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Entanglement

in Quantum Information Theory

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List of Publications

Journal Publications

[E1] J. Eisert and M. Wilkens, "Catalysis of Entanglement Manipulation for Mixed States", *Physical Review Letters* **85**, 437 (2000).

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See Chapter 3.

[E2] J. Eisert, T. Felbinger, P. Papadopolous, M.B. Plenio, and M. Wilkens, "Classical Information and Distillable Entanglement", *Physical Review Letters* **84**, 1611 (2000).

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See Chapter 5.

[E3] J. Eisert and H.-J. Briegel, "The Schmidt Measure as a Tool for Quantifying Multi-Particle Entanglement", *Physical Review A* **64**, 022306 (2000).

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See Section 2.3.

[E4] J. Eisert, K. Jacobs, P. Papadopoulos, and M.B. Plenio, "Optimal Local Implementation of Non-Local Quantum Gates", *Physical Review A* **62**, 052317 (2000).

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See Chapter 4.

[E5] J. Eisert and M. Wilkens, "Quantum Games", Journal of Modern Optics 47, 2543 (2000).

(Lanl e-print quant-ph/0004076)

See Chapter 6.

[E6] J. Eisert, M. Wilkens, and M. Lewenstein, "Quantum Games and Quantum Strategies", *Physical Review Letters* **83**, 3077 (1999).

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See Chapter 6.

[E7] J. Eisert and M.B. Plenio, "A Comparison of Entanglement Measures", Journal of Modern Optics 46, 145 (1999).

(Lanl e-print quant-ph/9807034)

[E8] R.A. Gonzales, J. Eisert, I. Koltracht, M. Neumann, and G. Rawitscher, "Integral Equation Method for the Continuous Spectrum Radial Schrödinger Equation", *Journal of Computational Physics* **134**, 134 (1997).

(Compare also lanl e-print nucl-th/9802022)

[E9] K. Audenaert, J. Eisert, E. Jane, M.B. Plenio, and S. Virmani, "The Asymptotic Relative Entropy of Entanglement", published after submission as *Physical Review Letters* **87**, 217902 (2001).

See Subsection 2.2.5.

[E10] J. Eisert, K. Audenaert, and M.B. Plenio, "Entanglement Measures and Non-Local State Distinguishability", published after submission as *Journal of Physics A* **36**, 5605 (2003).

See Subsection 2.2.5 and Appendix C.

Miscellaneous

- [E11] J. Eisert, invited review of "M.J. Canty, Konfliktlösungen mit Mathematica. Zweiper-sonenspiele (Springer, Heidelberg, 2000)" in Physikalische Blätter, October issue (2000).
- [E12] J. Eisert, invited review of "R. Erd, *OnlineRecht kompakt: Von der Domain zum Download. Leitfaden für Internetnutzer* (Fachhochschulverlag, Frankfurt/Main, 2000)" in *Physikalische Blätter*, scheduled for March issue (2001).

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Any storage, transmission, and processing of information relies on a physical carrier [1]. In a handwritten note the sheet of paper serves as the carrier of information, in a desk top computer it is the random access memory and the hard drive on which the relevant data are stored. Communication makes use of sound waves, radio waves, or light pulses. The new field of *quantum information* is based on the idea that single quantum systems can be used as the elementary carriers of information, such as single photons, atoms, and ions. Quantum theory – the theory that describes physical systems on the atomic scale – opens up new possiblities for information processing and communication [2, 3, 4, 5, 3, 6]. Envisioned applications range from the factorization of large numbers on a quantum computer to communication protocols, and key distribution in quantum cryptography.

Quantum theory may become relevant to technical development in information processing mainly for two reasons. On the one hand, the information processing and storage units in ordinary, "classical" computers are becoming smaller and smaller. The dimensions of transistor elements in silicon-based microchips are decreasing to the extent that they will be approaching scales in which quantum effects become relevant in the near future (see Fig. I.1). On the other hand, it has become possible to store and manipulate single quantum systems, e.g., with sophisticated methods from quantum optics and solid state physics [2, 4].

The superior "performance" of quantum systems in computation and communication applications is predominantly rooted in a property of quantum mechanical states called *entanglement*. Essentially, entanglement comes along with new kinds of correlations. Entangled quantum states may show stronger statistical correlations than those attainable in a classical composite system, where the correlation is produced by a classical random generator. ¹

The first prototol for quantum cryptography – proposed in the early 1980s by S. Wiesner, C.H. Bennett, and G. Brassard – did not yet rely on entanglement. It made use of the fact that the state of a transmitted quantum system can under no circumstances be measured without introducing noise. As a consequence, a secure key can be established [16]. The general idea of the quantum computer was born at around the same time. R. Feynman envisaged a quantum system the purpose of which was not so much universal computing, but rather the simulation of the dynamics of other quantum systems [17]. Perhaps the

¹In 1935 A. Einstein, B. Podolsky, and N. Rosen (EPR) published a seminal paper entitled "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?", which started a long lasting debate about the status of quantum theory [7]. On the basis of the predicted outcomes of measurements on two spacelike separated quantum particles in an entangled state, EPR came to the conclusion that quantum mechanics could not be a complete theory, suggesting the view that additional hidden variables should be appended to a quantum state in order to restore causality and locality. N. Bohr, one of the spokesmen of the so-called Copenhagen school of the interpretation of quantum mechanics, argued against the assumption of a more complete underlying local deterministic level [8]. It was not until 1964 that J. Bell presented a way to empirically test the two competing hypotheses [9, 10]. Bell's theorem is not strictly about quantum mechanics. It is a statement concerning correlations of measurement outcomes at distant events that any physical theory may predict under the assumption of an underlying local classical model [11, 12]. Starting from the 1980s many experiments were performed, each in favor of quantum mechanics and against local hidden variable models [13, 14, 15].

first theoretical proposal of a quantum computer in the "modern" sense was set forth in 1985 by D. Deutsch [18]. Based on the work of C.H. Bennett who had demonstrated that a universal classical computing machine can be made reversible [19], he introduced the concept of quantum networks [20], and showed that any unitary operation can be generated by appropriately putting together ingredients taken from a small set of operations, called quantum gates. The registers of such a quantum computer are not classical binary registers, but two-level quantum systems with two orthogonal states. Quantum computers with a few registers can already be built with present technology. However, the experimental realization of a large-scale quantum computer is extraordinarily difficult and is hindered by challenging practical problems.

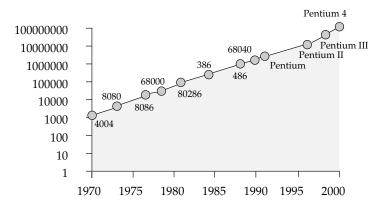


Figure I.1: This figure shows the number of transistors of several CPUs from 1970 to 2000. In 1965 G. Moore, later one of Intel's founders, formulated a hypothesis known as "Moore's Law". In retrospect the hypothesis proved to be in astonishing accordance with the technological development between 1970 and 2000. It states that "the number of transistors integrated on leading edge circuits in silicon-based chips would continue to double every 18 months". If the pace of development did not slow down (which does not seem very likely), one would have to expect quantum effects to become predominant from 2016 on, as the size of the information processing units would be of the same order of magnitude as the size of atoms.

Algorithms have been proposed in the framework of quantum computing which are capable of solving particular problems much more efficiently than any classical computer. The most prominent example is the celebrated polynomial-time algorithm of P. Shor for finding the prime factors of large integers [21, 3], an algorithm that astonished the scientific community in 1995. To date, no classical polynomial-time algorithm is known for this problem. L. Grover's quantum algorithm for search in a database offers a square root speedup compared to any classical algorithm [22]. It has become obvious that in order to be able to perform quantum information processing in the presence of noise, elaborate protection methods would be necessary. Such methods were developed under the names quantum error correction, fault tolerant quantum computing, quantum error correcting codes, and stabilizer codes, pioneered by the works of B.W. Schumacher, A.M. Steane, M. Nielsen, D.P. DiVincenzo, P.W. Shor, R. Calderbank, A. Ekert, E. Knill, R. Laflamme, and E. Rains. An overview of recent developments is given in the excellent comprehensive introductions in Refs. [2, 4, 5].

The performance of Shor's algorithm on a quantum computer can be traced back to properties of multi-particle entanglement, although entanglement enters in a rather subtle way. Several schemes in quantum cryptography rely on entanglement, and it plays a major role in considerations of quantum communication complexity [23]. It was only in recent

years that the significance of these correlations in quantum information theory was fully appreciated. To put it in a catchy manner, one may say that it is entanglement that makes quantum information theory different from its classical counterpart.

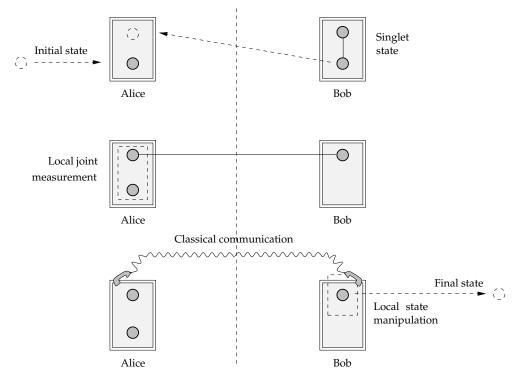


Figure I.2: The teleportation protocol as in Ref. [24].

On a purely theoretical level it is important to understand what kinds of tasks may be achieved with entangled quantum systems. It is impossible to transmit the particular "quantum information" of a quantum system through a classical channel. This means that the statistical predictions of quantum mechanics cannot fully be reproduced when trying to extract information about the preparation procedure from a state of one quantum system, transmitting the classical information through the channel, and preparing another quantum system in a certain state on the basis of this information. It is nevertheless possible to transfer a quantum state to a different quantum system at a distant location without physically transmitting the actual quantum system in the particular state – provided that the parties initially share a pair of two-level systems in a maximally entangled state. This transfer of a quantum state was named "teleportation", a term borrowed from science fiction literature.

The teleportation protocol, as illustrated in Fig. I.2, was proposed in 1993 by C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters. It represented a major breakthrough in the field [24]. Assume that one experimenter, from now on referred to as Alice, would like to send the unknown state of a given quantum system to Bob, another experimenter at a distant location. Bob prepares a bi-partite quantum system in a particular entangled state and gives one part of the system to Alice. In the next step Alice performs a local joint quantum measurement on both her quantum system and on the one she has received from Bob. Then she phones Bob and tells him about the outcome. Depending on the outcome of Alice's measurement, Bob can finally transform his part of the maximally

entangled system by use of a simple manipulation of the state. The state of his system is eventually identical to the state of Alice's original system: the state has been "sent" from Alice to Bob.

The significance of the proposal is not predominantly derived from its obvious practical implication. Rather, it has an immense paradigmatical value as it is a scheme that necessarily relies on entanglement. Another important proposal of this type, also by C.H. Bennett and S. Wiesner, is the dense coding protocol [25] concerning the transmission of classical information. A single quantum two-level system sent from Alice to Bob can carry a single bit of classical information. Surprisingly, if the two parties initially share a maximally entangled state, two bits of classical information can be transmitted with the same two-level system. Successful experimental quantum optical implementations of dense coding and teleportation were performed by A. Zeilinger [26] and by F. DeMartini [27], in addition to their respective co-workers. Many other applications of entanglement were suggested, the spectrum ranging from quantum cryptography using entangled quantum systems by B. Huttner and A. Ekert [28] to improved frequency standards [29] and clock synchronization [30].

In the wake of these developments, a more systematic theoretical exploration of entanglement has begun. To give a very fragmentary list of pioneering work in the field: The resource character of entanglement was first emphasized by C.H. Bennett, W. Wootters, and co-workers [24, 31, 32, 33]. They also presented ways of partly restoring the (extremely fragile) entanglement degraded in purity due to decoherence. These investigations have led to a more clearcut conception of how to quantify entanglement. The issue of quantification has also been systematically addressed by M.B. Plenio, V. Vedral, H.K. Lo, S. Popescu, N. Linden, and G. Vidal, to name just a few [34, 35, 32, 36, 37, 38]. It has become apparent that the state space of a composite quantum mechanical system contains a wealth of structures, and that not all entangled states can be transformed into maximally entangled states. For such states the term "bound entangled states" was coined, a concept introduced by the Horodecki family [39, 40]. The structure of the set of entangled states was further clarified by A. Peres, A. Sanpera, M. Lewenstein, I.C. Cirac, B. Terhal, and others (for a primer see Ref. [41]). M. Nielsen, G. Vidal, D. Jonathan, and M.B. Plenio investigated the question which transformations can be accomplished from one entangled state into another [42, 43, 44, 45]. The role of symmetry was emphasized and explored by R.F. Werner [46, 47]. Possibly triggered by the recent work on the theory of entanglement by physicists and mathematicians, there has also been a renewed interest in entanglement in the philosophy of physics; see, e.g., Refs. [48, 49] and the references therein.

This thesis deals with the resource character of quantum entanglement, in particular, with its quantification and its mathematical characterization. The introductory chapter reviews some important ideas that will be built on in later chapters. In particular, the concepts of states and operations in quantum mechanics will be explained, and the notions of separability and distillability will be introduced.

Chapter 2 deals with the quantification of entanglement. It will be investigated how entangled a given state of a bi-partite quantum system is. Related to this is the question of how well a particular task can be accomplished. Several entanglement monotones – proper measures of entanglement – will be proposed and their properties will be explored. The considerations will then be extended to multi-partite quantum systems.

In a sense, Chapter 2 already addresses the interconvertibility of the resource entanglement. Local operations alone supplemented with communication via a classical channel can only increase the classical correlations, but not the entanglement. But given a *single* quantum system in an entangled state, is it possible to transform the state into every other entangled state, provided that the "amount of entanglement" does not increase on average?

The answer is no, and this fact motivates the quest for criteria under what circumstances a particular transformation from one *known* initial state to a certain final state is possible with local quantum operations and classical communication. This is the topic of Chapter 3. Particular emphasis will be put on transformations of mixed quantum states, and on so-called entanglement-assisted local operations, where the two experimenters may borrow quantum systems in an entangled state, but they must not use up the entanglement.

The ensuing question is how non-local quantum operations can be implemented, if the experimenters *do not know* what state they share. Chapter 4 investigates how certain elementary quantum gates may be implemented in an optimal way in a distributed quantum computer. Optimality will be measured by the minimal amount of necessary resources.

The practically usable entanglement of a composite quantum system is related to the classical knowledge about the state of the system. In Chapter 5 a relation between the amount of accessible entanglement on the one hand and the classical information about the order of several entangled quantum systems on the other hand will be established. A physically relevant situation will be investigated in greater detail. Group theoretical methods will help to clarify the connection between information and entanglement in a more general set-up. Finally, Chapter 6 has a more visionary character. In this last chapter the relationship between quantum information theory and game theory will be explored.

Chapter 1

Quantum States, Operations, and Correlations

The questions that will be addressed in this chapter are: What is a state? What is a quantum operation? What is, very roughly, the structure of the state space of a bi-partite quantum system? These questions tell a lot about the emphasis of this field of research. It is of interest to see what kind of manipulations can be implemented in principle, and it turned out to be convenient to abstract from the actual implementation of the manipulation. Unitary operations, for example, correspond to Schrödinger dynamics, and this dynamics is governed by a Hamiltonian associated with a physical set-up. In the context of the thesis it is however more appropriate to speak about a unitary manipulation of the state, without bothering to investigate the Hamiltonian itself.

In the spirit of the recent development of quantum information theory it is only natural to think of entanglement in operational terms: Entanglement is conceived as a resource for computational and communicational tasks, and the emphasis is put on the usefulness of entanglement. Notions like distillability and separability of a state are defined in terms of local quantum operations. These two concepts will also be explained subsequently. This chapter is an introductory chapter which reviews important concepts that will be frequently used in the later chapters. It does not contain any original research material.

1.1 States in Quantum Mechanics

A state of a physical system collects the information which is available about the system that one has obtained in measurements. Associated with a quantum system is a complex Hilbert space $\mathcal H$ with scalar product $\langle\cdot|\cdot\rangle$ and corresponding norm $\|\cdot\|$. If one has maximal information about the state of a quantum system, in the sense that one has performed a preparation such that the values of a complete set of observables have been fixed, one can say that the system is in a *pure state* associated with a (state) vector $|\psi\rangle$ in Hilbert space satisfying $\|\psi\|=1$. This vector is uniquely determined except for a phase factor, and the state itself corresponds to the associated *unit ray* $\{\exp(i\phi)|\psi\rangle\mid\phi\in\mathbb{R}\}$.

The most elementary quantum system one can think of has an only two dimensional Hilbert space \mathbb{C}^2 : the two level system. The basis elements of the vector space can be labeled as $|0\rangle$ and $|1\rangle$. Any unit vector in this Hilbert space is of the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,\tag{1.1}$$

where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Such a quantum system may be in the state associated with $|0\rangle$ or $|1\rangle$, but according to Eq. (1.1) it can also occur in a coherent superposition of $|0\rangle$ and $|1\rangle$. Simple as this system is, it is of major importance in quantum information theory. It is the elementary physical carrier of information in an information processing device using quantum systems. Such a two-level system is called quantum bit in the context of quantum information theory, or, in short, *qubit* [50]. This term has been shaped in analogy to the *bit*, which is the fundamental unit of classical information: The qubit is the physical system that can be in two different orthogonal pure states. A possible implementation is a photon with its polarization degree of freedom, or a two-level system with a ground state and an excited state.

If one wants to include the possibility of partial information to the state concept, unit rays are not sufficient as a description of a state. The concept of mixed states also incorporates ignorance about a quantum state. For example, in a beam of unpolarized spin-1/2 particles the state of the quantum systems is given by the classical mixture with uniform distribution of particles in the state corresponding to the unit ray of $|1\rangle$ and $|0\rangle$, respectively. Another example of a mixed state is one which can be prepared in the following way. One takes a classical random number generator which produces an output labeled 1 with the classical probability p_1 , and the output labeled 2 with probability p_2 , where $p_1+p_2=1$. If one gets the outcome 1, the preparing procedure of the pure state corresponding to $|\psi_1\rangle$ is implemented. In the other case $|\psi_2\rangle$ is prepared. This procedure amounts to a preparation of the *mixed state*

$$\rho = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2|. \tag{1.2}$$

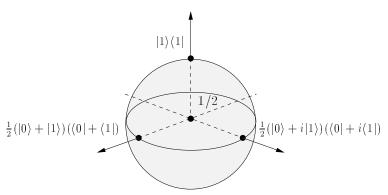


Figure 1.1: Schematic representation of the state space of a single qubit. Each point in the ball of radius 1 corresponds to a (mixed) state, pure states are represented by points on the sphere, the *Bloch sphere*. The coordinates of the point associated with a state ρ are in this representation given by $x=\langle 0|\rho|1\rangle+\langle 1|\rho|0\rangle,\ y=\langle 1|\rho|1\rangle-\langle 0|\rho|0\rangle,\ z=i(\langle 0|\rho|1\rangle-\langle 1|\rho|0\rangle)$. The *maximally mixed state* 1/2 corresponds to the center of the sphere.

A state of a quantum mechanical system with Hilbert space \mathcal{H} will from now on be identified with a (bounded) operator ρ fulfilling three requirements. Firstly, ρ and ρ^{\dagger} are defined on the entire Hilbert space and ρ is self-adjoint, $\rho = \rho^{\dagger}$. Secondly, it is a (semi)-positive operator $\rho \geq 0$, and, thirdly, due to the conditions imposed on probabilities $\mathrm{tr}[\rho] = 1$. A state ρ is a *pure state* if $\rho^2 = \rho$, otherwise it is said to be a *mixed state*. For a given Hilbert space, the associated set of all states is denoted by $\mathcal{S}(\mathcal{H})$ and referred to as *state space*. The state space is a convex set, that is, if ρ_1 is element of $\mathcal{S}(\mathcal{H})$ and the same is

¹This statement should of course not be understood in the sense that a mixed state corresponds to a mere classical probability distribution on the set of rays.

