Quantum II HW1

Vincent Baker

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1 Problem 1

We write the angular parts of the three wavefunctions as linear combinations of spherical harmonics, using the identity $\sin\phi = \frac{e^{i\phi}-e^{-i\phi}}{2i}$.

$$\psi_1(\theta,\phi) = \sin\theta\sin\phi = c_1 Y_1^1 + c_2 Y_1^{-1} \tag{1.1}$$
$$\psi_2(\theta,\phi) = \cos^2\theta = c_1 Y_2^0 + c_2 Y_0^0 \tag{1.2}$$
$$\psi_3(\theta,\phi) = \sin\theta\cos\theta\sin\phi = c_1 Y_2^1 + c_2 Y_2^{-1} \tag{1.3}$$

Solving for the constants and collecting terms we find:

$$\psi_1(\theta,\phi) = \sin\theta\sin\phi = i\sqrt{\frac{2\pi}{3}}\left(Y_1^1 + Y_1^{-1}\right) \tag{1.4}$$
$$\psi_2(\theta,\phi) = \cos^2\theta = \frac{1}{\sqrt{5\pi}} Y_2^0 + \frac{1}{\sqrt{4\pi}} Y_0^0 \tag{1.5}$$
$$\psi_3(\theta,\phi) = \sin\theta\cos\theta\sin\phi = i\sqrt{\frac{2\pi}{15}}\left(Y_2^1 + Y_2^{-1}\right) \tag{1.6}$$

Since $L^2 \left|\ell\ m\right\rangle = \hbar^2\ell(\ell+1)\left|\ell\ m\right\rangle$, $L_z\left|\ell\ m\right\rangle = \hbar m\left|\ell\ m\right\rangle$, and $\langle \ell m|\ell m\rangle = 1$, we find:

$$\langle\Psi_1|L^2|\Psi_1\rangle = i\sqrt{\frac{2\pi}{3}}4\hbar^2,\ \langle\Psi_1|L_z|\Psi_1\rangle = i\sqrt{\frac{2\pi}{3}}\hbar(1-1) = 0 \tag{1.7}$$
$$\langle\Psi_2|L^2|\Psi_2\rangle = \sqrt{\frac{1}{5\pi}}6\hbar^2,\ \langle\Psi_2|L_z|\Psi_2\rangle = i\sqrt{\frac{1}{4}}\hbar(0+0) = 0 \tag{1.8}$$
$$\langle\Psi_3|L^2|\Psi_3\rangle = i\sqrt{\frac{2\pi}{15}}12\hbar^2,\ \langle\Psi_3|L_z|\Psi_3\rangle = i\sqrt{\frac{2\pi}{15}}\hbar(1-1) = 0 \tag{1.9}$$

2 Problem 2

For J=1. we have $J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$, $J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$, and $J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Since $J_x = \frac{1}{2}(J_+ + J_-)$, $J_y = \frac{1}{2i}(J_+ - J_-)$, our matrices and their squares are:

$$J_x = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} J_x^2 = \frac{1}{4}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} \tag{2.1}$$

$$J_y = \frac{1}{2i}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix} J_y^2 = \frac{1}{4}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \tag{2.2}$$

$$J_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} J_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

We put the squared matrices into MATLAB and show directly that they commute (see p2.m). The sum of the squared matrices is $2\mathbf{I_3}$.

3 Problem 3

a) To find the probability that the total spin is S, we need the braket of the total state with the two component states $\langle {S \atop M} | {s_1\ s_2 \atop m_1 m_2} \rangle$. We first write the state $| {S \atop M} \rangle$ in terms of the component states.

$$| {S \atop M} \rangle = | {s_1\ s_2 \atop m_1\ m_2} \rangle \langle {s_1\ s_2 \atop m_1\ m_2} | {S \atop M} \rangle \tag{3.1}$$

$$\langle {S \atop M} | = \langle {s_1\ s_2 \atop m_1\ m_2} | {S \atop M} \rangle \langle {s_1\ s_2 \atop m_1\ m_2} | \tag{3.2}$$

$$\langle {S \atop M} | {s_1\ s_2 \atop m_1\ m_2} \rangle = \langle {s_1\ s_2 \atop m_1\ m_2} | {S \atop M} \rangle \langle {s_1\ s_2 \atop m_1\ m_2} | {s_1\ s_2 \atop m_1\ m_2} \rangle \tag{3.3}$$

$$\langle {S \atop M} | {s_1\ s_2 \atop m_1\ m_2} \rangle = \langle {s_1\ s_2 \atop m_1\ m_2} | {S \atop M} \rangle \tag{3.4}$$

b) For an "unpolarized" state the expectation value of the total spin is 0 since

$$< \sigma >= Trace(\rho\sigma) = \mathbf{a} \tag{3.5}$$

where $\mathbf{a}$ is the polarization vector.

