# PH.D. QUALIFYING EXAM SOLUTIONS
# 2000

MATT THIESSE

## Classical

**Problem 1.** *A projectile is fired from a gun at an angle $\theta$ from horizontal. What is the maximum firing angle $\theta_m$ for which the gun-projectile distance never decreases?*

Writing down the equations of motion in the $x$-direction,

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 = v_0 \cos\theta t$$

Likewise, in the $y$-direction,

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = v_0 \sin\theta t - \frac{1}{2}gt^2$$

In both cases, $v_0$ is the initial velocity of the projectile, and $\theta$ is the initial firing angle from the horizontal.

As a function of time, the distance between the gun and the projectile is

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
&= v_0^2 t^2 \cos^2\theta + (v_0 t \sin\theta - \frac{1}{2}gt^2)^2 \\
&= v_0^2 t^2 \cos^2\theta + v_0^2 t^2 \sin^2\theta - v_0 g t^3 \sin\theta + \frac{1}{4}g^2 t^4 \\
&= v_0^2 t^2 - v_0 g t^3 \sin\theta + \frac{1}{4}g^2 t^4 \\
&= t^2(v_0^2 - v_0 g t \sin\theta + \frac{1}{4}g^2 t^2) \\
r &= t\sqrt{v_0^2 - v_0 g t \sin\theta + \frac{1}{4}g^2 t^2}
\end{aligned}$$

The distance between the gun and the projectile will always be increasing when $\frac{dr}{dt} > 0$.

$$\begin{aligned}
\frac{dr}{dt} &= \frac{1}{2}t(v_0^2 - v_0 g t \sin\theta + \frac{1}{4}g^2 t^2)^{-1/2}(-v_0 g \sin\theta + \frac{1}{2}g^2 t) \\
&\quad + \sqrt{v_0^2 - v_0 g t \sin\theta + \frac{1}{4}g^2 t^2}
\end{aligned}$$

To find the limiting case, we set $\frac{dr}{dt} = 0$.

$$\begin{aligned}
0 &= \frac{1}{2}t(-v_0 g \sin\theta + \frac{1}{2}g^2 t) + v_0^2 - v_0 g t \sin\theta + \frac{1}{4}g^2 t^2 \\
&= -\frac{1}{2}v_0 g t \sin\theta + \frac{1}{4}g^2 t^2 + v_0^2 - v_0 g t \sin\theta + \frac{1}{4}g^2 t^2 \\
&= v_0^2 - \frac{3}{2}v_0 g t \sin\theta + \frac{1}{2}g^2 t^2
\end{aligned}$$

We want to find the maximum flight time of the projectile, so solving for $t$,

$$t_{max} = \frac{\frac{3}{2}g \sin\theta \pm \sqrt{\frac{9}{4}v_0^2 g^2 \sin^2\theta - 2g^2 v_0^2}}{g^2}$$

If there are two solutions for $t_{max}$, then the gun-projectile distance will decrease at some point during flight. We are looking for the limiting case, so we want only one solution for $t_{max}$ — the time when the projectile hits the ground. Therefore, we set the discriminent equal to 0.

$$\begin{aligned}
0 &= \frac{9}{4}\sin^2\theta - 2 \\
\frac{9}{4}\sin^2\theta &= 2 \\
\sin^2\theta &= \frac{8}{9} \\
\sin\theta &= \frac{\sqrt{8}}{9} \\
\theta &= 70.53^\circ
\end{aligned}$$

The maximum angle from which a projectile can be fired and always have the distance between the gun and the projectile increasing is $70.53^\circ$.

**Problem 2.** *An astronaut is on board a space-station that is shaped like a cylinder, and rotates about its cylindrical symmetry axis to provide "pseudo-gravity". The astronaut on earth can jump to a height $h$ that is one-fourth the radius $R$ of the cylinder. How high can she jump on the space-station?*

We will assume the astronaut has the same initial kinetic energy in both situations. On earth, her initial kinetic energy is

$$\frac{1}{2}mv_i^2 = mgh = mg\frac{R}{4}.$$

On the space-station, her initial kinetic energy is

$$\frac{1}{2}mv^2 = ma_s H$$

where $a_s$ is the acceleration felt on the space-station and $H$ is the total height jumped. Since her kinetic energy must be the same in both reference frames,

$$mgh = ma_sH$$
$$gh = a_sH$$

Since the space-station's acceleration is due to its rotation, $a_s = \omega^2 r$, where $\omega$ is the angular velocity of the space-station, and $r$ is the distance from the center of the rotational axis of the space-station. To find the total potential energy at the top of the astronaut's jump, we must integrate the acceleration from $R$ to $R - H$.

$$gh = \int_R^{R-H} \omega^2 r dr = \frac{1}{2}\omega^2[(R-H)^2 - R^2]$$

Solving for $H$, the total height jumped on the space-station is

$$H = R - \sqrt{\frac{2gh}{\omega^2} + R^2}$$

Since the space-station is rotating to provide pseudo-gravity, we can assume that $g = \omega^2 R$ or $\omega = \sqrt{g/R}$. Therefore, the total height the astronaut can jump on the space-station is

$$\boxed{H = R - \sqrt{2hR + R^2}}$$

**Problem 3.** *Baggage handlers are placing suitcases onto a rubber conveyor belt that is tilted at an angle $\theta$ above the horizontal. The motion of the belt is upward. Under what conditions would a suitcase slide down the belt, instead of being pulled upward?*