1.2. Operations

true for ρ_2 , then also all states on the straight line segment $\lambda \rho_1 + (1-\lambda)\rho_2$, $\lambda \in [0,1]$, are included in $\mathcal{S}(\mathcal{H})$. The above example involving the random number generator already points towards this property: The procedure of the convex combination corresponds to a *mixing* of two preparing procedures. One may prepare ρ_1 with probability λ and the state ρ_2 with probability $1-\lambda$, and then ignore the information about what preparation procedure has actually been chosen. ρ is then the state reflecting all the information which is finally available. The extreme points in this convex set are just one dimensional projectors $|\psi\rangle\langle\psi|$ with $\|\psi\|=1$.

Conversely, any mixed state ρ of a quantum system admits a representation of the form

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| \tag{1.3}$$

with a probability distribution $p_1,...,p_n$, and projections $|\psi_i\rangle\langle\psi_i|$, i=1,...,n. The state space $\mathcal{S}(\mathcal{H})$ of a quantum system is no *Choquet simplex*, i.e., not all elements are unique mixtures of the extremal boundary. Instead, for a given state there exists in general an infinite number of such representations, and both the weights and the projections may be different for two such decompositions.

In this thesis, all quantum systems that will be considered consist of at least two distinct parts. The Hilbert space of such a composite quantum system with parts A and B is appropriately constructed as the *tensor product* of the Hilbert spaces of its constituents,

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{1.4}$$

Almost all Hilbert spaces will be finite dimensional. In this case, if $\{|1\rangle_A,...,|N\rangle_A\}$, $N = \dim[\mathcal{H}_A]$, is a basis of \mathcal{H}_A and $\{|1\rangle_B,...,|M\rangle_B\}$, $M = \dim[\mathcal{H}_B]$, is a basis of \mathcal{H}_B , then $\{|i\rangle_A \otimes |j\rangle_B|i=1,...,N;j=1,...,M\}$ is a basis of \mathcal{H} . A product state vector is a vector of the form

$$|\psi\rangle = |\phi\rangle_A \otimes |\varphi\rangle_B = \left(\sum_{i=1}^N \alpha_i |i\rangle_A\right) \otimes \left(\sum_{j=1}^M \beta_j |j\rangle_B\right)$$
 (1.5)

with complex coefficients α_i and β_j , $\sum_{i=1}^N |\alpha_i| = 1$ and $\sum_{j=1}^M |\beta_j| = 1$. The tensor product symbol will often be omitted. The scalar product in \mathcal{H} is induced by the scalar products in \mathcal{H}_A and \mathcal{H}_B according to $\langle \psi_1 | \psi_2 \rangle = \langle \phi_1 |_A \phi_2 \rangle_A \langle \phi_2 |_B \varphi_2 \rangle_B$, where $|\psi_1 \rangle = |\phi_1 \rangle_A \otimes |\varphi_1 \rangle_B$ and $|\psi_2 \rangle = |\phi_2 \rangle_A \otimes |\varphi_2 \rangle_B$, and extended to \mathcal{H} by linearity.

A very useful tool is the so-called *Schmidt decomposition* of a pure state of a bi-partite system [51, 52]. Let $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, $\mathcal{H}_A=\mathcal{H}_B=\mathbb{C}^N$, and let $|\psi\rangle\in\mathcal{H}$ be a state vector. Then there exists an orthonormal basis $\{|1\rangle_A,...,|N\rangle_A\}$ of \mathcal{H}_A and an orthonormal basis $\{|1\rangle_B,...,|N\rangle_B\}$ of \mathcal{H}_B such that

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\alpha_i} |i\rangle_A |i\rangle_B, \tag{1.6}$$

where α_i , i=1,...,N, are real positive numbers satisfying $\sum_{i=1}^{N} \alpha_i = 1$. The numbers $\alpha_1,...,\alpha_N$ are called *Schmidt coefficients*; the *Schmidt rank* of the state is the number of non-vanishing Schmidt coefficients.

1.2 Operations

1.2.1 Elementary Quantum Operations

The dynamics of states of isolated quantum systems is governed by the *Schrödinger equation*. Time evolution of a state of a system with Hilbert space \mathcal{H} corresponds to the unitary

dynamical map

$$\rho \longmapsto \sigma = U\rho U^{\dagger},\tag{1.7}$$

where $U:\mathcal{H}\longrightarrow\mathcal{H}$ is a time-dependent unitary operator. States at a later time are hence unitarily equivalent to states at an earlier time. In particular, pure states remain pure throughout such a time evolution.

Also associated with an alteration of the state is the process of *measurement* in quantum mechanics 2 . Let i = 1, ..., K be the labels of the possible outcomes in a measurement. Each outcome of the measurement is furnished with a projector π_i ,

$$\pi_i \pi_j = \delta_{ij} \pi_i, \quad \sum_{i=1}^K \pi_i = 1.$$
 (1.8)

If the quantum system is initially in the state ρ , then the state immediately after the measurement is given by

$$\rho \longmapsto \sigma_i = \frac{\pi_i \rho \pi_i}{\operatorname{tr}[\pi_i \rho \pi_i]}.$$
 (1.9)

The outcome with label i is obtained with probability $p_i = \operatorname{tr}[\pi_i \rho] = \operatorname{tr}[\pi_i \rho \pi_i]$. This type of measurement will be referred to as *selective projective measurement*. It is called *complete* if all projectors π_i are one-dimensional. A *non-selective projective measurement* corresponds to a map

$$\rho \longmapsto \sum_{i=1}^{K} \pi_i \rho \pi_i. \tag{1.10}$$

In addition to unitary transformations and projective measurements there are two more actions one can take: If ρ_1 is a state of a quantum system with a Hilbert space \mathcal{H} , one can append a quantum system with Hilbert space \mathcal{K} in a state ω such that

$$\rho \longmapsto \rho \otimes \omega.$$
(1.11)

These auxiliary quantum systems are typically called *ancillae*. Similarly, one may *dismiss a local part* of the whole quantum system. This is taken into account by the partial trace operation: If a joint quantum system with Hilbert space $\mathcal{H} \otimes \mathcal{K}$ is in the state ρ , then

$$\rho \longmapsto \sigma = \operatorname{tr}_{\mathcal{K}}[\rho]. \tag{1.12}$$

 σ is the final state of the first system with Hilbert space \mathcal{H} alone.

Accordingly, one can apply any combination of these four basic ingredients. One may for example allow a coupling of the original quantum system to an auxiliary quantum system and let the two unitarily interact. After performing a non-selective projective measurement on the composite system one could eventually consider the original system only again

²The process of measurement in quantum mechanics is a subtle issue. While the basic formalism of quantum theory is very well-understood, and the applications of quantum mechanics are numerous, there are conceptual difficulties in quantum theory with the reconciliation of the measurement process and continuous Schrödingertype time evolution. The term "measurement problem" could be conceived as a collective term for several issues related to these difficulties: In Ref. [53] five different aspects of this problem are distinguished, among them the famous and-or-problem or the so-called pointer basis problem. The latter problem is considerably weakened by the insight that no quantum mechanical system is fully isolated from its environment, and the decoherence program has dealt with this issue [54, 55, 56]. Although one can learn much from such an approach about the classical limit of quantum mechanics, this viewpoint does not "solve" the problems with measurement, as the final mixture does correspond to an improper mixture in the words of Ref. [53], unless one is in the position to accept some kind of Everett-type many-world interpretation [52]. Also, there have been attempts to alter the standard description of time evolution in quantum mechanics, which try to give an explicit stochastic dynamical of the collapse of the state (see, e.g., Ref. [57]). For an overview about the topics related to quantum measurement see, e.g., Ref. [58]. Refs. [59, 60] describe the topic of measurement from the perspective of philosophy of physics, Ref. [61] with an emphasis on quantum optical experiments. This thesis - pragmatic in its scope - will not be concerned with these issues.

1.2. Operations

by taking a partial trace with respect to the auxiliary part. In each step well-defined states are mapped on other states, and the concatenation of these operations clearly amounts to a positive linear map from the state space onto itself.

1.2.2 Generalized Measurements

Alternatively, one can approach the issue of admissible quantum operations in a quite different spirit. From an axiomatic point of view one may look for all maps $\mathcal{E}: S(\mathcal{H}) \longrightarrow S(\mathcal{H}')$ which are consistent with the statistical interpretation of quantum theory. Certainly, \mathcal{E} must be linear, such that \mathcal{E} respects convex combinations of states as described above. Since the semi-positivity of states has to be preserved, \mathcal{E} is also required to be a positive map.

But \mathcal{E} being positive is not a sufficient criterion as will be argued subsequently. Any quantum system of interest can be conceived as being a part of a larger quantum system. Bearing this in mind, one can always append an additional system; in the Hilbert space \mathcal{K} of the additional system the map under consideration then simply acts as the identity operation, which leaves this part of the enlarged system in the same state. Hence, the map $\mathcal{E} \otimes \mathbb{1}_N$ has to be a valid operation, where N stands for the dimension of \mathcal{K} , and in particular, it must be positive.

This leads to the concept of complete positivity. A map \mathcal{E} is called *completely positive* if $\mathcal{E} \otimes \mathbb{1}_N$ is a positive map for all $N \in \mathbb{N}$. Quite surprisingly, this condition is stronger than mere positivity of \mathcal{E} . Indeed, there exists maps which are positive but fail to fulfil complete positivity, such as the transposition operation. In the course of this thesis, *quantum operations* are identified with linear completely positive maps on the state space.

Unitary operations, non-selective projective measurements, addition of uncorrelated systems, and the dismissal of parts of a compound system: all these operations can be cast into the form

$$\rho \longmapsto \mathcal{E}(\rho) = \sum_{i=1}^{K} E_i \rho E_i^{\dagger}, \tag{1.13}$$

where the so-called *Kraus operators* $E_i: \mathcal{H} \longrightarrow \mathcal{H}, i=1,...,K$, are not necessarily Hermitian operators [62]. Even more can be said about Eq. (1.13): Any linear completely positive map $\mathcal{E}: S(\mathcal{H}) \longrightarrow S(\mathcal{H})$ can be written in this form, and thus, Eq. (1.13) gives the most general possible trace-preserving operation in quantum mechanics. The *trace-preserving property* of \mathcal{E} manifests as

$$\sum_{i=1}^{K} E_i^{\dagger} E_i = 1, \tag{1.14}$$

because it implies that $\sum_{i=1}^K \operatorname{tr}[E_i \rho E_i^{\dagger}] = \operatorname{tr}[\rho] = 1$. As in general E_i^{\dagger} and E_i do not commute,

$$\sum_{i=1}^{K} E_i E_i^{\dagger} = 1 \tag{1.15}$$

is different from Eq. (1.14). The latter condition is equivalent with the statement that the identity is a fixed point of the map. A completely positive map satisfying Eq. (1.15) is called *unital*; if it is also trace-preserving it is said to be a *doubly stochastic operation*. Many operations in quantum mechanics are doubly stochastic – in particular in the context of quantum information theory – but not all. Doubly stochastic maps are those operations which increase the *von Neumann entropy*, that is, operations $\mathcal E$ for which $S(\mathcal E(\rho)) \geq S(\rho)$ for all states ρ , where $S(\rho) = -\mathrm{tr}[\rho \log_2 \rho]$. For properties of the von Neumann entropy and the relative entropy functional see Appendix A.

Quantum operations \mathcal{E}_i , i = 1, ..., K, may also be non-trace-preserving,

$$\rho \longmapsto \mathcal{E}_i(\rho) = \sum_j E_{i,j} \rho E_{i,j}^{\dagger}, \qquad (1.16)$$

with $\sum_{j} E_{i,j}^{\dagger} E_{i,j} \leq 1$. This corresponds to the fact that the operation is partly classical in the sense that each label i, i = 1, ..., K, belongs to a classical outcome in a measurement. The final state in an outcome associated with label i is then given by

$$\rho \longmapsto \frac{\sum_{j} E_{i,j} \rho E_{i,j}^{\dagger}}{\operatorname{tr}\left[\sum_{j} E_{i,j} \rho E_{i,j}^{\dagger}\right]},$$
(1.17)

which occurs with probability $p_i = \operatorname{tr}[\sum_j E_{i,j} \rho E_{i,j}^{\dagger}]$. Such an operation may be conceived as a generalization of a selective projective measurement. In order to guarantee that the corresponding non-selective operation is a valid one, the condition

$$\sum_{i=1}^{K} \sum_{j} E_{i,j}^{\dagger} E_{i,j} = \mathbb{1}$$
(1.18)

has to be satisfied.

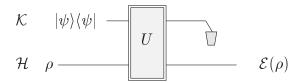


Figure 1.2: A general trace-preserving quantum operation \mathcal{E} .

As has been pointed out before, all quantum operations can be realized with unitary evolution, projective measurements, addition of ancillae, and discarding parts of the system. Interestingly, such a procedure can be done with a single step only. This is a direct consequence of the *Stinespring dilation theorem* [63], which has originally been formulated in the language of C*-algebras. To be specific, let $\mathcal H$ be a Hilbert space with dimension $\dim[\mathcal H]=N$, and let $\mathcal E:\mathcal S(\mathcal H)\longrightarrow\mathcal S(\mathcal H)$ be a trace-preserving quantum operation. Then there exists a Hilbert space $\mathcal K$ with $\dim[\mathcal K]\leq N^2$ and, for any fixed $|\psi\rangle\in\mathcal K$, there exists a unitary operator $U:\mathcal H\otimes\mathcal K\longrightarrow\mathcal H\otimes\mathcal K$ such that

$$\mathcal{E}(\rho) = \operatorname{tr}_{\mathcal{K}}[U(\rho \otimes |\psi\rangle\langle\psi|)U^{\dagger}]. \tag{1.19}$$

That is, every trace-preserving quantum operation may be realized by appending an appropriate ancilla once, applying a joint unitary operation on both systems and finally dismissing the ancilla (see Fig. 1.2).

1.2.3 Operations in Composite Quantum Systems

Most of the quantum systems that will be considered are *bi-partite systems* with two distinct parts *A* and *B*. Each part can be manipulated in an arbitrary way, but general joint quantum operations on both parts at the same time are not possible. To illustrate this setup imagine

1.2. Operations

two experimenters named Alice and Bob who are separated in space from each other, and they are each holding a part of a composite system. Alice may take actions in system A with associated Hilbert space \mathcal{H}_A while Bob is restricted to act in B with \mathcal{H}_B , where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

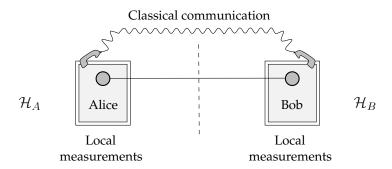


Figure 1.3: Schematic representation of a LOCC operation in a bi-partite setting. Alice performs a local generalized measurement in which she obtains one of the possible outcomes with labels i=1,...,K. The measurement result is transmitted via the classical channel to Bob, who implements a generalized measurement depending on the outcomes j=1,...,L. Again, Bob may transmit the classical information about the outcomes to Alice, and so on. In a one-local operation it is only possible to transmit information from one party to the other.

For bi-partite systems several natural classes of operations can be distinguished. A class of operations is defined as a closed set which includes the identity: "not to do anything at all" is always a valid option [64]. A local operation \mathcal{E}_A on system A can be formulated by means of Kraus operators $A_1,...,A_K$ acting on \mathcal{H}_A . In an operation of this type Alice performs an operation on the part of the state space belonging to her, while Bob remains inactive, i.e., he performs the identity operation. In the same way one can think of a local operation \mathcal{E}_B on Bob's side associated with Kraus operators $B_1,...,B_L$. A generic local operation \mathcal{E} acts as

$$\rho \longmapsto \mathcal{E}(\rho) = \sum_{i=1}^{K} \sum_{j=1}^{L} (A_i \otimes B_j) \rho (A_i \otimes B_j)^{\dagger}, \tag{1.20}$$

with operators $A_i: \mathcal{H}_A \longrightarrow \mathcal{H}_A$ and $B_i: \mathcal{H}_B \longrightarrow \mathcal{H}_B$. Local operations can also be generalized measurements. Two states σ_1 and σ_2 are said to be *locally distinguishable*, if $\operatorname{tr}_A[\sigma_1]$ and $\operatorname{tr}_A[\sigma_2]$ or $\operatorname{tr}_B[\sigma_1]$ and $\operatorname{tr}_B[\sigma_2]$ are orthogonal, such that there exists a local projective measurement discriminating between both states with unit probability.

A more general class of operations is one in which both Alice and Bob can only act locally, but they may communicate by classical means – such as ordinary telephone lines – to coordinate their actions. Such scenarios are ubiquitous in applications of quantum information theory [32, 31, 6, 4]. In a situation with one-way communication Alice performs a local operation including measurements and communicates the results of her measurements to Bob. Bob can then implement an operation depending on the values he has received from Alice via the classical channel. Such operations are called *one-local operations* [64, 32]. In terms of Kraus operators such an operation reads as

$$\rho \longmapsto \mathcal{E}(\rho) = \sum_{j,k} \sum_{i=1}^{K} (\mathbb{1}_A \otimes B_k^{(i)}) (A_{i,j} \otimes \mathbb{1}_B) \rho (A_{i,j} \otimes \mathbb{1}_B)^{\dagger} (\mathbb{1}_A \otimes B_k^{(i)})^{\dagger}. \tag{1.21}$$

Eq. (1.21) gives the most general one-local quantum operation, where the measurement results with labels 1, ..., K are transmitted classically from Alice to Bob: the Kraus operators $B_k^{(i)}$ of Bob's quantum operation may therefore depend on the actual outcome i.

In *two-local operations* any sequence of local operations and transmission of classical data about measurement outcomes is allowed (see Fig. 1.3). One assumes that communicating by classical means, that is, using the telephone, is rather cheap, and that there are no limits for classical communication. This class of operations is simply called *local quantum operations with classical communication*, typically abbreviated as *LOCC* or *LQCC*. Analogous to Eq. (1.21) LOCC operations can be represented with the help of Kraus operators, but in general LOCC operations any number of rounds of performing measurements and communicating the results from one party to the other is possible. LOCC operations are of paramount importance to issues related to quantum entanglement, and it will be the most important class of completely positive maps throughout this thesis.

The set of LOCC operations is included in the set of *separable operations* [64, 35, 34]. In these operations all Kraus operators correspond to product operations, and the full operation amounts to a mixing of states which have been manipulated in a classically correlated way. Thus, separable operations can be written in the general form

$$\rho \longmapsto \mathcal{E}(\rho) = \sum_{i=1}^{K} (A_i \otimes B_i) \rho (A_i \otimes B_i)^{\dagger}. \tag{1.22}$$

Since $E_i = A_i \otimes B_i$ in this case, the map \mathcal{E} is trace-preserving if both

$$\sum_{i=1}^{K} A_i^{\dagger} A_i = \mathbb{1}_A \text{ and } \sum_{i=1}^{K} B_i^{\dagger} B_i = \mathbb{1}_B.$$
 (1.23)

Note that not all trace-preserving separable operations can be implemented by means of LOCC operations [64], whereas the converse statement is obviously true (but see Ref. [65]).

The concept of separable operations can also be applied if Alice and Bob each hold more than one quantum system. Local operations by Alice still refer to operations in \mathcal{H}_A only, but they may include joint operations involving the different quantum systems on her side. Often an asymptotic limit $n \longrightarrow \infty$ is investigated of n equal copies $\rho^{\otimes n}$ of the same state ρ . These concepts can naturally also be extended to more than three parties. LOCC operations is the set of all local operations of all parties, Alice, Bob, Claire, ..., together with an arbitrary amount of two-way-classical communication.

