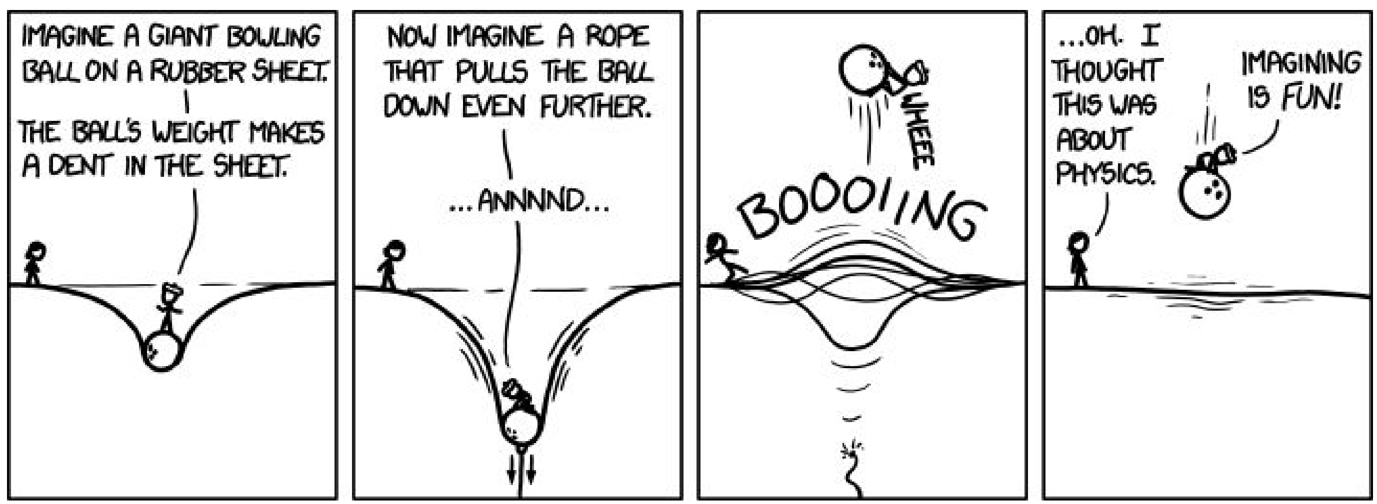
# Maxwell’s Equations in General Relativity

