



## DEPARTMENT OF PHYSICS

### PhD Qualifying Exam

Friday, January 12, 2007

### Modern Physics

1 PM – 4 PM

PRINT YOUR NAME\_\_\_\_\_

EXAM CODE\_\_\_\_\_

1. PUT YOUR EXAM CODE, **NOT YOUR NAME**, ON EACH PIECE OF PAPER YOU HAND IN. (This allows us to grade each student only on the work presented.)

2. Do each problem or question on a separate sheet of paper...even the short ones. (This allows us to grade them simultaneously.)

Answer 5 of 7 short answer questions in part I and 3 of 6 longer problems in part II, with at least one problem from Group A and one from Group B. *Circle* the numbers below to indicate which questions you have answered—write nothing on the lines.

| <i>Short questions</i> |       | <i>Long Problems</i> |       |
|------------------------|-------|----------------------|-------|
| <i>circle</i>          | grade | <i>circle</i>        | grade |
| 1.                     | _____ | A1.                  | _____ |
| 2.                     | _____ | A2.                  | _____ |
| 3.                     | _____ | A3.                  | _____ |
| 4.                     | _____ | B1.                  | _____ |
| 5.                     | _____ | B2.                  | _____ |
| 6.                     | _____ | B3.                  | _____ |
| 7.                     | _____ |                      |       |

## MODERN PHYSICS

### PART I: Short questions (25%)

#### ANSWER 5 OF 7 QUESTIONS

1. A particle of mass  $m$  is trapped between two infinitely high walls a distance  $L$  apart in one dimension.

- (a) Write down its eigenfunctions.
- (b) What are its energy eigenvalues?

2. (a) Write down the electronic ground state configuration of carbon.

(b) Using the spectroscopic notation  $^{2S+1}L_J$ , what terms arise in the ground state electronic configuration?

3. A particle moves in a coulomb potential. Classically, the particle moves in an elliptical orbit. Quantum mechanically, all the levels with fixed  $L$  (orbital angular momentum) are degenerate in energy. If the particle is treated relativistically, the "classical" motion is on a precessing ellipse. Quantum mechanically, the  $L$ -degeneracy is broken. Both effects arise from the "relativistic mass-velocity" correction, which can be effectively treated by a modification of the Coulomb potential,  $V_{\text{eff}}(r) = -K/r + f(r)$ , where  $f(r)$  is the perturbation. What is the  $r$ -dependence of the perturbing term  $f(r)$  and why?

4. Let  $A$ ,  $B$  and  $C$  be three operators. Express the commutator of the product  $AB$  with  $C$  in terms of the commutators  $[A,C]$  and  $[B,C]$ . Use the result of this calculation to derive a law for the time derivative of the product of two operators.

5. A black, non-reflective plane at temperature  $T_u$  is parallel to a black plane at temperature  $T_l$ . The net energy flux density in vacuum between the planes is  $J = \sigma(T_u^4 - T_l^4)$  where  $\sigma$  is Boltzmann's constant. A third black plane is inserted between the other two and is allowed to come to a steady state intermediate temperature  $T_m$ .

Find  $T_m$  in terms of  $T_u$  and  $T_l$  and show that the net energy flux density is cut in half. This result underlies the idea of heat shields, which effectively reduce heat flow.

6. In a system of  $N$  one-dimensional simple harmonic oscillators of frequency  $\nu$ , show that the number of oscillators with energy  $\geq \epsilon_n$  is equal to  $N \cdot \exp(-nh\nu/k_B T)$ , where  $\epsilon_n$  is the energy of the  $n$ th level of a single oscillator,  $h$  is the Planck's constant,  $k_B$  is the Boltzmann constant and  $T$  is the temperature. Assume that the oscillators are in thermal equilibrium.

7. The bulk modulus,  $K$ , defined by

$$\Delta P = -K \frac{\Delta V}{V} ,$$

measures the response in pressure due to a change in relative volume, essentially measuring the substance's resistance to uniform compression. Find the bulk modulus of an ideal gas.

## PART II: Long problems (75%)

ANSWER 3 OF 6 QUESTIONS, WITH AT LEAST ONE FROM GROUP A AND ONE FROM GROUP B.

### A1.

Consider the wavefunction of a system of two electrons where one is in an  $s$  state and the other in a  $d$  state.

Write down

- (a) all the possible configurations of total  $l, s, j$
- (b) the corresponding expectation value  $\langle \mathbf{L} \cdot \mathbf{S} \rangle$  for each possible configuration.
- (c) the spectroscopic term for each possible configuration,  $^{2s+1}L_J$

You should present your results as a small table, with columns labeled, in the order above.

Hint: recall that

$$\mathbf{L} \cdot \mathbf{S} = (1/2)(J^2 - L^2 - S^2)$$

### A2.

At some initial time, a quantum harmonic oscillator is in the state:

$$\psi(x, 0) = \frac{3}{5}|\phi_0\rangle + \frac{4}{5}|\phi_1\rangle$$

where  $\phi_n$  is a normalized eigenstate of the Hamiltonian:

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + 1/2)$$

and where  $\omega_0$  is the fundamental frequency of the mass-spring system.

