Math Phys II HW 2

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Abstract

1 Problem 1

We seek solutions of the Kortweg-deVries equation:

$$\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi}{\partial x} + \frac{\partial^3\psi}{\partial x^3} = 0 \tag{1.1}$$

We look for solutions $\psi(\xi)$, with $\xi = x - ct$. To write 1.1 in terms of ξ, we calculate the partial derivatives:

$$\frac{\partial\psi}{\partial t} = -c\frac{d\psi(\xi)}{d\xi}$$
$$\frac{\partial\psi}{\partial x} = \frac{d\psi(\xi)}{d\xi}$$
$$\frac{\partial^3\psi}{\partial x^3} = \frac{d^3\psi(\xi)}{d\xi^3}$$

We can now write 1.1 in terms of ξ:

$$-c\frac{d\psi}{d\xi} + \psi\frac{d\psi}{d\xi} + \frac{d^3\psi}{d\xi^3} = 0 \tag{1.2}$$

This simplifies to:

$$(\psi - c)\frac{d\psi}{d\xi} + \frac{d^3\psi}{d\xi^3} = 0 \tag{1.3}$$

We can integrate 1.3 to find:

$$\frac{d^2\psi}{d\xi^2} = c\psi - \frac{\psi^2}{2} \tag{1.4}$$

We then integrate again and multiply by $\frac{d\psi}{d\xi}$:

$$\frac{d\psi}{d\xi} = \int(c\psi - \frac{\psi^2}{2}) \tag{1.5}$$

$$(\frac{d\psi}{d\xi})^2 = \frac{\psi^2}{2}(c - \frac{\psi}{3}) \tag{1.6}$$

$$\frac{d\psi}{d\xi} = \frac{\psi}{\sqrt{2}}(c - \frac{\psi}{3})^{\frac{1}{2}} \tag{1.7}$$

We can now integrate for ξ using Wolfram Alpha:

$$\xi = \int \frac{d\psi}{\frac{\psi}{\sqrt{2}}(c-\frac{\psi}{3})^{\frac{1}{2}}} = \frac{2\sqrt{2}\ tanh^{-1}(\frac{\sqrt{c-\frac{\psi}{3}}}{\sqrt{c}})}{\sqrt{c}} \tag{1.8}$$

And then rearrange to find ψ as a function of ξ and c.

$$\xi\sqrt{c} = 2\sqrt{2}\ tanh^{-1}(\sqrt{1-\frac{\psi}{3c}}) \tag{1.9}$$

$$\frac{\xi\sqrt{c}}{2\sqrt{2}} = tanh^{-1}(\sqrt{1-\frac{\psi}{3c}}) \tag{1.10}$$

$$tanh(\frac{\xi\sqrt{c}}{2\sqrt{2}}) = \sqrt{1-\frac{\psi}{3c}} \tag{1.11}$$

$$tanh^2(\frac{\xi\sqrt{c}}{2\sqrt{2}}) = 1-\frac{\psi}{3c} \tag{1.12}$$

$$\psi = 3c\{1-tanh^2(\frac{\xi\sqrt{c}}{2\sqrt{2}})\} \tag{1.13}$$

$$\psi = \frac{3c}{cosh^2(\frac{\xi}{2}\sqrt{\frac{c}{2}})} \tag{1.14}$$

2 Problem 2

The general form of a second-order linear PDE is:

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x\partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} \tag{2.1}$$

The characteristic equation, with solutions $\xi(x,y)$ and $\eta(x,y)$, is:

$$A(\frac{dy}{dx})^2 + 2B(\frac{dy}{dx}) + C = 0 \tag{2.2}$$

We wish to write Eq. 1 in terms of ξ and η. We differentiate $\psi(\xi,\eta)$:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x} \tag{2.3}$$

Now we calculate the other partials with respect to η and ξ.

$$\frac{\partial}{\partial x}(\frac{\partial\psi}{\partial\xi}) = \frac{\partial^2\psi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x} \tag{2.4}$$

$$\frac{\partial}{\partial x}(\frac{\partial\psi}{\partial\eta}) = \frac{\partial^2\psi}{\partial\eta^2}\frac{\partial\eta}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x} \tag{2.5}$$

We use 3,4 and 5 to calculate $\frac{\partial^2\psi}{\partial x^2}$.