The theoretical study of operations is strongly linked to the theory of entanglement, and there is a deeply rooted connection between positive and completely positive maps and quantum entanglement (see, e.g., Refs. [66, 67, 41]). On the one hand several positive (but not completely positive) maps such as the reduction operation provide powerful tools to uncover nonseparability of mixed quantum states. On the other hand separable operations play an important role in the context of quantification of entanglement. One of the fundamental properties of entanglement monotones measuring the degree of entanglement in a given state is, e.g., that separable operations can never increase the amount of entanglement [32, 6, 35, 43, 4].

1.3 Correlated Quantum States

Assume that a bi-partite quantum system with parts A and B is in a pure product state. Then the probabilities associated with any local measurement performed in systems A and B factorize: the measurement outcomes are statistically independent. This comes to no

surprise, as the preparation of such a pure product state can be done locally by two independently acting experimenters.

This is not so if a pure state is no product state. Any pure state that is no product state cannot be prepared with local operations, and it will definitely violate some kind of *Bell's inequality*. There are several criteria for the non-locality of a quantum state, but all classify pure states along the lines of product and "correlated" states. Such correlated states are called *entangled states*, the singlet state with state vector $(|01\rangle - |10\rangle)/\sqrt{2}$ of two qubits being the prototypical example. The singlet is a particular maximally entangled state: A pure state of a bi-partite system is called *maximally entangled* if the reduced states of both parties are maximally mixed.

1.3.1 Separability

If the state of a bi-partite system is mixed, then new complications arise. It is no longer true that any state which cannot be prepared locally necessarily violates Bell-type inequalities. In mixed states both intrinsic quantum correlations and classical correlations may be present. It is the notion of separability that sharpens the distinction between these two types of correlations. A state of a bi-partite system is called *classically correlated* or *separable* [46] if it is a convex combination of product states. In technical terms, a state ρ is separable if it can be written in the form

$$\rho = \sum_{i=1}^{n} p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \tag{1.24}$$

where $0 \leq p_1,...,p_n \leq 1$ and $\sum_{i=1}^n p_i = 1$. The states $\rho_A^{(i)}$, i = 1,...,n, are taken from the state space $\mathcal{S}(\mathcal{H}_A)$, $\rho_B^{(i)}$ are elements of $\mathcal{S}(\mathcal{H}_B)$. A state of the form of Eq. (1.24) can be prepared with LOCC operations by locally producing one of the product states $\rho_A^{(i)} \otimes \rho_B^{(i)}$ with probability p_i and disregarding the information which one of the product states with label i = 1,...,n has been prepared. States that cannot be cast into the form Eq. (1.24) are called *entangled*.

The set of separable states shall be denoted as $\mathcal{D}(\mathcal{H})$. It is a convex set – as mixing of two separable states always produces again a separable state – it is compact and it includes the maximally mixed state. In the full state space the set $\mathcal{D}(\mathcal{H})$ is not a set of measure zero [68]. The product states on the right hand side of Eq. (1.24) can always be chosen to be pure states, according to

$$\rho = \sum_{i=1}^{n} \sum_{j,k} p_i \lambda_{i,j} \mu_{i,k} \left(|\psi^{(i,j)}\rangle \langle \psi^{(i,j)}|_A \otimes |\phi^{(i,k)}\rangle \langle \phi^{(i,k)}|_B \right), \tag{1.25}$$

where $\rho_A^{(i)} = \sum_j \lambda_{i,j} |\psi^{(i,j)}\rangle \langle \psi^{(i,j)}|_A$ and $\rho_B^{(i)} = \sum_k \mu_{i,k} |\phi^{(i,k)}\rangle \langle \phi^{(i,k)}|_B$. Due to the convexity property of the set of separable states the number of product terms in Eq. (1.25) can without loss of generality be restricted to $(NM)^2$, where $N = \dim[\mathcal{H}_A]$ and $M = \dim[\mathcal{H}_B]$, by virtue of a theorem by Caratheodory [69]. It is one of the major issues of quantum information theory to judge whether a given state ρ of an $N \times M$ dimensional bi-partite quantum system is separable or not [41, 70, 67, 71, 72]; it is a highly non-trivial task.

The set of separable states allows for a useful normal form of quantum states of bipartite systems. For any state ρ there exists a unique [73] decomposition

$$\rho = \lambda \rho_s + (1 - \lambda)\delta \rho, \tag{1.26}$$

where ρ_s is a separable state, $\delta \rho$ is a state with no product vector in its range, and λ is maximal. This decomposition is referred to as *best separable approximation* [74]. This normal form can be constructed using the method of maximally subtracting product vectors from

a state [74, 75], and has proven to be a practical tool in investigations of the structure of state space.

Included in the set of separable states $\mathcal{D}(\mathcal{H})$ is another important convex subset of $\mathcal{S}(\mathcal{H})$, the set $\mathcal{P}(\mathcal{H})$ of so-called *positive-partial-transpose states*, in short PPT states, which will be explained successively. The *partial transposition* with respect to system B is the transposition operation in \mathcal{H}_B . If $\rho_{m\mu,n\nu}$ is the matrix element of a state ρ in some orthonormal product basis, then the *partial transpose* of ρ with respect to B can be written as

$$\rho_{m\mu,n\nu}^{T_B} = \rho_{m\nu,n\mu}.\tag{1.27}$$

While the partial transposition is basis-dependent, this not true of the eigenvalues of the partial transpose. Since the transposition operation is no completely positive map, the partial transposition does not necessarily map states on states. Indeed, the partial transpose with respect to B, ρ^{T_B} , is not always positive. However, ρ^{T_B} is positive if and only if ρ^{T_A} is positive. A state ρ is called *PPT state*, if $\rho^{T_A} \geq 0$. In Ref. [70] it has been pointed out that for all states ρ

$$\rho$$
 is separable $\Longrightarrow \rho^{T_A} \ge 0$. (1.28)

The converse has been proven in Ref. [67] for the case of bi-partite quantum systems of dimension 2×2 and 2×3 : let $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\dim[\mathcal{H}_A] + \dim[\mathcal{H}_B] \leq 5$, then

$$\rho$$
 is separable $\iff \rho^{T_A} \ge 0$, (1.29)

such that $\mathcal{P}(\mathcal{H}) = \mathcal{D}(\mathcal{H})$, while in general $\mathcal{D}(\mathcal{H}) \subset \mathcal{P}(\mathcal{H})$ [76]. The statement of Eq. (1.29) implies an extraordinarily useful criterion for separability in bi-partite quantum systems of small dimensions. Another non-equivalent criterion is the so-called *reduction criterion* proposed in Ref. [77, 78],

$$\rho$$
 is separable \implies the map $\rho \longmapsto (\rho_A \otimes \mathbb{1}_B) - \rho$ is positive. (1.30)

where $\rho_A = \operatorname{tr}_B[\rho]$ denotes Alice's local state.

Recent progress in the quest for criteria for separability includes criteria for $2 \times N$ systems [79]. For quantum states which have a low rank (and are hence no generic states) surprisingly strong necessary and sufficient criteria can be derived even for $N \times M$ -systems with N, M > 2 [80]. It has turned out that the concept of *entanglement witnesses* [66, 81, 72, 71] is a powerful tool to study separability properties.

1.3.2 Distillability

Different from the question whether a state is separable is the question if it is *distillable*. Assume that a source produces pairs of quantum systems in a certain state ρ . This state ρ is distillable if one can – starting from a large number n of copies $\rho^{\otimes n}$ of the state – produce a smaller number k of approximately maximally entangled states by just applying LOCC operations. Such maximally entangled states are needed for many protocols in quantum information theory, and the issue is whether those desired maximally entangled states can be extracted from copies of some mixed state. This mixed state could be the state of an entangled composite quantum system that has in part been transmitted through a *noisy quantum channel*: noise cannot be avoided, and one tries to recover as much as possible of the original entanglement. The property to be distillable is in this sense practically much more important than separability.

In mathematical terms, a state $\rho \in \mathcal{S}(\mathcal{H})$ is *distillable* if there exists a $K \in \mathbb{N}$ and a state vector $|\psi\rangle$ taken from a 2×2 -dimensional subspace $\mathcal{C} \subset \mathcal{H}^{\otimes K} = (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes K}$ such that

$$\langle \psi | (\rho^{T_A})^{\otimes K} | \psi \rangle < 0. \tag{1.31}$$

It is not at all immediately obvious that this definition is compatible with the above intuitive definition. To clarify this connection two important results are necessary: Firstly, it has been demonstrated in Ref. [39] that in order to study distillability, not the full class of LOCC operations has to be considered, but just those quantum operations who are a concatenation of a projection on a 2×2 -dimensional subspace and further steps. Secondly, according to the Peres-Horodecki criterion states of 2×2 -systems are entangled if and only if their partial transpose is not positive. It is the central statement of Ref. [40] that by filtering and using the method proposed in Ref. [31] a large number of copies of any entangled state can be mapped onto a smaller number of approximate singlets with LOCC operations. The fidelity of the output states can be made arbitrarily close to 1, if the number of input states is increased.

In particular, it has been proved that if a state is distillable, then it must have a non-positive partial transpose [39],

$$\rho^{T_A} \ge 0 \Longrightarrow \rho \text{ is not distillable.}$$
(1.32)

Since there exist states which are entangled but for which the partial transpose is again a state, this fact implies that there exist entangled states which cannot be distilled. Motivated by the metaphor of distilling entanglement the term *bound entangled states* has been coined for these states [39, 82] (the entanglement is bound and cannot be set free by LOCC operations) in contrast to *free entangled states*.

It should however be clear that while Eq. (3.94) specifies a well-defined problem, it does not imply a constructive check for distillability. Essentially, the general distillability problem remains unsolved, although much progress was made in recent months [83, 41, 84]. In Ref. [83] the number number K in Eq. (3.94) is restricted to small numbers, and the so-defined K-distillability for K=1,2,... has been applied to the investigation of distillability properties of states with high symmetry. This concept provides strong evidence that certain states may be bound entangled, although they have a non-positive partial transpose [83, 84]. In a sense this is bad news, as this means that to check whether the partial transpose is positive is not sufficient to know whether a state is distillable.

Chapter 2

Quantification of Quantum Entanglement

2.1 Introduction

What is the degree of entanglement of a given state of a composite quantum system? It is one of the crucial issues of a theory of quantum entanglement to find a general answer to this basic question, and a lot of research effort has been devoted to this subject. Since entanglement is conceived as a resource, there is a wide range of questions related to quantification: On the one hand one could ask how much entanglement in pure, approximately maximally entangled form can be extracted from quantum systems in entangled states, on the other hand how much entanglement is actually needed to prepare a system in a particular state. Knowing about the amount of entanglement inherent in a quantum state also means knowing how well a certain task can be accomplished.

For bi-partite systems in pure states the quantification problem is essentially solved. The origin of the difficulties in the quantification of mixed-state entanglement of bi-partite systems is rooted in the fact that classical and quantum correlations are intertwined. A state may be classically correlated without being entangled. Also, if a state is entangled, it is not clear whether many copies of it can be transformed into copies of maximally entangled states by means of local operations and classical communication. A proper measure of entanglement should take this interplay of classical and quantum correlations into account. In particular, the value for the entanglement as given by some proper measure of entanglement should not increase on average, if two experimenters implement local operations only.

In multi-partite systems additional complications arise, and even the pure-state case is not well-understood yet. It is not obvious at all what "standard unit" to take in order to quantify the amount of entanglement in a multi-partite setting, as it became clear that several inequivalent "kinds of entanglement" exist. It is definitely not sufficient to consider only two-party entanglement in a quantum system consisting of more that two parts in order to characterize the entanglement of a general composite quantum system. This fact can probably most clearly be illustrated by investigating the properties of the so-called GHZ-state of three qubits: it is an entangled three-party state, but each two parties do not share any bi-partite entanglement at all. The reduced state of any two qubits is a separable state.

This chapter is concerned with several issues related to the quantification problem. Firstly, it will be clarified what it means to quantify the entanglement of a bi-partite system in a meaningful way, and several good measures of entanglement that can be found in the literature will be briefly reviewed. From Subsection 2.2.3 on new material will be presented. New entanglement measures will be proposed: this is done in order to simplify the technical difficulties in actually evaluating the degree of entanglement of a bi-partite system, and in order to get useful upper bounds for the amount of entanglement that can be distilled from many copies of the same state. Entropic and non-entropic quantities will be introduced, and their properties concerning continuity and additivity will be investigated. Thirdly, the last section of the chapter deals with multi-partite entanglement. After a short introduction to some issues of multi-particle entanglement a new measure will be introduced and its properties will be studied. Most of the material of this last section has been published in Ref. [E3].

2.2 Entanglement Monotones

Entanglement monotones are good measures of entanglement. They are functionals mapping states on positive numbers that appropriately quantify the amount of entanglement. It is the main feature of entanglement monotones, very roughly speaking, that they are capable of distinguishing between quantum and classical correlations. If a state is separable, the entanglement monotone gives the value 0, and one could say that the more entangled the state, the larger the value of the entanglement monotone. By no means is an entanglement monotone a uniquely defined functional. Instead, one would call any functional $E: \mathcal{S}(\mathcal{H}) \longrightarrow \mathbb{R}^+$ an entanglement monotone that satisfies a number of natural criteria that are a manifestation of physically motivated properties which any good measure of entanglement should have. They have been formulated in terms of the behavior under certain local quantum operations that can be implemented by remotely located parties. Conditions for acceptable functionals go back to Refs. [32, 31, 85, 34]. In the important paper Ref. [35] the major condition has been clarified: the monotonicity under local generalized measurements. In Refs. [38] and [86] these criteria have been cast into the elegant form that will be explained below.

2.2.1 General Properties

An *entanglement monotone* is a positive functional and it vanishes for separable states. These statements are put together in the first condition. The second property is concerned with mixing. As stated in the first chapter a convex combination of states corresponds to the mixing of preparation procedures. If one has prepared two particular states with certain values for the degree of entanglement, then the postulate states that the entanglement inherent in the state obtained from mixing can only be smaller than or equal to the weighted sum of the previous degrees of entanglement. In other words, mixing of preparing procedures alone never leads to an increased amount of entanglement. The other postulate deals with local generalized measurements. A local operation performed on one part of a bi-partite quantum system alone cannot on average increase the entanglement of the composite system. "On average" means that a particular outcome of the generalized measurement can well exhibit a larger amount of entanglement: this is, e.g., the basis of entanglement distillation procedures. However, the expected entanglement obtained from weighting the entanglement of the outcomes with the respective probabilities must not grow.

(i) $E: \mathcal{S}(\mathcal{H}) \longrightarrow \mathbb{R}$ is a positive functional, and $E(\sigma) = 0$ for any separable state $\sigma \in \mathcal{D}(\mathcal{H})$.

(ii) *E* is a convex functional, that is,

$$E(\sum_{i=1}^{n} p_i \sigma_i) \le \sum_{i=1}^{n} p_i E(\sigma_i)$$
(2.1)

for $p_i \in [0,1]$ and $\sigma_i \in \mathcal{S}(\mathcal{H})$, i = 1, ..., n, with $\sum_{i=1}^n p_i = 1$.

(iii) *E* is monotone under local operations: if one of the parties performs a local generalized measurement, then the expected entanglement cannot increase. This means that if, say, Alice implements a local operation leading to states

$$\sigma_i = \frac{\sum_j (A_{i,j} \otimes \mathbb{1}_B) \sigma(A_{i,j} \otimes \mathbb{1}_B)^{\dagger}}{p_i}, \quad i = 1, ..., K,$$
(2.2)

with probability $p_i = \text{tr}[\sum_j A_{i,j} \sigma A_{i,j}^{\dagger}]$, where $\sum_{i=1}^K \sum_j A_{i,j}^{\dagger} A_{i,j} = \mathbb{1}_A$, then

$$E(\sigma) \ge \sum_{i=1}^{K} p_i E(\sigma_i). \tag{2.3}$$

As any local trace-preserving completely positive map $\mathcal E$ amounts to an operation as specified in (iii) together with a mixing, that is, convex combination of different outcomes as in (ii), it follows that for such maps $E(\sigma) \geq E(\mathcal E(\sigma))$. In earlier writings on the quantification of entanglement the latter inequality was used as the starting point [34]. This implies that the degree of entanglement may not increase under a local non-selective generalized measurement. Condition (iii) on its own leads to an invariance under local unitary operations, that is, $E(U\rho U^{\dagger}) = E(\rho)$ for all $\rho \in \mathcal S(\mathcal H)$ and all local unitary operators $U: \mathcal H \longrightarrow \mathcal H$.

Even for pure states this set of conditions – together with an appropriate normalization – does not fully specify a measure for entanglement. However, if one appends two more conditions concerning the asymptotic regime, then E is completely determined for pure states according to the so-called *uniqueness theorem* [86, 38, 87]. These additional conditions are [86]

- (iv) E is weakly additive, meaning that $E(|\psi\rangle\langle\psi|^{\otimes n}) = nE(|\psi\rangle\langle\psi|)$ for all $|\psi\rangle \in \mathcal{H}$ and all $n \in \mathbb{N}$.
- (v) For a given $|\psi\rangle \in \mathcal{H}$ let $(\sigma_n)_{n\in\mathbb{N}}$ be a series of states $\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$ with the property that $\lim_{n\to\infty} ||\psi\rangle\langle\psi|^{\otimes n} \sigma_n|| = 0$, where ||.|| denotes the trace norm.¹ Then E satisfies

$$\lim_{n \to \infty} \frac{1}{n} \left| E(|\psi\rangle\langle\psi|^{\otimes n}) - E(\sigma_n) \right| = 0.$$
 (2.4)

This property is called *weak continuity*. Essentially, it is required that close to many products of pure states E is sufficiently continuous (see also Subsection 2.2.6).

The unique measure of entanglement for pure states of bi-partite systems is given by

$$E(|\psi\rangle\langle\psi|) = S(\operatorname{tr}_A[|\psi\rangle\langle\psi|]) = S(\operatorname{tr}_B[|\psi\rangle\langle\psi|]), \tag{2.5}$$

where S denotes the von Neumann entropy (see Appendix A).

¹The trace norm of a matrix A is defined as $||A|| = \text{tr}|A| = \text{tr}[\sqrt{A^{\dagger}A}]$.

2.2.2 Distillable Entanglement and Entanglement of Formation

The *distillable entanglement* grasps the resource character of entanglement in mathematical terms [32, 31]. It is the maximal number of maximally entangled states per copy that can be extracted from many copies of a given state σ by means of local operations and – possibly – classical communication. "Many copies" means that the asymptotic limit $n \to \infty$ of n identically prepared systems in a state σ is considered. As one transforms a certain number of non-maximally entangled states into a smaller number of approximately maximally entangled states with the use of LOCC operations, the metaphor of "distilling" entanglement has been used. With the procedure of distilling entanglement it is possible to partially reverse the process of decoherence: imagine that the degree of entanglement of many bipartite quantum systems has degraded due to channel noise. One can then extract a smaller number of approximately maximally entangled states from the larger ensemble [88, 89].

Let C be a class of operations, e.g., the set of LOCC operations. The C-distillable entanglement of a state σ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is then the optimal rate of maximally entangled quantum systems that can be achieved using generalized measurements from the set C of operations. As in Ref. [90] D_{\leftrightarrow} denotes the distillable entanglement with respect to LOCC operations, also called *two-way distillable entanglement*. If not otherwise specified, distillable entanglement is meant to be D_{\leftrightarrow} .

