Quantization of the Electromagnetic Field

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1 Quantization of the Electromagnetic Field

In this Chapter we quantized the electromagnetic field. Starting with Maxwell's equations, we introduce the scalar and vector potentials, choose a gauge, and find the wave equations satisfied by the vector potential. The vector potential is then resolved into its Fourier components and the energy is computed. It is a quadratic function of the coordinates. This quadratic form is compared with the quadratic form for the harmonic oscillator. The comparison facilitates the transformation of the energy into a sum of independent harmonic oscillator hamiltonians. Quantization follows by the standard rules: $[P,Q]=\hbar/i$. Linear combinations of the noncommuting coordinates P and Q lead directly to a representation of the vector potential in terms of creation and annihilation operators for photons of momentum $\hbar\mathbf{k}$ and polarization $\boldsymbol{\epsilon}_{\mathbf{k},\boldsymbol{\sigma}}$.

1.1 Maxwell's Equations

We begin as usual with the Maxwell equations in Gaussian coordinates

$$\begin{array}{rclrcl} & \text{Magnetic Eq.} & & & \text{Electric Eq.} & \\ \boldsymbol{\nabla}\cdot\mathbf{B} & = & 0 & \boldsymbol{\nabla}\cdot\mathbf{E} & = & 4\pi\rho \\ -\boldsymbol{\nabla}\times\mathbf{E}-\dfrac{1}{c}\dfrac{\partial\mathbf{B}}{\partial t} & = & 0 & \boldsymbol{\nabla}\times\mathbf{B}-\dfrac{1}{c}\dfrac{\partial\mathbf{E}}{\partial t} & = & \dfrac{4\pi}{c}\mathbf{j} \end{array} \tag{1}$$

The electric equations show that the electromagnetic field is generated by the motion of electrically charged particles. Magnetically charged particles (if they were to exist) would also be sources for the electromagnetic field, as indicated in the magnetic equations.

There is in adddition a conservation law. There is also (following Newton's Third Law) a reaction of the electromagnetic field on the charged particles that create the field. This back-reaction is called the Lorentz force law:

$$
\begin{array}{lrcl}
\text{Conservation Law :} & \nabla\cdot\mathbf{j}+\dfrac{\partial\rho}{\partial t} & = & 0 \\
\text{Lorentz Force Law :} & \mathbf{F} & = & e(\mathbf{E}+\dfrac{\mathbf{v}}{c}\times\mathbf{B})
\end{array}
\tag{2}
$$

1.2 Vector and Scalar Potentials

Since the magnetic equations are homogeneous (no magnetic monopoles), they can be used to introduce vector and scalar potentials as follows. Since $\nabla\cdot\mathbf{B}=\mathbf{0}$, it is always possible to find a vector field $\mathbf{A}$ with the property

$$
\nabla\cdot\mathbf{B}=\mathbf{0}\Rightarrow\mathbf{B}=\nabla\times\mathbf{A} \tag{3}
$$

The vector field $\mathbf{A}$ is not unique: the gradient of a scalar field ($\nabla\chi$) can be added to it, since $\nabla\times\nabla\chi=\mathbf{0}$.

The vector potential can be used in the second of the magnetic equations:

$$
\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\mathbf{A}=\nabla\times\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)=\mathbf{0} \tag{4}
$$

When the curl of something vanishes, then that something can be expressed as the gradient of a scalar function. Following convention, we set

$$
\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)=-\nabla\Phi\Rightarrow\mathbf{E}=-\nabla\Phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \tag{5}
$$

1.3 Gauge Transformations

The vector and scalar potentials are not unique. The transformation from one set of fields to another set that yield the same electric and magnetic fields under the operations identified in Eq. (3) and Eq. (5) are called gauge transformations.

$$
\begin{array}{rclcrcl}
\mathbf{A}\rightarrow\mathbf{A}' & = & \mathbf{A}+\nabla\chi & \qquad & \nabla\times\mathbf{A}' & = & \mathbf{B} \\
\Phi\rightarrow\Phi' & = & \Phi-\dfrac{1}{c}\dfrac{\partial\chi}{\partial t} & \qquad & -\nabla\Phi'-\dfrac{1}{c}\dfrac{\partial\mathbf{A}'}{\partial t} & = & \mathbf{E}
\end{array}
\tag{6}
$$

As usual, Newton's Third Law suggests that a change in the gauge of the (vector, scalar) potentials must see a response in the quantum wave function. The back reaction is

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi \Leftrightarrow \psi \to \psi' = e^{\frac{ie}{\hbar c}\chi}\psi \tag{7}$$

1.4 Dynamics of the Potential Fields

In this section we construct the wave equation satisfied by the vector potential **A** in the absence of charge sources in a convenient choice of gauge.

