Drexel Physics 2005 Modern Qual Solutions

2014 entering class

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1 Problem 1

2 Problem 3

Electron tunneling in transistors.

3 Problem A1

We see that $S_x = \frac{S_+ + S_-}{2}$. We find the eigenvalues in the usual manner:

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \tag{3.1}$$

$$\begin{vmatrix} -\lambda & \sqrt{2}/2\hbar & 0 \\ \sqrt{2}/2\hbar & -\lambda & \sqrt{2}/2\hbar \\ 0 & \sqrt{2}/2\hbar & -\lambda \end{vmatrix} = 0 \tag{3.2}$$

$$-\lambda(\lambda^2 - \hbar^2/2) - \hbar\sqrt{2}/2(-\lambda\hbar\sqrt{2}/2) = 0 \tag{3.3}$$

$$-\lambda^3 + \lambda\hbar^2 = 0 \tag{3.4}$$

$$\lambda = \{-\hbar, \hbar\} \tag{3.5}$$

We now solve $S_x\mathbf{e} = \lambda\mathbf{e}$ for the eigenvectors.

$$e_1 = e_3 \tag{3.6}$$

$$e_2 = \frac{1}{\lambda}\sqrt{2}\hbar e_1 \tag{3.7}$$

$$\lambda = -\hbar: \;\; \mathbf{e}_{-\hbar} = \{1, -\sqrt{2}, 1\}\frac{1}{2} \tag{3.8}$$

$$\lambda = \hbar: \;\; \mathbf{e}_{\hbar} = \{1, \sqrt{2}, 1\}\frac{1}{2} \tag{3.9}$$

We now calculate the probability of measuring $S_x = \hbar$. (The question says $S+x=1$, but 1 is not an eigenvalue of S_x.

$$P(S_x = \hbar) = \langle e_\hbar | u \rangle^2 = \left(\frac{1}{2\sqrt{2c}} (1*1 + 4*\sqrt{2} - 3*1) \right)^2 \tag{3.10}$$

$$P(S_x = \hbar) = \frac{1}{2c}(9 - 4\sqrt{2}) \tag{3.11}$$

After the measurement the system has a definite value of $S_x = \hbar$ and is in state $e_\hbar$. The eigenvector corresponding to $S_z = \hbar$ is $(1, 0, 0)$, so we calculate the probability the same way.

$$P(S_z = \hbar) = \langle \mathbf{e_1} | \mathbf{e}_\hbar \rangle^2 = \left(1 * \frac{1}{2} + 0 + 0 \right)^2 \tag{3.12}$$

$$P(S_z = \hbar) = \frac{1}{4} \tag{3.13}$$

4 Problem A2

a) We open up the brackets and find $\Psi(x,0) = 2x^3 - 10x_1^2 2x$.

$$\int_0^3 (C\Psi(x,0))^2 dx = 1 \tag{4.1}$$

$$\frac{1}{C^2} = \left(\int_0^3 2x^3 - 10x^2 + 12x dx \right)^2 \tag{4.2}$$

$$\frac{1}{C^2} = (-\frac{1}{2})^2 = \frac{1}{4} \tag{4.3}$$

$$C = 2 \tag{4.4}$$

$$\Psi(x,0) = 2(2x^3 - 10x^2 + 12x) \tag{4.5}$$

b) The wavefunction has a single 0 in the range at $x = 2$, so it most closely resembles the standard wavefunction $\sin \frac{2\pi}{3} x$. c) The wavefunctions can be derived from the Schrodinger equation:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = e\psi \tag{4.6}$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \tag{4.7}$$

With $k \equiv \frac{\sqrt{2mE}}{\hbar}$, the solutions are of the form $A\cos kx + B\sin kx$. The inifinte square well requires $\psi(0) = 0$, so $A = 0$. This square well also requires $\psi(3) = 0$, so k takes on discrete values determined by:

$$\sin\frac{n\pi x}{3} = 0 \tag{4.8}$$

$$k = \frac{n\pi}{3},\ n = 1, 2, 3... \tag{4.9}$$

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{3} \tag{4.10}$$

$$E = \frac{n^2\pi^2\hbar^2}{18m},\ n = 1, 2, 3... \tag{4.11}$$

So we can estimate the expectation value of the energy for $n = 2$ as $\frac{2\pi^2\hbar^2}{9m}$.

