Drexel Physics 2005 Modern Qual Solutions

2014 entering class

September 15, 2015

1 Problem 1

We need expected value of the particle energy in the well to be less than $E = h\nu$. The energy levels of the finite square well are XXX. We then compute the partition function and take the derivative to find $\langle E \rangle$ as a function of temperature.

2 Problem 2

With $S = kT \ln \Omega$ and $\frac{\partial S}{\partial E} = \frac{1}{T}$, we need to add:

$$E = kT^2 \ln d\Omega = (Joules) = (calories) \tag{2.1}$$

to increase the number of accessible states by 1000.

3 Problem 3

Electron tunneling in transistors.
Angular momentum quantization as measured in the Stern-Gerlach experiment. When a beam of spin $\frac{1}{2}$ particles passes through a magnetic field they are split into two distinct beams instead of smeared over an angular region.

4 Problem 4

To compute the normalization constant we need to solve:

$$C^2 \int_V R^2 = 1 \tag{4.1}$$

$$\int (e^{-r/a})^2 \; r^2 \; dr = \frac{1}{C^2} \tag{4.2}$$

$$\int r^2 \; e^{-2r/a} \; dr = \frac{1}{C^2} \tag{4.3}$$

$$\frac{2}{(2/a)^3} = \frac{1}{C^2} \tag{4.4}$$

$$C = 2\left(\frac{Z}{a_0}\right)^{3/2} \tag{4.5}$$

5 Problem 5

Free expansion of a gas is one irreversible process. The theory of microscopic reversability only holds for systems that remain at equilibrium.

6 Problem 6

The cloud of gas also has gravitational potential energy and electronic potential energy, so the total system energy may decrease even if the kinetic energy increases.

7 Problem 7

8 Problem A1

We see that $S_x = \frac{S_+ + S_-}{2}$. We find the eigenvalues in the usual manner:

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \tag{8.1}$$

$$\begin{vmatrix} -\lambda & \sqrt{2}/2\hbar & 0 \\ \sqrt{2}/2\hbar & -\lambda & \sqrt{2}/2\hbar \\ 0 & \sqrt{2}/2\hbar & -\lambda \end{vmatrix} = 0 \tag{8.2}$$

$$-\lambda(\lambda^2 - \hbar^2/2) - \hbar\sqrt{2}/2(-\lambda\hbar\sqrt{2}/2) = 0 \tag{8.3}$$

$$-\lambda^3 + \lambda\hbar^2 = 0 \tag{8.4}$$

$$\lambda = \{-\hbar, \hbar\} \tag{8.5}$$

We now solve $S_x\mathbf{e} = \lambda\mathbf{e}$ for the eigenvectors.

$$e_1 = e_3 \tag{8.6}$$

$$e_2 = \frac{1}{\lambda}\sqrt{2}\hbar e_1 \tag{8.7}$$

$$\lambda = -\hbar: \ \mathbf{e}_{-\hbar} = \{1, -\sqrt{2}, 1\}\frac{1}{2} \tag{8.8}$$

$$\lambda = \hbar: \ \mathbf{e}_{\hbar} = \{1, \sqrt{2}, 1\}\frac{1}{2} \tag{8.9}$$

We now calculate the probability of measuring $S_x = \hbar$. (The question says $S + x = 1$, but 1 is not an eigenvalue of S_x.

$$P(S_x = \hbar) = \langle e_\hbar | u\rangle^2 = \left(\frac{1}{2\sqrt{2c}}(1 * 1 + 4 * \sqrt{2} - 3 * 1)\right)^2 \tag{8.10}$$

$$P(S_x = \hbar) = \frac{1}{2c}(9 - 4\sqrt{2}) \tag{8.11}$$

After the measurement the system has a definite value of $S_x = \hbar$ and is in state $e_\hbar$. The eigenvector corresponding to $S_z = \hbar$ is $(1, 0, 0)$, so we calculate the probability the same way.

$$P(S_z = \hbar) = \langle \mathbf{e_1} | \mathbf{e}_\hbar\rangle^2 = \left(1 * \frac{1}{2} + 0 + 0\right)^2 \tag{8.12}$$

$$P(S_z = \hbar) = \frac{1}{4} \tag{8.13}$$

9 Problem A2

a) We open up the brackets and find $\Psi(x,0) = 2x^3 - 10x_1^2 2x$.

