PHYS 502: Mathematical Physics II

Winter 2014

Solutions to Homework #2

1. (a) Writing $\xi = x - ct$ and seeking solutions $\psi(\xi)$, we have

$$
\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{d\psi}{d\xi}\frac{\partial \xi}{\partial t} = -c\psi'(\xi) \\
\frac{\partial \psi}{\partial x} &= \psi'(\xi) \\
\frac{\partial^3 \psi}{\partial x^3} &= \psi'''(\xi),
\end{aligned}
$$

so the equation becomes

$$(\psi - c)\psi' + \psi''' = 0.$$

Integrating once, we have

$$\tfrac{1}{2}\psi^2 - c\psi + \psi'' = 0$$

so

$$\psi'' = c\psi - \tfrac{1}{2}\psi^2.$$

(b) Multiplying by ψ' and integrating again, we have

$$\left(\psi'\right)^2 = c\psi^2 - \tfrac{1}{3}\psi^3$$

or

$$\psi' = \psi\left(c - \tfrac{1}{3}\psi\right)^{1/2}.$$

Hence, writing $u = \psi/3c$, we have

$$
\begin{aligned}
\xi &= \int \frac{d\psi}{\psi(c - \frac{1}{3}\psi)^{1/2}} \\
&= \frac{1}{\sqrt{c}} \int \frac{du}{u(1-u)^{1/2}} \\
&= \frac{1}{\sqrt{c}} \log\left(\frac{1 - \sqrt{1-u}}{1 + \sqrt{1-u}}\right).
\end{aligned}
$$

Inverting, and after some algebra, we find

$$\psi = \frac{3c}{\cosh^2 \sqrt{c}\xi/2},$$

which represents a non-dispersive, traveling nonlinear wave.

2. For the PDE
$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} = 0,$$
the two solutions of the characteristic equation
$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$
are
$$\begin{aligned}\xi(x,y) &= \text{constant},\\ \eta(x,y) &= \text{constant}.\end{aligned}$$
Hence, along a characteristic,
$$\frac{dy}{dx} = -\frac{\partial \xi}{\partial x} \Big/ \frac{\partial \xi}{\partial y} = -\xi_x/\xi_y,$$
so ξ satisfies
$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0, \tag{1}$$
and similarly for η. We want to use ξ and η as coordinates and write the PDE in terms of them. We assume that the functions A, B, and C can always be written explicitly in terms of ξ and η (which is in principle true, but often difficult in practice!).

We start by expanding
$$\begin{aligned}\psi_x &= \psi_\xi\xi_x + \psi_\eta\eta_x,\\ \psi_{xx} &= (\psi_{\xi\xi}\xi_x + \psi_{\xi\eta}\eta_x)\xi_x + \psi_\xi\xi_{xx} + (\psi_{\xi\eta}\xi_x + \psi_{\eta\eta}\eta_x)\eta_x + \psi_\eta\eta_{xx}\\ &= \psi_{\xi\xi}\xi_x^2 + 2\psi_{\xi\eta}\xi_x\eta_x + \psi_{\eta\eta}\eta_x^2 + \psi_\xi\xi_{xx} + \psi_\eta\eta_{xx}.\end{aligned}$$
Similarly, we find
$$\begin{aligned}\psi_{yy} &= \psi_{\xi\xi}\xi_y^2 + 2\psi_{\xi\eta}\xi_y\eta_y + \psi_{\eta\eta}\eta_y^2 + \psi_\xi\xi_{yy} + \psi_\eta\eta_{yy},\\ \psi_{xy} &= \psi_{\xi\xi}\xi_x\xi_y + \psi_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \psi_{\eta\eta}\eta_x\eta_y + \psi_\xi\xi_{xy} + \psi_\eta\eta_{xy}.\end{aligned}$$
Combining terms, the coefficients of $\psi_{\xi\xi}$ and $\psi_{\eta\eta}$ are, respectively, $A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2$ and $A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2$, which are both zero, by Equation (1), so
$$\begin{aligned}A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} &= 2[A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y]\psi_{\xi\eta} + D(\xi,\eta,\psi_\xi,\psi_\eta)\\ &= 0,\end{aligned}$$
where the function D involves only first derivatives of ψ (and in fact is linear in them). Dividing through by the coefficient of $\psi_{\xi\eta}$ brings the equation into the desired form.

3. (a) In this case, $A = 1, B = 0, C = -c(x)^2$, and the characteristic equation is
$$\left(\frac{dx}{dt}\right)^2 = c(x)^2,$$
the solutions to which are
$$t = \pm\int_{x_0}^{x} \frac{ds}{c(s)}.$$

For $c(x) = c_0(1 + |x|/a)^{-1}$, we find

$$c_0 t = \pm \int^x ds\,(1 + |s|/a) = \pm\left[x + \mathrm{sign}(x)\frac{x^2}{2a}\right] + \mathrm{constant}.$$

In the language of the previous question, we have

$$\xi, \eta = x + \mathrm{sign}(x)\frac{x^2}{2a} \pm c_0 t.$$

(b) For $a \to \infty$, we have $c(x) = c_0$, and the characteristics are simply given by $x \pm c_0 t =$ constant. As discussed in class, the solution is $\psi(x,t) = f(\xi) + g(\eta)$, where $\xi = x + c_0 t, \eta = x - c_0 t$. Applying the initial conditions at $t = 0$, we have

$$\begin{aligned} f(x) + g(x) &= 0, \\ c_0 f'(x) - c_0 g'(x) &= e^{-|x|}, \end{aligned}$$

so

$$\begin{aligned} -g'(x) = f'(x) &= e^{-|x|}/2c_0, \\ -g(x) = f(x) &= \frac{1}{2c_0}\int^x e^{-|s|}ds \;=\; -\,\mathrm{sign}(x)e^{-|x|}/2c_0 + \mathrm{constant}, \end{aligned}$$

and hence

$$\psi(x,t) = f(x + c_0 t) - f(x - c_0 t) = \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} e^{-|s|}ds\,.$$