An object on an rough incline feels a force due to gravity, the normal force, and the frictional force. We first ignore the fact that the ramp is moving. The condition for the object to slide down the ramp under its own weight is such that

$$mg \sin\theta > \mu N = \mu mg \cos\theta$$

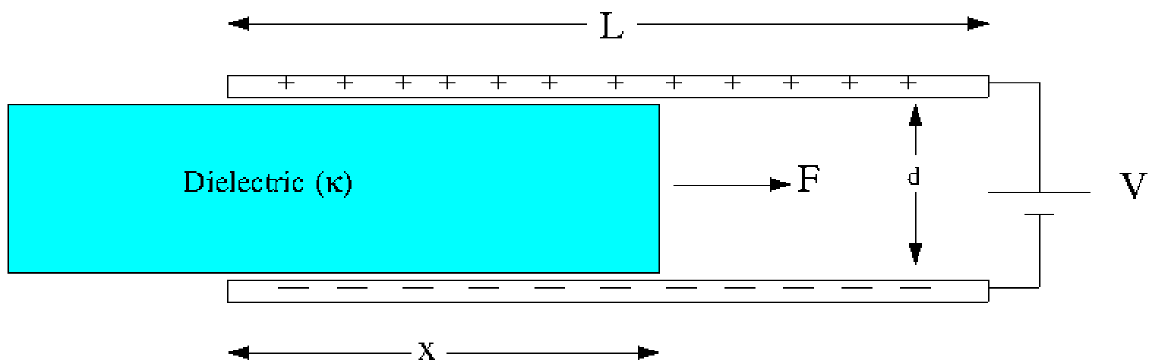
$$\mu < \tan\theta.$$

As long as this condition is met, the object will accelerate down the ramp (stationary with respect to the rubber belt) and eventually reach a certain speed $v_{critical}$. For $|v_{critical}| > |v_{ramp}|$, the object will actually be moving *down* the ramp.

**Problem 4.** *A mass is attached to the end of a string. The string passes through a hole in a frictionless table. The mass initially moves in a circle of radius $R$ with kinetic energy $E$; the string is held taut. If the string is now slowly pulled until the radius of the circle is $\frac{R}{2}$, how much work has been done pulling the string?*

The initial kinetic energy of the mass is $E_i = \frac{1}{2}mv_i^2 = \frac{1}{2}mR^2\omega_i^2$ and the final kinetic energy is $E_f = \frac{1}{2}m\left(\frac{R}{2}\right)^2\omega_f^2$. In addition, the angular momentum of the mass is conserved since there is no torque applied to the system. So, since $L_i = mR^2\omega_i$ and $L_f = m\left(\frac{R}{2}\right)^2\omega_f$ it is clear that $4\omega_i = \omega_f$. Therefore, the work done in pulling the string is $W = E_f - E_i = \frac{1}{2}mR^2\left(\frac{\omega_f^2}{4} - \omega_i^2\right) = \frac{1}{2}mR^2\left(\frac{16\omega_i^2}{4} - \omega_i^2\right) = \frac{3}{2}mR^2\omega_i^2 = 3E_i$.

**Problem 5.** *A parallel plate capacitor is connected to a battery whose terminal voltage is $V$. A dielectric (constant $\kappa$) is pulled into the space between the plates by a force $F$ as shown. Find the force $F$ in terms of $\kappa, \varepsilon_0, V, d$ and the width of the dielectric, $w$. Assume that $L - x >> d$ and neglect all fringing effects.*



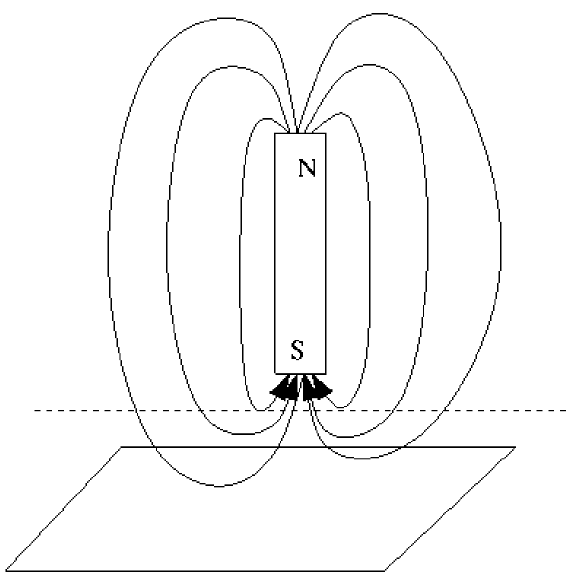

Just like a conductor is attracted to an electric field, a dielectric is attracted to a capacitor. The electric force inside causes a force pulling the dielectric inward. Say the dielectric moves an infinitesimal distance into the capacitor, the energy exerted by the capacitor on the dielectric is equal to the work done: $\mathrm{d}W = F\mathrm{d}x$ or $F = \frac{\mathrm{d}W}{\mathrm{d}x}$. The potential energy of a capacitor is $U = \frac{1}{2}CV^2$. While the voltage is held fixed due to the battery, the capacitance is different depending on how much of the capacitor is filled with the dielectric. The total capacitance is just the sum of the two capacitances (with and without the dielectric, think about adding the capacitances of two capacitors in parallel). Without the dielectric, $C_{w/o} = \frac{\varepsilon_0 w}{d}(L - x)$, and with the dielectric, $C_w = \frac{\varepsilon_0 x w}{d}\kappa$, so the total capacitance is $C = \frac{\varepsilon_0 w}{d}(L - x + x\kappa)$. Then, the potential energy (or Work), as a function of position, $x$, is