$$\int_0^3 (C\Psi(x,0))^2 dx = 1 \tag{9.1}$$

$$\frac{1}{C^2} = \left(\int_0^3 2x^3 - 10x^2 + 12x dx\right)^2 \tag{9.2}$$

$$\frac{1}{C^2} = (-\frac{1}{2})^2 = \frac{1}{4} \tag{9.3}$$

$$C = 2 \tag{9.4}$$

$$\Psi(x,0) = 2(2x^3 - 10x^2 + 12x) \tag{9.5}$$

b) The wavefunction has a single 0 in the range at $x = 2$, so it most closely resembles the standard wavefunction $\sin \frac{2\pi}{3}x$. c) The wavefunctions can be derived from the Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = e\psi \tag{9.6}$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \tag{9.7}$$

With $k \equiv \frac{\sqrt{2mE}}{\hbar}$, the solutions are of the form $A\cos kx + B\sin kx$. The inifinte square well requires $\psi(0) = 0$, so $A = 0$. This square well also requires $\psi(3) = 0$, so k takes on discrete values determined by:

$$\sin\frac{n\pi x}{3} = 0 \tag{9.8}$$

$$k = \frac{n\pi}{3},\ n = 1, 2, 3... \tag{9.9}$$

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{3} \tag{9.10}$$

$$E = \frac{n^2\pi^2\hbar^2}{18m},\ n = 1, 2, 3... \tag{9.11}$$

So we can estimate the expectation value of the energy for $n = 2$ as $\frac{2\pi^2\hbar^2}{9m}$.

10 Problem A3

The energies of states are $E_0 = \frac{1}{2}\hbar\omega$ and $E_2 = \frac{5}{2}\hbar\omega$ so the time-evolving wavefunction is:

$$|\Psi(t)\rangle = C_0 e^{i\frac{1}{2}\omega t}|0\rangle + C_2 e^{i\frac{5}{2}\omega t}|2\rangle \tag{10.1}$$

b) We can write the momentum and position operators in terms of the creation and annihilation operators, which come from "factoring" the Hamiltonian.

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{10.2}$$

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} + i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} \hat{P} \tag{10.3}$$

$$a^\dagger = -\left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} + i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} \hat{P} \tag{10.4}$$

$$\hat{P} = -\frac{i}{2}\sqrt{2m\omega\hbar}(a + a^\dagger) \tag{10.5}$$

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{10.6}$$

We now apply $\hat{P}$ to the wavefunction.

$$\langle\hat{P}\rangle = \langle\Psi(t)|\hat{P}|\Psi(t)\rangle \tag{10.7}$$

$$\langle\hat{P}\rangle = (-\frac{1}{2}m\omega\hbar)\Psi(t)\left((a + a^\dagger)(C_0 e^{i\frac{1}{2}\omega t}|0\rangle + C_2 e^{i\frac{5}{2}\omega t}|2\rangle)\right) \tag{10.8}$$

$$\langle\hat{P}\rangle = (-\frac{1}{2}m\omega\hbar)\Psi^*(t)\left(0 + C_2 e^{i\frac{5}{2}\omega t}(-\hbar\omega)|1\rangle + C_0 e^{i\frac{1}{2}\omega t}(\hbar\omega)|1\rangle + C_2 e^{i\frac{5}{2}\omega t}(\hbar\omega)|3\rangle\right) \tag{10.9}$$

$$\langle\hat{P}\rangle = 0 \tag{10.10}$$

In the last step we used the orthogonality of the harmonic oscillator eigenfunctions, since $\Psi(t)$ only includes states with energy levels 0 and 2. c) The expectation value of $\hat{X}$ is:

$$\langle\hat{X}\rangle = \langle\Psi(t)|\hat{X}|\Psi(t)\rangle \tag{10.11}$$

$$\tag{10.12}$$

Since $\hat{X} == \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$, we again see the combination of creation and annihilation operators. Once again the orthogoonality of the harmonic oscillator eigenfunctions and the fact that the initial wavefunction contains only states 0 and 2 cause the expectation value to be 0. The expectation of both $\hat{P}$ and $\hat{X}$ will be 0 whenever the initial state contains only odd levels or only even levels, corresponding to combinations of only odd/even wavefunctions.

11 Problem B1

a)
i) Each distinguishable particle may be in one of 4 states, so there are 16

total states. 4 states have energy 0, 8 states have energy ϵ, and 4 states have energy 2ϵ.

$$Z = \sum_1^8 e^{-\beta\epsilon} + \sum_1^8 e^{-2\beta\epsilon} + \sum_1^8 \tag{11.1}$$

$$Z = 4(1 + 2e^{-\beta\epsilon} + 1e^{-2\beta\epsilon}) \tag{11.2}$$

ii) For Bosons there are three spin combinations and three energy combinations for a total of 9 states. Three of the states have energy 0, three have energy ϵ and three have energy 2ϵ.

$$Z = 3(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}) \tag{11.3}$$

iii) For Fermions the Pauli exclusion principle will eliminate the states with both particles having the same spin. There are now 3 states, with one for each possible value of the energy.