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial\xi}{\partial x}(\frac{\partial^2\psi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x})$$
$$+\frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial\eta}{\partial x}(\frac{\partial^2\psi}{\partial\eta^2}\frac{\partial\eta}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x})$$
$$= \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2 + \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta} + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 + \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 + 2(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta}) \quad (2.6)$$

The calculation of $\frac{\partial^2\psi}{\partial y^2}$ is identical.

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\xi}{\partial y^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial y^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial y})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial y})^2 + 2(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2\psi}{\partial\xi\partial\eta}) \quad (2.7)$$

We now take $\frac{\partial}{\partial y}$ of equation 1:

$$\begin{aligned}\frac{\partial^2\psi}{\partial x\partial y} = {} & \frac{\partial^2\xi}{\partial x\partial y}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x\partial y}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}) \\ & + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}) + \frac{\partial^2\psi}{\partial\xi\partial\eta}(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y}) \quad (2.8)\end{aligned}$$

We can now write Eq. 1 in terms of ξ and η:

$$\begin{aligned}A\{ & \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 + 2(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta})\} \\ & + 2B\{\frac{\partial^2\xi}{\partial x\partial y}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x\partial y}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}) \\ & + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}) + \frac{\partial^2\psi}{\partial\xi\partial\eta}(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y})\} \\ & + C\{\frac{\partial^2\xi}{\partial y^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial y^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial y})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial y})^2 + 2(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2\psi}{\partial\xi\partial\eta})\} = 0\end{aligned} \quad (2.9)$$

We now take a break to stop Eq. 9 from giving us a migraine brought on by eye strain.

We collect the coefficients of all the derivates of ψ:

$$\frac{\partial\psi}{\partial\xi}(A\frac{\partial^2\xi}{\partial x^2} + 2B\frac{\partial^2\xi}{\partial x\partial y} + C\frac{\partial^2\xi}{\partial y^2}) \quad (2.10)$$

$$\frac{\partial\psi}{\partial\eta}(A\frac{\partial^2\eta}{\partial x^2} + 2B\frac{\partial^2\eta}{\partial x\partial y} + C\frac{\partial^2\eta}{\partial y^2}) \quad (2.11)$$

$$\frac{\partial^2\psi}{\partial\xi^2}(A(\frac{\partial\xi}{\partial x})^2 + 2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + C(\frac{\partial\xi}{\partial y})^2) \quad (2.12)$$

$$\frac{\partial^2\psi}{\partial\eta^2}(A(\frac{\partial\eta}{\partial x})^2 + 2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + C(\frac{\partial\eta}{\partial y})^2) \quad (2.13)$$

$$\frac{\partial^2\psi}{\partial\xi\partial\eta}(2A(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}) + 2B(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}) + 2C(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y})) \quad (2.14)$$

We recognize that since $\xi(x,y)$ and $\eta(x,y)$ are solutions to Eq. 2.2, along a characteristic curve:

$$\xi(x,y) = constant \tag{2.15}$$
$$\frac{dy}{dx} = \frac{\partial \xi}{\partial x}(-\frac{\partial \xi}{\partial y})^{-1} \tag{2.16}$$
$$A(\frac{\partial \xi}{\partial x}(-\frac{\partial \xi}{\partial y})^{-1})^2 + 2B(\frac{\partial \xi}{\partial x}(-\frac{\partial \xi}{\partial y})^{-1}) + C = 0 \tag{2.17}$$
$$A(\frac{\partial \xi}{\partial x})^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C(\frac{\partial \xi}{\partial y})^2 = 0 \tag{2.18}$$
$$\tag{2.19}$$

The same argument also implies:

$$A(\frac{\partial \eta}{\partial x})^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C(\frac{\partial \eta}{\partial y})^2 = 0 \tag{2.20}$$

So 2.12 and 2.13 are both equal to 0 and we've removed most second derivates of ψ. We now collect the $\frac{\partial^2}{\partial \xi \partial \eta}$ on the left and everything else on the right.