The first electric equation gives

$$\boldsymbol{\nabla} \cdot \left(-\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = -\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) = 4\pi\rho \tag{8}$$

The second electric equation gives

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}(-\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = \frac{4\pi}{c}\mathbf{j} \tag{9}$$

The vector identity $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = -\boldsymbol{\nabla}^{\mathbf{2}}\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})$ is useful for bringing Eq. (9) under control:

$$\begin{aligned} -\boldsymbol{\nabla}^{\mathbf{2}}\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\Phi) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} &= \frac{4\pi}{c}\mathbf{j} \\ -\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) &= \frac{4\pi}{c}\mathbf{j} \end{aligned} \tag{10}$$

Two commonly used gauges are

$$\begin{aligned} \text{Coulomb Gauge :} \quad \boldsymbol{\nabla} \cdot \mathbf{A} &= \mathbf{0} \\ \text{Maxwell Gauge :} \quad \boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} &= 0 \end{aligned} \tag{11}$$

We now make the two assumptions:

1. No sources: $\rho \to 0$, $\Phi \to 0$.
2. Maxwell Gauge: $\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$.

With these assumptions the **A** vector potential satisfies the wave equation:

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0 \tag{12}$$

1.5 Normal Mode Decomposition

The $\mathbf{A}$ vector potential is now written in terms of normal modes inside a box of volume V:

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k},\boldsymbol{\sigma}} \frac{N_{\mathbf{k},\boldsymbol{\sigma}}}{\sqrt{V}} \boldsymbol{\epsilon}_{\mathbf{k},\boldsymbol{\sigma}} \left\{ Q_{\mathbf{k},\boldsymbol{\sigma}} \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) - P_{\mathbf{k},\boldsymbol{\sigma}} \sin(\mathbf{k}\cdot\mathbf{x} - \omega t) \right\} \tag{13}$$

Here $N_{\mathbf{k},\boldsymbol{\sigma}}$ is a normalization constant, to be determined; $\boldsymbol{\epsilon}_{\mathbf{k},\boldsymbol{\sigma}}$ is a polarization index: it is a vector that describes the polarization of a mode propagating with direction $\mathbf{k}$. The index $\boldsymbol{\sigma} = 1, 2$ (3) describes the possible polarization directions: $\boldsymbol{\epsilon}_{\mathbf{k},\mathbf{1}}$ and $\boldsymbol{\epsilon}_{\mathbf{k},\mathbf{2}}$ are vectors perpendicular to the vector $\mathbf{k}$ and $\boldsymbol{\epsilon}_{\mathbf{k},\mathbf{3}}$ is a vector parallel to $\mathbf{k}$. Since $\boldsymbol{\nabla}\cdot\mathbf{A} = \mathbf{0}$, $\mathbf{k}\cdot\boldsymbol{\epsilon} = \mathbf{0}$ and there are only two polarization directions for each value of the mode index $\mathbf{k}$ ($\boldsymbol{\sigma} = 1, 2$, not 3 in the absence of sources). The wave equation Eq. (12) requires $\mathbf{k}\cdot\mathbf{k} - (\omega/c)^2 = 0$.

In terms of this normal mode decomposition, the magnetic and electric fields are:

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} &&= \sum_{\mathbf{k},\sigma} \frac{N_{\mathbf{k},\boldsymbol{\sigma}}}{\sqrt{V}} \boldsymbol{\epsilon}_{\mathbf{k},\boldsymbol{\sigma}} \times \mathbf{k} \times \{***\} \\
\mathbf{E} &= -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} &&= \sum_{\mathbf{k},\boldsymbol{\sigma}} \frac{N_{\mathbf{k},\boldsymbol{\sigma}}}{\sqrt{V}} \boldsymbol{\epsilon}_{\mathbf{k},\boldsymbol{\sigma}} \frac{\omega_{\mathbf{k}}}{c} \times \{***\} \\
\{***\} & &&= \{-Q_{\mathbf{k},\boldsymbol{\sigma}} \sin(\mathbf{k}\cdot\mathbf{x} - \omega t) - P_{\mathbf{k},\boldsymbol{\sigma}} \cos(\mathbf{k}\cdot\mathbf{x} - \omega t)\}
\end{aligned} \tag{14}$$

