PH.D. QUALIFYING EXAM SOLUTIONS
2002

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Classical

Problem 1. *Suppose you were to launch a space-craft from the Earth's equator by abruptly (and magically) transferring all of the rotational angular momentum (not orbital!) from Earth to the space-craft. What is the launch velocity of the spacecraft, in terms of the earth's mass M, its radius R, its rotational period T, and the spacecraft mass m?*

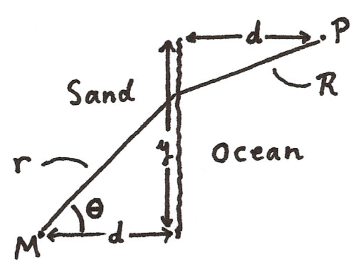
$$L = I\omega = I\frac{v}{R} = \frac{2}{5}MR^{\cancel{2}}\frac{v}{\cancel{R}}$$
$$L = \frac{2}{5}MR\left(\frac{2\pi R}{T}\right)$$
$$L' = I'\omega' = \left(mR^{\cancel{2}}\frac{v}{\cancel{R}}\right)$$
$$L = L'$$
$$\frac{4}{5}\pi MR^{\cancel{2}}\left(\frac{1}{T}\right) = m\cancel{R}v$$
$$v = \frac{4}{5}\pi\frac{M}{m}\frac{R}{T}$$

The moment of inertial I of a sphere ($\frac{2}{5}$MR^2) is used to find initial angular momentum L of Earth in terms of tangential velocity v on the surface and its radius R. The velocity is recast in terms of period T. This angular momentum is set equal to the angular momentum of the satellite with a moment of inertial that of a point rotating about the center of the Earth at a radius R with velocity v. The velocity is then isolated.

Problem 2. *Two identical race-cars approach a curve in the track that bends to the right. The right car follows the shortest possible path, hugging the inside curb, which has a radius r_{in}. The left car follows the longest possible path, hugging the outside curb which has radius r_{out}. Each car drives the turn at the maximum speed it can without sliding. By what factor does the time around the curve differ for the two cars?*

$$\text{Inside: } d = 2\pi r_{in}\left(\frac{1}{2}\right) t_{in} = \frac{d}{v} = \frac{\pi r_{in}}{v_{in}}$$
$$\text{Outside: } t_{out} = \frac{d}{v} = \frac{\pi r_{out}}{v_{out}}$$
$$\Sigma F = ma = \frac{\cancel{m}v^2}{R} = \mu\cancel{m}g$$
$$v_{max} = \sqrt{\mu g R} \ \leftarrow \text{with zero slippage}$$
$$\frac{t_{out}}{t_{in}} = \frac{\frac{\cancel{\pi}r_{out}}{\sqrt{\cancel{\mu g}r_{out}}}}{\frac{\cancel{\pi}r_{in}}{\sqrt{\cancel{\mu g}r_{in}}}}$$
$$\frac{t_{out}}{t_{in}} = \frac{\sqrt{r_{out}}}{\sqrt{r_{in}}}$$

Problem 3. *Suppose you are standing on the beach at position M at a distance d from the water. You see a friend struggling in the water at position F, d from shore, and a distance y along the shore (from the point where you would enter the water if you ran straight towards it.) At what angle should you run in order to reach your friend in the minimum time? Your speed on the sand is v=10 mph, and in the water is only V=5 mph.*



It is easiest to start with Snell's law here.

$$n_s \cos\theta_s = n_w \cos\theta_w$$

Remembering that the index of refraction is inversely proportional to your speed,

$$\frac{\cos\theta_s}{v_s} = \frac{\cos\theta_w}{v_w}$$
$$\frac{\cos\theta_s}{2v_w} = \frac{\cos\theta_w}{v_w}$$
$$\cos\theta_s = 2\cos\theta_w$$

Using geometry, we can substitute distances in for these trigonometric functions. Letting x be the distance along the shoreline from your starting position to where you enter the water,

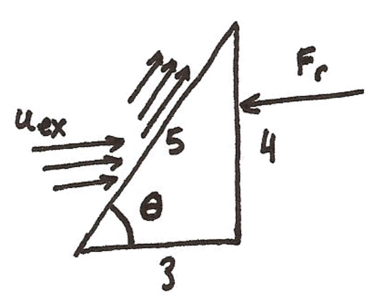
$$\frac{d}{\sqrt{d^2+x^2}} = \frac{2d}{\sqrt{d^2+(y-x)^2}}$$
$$0 = 3x^2 + 2yx + (3d^2 - y^2)$$
$$x = \frac{1}{3}\left(\pm\sqrt{4y^2-9d^2} - y\right)$$

Therefore, the angle at which you should run to reach your friend in the minimum time is

$$\tan\theta_s = \frac{x}{d}$$
$$\boxed{\theta_s = \tan^{-1}\left(\frac{1}{3d}\left(\pm\sqrt{4y^2-9d^2} - y\right)\right)}$$

Problem 4. *The jet exhaust from a military fighter plane, sitting at rest on the runway, is exhausting 4 kg/s at a velocity of* u_{ex}*=200 m/s, assumed to be horizontal, as shown in the drawing.*

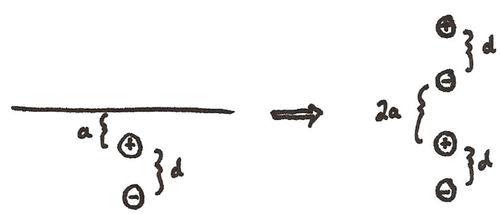
The jet is backed-up by a portable noise and heat deflector that deflects all of the exhaust gases upward, parallel to the ramp on the deflector, also as shown. The slope of the deflector is 4/3. Ignoring friction in the deflectors wheels, find the force, F_r*, required to maintain the deflector in the position shown?*



$$\begin{aligned}
\frac{\Delta p}{m} &= u_{ex} - u_{ex}\cos\theta \\
&= u_{ex} - u_{ex}\cos\frac{3}{5} \\
&= \frac{2}{5}u_{ex}
\end{aligned}$$

$$F = \frac{\Delta p}{\Delta t} = \left(200\frac{m}{s}\right)\left(\frac{2}{5}\right)\left(4\frac{kg}{s}\right) = 320N$$

