PHYS 502: Mathematical Physics II

Winter 2015

Solutions to Homework #3

1. (a) Separating $u(\rho, \phi) = R(\rho)\Phi(\phi)$, Laplace's equation becomes

$$\frac{\rho}{R}\left(\rho R'\right)' + \frac{\Phi''}{\Phi} = 0.$$

Hence $\Phi''/\Phi = -m^2$, where m is an integer (by the usual argument) and

$$\rho\left(\rho R'\right)' - m^2 R = 0,$$

or

$$\rho^2 R'' + \rho R' - m^2 R = 0.$$

Seeking a power-law solution $R \sim \rho^\alpha$ and substituting in, we find $\alpha = \pm m$.

(b) Hence the general solution is a sum of terms of the form

$$u_m(\rho, \phi) = \rho^m\left(a_m \cos m\phi + b_m \sin m\rho\right)$$

(aside from the second solution for $m = 0$, which is $\ln \rho$, as discussed in class, but we won't need that here since we are looking for regular solutions with $r < a$).

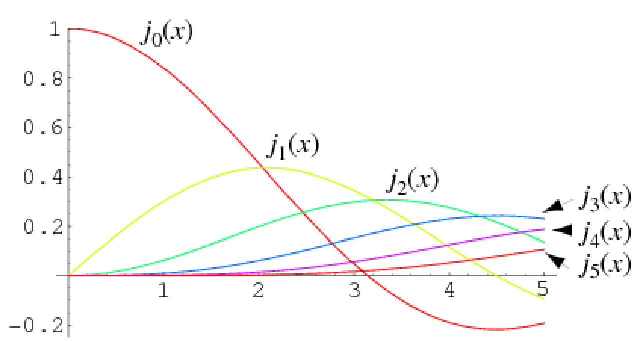
(c) The boundary condition is $u(a, \phi) = U \cos^2 \phi = \frac{1}{2}U(1 + \cos 2\phi)$, which picks out the cosine terms with $m = 0$ and $m = 2$. Hence the interior regular solution (non-negative powers of ρ) is

$$u(\rho, \phi) = \tfrac{1}{2}U\left(1 + \frac{\rho^2}{a^2}\cos 2\phi\right).$$

2. The solutions to the wave equation in a sphere are of the form

$$u(r, \theta, \phi) = j_l(kr)P_l^m(\cos\theta)e^{im\phi},$$

for integer l and m. The boundary condition $\partial u/\partial r = 0$ at $r = R$ requires $j_l'(kR) = 0$. As illustrated in the figure below, the three lowest allowed values of kR correspond, respectively, to the first zeros of j_1' and j_2', and the second zero of j_0'.



Since

$$j_0(x) = \frac{\sin x}{x},$$

we have

$$j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2},$$

so $j_0'(x) = 0 \to \tan x = x$, or $x = 4.49$. Similarly, since

$$\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, \\ j_2(x) &= \sin x \left(\frac{3}{x^2} - 1 \right) - \frac{3\cos x}{x^2}, \end{aligned}$$

$j_1'(x) = 0$ for $x = 2.08$, $j_2'(x) = 0$ for $x = 3.34$. (Note that the first zero of j_3' is at $x = 4.52$.) Thus, the three lowest frequencies are $\omega = kc = 2.08c/R, 3.34c/R, 4.49c/R$.

3. The equation to be solved is

$$\nabla^2 n + \lambda n = \frac{1}{\kappa} \frac{\partial n}{\partial t},$$

where $\lambda, \kappa > 0$ and $n = 0$ on the surface. For assumed time dependence $n \sim e^{\alpha t}$, the equation becomes

$$\nabla^2 n + k^2 n = 0,$$

where $k^2 = \lambda - \alpha/\kappa$. The critical case has $\alpha = 0$, or $k^2 = \lambda$.