4 Problem 4

We prove the identity:

$$(\sigma \cdot A)(\sigma \cdot B) = (A \cdot B)I_2 + i\sigma(A \times B) \tag{4.1}$$

The dot products of the Pauli matrices with A and B are:

$$\sigma \cdot A = \begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix} \tag{4.2}$$

$$\sigma \cdot B = \begin{bmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{bmatrix} \tag{4.3}$$

Multiplying the two matrices and simplifying:

$$\begin{aligned}(\sigma \cdot A)(\sigma \cdot B) = (A \cdot B)I_2 + (A_yB_z - A_zB_y) &\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\ +(A_zB_x - A_xB_z) &\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ +(A_xB_y - A_yB_x) &\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\end{aligned} \tag{4.4}$$

We reocgnize the terms of the cross product $A \times B$ in the last three terms. Pulling out a factor of i from all three terms, we have:

$$\begin{aligned}(\sigma \cdot A)(\sigma \cdot B) = (A \cdot B)I_2 + i\{(A_yB_z - A_zB_y) &\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ +(A_zB_x - A_xB_z) &\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \\ +(A_xB_y - A_yB_x) &\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}\end{aligned} \tag{4.5}$$

We have now recovered $\sigma_x, \sigma_y, \sigma_z$ and have proved 3.1.

5 Problem 5

Two spin $\frac{1}{2}$ particles interact through the potential $V(r) = V_1(r) + \sigma_1 \cdot \sigma_2 V_2(r)$. We will show that the spin-dependent potential can be split into two potentials based on addition of spin. We start with $\sigma = \frac{2}{\hbar}\mathbf{S}, \sigma \cdot \sigma = \frac{4}{\hbar^2}S^2$. We take the total spin $\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}$.

$$S_1 + S_2 = S \tag{5.1}$$

$$(S_1 + S_2)^2 = S^2 \tag{5.2}$$

$$S_1^2 + 2S_1S_2 + S_2^2 = S^2 \tag{5.3}$$

$$S_1 \cdot S_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right) \tag{5.4}$$

Since both particles are spin $\frac{1}{2}$ we have $S_1 \cdot S_1 = S_2 \cdot S_2 = \frac{3}{4}$. The values of $m_1 = m_2 = \pm\frac{1}{2}$, so the value of $M = \{1, 0, -1\}$ and therefore $S = 1, S \cdot S = \{2, 0\}$. Using 5.4 we find that

$$S_1 \cdot S_2 = \frac{1}{2}\left(\{2,0\} - \frac{3}{4} - \frac{3}{4}\right) \tag{5.5}$$

$$S_1 \cdot S_2 = \{\frac{1}{4}, -\frac{3}{4}\} \tag{5.6}$$

With $\sigma \cdot \sigma = \frac{4}{\hbar^2}S^2$ we have therefore shown that $V(r) = V_1(r) + \sigma_1 \times \sigma_2 V_2(r)$ can be written as two equations:

$$V(r) = V_1(r) + V_2(r) \tag{5.7}$$

$$V(r) = V_1(r) - 3V_2(r) \tag{5.8}$$

6 Problem 6

With $J = 0$ the system has a single eigenstate $|00\rangle$ and is therefore spherically symmetric. With $J = \frac{1}{2}$ the system has a 2D space defined by eigenstates $|\frac{1}{2}\ \frac{1}{2}\rangle$, $|\frac{1}{2}\ -\frac{1}{2}\rangle$. With a 2D space the system cannot exhibit an electric quadrupole moment, only a dipole moment.

7 Problem 7

a) Starting with a Hamiltonian that couples an electric quadrupole moment to the gradient of the electric field:

$$H_p = C\{S_i S_j \Phi_{ij}\} \tag{7.1}$$

Transforming to the principal axes coordinate system the cross-derivative terms are zero, so only terms with i=j survive.

$$H_p = C\{S_x^2\Phi_{xx} + S_y^2\Phi_{yy} + S_z^2\Phi_{zz}\} \tag{7.2}$$

b) We can also write this in the form:

$$H_p = A(3S_z^2 - \mathbf{S}\cdot\mathbf{S}) + B(S_+^2 + S_-^2) \tag{7.3}$$

$$H_p = A(2S_z^2 - S_x^2 - S_y^2) + B(2S_x^2 - 2S_y^2) \tag{7.4}$$

$$H_p = S_x^2(2B - A) + S_y^2(-2B - A) + S_z^2(2A) \tag{7.5}$$

Equating coefficients between 5.2 and 5.5 we have:

$$2B - A = C\Phi_{xx} \tag{7.6}$$
$$-2B - A = C\Phi_{yy} \tag{7.7}$$
$$2A = C\Phi_{zz} \tag{7.8}$$

We find that $A = \frac{C}{2}\Phi_{zz}$, $B = \frac{C}{4}(\Phi_{xx} - \Phi_{yy})$.

c) For a spin $\frac{3}{2}$ system the 4x4 matrix representations of S^2, S_z, S_+, S_- are:

$$S^2 = \frac{15}{4}\mathbf{I_4} \tag{7.9}$$
$$S_z = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} \tag{7.10}$$
$$S_z^2 = \begin{bmatrix} \frac{9}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{9}{4} \end{bmatrix} \tag{7.11}$$
$$S_+ = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.12}$$
$$S_- = S_+^\dagger \tag{7.13}$$
$$S_+^2 + S_-^2 = \begin{bmatrix} 0 & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{3} & 0 & 0 \end{bmatrix} \tag{7.14}$$

The energy eigenvalues are $3A - 2\sqrt{3}B, -\frac{11}{4}A - 2\sqrt{3}B, -\frac{11}{4}A + 2\sqrt{3}B, 3A + 2\sqrt{3}B$.