Problem 5. *A perfectly conducting plane stretches horizontally in the x-y direction. Below it is a dipole, composed of charges +Q and Q, separated by distance d. The dipole is perpendicular to the plane (and thus in the z direction), and the + charge is nearest the plane, at distance a. If the mass of the dipole is m, for what values of a will the dipole move downward, away from the plane.*



Using the method if images, this problem is reduced to the interaction between two dipoles.

$$\begin{aligned}
F_g &= F_c \\
mg &= \frac{1}{4\pi\varepsilon_o}\frac{p_1\cdot p_2}{d_1 d_2}\frac{1}{r^2} = \frac{1}{4\pi\varepsilon_o}\frac{p^2}{d^2}\left(\frac{1}{(2a+d)^2}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q^2\cancel{d^2}}{\cancel{d^2}}\left(\frac{1}{(2a+d)^2}\right) \\
a &= \frac{1}{2}\left(\frac{q}{\sqrt{4\pi\varepsilon_o mg}} - d\right)
\end{aligned}$$

Problem 6. *A conducting sphere of radius a is placed in a uniform electric field, E=$E_o k$ and the sphere is grounded. The resulting potential of the system outside the sphere in polar coordinates is*

$$\phi(r,\theta) = -E_o r(1-(a/r)^3)\cos\theta$$

Find the surface charge density induced on the sphere.

$$\begin{aligned}
\sigma &= -\varepsilon_o \frac{\partial V}{\partial \hat{n}} \\
&= -\varepsilon_o \frac{\partial}{\partial r}\left(-E_o r(1-(a/r)^3)\cos\theta\right) \\
&= \varepsilon_o E_o \cos\theta \left(1 + \frac{2a^3}{r^3}\right) \\
&= 3\varepsilon_o E_o \cos\theta
\end{aligned}$$

The potential is evaluated at the surface with $r = a$.

Problem 7. *Special relativity leaves the form of Maxwell's equations unchanged in all inertial frames of reference. Thus, the field equations can be expressed in terms of a tensor field that unites Eand B. In cylindrical coordinates, this implies that if E and B are the electric and magnetic fields in a frame S, then the radial electric field in S ′ in cylindrical coordinates is*

$$E'_r = \gamma(E_r - vB_\theta)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and v is the speed of S ′ relative to S, along the z axis.

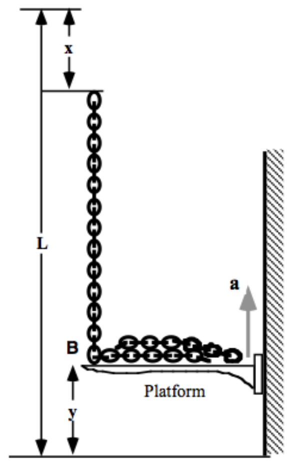
Now consider an infinite cylinder of charge density λ and current I, at rest in frame S. Find the speed v of S ′ where the electric field becomes zero.

Solve for v by setting S′=0. E and B fields are found using Gauss' and Ampere's Law, respectively. The velocity v is then determined.

$$\begin{aligned}
\int E \cdot dA &= \frac{q}{\varepsilon_o} & \qquad \int B \cdot dS &= \mu_o I \\
E\,(2\pi r \cancel{h}) &= \frac{\lambda \cancel{h}}{\varepsilon_o} & \qquad B\,(2\pi r) &= \mu_o \\
E &= \frac{\lambda}{\varepsilon_o \mu_o I} & \qquad B &= \frac{\mu_o I}{2\pi r}
\end{aligned}$$

$$v = \frac{E_r}{B_\theta} = \frac{\lambda}{\varepsilon_o \mu_o I}$$

A1. Cheeky child chained to the platform. *An open-link chain of total length L and of mass ρ per unit length hangs so that it just reaches a platform below without resting on it. At time t=0 two things happen at once. The chain is released from rest (where x=0). The platform starts from rest (where y=0) and moves up with a constant acceleration a. Determine an expression for the total force F exerted on the platform by the chain t seconds after the motion starts.*

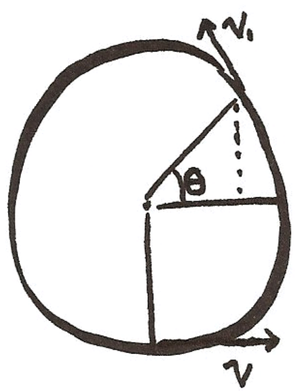


$$\begin{aligned}
y &= \frac{1}{2}at^2 \\
x &= \frac{1}{2}gt^2 \\
D \equiv x + y &= \frac{1}{2}(a+g)\,t^2 \\
m = D\rho &= \frac{1}{2}(a+g)\,t^2\rho \\
p = mv &= \frac{1}{2}(a+g)\,t^2\rho\,((a+g)\,t) \\
F = \frac{dp}{dt} &= \frac{d}{dt}\left(\frac{1}{2}(a+g)^2\,t^3\rho\right) \\
&= \frac{3}{2}\rho\,(a+g)^2\,t^2
\end{aligned}$$