For pure states the distillable entanglement can be evaluated: $S(\operatorname{tr}_A[|\psi\rangle\langle\psi|])$ quantifies the amount of EPR entanglement contained asymptotically in the state $|\psi\rangle\langle\psi|$. That is, the optimal rate in a distillation that can be achieved is given by

$$D_{\leftrightarrow}(|\psi\rangle\langle\psi|) = S(\operatorname{tr}_A[|\psi\rangle\langle\psi|]) = S(\operatorname{tr}_B[|\psi\rangle\langle\psi|]). \tag{2.8}$$

It is not known how to evaluate D_{\leftrightarrow} for general mixed states [92], and only a single example for 2×2 systems is known, and some examples in higher dimensional systems that will be presented in Chapter 5. Bound entangled states are defined as entangled states for which the distillable entanglement D_{\leftrightarrow} vanishes.

Given the fact that it is so extraordinarily difficult to deal with this quantity, powerful upper bounds are necessary. Such an upper bound is a functional introduced in Refs. [35, 34, 34, 6] and slightly modified in Ref. [91]. This is the *relative entropy of entanglement*, defined as

$$E_R(\sigma) = \min_{\rho \in \mathcal{D}(\mathcal{H})} S(\sigma||\rho)$$
 (2.9)

for states σ ; it is an entanglement monotone [35]. In this expression the minimum is attained (and hence the expression does not have to be formulated with an infimum) as $\mathcal{D}(\mathcal{H})$ is a compact set and due to the lower semi-continuity of the relative entropy functional. The properties of the relative entropy functional will be explained in detail in Appendix A.

This bound can be made stronger – and this is the modification of Ref. [91] – in that the variation is not performed over the set $\mathcal{D}(\mathcal{H})$ of separable states, but over the set $\mathcal{P}(\mathcal{H})$ of

$$\frac{1}{n_j} \sum_{i} p_i^{(j)} \log_2(\dim[\mathcal{H}_i^{(j)}]) \quad \to \quad D_C(\sigma), \tag{2.6}$$

$$\frac{1}{n_j} \sum_{i} p_i^{(j)} (1 - F_i^{(j)}) \log_2(\dim[\mathcal{H}_i^{(j)}]) \quad \to \quad 0.$$
 (2.7)

 $F_i^{(j)} = \langle \psi_i^{(j)} | \rho_i^{(j)} | \psi_i^{(j)} \rangle$ denotes the fidelity of $\rho_i^{(j)}$ with respect to a maximally entangled state on $\mathcal{H}_i^{(j)} \otimes \mathcal{H}_i^{(j)}$. This is the definition that will be used in subsequent considerations.

 $^{^2}$ In accordance with Refs. [64, 91] a more precise definition of distillable entanglement can be given. Let C be a class of operations, e.g., the set of LOCC operations. The C-distillable entanglement of a state σ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is the maximum number $D_C(\sigma)$ such that there exists a sequence of generalized measurements from C with labels j=1,2,... with the following properties: For each j the generalized measurement takes the input state $\sigma^{\otimes n_j}$, that is, n_j copies of σ , and maps it on $\rho_i^{(j)}$ with probability $p_i^{(j)}$, where $\rho_i^{(j)} \in \mathcal{S}(\mathcal{H}_i^{(j)} \otimes \mathcal{H}_i^{(j)})$. In the limit $j \to \infty$, $n_j \to \infty$ and

PPT states,

$$B_R(\sigma) = \min_{\rho \in \mathcal{P}(\mathcal{H})} S(\sigma||\rho). \tag{2.10}$$

In particular, this has the implication that bound entangled states that are included in $\mathcal{P}(\mathcal{H})$ are mapped on zero, just as the distillable entanglement of such bound entangled states vanishes.

The distillable entanglement specifies how much entanglement can be extracted from copies of a certain mixed state by means of LOCC operations. The entanglement of formation is in a sense the dual measure to the distillable entanglement D_{\leftrightarrow} . It gives an answer to the question: how many maximally entangled states are needed in order to prepare copies of a particular state, again meant in the asymptotic limit.³ The *entanglement of formation* is defined by [85]

$$E_F(|\psi\rangle\langle\psi|) = S(\operatorname{tr}_A[|\psi\rangle\langle\psi|]) \tag{2.11}$$

for pure states $|\psi\rangle\langle\psi|$ and extended to mixed states according to

$$E_F(\sigma) = \min \sum_{i} \mu_i E(|\psi_i\rangle\langle\psi_i|), \qquad (2.12)$$

where the minimum is taken over all possible decompositions

$$\sigma = \sum_{i} \mu_{i} |\psi_{i}\rangle\langle\psi_{i}| \tag{2.13}$$

of σ in terms of pure states $|\psi_1\rangle\langle\psi_1|$, $|\psi_2\rangle\langle\psi_2|$, with a probability distribution $\mu_1,\mu_2,...$ An extension of this type to mixed states is also called *convex roof extension* to mixed states [94, 95]. According to this construction the entanglement of formation is the largest convex function that is consistent with the von Neumann entropy of the local state for pure states. Therefore, it can only be larger than the relative entropy of entanglement, and

$$D_{\leftrightarrow}(\sigma) \le E_R(\sigma) \le E_F(\sigma)$$
 (2.14)

holds for all $\sigma \in \mathcal{S}(\mathcal{H})$, implying that the entanglement of formation is also an upper bound for distillable entanglement, although a strictly less tight one than the relative entropy of entanglement. A very useful lower bound of $E_R(\sigma)$ is given by $\max\{S(\operatorname{tr}_A[\sigma]) - S(\sigma), S(\operatorname{tr}_B[\sigma]) - S(\sigma)\}$ [96].

For pure states, D_{\leftrightarrow} , E_R , and E_F all coincide [32, 35]. It is extremely hard to directly evaluate any of these quantities for general mixed states, the spectrum ranging from difficult to hopeless. For quantum systems consisting of two qubits a general formula for the entanglement of formation is known [33, 97].

2.2.3 Non-Entropic Entanglement Monotones

The use of the relative entropy functional is motivated by an interpretation in terms of statistical distinguishability [35, 98]. Moreover, the relative entropy of entanglement provides a tight bound for distillable entanglement. The entanglement of formation is important due to its interpretation as a measure of the cost of the preparation of a state. The major drawback of these entanglement monotones is that a minimization over a set of high

 $^{^3}$ This statement is only correct if E_F is weakly additive, in the sense that $E_F(\sigma^{\otimes n}) = nE_F(\sigma)$ for all states σ . Then it can be rigorously shown that E_F quantifies the asymptotic entanglement cost of preparing a mixed state of a bi-partite system [93]. So far, however, no proof of additivity is available for the entanglement of formation. Without using the additivity conjecture one can still identify the entanglement of formation with the asymptotic entanglement cost in a preparation procedure, but one has to replace $E_F(\sigma)$ by the regularized version of it, $E_F^\infty(\sigma) = \limsup_{n \to \infty} E_F(\sigma^{\otimes n})/n$, see Subsection 2.2.5.

dimension is always necessary, and analytically, this task can hardly be performed. In systems consisting of two qubits the Bell basis with its particular properties is available, which makes the task of evaluating entanglement measures more accessible, the most spectacular example being the analytical formula for the entanglement of formation [33, 97]. In higher dimensional Hilbert spaces, however, there is no equivalent of the Bell basis [99].

It would therefore be desirable to have a very simple quantity at hand which is a good measure of entanglement in the sense that is fulfils the above conditions (i), (ii), and (iii). This quantity should not involve a minimization over a high dimensional set.⁴ It is the purpose of this subsection to show that a quantity first investigated in Ref. [68] fulfils those conditions. This quantity has been given the name *negativity* E_N , as it quantifies the "negativity" of the partial transpose of a state. If a state is a PPT state, it is assigned a positive value, otherwise the negativity vanishes. As the entanglement measure involves just a trace norm of the partial transpose, it may well be calculated with paper and pencil, and a complicated minimization is not necessary. ⁵

The significance of this measure stems also from the fact that the logarithm of the trace norm of the partial transpose is known to be a useful upper bound for distillable entanglement D_{\leftarrow} [92, 86], although certainly not a particularly tight one. It gives a simple answer to the question how much entanglement can at most be distilled from copies of a certain state. As the logarithm is no convex function, the logarithm of the trace norm of the partial transpose cannot be guaranteed to be an entanglement monotone any more.

Proposition 2.1. – Let
$$\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$$
, and let for $\sigma \in \mathcal{S}(\mathcal{H})$
$$E_N(\sigma) = \|\sigma^{T_B}\| - 1, \tag{2.15}$$

where $\|.\|$ denotes the trace norm. Then E_N is an entanglement monotone.

Proof: In Eq. (2.15) the negativity is defined via the trace norm of the partial transpose with respect to system B. Equivalently, one could define it as $E_N(\sigma) = \|\sigma^{T_A}\| - 1$, since $\|\sigma^{T_B}\| = \|\sigma^{T_A}\|$ for all $\sigma \in \mathcal{S}(\mathcal{H})$. To see that the first condition is satisfied, note that $\|\sigma^{T_B}\| \geq \operatorname{tr}[\sigma^{T_B}] = 1$ for all $\sigma \in \mathcal{S}(\mathcal{H})$, and hence, E_N is a positive functional. For a separable state $\rho \in \mathcal{D}(\mathcal{H})$ the partial transpose ρ^{T_B} is again a state, which results into $E_N(\rho) = 0$. That is, condition (i) is satisfied. The convexity of E_N (ii) follows from the triangle inequality with respect to the trace norm. The remaining task is to prove the validity of condition (iii). Assume that Alice performs a local generalized measurement on a system prepared in the state σ . Any final state σ_i , i=1,...,K, in a local generalized measurement can be represented with the help of Kraus operators $A_{i,j}$, i=1,...,K, j=1,2,..., acting in \mathcal{H}_B as the identity as

$$\sigma_i = \frac{\sum_j A_{i,j} \sigma A_{i,j}^{\dagger}}{p_i},\tag{2.16}$$

where

$$p_i = \operatorname{tr}\left[\sum_j A_{i,j} \sigma A_{i,j}^{\dagger}\right], \tag{2.17}$$

 $^{^4}$ The main advantage of E_N measure is the fact that no minimization is necessary to evaluate the degree of entanglement of a given quantum state. There exist, however, also non-entropic entanglement monotones that are defined via a minimization of a high dimensional set. Such an entanglement monotone will be investigated in Appendix C.

⁵Note that in independent research an alternative proof of this statement has been found by G. Vidal and R.F. Werner. However, the results are unpublished and they were not available to the author of this thesis.

 $\sum_{ij} A_{i,j}^{\dagger} A_{i,j} = \mathbb{1}_A$. Acting as the identity means that the Kraus operators can be written in the form $A_{i,j} \otimes \mathbb{1}_B$. Since the sum over i corresponds to a mixing which – due to the convexity property – can only reduce E_N , it suffices to consider final states of the form

$$\sigma_i = \frac{A_i \sigma A_i^{\dagger}}{p_i} \tag{2.18}$$

with $p_i = \text{tr}[A_i \sigma A_i^{\dagger}]$, where the Kraus operators $A_1, ..., A_K$ satisfy $\sum_{i=1}^K A_i^{\dagger} A_i = \mathbb{1}_A$ according to the trace-preserving property of the quantum operation. Let

$$\sigma^{T_B} = T^+ - T^- \tag{2.19}$$

be the *Jordan decomposition* [100] of the partial transpose of a state σ . Both T^+ and T^- are positive and Hermitian, and $|\sigma^{T_B}| = ((\sigma^{T_B})^2)^{1/2} = T^+ + T^-$. Then

$$\sum_{i=1}^{K} p_i E_N(\sigma_i) = \sum_{i=1}^{K} p_i \left(\frac{\left\| (A_i \sigma A_i^{\dagger})^{T_B} \right\|}{p_i} - 1 \right) = \sum_{i=1}^{K} \|A_i \sigma^{T_B} A_i^{\dagger}\| - 1.$$
 (2.20)

For each i = 1, ..., K

$$||A_i(T^+ - T^-)A_i^{\dagger}|| \le ||A_iT^+A_i^{\dagger}|| + ||A_iT^-A_i^{\dagger}|| = \operatorname{tr}[A_iT^+A_i^{\dagger} + A_iT^-A_i^{\dagger}].$$
 (2.21)

As
$$\sum_{i=1}^K A_i^{\dagger} A_i = \mathbb{1}_A$$
,

$$\sum_{i=1}^{K} p_i E_N(\sigma_i) \le \operatorname{tr}[T^+ + T^-] - 1 = \|\sigma^{T_B}\| - 1 = E_N(\sigma). \tag{2.22}$$

Hence, E_N is an entanglement monotone.

It is an immediate consequence of Proposition 2.1 that a quantity considered in Ref. [67] is also a good measure of entanglement. This measure is defined only for two-qubit-systems. It is proportional to the absolute value of the smallest eigenvalue of the partial transpose of a state. More precisely, for a state σ this measure of entanglement is given by $2 \max\{-\lambda_4, 0\}$, where λ_4 is the smallest eigenvalue of σ^{T_B} . It turns out that this measure of entanglement and E_N are simply identical for systems with $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$.

Remark 2.2. – Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. Then

$$E_N(\sigma) = 2\max\{-\lambda_4, 0\}$$
 (2.23)

for states $\sigma \in \mathcal{S}(\mathcal{H})$, where λ_4 is the smallest eigenvalue of σ^{T_B} .

Proof: Let $\lambda_1,...,\lambda_4$ with $\lambda_1 \geq ... \geq \lambda_4$ be the ordered list of eigenvalues of σ^{T_B} . It has been shown in Ref. [75] that $\lambda_1,\lambda_2,\lambda_3 \geq 0$ for all states σ . If σ is separable, $\lambda_4 \geq 0$ according to the Peres-Horodecki-criterion [70, 67]. In the other case $E_N(\sigma) = \|\sigma^{T_B}\| - 1 = \lambda_1 + \lambda_2 + \lambda_3 + |\lambda_4| - 1 = 2|\lambda_4|$.

In 2×2 -systems the negativity E_N coincides with another useful quantity for pure states and for *Werner states* [46], i.e., states of the form

$$\rho_W = \lambda |\psi^-\rangle \langle \psi^-| + (1-\lambda)\mathbb{1}/4, \tag{2.24}$$

where $\lambda \in [0,1]$. This quantity is the so-called *concurrence* E_C , which is defined as follows: let σ be a state of a 2×2 -system, and let $\mu_1, ..., \mu_4$ with $\mu_1 \geq ... \geq \mu_4$ be the ordered list of eigenvalues of $(\sqrt{\sigma}\tilde{\sigma}\sqrt{\sigma})^{1/2}$, where $\tilde{\sigma} = (\sigma_y \otimes \sigma_y)\sigma^*(\sigma_y \otimes \sigma_y)$, and σ_y is one of the Pauli-matrices. The asterisk denotes complex conjugation. Then

$$E_C(\sigma) = \max\{0, \mu_1 - \mu_2 - \mu_3 - \mu_4\}. \tag{2.25}$$

The concurrence is – according to Wootters' formula [33] – related to the entanglement of formation as $E_F(\sigma) = -\nu_1 \log_2(\nu_1) - \nu_2 \log_2(\nu_2)$, where $\nu_{1,2} = (1 \pm \sqrt{1 - E_C(\sigma)^2})/2$.

Remark 2.3. – Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, and let $\sigma \in \mathcal{S}(\mathcal{H})$ either be a pure state or a Werner state. Then $E_N(\sigma) = E_C(\sigma)$.

Proof: The proof follows directly from the definition of the concurrence. For pure states one can make use of the fact that any state vector can be written in the Schmidt decomposition as $|\psi\rangle = \sqrt{\alpha}|00\rangle + \sqrt{\beta}|11\rangle$, with $1 \ge \alpha, \beta \ge 0$ and $\alpha + \beta = 1$.

The quantity $\log_2(\|\sigma^{T_B}\|) = \log_2(E_N(\sigma) + 1)$ is a useful upper bound for distillable entanglement $D_{\hookrightarrow}(\sigma)$ of a state σ [86]. This fact will later be made use of when it comes to a numerical investigation concerning the additivity properties of the relative entropy of entanglement. For some states this bound is tight, e.g., for the singlet state on $\mathbb{C}^2 \otimes \mathbb{C}^2$ with state vector $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. The bound $\log_2(\|\sigma^{T_B}\|)$ will be referred to as \log negativity.

2.2.4 Entropic Entanglement Monotones

The relative entropy of entanglement with respect to the set $\mathcal{P}(\mathcal{H})$ of PPT states is a good upper bound for D_{\leftarrow} . Unfortunately, for most states it is rather difficult to evaluate. Say, for a quantum system consisting of two systems with a Hilbert space of dimension N one has to find a minimum in an N^4-1 -dimensional set. Fortunately, both the set of separable states and the set of PPT states are convex sets with non-empty interior, and as the relative entropy functional is convex with respect to both arguments one may use efficient algorithms to find the minimum in these sets numerically. However, in higher dimensional Hilbert spaces even a numerical optimization is an extraordinary expensive procedure.

One very fruitful way out is to consider only particular states which exhibit a certain symmetry, and hope that characteristic features of the general picture remain, an idea going back to Ref. [46]. Historically – if one can speak of historical events in a topic just a decade old – such states with high symmetry were the first for which entanglement measures could be computed [46, 32, 31, 34]. The problem of quantifying the entanglement of *isotropic states* and so-called *Werner states* is much more feasible than that of general states, and a general strategy has been outlined in Ref. [47].

For general quantum states it seems appropriate to try to restrict the set of reference states while keeping certain desired features of the relative entropy of entanglement. The aim is to lessen the dimension of the set over which the variation has to be performed, but the restricted quantity should still be an entanglement monotone. Let $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$, and let for a given $\sigma \in \mathcal{S}(\mathcal{H})$

$$\mathcal{D}_{\sigma}(\mathcal{H}) = \left\{ \rho \in \mathcal{D}(\mathcal{H}) \middle| \operatorname{tr}_{A}[\rho] = \operatorname{tr}_{A}[\sigma], \operatorname{tr}_{B}[\rho] = \operatorname{tr}_{B}[\sigma] \right\}$$
(2.26)

be the subset of separable states that are locally identical to σ . In the same way one can define

$$\mathcal{P}_{\sigma}(\mathcal{H}) = \left\{ \rho \in \mathcal{P}(\mathcal{H}) \middle| \operatorname{tr}_{A}[\rho] = \operatorname{tr}_{A}[\sigma], \operatorname{tr}_{B}[\rho] = \operatorname{tr}_{B}[\sigma] \right\}$$
(2.27)

as a subset of the set $\mathcal{P}(\mathcal{H})$ of PPT states. For each σ both sets are convex and compact subsets of the state space and both include the maximally mixed state $1/N^2$. Now let

$$E_M(\sigma) = \min_{\rho \in \mathcal{D}_{\sigma}(\mathcal{H})} S(\sigma||\rho). \tag{2.28}$$

This functional is referred to as *modified relative entropy of entanglement*. Analogously, one can define⁶

$$B_M(\sigma) = \min_{\rho \in \mathcal{P}_{\sigma}(\mathcal{H})} S(\sigma||\rho). \tag{2.29}$$

Quite surprisingly, E_R and B_R , respectively, do not lose their monotonicity property by this restriction:

Proposition 2.4. – E_M and B_M are entanglement monotones.