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta}(2A(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}) + 2B(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}) + 2C(\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y})) =$$
$$\frac{\partial \psi}{\partial \xi}(A\frac{\partial^2 \xi}{\partial x^2} + 2B\frac{\partial^2 \xi}{\partial x \partial y} + C\frac{\partial^2 \xi}{\partial y^2}) + \frac{\partial \psi}{\partial \eta}(A\frac{\partial^2 \eta}{\partial x^2} + 2B\frac{\partial^2 \eta}{\partial x \partial y} + C\frac{\partial^2 \eta}{\partial y^2}) \tag{2.21}$$

Dividing the awful mess on the right side by the slightly-less-awful mess on the left:

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta} =$$
$$\frac{\frac{\partial \psi}{\partial \xi}(A\frac{\partial^2 \xi}{\partial x^2} + 2B\frac{\partial^2 \xi}{\partial x \partial y} + C\frac{\partial^2 \xi}{\partial y^2}) + \frac{\partial \psi}{\partial \eta}(A\frac{\partial^2 \eta}{\partial x^2} + 2B\frac{\partial^2 \eta}{\partial x \partial y} + C\frac{\partial^2 \eta}{\partial y^2})}{2(A(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}) + B(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}) + C(\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}))} \tag{2.22}$$

The right-hand side depends only on first derivates of ψ, the known functions A/B/C, and known derivatives of ξ and η.

3 Problem 3

We are solving the characteristic equation for:

$$\frac{\partial^2 \psi}{\partial t^2} - c(x)^2\frac{\partial^2 \psi}{\partial x^2} = 0$$

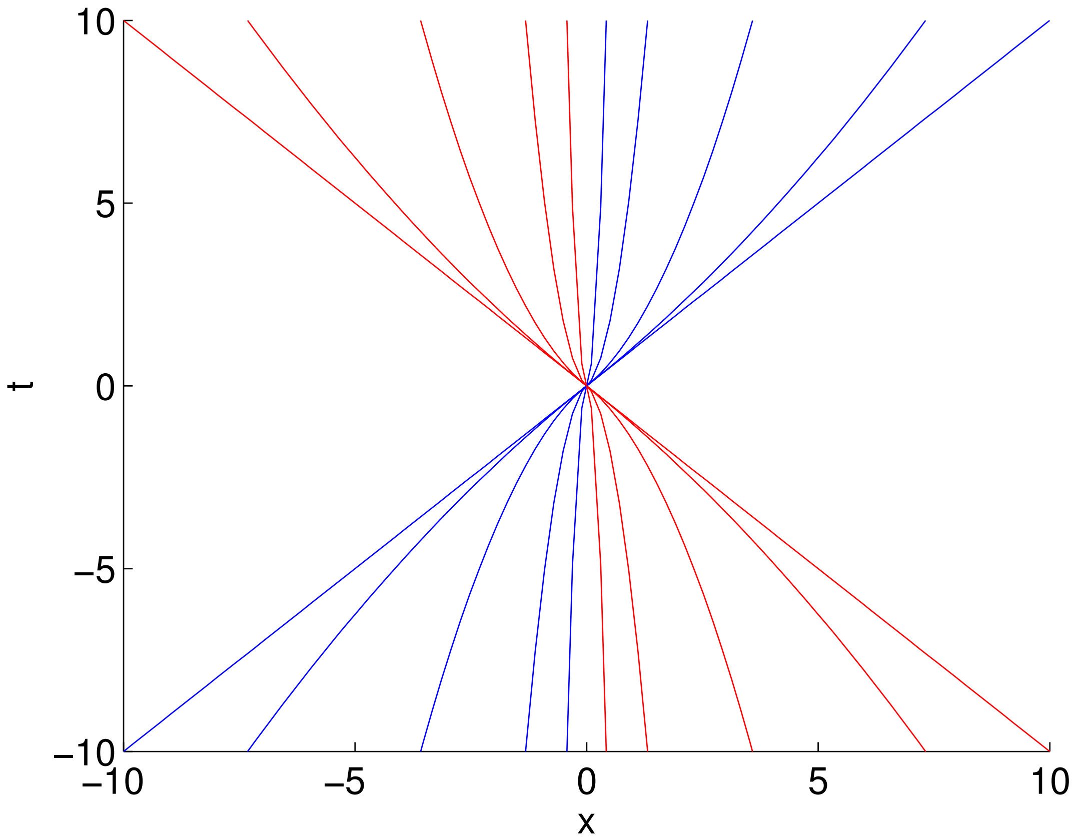
With A=1, B=0, and $c = -c(x)^2$, the characteristic equation is:

