Quantum III HW2

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1 Problem 1

We first examine the inner product $\langle\psi_n|W|\psi_i\rangle$, using the ladder operator relation $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$.

$$\langle\psi_n|W|\psi_i\rangle = -qE\sqrt{\frac{\hbar}{2m\omega}}\,\langle\psi_n|(a + a^\dagger)|\psi_n\rangle \tag{1.1}$$

$$\langle\psi_n|W|\psi_i\rangle = -qE\sqrt{\frac{\hbar}{2m\omega}}(\langle\psi_n|\psi_{i-1}\rangle + \langle\psi_n|\psi_{i+1}\rangle) \tag{1.2}$$

$$\langle\psi_n|W|\psi_i\rangle = -qE\sqrt{\frac{\hbar}{2m\omega}}(\delta_{n,n-1} + \delta_{n,n+1}) \tag{1.3}$$

a) We examine the first-order perturbation theory result for C_{10}:

$$C^1_{01} = -\frac{i}{\hbar}\int_0^t dt' e^{i\omega_{10}t'}\,\langle\psi_1|W|\psi_0\rangle \tag{1.4}$$

$$C^1_{01} = qE\sqrt{\frac{\hbar}{2m\omega_0}}\frac{1}{E_1 - E_0}\left(e^{\frac{i}{\hbar}(E_1-E_0)\tau} - 1\right) \tag{1.5}$$

$$C^1_{01} = qE\sqrt{\frac{1}{2m\hbar\omega_0^3}}\left(e^{i\omega_0\tau} - 1\right) \tag{1.6}$$

$$|C^1_{01}|^2 = \frac{(qE)^2}{2m\hbar\omega_0^3}\left(2 - (e^{i\omega_0\tau} + e^{-i\omega_0\tau}\right) \tag{1.7}$$

$$|C^1_{01}|^2 = \frac{(qE)^2}{m\hbar\omega_0^3}\left(1 - \cos\omega_0\tau\right) \tag{1.8}$$

The probabiltiy varies at the harmonic ocsillator amgular frequency.

b) We now examine C_{20}. We have seen that $\langle\psi_2|W|\psi_0\rangle = 0$, so the first-order approximation will be 0. For the second-order perturbation results, $\langle\psi_2|W|\psi_1\rangle = \langle\psi_1|W|\psi_0\rangle = -qE\sqrt{\frac{\hbar}{2m\omega}}$. So only the transition through intermediate state $|1\rangle$ will contribute.

$$C_{01}^2 = -\frac{1}{\hbar^2}\int_0^\tau dt' e^{i\omega_{21}t'} W_{21} \int_0^{t'} dt' e^{i\omega_{10}t'} W_{10} \tag{1.9}$$

$$C_{01}^2 = \frac{iq^2E^2}{2m\omega_0^2\hbar}\int_0^\tau dt' e^{i2\omega_0 t'} - e^{i\omega_0 t'} \tag{1.10}$$

$$C_{01}^2 = \frac{q^2E^2}{2m\omega_0^3\hbar}\left(\frac{e^{i2\omega_0\tau}-1}{2} - e^{i\omega_0\tau}+1\right) \tag{1.11}$$

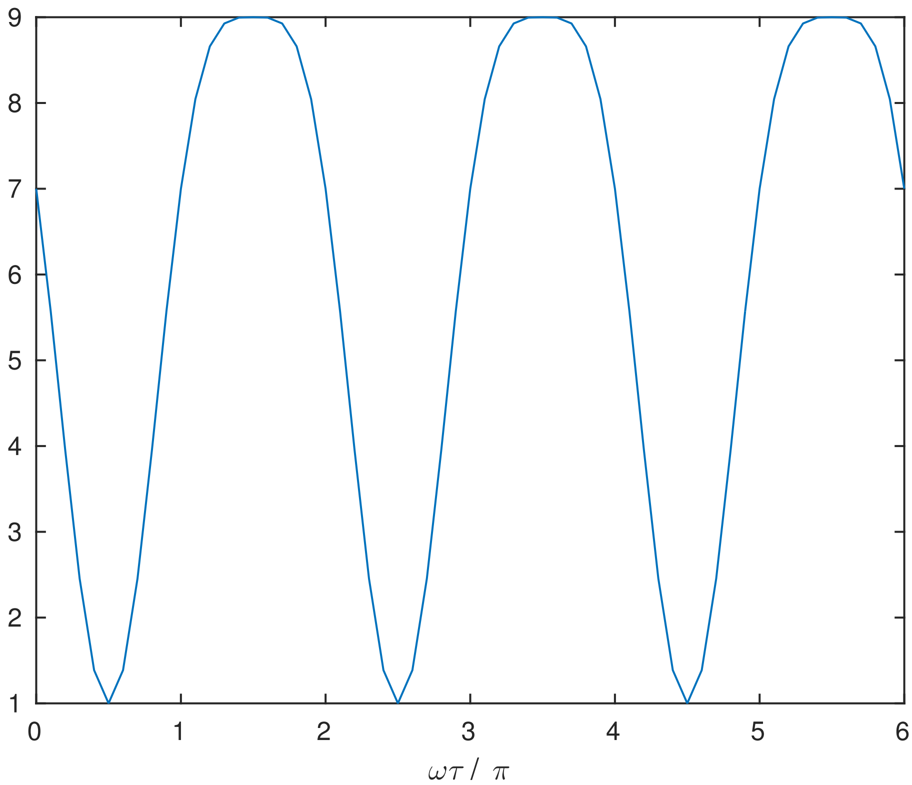
$$C_{01}^2 = \frac{q^2E^2}{4m\omega_0^3\hbar}\left(e^{i\omega_0\tau}-1\right)^2 \tag{1.12}$$

