Guitar Pickups

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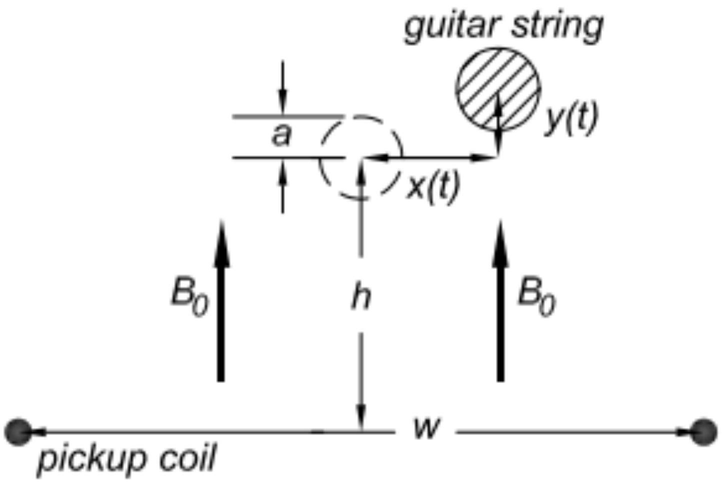
The Setup

- Guitar pickups consist of a magnet and a coil of wire
- Moving, magnetized guitar strings → Induced voltage!
- The voltage can then be fed to an amplifier & speaker.
- TA DA, MUSIC!

The Setup

- Steel string
  - radius a
  - magnetic permeability μ_{rel}
- Pickup Coil
  - Surface area $w*w$
- Start at height h
- Constant Mag. Field $B_0\hat{y}$
- Position $(x(t),y(t))$

Find the induced voltage!



Calculation Road Map

- Find the magnetic scalar potential

$$\mathbf{H} = -\boldsymbol{\nabla}\phi$$

- Find the B-field

$$\mathbf{B} = \mu\mathbf{H}$$

- Find the magnetic flux through the pickup

$$\Phi_B = \iint_S \mathbf{B} \cdot d\mathbf{S}$$

- Find the potential!

$$V = -\dot{\Phi}$$

Finding the Scalar Potential

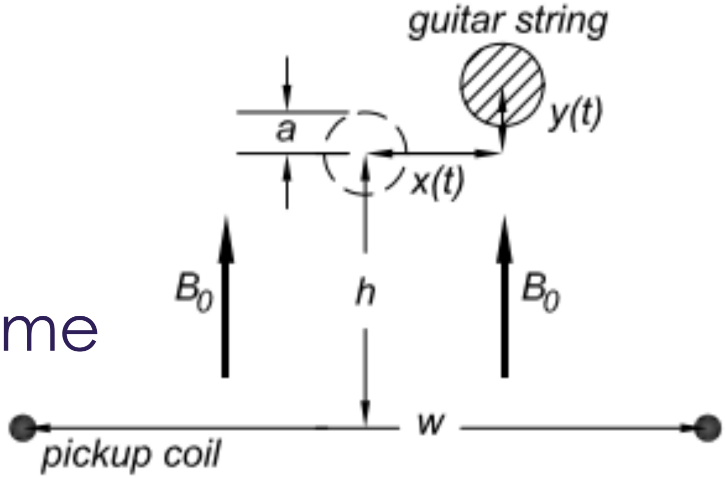
- No conduction currents in the string or magnet:

$$\nabla \times \mathbf{H} = 0$$

- So we can use $\mathbf{H}$ to find a scalar potential:

$$\mathbf{H} = -\nabla \phi.$$

(we're doing this in the frame of the vibrating string)