$$(\frac{dx}{dt})^2 - c(x)^2 = 0 \tag{3.1}$$
$$\frac{dx}{dt} = \pm c(x) \tag{3.2}$$
$$dt = \pm\frac{1}{c(x)}dx \tag{3.3}$$

With $c(x) = c_0(1 + \frac{|x|}{a})$ the characteristic curve can be written:

$$t = \pm\frac{1}{c_0}(x + sgn(x)\frac{x^2}{2a}) + C \tag{3.4}$$

Several characteristic curves are shown below for a-values 0.01, 0.1, 1, 10 and 10^6. The positive curves are shown in blue, the negative curves in red. The large value of a produces nearly straight lines.



We now look to find a solution given the initial conditions:

$$\psi(x, 0) = 0 \tag{3.5}$$

$$\frac{\partial\psi}{\partial t}\,|_{t=0}= e^{-|x|} \tag{3.6}$$

When $a = \infty$ the characteristic solutions become:

$$\xi = x + c_0t \tag{3.7}$$

$$\eta = x - c_0t \tag{3.8}$$

These characteristic curves are consistent with the large value of a plotted above. The solution can be written as a combination $\psi = f(\xi) + g(\eta)$. Using the initial condition $\psi(x, 0) = 0$ we see that $f(x) + g(x) = 0$, so that $g(x) = -f(x)$. We also note that, since $x > 0$, $e^{-|x|} = e^{-x}$. We differentiate the combined solution with respect to t and use the second boundary condition:

$$-v\frac{df}{dt} + c_0\frac{dg}{dt} = e^{-x} \tag{3.9}$$
$$\int -2\frac{df}{dt} = \int \frac{1}{c_0}e^{-x} \tag{3.10}$$
$$f = \frac{-1}{2c_0}e^{-x} \tag{3.11}$$

We can now write the combined solution:

$$\psi(x,t) = \frac{1}{2c_0}(e^{-x+vt} - e^{-x-vt}) \tag{3.12}$$

4 Problem 4

We solve the diffusion equation for the temperature in a uniform cube of side L.

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{4.1}$$

We first separate T into time and space terms.

$$T = \Gamma(t)\Psi(\vec{r}) \tag{4.2}$$
$$\nabla^2 T = \nabla^2\Psi\Gamma \tag{4.3}$$
$$\frac{\partial T}{\partial t} = \Psi\Gamma' \tag{4.4}$$
$$\frac{\nabla^2\Psi}{\Psi} - \frac{1}{\kappa}\frac{\Gamma'}{\Gamma} = 0 \tag{4.5}$$
$$\frac{\nabla^2\Psi}{\Psi} = -\frac{1}{\kappa}\frac{\Gamma'}{\Gamma} = v^2 \tag{4.6}$$
$$\tag{4.7}$$

We can now solve the time dependent part.

$$\Gamma' = \Gamma k v^2 \tag{4.8}$$
$$\Gamma' - kv^2\Gamma = 0 \tag{4.9}$$
$$\Gamma = \alpha e^{-tkv^2} \tag{4.10}$$

The boundary condition $\Gamma = 0$ at $t = 0$ can't be satisfied by this equation, so we shift the temperature scale so that the system starts at $T = -T_0$ and the heat bath is at a temperature of 0. We can then solve for α.

$$\alpha = -T_0 \tag{4.11}$$
$$\Gamma = -T_0 e^{-tkv^2} \tag{4.12}$$

We now separate the X/Y/Z components of the spatial function.