$$Z = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} \tag{11.4}$$

b) As $T \to \infty$, $\beta = \frac{1}{kT} \to 0$ and $e^{-\beta\epsilon} \to 1$. For the classical particles, each state will have probability $\frac{1}{16}$, so a quarter of the systems will have $E = 0$.
For the bosons, each state will have probability $\frac{1}{9}$, so one third of the systems will have $E = 0$.
For the fermions each state will have probability $\frac{1}{3}$, so one third of the systems will have $E = 0$.
c) We can't use $\langle E \rangle = \frac{\partial \ln Z}{\partial(-\beta)}$ since we have a small number of states (?? I'm not actually sure if this is correct, I may have just screwed up the calculation). Instead, we calculate $\langle E \rangle$ directly.
For the classical particles:

$$\langle E \rangle = \frac{\sum E_s e^{-\beta E_s}}{Z} \tag{11.5}$$

$$= \frac{4 \cdot 0 + 8\epsilon e^{-\beta\epsilon} + 4 \cdot 2\epsilon e^{-2\beta\epsilon}}{4(1 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon})} \tag{11.6}$$

$$= 2\epsilon \frac{e^{-\beta\epsilon} + e^{-2\beta\epsilon}}{1 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{11.7}$$

For the bosons:

$$\langle E \rangle = \frac{\sum E_s e^{-\beta E_s}}{Z} \tag{11.8}$$

$$= \frac{3 \cdot 0 + 3\epsilon e^{-\beta\epsilon} + 3 \cdot 2\epsilon e^{-2\beta\epsilon}}{3(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon})} \tag{11.9}$$

$$= \epsilon \frac{e^{-\beta\epsilon} + 2e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{11.10}$$

For the fermions:

$$\langle E \rangle = \frac{\sum E_s e^{-\beta E_s}}{Z} \tag{11.11}$$

$$= \frac{0 + \epsilon e^{-\beta\epsilon} + 2\epsilon e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{11.12}$$

$$= \epsilon \frac{e^{-\beta\epsilon} + 2e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \tag{11.13}$$

d) In the high temperature limit, $\langle E \rangle = \epsilon$ for all types of particles.

12 Problem B2

a) The heat capacity $C_v = (\frac{\partial U}{\partial T})_V$. If the heat capacity is constant then $U \sim T$. Then $A = U - TS = CT - TS = T(C - S)$ with C some constant.
b) For a monatomic ideal gas the Hamiltonian is $H = \frac{p^2}{2m}$. The partition function of one atom is:

$$Z_1 = \frac{1}{h^3} \int d^3x \; d^3p \; e^{-\beta \frac{p^2}{2m}} \tag{12.1}$$

$$Z_1 = \frac{V}{h^3} \int d^3p \; e^{-\beta \frac{p^2}{2m}} \tag{12.2}$$

Each momentum component integral $\int_0^\infty e^{-\beta \frac{p^2}{2m}} dp = \sqrt{2m\pi kT}$. With three independent components, the total partition function of one atom is:

$$Z_1 = \frac{V}{h^3}(2m\pi kT)^{3/2} \tag{12.3}$$

$$\tag{12.4}$$

The system paritition function (indistinguishable particles) is:

$$Z_N = \frac{1}{N!}(\frac{V}{h^3})^N (2m\pi kT)^{3N/2} \tag{12.5}$$

The Helmholtz free energy A is $A = -kT \ln Z_N$.

$$A = -NkT(\ln \frac{V}{N!} + \frac{3}{2} \ln \frac{2m\pi kT}{h^2}) \tag{12.6}$$

c) The energy levels of the harmonic oscillator are $(n + \frac{1}{2})\hbar\omega$. The partition function for one atom is:

$$\sum_{s=0}^{\infty} e^{-\beta(s+\frac{1}{2})\hbar\omega} = e^{-\beta\frac{1}{2}\hbar\omega}(1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + ...) \tag{12.7}$$

$$= e^{-\beta\frac{1}{2}\hbar\omega}(\frac{1}{1 - e^{\beta\hbar\omega}}) \tag{12.8}$$

The partition function for N atoms is $Z_1^N = \left(e^{-\beta\frac{1}{2}\hbar\omega}(\frac{1}{1-e^{\beta\hbar\omega}})\right)^N$. The Helmholtz free energy is:

$$A = -kT \ln Z_N \tag{12.9}$$

$$A = -NkT(-\beta\frac{1}{2}\hbar\omega + \beta\hbar\omega) \tag{12.10}$$

$$A = -\frac{N}{2}\hbar\omega \tag{12.11}$$