the 3n-j symbols

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I. the 3-j symbols

Definition of the 3-j Symbols

The 3-j symbol of Wigner[1] is denoted by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

and is related to the Clebsch-Gordan coefficient in Condon and Shortley[2] by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{(2j_3+1)^{1/2}} (j_1\, m_1\, j_2\, m_2 | j_1\, j_2\, j_3\, -m_3). \tag{1.1}$$

There are many notations and phase factors that are in current use. For example, Condon and Shortley[2] use

$$(j_1\, m_1\, j_2\, m_2 | j_1\, j_2\, j_3\, m_3) = (2j_3+1)^{1/2}(-1)^{-j_1+j_2-m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \tag{1.1a}$$

Schwinger's[3] phase is the same as ours:

$$X(j_1\, j_2\, j_3;\, m_1\, m_2\, m_3) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{1.2}$$

Fano[4] and Landau and Lifschitz[5] use

$$\langle j_1\, m_1, j_2\, m_2, j_3\, m_3 | 0 \rangle = S_{j_1 m_1; j_2 m_2; j_3 m_3} = (-1)^{j_1-j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{1.3}$$

respectively. Racah's[6] symmetrized vector-coupling coefficient is

$$V(j_1\, j_2\, j_3;\, m_1\, m_2\, m_3) = (-1)^{j_1-j_2-j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{1.4}$$

the 3-j and 6-j symbols

The symmetric formula given by Racah[6] for the Clebsch-Gordan coefficient can be modified according to Eq. 1.1 to give

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}$$
$$\times\left(\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!}{(j_1+j_2+j_3+1)!}\right)^{1/2}$$
$$\sum_k \frac{(-1)^k}{k!(j_1+j_2-j_3-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j_3-j_2+m_1+k)!(j_3-j_1-m_2+k)!}. \tag{1.5}$$

Computations were made using this formula only, as described on page 37.

Properties of the 3-j Symbols

Symmetry Properties. The symmetry properties of the 3-j symbol are given by[7,8]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_k & j_l & j_n \\ m_k & m_l & m_n \end{pmatrix} \tag{1.6}$$

for (k, l, n), an even permutation of $(1, 2, 3)$, and

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_k & j_l & j_n \\ m_k & m_l & m_n \end{pmatrix} \tag{1.7}$$

for (k, l, n), an odd permutation of $(1, 2, 3)$.

It is also true that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{1.8}$$

Written out in detail, Eq. 1.6 is

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_2 & m_3 & m_1 \end{pmatrix}, \tag{1.6a}$$

and Eq. 1.7 is

$$(-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}. \tag{1.7a}$$

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The 3-j symbol automatically equals zero unless both $m_1 + m_2 + m_3 = 0$, and the vectors $\mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3 = 0$. The last equality implies that j_1, j_2, and j_3 satisfy the triangle conditions:

$$j_1 + j_2 - j_3 \geq 0; j_1 - j_2 + j_3 \geq 0; -j_1 + j_2 + j_3 \geq 0, \tag{1.9}$$

and the sum $j_1 + j_2 + j_3$ must be an integer.

Symmetries in addition to those shown in Eqs. 1.6, 1.7, and 1.8 have been recently discovered by Regge.[9] Write the 3-j symbol as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{bmatrix} -j_1 + j_2 + j_3 & j_1 - j_2 + j_3 & j_1 + j_2 - j_3 \\ j_1 - m_1 & j_2 - m_2 & j_3 - m_3 \\ j_1 + m_1 & j_2 + m_2 & j_3 + m_3 \end{bmatrix}. \tag{1.10}$$

The following symmetries exist:

(a) The columns of the square-bracket symbol may be permuted. The symbol is multiplied by $(-1)^{j_1+j_2+j_3}$ if the permutation is odd. This symmetry corresponds to the ones shown in Eqs. 1.6 and 1.7.

(b) The rows may be permuted. The symbol is again multiplied by $(-1)^{j_1+j_2+j_3}$ if the permutation is odd. Note that the interchange of the bottom two rows corresponds to the symmetry shown in Eq. 1.8, but the others are new.

(c) The symbol, regarded as a matrix, may be transposed about the major diagonal. This symmetry is also new.