$$\Psi(\vec{r}) = X(x)Y(y)Z(z) \tag{4.13}$$

$$\frac{\nabla^2\Psi}{\Psi} = v^2 \tag{4.14}$$

$$\nabla^2\Psi - v^2\Psi = 0 \tag{4.15}$$

$$X^{''}YZ + XY^{''}Z + XYZ^{''} + v^2XYZ = 0 \tag{4.16}$$

$$\frac{X^{''}}{X} + \frac{Y^{''}}{Y} + \frac{Z^{''}}{Z} - v^2 = 0 \tag{4.17}$$

$$X^{''} + a^2X = 0 \tag{4.18}$$

$$Y^{''} + b^2Y = 0 \tag{4.19}$$

$$Z^{''} + c^2Z = 0 \tag{4.20}$$

$$a^2 + b^2 + c^2 = v^2 \tag{4.21}$$

The spatial functions are all zero at both 0 and L (since we've shifted the temperature scale so the heat bath is at 0). The solutions are:

$$X = sin\frac{\ell\pi}{L}x \tag{4.22}$$

$$Y = sin\frac{m\pi}{L}y \tag{4.23}$$

$$Z = sin\frac{n\pi}{L}z \tag{4.24}$$

$$a = \frac{\ell L}{\pi},\ b = \frac{mL}{\pi},\ \gamma = \frac{nL}{\pi} \tag{4.25}$$

We can now write the general solution to the equation as a linear combination of the separate solutions.

$$T = \sum_{\ell,m,n=0}^{\infty} v^2_{\ell mn}(-T_0)e^{-tkv^2_{\ell mn}}\ sin(\frac{\ell\pi}{L}x)\ sin(\frac{m\pi}{L}y)\ sin(\frac{n\pi}{L}z) \tag{4.26}$$

With ℓ, m, n integers and subject to $\frac{\pi^2}{L^2}(\ell^2 + m^2 + n^2) = v^2$. We can use the boundary condition at t=0 to find $v^2_{\ell mn}$. Since the XYZ terms are all sines we use the tyical method for finding coefficients of a Fourier series.

$$v^2_{\ell mn} = (\frac{2}{L})^3 \int_0^L \int_0^L \int_0^L dx\ dy\ dz\ \ -T_0 sin(\frac{\ell\pi}{L}x)\ sin(\frac{m\pi}{L}y)\ sin(\frac{n\pi}{L}z) \tag{4.27}$$

$$v^2_{\ell mn} = \frac{-T_0 8}{L^2\pi}\{\frac{1}{\ell}(1-(-1)^\ell)\ \frac{1}{m}(1-(-1)^m)\ \frac{1}{n}(1-(-1)^n)\} \tag{4.28}$$

5 Problem 5

We solve the wave equation for a particle confined to a cylinder of radius R_0 and height H.

$$\frac{-\hbar^2}{2m}\nabla^2\psi = E\psi \tag{5.1}$$

$$\nabla^2\psi + k^2\psi = 0, k \equiv \sqrt{\frac{2mE}{\hbar^2}} \tag{5.2}$$

We separate the equation into functions in cylindrical coordinates.

$$\psi = R(r)\Phi(\phi)Z(z) \tag{5.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{dR}{dr})\Phi Z + \frac{1}{r^2}\frac{d\Phi^2}{d\phi}RZ + \frac{d^2Z}{dz}R\Phi + k^2 r\Phi Z = 0 \tag{5.4}$$

$$\frac{1}{rR}\frac{d}{dr}(rR^{'}) + \frac{1}{r^2}\frac{\Phi^{''}}{\Phi} + \frac{Z^{''}}{Z} + k^2 = 0 \tag{5.5}$$

We can now solve for Z incorporating the boundary conditions that $\psi = 0$ at z=0,H.

$$\frac{Z^{''}}{Z} = const. = a^2 \tag{5.6}$$

$$Z^{''} - a^2 Z = 0 \tag{5.7}$$

$$Z = sin\frac{\ell\pi}{H}z,\ a^2 = (\frac{\ell\pi}{H})^2 \tag{5.8}$$

Substituing the constant value of $\frac{Z^{''}}{Z}$ into 5.5 and multiplying by r^2, we get:

$$\frac{r}{R}\frac{d}{dr}(rR^{'}) + \frac{\Phi^{''}}{\Phi} + r^2(a^2 + k^2) = 0 \tag{5.9}$$

we can now separate the Φ term.