With these expressions it is possible to compute the hamiltonian as a function of the amplitudes P and Q:

$$H = \frac{1}{8\pi}\int (\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})\, dV = \sum_{\mathbf{k},\boldsymbol{\sigma}} \frac{1}{8\pi}\frac{N^2_{\mathbf{k},\boldsymbol{\sigma}}}{V} \left\{ \mathbf{k}\cdot\mathbf{k} + \frac{\omega^2}{c^2} \right\} \frac{V}{2} \left(Q^2_{\mathbf{k},\boldsymbol{\sigma}} + P^2_{\mathbf{k},\boldsymbol{\sigma}} \right) \tag{15}$$

In order to identify this sum with a standard harmonic oscillator hamiltonian $\frac{1}{2}(Q^2 + P^2)\hbar\omega$ we make the identifications

$$\frac{1}{8\pi} N_{\mathbf{k},\boldsymbol{\sigma}}^2 \frac{2\omega^2}{c^2} = \hbar\omega \tag{16}$$

This identification serves to define the normalization constant $N_{\mathbf{k},\boldsymbol{\sigma}}$. Collecting all this information yields an expression for the $\mathbf{A}$ vector potential

$$\mathbf{A} = \sum_{\mathbf{k},\boldsymbol{\sigma}} \sqrt{\frac{4\pi\hbar c^2}{V\omega_{\mathbf{k}}}} \boldsymbol{\epsilon}_{\mathbf{k},\boldsymbol{\sigma}} \left\{ Q_{\mathbf{k},\boldsymbol{\sigma}} \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) - P_{\mathbf{k},\boldsymbol{\sigma}} \sin(\mathbf{k}\cdot\mathbf{x} - \omega t) \right\} \tag{17}$$

1.6 Quantization

In order to quantize the electromagnetic field we impose canonical commutation relations on Q and P: $[P, Q] = 1/i$, transforming them from amplitudes to operators (the $\hbar$ is absorbed in definition in Eq. (16)). Then we relate the noncommuting operators $Q_{\mathbf{k},\boldsymbol{\sigma}}, P_{\mathbf{k},\boldsymbol{\sigma}}$ with harmonic oscillator creation and annihilation operators $a_{\mathbf{k},\boldsymbol{\sigma}}^\dagger, a_{\mathbf{k},\boldsymbol{\sigma}}$ as follows:

$$\begin{array}{rclcrcl}
a_{\mathbf{k},\boldsymbol{\sigma}} &=& \frac{1}{\sqrt{2}} \left(Q_{\mathbf{k},\boldsymbol{\sigma}} + iP_{\mathbf{k},\boldsymbol{\sigma}} \right) & & Q_{\mathbf{k},\boldsymbol{\sigma}} &=& \frac{1}{\sqrt{2}} \left(a_{\mathbf{k},\boldsymbol{\sigma}} + a_{\mathbf{k},\boldsymbol{\sigma}}^\dagger \right) \\
a_{\mathbf{k},\boldsymbol{\sigma}}^\dagger &=& \frac{1}{\sqrt{2}} \left(Q_{\mathbf{k},\boldsymbol{\sigma}} - iP_{\mathbf{k},\boldsymbol{\sigma}} \right) & & P_{\mathbf{k},\boldsymbol{\sigma}} &=& \frac{-i}{\sqrt{2}} \left(a_{\mathbf{k},\boldsymbol{\sigma}} - a_{\mathbf{k},\boldsymbol{\sigma}}^\dagger \right) \\
\left[a_{\mathbf{k}',\boldsymbol{\sigma}'}, a_{\mathbf{k},\boldsymbol{\sigma}}^\dagger \right] &=& 1\ \delta(\mathbf{k}', \mathbf{k}) \delta_{\boldsymbol{\sigma}',\boldsymbol{\sigma}} & \leftrightarrow & \left[P_{\mathbf{k}',\boldsymbol{\sigma}'}, Q_{\mathbf{k},\boldsymbol{\sigma}} \right] &=& \frac{1}{i}\ \delta(\mathbf{k}', \mathbf{k}) \delta_{\boldsymbol{\sigma}',\boldsymbol{\sigma}}
\end{array} \tag{18}$$

