3}{5}\frac{5}{3}\frac{(1+\frac{5\pi^2}{8}\ln z^{-2}-\frac{\pi^2}{24}\ln z^{-2}+...)}{(1+2\frac{\pi^2}{8}\ln z^{-2}+...)} \tag{3.8}$$

$$\gamma \simeq 1+\frac{\pi^2}{3}(\ln z)^{-2} \tag{3.9}$$

Substituting $\ln z = \frac{e_f}{kT}$ we prove $\gamma \simeq 1+\frac{\pi^2}{3}(\frac{kT}{e_f})^2$.

4 8.4

a) We prove the following relations for the isothermal compressibility and adiabatic compressibility of an ideal Fermi gas:

$$\kappa_T = \frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \tag{4.1}$$

$$\kappa_S = \frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S \tag{4.2}$$

Using the expressions for P and n and taking the derivatives with T held constant and with z held constant:

$$P = kT\frac{g}{\lambda^3}f_{5/2}(z) \tag{4.3}$$

$$\frac{\partial P}{\partial z} = kT\frac{g}{\lambda^3}\frac{1}{z}f_{3/2}(z) \tag{4.4}$$

$$\frac{\partial P}{\partial T} = \frac{5}{2}\frac{(2\pi m)^{3/2}}{gh^3}k^{5/2}T^{3/2}f_{5/2}(z) \tag{4.5}$$

$$n = \frac{g}{\lambda^3}f_{3/2}(z) \tag{4.6}$$

$$\frac{\partial n}{\partial z} = \frac{g}{\lambda^3}\frac{1}{z}f_{1/2}(z) \tag{4.7}$$

$$\frac{\partial n}{\partial T} = \frac{3}{2}\frac{(2\pi mk)^{3/2}}{gh^3}T^{1/2}f_{3/2}(z) \tag{4.8}$$

Writing the compressibility expressions as functions of n and using the appropriate derivatives, we show:

$$\kappa_T = \frac{N}{V}\left(\frac{\partial(\frac{N}{V})}{\partial P}\right)_T = \frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_T \tag{4.9}$$

$$\kappa_T = \frac{1}{nkT}\frac{f_{1/2}(z)}{f_{3/2}(z)} \tag{4.10}$$

$$\kappa_S = \frac{N}{V}\left(\frac{\partial(\frac{N}{V})}{\partial P}\right)_S = \frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_S \tag{4.11}$$

$$\kappa_S = \frac{3}{5nkT}\frac{f_{3/2}(z)}{f_{5/2}(z)} \tag{4.12}$$

Using the expansions of the Fermi integrals in powers of $\ln z$ up to second order (eqns 8.1.30-8.1.32 of Pathria):

$$\kappa_T \simeq \frac{1}{nkT}\frac{3}{2\ln z}(1-\frac{\pi^2}{24}(\ln z)^{-2}+...)/(1+\frac{\pi^2}{8}(\ln z)^{-2}+...) \tag{4.13}$$

$$\kappa_T \simeq \frac{1}{nkT}\frac{3}{2\ln z}(1-\pi^2(\frac{1}{24}+\frac{1}{8})(\ln z)^{-2}+...) \tag{4.14}$$

$$\kappa_T \simeq \frac{3}{2nkT\ln z}(1-\frac{\pi^2}{6}(\ln z)^{-2}+...) \tag{4.15}$$

We substitute Pathria 8.1.35 for $\ln z$ and keep only the first-order part in the exponential term. Working up to second order:

$$\kappa_T \simeq \frac{3}{2nkT\frac{e_f}{kT}(1-\frac{\pi^2}{12}(\frac{kT}{e_f})^2)}(1-\frac{\pi^2}{6}(\frac{kT}{e_f})^2) \tag{4.16}$$

$$\kappa_T \simeq \frac{3}{2ne_f}(1-\frac{\pi^2}{6}(\frac{kT}{e_f})^2)(1+\frac{\pi^2}{12}(\frac{kT}{e_f})^2) \tag{4.17}$$

$$\kappa_T \simeq \frac{3}{2ne_f}\left(1-\frac{\pi^2}{12}(\frac{kT}{e_f})^2\right) \tag{4.18}$$