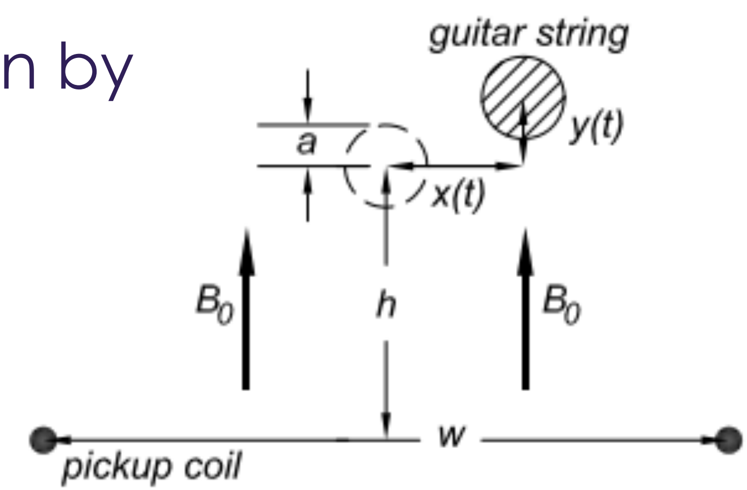
Finding the Scalar Potential

- The permanent magnet is in the y-direction, so we can write down its field:

$$\mathbf{H}_0 = \frac{B_0}{\mu_0}\hat{\mathbf{y}} = \frac{d}{dy}\frac{B_0 y}{\mu_0}\hat{\mathbf{y}}$$

- Which is also easily given by

$$\mathbf{H}_0 = -\nabla\frac{-B_0 y}{\mu_0}$$



Finding the Scalar Potential

$$\mathbf{H}_0 = -\boldsymbol{\nabla}\frac{-B_0 y}{\mu_0}$$

- Let's convert this into cylindrical via $y = r\sin(\theta)$

$$\mathbf{H}_0 = -\boldsymbol{\nabla}\frac{-B_0 r\sin\theta}{\mu_0}$$

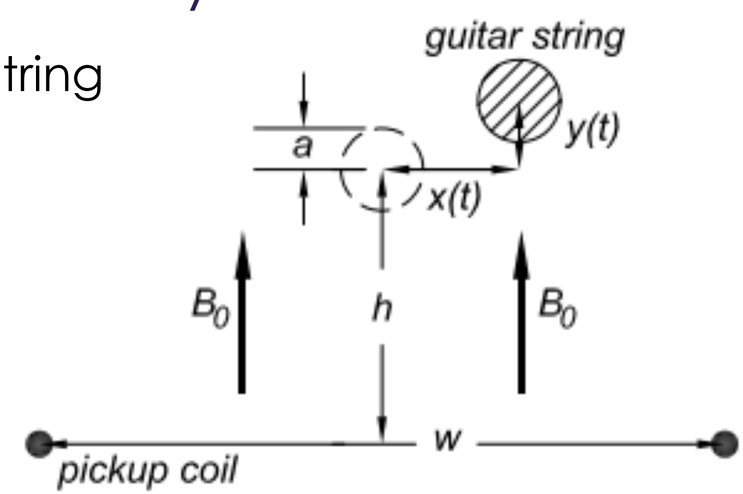
- The scalar potential is given by

From perm. magnet

From string

$$\phi(r<a) = \frac{-B_0 r\sin\theta}{\mu_0} + A\frac{r}{a}\sin\theta$$

$$\phi(r>a) = \frac{-B_0 r\sin\theta}{\mu_0} + A\frac{a}{r}\sin\theta$$



Boundary Conditions!

$$\phi(r < a) = \frac{-B_0 r \sin\theta}{\mu_0} + A\frac{r}{a}\sin\theta$$
$$\phi(r > a) = \frac{-B_0 r \sin\theta}{\mu_0} + A\frac{a}{r}\sin\theta$$

- Let's match some boundary conditions at $r = a$!

$$B_r(r = a^-) = -\mu_{\rm rel}\mu_0 \frac{\partial \phi(r = a^-)}{\partial r}$$
$$= \mu_{\rm rel}\left(B_0 + \frac{\mu_0 A}{a}\right)\sin\theta$$

$$B_r(r = a^+) = -\mu_0 \frac{\partial \phi(r = a^+)}{\partial r}$$
$$= \left(B_0 - \frac{\mu_0 A}{a}\right)\sin\theta$$

$$\mathbf{B} = \mu\mathbf{H}$$

The normal ($\boldsymbol{n} = \boldsymbol{r}$) B-fields must be continuous at $r = a$!