The $\mathbf{A}$ vector potential, expressed in terms of the creation and annihilation operators for photon modes with indices $(\mathbf{k}, \boldsymbol{\sigma})$, is

$$\mathbf{A} = \sum_{\mathbf{k},\boldsymbol{\sigma}} \sqrt{\frac{2\pi\hbar c^2}{V\omega_{\mathbf{k}}}}\ \boldsymbol{\epsilon}_{\mathbf{k},\boldsymbol{\sigma}} \left\{ a_{\mathbf{k},\boldsymbol{\sigma}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + a_{\mathbf{k},\boldsymbol{\sigma}}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \right\} \tag{19}$$

1.7 Density of States

If we quantize the $\mathbf{A}$ vector potential in a box (cube) of edge length L, the spacial part of the wavefunction must satisfy periodic boundary conditions: $e^{i\mathbf{k}\cdot\mathbf{x}} = e^{i\mathbf{k}\cdot(\mathbf{x}+(\mathbf{L},\mathbf{0},\mathbf{0}))}$. This requires $e^{ik_xL} = 1$, so that $k_x = 2\pi n_x/L$, $n_x = n_1 = 0, \pm1, \pm2, \cdots$. Similar arguments hold for the y and z directions. The result is

$$\mathbf{k} = \frac{2\pi}{L}\,(n_1, n_2, n_3) \tag{20}$$

The number of spacial modes with values of $\mathbf{n}$ less than some value $n = \sqrt{n_1^2 + n_2^2 + n_3^2}$ is the volume of a sphere of radius n. The number within n and $n + dn$ is

$$\rho(n)dn = 4\pi n^2 dn \tag{21}$$

The energy of a photon of momentum $\hbar\mathbf{k}$ is $E = \hbar c|\mathbf{k}| = \hbar c\frac{2\pi}{L}|\mathbf{n}|$, so that

$$4\pi n^2 dn = 4\pi \left(\frac{L}{2\pi\hbar c}\right)^3 E^2 dE \tag{22}$$

Each spacial photon mode has two possible polarization directions, so that

$$\rho(E)dE = 2 \times 4\pi \times \frac{V}{(2\pi\hbar c)^3} E^2\ dE \tag{23}$$

This expression (for $\rho(E)$) counts the number of distinct photon modes within energy dE of the energy $E = \hbar\omega$.

1.8 Remarks on Magnetic Sources

In the event that magnetic sources exist, Maxwell's equations must be modified. In Gaussian units they become

$$\begin{array}{rclrcl} & \text{Magnetic Eq.} & & & \text{Electric Eq.} & \\ \boldsymbol{\nabla}\cdot\mathbf{B} & = & 4\pi\rho_m & \boldsymbol{\nabla}\cdot\mathbf{E} & = & 4\pi\rho_e \\ -\boldsymbol{\nabla}\times\mathbf{E} - \dfrac{1}{c}\dfrac{\partial\mathbf{B}}{\partial t} & = & \dfrac{4\pi}{c}\mathbf{j}_m & \boldsymbol{\nabla}\times\mathbf{B} - \dfrac{1}{c}\dfrac{\partial\mathbf{E}}{\partial t} & = & \dfrac{4\pi}{c}\mathbf{j}_e \end{array} \tag{24}$$

There are now two conservation laws:

$$\begin{array}{lrcl} \text{Conservation Law, electric charge}: & \boldsymbol{\nabla}\cdot\mathbf{j}_e + \dfrac{\partial\rho_e}{\partial t} & = & 0 \\ \text{Conservation Law, magnetic charge}: & \boldsymbol{\nabla}\cdot\mathbf{j}_m + \dfrac{\partial\rho_m}{\partial t} & = & 0 \end{array} \tag{25}$$

The Lorentz Force Law now has two contributions:

$$\text{Lorentz Force Law}: \qquad \mathbf{F} = q_e(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}) + q_m(\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E}) \tag{26}$$

1.9 Electric - Magnetic Duality

There is a duality between the electric and magnetic equations that can be expressed as a $U(1)$ symmetry. The rotation occurs in the space of four-vector currents and the electric and magnetic fields:

$$\begin{aligned} \begin{bmatrix} j_e^\mu \\ j_m^\mu \end{bmatrix}' &= \begin{bmatrix} \cos\xi I_4 & -\sin\xi I_4 \\ \sin\xi I_4 & \cos\xi I_4 \end{bmatrix} \begin{bmatrix} j_e^\mu \\ j_m^\mu \end{bmatrix} \\ \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix}' &= \begin{bmatrix} \cos\xi I_3 & -\sin\xi I_3 \\ \sin\xi I_3 & \cos\xi I_3 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} \end{aligned} \tag{27}$$

These transformations leave the Maxwell equations invariant.