$$|C_{01}^2|^2 = \left(\frac{q^2E^2}{4m\omega_0^3\hbar}\right)^2\left(e^{i\omega_0\tau}-1\right)^2 \tag{1.13}$$

$$|C_{01}^2|^2 = \left(\frac{q^2E^2}{4m\omega_0^3\hbar}\right)^2\left(e^{i2\omega_0\tau}+e^{-i2\omega_0\tau}-4(e^{i\omega_0\tau}+e^{-i\omega_0\tau})+6\right) \tag{1.14}$$

$$|C_{01}^2|^2 = \left(\frac{q^2E^2}{4m\omega_0^3\hbar}\right)^2\left(2\cos 2\omega_0\tau - 8\cos\omega_0\tau+6\right) \tag{1.15}$$

A scaled version of this function is plotted below. The probability again oscillates at the harmoic oscillator frequency ω_0.



2 Problem 2

Wigner and Weisskopf analyze a time-dependent perturbation of a stationary system in the interaction picture. They find an energy shift as well as a line broadening using second-order results. Fano found the stationary states of a system with an interaction potential between a discrete state and a comtinuum of states using the Schrodinger picture. Fano's exact result using the Schordinger picture determines that the discrete state is diluted across a range of continuum states near the discrete-state energy.
If we consider the discrete-continuum interaction terms in the Hamiltonian as a time-dependent perturbation we can view the Wigner-Weisskopf results as an approximation of Fano's results. However, the time-dependent perturbation results as presented by Wigner-Weisskopf has a decay width term:

$$\Gamma = 2\pi \sum_{m \neq i} |V_{mi}|^2 \delta(E_i - E_m) \tag{2.1}$$

Unlike Fano's result, there is no sudden phase change as the continuum energy passes through the discrete state energy, so Wigner-Weisskopf does not predict the asymmetric absorption peaks that Fano explained.

3 Problem 3

We follow Fano's derivation of the factorization:

$$\begin{aligned}\frac{1}{(\bar{E} - E')(E - E')} =& \frac{1}{\bar{E} - E'} \left(\frac{1}{E - E'} - \frac{1}{\bar{E} - E'} \right) \\ &+ \pi^2 \delta(\bar{E} - E) \delta(E' - \frac{1}{2}(\bar{E} + E))\end{aligned} \tag{3.1}$$

We begin by determing the Fourier expansion of $\frac{1}{E - E'}$.

$$\frac{1}{E - E'} = \int_{-\infty}^{\infty} \hat{f}(k) e^{i2\pi k E'} dk \tag{3.2}$$

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{1}{E - E'} e^{-i2\pi k E'} dE' \tag{3.3}$$

We evaluate 3.3 with its single pole using the residue theorem. When $k > 0$ we will close the contour in the upper half of the complex plane, and we will avoid the pole using a counter-clockwise circle. When $k < 0$ we will close the contour in the lower half of the complex plane, and we will avoid the pole using a clockwise circle. This leads to:

$$P(\int) - i\pi Res = 0 \ (k < 0) \tag{3.4}$$

$$P(\int) + i\pi Res = 0 \ (k > 0) \tag{3.5}$$

The residue at the simple pole can be evaluated using $lim(E' \to E)(E - E')\frac{e^{-i2\pi kE'}}{E-E'} = e^{-i2\pi kE}$. Writing the sign of k as $\frac{k}{|k|}$, and noting that $ie^{i\theta} = e^{-i\theta}$, we have proved Fano's equation A1:

$$\frac{1}{E - E'} = -i\pi \int_{-\infty}^{\infty} \frac{k}{|k|} e^{i2\pi k(E-E')} dk \tag{3.6}$$

