

PHYS 502: Mathematical Physics II

Winter 2015, Homework #2

(Due February 2, 2015)

1. The Kortweg-de Vries (KdV) equation

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0$$

is a well-known two-dimensional nonlinear partial differential equation. In general, the solutions are *dispersive*, meaning that, in a solution of the form $e^{ikx-i\omega t}$, the frequency ω depends on the wavenumber k . However, non-dispersive *soliton* solutions also exist.

(a) Let $\xi = x - ct$ and seek a traveling wave solution $\psi(\xi)$. Rewrite the above equation in terms of ξ and show that it implies

$$(\psi - c) \frac{d\psi}{d\xi} + \frac{d^3 \psi}{d\xi^3} = 0,$$

and integrate to find

$$\frac{d^2 \psi}{d\xi^2} = c\psi - \frac{1}{2}\psi^2.$$

(b) Hence show that

$$\left(\frac{d\psi}{d\xi} \right)^2 = c\psi^2 - \frac{1}{3}\psi^3,$$

and solve this equation to find the solution $\psi(\xi)$.

2. A second-order linear partial differential equation in two dimensions (x, y) has the form

$$A(x, y) \frac{\partial^2 \psi}{\partial x^2} + 2B(x, y) \frac{\partial^2 \psi}{\partial x \partial y} + C(x, y) \frac{\partial^2 \psi}{\partial y^2} = 0.$$

As discussed in class, the characteristic equation for this system is

$$A \left(\frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0.$$

Assume that the system is hyperbolic, and denote the two solution families (corresponding to the two roots of the above quadratic equation) of this ODE by

$$\begin{aligned} \xi(x, y) &= \text{constant}, \\ \eta(x, y) &= \text{constant}. \end{aligned}$$

Transform the PDE to the (ξ, η) coordinate system, and show that it takes the form

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta} = \dots$$

where the (rather ugly) right-hand side depends only on known functions and first derivatives of ψ .

3. (a) Write down and solve the characteristic equation for the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c(x)^2 \frac{\partial^2 \psi}{\partial x^2} = 0.$$

where the signal speed is $c(x) = c_0 (1 + |x|/a)^{-1}$. Sketch some representative characteristic curves.

- (b) For the case $c = \text{constant}$ ($a \rightarrow \infty$) find, using the method of characteristics, the solution to the equation satisfying the initial conditions

$$\psi(x, 0) = 0, \quad \left. \frac{\partial \psi}{\partial t} \right|_{t=0} = e^{-|x|},$$

for $x > 0, t > 0$.

4. A uniform cube of side L initially is at temperature $T = 0$. At time $t = 0$ the cube is immersed in a heat bath of temperature $T_0 > 0$. The temperature within the cube obeys the diffusion equation

$$\nabla^2 T = \frac{1}{\kappa} \frac{\partial T}{\partial t}.$$

Write down a general expression for the temperature in the cube, fit it to the boundary and initial conditions, and hence derive a formula (in the form of an infinite sum) for the temperature at any point within the cube at any subsequent time.

5. (a) A particle of mass m is contained in a cylinder of radius R and height H . The particle is described by a wavefunction $\psi(\rho, \phi, z)$ satisfying

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E\psi,$$

where ρ, ϕ, z are cylindrical polar coordinates and $\psi = 0$ on the surface of the cylinder ($\rho = R, z = 0, H$). Find the ground-state energy of the system, and write down an explicit expression for the (unnormalized) lowest-energy wavefunction.

- (b) Repeat part (a), but now for a particle moving in *two* dimensions, within a semicircular region of radius R , again with $\psi = 0$ on the boundary of the region.