1.10 Covariant Formulation

Maxwell's equations without magnetic sources are elegantly written in covariant form. Introduce coordinates in their contravariant form $x^\alpha = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and their covariant counterparts $x_\alpha = g_{\alpha,\beta}x^\beta = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$. The metric tensor used to raise and lower indices is diagonal: $g_{\alpha,\beta} = \text{diag.}(+1, -1, -1, -1) = g^{\alpha,\beta}$. Dereivatives are defined by $\partial_\alpha = \frac{\partial}{\partial x^\alpha} = (c^{-1}\partial_t, \partial_x, \partial_y, \partial_z)$ and the contravariant derivative is $\partial^\alpha = (c^{-1}\partial_t, -\boldsymbol{\nabla})$. The relativistic velocity vector is $u^\alpha = \frac{dx^\alpha}{d\tau}$, where $(cd\tau)^2 = (cdt)^2 + d\mathbf{x}\cdot d\mathbf{x}$. The current carried by an electric charge q is $j^\alpha = qu^\alpha = (\rho, \mathbf{j}/c)$.

The contravariant components of the four-vector potential are $A^\alpha = (\Phi, \mathbf{A})$ and the covariant components are $A_\alpha = (\Phi, -\mathbf{A})$, so that $(A^1, A^2, A^3) = (A_x, A_y, A_z)$ and $(A_1, A_2, A_3) = (-A_x, -A_y, -A_z)$. The four-vector potential is used as a means to construct the electric and magnetic fields by constructing an anitsymmetric second order covariant tensor:

$$F_{\alpha,\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = -F_{\beta,\alpha} \tag{28}$$

Three other second order antisymmetric tensors can be constructed from this tensor. One of these is $F^{\alpha,\beta}$, where the indices are raised using the metric tensor g^{**}. The other two are *Hodge duals* to this first pair, obtained by raising/lowering indices using the fully antisymmetric fourth order tensor $\epsilon_{\alpha\beta\gamma\delta}$.

$$F^{\mu,\nu} = g^{\mu,\alpha}g^{\nu,\beta}F_{\alpha,\beta} \qquad {}^*F^{\mu,\nu} = \frac{1}{2}\epsilon^{\mu\nu,\alpha\beta}F_{\alpha,\beta} \qquad {}^*F_{\mu,\nu} = \frac{1}{2}\epsilon_{\mu\nu,\alpha\beta}F^{\alpha,\beta} \tag{29}$$

The electric and magnetic fields are related to these antysymmetric tensors as follows:

$$
\begin{aligned}
F_{\alpha,\beta} &= \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix} \leftrightarrow (+\mathbf{E}, -\mathbf{B}) \\
F^{\alpha,\beta} &= \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix} \leftrightarrow (-\mathbf{E}, -\mathbf{B}) \\
{}^*F^{\alpha,\beta} &= \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix} \leftrightarrow (-\mathbf{B}, +\mathbf{E}) \\
{}^*F_{\alpha,\beta} &= \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix} \leftrightarrow (+\mathbf{B}, +\mathbf{E})
\end{aligned}
\tag{30}
$$

In this notation the Maxwell Equations are

$$\partial_\alpha F^{\alpha,\beta} = 4\pi j^\beta \qquad \partial_\alpha\ {}^*F^{\alpha,\beta} = 0 \tag{31}$$

The Conservation Law for electric charge is

$$\partial_\beta \left(4\pi j^\beta\right) = \partial_\beta \partial_\alpha F^{\alpha,\beta} = 0 \tag{32}$$

This last is an identity since a symmetric tensor $\partial_\beta \partial_\alpha$ is contracted against an antisymmetric tensor $F^{\alpha,\beta}$. The Lorenz gauge is

$$\partial_\alpha A^\alpha = 0 \tag{33}$$

Finally, the Lorentz force law on a charge q is

$$\frac{dp_\alpha}{d\tau} = m\frac{du_\alpha}{d\tau} = qF_{\alpha,\beta} u^\beta \tag{34}$$