A2. Loop-de-loop, loop-de-lie. *With one hand, a small child is launching a ball tied to the end of a string and hanging from her other hand. (i) At first, her push gets the ball to oscillate. (ii) She discovers then that she is able to get the ball to complete a circle if she gives it a certain angular velocity. (iii) For another range she can get an intermediate result in which the ball comes above the level of the hand clutching the string, before the ball leaves its circular path (with the string slackening) and then becoming taut again when it comes back to another part of the circle.*

If the string length is L, what velocities correspond to situations (i), (ii) and (iii) above? With what velocity should the ball be launched so that it will land squarely in the hand holding the string?

$$
\begin{aligned}
(i)\!: &\ \cancel{m}gR = \frac{1}{2}\cancel{m}v^2 \\
&\ 0 < v < \sqrt{2gR} \\
(ii)\!: &\ \frac{\cancel{m}v_{top}^2}{R} = \cancel{m}g \\
&\ v_{top} = \sqrt{gR} \\
&\ \frac{1}{2}\cancel{m}v^2 = \cancel{m}g\,(2R) + \frac{1}{2}\cancel{m}\,(gR) \\
&\ v > \sqrt{5gR} \\
(iii)\!: &\ \sqrt{2gR} < v < \sqrt{5gR}
\end{aligned}
$$



$$x_f = x_i - v_{1x}t$$

$$0 = R\cos\theta - v_1\sin\theta t$$

$$t = \frac{R\cos\theta}{v_1\sin\theta}$$

$$v_{fy} = v_{1y} - gt$$

$$v_{fy} = v_1\cos\theta - \frac{R\cos\theta}{v_1\sin\theta}g$$

$$v_{fy}^2 = v_{1y}^2 - 2g\left(y_f - y_i\right)$$

$$\left(v_1\cos\theta - \frac{R\cos\theta}{v_1\sin\theta}g\right)^2 = v_1^2\cos^2\theta - 2g\left(\cancel{R} - R\left(\cancel{1} + \sin\theta\right)\right)$$

$$v_1^2 = \frac{Rg\cos^2\theta}{2\sin\theta}$$

$$F_c = F_g$$

$$\frac{mv_1^2}{R} = mg\sin\theta$$

$$v_1^2 = Rg\sin\theta$$

$$Rg\sin\theta = \frac{Rg\cos^2\theta}{2\sin\theta}$$

$$\theta = \tan^1\frac{1}{\sqrt{2}}$$

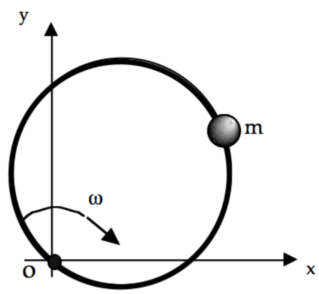
$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + mgR\left(1 + \sin\theta\right)$$

$$v^2 = \frac{Rg\cos^2\theta}{2\sin\theta} + 2Rg\left(1 + \sin\theta\right)$$

$$v = \sqrt{3.732Rg}$$

A3. Hoop-la. *A bead of mass m can slide without friction on a circular wire of radius R. The z-axis is vertical (out of the page). The circular wire rotates clockwise in a horizontal (x-y) plane about O with a constant angular velocity $\dot{\phi}$.*

(a) Choose appropriate co-ordinates, and write down the Lagrangian for the bead.



$$x = R\left(\cos\phi + \cos\theta\right) \qquad y = R\left(\sin\phi + \sin\theta\right)$$
$$\dot{x} = R\left(-\sin\phi\dot{\phi} - \sin\theta\dot{\theta}\right) \qquad \dot{y} = R\left(\cos\phi\dot{\phi} + \cos\theta\dot{\theta}\right)$$

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{1}{2}mR^2\left[\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\theta}\dot{\phi}\cos\left(\phi - \theta\right)\right]\end{aligned}$$

(b) By solving the Lagranges equation(s), describe the motion of the bead.

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \\
&= \frac{1}{\cancel{2}} mR^2 \cancel{2}\dot{\phi}\dot{\theta}\sin(\phi-\theta) - \frac{1}{2}mR^2\frac{d}{dt}\left(2\dot{\theta} + 2\dot{\phi}\cos(\phi-\theta)\right) \\
&= \cancel{mR^2}\dot{\phi}\dot{\theta}\sin(\phi-\theta) - \cancel{mR^2}\left(\ddot{\theta} + \ddot{\phi}\cos(\phi-\theta) + \dot{\phi}\left(-\sin(\phi-\theta)\right)\left(\dot{\phi}-\dot{\theta}\right)\right) \\
&= \dot{\phi}\dot{\theta}\sin(\phi-\theta) - \ddot{\theta} + \dot{\phi}\sin(\phi-\theta)\left(\dot{\phi}-\dot{\theta}\right) \\
\ddot{\theta} &= \dot{\phi}\left[\sin(\phi-\theta)\right]\left[\dot{\phi} - \cancel{\dot{\theta}} + \cancel{\dot{\theta}}\right] \\
&= \dot{\phi}^2\sin(\phi-\theta)
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \\
&= -\frac{1}{\cancel{2}} mR^2 \cancel{2}\dot{\phi}\dot{\theta}\sin(\phi-\theta) - \frac{d}{dt}\left(\cancel{2}\frac{1}{\cancel{2}}mR^2\dot{\phi} + \cancel{2}\frac{1}{\cancel{2}}mR^2\dot{\theta}\cos(\phi-\theta)\right) \\
&= -mR^2\dot{\phi}\dot{\theta}\sin(\phi-\theta) - mR^2\ddot{\phi} - mR^2\ddot{\theta}\cos(\phi-\theta) - mR^2\dot{\theta}\left(-\sin(\phi-\theta)\right)\left(\dot{\phi}-\dot{\theta}\right) \\
&= \cancel{-\dot{\phi}\dot{\theta}\sin(\phi-\theta)} - \ddot{\theta}\cos(\phi-\theta) + \dot{\theta}\sin(\phi-\theta)\left(\cancel{\dot{\phi}} - \dot{\theta}\right) \\
\ddot{\theta} &= -\dot{\theta}^2\tan(\phi-\theta)
\end{aligned}$$