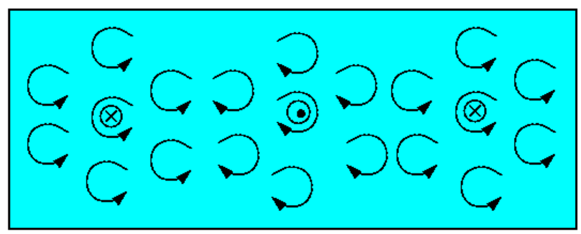
$$U = W = \frac{1}{2}\frac{\varepsilon_0 w}{d}(L - x + x\kappa)V^2.$$

Then the force is

$$F = \frac{\mathrm{d}W}{\mathrm{d}x} = \frac{1}{2}\frac{\varepsilon_0 w V^2}{d}(\kappa - 1).$$

**Problem 6.** *A bar magnet is oriented vertically and dropped toward a flat, horizontal copper plate. Why is there a repulsive force between the bar and the plate? Is the force elastic (that is, if the magnet is strong enough, will it bounce back)?*



When the bar magnet drops (South first), its magnetic field is changing in the copper plate. In particular, if we looked down on the copper plate from the perspective of the dropping bar magnet, at radii far away from the impact point, the bar magnet's magnetic field is increasing into the plate as it drops. Just under the magnet, in the near vicinity of the impact point, the magnet's magnetic field is increasing out of the plate. As we all know, a changing magnetic field induces a current which counteracts the change in external magnetic field. So, eddy currents will be induced in the copper plate. In the vicinity of the impact point, those eddy currents will be clockwise, and far away from the impact point, those eddy currents will be counterclockwise. The opposing changing bar magnetic field and the induced eddy currents in the copper plate will repel each other, thereby slowing the acceleration of the bar magnet. In reality, this opposing force will never be enough to actually *bounce* the magnet. But, hypothetically, even if the magnetic is strong enough, the collision cannot be elastic. The forces between the bar magnet and the induced currents are nonconservative since there is always a force opposing the motion (like kinetic friction). So, the kinetic energy of the bar magnet is constantly decreasing during the fall (and the rebound, if possible), so the collision can never be elastic.

**Problem 7.** *An electromagnetic wave crosses a boundary: what are the conditions on the vectors* ***E, D, B*** *and* ***H****?*

From Griffiths, pg. 331-333:

In general, the fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$ will be discontinuous at a boundary between two different media, or at a surface that carries charge density $\sigma$ or current density $\mathbf{K}$. The explicit form of these discontinuities can be deduced from Maxwell's equations,

$$\nabla \cdot \mathbf{D} = \rho_f, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t},$$

in their integral form

$$\left.\begin{array}{ll} \text{(i.)} & \oint_{\mathcal{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{f_{\text{enc}}} \\ \text{(ii.)} & \oint_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0 \end{array}\right\} \text{over any closed surface } \mathcal{S}.$$

$$\left.\begin{array}{ll} \text{(iii.)} & \oint_{\mathcal{P}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{a} \\ \text{(iv.)} & \oint_{\mathcal{P}} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{f_{\text{enc}}} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{a} \end{array}\right\} \text{for any surface } \mathcal{S} \text{ bounded by the closed loop } \mathcal{P}.$$

Applying (i.) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary, we obtain:

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f a.$$

(The positive direction for $\mathbf{a}$ is *from* 2 *toward* 1. The edge of the wafer contributes nothing in the limit as the thickness goes to zero, nor does any *volume* charge density.) Thus, the component of $\mathbf{D}$ that is perpendicular to the interface is discontinuous in the amount

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.}$$

Identical reasoning applied to equation (ii.), yields

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0.}$$

Turning to (iii.), a very thin Amperian loop straddling the surface gives

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{a}.$$

But in the limit as the width of the loop goes to zero, the flux vanishes. (I have already dropped the contribution of the two ends to $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l}$, on the same grounds.)

Therefore,

$$\boxed{\mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0.}$$

That is, the components of $\mathbf{E}$ *parallel* to the interface are continuous across the boundary. By the same token, (iv.) implies

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}},$$

where $I_{f_{\text{enc}}}$ is the free current passing through the Amperian loop. No *volume* current density will contribute (in the limit of infinitesimal width) but a *surface* current can. In fact, if $\hat{\mathbf{n}}$ is a unit vector perpendicular to the interface (pointing from 2 toward 1), so that $(\hat{\mathbf{n}} \times \mathbf{l})$ is normal to the Amperian loop, the

$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l},$$

and hence

$$\boxed{\mathbf{H}_1^{\|} - \mathbf{H}_2^{\|} = \mathbf{K}_f \times \hat{\mathbf{n}}.}$$

So the *parallel* components of $\mathbf{H}$ are discontinuous by an amount proportional to the free surface current density.
The boxed equations above are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of $\mathbf{E}$ and $\mathbf{B}$ alone:

$$\text{(i.)}\quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad \text{(iii.)}\quad \mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0,$$

$$\text{(ii.)}\quad B_1^{\perp} - B_2^{\perp} = 0, \qquad \text{(iv.)}\quad \frac{1}{\mu_1}\mathbf{B}_1^{\|} - \frac{1}{\mu_2}\mathbf{B}_2^{\|} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