Also about minor diagonal

minor

major

For example, symmetry (b), for interchange of the top two rows, implies that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= (-1)^{j_1+j_2+j_3} \begin{pmatrix} \frac{1}{2}(j_2+j_3-m_1) & \frac{1}{2}(j_1+j_3-m_2) & \frac{1}{2}(j_1+j_2-m_3) \\ j_1-\frac{1}{2}(j_2+j_3+m_1) & j_2-\frac{1}{2}(j_1+j_3+m_2) & j_3-\frac{1}{2}(j_1+j_2+m_3) \end{pmatrix}. \tag{1.11}$$

Interchange of top and bottom rows gives the same result. Symmetry (c) yields

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & \frac{1}{2}(j_2 + j_3 + m_1) & \frac{1}{2}(j_2 + j_3 - m_1) \\ j_2 - j_3 & m_3 - \frac{1}{2}(j_2 - j_3 - m_1) & -m_3 - \frac{1}{2}(j_2 - j_3 + m_1) \end{pmatrix}. \tag{1.12}$$

Note that all 72 symmetries leave the sum of each row of the 3-j symbol invariant.

Orthogonality Properties. The orthogonality properties of the 3-j symbols are

$$\sum_{j_3 m_3} (2j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta(m_1, m_1')\delta(m_2, m_2') \tag{1.13}$$

and

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta(j_3, j_3')\delta(m_3, m_3')}{(2j_3 + 1)}. \tag{1.14}$$

Since the m's must sum to zero, some of the m summations in Eqs. 1.13 and 1.14 are purely formal.

Note that the 3-j symbols are not matrix elements of a unitary transformation.

Uses of the 3-j Symbols

Coupling of Two Angular Momenta. The 3-j symbols play their most fundamental role in physical systems where two angular momenta vector-couple to form a resultant that is a good quantum number. The two angular momenta may involve the orbital angular momenta of two particles or the orbital angular momentum and the intrinsic spin of the same particle. In gamma emission or absorption, the two coupled systems are the nucleus and the photon that it loses or gains.

The system in the representation $(\mathbf{j}_1^2, \mathbf{j}_2^2, j_{1z}, j_{2z})$ is denoted by $|j_1 m_1 j_2 m_2)$. The coupled system in the representation $(\mathbf{j}_1^2, \mathbf{j}_2^2, \mathbf{j}^2, j_z)$ is denoted by $|j_1 j_2 jm)$.

The transformation from the representation $|j_1 j_2 m_1 m_2)$ to $|j_1 j_2 jm)$ is (see Eq. 1.1a)

$$|j_1 j_2 jm) = (-1)^{j_2 - j_1 - m} \sum_{m_1 m_2} (2j + 1)^{1/2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} |j_1 m_1 j_2 m_2) \tag{1.15}$$

where $m_1 + m_2 = m$. Conversely,

$$|j_1 m_1 j_2 m_2) = \sum_{j m} (-1)^{j_2 - j_1 - m} (2j + 1)^{1/2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} |j_1 j_2 jm) \tag{1.16}$$

where, again, $m_1 + m_2 = m$, and $|j_1 - j_2| \leq j \leq j_1 + j_2$.

It will be convenient later to speak of one representation being expanded in terms of the other. In the case at hand, the expansion coefficients are, obviously,

$$(j_1 j_2 jm | j_1 m_1 j_2 m_2) = (-1)^{j_2 - j_1 - m} (2j + 1)^{1/2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}. \tag{1.17}$$

the 3-j symbols

The Matrix Elements of Spherical Harmonics and Tensor Operators. A common integral in atomic and nuclear theory is

$$(l'\ m'|Y_{LM}|l\ m) = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, Y^*_{l'm'}\, Y_{LM}\, Y_{lm} \tag{1.18}$$

where $Y_{lm} = Y_{lm}(\theta, \varphi)$ is the usual spherical harmonic.[7,10] These integrals are sometimes called Gaunt's coefficients.[11] They are easily expressed in terms of the 3-j symbols:

$$(l'\ m'|Y_{LM}|l\ m) = (-1)^{-m'} \left[\frac{(2l'+1)(2L+1)(2l+1)}{4\pi}\right]^{1/2} \times \begin{pmatrix} l' & L & l \\ -m' & M & m \end{pmatrix} \begin{pmatrix} l' & L & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.19}$$

Since $Y_{l0}(\theta, \varphi) = [(2l+1)/4\pi]^{1/2} P_l\,(\cos\theta)$, Eq. 1.19 can be specialized to an integral over three Legendre polynomials. This integral is expressed in Eq. 1.44.