Note: $\ddot{\phi} = 0$

B1. A ringer! *(a) A loop of wire of radius a is centered in the x-y plane, and carries a current I. Find the magnetic field on the axis of the loop, a distance of z from the plane.*

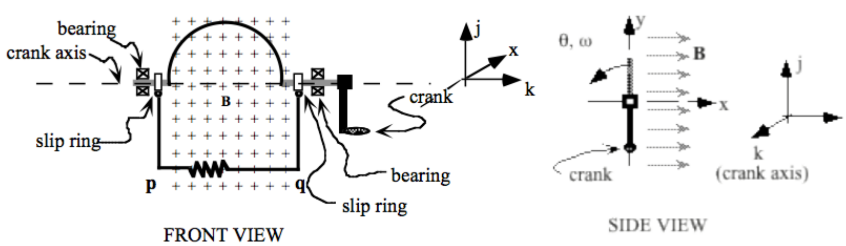
$$\begin{aligned}
B(r) &= \frac{\mu_o I}{4\pi}\int \frac{dl' \times r'}{r^2} \\
&= \frac{\mu_o I}{4\pi}\int_0^{2\pi a} \frac{dl' \times r'}{(a^2+z^2)} \\
&= \frac{\mu_o I}{4\pi}\int_0^{2\pi a} \frac{dl}{(a^2+z^2)}\sin\theta \\
&= \frac{\mu_o I}{4\pi}\frac{2\pi a}{(a^2+z^2)}\frac{a}{\sqrt{a^2+z^2}} \\
&= \frac{\mu_o I}{2}\frac{a^2}{(a^2+z^2)^{\frac{3}{2}}}
\end{aligned}$$

(b) Suppose we put N such rings concentrically, each carrying a current I, so that a kind of disk is formed in the x-y plane. Suppose the outer radius of the "disk" is b, and the inner radius is a. Now what is the magnetic field at distance z from the plane?

$$B(r) = \int_a^b \frac{\mu_o I}{2} \frac{r dr}{(r^2 + z^2)^{\frac{3}{2}}}$$
$$= -\frac{\mu_o I}{2} \left[\left(b^2 + z^2\right)^{-\frac{1}{2}} - \left(a^2 + z^2\right)^{-\frac{1}{2}} \right]$$

This is in the limit of large N; otherwise, a summation would have to be used.

B2. See how it turns out. *An electrical circuit consists of a conducting wire bent into a semi-circular shape (radius r_o) and a resistor R. A constant magnetic field B (perpendicular and inward the to paper along positive x direction) is applied in the region. The conducting wire, bent like a semi-circular shape, is rotated by a hand crank with a constant angular velocity ω in the counterclockwise direction, as viewed from the crank end, and the angle θ' is measured from the vertical position. The area A_o of the segment of the circuit below the crank axis of rotation (i.e. the zaxis) is a constant. Assume that at $t = 0$, the angle θ is zero.*



(a) Assuming that the angle θ=0 at t=0, what is magnetic flux Φ linked to the circuit at t=0? Express your answer in terms of the parameters r_o, B, and A_o.

$$\Phi = B\left(\frac{\pi r_o^2}{2} + A_o\right)$$

(b) Assuming that the angle θ= zero at t=0, what is magnetic flux Φ linked to the circuit at t =π/ω? Express your answer in terms of the parameters r_o, B, and A_o.

$$\Phi = B\left(A_o - \frac{\pi r_o^2}{2}\right)$$

(c) What is the emf (ε(t) as a function of time) induced across the terminals (at the slip ring) of the conducting wire?

$$\mathcal{E} = -\frac{d\Phi}{dt}$$
$$\Phi\left(t\right) = B\frac{\pi r_o^2}{2}\cos\left(\omega t\right) + BA_o$$
$$\mathcal{E} = B\omega\left(\frac{\pi r_o^2}{2}\sin\left(\omega t\right)\right)$$

(d) Between t=0 and t=π/ω, the area bounded by the circuit decreases. What is the direction of the induced current in the resistor during this time interval? Give reasons to substantiate your answer. What is the magnitude of this induced current?

The direction of the induced current is clockwise. The flux is decreasing, so in order to maintain a constant flux, current is induced in the clockwise direction to create a magnetic field that will counteract the loss of magnetic field lines passing through the area.

$$I\left(t\right) = \frac{V}{R} = B\omega\left(\frac{\pi r_o^2}{2R}\sin\left(\omega t\right)\right)$$

(e) What is the smallest time does the clock read when the induced emf is at its maximum value?

$$\omega t = \frac{\pi}{2} \rightarrow t = \frac{\pi}{2\omega}$$

The maximum is found by inspecting the functional form of the emf. It is achieved when the value of the sine argument is an odd integer multiple of $\pi/2$. The shortest time corresponds to the first value.