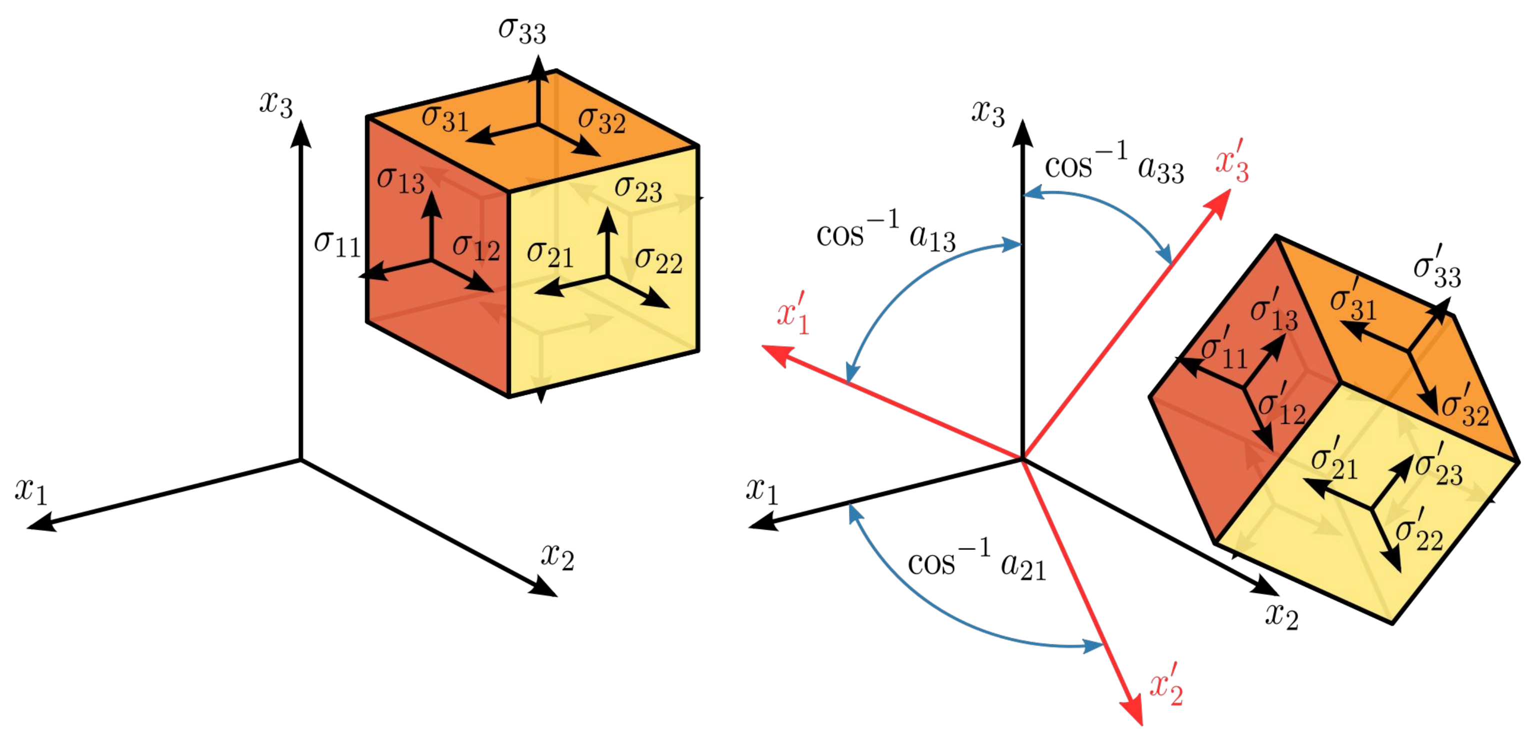
Joseph P. Glaser

# Background on Tensors



# Tensor Definition

*Tensors* are simply arrays of numbers, or functions, that transform according to certain rules under a change of coordinates.



# Examples of Tensors

Tensors are classified as: $(n, m)$ Tensor of Rank $(n + m)$, where:

- n is the number of contravariant indices
- m is the number of covariant indices

| $n, m$ | $m = 0$ | $m = 1$ | $m = 2$ |
|---|---|---|---|
| $n = 0$ | Scalar | Covector | Metric Tensor |
| $n = 1$ | Vector | Linear Transform | |
| $n = 2$ | Inverse Metric Tensor | | Elasticity Tensor |

# Useful Operations on Tensors

Einstein Summation Notation:

$$y = \sum_{i=0}^{N} v_i w^i = v_i w^i$$

Raising & Lowering a Tensor's Index:

$$g^{ij} A_j = A^i \qquad g_{ij} A^j = A_i$$

# The Electromagnetic Field Tensor

# Maxwell's Equations

We want the following to hold in General Relativity:

1) Gauss's Law [Electricity]
2) Gauss's Law [Magnetism]
3) Faraday's Law
4) Ampere's Law

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

# Introducing the Four-Current

Assume a charge density, $\rho$, is moving with some velocity, $\boldsymbol{v}$.

The four-current density is thus:

$$J^i = \gamma[\rho, \vec{J}]$$

where the electrical current density as seen in the lab-frame is:

$$\vec{J} = \vec{\beta}\rho$$

# Satisfying Gauss's Law

$$\nabla \cdot \vec{E} = 4\pi k \rho$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi k \rho$$

$$\frac{\partial F^{00}}{\partial t} + \frac{\partial F^{01}}{\partial x} + \frac{\partial F^{02}}{\partial y} + \frac{\partial F^{03}}{\partial z} = 4\pi k J^0$$

$$\partial_j F^{0j} = 4\pi k J^0$$

# Satisfying the Coulomb Force

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\vec{E} \qquad \frac{\mathrm{d}p^i}{\mathrm{d}\tau} = qF^{ij}u_j$$

where:

$$p^i = mu^i \qquad u^i = \gamma[1, \vec{v}]$$

Noting the relationship:

$$2p_i\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = \frac{\mathrm{d}\vec{p}\cdot\vec{p}}{\mathrm{d}\tau} = 0$$

