Electromagnetic Theory II HW6

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8.4

The general wave equation for a cylindrical waveguide is:

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \tag{0.1}$$
$$\psi = E_z \textbf{ (TM) },B_z \textbf{ (TE)} \tag{0.2}$$

The solutions that are regular at $\rho = 0$ are:

$$\psi = A_m J_m(\gamma\rho)e^{\pm im\phi} \tag{0.3}$$

The boundary conditions on the conductor surface $\rho = R$ will be different for TE and TM modes. For TM modes the E field vanishes at the surface, so the eigenvalues are the zeros of $J_m(\gamma R)$. For TE modes the normal derivative of the magnetic field vanishes at the surface, so the eigenvalues are the zeros of $J'_m(\gamma R)$. Therefore the mode frequencies are:

$$\omega_{m,n} = \frac{Z_m(n)}{\sqrt{\mu\epsilon}R} \textbf{ (TM)} \tag{0.4}$$
$$\omega_{m,n} = \frac{Z'_m(n)}{\sqrt{\mu\epsilon}R} \textbf{ (TE)} \tag{0.5}$$

Where we have defined $Z_m(n)$ as the Nth zero of J_m and $Z'_m(n)$ as the Nth zero of J'_m. The first few roots are:

$$Z_0 = 2.41, 5.52, 8.65 \tag{0.6}$$
$$Z_1 = 3.83, 7.02, 10.17 \tag{0.7}$$
$$Z_2 = 5.14, 8.41, 11.62 \tag{0.8}$$
$$Z'_0 = 3.83, 7.02, 10.17 \tag{0.9}$$
$$Z'_1 = 1.84, 5.33, 8.54 \tag{0.10}$$
$$Z'_2 = 3.05, 6.71, 9.97 \tag{0.11}$$

The lowest root is $Z_1'(1)$, so the the dominant mode is TE_{11}. Listing the dominant mode and the next four higher modes:

Mode	Frequency	Ratio
TE_{11}	1.84	1
TM_{01}	2.41	1.31
TE_{21}	3.05	1.66
TE_{01}, TM_{11}	3.83	2.08
TM_{21}	5.14	2.79

b) The attenuation coefficients are found from (Jackson 8.57):

$$\beta_\lambda = -\frac{1}{2P}\frac{dP}{dz} \tag{0.12}$$

The power and power loss for TE and TM modes are:

$$P_{TM} = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\epsilon\int_A \psi^*\psi\ da \tag{0.13}$$

$$\frac{dP}{dz}_{TM} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2\oint_C \frac{1}{\mu^2\omega_\lambda^2}\left|\frac{\partial\psi}{\partial n}\right|^2\ d\ell \tag{0.14}$$

$$P_{TE} = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\mu\int_A \psi^*\psi\ da \tag{0.15}$$

$$\frac{dP}{dz}_{TM} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2\oint_C \frac{1}{\mu\epsilon\omega_\lambda^2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)|\mathbf{n}\times\nabla_t\psi|^2 + \left|\frac{\partial\psi}{\partial n}\right|^2\ d\ell \tag{0.16}$$

We can evaluate the power expressions using the orthogonality of the Bessel functions:

$$\int_0^1 xJ_m(xu)J_m(xu)\ dx = \frac{1}{2}\left(J_{m+1}(u)\right)^2 \tag{0.17}$$

For the TM modes, integrating from $\rho = 0$ to R with $u \equiv Z_m(n)/R$ making the substitution $\rho' = \rho/R$ we find:

$$P_{TM} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} 2\pi\int_0^1 R^2\rho'\left[J_m(\rho' Z_m(n))\right]^2\ d\rho' \tag{0.18}$$

$$P_{TM} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\pi R^2\left[J_{m+1}(Z_m(n))\right]^2 \tag{0.19}$$

For the TE modes we are working with the zeros of the derivatives of the Bessel functions.

8.6

We have solved the cylindrical waveguide in the previous problem, the cylindrical cavity frequencies come from the modified expression for γ in a cavity:

$$\gamma^2 = \mu\epsilon\omega^2 - \left(\frac{p\pi}{d}\right)^2 \tag{0.20}$$

$$\omega_{\lambda p}^2 = \frac{1}{\mu\epsilon}\left\{\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2\right\} \tag{0.21}$$

Where $p = 0, 1, 2..$ for TM modes (cosine solutions in z) and $p = 1, 2, 3...$ for TE modes (sine solutions in z). Writing the modes in terms of the zeros of the Bessel functions and their derivatives, and pulling out a factor of $1/R^2$:

$$\omega_{m,n,p} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{\left(Z_m(n)^2 + \left(\frac{p\pi R}{d}\right)^2\right)}\ \textbf{(TM)} \tag{0.22}$$

$$\omega_{m,n,p} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{\left(Z_m'(n)^2 + \left(\frac{p\pi R}{d}\right)^2\right)}\ \textbf{(TE)} \tag{0.23}$$

$$\tag{0.24}$$

9.3