(f) How do your answers to the above questions change if the semi-circular segment of the circuit is kept fixed and the rectangular segment of the circuit shown below the crank axis is rotated at a constant rate?

$$(a) \rightarrow \text{same}$$
$$(b) \rightarrow \text{same}$$
$$(c) \rightarrow \Phi\left(t\right) = B\frac{\pi r_o^2}{2} + BA_o\cos\left(\omega t\right) \rightarrow \mathcal{E} = B\omega\left(A_o\sin\left(\omega t\right)\right)$$
$$(d) \rightarrow I\left(t\right) = \frac{BA_o\omega}{R}\sin\left(\omega t\right) \text{ in CW direction}$$
$$(e) \rightarrow \text{same}$$

Modern

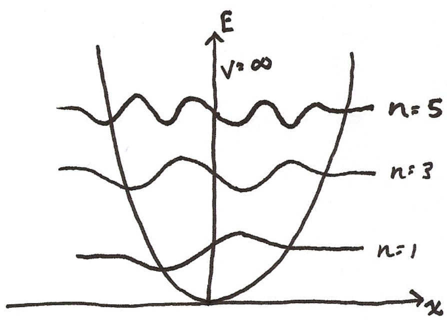
Problem 1. *Consider a particle in the potential*

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & x > 0 \\ \infty & x \leq 0 \end{cases}$$

The only difference between this problem and the standard simple harmonic oscillator is the presence of the infinite barrier, which causes the wave function to vanish at $x = 0$. With the guidance of the solution of the simple harmonic oscillator problem, find the new eigenvalues. Note: No calculations are needed to answer this question.

$$E_n = \left(\frac{1}{2} + n\right)\hbar\omega \text{ where } n = 1, 3, 5...$$

Since only even wave functions are permitted, the odd integer solutions to the harmonic oscillator energies prevail.

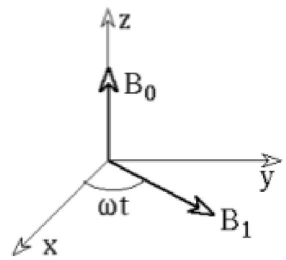


Problem 2. *What is the nature and physical origin of the force exerted on a neutral atom by a nearby conducting surface with a zero net charge? What is the essential difference between this force and the Van der Waals force between two neutral atoms?*

The Van der Waals force refers to force between two atoms in close proximity and their permanent or induced dipole moments; it can be either an attraction or repulsion. Though, the other force presented is always positive. The atom induces a net charge on the surface that is always attractive (the orientation is similar to that of **Problem 5** on the classical portion of the exam).

Problem 3. *A particle is under the action of two magnetic fields of constant magnitude. One (B_0) is oriented along the z axis and the second (B_1) rotates in the x-y plane at a constant angular frequency ω, as sketched in the diagram.*

In the rotating reference system where B_1 is stationary, give a qualitative description of the evolution of a magnetic moment originally oriented along the positive direction of the z axis.



The magnetic moment is initially aligned with the positive z axis. It wants to orient itself parallel to the vector sum of $B_0 + B_1$. In a rotating reference frame (moving with ω), the magnetic moment appears to slowly fall (relative time scale, but finite relaxation time nonetheless) towards $B_0 + B_1$. Once it reaches it, it will be stationary in this frame.

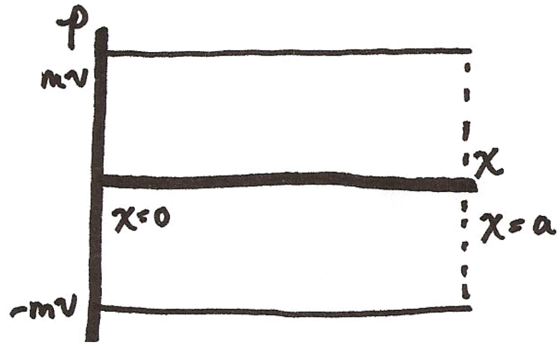
Problem 4. *A particle is confined in a 2-d rectangular box with infinite potential at the walls, with sides of length L and $\frac{L}{2}$. What are the energies corresponding to the 4 lowest states?*

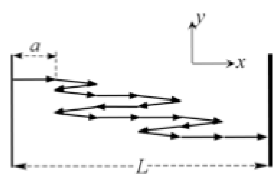
The energy of a 2-d rectangular box is the sum of two 1-d infinite square wells with the respective widths: $E = \frac{\hbar^2}{2m}\left(\frac{n_x^2}{L^2} + \frac{n_y^2}{\left(\frac{L}{2}\right)^2}\right)$

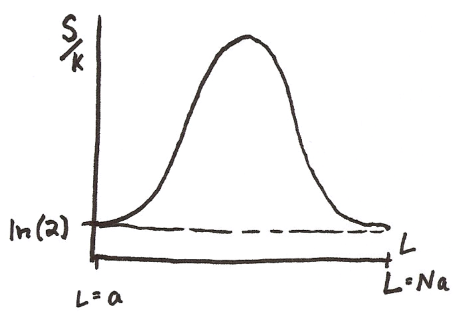
$$n_x = 1, n_x = 1 \rightarrow E = \frac{5}{2}\frac{\pi^2\hbar^2}{mL^2}$$
$$n_x = 2, n_x = 1 \rightarrow E = \frac{8}{2}\frac{\pi^2\hbar^2}{mL^2}$$
$$n_x = 3, n_x = 1 \rightarrow E = \frac{13}{2}\frac{\pi^2\hbar^2}{mL^2}$$
$$n_x = 1, n_x = 2 \rightarrow E = \frac{17}{2}\frac{\pi^2\hbar^2}{mL^2}$$

Problem 5. *A particle of mass m moves in one-dimensional space within the range $0 \leq x \leq a$, and is reflected (via elastic collision) by walls at $x = 0$ and $x = a$. Sketch the trajectory of this particle in the phase space (x, p), where p is the momentum. Assume that the particle obeys classical mechanics.*

Problem 6. *A 1-dimensional chain has N (> 1) elements of length a, and the angle between adjacent elements can only be $0°$ or $180°$. The joints can turn freely and the two ends of the chain are fixed at a distance L. If the entropy of chain is S, make a rough sketch of the dependence of S on L, and briefly justify your sketch. No mathematical expressions or details are required. (The elements in the drawing are displaced in the y direction for clarity.)*







The system is symmetrical. The most ordered the state can be is either at $L = a$ or $L = Na$; so the entropy, in natural units, is $\ln(2)$. The functional form is Gaussian.