Matching BCs

$$\phi(r > a) = \frac{-B_0 r \sin\theta}{\mu_0} + A\frac{a}{r}\sin\theta$$

- Solve for A!

$$A = \frac{aB_0}{\mu_0}\frac{1-\mu_{\text{rel}}}{1+\mu_{\text{rel}}}$$

- Hooray we now have our scalar potential!

$$\phi(r > a) = -\frac{B_0}{\mu_0}\left(r + \frac{a^2}{r}\frac{\mu_{\text{rel}}-1}{\mu_{\text{rel}}+1}\right)\sin\theta.$$

Total B-Field

- As a reminder

$$\mathbf{B} = \mu\mathbf{H} = -\mu\boldsymbol{\nabla}\phi$$

- We have two components to our magnetic scalar field, so we have 2 B-field components

$$B_r(r > a) = -\mu_0 \frac{\partial\phi(r > a)}{\partial r} = B_0\left(1 - \frac{a^2}{r^2}\frac{\mu_{\rm rel} - 1}{\mu_{\rm rel} + 1}\right)\sin\theta$$

$$B_\theta(r > a) = -\frac{\mu_0}{r}\frac{\partial\phi(r > a)}{\partial\theta} = B_0\left(1 + \frac{a^2}{r^2}\frac{\mu_{\rm rel} - 1}{\mu_{\rm rel} + 1}\right)\cos\theta$$

Back to Cartesian

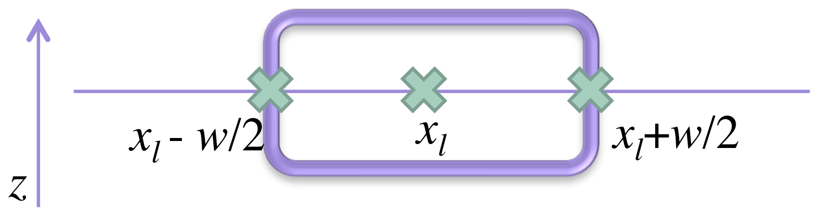
- The pickup coil is at a fixed y coordinate

$$B_y(r > a) = B_r \sin\theta + B_\theta \cos\theta$$

$$= B_0 \left(1 + a^2 \frac{\mu_{\text{rel}} - 1}{\mu_{\text{rel}} + 1} \frac{y^2 - x^2}{(x^2 + y^2)^2}\right)$$

- So we only have to integrate over our x-loop!

$$\Phi_B = \iint_S \mathbf{B} \cdot d\mathbf{S}$$



$$\Phi = w \int_{x_l - w/2}^{x_l + w/2} B_y(x, y_l)\, dx$$

Wolfram Alpha!!

- Ta-da!

$$\Phi = B_0 w \left[w + a^2 \frac{\mu_{\text{rel}} - 1}{\mu_{\text{rel}} + 1} \left(\frac{x_l + w/2}{(x_l + w/2)^2 + y_l^2} - \frac{x_l - w/2}{(x_l - w/2)^2 + y_l^2} \right) \right]$$

- Now remember we are still in the frame of the string. Transform back into the lab frame by:

$$x_l = x \qquad y_l = y + h$$

Magnetic Flux!

$$\Phi \;=\; B_0 w\left[1 + a^2\frac{\mu_{\rm rel}-1}{\mu_{\rm rel}+1}\frac{(y+h)^2 - x^2 + w^2/4}{(x^2-w^2/4)^2 + 2(x^2+w^2/4)(y+h)^2 + (y+h)^4}\right]$$