In particular, if there is no free charge or free current at the interface, then

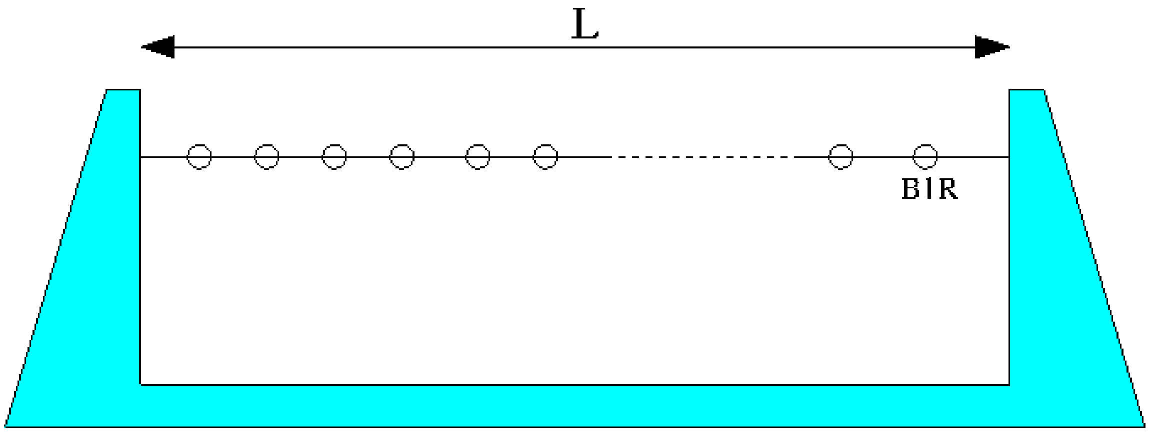
$$\text{(i.)}\quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, \qquad \text{(iii.)}\quad \mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0,$$

$$\text{(ii.)}\quad B_1^{\perp} - B_2^{\perp} = 0, \qquad \text{(iv.)}\quad \frac{1}{\mu_1}\mathbf{B}_1^{\|} - \frac{1}{\mu_2}\mathbf{B}_2^{\|} = 0.$$

**Problem A1: Drawing a bead on kinetic theory.** *$N$ identical balls of mass $m$ and radius $r$ are strung like beads along a smooth horizontal rod of length $L$, mounted between two immovable supports. The balls can slide along the rod, and initially are at rest. Collisions between balls as well as collisions between balls and supports are perfectly elastic while friction between balls and the rod is absent.*

*A. Suppose the balls are equally spaced along the rod. What is the average force felt at each support if the ball nearest the right support (B1R) is struck horizontally so as to acquire a speed of $V$ towards the right?*

The average force felt at each support is $F_{\text{avg}} = \frac{\Delta p}{\Delta t}$ since the walls provide a momentum transfer $\Delta p$ every time interval $\Delta t$. $\Delta p$ is easy; $2mV$. When we kick the B1R ball to the




left, it travels a certain distance before colliding with the next ball. Since the collisions are perfectly elastic, the momentum is transferred instantaneously. In fact, before the collision, the momentum is centered at the center of mass of B1R. Immediately after the collision, B1R stops and the momentum is transferred to the center of mass of B2R. In this process, there is a distance of $2r$ lost due to the finite size of the balls. Now, to figure out $\Delta t$, we see that we can treat the string of balls as a single ball that travels a certain distance. That distance is not $L$ but $L$ minus the sum of the diameters of all the balls. Let us imagine squishing all the balls against one wall. The total distance for the balls to travel is $L-2Nr$ since $2Nr$ is the total diameter of all the balls squished together. So, $\Delta t$ is $\frac{L-2Nr}{V}$. This is the time it takes one ball to travel from one support to the other, so the $\Delta t$ needed to calculate the average force on a single support will be twice this value. Therefore,

$$F_{\text{avg}} = \frac{\Delta p}{2\Delta t} = \frac{mV^2}{L-2Nr}.$$

And the units check out to newtons.

*Enumerate your answers when $L = 2m$, $N = 10$, $m = 0.1kg$, $r = 0.02m$, and $V = 4m/s$.*

Okay,

$$F_{\text{avg}} = \frac{(0.1)(4)^2}{(2) - 2(10)(0.02)} = 1\text{N}.$$

*B. What would happen to your answers if B1R was struck towards the left?*

If B1R was struck towards the left, there would be no change in the average force calculated above. The only thing that would happen is a "phase shift." The balls will hit the walls in the same time interval but different absolute times.

*C. Suppose now B1R is at the same position but the others are randomly spaced to the left. Describe qualitatively how your answers would be affected.*

Again, no change. The instantaneous momentum transfers result in subtracting out the distance corresponding to the diameter of all the balls. It doesn't matter how they are spaced, there is $L-2Nr$ distance for movement to occur regardless of the initial placement of the balls.

*D. If friction is present and introduces a constant deceleration of 1 m/s$^2$ on any moving ball, and initial conditions are those of A., what is the total number of collisions at each wall?*

Let us imagine, again, that one ball travels a distance of $L-2Nr$ with initial velocity $V = 4$m/s. Its final velocity after one trip across the wire is

$$\begin{aligned} V_f^2 &= V_i^2 + 2ad = 4^2 - 2(L-2Nr) \\ &= 16 - 3.2 = 12.8\text{m/s} \\ V_f &= 3.58\text{m/s}. \end{aligned}$$

So, for every trip we subtract 3.2 from the initial velocity squared. We can do this 5 times before the velocity reaches zero. Therefore, the left wall is hit three times and the right wall is hit twice.