Problem 7. Suppose a single particle has two possible energy states: -$\frac{1}{2}\varepsilon$ and $\frac{1}{2}\varepsilon$. Show that the average energy at temperature T is simply $\frac{1}{2}\varepsilon\tanh\left(\frac{\varepsilon}{2kT}\right)$.

$$
\begin{aligned}
\langle E\rangle &= \sum_i p_i E_i \\
&= \left(\frac{\frac{1}{2}\varepsilon e^{-\frac{\varepsilon}{2k_BT}}}{e^{-\frac{\varepsilon}{2k_BT}}+e^{\frac{\varepsilon}{2k_BT}}}\right) - \left(\frac{\frac{1}{2}\varepsilon e^{\frac{\varepsilon}{2k_BT}}}{e^{-\frac{\varepsilon}{2k_BT}}+e^{\frac{\varepsilon}{2k_BT}}}\right) \\
&= \frac{\frac{1}{2}\left(\varepsilon e^{-\frac{\varepsilon}{2k_BT}} - \varepsilon e^{\frac{\varepsilon}{2k_BT}}\right)}{e^{-\frac{\varepsilon}{2k_BT}}+e^{\frac{\varepsilon}{2k_BT}}} \\
&= \frac{1}{2}\varepsilon\tanh\left(\frac{\varepsilon}{2k_BT}\right)
\end{aligned}
$$

A1. Found around the quad. *A system with angular momentum $l = 1$ is described by the quadrupolar Hamiltonian*

$$
\mathcal{H} = \frac{\omega_0}{\hbar}\left(L_x^2 - L_z^2\right)
$$

a) Derive a matrix representation for $\mathcal{H}$.

$$
\begin{aligned}
\mathcal{H} &= \frac{\hbar^2}{2}\left(\frac{\omega_0}{\hbar}\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix} - \frac{\omega_0}{\hbar}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\right) \\
&= \frac{\hbar\omega_0}{2}\begin{pmatrix}0&0&1\\0&2&0\\1&0&0\end{pmatrix}
\end{aligned}
$$

b) Calculate the eigenvalues and the stationary states of this system.

$$
\begin{aligned}
\mathcal{H}\psi &= E\psi \\
0 &= |\mathcal{H} - IE| \\
&= \left| \frac{\hbar\omega_0}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| \\
&= \begin{vmatrix} -\lambda & 0 & \frac{\hbar\omega_0}{2} \\ 0 & \hbar\omega_2 - \lambda & 0 \\ \frac{\hbar\omega_0}{2} & 0 & -\lambda \end{vmatrix} \\
&= \left[-\lambda \left(\left(\hbar\omega_0 - \lambda \right) \left(-\lambda \right) \right) - \frac{\hbar\omega_0}{2} \left(\hbar\omega_0 - \lambda \right) \left(\frac{\hbar\omega_0}{2} \right) \right] \\
\lambda &= \pm \frac{\hbar\omega_0}{2}, \hbar\omega_0
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{H}\psi = E\psi \\
&\mathcal{H} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \\
&\frac{\hbar\omega_0}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \frac{\hbar\omega_0}{2} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \\
&\psi_3 = \psi_1, 2\psi_2 = \psi_2, \psi_1 = \psi_3 \\
&E_1 = \frac{\hbar\omega_0}{2}, \psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Repeat with other eigenvalues.

$$
\begin{aligned}
E_2 &= -\frac{\hbar\omega_0}{2}, \psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
E_3 &= \hbar\omega_0, \psi_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

c) At time $t = 0$ the system is prepared in the state

$$|\psi(0)\rangle = |l = 1, m = 1\rangle \equiv |1\rangle$$

Calculate the state of the system at time $t > 0$

$$\begin{aligned}
|\psi(t)\rangle &= c_1 |1,1\rangle e^{\frac{-iE_1 t}{\hbar}} + c_2 |1,-1\rangle e^{\frac{-iE_2 t}{\hbar}} + c_3 |1,0\rangle e^{\frac{-iE_3 t}{\hbar}} \\
&= c_1 |1,1\rangle e^{\frac{-i\omega_0 t}{2}} + c_2 |1,-1\rangle e^{\frac{i\omega_0 t}{2}} + c_3 |1,0\rangle e^{i\omega_0 t} \\
|1,1\rangle &= c_1 |1,1\rangle e^{\frac{-i\omega_0 t}{2}} + c_2 |1,-1\rangle e^{\frac{i\omega_0 t}{2}} + c_3 |1,0\rangle e^{i\omega_0 t} \\
c_1 &= 1 \\
c_2 &= c_3 = 0 \\
|\psi(t)\rangle &= |1,1\rangle e^{-\frac{i\omega_0 t}{2}} = \frac{1}{\sqrt{2}} (e_1 + e_2) e^{-\frac{i\omega_0 t}{2}}
\end{aligned}$$

d) At some time t a measurement of L_z is carried out. What are the possible outcomes and the respective probabilities.

$$\textit{Note: } L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Possible outcomes: $\hbar$ with probability $\frac{1}{2}$, -$\hbar$ with probability $\frac{1}{2}$

A2. Out of whack. *In a particular basis, the spin operators for a spin $\frac{1}{2}$ particle (an electron with an intrinsic magnetic moment μ_B) may be expressed as the Pauli Spin Matrices:*

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y = i\frac{\hbar}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

At time $t = 0$, a constant magnetic field of amplitude B is induced in the x direction.

a) What is the matrix representation of the Hamiltonian of the system in this basis?