- Now to find the potential difference, do a BUNCH OF ALGEBRA

$$V = -\dot{\Phi}$$

$$= \;B_0 a^2 w^2\frac{\mu_{\rm rel}-1}{\mu_{\rm rel}+1}\left\{\frac{2x\dot{x}[3(y+h)^4 + 2(x^2+w^2/4)(y+h)^2 - (x^2-w^2/4)^2]}{[(x^2-w^2/4)^2 + 2(x^2+w^2/4)(y+h)^2 + (y+h)^4]^2}\right.$$

$$\left.\frac{\dot{y}[2(y+h)^5 - 4(x^2-w^2/4)(y+h)^3 + (6x^4 - x^2w^2/2 - w^4/8)(y+h)]}{[(x^2-w^2/4)^2 + 2(x^2+w^2/4)(y+h)^2 + (y+h)^4]^2}\right\}$$

Voltage

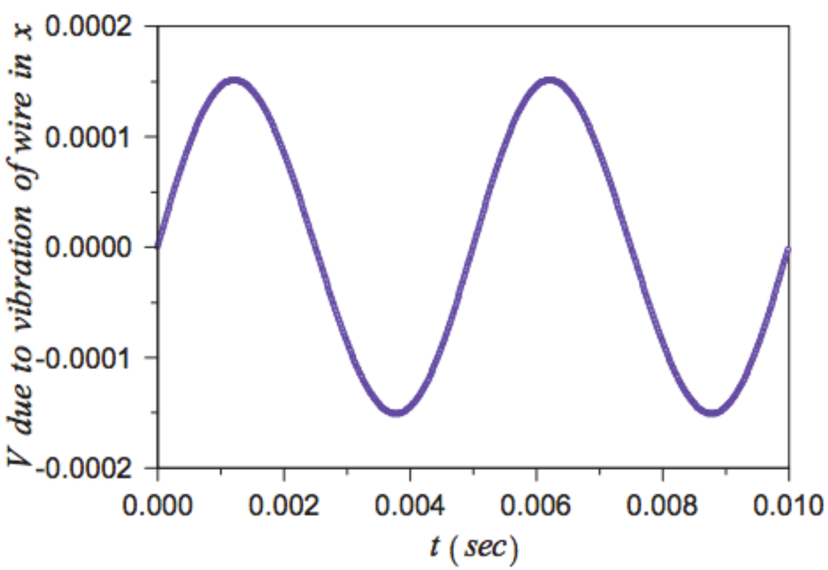
- We can approximate for $x, y << h, w$

$$V \approx B_0 a^2 w^2 \frac{\mu_{\rm rel} - 1}{\mu_{\rm rel} + 1}\left[2x\dot{x}\frac{3h^2 - w^2/4}{(h^2 + w^2/4)^3} + \frac{2h\dot{y}}{(h^2 + w^2/4)^2}\right]$$

