PHYS 501: Mathematical Physics I

Fall 2014

Homework #6
(Due: December 5, 2014)

1. Use contour integration to find the inverse Fourier transform $f(t)$ of the function

$$F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin \omega a}{\omega}$$

(where $a > 0$), for all values of t. Recall that F was obtained as the Fourier transform of a step function with a discontinuity at $|t| = a$. What is the value of $f(a)$? (Determine $f(a)$ from the integral—don't appeal to the general properties of Fourier transforms!)

2. Find the (3-D) Fourier transform of the wave function for a $2p$ electron in a hydrogen atom:

$$\psi(\mathbf{x}) = (32\pi a_0^5)^{-1/2}\, z e^{-r/2a_0}\,,$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius, r is radius, and z is a rectangular coordinate.

3. Consider the solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + xy \;=\; 0$$

for which $|y| \to 0$ as $|x| \to \infty$. (This is the *Airy equation.* It appears in the theory of the diffraction of light.)

(a) Sketch this solution. Don't use Mathematica! Specifically, what behavior do you expect as $x \to -\infty$ and $x \to +\infty$?

(b) By Fourier transforming the above equation, determine $Y(\omega)$, the Fourier transform of $y(x)$, and hence write down an integral expression for $y(x)$. (Hint: What is the inverse transform of $Y'(\omega)$?)

4. Poisson's equation (in <u>three</u> dimensions) is

$$\nabla^2\phi = 4\pi G\rho.$$

(a) Let $\tilde{\phi}(\mathbf{k})$ and $\tilde{\rho}(\mathbf{k})$ be the Fourier transforms of $\phi(\mathbf{x})$ and $\rho(\mathbf{x})$, respectively. Show that

$$\tilde{\phi} = -\frac{4\pi G\tilde{\rho}}{k^2},$$

and hence write down an integral expression for $\phi(\mathbf{x})$.

(b) For a point mass at the origin, $\rho(\mathbf{x}) = m\delta(\mathbf{x})$. Use the result of part (a) to determine the solution for ϕ.