Proof: The reasoning for E_M and B_M is fully analogous. For brevity, only E_M is considered in this proof. Condition (i) is satisfied due to the nilpotence property of the relative entropy functional. In order to show that E_M is convex, let $\sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H})$ and $\lambda \in [0, 1]$. Let

$$\rho_1 \in \mathcal{D}_{\sigma_1}(\mathcal{H}) \text{ and } \rho_2 \in \mathcal{D}_{\sigma_2}(\mathcal{H})$$
(2.30)

be the separable states for which the respective minimum in Eq. (2.28) is attained. This minimum is assumed due to the compactness of the sets \mathcal{D}_{σ_1} and \mathcal{D}_{σ_1} and since the relative entropy functional is lower semi-continuous. Then, as the relative entropy is joint convex,

$$\lambda E_{M}(\sigma_{1}) + (1 - \lambda)E_{M}(\sigma_{2}) = \lambda S(\sigma_{1}||\rho_{1}) + (1 - \lambda)S(\sigma_{2}||\rho_{2})$$

$$\geq S(\lambda \sigma_{1} + (1 - \lambda)\sigma_{2}||\lambda \rho_{1} + (1 - \lambda)\rho_{2}). \quad (2.31)$$

The convex combination of ρ_1 and ρ_2 with weight λ is locally identical to the corresponding mixture of σ_1 and σ_2 with the same weight,

$$\lambda \rho_1 + (1 - \lambda)\rho_2 \in \mathcal{D}_{\lambda \sigma_1 + (1 - \lambda)\sigma_2}(\mathcal{H}), \tag{2.32}$$

and hence,

$$\lambda E_M(\sigma_1) + (1 - \lambda)E_M(\sigma_2) \ge E_M(\lambda \sigma_1 + (1 - \lambda)\sigma_2) \tag{2.33}$$

holds.

Let Alice perform a local generalized measurement on \mathcal{H}_A . As again, mixing can only reduce the value of E_M due to the convexity property, one may without loss of generality assume that the posterior states can be written in the form $\eta_i = A_i \sigma A_i^\dagger/p_i$, where $p_i = \text{tr}[A_i \sigma A_i^\dagger]$ for i = 1, 2, ..., K, $\sum_{i=1}^K A_i^\dagger A_i = \mathbb{I}_A$.

Let $\sigma_i = A_i \sigma A_i^{\dagger}$ and $\rho_i = A_i \rho A_i^{\dagger}$ for i = 1, ..., K. Note that in general $tr[\sigma_i] \leq 1$ and $tr[\rho_i] \leq 1$. It follows from the monotonicity of the relative entropy that

$$\sum_{i} \operatorname{tr}[\sigma_{i}] S(\sigma_{i}/\operatorname{tr}[\sigma_{i}] | |\rho_{i}/\operatorname{tr}[\rho_{i}]) \leq S(\sigma||\rho), \tag{2.34}$$

⁶In Appendix C a very similar measure of entanglement will be defined that is based on the trace norm distance.

see Ref. [35, 101]. Let ω be the state that attains the minimum in Eq. (2.28) with respect to σ . The fact that $\omega \in \mathcal{D}_{\sigma}(\mathcal{H})$ implies that both $\operatorname{tr}_A[A_i\sigma A_i^{\dagger}] = \operatorname{tr}_A[A_i\omega A_i^{\dagger}]$ and $\operatorname{tr}_B[A_i\sigma A_i^{\dagger}] = \operatorname{tr}_B[A_i\omega A_i^{\dagger}]$ hold, as the quantum operation acts only locally, such that

$$\omega_i/\operatorname{tr}[\omega_i] \in \mathcal{D}_{\sigma_i/\operatorname{tr}[\sigma_i]}(\mathcal{H})$$
 (2.35)

for all i = 1, 2, ..., K, where $\omega_i = A_i \omega A_i^{\dagger}$. Therefore,

$$E_{M}(\sigma) = S(\sigma||\omega) \geq \sum_{i} \operatorname{tr}[\sigma_{i}] S(\sigma_{i}/\operatorname{tr}[\sigma_{i}] | |\omega_{i}/\operatorname{tr}[\omega_{i}])$$

$$\geq \sum_{i} \operatorname{tr}[\sigma_{i}] E_{M}(\sigma_{i}/\operatorname{tr}[\sigma_{i}]). \tag{2.36}$$

The same argument applies to the other party. On average E_M can only decrease when performing a generalized local measurement.

By construction, $E_M \geq E_R \geq D_{\leftrightarrow}$, meaning that the modified relative entropy of entanglement is a weaker bound than E_R itself. Similarly, $B_M \geq B_R \geq D_{\leftrightarrow}$. For a large class of states it can nevertheless be shown that E_R and E_M give the same value.

Proposition 2.5. – For pure states $|\psi\rangle\langle\psi|\in\mathcal{S}(\mathbb{C}^N\otimes\mathbb{C}^N)$

$$E_M(|\psi\rangle\langle\psi|) = E_R(|\psi\rangle\langle\psi|) = D_{\leftrightarrow}(|\psi\rangle\langle\psi|). \tag{2.37}$$

In $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ also $E_M(\sigma) = E_R(\sigma)$ for

- (i) Bell diagonal states,⁷
- (ii) states of the form $\sigma = \lambda |\phi^+\rangle \langle \phi^+| + (1-\lambda)|01\rangle \langle 01|$,
- (iii) $\sigma = \lambda |\phi^+\rangle \langle \phi^+| + (1-\lambda)|00\rangle \langle 00|$
- (iv) $\sigma = \lambda |00\rangle\langle 00| + \mu |00\rangle\langle 11| + \mu^* |11\rangle\langle 00| + (1-\lambda)|11\rangle\langle 11|$, and
- (v) $\sigma = a|00\rangle\langle00| + b|00\rangle\langle11| + b^*|11\rangle\langle00| + (1-2a)|01\rangle\langle01| + a|11\rangle\langle11|$,

where $\lambda \in [0,1]$, $\mu \in \mathbb{C}$ with $\lambda(1-\lambda) - |\mu|^2 \ge 0$, and $a \in [0,1/2]$, $b \in \mathbb{C}$ with $a^2 - |b|^2 \ge 0$.

Proof: In the Schmidt decomposition any $|\psi\rangle\in\mathcal{H}$ can be written as $|\psi\rangle=\sum_{i=1}^N\sqrt{\alpha_i}\,|ii\rangle$. A state $\rho\in\mathcal{D}(\mathcal{H})$ that minimizes the relative entropy functional is given by $\rho=\sum_{i=1}^N\alpha_i\,|ii\rangle\langle ii|$, which is included in $\mathcal{D}_{|\psi\rangle\langle\psi|}(\mathcal{H})$. In Ref. [35] closest separable states of the classes of states (i) – (v) are computed. For each σ the investigation of $\mathrm{tr}_A[\sigma]$, $\mathrm{tr}_B[\sigma]$ and $\mathrm{tr}_A[\rho]$, $\mathrm{tr}_B[\rho]$ of the respective separable reference state ρ shows that in all cases $\rho\in\mathcal{D}_\sigma(\mathcal{H})$.

⁷ Bell diagonal states on $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2$ are states which are diagonal in the basis consisting of the four state vectors $|\psi^+\rangle=(|01\rangle+|10\rangle)/\sqrt{2}, |\psi^-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}, |\phi^+\rangle=(|00\rangle+|11\rangle)/\sqrt{2}, |\phi^-\rangle=(|00\rangle-|11\rangle)/\sqrt{2}$. The corresponding states are called Bell states, the basis is the Bell basis.

In this spirit one can restrict the set of separable reference states also to other subsets of the separable states in order to make the minimizing procedure more accessible, or to simplify symmetry arguments. Let $\mathcal{C}_{\sigma}(\mathcal{H})$ be a compact and convex subset of $\mathcal{P}(\mathcal{H})$. Then one may – under certain circumstances – minimize the relative entropy over the set $\mathcal{C}_{\sigma}(\mathcal{H})$ without losing the monotonicity property:

Proposition 2.6. – Let $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$ and let Φ be a map

$$\sigma \longmapsto \Phi(\sigma) = \mathcal{C}_{\sigma}(\mathcal{H}) \tag{2.38}$$

mapping states $\sigma \in \mathcal{S}(\mathcal{H})$ on compact sets $\mathcal{C}_{\sigma}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$. If Φ has the property that

$$A\rho A^{\dagger}/tr[A\rho A^{\dagger}] \in \mathcal{C}_{A\sigma A^{\dagger}/tr[A\sigma A^{\dagger}]}(\mathcal{H})$$
 (2.39)

$$\lambda \rho_1 + (1 - \lambda)\rho_2 \in \mathcal{C}_{\lambda \sigma_1 + (1 - \lambda)\sigma_2}(\mathcal{H})$$
 (2.40)

for all $\sigma, \sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H})$, for all $\rho \in \mathcal{C}_{\sigma}(\mathcal{H})$, $\rho_1 \in \mathcal{C}_{\sigma_1}(\mathcal{H})$, $\rho_2 \in \mathcal{C}_{\sigma_2}(\mathcal{H})$, and all $\lambda \in [0, 1]$ and all $A : \mathbb{C}^N \longrightarrow \mathbb{C}^N$, then

$$E_G(\sigma) = \min_{\rho \in \mathcal{C}_{\sigma}(\mathcal{H})} S(\sigma||\rho)$$
 (2.41)

is an entanglement monotone.

Proof: One may proceed as in Proposition 2.5. Eq. (2.40) guarantees that if ρ_1 and ρ_2 are optimal separable states achieving the respective minima in Eq. (2.41) for two states σ_1 and σ_2 , then $\lambda \rho_1 + (1-\lambda)\rho_2$ is a possible (but not necessarily optimal) separable reference state for $\lambda \sigma_1 + (1-\lambda)\sigma_2$. Hence, E_G is convex. A similar argument can be used to show that on average, E_G can only decrease in the course of a local general quantum measurement. Let A_i , i=1,...,K, satisfying $\sum_{i=1}^K A_i^\dagger A_i = \mathbbm{1}_A$ be the Kraus operators associated with a local generalized measurement performed by, say, Alice. If ρ is the optimal separable state corresponding to σ taken from the set $\mathcal{C}_{\sigma}(\mathcal{H})$, then Eq. (2.39) ensures that $A_i\rho A_i^\dagger/\mathrm{tr}[A_i\rho A_i^\dagger]$ is an allowed separable reference state of the posteriori state $A_i\sigma A_i^\dagger/\mathrm{tr}[A_i\sigma A_i^\dagger]$. Using the same argument as in Proposition 2.5 one can conclude that E_G is an entanglement monotone.

So far, the considered entropic entanglement monotones have been defined in such a way that the reference state σ appeared in the first argument of the relative entropy functional. In the subsequent quantity σ is part of the second argument. Depending on a parameter $\mu \in [0,1]$ let

$$E_{\mu}(\sigma) = \min_{\omega \in \mathcal{D}_{\sigma}(\mathcal{H})} \min_{\rho \in \mathcal{D}_{\sigma}(\mathcal{H})} S(\rho || \mu \sigma + (1 - \mu)\omega). \tag{2.42}$$

Proposition 2.7. – E_{μ} is an entanglement monotone for all $\mu \in [0,1]$.

Proof: E_{μ} is clearly positive. Also, if $\sigma \in \mathcal{D}(\mathcal{H})$, then also $\mu \sigma + (1 - \mu)\omega \in \mathcal{D}(\mathcal{H})$ for all $\mu \in [0, 1]$ and all $\omega \in \mathcal{D}_{\sigma}(\mathcal{H})$. Let $\sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H})$, $\lambda \in [0, 1]$, and $\sigma = \lambda \sigma_1 + (1 - \lambda)\sigma_2$. Then

$$\lambda E_{\mu}(\sigma_{1}) + (1 - \lambda)E_{\mu}(\sigma_{2}) = \lambda S(\rho_{1}||\mu\sigma_{1} + (1 - \mu)\omega_{1}) + (1 - \lambda)S(\rho_{2}||\mu\sigma_{2} + (1 - \mu)\omega_{2})$$

$$\geq S(\lambda\rho_{1} + (1 - \lambda)\rho_{2}||\mu\sigma + (1 - \mu)(\lambda\omega_{1} + (1 - \lambda)\omega_{2}))$$

$$\geq E_{\mu}(\sigma). \tag{2.43}$$

That is, E_{μ} is a convex functional for all $\mu \in [0, 1]$.

Again, to show that on average E_{μ} does not increase under local generalized measurements let A_i , i=1,...,K, $\sum_{i=1}^K A_i^{\dagger}A_i=\mathbbm{1}_A$, be the Kraus operators of the local generalized measurement implemented by Alice. The first observation is that

$$\sum_{i=1}^{K} \operatorname{tr}[A_{i}\rho A_{i}^{\dagger}] S\left(\frac{A_{i}\rho A_{i}^{\dagger}}{\operatorname{tr}[A_{i}\rho A_{i}^{\dagger}]} \middle| \frac{\mu A_{i}\sigma A_{i}^{\dagger} + (1-\mu)A_{i}\omega A_{i}^{\dagger}}{\operatorname{tr}[\mu A_{i}\sigma A_{i}^{\dagger} + (1-\mu)A_{i}\omega A_{i}^{\dagger}]}\right) \leq S(\rho||\mu\sigma + (1-\mu)\omega)$$
(2.44)

(see Proposition 2.4). Let then $\rho, \omega \in \mathcal{D}_{\sigma}(\mathcal{H})$. The Kraus operators act in the Hilbert space of one party only and therefore,

$$\operatorname{tr}[A_i \rho A_i^{\dagger}] = \operatorname{tr}[A_i \sigma A_i^{\dagger}] = \operatorname{tr}[A_i \omega A_i^{\dagger}] \tag{2.45}$$

for all i=1,...,K, because $\operatorname{tr}[A_i\rho A_i^{\dagger}]=\operatorname{tr}_A[A_i\operatorname{tr}_B[\rho]A_i^{\dagger}]=\operatorname{tr}_A[A_i\operatorname{tr}_B[\sigma]A_i^{\dagger}]=\operatorname{tr}[A_i\sigma A_i^{\dagger}]$ and similarly for ω . It follows that Eq. (2.44) can be written as

$$\sum_{i=1}^{K} \operatorname{tr}[A_{i}\sigma A_{i}^{\dagger}] S\left(\frac{A_{i}\rho A_{i}^{\dagger}}{\operatorname{tr}[A_{i}\rho A_{i}^{\dagger}]} \middle| \left| \mu \frac{A_{i}\sigma A_{i}^{\dagger}}{\operatorname{tr}[A_{i}\sigma A_{i}^{\dagger}]} + (1-\mu) \frac{A_{i}\omega A_{i}^{\dagger}}{\operatorname{tr}[A_{i}\omega A_{i}^{\dagger}]} \right| \le S(\rho||\mu\sigma + (1-\mu)\omega), \tag{2.46}$$

such that

$$\sum_{i=1}^{K} \operatorname{tr}[A_{i}\sigma A_{i}^{\dagger}] E_{\mu} \left(\frac{A_{i}\sigma A_{i}^{\dagger}}{\operatorname{tr}[A_{i}\sigma A_{i}^{\dagger}]} \right) \leq E_{\mu}(\sigma). \tag{2.47}$$

This is the desired monotonicity property.

Practically, this measure is not very useful. The interesting aspect is that for $\mu=1$ it is a monotone that is fully additive, see the subsequent section, disproving the conjecture that fully additive entanglement monotones do not exist.

2.2.5 Additivity of Entanglement Measures

Problems of additivity occur in different contexts in quantum information theory, most notably in the characterization of quantum channels [102] and in the theory of the quantification of entanglement. In the latter case it is the additivity of entanglement measures that attracts major interest. The *additivity property* of a functional quantifying the degree of entanglement is – in a sense – a manifestation of the idea that entanglement is an extensive quantity: Assume two parties holding a pair of quantum systems in a certain mixed state σ . The entanglement – as quantified by an appropriate measure of entanglement E – is given by $E(\sigma)$. Later, the parties get another copy of the same state which has been prepared by the same source. The question that arises is whether they now share two times $E(\sigma)$ units of entanglement? Note that the two states of the pairs of quantum systems do not show any correlations. Intuitively, one might be tempted to assume that every entanglement measure automatically has this property, or that the structure of tensor products of Hilbert spaces is simple enough such that the decision is trivial whether a given measure quantifies entanglement as extensive in this sense.

It turned out that the problem of additivity is among the most notorious key problems of a theory of entanglement [47, 103]. Suppose the underlying Hilbert space has the structure $\mathcal{H} = (\mathcal{H}_A^{(1)} \otimes \mathcal{H}_A^{(2)}) \otimes (\mathcal{H}_B^{(1)} \otimes \mathcal{H}_B^{(2)})$. Local operations of Alice act in $\mathcal{H}_A^{(1)} \otimes \mathcal{H}_A^{(2)}$, whereas local operations of Bob act in $\mathcal{H}_B^{(1)} \otimes \mathcal{H}_B^{(2)}$. A measure of entanglement E is called *fully additive* [6, 35], if

$$E(\sigma \otimes \rho) = E(\sigma) + E(\rho), \tag{2.48}$$

for a state $\sigma \otimes \rho$ with $\sigma \in \mathcal{S}(\mathcal{H}_A^{(1)} \otimes \mathcal{H}_B^{(1)})$ and $\rho \in \mathcal{S}(\mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)})$ (see Fig. 2.1). Subadditivity is equivalent with $E(\sigma \otimes \rho) \leq E(\sigma) + E(\rho)$ for all such states σ and ρ . It is said to be *weakly additive* [86], if

$$E(\sigma^{\otimes n}) = nE(\sigma), \tag{2.49}$$

for all $n \in \mathbb{N}$, that is, if the state $\sigma^{\otimes n}$ is the state of n quantum systems prepared in a state σ by a stationary memoryless source. Strong additivity actually implies weak additivity.

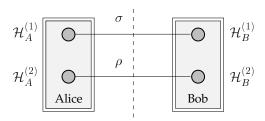


Figure 2.1: An entanglement monotone E is fully additive, if a state of this structure is assigned a value $E(\sigma \otimes \rho) = E(\sigma) + E(\rho)$, where $\sigma \in \mathcal{S}(\mathcal{H}_A^{(1)} \otimes \mathcal{H}_B^{(1)})$ and $\rho \in \mathcal{S}(\mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)})$.

The unique measure of entanglement for pure states – the von Neumann entropy of a local state – is fully additive on pure states. This is due to the fact that the von Neumann entropy satisfies $S(\sigma^{(1)}\otimes\sigma^{(2)})=S(\sigma^{(1)})+S(\sigma^{(2)})$ for all states $\sigma^{(1)}$ and $\sigma^{(2)}$. The interpretation of the additivity of the unique measure for pure states can be stated as follows: If two parties share n copies (n very large) of a system in a not maximally entangled state $|\psi\rangle\langle\psi|$, then they convert them into $nE_F(|\psi\rangle\langle\psi|)$ copies of systems in a maximally entangled state. Also, to prepare n copies of $|\psi\rangle\langle\psi|$, approximately $nE_F(|\psi\rangle\langle\psi|)$ copies of systems in a maximally entangled state are needed.

For mixed states the situation is very different. The distillable entanglement is by definition weakly additive. D_{\leftrightarrow} is nevertheless not known to be fully additive. However, for no entanglement monotone that is not defined via a limit of infinitely many copies of a state a proof of weak additivity is known (but see Proposition 2.13). In order to achieve weak additivity for a generally subadditive entanglement monotone E one may consider the *regularized* version of it. This is defined as

$$E^{\infty}(\sigma) = \limsup_{n \to \infty} \frac{E(\sigma^{\otimes n})}{n},$$
(2.50)

which is the mean entanglement of several copies of the state, evaluated in the limit of infinitely many copies. In practice it is hardly possible to calculate this quantity for any entanglement monotone for a given mixed state. Proposition 2.14, however, will give a first example of the value of a regularized measure of entanglement.