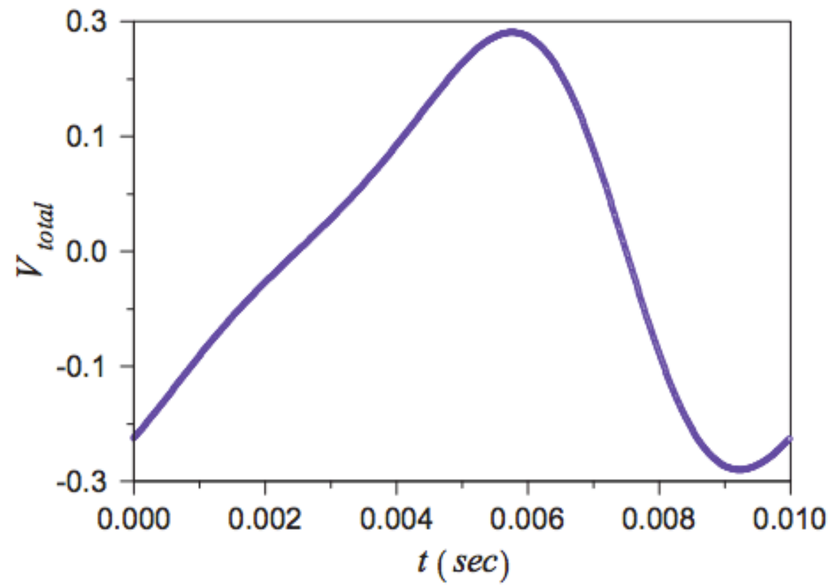
Voltage

$$V \approx B_0 a^2 w^2 \frac{\mu_{\text{rel}} - 1}{\mu_{\text{rel}} + 1} \left[2x\dot{x} \frac{3h^2 - w^2/4}{(h^2 + w^2/4)^3} + \frac{2h\dot{y}}{(h^2 + w^2/4)^2} \right]$$

x creates harmonic structure



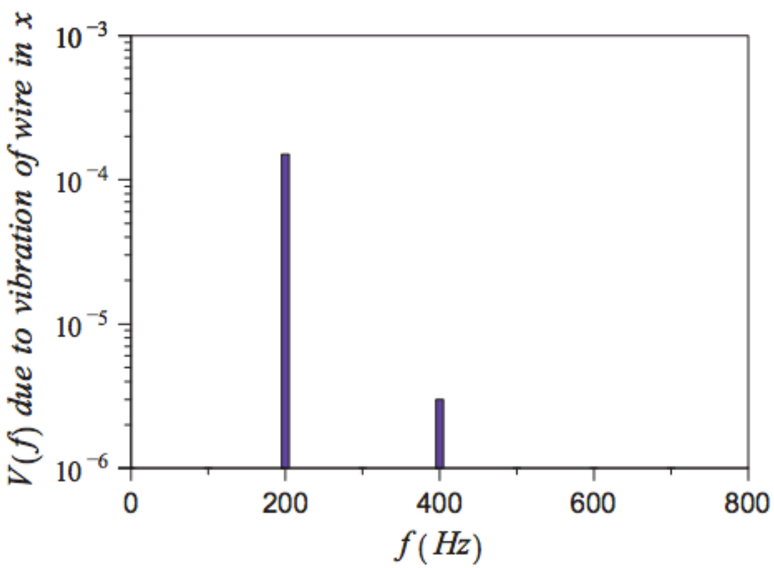
But y dominates the waveform



Voltage

$$V \approx \quad B_0 a^2 w^2 \frac{\mu_{\rm rel} - 1}{\mu_{\rm rel} + 1} \left[2x\dot{x} \frac{3h^2 - w^2/4}{(h^2 + w^2/4)^3} + \frac{2h\dot{y}}{(h^2 + w^2/4)^2} \right]$$

x creates harmonic structure



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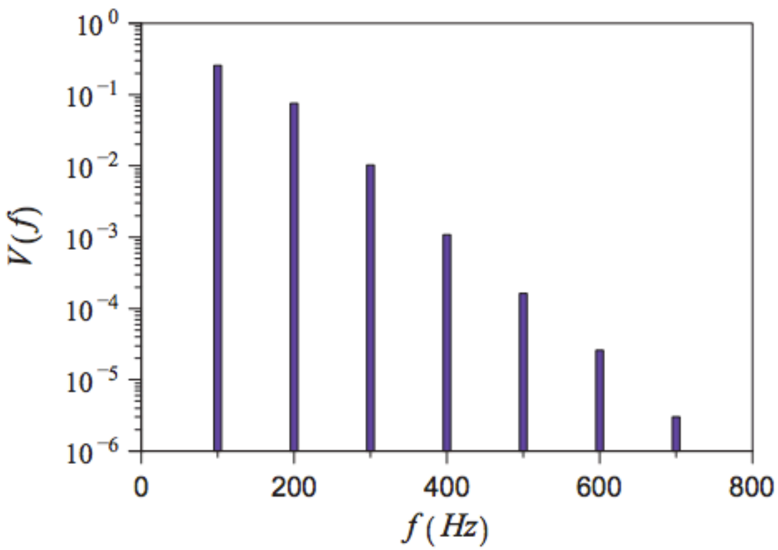

Questions?