$$\begin{aligned}
\mathcal{H} &= -\vec{\mu} \cdot \vec{B} \\
&= \frac{e}{m} \vec{S} \cdot \vec{B} \\
&= \frac{e}{m} S_x B \\
&= \frac{eB\hbar}{2m} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

b) Assume that the initial state of the electron is expressed as $\psi(0) = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$
What is the probability of measuring the S_z as $\frac{\hbar}{2}$ at time t?

$$\begin{aligned}
\mathcal{H}\psi &= E\psi \\
0 &= |\mathcal{H} - IE| \\
&= \left|\frac{eB\hbar}{2m}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right| \\
&= \begin{vmatrix}-\lambda & \frac{eB\hbar}{2m}\\ \frac{eB\hbar}{2m} & -\lambda\end{vmatrix} \\
\lambda &= \pm\frac{eB\hbar}{2m}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}\psi &= E\psi \\
\frac{eB\hbar}{2m}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} &= \frac{eB\hbar}{2m}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\
\beta &= \alpha \\
E_1 = \frac{eB\hbar}{2m}, \psi_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}(e_1 + e_2) \\
\frac{eB\hbar}{2m}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} &= -\frac{eB\hbar}{2m}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\
\beta &= -\alpha \\
E_2 = \frac{eB\hbar}{2m}, \psi_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}(e_1 - e_2)
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix}1/2\\ \sqrt{3}/2\end{pmatrix} &= \frac{a}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} \\
a &= \frac{\sqrt{2}+\sqrt{6}}{4}, b = \frac{\sqrt{2}-\sqrt{6}}{4} \\
\psi(t) &= \frac{\sqrt{2}+\sqrt{6}}{4\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}e^{-i\frac{eBt}{2m}} + \frac{\sqrt{2}-\sqrt{6}}{4\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}e^{i\frac{eBt}{2m}} \\
c_1 &= \langle e_1|\psi(t)\rangle \\
\left|c_1^2\right| &= \frac{1}{2} - \frac{1}{4}\cos\left(\frac{eBt}{m}\right)
\end{aligned}$$

c) What is the probability of measuring the S_x as $\frac{\hbar}{2}$ at time t?

$$\text{The probability is } \frac{1}{4}$$

d) Imagine that you measure the x-spin in part c and immediate measure S_x. What then is the probability of measuring $\frac{\hbar}{2}$?

Making a measurement on the system "collapses" the wave function. Any immediate measurements after this will result in the same value. So the probability of measuring $\frac{\hbar}{2}$ again is 1.

A3. Out of state residence? *Let $|b'\rangle$ and $|b''\rangle$ be the eigenstates of a Hermitian operator B with the eigenvalues b' and b", respectively (b' is not equal to b"). The Hamiltonian operator is given by*

$$\mathcal{H} = |b'\rangle C \langle b''| + |b''\rangle C \langle b'|$$

where C is just a real number.
a) Calculate the eigenvalues and eigenvectors of the Hamiltonian.

$$\begin{aligned}
B\,|b'\rangle &= b'\,|b'\rangle \\
B\,|b''\rangle &= b''\,|b''\rangle \\
|\psi\rangle &= \alpha\,|b'\rangle + \beta\,|b''\rangle \\
\mathcal{H}\,|\psi\rangle &= |b'\rangle C \langle b''| \left(\alpha\,|b'\rangle + \beta\,|b''\rangle\right) + |b''\rangle C \langle b'| \left(\alpha\,|b'\rangle + \beta\,|b''\rangle\right) \\
&= |b'\rangle C \cancel{\langle b''|\,\alpha\,|b'\rangle} + |b'\rangle C \langle b''|\,\beta\,|b''\rangle + |b''\rangle C \langle b'|\,\alpha\,|b'\rangle + \cancel{|b''\rangle C \langle b'|\,\beta\,|b''\rangle} \\
C &= \pm 1 \\
\alpha &= \pm\beta \\
\mathcal{H}\psi &= E\psi \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|b'\rangle + |b''\rangle\right) \\
\mathcal{H}\,|\psi\rangle &= |b'\rangle C \langle b''| \left(\frac{1}{\sqrt{2}}\left(|b'\rangle + |b''\rangle\right)\right) + |b''\rangle C \langle b'| \left(\frac{1}{\sqrt{2}}\left(|b'\rangle + |b''\rangle\right)\right) \\
E_1 &= \frac{C}{\sqrt{2}} \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}\left(|b'\rangle - |b''\rangle\right) \\
E_2 &= -\frac{C}{\sqrt{2}} \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}\left(|b''\rangle - |b'\rangle\right) \\
E_3 &= -\frac{C}{\sqrt{2}}
\end{aligned}$$

b) Suppose the system is known to be in state $|b'\rangle$ at $t = 0$, what is the state vector in the Schrödinger picture for $t > 0$?

$$
\begin{aligned}
|\psi\rangle &= C_1|\psi\rangle + C_2|\psi_2\rangle + C_3|\psi_3\rangle \\
&= C_1\left(\frac{1}{\sqrt{2}}\left(|b'\rangle + |b''\rangle\right)\right)e^{-\frac{iCt}{\sqrt{2}\hbar}} + C_2\left(\frac{1}{\sqrt{2}}\left(|b'\rangle - |b''\rangle\right)\right)e^{\frac{iCt}{\sqrt{2}\hbar}} + C_3\left(\frac{1}{\sqrt{2}}\left(|b''\rangle - |b'\rangle\right)\right)e^{\frac{iCt}{\sqrt{2}\hbar}} \\
|b'\rangle &= \frac{1}{\sqrt{2}}\left((C_1 + C_2 - C_3)|b'\rangle + (C_1 - C_2 + C_3)|b''\rangle\right) \\
C_1 &= \frac{\sqrt{2}}{2} \\
C_3 &= C_2 - \frac{\sqrt{2}}{2} \\
|\psi\rangle &= \frac{1}{2}\left(e^{-\frac{iCt}{\sqrt{2}\hbar}} + e^{\frac{iCt}{\sqrt{2}\hbar}}\right)|b'\rangle + \frac{1}{2}\left(e^{-\frac{iCt}{\sqrt{2}\hbar}} - e^{\frac{iCt}{\sqrt{2}\hbar}}\right)|b''\rangle
\end{aligned}
$$

c) What is the probability of finding the system in $|b''\rangle$ for $t > 0$, if the system is known to be in state $|b'\rangle$ at $t = 0$?