It has been shown [86] that all weakly additive and weakly continuous entanglement monotones are confined by the distillable entanglement D_{\leftrightarrow} and the regularized entanglement of formation,

$$D_{\leftrightarrow}(\sigma) \le E(\sigma) = \limsup_{n \to \infty} \frac{E(\sigma^{\otimes n})}{n} \le E_F^{\infty}(\sigma), \tag{2.51}$$

As a corollary it follows that E_M^{∞} is a lower bound for E_F^{∞} , and, as the entanglement of formation can only be subadditive, it can be concluded that

$$E_M^{\infty}(\sigma) \le E_F(\sigma) \tag{2.52}$$

holds for any state σ . E_N is not additive in any sense; $\log_2(E_N+1)$ is fully additive, as $\|\sigma^{T_A}\otimes\rho^{T_B}\|=\|\sigma^{T_A}\|\|\rho^{T_B}\|$ for all $\rho\in\mathcal{H}^{(1)}$ and all $\sigma\in\mathcal{H}^{(2)}$, but it is no entanglement monotone. Another fully additive quantity is E_1 defined in Eq. (2.42):

Proposition 2.8. – E_1 is fully additive.

Proof: The additivity of E_1 follows from Lemma 2.9 together with the additivity property of the relative entropy [104], i.e.,

$$S(\rho^{(1)} \otimes \rho^{(2)} || \sigma^{(1)} \otimes \sigma^{(2)}) = S(\rho^{(1)} || \sigma^{(1)}) + S(\rho^{(2)} || \sigma^{(2)})$$
(2.53)

for all $\sigma^{(1)}$, $\rho^{(1)} \in \mathcal{S}(\mathcal{H}^{(1)})$ and $\sigma^{(2)}$, $\rho^{(2)} \in \mathcal{S}(\mathcal{H}^{(2)})$.

Lemma 2.9. – Let $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$ and let \mathcal{C} be the (compact and convex) subset of $\mathcal{D}(\mathcal{H})$ of all states which can be written in the form $\rho^{(1)} \otimes \rho^{(2)}$, where $\rho^{(1)} \in \mathcal{D}(\mathcal{H}^{(1)})$ and $\rho^{(2)} \in \mathcal{D}(\mathcal{H}^{(2)})$. Then

$$\min_{\rho \in \mathcal{D}(\mathcal{H})} S(\rho \| \sigma^{(1)} \otimes \sigma^{(2)}) = \min_{\rho \in \mathcal{C}} S(\rho \| \sigma^{(1)} \otimes \sigma^{(2)})$$
 (2.54)

for all $\sigma^{(1)} \in \mathcal{S}(\mathcal{H}^{(1)})$ and $\sigma^{(2)} \in \mathcal{S}(\mathcal{H}^{(2)})$.

Proof: From the *conditional expectation property* of the relative entropy [104] with respect to the partial trace projection ⁸ it follows that

$$S(\rho||\sigma^{(1)} \otimes \sigma^{(2)}) = S(\rho^{(1)}||\sigma^{(1)}) + S(\rho||\rho^{(1)} \otimes \sigma^{(2)})$$
(2.56)

for all $\sigma^{(1)} \in \mathcal{S}(\mathcal{H}^{(1)})$, $\sigma^{(2)} \in \mathcal{S}(\mathcal{H}^{(2)})$, and $\rho \in \mathcal{S}(\mathcal{H})$, where $\rho^{(1)} = \operatorname{tr}_2[\rho]$ and $\rho^{(2)} = \operatorname{tr}_1[\rho]$ are the reduced density operators, such that

$$S(\rho||\sigma^{(1)} \otimes \sigma^{(2)}) = S(\rho^{(1)}||\sigma^{(1)}) + S(\rho^{(2)}||\sigma^{(2)}) + S(\rho||\rho^{(1)} \otimes \rho^{(2)}), \tag{2.57}$$

and hence

$$S(\rho||\sigma^{(1)} \otimes \sigma^{(2)}) \ge S(\rho^{(1)} \otimes \rho^{(2)}||\sigma^{(1)} \otimes \sigma^{(2)}).$$
 (2.58)

This in turn implies that the state $\rho \in \mathcal{D}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ which minimizes $S(\rho \| \sigma^{(1)} \otimes \sigma^{(2)})$ can always be taken out of the smaller subset \mathcal{C} .

$$S(\sigma||\rho) = S(\sigma|_{\mathcal{B}}||\rho|_{\mathcal{B}}) + S(\sigma||\sigma \circ E)$$
(2.55)

holds. In the words of Ref. [104], the interpretation of this equality is that the "informational divergence" of σ from ρ on the algebra $\mathcal A$ is given by the sum of the corresponding divergence on the subalgebra $\mathcal B$ and the divergence of σ from $\sigma \circ E$, which could be conceived as the extension of $\sigma|_{\mathcal B}$ to the full algebra $\mathcal A$.

⁸Originally, the conditional expectation property has been formulated in terms of finite dimensional C^* -algebras [104]. On a finite dimensional C^* -algebra $\mathcal A$ there exists a (unique) trace functional tr with the property that $\operatorname{tr}[AB]=\operatorname{tr}[BA]$ for all $A,B\in\mathcal A$ (isomorphic algebras are not distinguished). Associated with every functional σ on the algebra $\mathcal A$ is a density operator $D_\sigma\in\mathcal A$ via $\sigma(A)=\operatorname{tr}[D_\sigma A]$. That is, states are identified with functionals on the algebra $\mathcal A$. A conditional expectation is now defined as follows. Let $\mathcal B\subset\mathcal A$ be a $\mathbb C^*$ -algebra. A conditional expectation is a linear map $E:\mathcal A\longrightarrow\mathcal B$ with the properties that (i) for all $B\in\mathcal B$ E(B)=B, (ii) if $A\in\mathcal A^+$ (the positive part of $\mathcal A$) then $E(A)\in\mathcal B^+$, (iii) E(AB)=E(A)B for all $A\in\mathcal A$ and $B\in\mathcal B$.

The conditional expectation property can be formulated with the help of such maps E. Let $\mathcal{B} \subset \mathcal{A}$ be a subalgebra of \mathcal{A} (also a C*-algebra), and let ρ be a state of \mathcal{A} with an invertible density D_{ρ} . If there exists a conditional expectation $E: \mathcal{A} \to \mathcal{B}$ with $\rho \circ E = \rho$, then for any state σ of \mathcal{A} the equality

This is the only known example of an entanglement monotone that is (i) additive and (ii) that is not defined as a regularized entanglement monotone in an asymptotic limit as in Eq. (2.50). In particular, it is the only fully additive entanglement monotone. However, from a practical point of view E_1 is useless: this is because pure states are not mapped on real numbers; it diverges on pure states. E_1 maps $\mathcal{S}(\mathcal{H})$ on $\mathbb{R} \cup \{\infty\}$ (see also Ref. [35] in this context).

Conjecture 2.10. – *Under the assumptions of Lemma 2.9*

$$S(\sigma^{(1)} \otimes \sigma^{(2)} || \rho) \ge S(\sigma^{(1)} \otimes \sigma^{(2)} || \rho^{(1)} \otimes \rho^{(2)})$$
 (2.59)

holds, where $\rho^{(1)} = tr_2[\rho]$ and $\rho^{(2)} = tr_1[\rho]$.

If this conjecture was true, there would be wide-reaching consequences: It would imply that the relative entropy of entanglement was fully additive, and that the regularized relative entropy of entanglement was identical to E_R itself. Unfortunately, the statement of Conjecture 2.10 is wrong in general. A random matrix test provides counterexamples to this statement. 9

Quite recently, it has been shown that E_R is not weakly additive, refuting a common belief [106, 47]. Instead, the relative entropy of entanglement with respect to separable states can be shown to be strictly subadditive. In the remainder of this subsection a numerical investigation of the asymptotic limit of infinitely many copies of a state will be presented which goes along the lines of the counterexample of Ref. [47]. The considered state is a state with high symmetry: a Werner state [46, 47]. The result of the numerical investigation will be summarized in Proposition 2.14.

The Hilbert space is taken to be $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, $\mathcal{H}_A=\mathcal{H}_B=\mathbb{C}^N$, and later restricted to $\mathbb{C}^3\otimes\mathbb{C}^3$. Let π be the *permutation operator* that interchanges the states of both parties. In terms of the permutation operator the projections on the *symmetric* and the *antisymmetric* subspaces of $\mathbb{C}^N\otimes\mathbb{C}^N$ can be written as

$$\pi_s = (1+\pi)/2, \qquad \pi_a = (1-\pi)/2,$$
 (2.62)

respectively. The trace of these operators is given by $tr[\pi_s] = N(N+1)/2$ and $tr[\pi_a] = N(N-1)/2$. Let

$$\sigma_s = \pi_s/\text{tr}[\pi_s], \qquad \sigma_a = \pi_a/\text{tr}[\pi_a].$$
 (2.63)

It is obvious that σ_a and σ_s are invariant under the map $\omega \longmapsto (U \otimes U)\omega(U \otimes U)^{\dagger}$, where $U:\mathcal{H}_A \to \mathcal{H}_A$ is a unitary operator. In fact, all states that are invariant under a random local unitary operation with operators $U \otimes U$ are a convex combination of these two states σ_a and σ_s . More precisely, let the projection $\Pi: \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H})$ be

$$\Pi(\rho) = \int d\mu_U (U \otimes U) \rho(U \otimes U)^{\dagger}, \qquad (2.64)$$

⁹Let $\mathcal{H} = \mathbb{C}^4 \otimes \mathbb{C}^4$. Every state $\rho \in \mathcal{S}(\mathcal{H})$ can be represented according to

$$\rho = UDU^{\dagger},\tag{2.60}$$

where U is a unitary 16×16 -matrix and D is a diagonal 16×16 matrix, $D_{ij} = p_i \delta_{ij}$. A random state may now be drawn as follows [68], [E7]: A plausible (but by no means the only) choice for the ensemble of random unitaries is the one with a uniform distribution on unitaries of the above type (corresponding to the Haar measure on the group U(16)), which is called *circular unitary ensemble* [105]. For the diagonal matrix D a uniform distribution is chosen on the manifold defined by $\sum_i p_i = 1$. If one checks the validity of Conjecture 2.10 with the help of random matrices drawn from this ensemble, one can easily find counterexamples. The relative frequency of a violation of Eq. (2.59) in a test with N=1000 runs yielded as an estimate for the probability of a violation the (surprisingly small) value

$$p = 0.007 \pm 0.001. \tag{2.61}$$

where the integral is performed with respect to the normalized invariant measure of the unitary group (*Haar measure*) [46, 83]. This quantum operation is typically referred to as *twirling operation*; it is a trace-preserving completely positive unital map mapping arbitrary states on *Werner states* [46]. Then operators O that satisfy $\Pi(O) = O$ are just those operators for which $[U \otimes U, O] = 0$: they form the *commutant* of the group G having unitaries of the type $U \otimes U$ as elements. It turns out that the commutant is a vector space spanned by $\mathbbm{1}$ and π [46, 47], and all states ρ that are invariant under Π are a mixture of σ_a and σ_s . The weights of the convex combination are given by $\Pi(\rho) = \sigma_a \operatorname{tr}[\rho \pi_a] + \sigma_s \operatorname{tr}[\rho \pi_s]$ [83].

Of interest to the issue of this subsection is a Hilbert space of the structure $\mathcal{H}^{\otimes n}$, $n \geq 2$. Suppose that the two parties share n copies of the state $\sigma = \lambda \sigma_s + (1 - \lambda)\sigma_a$. To be specific, let $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3$. The state $\sigma^{\otimes n}$ is invariant under unitary operations of the type

$$\omega \longmapsto \left(U^{(1)} \otimes U^{(1)} \right) \otimes \dots \otimes \left(U^{(n)} \otimes U^{(n)} \right) \omega \left(U^{(1)} \otimes U^{(1)} \right)^{\dagger} \otimes \dots \otimes \left(U^{(n)} \otimes U^{(n)} \right)^{\dagger}. \tag{2.65}$$

For each copy of the state the same unitary local operations are applied on both Alice's part and Bob's part of the composite quantum system. The group of local unitaries is hence the group $G^{\otimes n}$, where G is as before the group with unitary operators $U \otimes U$ as elements. By iterating the argument of Ref. [47] one finds that states that are invariant under this operation are necessarily a convex combination of states of the form $\sigma_{i_1} \otimes ... \otimes \sigma_{i_n}$, where $i_1,...,i_n \in \{a,s\}$.

Due to the symmetry of the state the evaluation of the relative entropy of entanglement is simplified by large: If σ is invariant under a group G, then the variation over the set $\mathcal{D}(\mathcal{H})$ or $\mathcal{P}(\mathcal{H})$ can also be restricted to the subset which is also invariant under the same group G [91, 107, 47]. For $\lambda=0$ and n=2 this reasoning leads to the counterexample of Ref. [47]:

Example (Werner and Vollbrecht). – For a single copy of the system in the state σ_a the relative entropy of entanglement with respect to separable states can be evaluated as

$$E_R(\sigma_a) = S(\sigma_a||(\sigma_a + \sigma_s)/2), \tag{2.66}$$

leading to $E_R(\sigma_a) = 1$. For two copies of the same state σ_a one finds that $E_R(\sigma_a \otimes \sigma_a) \leq S(\sigma_a \otimes \sigma_a || \rho)$, where $\rho = (1/3)\sigma_a + (3/4)\sigma_s$, and hence,

$$2 = 2E_R(\sigma_a) > S(\sigma_a \otimes \sigma_a || \rho) = \log_2(3) \ge E_R(\sigma_a \otimes \sigma_a). \tag{2.67}$$

The statement that ρ is a separable state can be proved by designing a protocol preparing the state locally starting from a product state. Therefore, for σ_a the relative entropy of entanglement is subadditive.

A consequence is that E_M has the same property. Let ρ be defined as in the previous example, then the subadditivity follows from the fact that $\operatorname{tr}_A[\rho] = \operatorname{tr}_A[\sigma_a \otimes \sigma_a] = \mathbb{1}_B$, $\operatorname{tr}_B[\rho] = \operatorname{tr}_B[\sigma_a \otimes \sigma_a] = \mathbb{1}_A$, and $\operatorname{tr}_A[(\sigma_a + \sigma_s)/2] = \operatorname{tr}_A[\sigma_a] = \mathbb{1}_B$, $\operatorname{tr}_B[(\sigma_a + \sigma_s)/2] = \operatorname{tr}_B[\sigma_a] = \mathbb{1}_A$:

Corollary 2.11. – E_M is truly subadditive.

From now on let n=1,2,... be arbitrary. The numerical investigation does not involve the relative entropy of entanglement with respect to the separable states E_R , but the tighter bound B_R , that is, the relative entropy with respect to PPT states. It has already been pointed out that this quantity is subadditive [91]. However, in this subsection the case of many copies will be addressed. Let $\sigma=\lambda\sigma_s+(1-\lambda)\sigma_a$. The elements of series $(e_n)_{n\in\mathbb{N}}$ with

$$e_n(\lambda) = \frac{B_R(\sigma^{\otimes n})}{n} \tag{2.68}$$

are the average amounts of entanglement of n copies of the state σ . The regularized quantity $B_R^\infty(\sigma)$ can then be formulated as

$$B_R^{\infty}(\sigma) = \limsup_{n \to \infty} e_n(\lambda). \tag{2.69}$$

The series $(e_n(\lambda))_{n\in\mathbb{N}}$ is actually a convergent series: It has been proved in Ref. [108] that if $(a_n)_{n\in\mathbb{N}}$ is a series with the property $a_n+a_m\geq a_{n+m}$ for all $n,m\geq 0$ and for which a $C\geq 0$ exists such that $a_n\leq Cn$ for all n, then $(a_n/n)_{n\in\mathbb{N}}$ is a convergent series. If follows from the subadditivity of E_R that the sequence with elements $a_n=ne_n(\lambda)$ satisfies these criteria.

In order to show that every known element of the series is also an upper bound for the limit, assume that the value of $e_m(\lambda)$ for an $m \in \mathbb{N}$ is known. Then the sequence $e_m(\lambda)$, $e_{2m}(\lambda)$, $e_{3m}(\lambda)$, ... is a subsequence satisfying $e_m(\lambda) \geq e_{km}(\lambda)$ for all $k \in \mathbb{N}$, and hence,

$$e_m(\lambda) \ge \liminf_{n \to \infty} e_n(\lambda).$$
 (2.70)

Since $(e_n)_{n\in\mathbb{N}}$ is convergent, also $e_m(\lambda) \geq \lim_{n\to\infty} e_n(\lambda)$ holds.

Once it is clear that the limit is well defined, a recipe for calculating the value of $e_n(\lambda) = B_R(\sigma^{\otimes n})/n$ has to be developed. Again, the symmetry of the state simplifies the problem considerably. First, note that due to the invariance of $\sigma^{\otimes n}$ under $G^{\otimes n}$, the closest PPT state must be a mixture of $\sigma_{i_1} \otimes ... \otimes \sigma_{i_n}$, $i_1,...,i_n \in \{a,s\}$. Second, $\sigma^{\otimes n}$ is invariant under the map

$$\omega \longmapsto (\pi \otimes \pi)\omega(\pi \otimes \pi), \tag{2.71}$$

where $\pi \in S_n$. S_n is the *symmetric group* of degree n, whose elements are the *permutation operators of degree* n. π denotes both a permutation and the associated unitary. Note that $\pi \otimes \pi$ corresponds to a local operation, as each permutation operator acts in Alice's or Bob's system only. Hence, in the variation over PPT states it suffices to consider states that are invariant under both groups:

Lemma 2.12. – Let $e_n(\lambda) = B_R(\sigma^{\otimes n})/n$ be defined as above. Then

$$B_R(\sigma^{\otimes n}) = S(\sigma^{\otimes n}||\rho_n), \tag{2.72}$$

where the state $\rho_n \in \mathcal{P}(\mathcal{H})$ is of the form

$$\rho_n = \sum_{k=0}^n \frac{p_k}{\binom{n}{k}} \sum_{\pi \in S_n} (\pi \otimes \pi) \left(\sigma_a^{\otimes k} \sigma_s^{\otimes (n-k)} \right) (\pi \otimes \pi). \tag{2.73}$$

 $p_0, ..., p_n$ is a probability distribution. The second sum is performed over all elements of the symmetric group S_n of degree n.