**Problem A2: Crank calls.** *The two mechanisms shown are very common mechanisms that convert rotational movement into linear translational movement. The first is called a "Slider-Crank" mechanism (such as is found in pumps, internal combustion engines, etc.) and the second is a "Scotch-Yoke" mechanism. In both mechanisms the point "C" is fixed to the reference frame, point "B" translates in a horizontal direction only, link "AB" of the slider crank is "2R" long, and link "AC", called the CRANK, rotates with angular values of $\theta$, $\omega$, and $\alpha$. See figure below.*

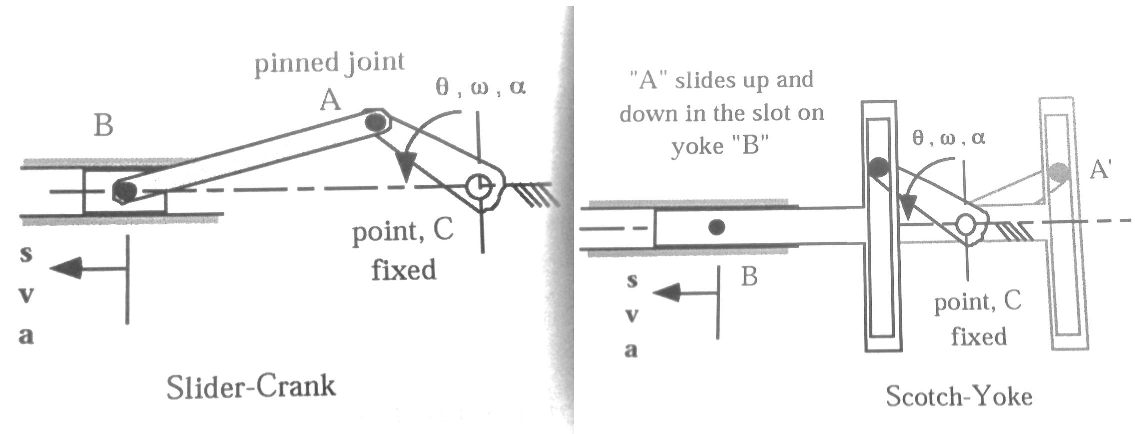
*For each of these mechanisms, assuming $\theta = 0$ and $s = 0$ when $AC$ is pointing vertically upward, determine the expressions for the position, s, velocity, v, and acceleration, a, of point "B" in terms of the crank's length "R", and the angular values of $\theta$, $\omega$, and $\alpha$.*

For the Scotch-Yoke mechanism, the position of point B is only dependent on the length of the crank and the angle, $\theta$. In particular, the $x$ component of the crank's length is the position,

$$s = R\sin\theta.$$

The velocity is just the time derivative of the position,

$$v = \dot{s} = R\cos\theta\dot{\theta} = R\omega\cos\theta.$$




Similarly, the acceleration is the time derivative of the velocity,

$$a = \dot{v} = R\cos\theta\dot{\omega} - R\omega\sin\theta\dot{\theta} = R\alpha\cos\theta - R\omega^2\sin\theta.$$

Alrighty, the slider crank is much more difficult and ugly. Let us first examine the geometry more closely. We first see that

$$AB\sin\gamma = R\cos\theta$$

and, hence,

$$\gamma = \sin^{-1}\left(\frac{R}{AB}\cos\theta\right).$$

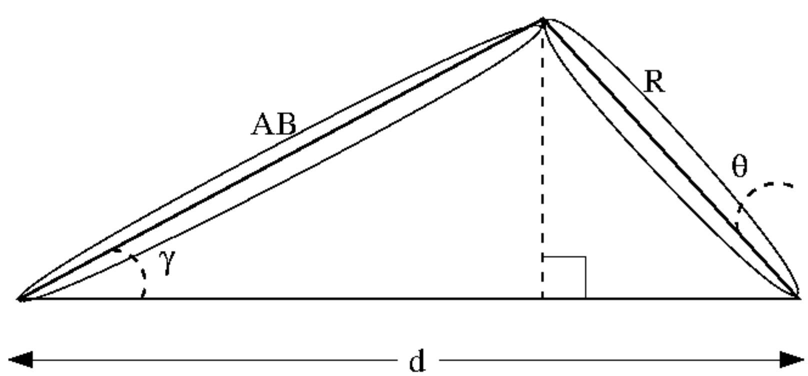
In addition,

$$d = AB\cos\gamma + R\sin\theta.$$

So, combining these we get

$$d = AB\cos\left[\sin^{-1}\left(\frac{R}{AB}\cos\theta\right)\right] + R\sin\theta.$$

But we don't want $d$, we want $s = d - AB\cos\gamma_m$ since $AB\cos\gamma_m$ is the starting position of the left side of the $AB$ bar when $R$ is vertical ($\gamma_m = \sin^{-1}\frac{R}{AB}$ is the max $\gamma$). Also, we can simplify since, by definition, $\cos\left(\sin^{-1} x\right)$



$= \sqrt{1-x^2}$. We find the position $s$ to be

$$s = \sqrt{(AB)^2 - (R\cos\theta)^2} + R\sin\theta - AB\cos\gamma_m.$$

The time derivative of the position is the velocity:

$$v = \dot{s} = \frac{1}{2}\left((AB)^2 - R^2\cos^2\theta\right)^{-\frac{1}{2}}\left(-2R^2\cos\theta\right)\left(-\sin\theta\dot{\theta}\right) + R\cos\theta\dot{\theta}$$

$$v = \frac{R^2\sin 2\theta\dot{\theta}}{2\sqrt{(AB)^2 - R^2\cos^2\theta}} + R\cos\theta\dot{\theta}.$$