# Satisfying the Coulomb Force

$$2p_i\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = 2(mu_i)qu_jF^{ij} = 0$$

If we suppose that ***F*** is anti-symmetric and zero along the diagonal, the above equation holds for all four-velocities.

$$F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & a & b \\ -E_y & -a & 0 & c \\ -E_z & -b & -c & 0 \end{bmatrix}$$

# Satisfying Lorentz Force

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = qF^{ij}u_j$$

$$\begin{aligned}\frac{\mathrm{d}p^x}{\mathrm{d}\tau} &= q(-E_x u_0 + 0 + au_2 + bu_3) \\ &= q\gamma(E_x + av_y + bv_z) \\ &= q\gamma(E_x + B_z v_y + (-B_y)v_z) \\ &= q\gamma(E_x + [\vec{v} \times \vec{B}]_x)\end{aligned}$$

# The Electromagnetic Field Tensor

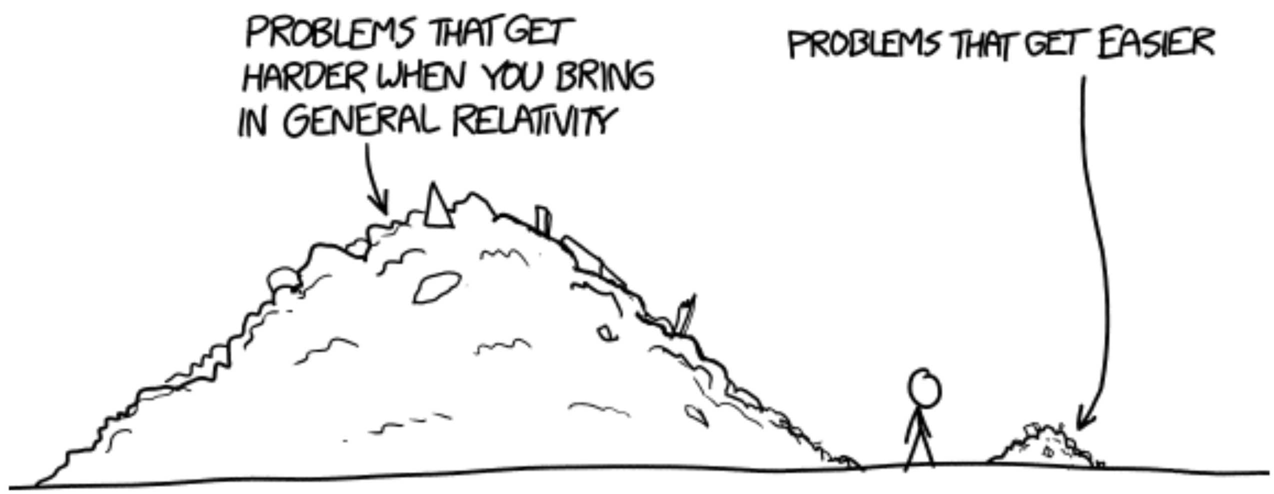
$$F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

# Confirming Ampere's Law

Try out the x-component of the four-current density:

$$\begin{aligned}
4\pi k J^1 &= \partial_j F^{1j} \\
&= \partial_j F^{10} + \partial_j F^{12} + \partial_3 F^{13} \\
&= \partial_0 E_x + \partial_2 B_z - \partial_3 B_y \\
&= -\frac{\partial E_x}{\partial t} + [\nabla \times \vec{B}]_x
\end{aligned}$$

# The Reissner-Nordström Charged Mass Solution



# The R-N Metric

$$g^{ij} = \begin{bmatrix} -f(r) & 0 & 0 & 0 \\ 0 & 1/f(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin(\theta) \end{bmatrix}$$