For simplicity of notation assume from now on that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^3$. Explicitly, if the basis elements of \mathcal{H}_A and \mathcal{H}_B are labeled $\{|1\rangle, |2\rangle, |3\rangle\}$, the states σ_a and σ_s can be written as

$$\sigma_s = \frac{1}{6} \left(\sum_{i=1}^3 |ii\rangle\langle ii| + \sum_{i,j=1(i< j)}^3 (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|) \right), \tag{2.74}$$

$$\sigma_a = \frac{1}{3} \left(\sum_{i,j=1(i < j)}^{3} (|ij\rangle - |ji\rangle) (\langle ij| - \langle ji|) \right). \tag{2.75}$$

The operators $\sigma_a^{T_A}$ and $\sigma_s^{T_A}$ commute and can be simultaneously diagonalized. In an appropriate basis they can be represented according to

where diag denotes a diagonal matrix with the vector entries as main diagonal elements. When the task is to find a criterion under what circumstances a convex combination of products of these states is a PPT state, the degeneracy of the largest eigenvalue is of no relevance, and one can treat $\sigma_a^{T_A}$ and $\sigma_s^{T_A}$ as if they were operators ω_a and ω_s with a matrix representation diag(1/6, -1/3) and diag(1/12, 1/3), respectively. That is, ρ_n is a PPT state if and only if

$$\sum_{k=0}^{n} \frac{p_k}{\binom{n}{k}} \sum_{\pi \in S_n} \pi \left(\omega_a^{\otimes k} \omega_s^{\otimes (n-k)} \right) \pi \ge 0.$$
 (2.78)

For n=1 the state ρ_n is a PPT state if and only if $p_0+2p_1\geq 0$ and $p_0-p_1\geq 0$. For larger none can explicitly show [109] that Eq. (2.78) is equivalent with

$$\sum_{k=0}^{n} \frac{p_k}{\binom{n}{k}} \sum_{l=0}^{k} \left(-\frac{1}{2}\right)^l \binom{n-s}{k-l} \binom{s}{l} 2^k \ge 0 \tag{2.79}$$

for all s = 0, ..., n. The sum can be evaluated as

$$\sum_{r=1}^{n} p_r \frac{2^r (n-r)!}{n!(n-s)!(n-r-s)!} {}_2F_1\left(-r, -s, 1+n-r-s, -1/2\right) + p_0 \ge 0, \quad (2.80)$$

where ${}_{2}F_{1}$ denotes the hypergeometric function. This condition, together with the general form of the subset of PPT states that has to be considered of Lemma 2.12 allows for a numerical evaluation of the regularized entanglement measure $B_R^\infty(\sigma)$ with arbitrary accuracy. The elements of the series $(e_n)_{n\in\mathbb{N}}$ are given by

$$e_n(\lambda) = \frac{S(\sigma^{\otimes n}||\rho_n)}{n} = \frac{1}{n} \sum_{k=1}^n \binom{n}{k} \lambda^{n-k} (1-\lambda)^k \log_2 \left(\binom{n}{k} \lambda^{n-k} (1-\lambda)^k / p_k \right) + \frac{1}{n} \lambda^n \log_2 \left(\lambda^n / p_0 \right).$$

$$(2.81)$$

Example 2.13. – For $\lambda = 0$, that is, $\sigma = \sigma_a$, the remaining minimization can be performed numerically in an efficient way. Then Eq. (2.81) reduces to

$$\frac{S(\sigma_a^{\otimes n}||\rho_n)}{n} = \frac{1}{n}\log_2(p_n). \tag{2.82}$$

In order to minimize the relative entropy functional of $\sigma_a^{\otimes n}$ with respect to a state of the form given by Eq. (2.73) one has to maximize p_n under the affine constraints given by Eq. (2.2.5), since the logarithm is a monotone increasing function. Minimization problems in which the objective function is linear and the constraint functions are affine are linear programming problems. For such linear programming problems efficient and stable numerical algorithms are available.¹⁰ The values of $(e_n(0))_{n\in\mathbb{N}}$ can therefore be evaluated with high accuracy. The first 7 values $e_1(0),...,e_7(0)$ are given by

$$e_1(0) = 1,$$
 (2.84)

$$e_2(0) = \log_2(3)/2,$$
 (2.85)

$$e_3(0) = \log_2(5)/3,$$
 (2.86)

$$e_4(0) = 3/4,$$
 (2.87)

$$e_5(0) = \log_2(13.25)/5,$$
 (2.88)

$$e_6(0) = \log_2(21.75)/6,$$
 (2.89)

$$e_7(0) = \log_2(36)/7,$$
 (2.90)

see Fig. 2.2. The corresponding closest PPT states will be presented in Appendix B. With not too much numerical effort the program can evaluate the average value of B_R for 40 copies of the state σ_a . The series converges quickly, and the values for 20 and 30 copies are identical to four significant digits. The main result can be stated as follows.

Proposition 2.14. – Let $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3$, and let $\sigma_a = \pi_a/tr[\sigma_a]$ be the state that is proportional to the projector on the antisymmetric subspace of \mathcal{H} as defined above. Then $B_R^{\infty}(\sigma_a)$ satisfies $B_R^{\infty}(\sigma_a) \leq B_R(\sigma_a^{\otimes n})/n$ for all $n \in \mathbb{N}$. In particular,

$$B_R^{\infty}(\sigma_a) \le B_R(\sigma_a^{\otimes 40})/40 = 0.73697.$$
 (2.91)

Interestingly, this value strongly suggests that $B_R^{\infty}(\sigma_a)$ is identical with another upper bound for distillable entanglement D_{\leftrightarrow} (see also Ref. [112]), namely, the log negativity. For the state σ_a the log negativity is given by

$$\log_2(E_N(\sigma_a) + 1) = \log_2(\|\sigma_a^{T_A}\|) = \log_2(5/3) = 0.73697.$$
(2.92)

The log negativity is fully additive, and hence, $\lim_{n\to\infty} \log_2(\|(\sigma_a^{T_A})^{\otimes n}\|)/n = \log_2(\|\sigma_a^{T_A}\|)$.

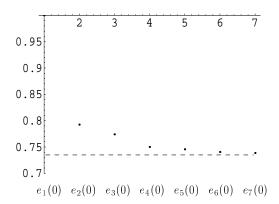


Figure 2.2: The average entanglement per copy of σ_a for n = 1, ..., 7 copies. The dashed line corresponds to $B_R(\sigma_a^{\otimes 40})/40$.

$$z = a_{01}x_1 + \dots + a_{0m}x_m (2.83)$$

with primary constraints $x_1 \ge 0, \dots, x_m \ge 0$ and additional constraints of the form $a_{k1}x_1 + \dots + a_{km}x_m = b_k, \quad k = 1, \dots, l$, where $(a_{ij})_{i=1,\dots,l;j=1,\dots,m}$ is a real matrix and b_1,\dots,b_l are real positive numbers. The most prominent algorithm for solving such a linear programming problem in restricted normal form is the *Simplex method* [110, 111]. The subsequent analysis has been carried out using this method.

¹⁰The above problem can be cast into the so-called *restricted normal form*. This means that by introducing auxiliary variables the function that has to be maximized can be written as

This result has an implication on the issue of "irreversibility" of asymptotic manipulation of entanglement. In Ref. [92] it is stated that the process of distillation is truly irreversible, meaning that there exist states for which the distillable entanglement D_{\leftrightarrow} and the regularized entanglement of formation E_F^{∞} are different from each other: More resources are needed in a preparation procedure even in the asymptotic limit than can be distilled from many copies of the same state. The argument goes as follows: if one has two quantities E_1 and E_2 satisfying

$$D_{\leftrightarrow}(\sigma) \le E_1(\sigma), \ E_1(\sigma) < E_2(\sigma), \ E_2(\sigma) \le E_F^{\infty}(\sigma)$$
 (2.93)

for a mixed state σ , then existence of irreversibility is confirmed. The two considered quantities are $E_1=B_R^\infty$ and $E_2=\log_2(E_N+1)$, and the crucial ingredient is that $B_R^\infty(\sigma)<\log_2(E_N(\sigma)+1)$, where σ is taken from a small subset of Werner states. Unfortunately, a theorem from Ref. [91] has been used in Ref. [92] that has turned out to be wrong in general. For the case $\sigma=\sigma_a$ the above numerical investigation even suggests that these quantities are equal, which leaves the question again open whether this type of irreversibility exists in quantum theory. It is conceivable that D_{\leftrightarrow} and E_F^∞ are identical for all states, meaning that one can extract the same amount of entanglement from many copies of a mixed state as one has to invest in order to "form" copies of this state.

It would also be very interesting to carry out a similar analysis in the case that $\sigma = \lambda \sigma_s + (1-\lambda)\sigma_a$ with $0 < \lambda < 1$. Of particular interest is the regime $\lambda \in [2/5,1/2]$. It has been conjectured on indeed well-footed grounds [83] that in this regime σ is a bound entangled state, meaning that $D_{\leftrightarrow}(\sigma) = 0$, despite of the fact that σ is by definition not a PPT state (see also [113]). This state would be a counterexample to the statement "all states with a non-positive partial transpose are distillable". As $B_R^{\infty}(\sigma)$ is a tight upper bound for $D_{\leftrightarrow}(\sigma)$, it might well be that a similar analysis as above yields the value $B_R^{\infty}(\sigma) = 0$. The implication would be that testing whether the partial transpose is positive is not enough to find out whether copies of a state can be distilled into a useful form.

2.2.6 Continuity Properties

As has already been mentioned, there is yet another property of an entanglement monotone that is important when considering the asymptotic limit: it is the appropriate *continuity* of the entanglement measure. Take, say, a distillation process in which one tries to distill a certain pure state $|\psi\rangle\langle\psi|$ from a large number of identically prepared quantum systems in a mixed state σ . More precisely, one has a number m of copies of quantum systems in a state σ at hand. By applying LOCC operations, one maps the state $\sigma^{\otimes m}$ with unit probability on ρ_n . This state can be made arbitrarily close to the desired state $|\psi\rangle\langle\psi|^{\otimes n}$, and the rate n/m approaches the optimal rate to which this transformation is possible as $n\to\infty$.

Against the backdrop of these considerations, a good measure of entanglement E should fulfil the following condition that equals the weak continuity introduced in Eq. (2.4): the difference in the degree of entanglement per copy $(E(\rho_n) - E(|\psi\rangle\langle\psi|^{\otimes n}))/n$ tends to zero as $n \to \infty$. Roughly speaking, the requirement is that the measure of entanglement is sufficiently continuous close to many copies of products of pure states.

This form of continuity contrasts with the *strong continuity* of the unique measure of entanglement for pure states [114] via Fannes inequality [115]¹¹: For *pure states* σ_1 and σ_2

$$|S(\sigma) - S(\rho)| \le \|\sigma - \rho\| \log_2(d) - \|\sigma - \rho\| \log_2(\|\sigma - \rho\|)$$
 (2.94)

for two states σ and ρ satisfying $\|\sigma - \rho\| < 1/3$, d is the dimension of the underlying Hilbert space.

¹¹Fannes' inequality states that

with $\|\sigma_1 - \sigma_2\| < 1/3$

$$|E_F(\sigma_1) - E_F(\sigma_2)| \le \log_2(d) \|\sigma_1 - \sigma_2\| - \|\sigma_1 - \sigma_2\| \log_2(\|\sigma_1 - \sigma_2\|)$$
 (2.95)

holds, where d is the dimension of the Hilbert space of the quantum system. For the relative entropy of entanglement E_R and B_R [116] and the entanglement of formation E_F [114] similar results to Eq. (2.95) hold for mixed states.

In this subsection a single statement will be presented: The subsequent proposition shows that the modified relative entropy of entanglement E_M is weakly continuous. The proof is technical, and therefore, the major part of the proof is presented in Lemma 2.16. The result will not be necessary for the understanding of the later considerations, so the proof may be skipped.

Proposition 2.15. – E_M is weakly continuous: Let $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$, and let $|\psi\rangle \in \mathcal{H}$. Let $(\sigma_n)_{n \in \mathbb{N}}$ be a series of states $\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$ with the property $\lim_{n \to \infty} ||\psi\rangle\langle\psi|^{\otimes n} - \sigma_n|| = 0$, where ||.|| denotes the trace norm. Then E_M satisfies

$$\lim_{n \to \infty} \frac{1}{n} \left| E_M(|\psi\rangle\langle\psi|^{\otimes n}) - E_M(\sigma_n) \right| = 0.$$
 (2.96)

Proof: Let $|\psi\rangle \in \mathcal{H}$ and $(\sigma_n)_{n\in\mathbb{N}}$ be a series of states $\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$ as above. The first step is to introduce a certain appropriate series of pure states: There exists a series $(|\psi\rangle_n)_{n\in\mathbb{N}}$ with $|\phi_n\rangle \in \mathcal{H}^{\otimes n}$ for n=1,2,... with the properties (i)

$$\operatorname{tr}_{A}[|\phi_{n}\rangle\langle\phi_{n}] = \operatorname{tr}_{A}[\sigma_{n}], \ \operatorname{tr}_{B}[|\phi_{n}\rangle\langle\phi_{n}] = \operatorname{tr}_{B}[\sigma_{n}], \tag{2.97}$$

and (ii) $\lim_{n \to \infty} \| |\phi_n\rangle \langle \phi_n| - |\psi\rangle \langle \psi|^{\otimes n} \| = 0, \quad \lim_{n \to \infty} \| |\phi_n\rangle \langle \phi_n| - \sigma_n \| = 0. \tag{2.98}$

That is, $|\phi_n\rangle\langle\phi_n|$ is locally identical to σ_n . Note that the trace distance of two states is non-increasing under trace-preserving completely positive maps \mathcal{E} [117],

$$\|\sigma - \rho\| \ge \|\mathcal{E}(\sigma) - \mathcal{E}(\rho)\| \tag{2.99}$$

for all states σ , ρ . In particular, this statements holds for the partial trace operation. Thus, such a sequence always exists. As a consequence,

$$\frac{|E_{M}(|\psi\rangle\langle\psi|^{\otimes n}) - E_{M}(\sigma_{n})|}{n} \leq \frac{|E_{M}(|\psi\rangle\langle\psi|^{\otimes n}) - E_{M}(|\phi_{n}\rangle\langle\phi_{n}|)|}{n} + \frac{|E_{M}(|\phi_{n}\rangle\langle\phi_{n}|) - E_{M}(\sigma_{n})|}{n}.$$
(2.100)

The first term on the right hand side of Eq. (2.100) will vanish in the limit $n \to \infty$. According to Lemma 2.6 E_M and the von Neumann entropy of a local state (see Eq. (2.5)) coincide for pure states. Therefore, the result for pure states given by Eq. (2.95) can be applied. It follows that

$$\frac{|E_{M}(|\psi\rangle\langle\psi|^{\otimes n}) - E_{M}(\sigma_{n})|}{n} = \frac{|S(\operatorname{tr}_{A}[|\psi\rangle\langle\psi|^{\otimes n}]) - S(\operatorname{tr}_{A}[|\phi_{n}\rangle\langle\phi_{n}|])|}{n} \\ \leq x \log_{2}(nN^{2}) - x \log_{2}(x)$$
 (2.101)

for all $n \ge n_0$, where $n_0 \in \mathbb{N}$ is sufficiently large, as $\dim[\mathcal{H}^{\otimes n}] = nN^2$. In this step, both Eq. (2.99) and Fannes' inequality [115, 104] have been used.

The technical part of the proof is to show that the second term on the right hand side of Eq. (2.100) tends to zero in the limit $n \to \infty$ as well. It will be shown in Lemma 2.16 that

$$\lim_{n \to \infty} \frac{|E_M(|\phi_n\rangle\langle\phi_n|) - E_M(\sigma_n)|}{n} = 0.$$
 (2.102)

That is, E_M is weakly continuous.

Again, this observation is consistent with the uniqueness theorem for entanglement measures for pure states. E_M satisfies all the criteria of the theorem, and hence,

$$E_M(|\psi\rangle\langle\psi|) = D_{\leftrightarrow}(|\psi\rangle\langle\psi|) \tag{2.103}$$

for all $|\psi\rangle\in\mathcal{H}$. The same argument can be applied to B_M . The subsequent lemma provides part of the proof of Proposition 2.15. The first part of the proof of Lemma 2.16 is closely related to a proof of the strong continuity of the relative entropy of entanglement given in Ref. [116]. However, an important assumption of the proof in Ref. [116] is not available in this case. The rather elaborate detour in the proof is necessary due to the additional constraint in E_M compared to the relative entropy of entanglement.

Lemma 2.16. – Let $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$, and let $|\psi\rangle \in \mathcal{H}$. Let $(\sigma_n)_{n \in \mathbb{N}}$ be a series of states $\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$ with the property that $\lim_{n \to \infty} ||\psi\rangle\langle\psi|^{\otimes n} - \sigma_n|| = 0$. Let $(|\phi_n\rangle\langle\phi_n|)_{n \in \mathbb{N}}$, $|\phi_n\rangle \in \mathcal{H}^{\otimes n}$ for n = 1, 2, ..., be a series of pure states satisfying

$$tr_A[|\phi_n\rangle\langle\phi_n] = tr_A[\sigma_n], \ tr_B[|\phi_n\rangle\langle\phi_n] = tr_B[\sigma_n],$$
 (2.104)

and

$$\lim_{n \to \infty} \left\| |\phi_n\rangle \langle \phi_n| - |\psi\rangle \langle \psi|^{\otimes n} \right\| = 0, \quad \lim_{n \to \infty} \left\| |\phi_n\rangle \langle \phi_n| - \sigma_n \right\| = 0. \tag{2.105}$$

Then the modified relative entropy of entanglement E_M satisfies

$$\lim_{n \to \infty} \frac{|E_M(|\phi_n\rangle\langle\phi_n|) - E_M(\sigma_n)|}{n} = 0.$$
 (2.106)

Proof: For brevity, call $\eta_n = |\phi_n\rangle\langle\phi_n|$. Let $\eta_n^* \in \mathcal{D}_{\eta_n}(\mathcal{H})$ and $\sigma_n^* \in \mathcal{D}_{\sigma_n}(\mathcal{H})$ be states that satisfy

$$E_M(\eta_n) = S(\eta_n | | \eta_n^*), \quad E_M(\sigma_n) = S(\sigma_n | | \sigma_n^*),$$
 (2.107)

respectively. Application of the triangle inequality yields

$$|E(\eta_n) - E(\sigma_n)| \le |S(\eta_n) - S(\sigma_n)| + |\text{tr}[\eta_n \log_2(\eta_n^*)] - \text{tr}[\sigma_n \log_2(\sigma_n^*)]|.$$
 (2.108)

The first step is to use Eq. (2.95) to get

$$|S(\eta_n) - S(\sigma_n)| \le ||\eta_n - \sigma_n|| \log_2(nN^2) - ||\eta_n - \sigma_n|| \log_2(||\eta_n - \sigma_n||)$$
 (2.109)

for $n \ge n_0$, $n_0 \in \mathbb{N}$ sufficiently large, as $\dim[\mathcal{H}^{\otimes n}] = nN^2$. It follows that

$$\lim_{n \to \infty} |S(\eta_n) - S(\sigma_n)|/n = 0.$$
 (2.110)

The remaining task is to find an appropriate upper bound for the right hand side in Eq. (2.108). In order to construct this upper bound an auxiliary quantity is helpful. Let

$$M_{\lambda}^{n}(\sigma) = -\min_{\omega \in \mathcal{D}_{\eta_{n}}(\mathcal{H})} \operatorname{tr}[\sigma \log_{2}(\lambda \omega + (1 - \lambda)\tau_{n})], \tag{2.111}$$

which depends on n and on an additional parameter $\lambda \in (0,1)$ that will later be fixed. τ_n stands in this definition for

$$\tau_n = \operatorname{tr}_B[\eta_n] \otimes \operatorname{tr}_A[\eta_n]. \tag{2.112}$$

That is, τ_n is the mixed product state that is locally identical to η_n for each n. The states taken from $\mathcal{D}_{\eta_n}(\mathcal{H})$ which are optimal in M_{λ}^n for η_n and σ_n will be called η'_n and σ'_n , respectively. Then the right hand side of Eq. (2.108) becomes

$$|-\operatorname{tr}[\eta_n \log_2(\eta_n^*)] + \operatorname{tr}[\sigma_n \log_2(\sigma_n^*)]|$$

$$\leq |-\operatorname{tr}[\eta_n \log_2(\eta_n^*)] - M_{\lambda}(\eta_n)| + |M_{\lambda}(\eta_n) - M_{\lambda}(\sigma_n)| + |M_{\lambda}(\sigma_n) + \operatorname{tr}[\sigma_n \log_2(\sigma_n^*)]|.$$
(2.113)

Now set $\lambda = 1 - \|\eta_n - \sigma_n\|$. Then

$$|-\operatorname{tr}[\eta_n \log_2(\eta_n^*)] - M_{\lambda}(\eta_n)| = |-\operatorname{tr}[\eta_n (\log_2(\eta_n^*) + \log_2(\lambda \eta_n' + (1-\lambda)\tau_n))]|. \quad (2.114)$$

The logarithm is an operator monotone function, that is, if A and B are Hermitian matrices, then $A \leq B$ implies that $\log_2(A) \leq \log_2(B)$.

$$-\operatorname{tr}[\eta_n \log_2(\lambda \eta_n' + (1 - \lambda)\tau_n)] \le -\operatorname{tr}[\eta_n \log_2(\lambda \eta_n^* + (1 - \lambda)\tau_n)] \le \operatorname{tr}[\eta_n \log_2(\eta_n^*)] - \log_2(\lambda). \tag{2.115}$$