In terms of Racah's irreducible tensor operators[6] where $C_M^{(J)}$ can be defined as

$$C_M^{(J)} = \left(\frac{4\pi}{2J+1}\right)^{1/2} Y_{JM}, \tag{1.20}$$

the matrix elements are

$$(l'\ m'|C_M^{(L)}|l\ m) = (-1)^{-m'}\,[(2l'+1)(2l+1)]^{1/2} \begin{pmatrix} l' & L & l \\ -m' & M & m \end{pmatrix} \begin{pmatrix} l' & L & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.21}$$

Indiscriminate use of Eq. 1.21 will lead to difficulties. *Pseudo*tensors such as the angular momentum operator J must be composed properly through the Clebsch-Gordan series. Tensor products of operators are discussed on p. 22.

The Wigner-Eckart theorem[12,13] states that Eq. 1.19 or Eq. 1.21 can be factored into two parts, one of which is independent of the magnetic numbers m', m, and M. This factor is known as the *reduced matrix element* or the *double-bar matrix element.*† It is expressed as

$$(l'\|Y_L\|l) = (-1)^{l'} \left[\frac{(2l'+1)(2L+1)(2l+1)}{4\pi}\right]^{1/2} \begin{pmatrix} l' & L & l \\ 0 & 0 & 0 \end{pmatrix} \tag{1.22}$$

† The reduced matrix element is usually expressed in terms of the Clebsch-Gordan coefficient: $(j'\ m'|C_M^{(J)}|j\ m) = C(j\,J\,j';\, m\,M\,m')(j'\|C^{(J)}\|j)$. The definition in Eq. 1.22 therefore differs from the usual convention by a factor and phase, but it is more in keeping with the use of the 3-j symbols. Our convention is the same as Racah's[6] and Wigner's[1] but differs from that of Condon and Shortley.[2]

or

$$(l'\|C^{(L)}\|l) = (-1)^{l'}[(2l'+1)(2l+1)]^{1/2}\begin{pmatrix} l' & L & l \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.23}$$

so that the matrix elements of Eqs. 1.19 and 1.21 may be written as

$$(l'\,m'|Y_{LM}|l\,m) = (-1)^{l'-m'}(l'\|Y_L\|l)\begin{pmatrix} l' & L & l \\ -m' & M & m \end{pmatrix} \tag{1.24}$$

and similarly,

$$(l'\,m'|C_M^{(L)}|l\,m) = (-1)^{l'-m'}(l'\|C^{(L)}\|l)\begin{pmatrix} l' & L & l \\ -m' & M & m \end{pmatrix}. \tag{1.25}$$

The equations analogous to Eqs. 1.22 and 1.23 for half-integer j' and j are, respectively,

$$((l', \tfrac{1}{2})j'\|Y_L\|(l, \tfrac{1}{2})j)$$
$$= (-1)^{j'+1/2}\left[\frac{(2j'+1)(2L+1)(2j+1)}{4\pi}\right]^{1/2}\begin{pmatrix} j' & L & j \\ \tfrac{1}{2} & 0 & -\tfrac{1}{2} \end{pmatrix}\tfrac{1}{2}[1+(-1)^{l'+L+l}]$$

and

$$((l', \tfrac{1}{2})j'\|C^{(L)}\|(l, \tfrac{1}{2})j)$$
$$= (-1)^{j'+1/2}[(2j'+1)(2j+1)]^{1/2}\begin{pmatrix} j' & L & j \\ \tfrac{1}{2} & 0 & -\tfrac{1}{2} \end{pmatrix}\tfrac{1}{2}[1+(-1)^{l'+L+l}].$$

Here it is assumed that j' and j have been formed by coupling l' and l to spin one-half particles.

The Gradient Formula. The following derivation of tne gradient formula was obtained by Schwartz.[14] Other derivations may be found in Rose[10] and in Edmonds.[7]

The derivation centers about the commutator identity,

$$\nabla = \tfrac{1}{2}[\nabla^2, \mathbf{r}]. \tag{1.26}$$

In what follows, the notations ∇_μ and r_μ refer to components of the rank-one spherical tensors

$$\nabla_0 = \frac{\partial}{\partial z}, \qquad \nabla_{\pm 1} = \mp\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right), \tag{1.27}$$

and similarly,

$$r_0 = z, \qquad r_{\pm 1} = \mp(x \pm iy). \tag{1.28}$$

That is,

$$r_\mu = rC_\mu^{(1)} \tag{1.29}$$

where r (no subscript) is the scalar length and $C_\mu^{(1)}$ is the tensor operator of rank one.