5 Problem B1

a)
i) Each distinguishable particle may be in one of 4 states, so there are 16 total states. 4 states have energy 0, 8 states have energy ϵ, and 4 states have energy 2ϵ.

$$Z = \sum_1^8 e^{-\beta\epsilon} + \sum_1^8 e^{-2\beta\epsilon} + \sum_1^8 \tag{5.1}$$

$$Z = 4(1 + 2e^{-\beta\epsilon} + 1e^{-2\beta\epsilon}) \tag{5.2}$$

ii) For Bosons there are three spin combinations and three energy combinations for a total of 9 states. Three of the states have energy 0, three have energy ϵ and three have energy 2ϵ.

$$Z = 3(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}) \tag{5.3}$$

iii) For Fermions the Pauli exclusion principle will eliminate the states with both particles having the same spin. There are now 3 states, with one for each possible value of the energy.

$$Z = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} \tag{5.4}$$

b) As $T \to \infty$, $\beta = \frac{1}{kT} \to 0$ and $e^{-\beta\epsilon} \to 1$. For the classical particles, each state will have probability $\frac{1}{16}$, so a quarter of the systems will have $E = 0$.
For the bosons, each state will have probability $\frac{1}{9}$, so one third of the systems will have $E = 0$.
For the fermions each state will have probability $\frac{1}{3}$, so one third of the systems will have $E = 0$.
c) We can't use $\langle E\rangle = \frac{\partial \ln Z}{\partial(-\beta)}$ since we have a small number of states. Instead,

we calculate $\langle E \rangle$ directly.
For the classical particles:

$$\langle E \rangle = \frac{\sum E_s e^{-\beta E_s}}{Z} \tag{5.5}$$

$$= \frac{4 \cdot 0 + 8\epsilon e^{-\beta\epsilon} + 4 \cdot 2\epsilon e^{-2\beta\epsilon}}{4(1 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon})} \tag{5.6}$$

$$= 2\epsilon \frac{e^{-\beta\epsilon} + e^{-2\beta\epsilon}}{1 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{5.7}$$

For the bosons:

$$\langle E \rangle = \frac{\sum E_s e^{-\beta E_s}}{Z} \tag{5.8}$$

$$= \frac{3 \cdot 0 + 3\epsilon e^{-\beta\epsilon} + 3 \cdot 2\epsilon e^{-2\beta\epsilon}}{3(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon})} \tag{5.9}$$

$$= \epsilon \frac{e^{-\beta\epsilon} + 2e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{5.10}$$

For the fermions:

$$\langle E \rangle = \frac{\sum E_s e^{-\beta E_s}}{Z} \tag{5.11}$$

$$= \frac{0 + \epsilon e^{-\beta\epsilon} + 2\epsilon e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{5.12}$$

$$= \epsilon \frac{e^{-\beta\epsilon} + 2e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{5.13}$$

d) In the high temperature limit, $\langle E \rangle = \epsilon$ for all types of particles.

6 Problem B2

a) The heat capacity $C_v = (\frac{\partial U}{\partial T})_V$. If the heat capacity is constant then $U \sim T$. Then $A = U - TS = CT - TS = T(C - S)$ with C some constant.
b) For a monatomic ideal gas the Hamiltonian is $H = \frac{p^2}{2m}$. The partition function of one atom is:

$$Z_1 = \frac{1}{h^3} \int d^3x \; d^3p \; e^{-\beta \frac{p^2}{2m}} \tag{6.1}$$

$$Z_1 = \frac{V}{h^3} \int d^3p \; e^{-\beta \frac{p^2}{2m}} \tag{6.2}$$

Each momentum component integral $\int_0^\infty e^{-\beta\frac{p^2}{2m}}dp = \sqrt{2m\pi kT}$. With three independent components, the total partition function of one atom is:

$$Z_1 = \frac{V}{h^3}(2m\pi kT)^{3/2} \tag{6.3}$$

$$Z_N = \frac{1}{N!}(\frac{V}{h^3})^N(2m\pi kT)^{3N/2} \tag{6.4}$$

The Helmholtz free energy A is $A = -kT\ln Z_N$.

$$A = -NkT(\ln\frac{V}{N!} + \frac{3}{2}\ln\frac{2m\pi kT}{h^2}) \tag{6.5}$$

c) The energy levels of the harmonic oscillator are $(n+\frac{1}{2})\hbar\omega$. The partition function for one atom is:

$$\sum_{s=0}^{\infty} e^{-\beta(s+\frac{1}{2})\hbar\omega} = e^{-\beta\frac{1}{2}\hbar\omega}(1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + ...) \tag{6.6}$$

$$= e^{-\beta\frac{1}{2}\hbar\omega}(\frac{1}{1-e^{\beta\hbar\omega}}) \tag{6.7}$$

The partition function for N atoms is $Z_1^N = \left(e^{-\beta\frac{1}{2}\hbar\omega}(\frac{1}{1-e^{\beta\hbar\omega}})\right)^N$. The Helmholtz free energy is:

$$A = -kT\ln Z_N \tag{6.8}$$

$$A = -NkT(-\beta\frac{1}{2}\hbar\omega + \beta\hbar\omega) \tag{6.9}$$

$$A = -\frac{N}{2}\hbar\omega \tag{6.10}$$