$$\left|\frac{1}{2}\right|^2 + \left|-\frac{1}{2}\right|^2 = \frac{1}{2}$$

B1. Perfect Harmony? *A quantum harmonic oscillator has energy levels $E_n = \hbar\omega\left(n + {}^1\!/_2\right)$ where $n = 0, 1, 2...$*
a) Show that the probability for finding the oscillator in its n-th quantum state at temperature T is

$$P_n = \left(1 - e^{-\hbar\omega/kT}\right)e^{-n\hbar\omega/kT}$$

$$
\begin{aligned}
P_n &= \frac{e^{-E_n\beta}}{\sum_{n=0}^{\infty} e^{-E_n\beta}} \\
&= \frac{e^{-\hbar\omega(n+1/2)\beta}}{\sum e^{-\hbar\omega(n+1/2)\beta}} \\
&= \frac{\cancel{e^{-\hbar\omega\beta/2}}e^{-\hbar\omega n\beta}}{\cancel{e^{-\hbar\omega\beta/2}}\sum e^{-\hbar\omega n\beta}} \\
&= \frac{e^{-\hbar\omega n\beta}}{1 + \sum_{n=1}^{\infty} e^{-\hbar\omega n\beta}} \\
&= \left(1 - e^{-\hbar\omega/kT}\right)e^{-n\hbar\omega/kT}
\end{aligned}
$$

b) What is the average internal energy of this harmonic oscillator at temperature T? What are its limiting values at very low and very high temperatures?

$$
\begin{aligned}
U &= -\frac{\partial}{\partial\beta}\ln z \\
&= -\frac{\partial}{\partial\beta}\ln\left[\sum_{n=0}^{\infty} e^{-\hbar\omega n\beta} e^{-\frac{\hbar\omega\beta}{2}}\right] \\
&= -\frac{\partial}{\partial\beta}\ln\left[e^{-\frac{\hbar\omega\beta}{2}}\frac{1}{(1-e^{-\hbar\omega\beta})}\right] \\
&= \frac{\hbar\omega}{2} + \hbar\omega\left[\frac{e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}}\right] \\
&T \to 0, U \to \frac{\hbar\omega}{2} \\
&T \to \infty, U \to \infty
\end{aligned}
$$

c) What is its specific heat at temperature T? What are its limiting values at very low and very high temperatures?

$$\textit{Notes: for } x < 1, \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

$$
\begin{aligned}
C_p &= \frac{\partial U}{\partial T} \\
&= \frac{\partial}{\partial T}\left(\frac{\hbar\omega}{2} + \hbar\omega\left[\frac{e^{-\frac{\hbar\omega}{kT}}}{1-e^{-\frac{\hbar\omega}{kT}}}\right]\right) \\
&= \frac{\hbar^2\omega^2}{kT^2}\left[\frac{e^{-\hbar\omega\beta} - 2e^{-2\hbar\omega\beta}}{[1-e^{-\hbar\omega\beta}]^2}\right] \\
&T \to 0, C_p \to 0 \\
&T \to \infty, C_p \to \infty
\end{aligned}
$$

B2. Heavier and heavier. *An ideal gas is introduced into a "test tube" (actually a sealed cylinder). The gas density is ρ_0, and the mass of each molecule is M. The test tube is placed into a centrifuge, which then spins the tube at an angular velocity ω. The equilibrium temperature is T.*
a) Compute the density of the gas in the test tube as a function of height, h, above the bottom. (The bottom of the tube, $h = 0$, is the point farthest from the rotation axis of the centrifuge.)

$$\frac{F}{V} = \rho\frac{v^2}{r} = \rho\omega^2 (R-h)$$
$$-\frac{dP}{dH} = \rho\omega^2 (R-h)$$

$$PV = NkT$$
$$P = NkT = \frac{\rho KT}{M}$$
$$\frac{dP}{dh} = \frac{kT}{M}\frac{d\rho}{dh}$$
$$\frac{kT}{M}\frac{d\rho}{dh} = -\rho\omega^2 (R-h)$$
$$\int_0^\rho \frac{d\rho}{\rho} = \int_0^h -\frac{M\omega^2}{kT}(R-h)\,dh$$
$$\rho(h) = \rho_0 e^{-\frac{M\omega^2}{kT}\left(Rh-\frac{1}{2}h^2\right)}$$

b) Describe how you would use a centrifuge to separate the isotope $^{235}_{92}U$ *from* $^{238}_{92}U$

The heavier isotope would be on the outermost end of the centrifuge and the lighter ones would be towards the inside. If spun long enough, it would be fairly easy to separate the two isotopes.

c) Suppose the naturally occurring ratio of $^{235}_{92}U/^{238}_{92}U = 0.7$*. Assume the Uranium is gasified by allowing it to react with fluorine* $\left(^{19}_{1}UF\right)$*, forming the gas* UF_6 *. Assuming that* $R = 100cm$*,* $T = 300K$*, and* $\omega = 60000rpm$*, estimate the ratio* $^{235}_{92}U/^{238}_{92}U$ *at the top of the "test-tube".*

Plug in values to answer from part a, but with $M = 3m_p$ for the difference in masses.

$$ratio = 0.7e^{-1.210(h-0.5h^2)}$$