The time derivative of the velocity is the acceleration:

$$a = \dot{v} = \frac{2\sqrt{(AB)^2 - R^2\cos^2\theta}R^2\left(\sin(2\theta)\ddot{\theta} + 2\cos(2\theta)\dot{\theta}^2\right) - R^4\sin^2(2\theta)\dot{\theta}^2\left((AB)^2 - R^2\cos^2\theta\right)^{-\frac{1}{2}}}{4\left((AB)^2 - R^2\cos^2\theta\right)}$$

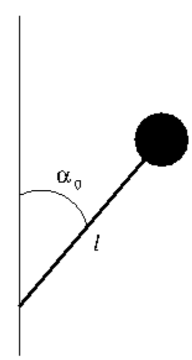
$$+R\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right).$$

Do your own substitutions to get the final answer, my hands hurt:

$$\dot{\theta} = \omega,\ \ddot{\theta} = \alpha,\ R^2 = s^2 + (AB)^2 - 2s(AB)\cos\gamma.$$

**Problem A3: Enjoy the fall.** *A mass is attached at one end of a massless, rigid rod of length $l$, and the rod is suspended at its other end by a frictionless pivot, as illustrated. The rod is released from rest at an angle $\alpha_0 < \frac{\pi}{2}$ with the vertical. At what angle $\alpha$ does the force in the rod change from compression (i.e. force toward the pivot) to tension (force away from the pivot)?*
*Also describe this physical situation in words*

From conservation of energy, the initial potential energy is equal to the final potential energy at some height $h$ below the initial height plus the final kinetic energy at the moment the force in the rod changes direction.




$$mgl = mg(l-h) + \frac{1}{2}mv^2$$
$$mgh = \frac{1}{2}ml^2\omega^2$$
$$\frac{2g}{l^2}h = \omega^2$$

From kinematics, the force on the rod changes direction when the normal (compression) force is equal to the centripetal force.

$$mg\cos\alpha = m\frac{v^2}{l}$$
$$g\cos\alpha = l\omega^2$$

So, combining these two equations, we eliminate $\omega$ and get

$$\frac{2h}{l} = \cos\alpha.$$

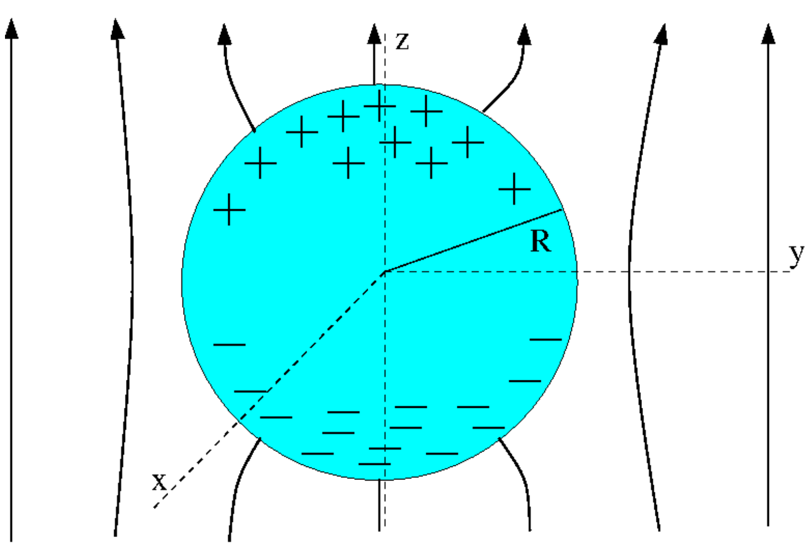
The way we have defined $h$ is such that $h = l - l\cos\alpha$, so the critical $\alpha$ is

$$2 - 2\cos\alpha = \cos\alpha$$
$$\alpha = \cos^{-1}\frac{2}{3}$$
$$\alpha \approx 48.2^\circ.$$

**Problem B1: A charge will apply.** *A neutral spherical conductor is introduced to a uniform electric field pointing to the positive z axis. Let the radius of the conductor be R.*

See Griffiths, Example 3.8, pg 141-142, and pg 149 for dipole moment stuff.

*A. Sketch the charge distribution within and on the conductor.*



*B. Calculate the charge density within the conductor and the surface charge density.*

The charge density within the conductor is easy, $\rho_{\text{in}} = 0$. The surface charge density is

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n},$$

where $\partial V/\partial n$ is the normal derivative of $V$ at the surface. In the case of the sphere, the surface charge density is

$$\sigma(\theta) = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R}.$$

So, we need to find the potential. While the sphere is an equipotential, we may as well set it to be zero. Then, by symmetry, the entire $xy$ plane is at potential zero. Then, the potential goes to

$$V \to -E_0 z.$$

So the boundary conditions are

$$\begin{cases} V = 0 & \text{when } r = R \\ V \to -E_0 r\cos\theta & \text{for } r \gg R. \end{cases}$$

Since we are going to use spherical coordinates with azimuthal symmetry ($\phi$ independent), the potential on the surface of the sphere is a solution of Laplace's equation, $\nabla^2 V = 0$,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

We look for a solution of the form

$$V(r,\theta) = R(r)\Theta(\theta).$$

Separating the variables, we get two equations:

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = l(l+1), \quad \frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = -l(l+1).$$

The radial equation has the solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}},$$

and the angular equation has the solution

$$\Theta(\theta) = P_l(\cos\theta)$$

where $P_l$ are the Legendre polynomials. Therefore, the general solution of the potential is the linear combination of separable solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$