4. It is most convenient to work in terms of $T' = T - T_0$, so

$$\nabla^2 T' = \frac{1}{\kappa}\frac{\partial T'}{\partial t}\,,$$

with $T' = -T_0$ initially inside the cube and $T' = 0$ on the surface. As usual, we separate out the time dependence $e^{-\alpha\kappa t}$, so the spatial part of the solution $\chi(x,y,z)$ satisfies

$$\nabla^2\chi + \alpha\chi = 0.$$

Separating in x, y, and z, we find that, to satisfy the boundary conditions at $x, y, z = 0$, χ must be a sum of terms of the form

$$\chi \sim \sin ax\,\sin by\,\sin cz\,.$$

Applying the boundary conditions at $x, y, z = L$ gives

$$a = \frac{k\pi}{L}, \quad b = \frac{l\pi}{L}, \quad c = \frac{m\pi}{L},$$

and

$$\alpha = \alpha_{klm} = a^2 + b^2 + c^2 = \frac{\pi^2}{L^2}\left(k^2 + l^2 + m^2\right).$$

Thus the general solution satisfying the differential equation and the boundary conditions is

$$T = T_0 + \sum_{k,l,m} a_{klm}\,\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{l\pi y}{L}\right)\sin\left(\frac{m\pi z}{L}\right)e^{-\alpha_{klm}\kappa t}\,.$$

We determine the coefficients a_{klm} by enforcing the initial condition, $T = 0$, or $T' = -T_0$, so

$$
\begin{aligned}
a_{klm} &= \frac{8}{L^3} \int_0^L dx \int_0^L dy \int_0^L dz\, (-T_0) \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \\
&= -\frac{8T_0}{L^3} \left[\frac{L}{k\pi}\left\{1-(-1)^k\right\}\right] \left[\frac{L}{l\pi}\left\{1-(-1)^l\right\}\right] \left[\frac{L}{m\pi}\left\{1-(-1)^m\right\}\right] \\
&= \begin{cases} -\dfrac{64T_0}{klm\pi^3} & (k,l,m \text{ all odd}) \\ 0 & (\text{otherwise}) \end{cases}
\end{aligned}
$$

and hence

$$
T(x,y,z,t) = T_0 \left[1 - \frac{64}{\pi^3} \sum_{\substack{k,l,m \\ \text{odd}}} \frac{1}{klm} \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) e^{-\alpha_{klm}\kappa t}\right].
$$

5. (a) Schrödinger's equation is

$$
(\nabla^2 + k^2)\psi = 0,
$$

where $k^2 = 2mE/\hbar^2$. The boundary conditions are that $\psi = 0$ on all surfaces of a cylinder of radius R and height H. Take the axis of the cylinder to have $r = 0$ in cylindrical polar coordinates, and the flat faces to lie at $z = 0$ and $z = H$. The general form of the solution is a sum of terms of the form

$$
\psi \sim J_m(\beta r) e^{im\phi} \sin lz,
$$

where $\beta^2 + l^2 = k^2$ and the $\sin lz$ term is chosen to satisfy the boundary condition at $z = 0$. The boundary condition at $z = H$ then implies $lH = n\pi$, for integral n. The boundary condition at $r = R$ is $J_m(\beta R) = 0$, so $\beta R = \alpha_{mq}$, the q-th root of J_m. Hence

$$
E_{mqn} = \frac{\hbar^2 k_{mqn}^2}{2m} = \frac{\hbar^2}{2m}\left[\beta^2 + l^2\right] = \frac{\hbar^2}{2m}\left[\left(\frac{\alpha_{mq}}{R}\right)^2 + \left(\frac{n\pi}{H}\right)^2\right]
$$

for integral m, q, and n. Clearly the minimum energy corresponds to $m = 0, q = 1, n = 1$, so

$$
E_{min} = \frac{\hbar^2}{2m}\left[\left(\frac{\alpha_{01}}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2\right].
$$

Here, $\alpha_{01} = 2.405$. The corresponding (unnormalized) wavefunction is

$$
\psi \sim J_0\left(\frac{\alpha_{01}\, r}{R}\right) \sin\left(\frac{\pi z}{H}\right)
$$

(b) In two dimensions, similar reasoning to that in part (a) leads to the conclusion that the wavefunction must have the form

$$
\psi \sim J_m(kr)\, e^{im\theta}.
$$

The boundary condition $\psi = 0$ at $r = R$ implies $J_m(kR) = 0$. The boundary condition at $\theta = 0, \pi$ implies that the appropriate $\sim e^{im\theta}$ term is actually $\sin m\theta$, where m is a positive integer. The minimum k, and hence E, occurs at the lowest nonzero root of J_m for $m > 0$, corresponding to the first root of J_1, $\alpha_{11} = 3.83$. Hence the ground-state solution (again unnormalized) has

$$
\psi \sim J_1\left(\frac{\alpha_{11}\, r}{R}\right) \sin\theta, \qquad E = \frac{\hbar^2}{2m}\left(\frac{\alpha_{11}}{R}\right)^2.
$$