We can now write the double-pole expression as:

$$\frac{1}{(\bar{E} - E')(E - E')} = -\pi^2 \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \frac{kk'}{|kk'|} e^{i2\pi[k(\bar{E}-E')+k'(E-E')]} \tag{3.7}$$

Fano now makes the substitution $u = k + k', \ v = \frac{1}{2}(k - k')$ and finds that:

$$u^2 = k^2 + 2kk' + k'^2 \tag{3.8}$$

$$4v^2 = k^2 - 2kk' + k'^2 \tag{3.9}$$

$$\frac{kk'}{|kk'|} = \frac{u^2 - 4v^2}{|u^2 - 4v^2|} \tag{3.10}$$

This expression will be -1 for $u^2 < 4v^2$ and +1 for $u^2 > 4v^2$, so it is equivalent to $2St(u^2 - 4v^2) - 1$ where $St()$ is the step function. This substitution allows us to write equation7 as:

$$\begin{aligned} \frac{1}{(\bar{E} - E')(E - E')} =& \pi^2 \int_{-\infty}^{\infty} du \ e^{i2\pi u[\frac{1}{2}(\bar{E}+E)-E')]} \\ & \times \left(\int_{-\infty}^{\infty} dv - 2 \int_{-\frac{1}{2}|u|}^{\frac{1}{2}|u|} dv \right) e^{i2\pi v(\bar{E}-E)} \end{aligned} \tag{3.11}$$

Using the exponential delta function $\delta(x - a) = \int e^{x-a} dx$

4 Problem 4

We start from Fano eq. 10:

$$\begin{aligned} a^*(\bar{E})\{1 + \int dE' V_{E'}^* &\left(\frac{1}{\bar{E} - E'} + z(\bar{E})\delta(\bar{E} - E' \right) \\ &\times \left(\frac{1}{E - E'} + z(E)\delta(E - E') \right) V_{E'}\} a(E) \\ &= \delta(\bar{E} - E) \end{aligned} \tag{4.1}$$

Opening up the expression under the integral we find:

$$\begin{aligned} a^*(\bar{E})a(E)\{1 &+ \int |V_{E'}|^2 \frac{1}{(\bar{E}-E')(E-E')} dE' \\ &+ \int |V_{E'}|^2 \frac{z(\bar{E})\delta(\bar{E}-E')}{\bar{E}-E'} dE' \\ &+ \int |V_{E'}|^2 \frac{z(E)\delta(E-E')}{E-E'} dE' \\ &+ \int |V_{E'}|^2 z(\bar{E})z(E)\delta(\bar{E}-E')\delta(E-E')dE' \\ &= \delta(\bar{E}-E) \end{aligned} \tag{4.2}$$

The first expression is expanded using the result of problem 3 and the definition of $F(E) = P\int dE' \frac{|V_{e'}|^2}{E-E'}$.

$$\begin{aligned} \int |V_{E'}|^2 \frac{1}{(\bar{E}-E')(E-E')} dE' = \frac{|V_E|^2}{\bar{E}-E} \left(F(E) - F(\bar{E})\right) \\ + \pi^2 \delta(\bar{E}-E) \end{aligned} \tag{4.3}$$

The second and third expressions in 4.2 can be combined into $\frac{1}{\bar{E}-E}\left(z(E)|V_E^2 - z(\bar{E})|V_{\bar{E}}|^2\right)$. Using $\delta(\bar{E}-E')\delta(E-E') = \delta(\bar{E}-E)\delta(E' - \frac{1}{2}(\bar{E}+E))$ the fourth expression is:

$$\int |V_{E'}|^2 z(\bar{E})z(E)\delta(\bar{E}-E')\delta(E-E')dE' = |V_E|^2 z(E)^2 \delta(\bar{E}-E) \tag{4.4}$$

We can now collect terms:

$$\begin{aligned} &|a(E)|^2|V_E|^2(\pi^2 + z(E)^2)\delta(\bar{E}-E) + a^*(\bar{E})a(E) \\ &\quad \times \left\{1 + \frac{1}{\bar{E}-E}\left(F(E) - F(\bar{E}) + z(E)|V_E|^2 - z(\bar{E})|V_{\bar{E}}|^2\right)\right\} \\ &= \delta(\bar{E}-E) \end{aligned} \tag{4.5}$$

Since $F(E) = E - z(E)|V_E|^2 - E_\phi$, the term inside the brackets reduces to $1 + \frac{E-\bar{E}}{\bar{E}-E} = 0$ so the second term vanishes. We then have:

$$|a(E)|^2 = \frac{1}{|V_E|^2(\pi^2 + z(E)^2)} = \frac{|V_E|^2}{(E - E_\phi - F(E))^2 + \pi^2|V_E|^4} \tag{4.6}$$

5 Problem 5