(a) For a sphere, the general solution is $n \sim j_l(kr) P_l^m(\cos\theta)\, e^{im\phi}$. The surface boundary condition is $j_l(kR) = 0$, and the minimum k corresponds to the first root of j_0, so $l = m = 0$. Since $j_0(x) \sim \sin x/x$, we find $kR = \pi$ and the critical radius is

$$R_0 = \frac{\pi}{k} = \frac{\pi}{\sqrt{\lambda}}.$$

Note that, in order to satisfy the boundary condition, increasing R has the effect of decreasing k and hence of increasing $\alpha = \kappa(\lambda - k^2)$. Thus the sphere is unstable for $R > R_0$.

(b) For a *hemisphere*, the extra boundary condition at $\theta = \pi/2$ means that the $l = 0$ mode is not a solution. We now require $P_l^m(\cos\theta) = 0$ at $\theta = \pi/2$ (where we have assumed that the z axis is the axis of symmetry of the hemisphere). The lowest-order P_l^m satisfying the boundary condition is $P_1^0 = \cos\theta$, so $l = 1$ and the radial boundary condition becomes $j_1(kR) = 0$. Since $j_1(x) \sim \sin x/x^2 - \cos x/x$, the first zero has $x = \tan x$, or $x = 1.43\pi = 4.49$. The critical ($\alpha = 0$) radius for this geometry then is

$$R_1 = \frac{1.43\pi}{k} = \frac{1.43\pi}{\sqrt{\lambda}} = 1.43 R_0.$$

(c) Now the system is spherical again, but the radius is $R_1 > R_0$ and the system is unstable. Writing $\beta = 1.43$, the boundary condition now implies

$$\begin{aligned} kR_1 = \left(\lambda - \frac{\alpha}{\kappa}\right)^{1/2} R_1 &= \pi \\ \Rightarrow \quad \alpha &= \kappa\lambda\left(1 - \beta^{-2}\right). \end{aligned}$$

The growth time scale therefore is

$$\tau = \alpha^{-1} = \left(\frac{\beta^2}{\beta^2 - 1}\right) \frac{1}{\kappa\lambda} = \frac{1.96}{\kappa\lambda}.$$

4. (a) The general regular solution (in polar coordinates) to the 2-D Helmholtz equation is

$$u(r,\theta) = \sum_m J_m(kr)\,(a_m \cos m\theta + b_m \sin m\theta).$$

The boundary condition $u(R,\theta) = f(\theta)$ implies

$$\begin{aligned} \sum_m J_m(kR)\,(a_m \cos m\theta + b_m \sin m\theta) &= f(\theta) \\ &= \sum_m (A_m \cos m\theta + B_m \sin m\theta), \end{aligned}$$

where

$$\begin{aligned} A_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta')\cos m\theta'\,d\theta' \qquad (\times\tfrac{1}{2} \text{ for } m=0) \\ B_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta')\sin m\theta'\,d\theta' \qquad (m>0). \end{aligned}$$

Hence

$$a_m = \frac{A_m}{J_m(kR)}, \qquad b_m = \frac{B_m}{J_m(kR)},$$

and so

$$u(r,\theta) = \int_0^{2\pi} K(r,\theta,\theta')f(\theta')\,d\theta',$$

where

$$\begin{aligned} K(r,\theta,\theta') &= = \sum_m \frac{J_m(kr)}{\pi J_m(kR)}\big(\cos m\theta\cos m\theta' + \sin m\theta \sin m\theta'\big) \\ &= \frac{1}{\pi}\sum_m \frac{J_m(kr)}{J_m(kR)}\cos m(\theta-\theta') \end{aligned}$$

(again with an extra factor of $\frac{1}{2}$ in the $m=0$ term).

(b) For $f(\theta) = \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, we have $A_0 = A_2 = \frac{1}{2}$, and all the other As and Bs are zero. Hence

$$u(r,\theta) = \tfrac{1}{2}\left[\frac{J_0(kr)}{J_0(kR)} + \frac{J_2(kr)}{J_2(kR)}\cos 2\theta\right].$$