1.11 Magnetic Potential

If magnetic charges exist a second (pseudo)vector potential P must be introduced and used to construct a second order tensor:

$$P^{\alpha,\beta} = \frac{\partial P^\beta}{\partial x_\alpha} - \frac{\partial P^\alpha}{\partial x_\beta} = \partial^\alpha P^\beta - \partial^\beta P^\alpha = -P^{\beta,\alpha} \tag{35}$$

The Maxwell equations can now be constructed using

$$\mathcal{F}_{\alpha,\beta} = F_{\alpha,\beta} + \epsilon_{\alpha,\beta,\mu,\nu}\partial^\mu P^\nu = F_{\alpha,\beta} + \frac{1}{2}\epsilon_{\alpha,\beta,\mu,\nu}P^{\mu,\nu} \tag{36}$$

These equations are

$$\partial_\alpha \mathcal{F}^{\alpha,\beta} = 4\pi j_e^\beta \qquad \partial_\alpha\ {}^*\mathcal{F}^{\alpha,\beta} = 4\pi j_m^\beta \tag{37}$$

the magnetic current is defined in the same was as the electric current. The Lorentz force Law is

$$\frac{dp_\alpha}{d\tau} = \left(q_e\mathcal{F}_{\alpha,\beta} + q_m\ {}^*\mathcal{F}_{\alpha,\beta}\right)u^\beta \tag{38}$$

1.12 Dirac Monopole

In an early and very elegant argument, Dirac proposed that magnetic monopoles are consistent with the formulation of Quantum Mechanics. He argues that if a particle is transported along a closed path in space, its probability distribution remains unchanged at the beginning and end points. This means that around a closed path the wavefunction can change only by a phase factor, $e^{i\beta}$. He continues to argue that this phase factor can only be $+1$, so that β is an integer multiple of 2π. If the particle is charged then $\beta = \frac{q_e}{\hbar c}\oint \mathbf{A}\cdot d\mathbf{s}$.

If a magnetic monopole exists it is possible to represent the radial field that it produces by introducing a vector potential $\mathbf{A}$, but this can't be done globally because of a singularity at the origin: that is, $\boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi q_m\delta(\mathbf{x})$ is not consistent with a globally defined vector potential $\mathbf{A}$ that satisfies $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. So if the monopole is located at the origin, one vector function, $\mathbf{A}_u$, can be defined on the upper half of a spherical surface centered on the origin and another function, $\mathbf{A}_l$, can be defined on the lower surface. Neither function can be extended over the entire sphere, and their overlap must be nonzero. In fact, their overlap must contain a closed loop like the equator, along which the two local vector potential fields can be related.

Then he imagines enlarging one of the components, say $\mathbf{A}_u$, so that its boundary extends almost, but not quite, to the negative z axis, along which a singularity extends. If an electrically charged particle is moved around a small closed loop containing the singularity, its phase will change by $2\pi n$. On the other hand it will change by $\frac{q_e}{\hbar c}\int\int(\nabla\times\mathbf{A})\cdot d\mathbf{S}$ by Stokes' Theorem. The surface integral measures the charge contained inside the sphere, which is $4\pi q_m$. As a result, Dirac obtains

$$2\pi n = \frac{q_e}{\hbar c}4\pi q_m \tag{39}$$

So if magnetic monopoles exist:

1. Its smallest allowed value satisfies $q_m q_e = \hbar c/2$. Since $\alpha = q_e^2/\hbar c$, $q_m/q_e = 1/(2\alpha) \simeq 137/2$.

2. If a single magnetic monopole exists in the universe, then electric charge must be quantized. And vice versa, but since we can see plenty of electric charges, magnetic charges, if they exist, must be quantized.

3. The magnetic charge interaction is much stronger than the electric charge interaction: $\frac{q_m^2}{q_e^2} = \left(\frac{1}{2\alpha}\right)^2 \simeq (68)^2$, which could be a contributing factor for not (yet) having seen magnetic monopoles.

So far, searches for magnetic monopoles have provided an upper bound of less than 1 monopole per 10^{29} nucleons and a lower bound on monopole masses of 600 GeV/c^2. One of the objectives of the Inflation stage of the current cosmological paradigm is to expand the small number of the original magnetic monopoles produced in the Big Bang to such an extent that their current dilution renders them experimentally invisible.