QUANTUM MECHANICS II

PHYS 517

Problem Set # 3

Distributed: April 17, 2015

Due: April 27, 2015

Density Matrices

1. Invariance Under Cyclic Permutation: $A, B, C, \cdots$ are matrices for which all the products below make sense.

a. Show $Tr(AB) = Tr(BA)$.

b. Show $Tr(ABC \ldots Z) = Tr(BC \ldots ZA) = Tr(C \ldots ZAB) = \cdots$.

c. Show that in general $Tr(ABC) \neq Tr(ACB)$.

d. Assume $|u\rangle, |v\rangle$ are $n \times 1$ column vectors. Show $Tr|u\rangle\langle v| = \langle v|u\rangle$.

2. Density Matrices ?: Below are five matrices. State (a) which are suitable density matrices; (b) if Y, does it represent a pure state?; if Y, compute the pure state.

- eig   Y [0,1]

$$\rho_1 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix} \qquad \rho_2 = \begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

$\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$ 0, $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ 1 $\begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$

$$\rho_3 = \frac{1}{3}|u\rangle\langle u| + \frac{2}{3}|v\rangle\langle v| + \frac{\sqrt{2}}{3}|v\rangle\langle u| + \frac{\sqrt{2}}{3}|u\rangle\langle v|$$

no, cyclic perm, $\langle v|\frac{1}{3}|u\rangle$

where $\langle u|u\rangle = \langle v|v\rangle = 1, \langle v|u\rangle = 0.$

N $\# |u\rangle\langle v|$
$= \# \langle v|u\rangle = 0$

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$$\rho_4 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 \end{bmatrix} \qquad \rho_5 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}$$

3. Two State Systems: Consider a dynamical variable that can take only two values: $+1$ and -1. The eigenvectors of the corresponding operator are denoted $|+\rangle$ and $|-\rangle$. Now consider the following states:

$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ evecs: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$|+\rangle$ $|-\rangle$

$(1-\lambda)(-1-\lambda)$

$\lambda = 1, -1$

the one-parameter family of pure states that are represented by the vectors $|\theta\rangle = \sqrt{\frac{1}{2}}\left(|+\rangle + e^{i\theta}|-\rangle\right)$ for arbitrary angle θ; and the nonpure state $\rho = \frac{1}{2}\left(|+\rangle\langle+| + |-\rangle\langle-|\right)$. Pauli matrices

a. Show that $\langle\sigma\rangle = 0$ for all these states.

b. What, if any, are the physical differences between these states, and how could they be measured?

4. Spin: $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. For a state represented by $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ with as usual $\alpha^*\alpha + \beta^*\beta = 1$, calculate the probability that the spin component is positive.

5. Conditioned Probabilities: Suppose that the operator

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

evecs $\begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}_{-\sqrt{2}}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}_{0}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}_{\sqrt{2}}$

represents a dynamical variable. Calculate the probability $P(M = 0|\rho_i)$ for the following state operators:

$\langle e_2|\rho_1|e_2\rangle = \frac{3}{8}$

$\langle e_2|\rho_2|e_2\rangle = 0$

$\langle e_2|\rho_3|e_2\rangle = \frac{1}{2}$

$P(M=0|\rho_1) = \frac{1}{4}$ $\quad P(M=0|\rho_2) = 0$ $\quad P(M=0|\rho_3) = \frac{1}{2}$

$$\rho_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \qquad \rho_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad \rho_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

6. Two Ways to Compute: Let $R = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$ represent a dynamical variable, and $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ be an arbitrary state vector with $|a|^2 + |b|^2 = 1$. Calculate $\langle R^2\rangle$ in two ways:

a. Evaluate $\langle R^2\rangle = \langle\psi|R^2|\psi\rangle$ directly.

b. Find the eigenvalues and eigenvectors of R, $R|\phi_n\rangle = r_n|\phi_n\rangle$, expand the state vector as a linear combination of these eigenvectors $|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$, and evaluate

$$\langle R^2\rangle = |c_1|^2 r_1^2 + |c_2|^2 r_2^2.$$

Are the results from **a** and **b** the same? (Hint: Yes or bust!)