The first boundary condition, $V = 0$ for $r = R$, forces

$$B_l = -A_l R^{2l+1}.$$

And for the second boundary condition, $V \to -E_0 r\cos\theta$ for $r \gg R$, ensures that $\frac{B_l}{r^{l+1}}$ is negligible, and that

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta.$$

Therefore, the only term possible is for $l = 1$. Then, we can read off

$$A_1 = -E_0.$$

The final solution for the potential is

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

So, the induced surface charge density is

$$\sigma(\theta) = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} = \epsilon_0 E_0 \left.\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\right|_{r=R} = 3\epsilon_0 E_0\cos\theta.$$

*C. Calculate the dipole moment of the charge distribution.*

The dipole moment of a charge distribution is

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\mathrm{d}\tau'$$

where we are integrating over the distribution. In our case, our distribution is on the surface of a sphere so our integral will be a surface integral:

$$\mathbf{p} = \int (z\hat{\mathbf{z}})\sigma\mathrm{d}a.$$

Therefore, the dipole moment is (using Schaum's Mathematical Handbook eq. 17.19.4)

$$\begin{aligned}\mathbf{p} &= \int_0^{\pi}\int_0^{2\pi}(R\cos\theta\hat{\mathbf{z}})(3\epsilon_0 E_0\cos\theta)R^2\sin\theta\mathrm{d}\phi\mathrm{d}\theta \\ &= 6\pi\epsilon_0 E_0 R^3\hat{\mathbf{z}}\int_0^{\pi}\cos^2\theta\sin\theta\mathrm{d}\theta \\ &= 4\pi\epsilon_0 E_0 R^3\hat{\mathbf{z}}\end{aligned}$$

**Problem B2: How can you resist?** *In the circuit shown to the right the resistors are $R_1 = 3K\Omega$, $R_2 = 12K\Omega$, and $R_3$ (unknown), the capacitor is $C = 40\mu F$ and the inductor is L of unknown value. Emf of the source is $\varepsilon = 45\,V$. There are three switches $S_1$, $S_2$, and $S_3$. Assume switch $S_3$ is open throughout whereas switch $S_2$ is kept closed all the time. At $t = 0$, with no current through the inductor, switch $S_1$ is closed.*

*a. What are the* **currents** *(in mA)* $I_1, I_2,$ *and* $I_3$ *through the resistors* $R_1, R_2,$ *and* $R_3$ *respectively immediately after* $S_1$ *is closed?*

Immediately after the switch $S_1$ is closed, the inductor acts like a break in the wire, so $I_3 = 0$.

From Kirchoff's current law, $I_1 = I_2 + I_3$. Since $I_3 = 0$, $I_1 = I_2$. Applying Kirchoff's voltage law around the left-hand loop, we find that

$$\begin{aligned} -I_1R_1 + \varepsilon - I_2R_2 &= 0 \\ -I_1R_1 + \varepsilon - I_1R_1 &= \\ I_1(R_1 + R_2) &= \varepsilon \\ I_1 &= \frac{\varepsilon}{R_1 + R_2} = 0.003A \end{aligned}$$

So $\boxed{I_1 = 3 \text{ mA}, I_2 = 3 \text{ mA}, \text{ and} I_3 = 0 \text{ mA}.}$

*b. What is the* **voltage** $V_{ab}$ *(in V) immediately after* $S_1$ *is closed?*

From Ohm's law,

$$V_{ab} = I_2R_2 = \boxed{36 \text{ V} = V_{ab}}$$

*c. If the* **voltage** $V_{ab} = 30$ *V a long time after* $S_1$ *is closed, what is the* **voltage** $V_1$ *across the resistor* $R_1$*? What is the* **current** $I_1$ *(in mA) through the resistor* $R_1$*?*

Again, using Kirchoff's loop rule for the left-hand loop,

$$\begin{aligned} -I_1R_1 + \varepsilon - I_2R_2 &= 0 \\ -V_1 + 45 \text{ V} - 30 \text{ V} &= \\ &\boxed{V_1 = 15 \text{ V}} \end{aligned}$$

Then, using Ohm's law, the current through $R_1$ is

$$I_1 = \frac{V_1}{R_1} = \frac{15 \text{ V}}{3 \text{ k}\Omega} = \boxed{5 \text{ mA} = I_1}$$

*d. A long time after* $S_1$ *is closed, if the current* $I_3$ *through* $R_3$ *is 2.5 mA, find the resistor* $R_3$ *(in k*$\Omega$*). What is the current (in mA)* $I_2$ *through the resistor* $R_2$*?*

Remembering the identity $I_1 = I_2 + I_3$ from part (a), we can rewrite the voltages through the left-hand loop as

$$\begin{aligned} \varepsilon &= R_1(I_2 + I_3) + R_2 I_2 \\ &= I_2(R_1 + R_2) + R_1 I_3 \end{aligned}$$

$$I_2 = \frac{\varepsilon - R_1 I_3}{R_1 + R_2} \tag{1}$$

After a long time with switch $S_1$ closed, the inductor acts like a bare wire, so $\frac{dI_3}{dt} = 0$. Applying Kirchoff's loop rule to the right-hand loop,

$$-L\frac{dI_3}{dt} - I_3 R_3 + I_2 R_2 = 0 \tag{2}$$

$$R_3 I_3 = R_2 I_2 \tag{3}$$

Making the substitution for $I_2$ from Eqn. 1,

$$\begin{aligned} R_3 I_3 &= \frac{R_2(\varepsilon - R_1 I_3)}{R_1 + R_2} \\ R_3 &= \frac{R_2}{I_3}\frac{\varepsilon - R_1 I_3}{R_1 + R_2} = \frac{12\ \text{k}\Omega}{2.5\ \text{mA}}\frac{45\ \text{V} - (3\ \text{k}\Omega)(2.5\ \text{mA})}{3\ \text{k}\Omega + 12\ \text{k}\Omega} \\ &\boxed{R_3 = 12\ \text{k}\Omega} \end{aligned}$$