We now follow the same procedure to calculate κ_S.

$$\kappa_S \simeq \frac{3}{5nkT}\frac{5}{2\ln z}(1+\frac{\pi^2}{8}(\ln z)^{-2}+...)/(1+\frac{5\pi^2}{8}(\ln z)^{-2}+...) \tag{4.19}$$

$$\kappa_S \simeq \frac{3}{5nkT}\frac{5}{2\ln z}(1+\pi^2(\frac{1}{8}-\frac{5}{8})(\ln z)^{-2}+...) \tag{4.20}$$

$$\kappa_S \simeq \frac{3}{2nkT\ln z}(1-\frac{\pi^2}{2}(\ln z)^{-2}+...) \tag{4.21}$$

We substitute Pathria 8.1.35 for $\ln z$ and keep only the first-order part in the exponential term. Working up to second order:

$$\kappa_S \simeq \frac{3}{2nkT\frac{e_f}{kT}(1-\frac{\pi^2}{12}(\frac{kT}{e_f})^2)}(1-\frac{\pi^2}{2}(\frac{kT}{e_f})^2) \tag{4.22}$$

$$\kappa_S \simeq \frac{3}{2ne_f}(1-\frac{\pi^2}{2}(\frac{kT}{e_f})^2)(1+\frac{\pi^2}{12}(\frac{kT}{e_f})^2) \tag{4.23}$$

$$\kappa_S \simeq \frac{3}{2ne_f}\left(1-\frac{5\pi^2}{12}(\frac{kT}{e_f})^2\right) \tag{4.24}$$

b) Starting with the given relation for $C_P - C_V$:

$$C_P - C_V = TV\kappa_T \left(\frac{\partial P}{\partial T}\right)_V^2 \tag{4.25}$$

$$PV = \frac{2}{3}U \tag{4.26}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{2}{3V}\left(\frac{\partial U}{\partial T}\right)_V = \frac{2}{3V}C_V \tag{4.27}$$

$$\frac{C_P - C_V}{C_V} = \frac{4}{9}\frac{1}{V}C_V\frac{1}{nk}\frac{f_{1/2}(z)}{f_{3/2}(z)} = \frac{4}{9}\frac{C_V}{Nk}\frac{f_{1/2}(z)}{f_{3/2}(z)} \tag{4.28}$$

Using $C_V = \frac{\pi^2}{2}\frac{kT}{e_f}$ and again substituting the $\ln z$ series expansion:

$$\frac{f_{1/2}}{f_{3/2}} = \frac{3}{2\ln z}\left(1 - \frac{\pi^2}{24}(\ln z)^{-2}\right)\left(1 - \frac{\pi^2}{8}(\ln z)^{-2}\right) \tag{4.29}$$

$$\frac{f_{1/2}}{f_{3/2}} = \frac{3}{2}\frac{kT}{e_f}\left(1 - \frac{\pi^2}{6}(\frac{kT}{e_f})^2\right) \tag{4.30}$$

$$\frac{C_P - C_V}{C_V} = \frac{4}{9}\frac{\pi^2}{2}\frac{kT}{e_f}\frac{3}{2}\frac{kT}{e_f}\left(1 - \frac{\pi^2}{6}(\frac{kT}{e_f})^2\right) \tag{4.31}$$

$$\frac{C_P - C_V}{C_V} \simeq \frac{\pi^2}{3}\left(\frac{kT}{e_f}\right)^2 \tag{4.32}$$

c) Using the relation $\gamma = \frac{\kappa_T}{\kappa_S}$, keeping terms to second order, we find:

$$\gamma \simeq \left(1 - \frac{\pi^2}{12}(\frac{kT}{e_f})^2\right) / \left(1 - \frac{5\pi^2}{12}(\frac{kT}{e_f})^2\right) \tag{4.33}$$

$$\gamma \simeq \left(1 - \frac{\pi^2}{12}(\frac{kT}{e_f})^2\right)\left(1 + \frac{5\pi^2}{12}(\frac{kT}{e_f})^2\right) \tag{4.34}$$

$$\gamma \simeq 1 - \frac{\pi^2}{3}(\frac{kT}{e_f})^2 \tag{4.35}$$

So we have verified the result in part A for γ at low temperatures.