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$$

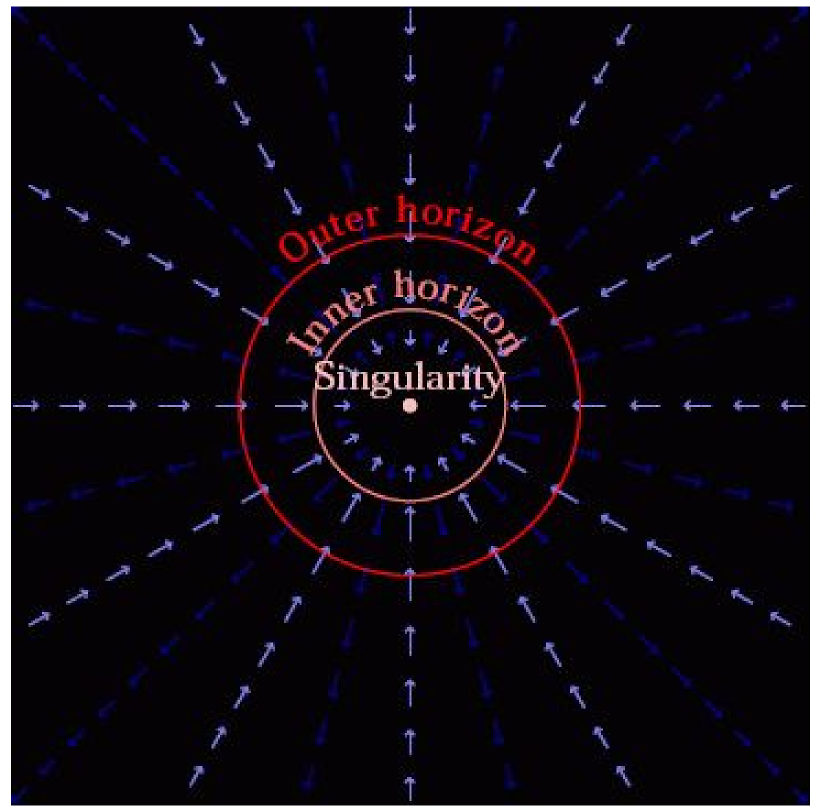
# Application to Charged Black Holes

$$r_{outer} = M + \sqrt{M^2 - Q^2}$$

$$r_{inner} = M - \sqrt{M^2 - Q^2}$$

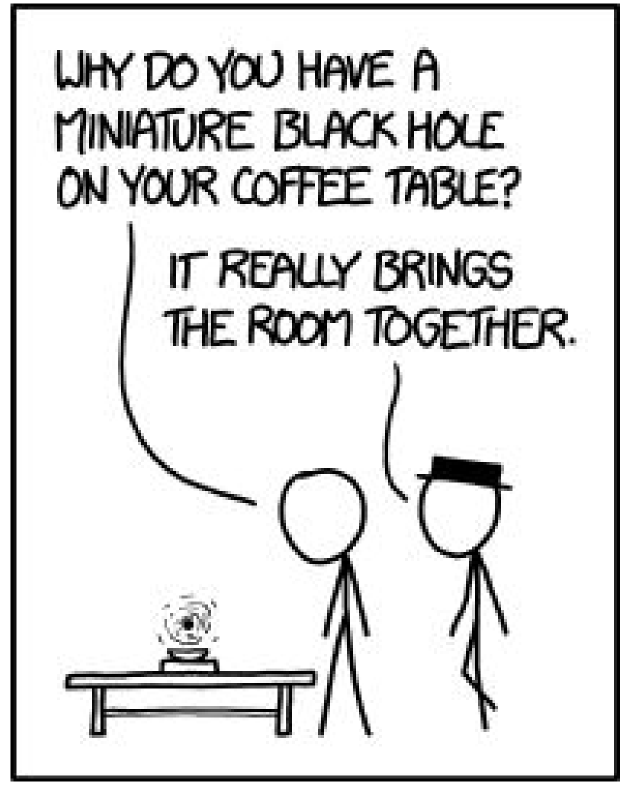
$$v = \sqrt{\frac{2M(r)}{r}} = \frac{1}{\sqrt{1 - f(r)^2}}$$

$$r_{stop} = \frac{Q^2}{2M}$$




*Andrew Hamilton, University of Colorado Boulder*

# Thanks!



WHY DO YOU HAVE A MINIATURE BLACK HOLE ON YOUR COFFEE TABLE?
IT REALLY BRINGS THE ROOM TOGETHER.