You may find it useful to recall that

$$\begin{aligned}\hat{p} &= \frac{m\omega_0}{i\sqrt{2}\beta}(\hat{a} - \hat{a}^\dagger) \\ \hat{x} &= \frac{1}{\sqrt{2}\beta}(\hat{a} + \hat{a}^\dagger) \\ \beta &\equiv \sqrt{\frac{m\omega_0}{\hbar}}\end{aligned}$$

(A2, continued)

though you should feel free to leave your answer in terms of  $\beta$  in any of your answers.

- (a) What is the expectation value of the energy of this system?
- (b) Sketch the wavefunction at  $t=0$ .
- (c) What is the expectation value of position  $\langle x \rangle$  at time  $t=0$ ?
- (d) Find the expectation value  $\langle x(t) \rangle$  for all times in the future.

### A3.

An atom consists of one electron bound to an infinitely heavy particle having one unit of positive charge. The particle instantaneously changes its charge from  $+1e$  to  $+2e$  (for example, by emission of a negatively charged electron).

- (a) Outline the method by which you would compute the probability that an initially ground state electron would remain in the ground state of the new atom.
- (b) Set up the integral for calculating the probability.
- (c) Evaluate the integral.

Some useful information for this problem:

$$\int x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2)$$

#### Normalized Eigenstates of Hydrogen Atom

$$\begin{aligned}\varphi_{100} &= \frac{1}{\sqrt{\pi}a_0^{3/2}} e^{-r/a_0} \\ \varphi_{200} &= \frac{1}{(2a_0)^{3/2} \sqrt{\pi}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \\ \varphi_{210} &= \frac{1}{(2a_0)^{3/2} \sqrt{\pi}} \left(\frac{r}{2a_0}\right) e^{-r/2a_0} \cos \theta \\ \varphi_{21\pm 1} &= \frac{1}{8a_0^{3/2}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \sin \theta e^{\pm i\phi} \\ a_0 &\equiv \frac{\hbar^2}{\mu e^2}\end{aligned}$$

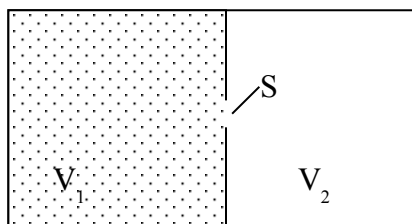
## B1.

- (a) Consider a solution of  $N$  molecules in water, at a concentration of  $N/V$ . Let each molecule interact with its surroundings (i.e. the water) with an average potential  $U$ .  $U$  may be a function of  $T$ . Beginning with the canonical partition function, compute the Helmholtz Free Energy per molecule.
- (b) Suppose the solution forms a kind of “glass,” so that each molecule can be considered in its own “cell,” but doesn't exchange with other cells. Now compute the Helmholtz Free Energy per molecule, beginning with the canonical partition function. Assume the interaction with surroundings is still an average, but call it  $U'$ .
- (c) Give expressions for the entropic and enthalpic contribution from these molecules to the energy associated with forming the glass (i.e. distinct from the contribution of the water itself). Suppose we had begun in part (a) with the molecules in an ideal gas state. How would your answer to part (a) change?

## B2.

A box made of perfectly reflecting walls is partitioned into two compartments 1 and 2. Initially, all of the (ideal) gas is in compartment 1 at temperature  $T_1$ . A small hole of area  $S$  is made in the partition with dimensions much smaller than the mean free path of the molecules in the gas. This hole is held open for a time  $t$  and then sealed. The gas that escapes into compartment 2 then comes to equilibrium.

- (a) Find the number of molecules in compartment 2.
- (b) Find the final temperature in compartment 2.



Notes: The Maxwell distribution is  $f_v = \left( \frac{m}{2\pi k_B T} \right)^{3/2} e^{-\frac{mv^2}{2k_B T}}$

It can be assumed that the number of particles escaping the first compartment is relatively small and the speed distribution doesn't change, and that the time  $t$  is much shorter than the average time between collisions.

These integrals might be useful:

(B2, continued)

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^2 e^{-ax} dx = \frac{1}{4a} \sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

### B3.

(a) Using the fact that the entropy  $S(N,V,E)$  is extensive for large systems, show that

$$N \left( \frac{\partial S}{\partial N} \right)_{V,E} + V \left( \frac{\partial S}{\partial V} \right)_{N,E} + E \left( \frac{\partial S}{\partial E} \right)_{V,N} = S.$$

Show from this that in general

$$S = (E + PV - \mu N)/T \quad (1)$$

and hence derive the Euler relation.

(b) Test this explicitly for the ideal gas. Use the ideal gas entropy

$$S(N, V, E) = \frac{5}{2} Nk_B + Nk_B \ln \left[ \frac{V}{Nh^3} \left( \frac{4\pi mE}{3N} \right)^{\frac{3}{2}} \right],$$

to derive formulas for the intensive parameters,  $T$ ,  $P$ , and  $\mu$  in terms of  $E$ ,  $N$ , and  $V$ , and verify Equation (1).