$$\frac{\Phi^{''}}{\Phi} = const. = -m^2 \tag{5.10}$$

$$\Phi^{''} + m^2\Phi = 0 \tag{5.11}$$

There is no explicit boundary condition on ϕ. However, since ϕ is an angular coordinate Φ must be single-valued on multiples of 2π. With $\Phi(0) = \Phi(2\pi)$, the solution is a general exponential with m confined to integer values.

$$\Phi = e^{im\phi},\ m = 0, 1, 2... \tag{5.12}$$

Using the constant value of $\frac{\Phi^{''}}{\Phi}$ in 5.7, we now solve the radial part.

$$\frac{r}{R}\frac{d}{dr}(rR^{'}) + \frac{\Phi^{''}}{\Phi} + r^2(a^2 + k^2) = 0 \tag{5.13}$$

$$\frac{r}{R}\frac{d}{dr}(rR^{'}) - m^2 + r^2n^2 = 0,\ n^2 \equiv a^2 + k^2 \tag{5.14}$$

$$\frac{r}{R}(R^{'} + rR'') - m^2 + r^2n^2 = 0 \tag{5.15}$$

$$r^2R^{''} + rR^{'} + R(r^2n^2 - m^2) = 0 \tag{5.16}$$

This is not quite the right form for Bessel's equation, so we make the substitution $x = nr$.

$$x = nr \tag{5.17}$$

$$x\frac{dR}{dx} = nr\frac{1}{n}\frac{dR}{dr} = r\frac{dR}{dr} \tag{5.18}$$

$$x^2\frac{d^2R}{dx^2} = (nr)^2(\frac{1}{n})^2\frac{d^2R}{dr^2} = r^2\frac{d^2R}{dr^2} \tag{5.19}$$

$$x^2R^{''} + xR^{'} + R(x^2 - m^2) = 0 \tag{5.20}$$

The solutions are Bessel's functions $J_m(x) = J_m(nr)$. The boundary condition $\Psi = 0$ at R_0 restricts the values of n to $\frac{\alpha_{ms}}{R_0}$, with α_{ms} the s^{th} root of J_m. The energy levels can be found from eq. 5.2:

$$n = \frac{\alpha_{ms}}{R_0} \tag{5.21}$$

$$k^2 = \frac{2mE}{\hbar^2} = n^2 + a^2 = (\frac{\alpha_{ms}}{R_0})^2 + (\frac{\ell\pi}{H})^2 \tag{5.22}$$

$$E = \frac{\hbar^2}{2m}\{(\frac{\alpha_{ms}}{R_0})^2 + (\frac{\ell\pi}{H})^2\} \tag{5.23}$$

The minimum energy level is $\alpha_{01} = 2.4048$, $\ell = 1$. The minimum energy is therefore:

$$E_0 = \frac{\hbar^2}{2m}\{(\frac{2.4048}{R_0})^2 + (\frac{\pi}{H})^2\} \tag{5.24}$$

In the ground state m=0 so the ϕ term is 1. The ground state wavefunction is:

$$\Psi_{01} = J_0(\frac{\alpha_{01}r}{R_0})\ sin(\frac{\pi}{H}z) \tag{5.25}$$

We solve a 2-D semicircle of radius R_0 using the same method. There is no z component, so $a^2 = 0,\ n^2 = k^2$.

The boundary conditions on ϕ have changed so that $\Phi(\pi) = \Phi(0) = 0$. The solution for Φ is no longer a general exponential, but is:

$$\Phi(\phi) = sin(m\phi),\ m = 1, 2, 3... \tag{5.26}$$

The energy levels are:

$$k^2 = \frac{2mE}{\hbar^2} = n^2 = (\frac{\alpha_{ms}}{R_0})^2 \tag{5.27}$$

$$E = \frac{\hbar^2}{2m}(\frac{\alpha_{ms}}{R_0})^2 \tag{5.28}$$

The minimum energy level will occur at $\alpha_{11} = 3.8317$. The ground state wavefunction is:

$$\Psi_{11} = J_1(\frac{\alpha_{11}r}{R_0})\ sin(\phi) \tag{5.29}$$