To find the current $I_2$, we go back to Eqn. 3,

$$\begin{aligned} I_2 R_2 &= I_3 R_3 \\ I_2 &= \frac{I_3 R_3}{R_2} = \frac{(2.5\ \text{mA})(12\ \text{k}\Omega)}{12\ \text{k}\Omega} \\ &\boxed{I_2 = 2.5\ \text{mA}} \end{aligned}$$

*e. If the energy stored in the inductor a long time after $S_1$ is closed is 156.3 nJ, what is the value of the inductor $L$?*

$$\begin{aligned} U &= \frac{1}{2} L I_3^2 \\ L &= \frac{2U}{I_3^2} = \frac{2 \cdot 156.3\ \text{nJ}}{(2.5\ \text{mA})^2} \\ &\boxed{L = 0.050016\ \text{H}} \end{aligned}$$

*f. What is the rate (A/s) at which the current $I_3$ through the inductor changes when $t = t_L \ln 2$ s after $S_1$ is closed?*

Starting with Eqn. 2,

$$\begin{aligned}
-L\frac{dI_3}{dt} - R_3 I_3 + I_2 R_2 = 0 \\
\frac{dI_3}{dt} = I_3\frac{R_3}{L} - I_2\frac{R_2}{L}
\end{aligned}$$

Unfortunately, we cannot stop here, as this expression is not a function of time (which we need...). Solving for $dI_3$,

$$\begin{aligned}
dI_3 &= \left(I_3\frac{R_3}{L} - I_2\frac{R_2}{L}\right) dt \\
\int \frac{\mathrm{d}I_3}{I_3\frac{R_3}{L} - I_2\frac{R_2}{L}} &= \int_0^t \mathrm{d}t \\
L\int \frac{\mathrm{d}I_3}{I_3 R_3 - I_2 R_2} &= t \\
-\frac{L}{R_3}\int_0^{I_3} \frac{\mathrm{d}I_3'}{-I_3' + I_2\frac{R_2}{R_3}} &= \\
-\frac{L}{R_3}\ln\left(-I_3' + I_2\frac{R_2}{R_3}\right)\Bigg|_0^{I_3} &= \\
\ln\left(-I_3 + I_2\frac{R_2}{R_3}\right) - \ln\left(I_2\frac{R_2}{R_3}\right) &= -\frac{R_3}{L}t \\
\ln\left(\frac{-I_3 + I_2\frac{R_2}{R_3}}{I_2\frac{R_2}{R_3}}\right) &= \\
\frac{-I_3 + I_2\frac{R_2}{R_3}}{I_2\frac{R_2}{R_3}} &= e^{-R_3 t/L} \\
-I_3 + I_2\frac{R_2}{R_3} &= I_2\frac{R_2}{R_3}e^{-R_3 t/L} \\
I_3 &= I_2\frac{R_2}{R_3}\left(1 - e^{-R_3 t/L}\right)
\end{aligned}$$

Since we are looking for the rate of change of $I_3$, we need to take the derivative of the above expression and evaluate at $t = t_L \ln 2$.

$$\frac{dI_3}{dt} = I_2 \frac{R_2}{R_3} \frac{R_3}{L} e^{-R_3 t/L}$$
$$\frac{dI_3}{dt} = I_2 \frac{R_2}{L} e^{-R_3 t/L}$$

*g. What is the emf (in V) induced in the inductor when $t = t_L \ln 2$ s after $S_1$ is closed?*

$$V_L = L\frac{dI_3}{dt} = I_2 R_2 e^{-R_3 t/L}$$

*h. Find the rate (in nJ/s) at which the magnetic field energy is stored in the inductor when $t = t_L \ln 2$ s after $S_1$ is closed.*

The energy stored in an inductor is equal to

$$U = \frac{1}{2} L I_3^2$$

The rate of storage of this energy is the time derivative, so

$$\begin{aligned} \frac{dU}{dt} &= L I_3 \frac{dI_3}{dt} \\ &= L I_2 \frac{R_2}{R_3}\left(1 - e^{-R_3 t/L}\right) I_2 \frac{R_2}{L} e^{-R_3 t/L} \\ \frac{dU}{dt} &= \frac{(I_2 R_2)^2}{R_3}\left(1 - e^{-R_3 t/L}\right) e^{-R_3 t/L} \end{aligned}$$

*j. Find the rate (in mW) at which energy is input to the circuit by the source when $t = (L/R) \ln 2$ s after $S_1$ is closed.*

*k. What is the time constant (in $\mu$s) of the circuit?*

$$\tau = \frac{L}{R_3} = \frac{0.050016 \text{ H}}{12 \text{ k}\Omega} = \boxed{4.168\mu\text{s} = \tau}$$

*l. A long time after $S_1$ is closed, the current through the inductor has reached a steady value given in part (d). Now the switches $S_1$ and $S_2$ are opened and $S_3$ is closed simultaneously. Describe (qualitatively) the phenomenon that occurs in the closed part of the circuit. How does this compare to an almost similar situation in mechanics?*

The current in the LC circuit will oscillate, mimicking a simple harmonic oscillator.