The first inequality in Eq. (2.115) holds since η_n^* is included in $\mathcal{D}_{\eta_n}(\mathcal{H})$, but it is not necessarily optimal in M_{λ}^n for η_n . Therefore,

$$|-\operatorname{tr}[\eta_n \log_2(\eta_n^*)] - M_{\lambda}(\eta_n)| \le |\log_2(\lambda)| \le 2\|\eta_n - \sigma_n\|.$$
 (2.116)

The same argument applies to $|-\text{tr}[\sigma_n \log_2(\sigma_n^*)] - M_{\lambda}(\sigma_n)|$, and hence, $|-\text{tr}[\sigma_n \log_2(\sigma_n^*)] - M_{\lambda}(\sigma_n)| \le 2\|\eta_n - \sigma_n\|$. This means that

$$\lim_{n\to\infty} \frac{|-\operatorname{tr}[\eta_n \log_2(\eta_n^*)] - M_{\lambda}(\eta_n)|}{n} = 0, \quad \lim_{n\to\infty} \frac{|-\operatorname{tr}[\sigma_n \log_2(\sigma_n^*)] - M_{\lambda}(\sigma_n)|}{n} = 0.$$
(2.117)

The last step is to find an upper bound for $|M_{\lambda}(\eta_n) - M_{\lambda}(\sigma_n)|$. In this last step it will be used that $\lim_{n\to\infty} \|\eta_n - |\psi\rangle\langle\psi|^{\otimes n}\| = 0$, and employing the fact that the trace norm can only decrease under the application of the partial trace operation [117] it follows that also

$$\lim_{n \to \infty} \left\| |\operatorname{tr}_{A}[\eta_{n}] - \operatorname{tr}_{A}[|\psi\rangle\langle\psi|^{\otimes n}] \right\| = 0 = \lim_{n \to \infty} \left\| |\operatorname{tr}_{A}[\tau_{n}] - \operatorname{tr}_{A}[|\psi\rangle\langle\psi|^{\otimes n}] \right\|. \tag{2.118}$$

In particular, this means that there exists a constant $1 \ge C > 0$ independent of n and an $n_1 \in \mathbb{N}$ such that

$$\tau_n^* \ge C^n \mathbb{1} \tag{2.119}$$

for all $n \geq n_1$, where τ_n^* is the state that is given by τ_n restricted to range $[\sigma_n] \cup \text{range}[\eta_n]$. Using again the operator monotonicity of \log_2 one can conclude that

$$(\log_{2}(1-\lambda) + n\log_{2}(C)) \mathbb{1} \leq \log_{2}(1-\lambda)\mathbb{1} + \log_{2}(\tau_{n}^{*}) = \log_{2}((1-\lambda)\tau_{n}^{*})$$

$$\leq \log_{2}(\lambda\sigma_{n}' + (1-\lambda)\tau_{n}^{*}) \leq 0.$$
 (2.120)

Equipped with Eq. (2.120) one can find an appropriate upper bound for $|M_{\lambda}(\eta_n) - M_{\lambda}(\sigma_n)|$. It is given by

$$|M_{\lambda}(\eta_{n}) - M_{\lambda}(\sigma_{n})| \leq |-\operatorname{tr}[\eta_{n} \log_{2}(\lambda \sigma'_{n} + (1 - \lambda)\tau_{n})] + \operatorname{tr}[\sigma_{n} \log_{2}(\lambda \sigma'_{n} + (1 - \lambda)\tau_{n})]|$$

$$= |\operatorname{tr}[(\sigma_{n} - \eta_{n})(\log_{2}(\lambda \sigma'_{n} + (1 - \lambda)\tau_{n}^{*}))]|$$

$$\leq ||\eta_{n} - \sigma_{n}|| (-\log_{2}(1 - \lambda) - n\log_{2}(C))$$
(2.121)

for $n \geq n_1$. Thus

$$\lim_{n \to \infty} \frac{|M_{\lambda}(\eta_n) - M_{\lambda}(\sigma_n)|}{n} \leq \lim_{n \to \infty} \left(-\frac{1}{n} \|\eta_n - \sigma_n\| \log_2 (\|\eta_n - \sigma_n\|) - \log_2(C) \|\eta_n - \sigma_n\| \right)$$

$$= 0. \tag{2.122}$$

Combining Eqs. (2.110), (2.117), and (2.122) it follows that

$$\lim_{n \to \infty} \frac{|E_M(|\phi_n\rangle\langle\phi_n|) - E_M(\sigma_n)|}{n} = 0.$$
 (2.123)

This is the statement of the lemma.

2.3 Quantification of Multi-Partite Quantum Entanglement

In this subsection multi-partite entanglement will be investigated. As has been pointed out in the introduction of this chapter, new complex structures emerge in the case when more than two parties are present [118, 119, 120]. In a bi-partite setting all pure-state entanglement is essentially equivalent to the entanglement of a Bell state of two qubits: A large number n of copies of a particular pure state with a state vector $|\psi\rangle$ can be transformed reversibly into a smaller number m of Bell states with local operations and classical communication. In the asymptotic limit the ratio of the numbers m/n is given by $S(\operatorname{tr}_A[|\psi\rangle\langle\psi|]) = D_{\leftrightarrow}(|\psi\rangle\langle\psi|) = E_F(|\psi\rangle\langle\psi|)$ [32, 119]. Such a reasoning is not possible in the multi-partite domain. There is no single "unit" of entanglement like the entanglement of the singlet in the bi-partite case [38, 121, 122, 123]. Accordingly, if more than two parties are present, a single number indicating the amount of entanglement of a pure state is not sufficient: it has been shown that several inequivalent "kinds of entanglement" have to be distinguished [124, 125, 126].

The system which will be considered in this section is a general N-partite quantum system with parties $A_1, ..., A_N$ holding quantum systems with dimension $d_1, ..., d_N$, that is, the state space of the composite system is given by $\mathcal{S}(\mathcal{H})$, where

$$\mathcal{H} = \mathcal{H}_{A_1} \otimes ... \otimes \mathcal{H}_{A_N}, \qquad \mathcal{H}_{A_i} = \mathbb{C}^{d_i}, \qquad i = 1, ..., N.$$
 (2.124)

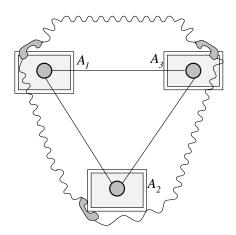


Figure 2.3: LOCC operations of three parties A_1 , A_2 , and A_3 .

If N=2, that is, for bi-partite systems, the entanglement of pure states can be described by considering the eigenvalues of the two reduced states. Many of the particular results about entanglement manipulation in the bi-partite case can in some way be traced back to the high symmetry of the Schmidt decomposition. In the case of $N\geq 2$ a general Schmidt decomposition is not available, and the pure states which are Schmidt decomposable form a small subset of all pure states. Recently, considerable effort has been devoted to a better understanding of equivalence classes of pure states, where two states are called equivalent if they can be transformed into each other by local unitary operations [37, 127, 128, 129, 130]. Canonical forms of representants of these *equivalence classes* which may be conceived as generalizations of Schmidt decompositions to multi-partite settings can be found in Refs. [127] and [129].

The structure of multi-particle entanglement cannot most appropriately be described by studying only the full composite system. Instead, several properties are revealed only when one conceives the full system as a composite system of several parts, taking into account that each part may well include several quantum systems held by different parties. These parts are taken to be systems on their own when examining entanglement properties. The following investigations involve arbitrary partitions of the N-partite system with parties $A_1, ..., A_N$ into k parts, k = 2, ..., N. In accordance with Ref. [126] a division of the original system into two parts will be called a 2-split, and a division into k parts a k-split. For a three-party system with parts A_1, A_2 , and A_3 the 3-split $A_1A_2A_3$ and the three 2-splits $(A_1A_2)A_3, (A_2A_3)A_1$, and $(A_3A_1)A_2$ are possible.

These splits reveal the *separability structure* of a state. Let $\rho \in \mathcal{S}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_3})$ be an arbitrary state of such a tri-partite system. Following Ref. [125, 126], several classes of separable states can be distinguished for a tri-partite system.

$$\rho = \sum_{i} p_{i} |\psi^{(i)}\rangle\langle\psi^{(i)}|_{A_{1}} \otimes |\phi^{(i)}\rangle\langle\phi^{(i)}|_{A_{2}} \otimes |\varphi^{(i)}\rangle\langle\varphi^{(i)}|_{A_{3}}$$
 (2.125)

$$\rho = \sum_{i} p_{i} |\psi^{(i)}\rangle\langle\psi^{(i)}|_{A_{1}} \otimes |\phi^{(i)}\rangle\langle\phi^{(i)}|_{A_{2}A_{3}}$$
(2.126)

$$\rho = \sum_{i} p_{i} |\psi^{(i)}\rangle\langle\psi^{(i)}|_{A_{2}} \otimes |\phi^{(i)}\rangle\langle\phi^{(i)}|_{A_{1}A_{3}}$$

$$(2.127)$$

$$\rho = \sum_{i} p_i |\psi^{(i)}\rangle \langle \psi^{(i)}|_{A_3} \otimes |\phi^{(i)}\rangle \langle \phi^{(i)}|_{A_1 A_2}, \tag{2.128}$$

where $p_1, p_2, ...$ is a probability distribution. If the state ρ can be put into the form of Eq. (2.125) then it is called *fully separable*. For party A_1 this state is *one-system bi-separable* if it is of the form of Eq. (2.126), but not of the form of Eq. (2.127) and Eq. (2.128). It is a *two-system bi-separable state* if both Eq. (2.126) and Eq. (2.127) may be satisfied, but not Eq. (2.128). *Three-system bi-separable states* are states of the form of Eq. (2.126), Eq. (2.127), and Eq. (2.128), but which fail to fulfil Eq. (2.125). Finally, *fully inseparable states* can be formulated in any of these forms. Examples for each class can be found in Ref. [126]. Such a terminology leads to a classification of states with respect to separability properties.

2.3.1 The Schmidt Measure

In Ref. [131] a certain class of multi-qubit states has been introduced, the so-called N-party cluster states $|\phi_N\rangle\langle\phi_N|$. These states are of practical relevance in the context of quantum computing: It has turned out that such cluster states of a quantum Ising model may provide the "carrier" of a new type of quantum computing, as has been shown in Ref. [132]. A quantum Ising system in a cluster state is a resource with which quantum computations can be performed by 1-qubit measurements only. Any computation can be realized by a proper sequence of 1-qubit measurements on the cluster. It is generally hoped that this

approach might amount to a new way of lessening the difficulties in the realization of a large-scale universal quantum computer. It has been demonstrated that the minimal number of product terms for such a cluster state $|\phi_N\rangle\langle\phi_N|$ is given by $2^{\lfloor N/2\rfloor}$, if one expands $|\phi_N\rangle$ in product state vectors of N qubits. The observation concerning the minimal number of product states is related to the findings of Ref. [124], in which such numbers have first been considered: it has been shown that there are two classes of tripartite entangled pure states of three qubits which cannot be transformed into each other with nonvanishing probability, the so-called W-state [124] and the GHZ-state [133] being representatives. One has three and the other one has two product states in the minimal decomposition in terms of product states. This statement is also made stronger in that it is pointed out that this minimal number of product terms can never be increased by means of invertible local operations. Building upon these observations one can define an entanglement monotone on the entire state space, containing the mixed states, of an arbitrary multi-partite system:

Any $|\psi\rangle \in \mathcal{H} = \mathbb{C}^{d_1} \otimes ... \otimes \mathbb{C}^{d_N}$ can be written in the form

$$|\psi\rangle = \sum_{i=1}^{R} \alpha_i |\psi^{(i)}\rangle_{A_1} \otimes \dots \otimes |\phi^{(i)}\rangle_{A_N}, \tag{2.129}$$

where $\alpha_i \in \mathbb{C}$, i = 1, ..., R, with some $R \in \mathbb{N}$. Let r be the minimal number of product terms R in such a decomposition of $|\psi\rangle$. The *Schmidt measure* is then defined as

$$E_S(|\psi\rangle\langle\psi|) = \log_2(r). \tag{2.130}$$

In the case of a bi-partite system with parties A_1 and A_2 the minimal number of product terms r is given by the Schmidt rank of the state, which is in turn identical to the rank of the local state of the respective party.

The definition of E_S can be extended to the full state space in a natural way. This is done by using a convex roof construction [85, 94] as in the entanglement of formation. For a $\sigma \in \mathcal{S}(\mathcal{H})$ let

$$E_S(\sigma) = \min \sum_{i} \lambda_i E_S(|\psi_i\rangle\langle\psi_i|), \qquad (2.131)$$

where the minimum is taken over all possible convex combinations of the form

$$\sigma = \sum_{i} \lambda_i |\psi_i\rangle\langle\psi_i| \tag{2.132}$$

in terms of pure states $|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, \dots$, with $0 \le \lambda_i \le 1$ for all i.

The Schmidt measure serves as an entanglement monotone in the sense of Ref. [38] (see also Ref. [86]) and is hence a proper measure of mixed state entanglement. In particular, it can neither increase on average under LOCC operations nor under mixing. It vanishes for fully separable states, that is, for states $\sigma \in \mathcal{S}(\mathcal{H})$ that can be cast into the form

$$\sigma = \sum_{i=1}^{n} p_i |\psi^{(i)}\rangle \langle \psi^{(i)}|_{A_1} \otimes \dots \otimes |\phi^{(i)}\rangle \langle \phi^{(i)}|_{A_N}$$
(2.133)

where $p_1, ..., p_n$ is a probability distribution. In this multi-partite setting the Schmidt measure is said to be an entanglement monotone because the following three conditions are satisfied

- (i) $E_S \ge 0$, and $E_S(\sigma) = 0$ if σ is fully separable.
- (ii) E_S is a convex functional.

(iii) E_S is monotone under local generalized measurements: Let σ be the initial state, and let *one* of the parties $A_1, ..., A_N$ perform a (partly selective) local generalized measurement leading to the final states $\sigma_1, ..., \sigma_K$ with respective probabilities $p_1, ..., p_K$. Then

$$E_S(\sigma) \ge \sum_{i=1}^K p_i E_S(\sigma_i). \tag{2.134}$$

Proposition 2.17. – The Schmidt measure E_S is an entanglement monotone.

Proof: Condition (i) follows immediately from the definition. Due to the convex roof construction E_S is also a convex functional (condition (ii)): let σ_1 and σ_2 be states from $\mathcal{S}(\mathcal{H})$, and let $\sigma_1 = \sum_j \mu_j |\phi_j\rangle \langle \phi_j|$ and $\sigma_2 = \sum_j \eta_k |\varphi_k\rangle \langle \varphi_k|$ be the two decompositions for which the respective minima in Eq. (2.131) are attained. Then

$$\sum_{j} \lambda \mu_{j} |\phi_{j}\rangle \langle \phi_{j}| + \sum_{k} (1 - \lambda) \eta_{k} |\varphi_{k}\rangle \langle \varphi_{k}|$$
 (2.135)

is a valid decomposition of $\sigma = \lambda \sigma_1 + (1 - \lambda)\sigma_2$, but it is not necessarily the optimal one. Hence, $E_S(\lambda \sigma_1 + (1 - \lambda)\sigma_2) \leq \lambda E_S(\sigma_1) + (1 - \lambda)E_S(\sigma_2)$.

The local measurement of condition (iii) can be assumed to be performed by party A_1 . Again, it suffices to consider the posterior state in a local generalized measurement that does not involve mixing. That is, it is sufficient to consider final states of the form

$$\sigma_i = \frac{E_i \sigma E_i^{\dagger}}{p_i},\tag{2.136}$$

where $p_i = \operatorname{tr}[E_i \sigma E_i^{\dagger}]$, $\sum_{i=1}^K E_i^{\dagger} E_i = \mathbb{1}$. The Kraus operators $E_1,...,E_K$ act in $\mathcal{H}_{A_2} \otimes ... \otimes \mathcal{H}_{A_N}$ as the identity. For any pure state $|\psi\rangle\langle\psi|\in\mathcal{S}(\mathcal{H})$

$$E_S\left(\frac{E_i|\psi\rangle\langle\psi|E_i^{\dagger}}{\operatorname{tr}[E_i|\psi\rangle\langle\psi|E_i^{\dagger}]}\right) \le E_S(|\psi\rangle\langle\psi|) \tag{2.137}$$

for all i=1,...,K. This can be seen as follows. Let $|\psi\rangle=\sum_{i=1}^r\alpha_i|\psi^{(i)}\rangle_{A_1}\otimes...\otimes|\psi^{(i)}\rangle_{A_N}$ be the decomposition of $|\psi\rangle$ into products as in Eq. (2.129) with the minimal number of terms r. Then E_i is either invertible, and then $E_i|\psi\rangle$ has the same minimal number of product terms r'=r, or it is not invertible, such that $r'\leq r$. Moreover, if Eq. (2.137) holds for pure states $|\psi\rangle\langle\psi|$, it is also valid for arbitrary states $\sigma\in\mathcal{S}(\mathcal{H})$:

Let $\sigma = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$ be the optimal decomposition of σ belonging to the minimum in Eq. (2.131), then

$$E_{S}(\sigma) = \sum_{k} \lambda_{k} E_{S}(|\psi_{k}\rangle\langle\psi_{k}|)$$

$$\geq \sum_{k} \lambda_{k} E_{S}(E_{i}|\psi_{k}\rangle\langle\psi_{k}|E_{i}^{\dagger}/\text{tr}[E_{i}|\psi_{k}\rangle\langle\psi_{k}|E_{i}^{\dagger}])$$

$$\geq E_{S}(E_{i}\sigma E_{i}^{\dagger}/\text{tr}[E_{i}\sigma E_{i}]) \qquad (2.138)$$

for all i = 1, ..., K. The statement of condition (iii) then follows from the fact that

$$\sum_{i=1}^{K} p_i E_S(E_i \sigma E_i^{\dagger} / \text{tr}[E_i \sigma E_i]) \le \sum_{i=1}^{K} p_i E_S(\sigma) = E_S(\sigma). \tag{2.139}$$

The Schmidt measure cannot be increased on average under LOCC. It can be used as a functional appropriately quantifying the entanglement of a given state of a N-partite quantum system. A number of noteworthy properties are listed below. The normalization and the additivity on pure states follow immediately from the definition of the Schmidt measure for pure states. For mixed states E_S is subadditive.

Lemma 2.18. – The Schmidt measure is normalized,

$$E_S(|\psi\rangle\langle\psi|) = 1 \tag{2.140}$$

for all N-party GHZ-states, i.e., states with state vectors of the form

$$|\psi\rangle = (|0^{\otimes N}\rangle + |1^{\otimes N}\rangle)/\sqrt{2}.\tag{2.141}$$

Lemma 2.19. – E_S is fully additive on pure states: If the parties $A_1,...,A_N$ share n N-partite quantum systems in the state $|\psi_1\rangle\langle\psi_1|$ and m N-partite systems in the state $|\psi_2\rangle\langle\psi_2|$, then it follows that

$$E_S(|\psi_1\rangle\langle\psi_1|^{\otimes n}\otimes|\psi_2\rangle\langle\psi_2|^{\otimes m}) = nE_S(|\psi_1\rangle\langle\psi_1|) + mE_S(|\psi_2\rangle\langle\psi_2|). \tag{2.142}$$

Lemma 2.20. – Let $\sigma \in \mathcal{S}(\mathcal{H})$ be a fully separable state, then $E_S(\sigma^{\otimes n} \otimes |\psi\rangle\langle\psi|^{\otimes m}) = mE_S(|\psi\rangle\langle\psi|)$ holds for all pure states $|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H})$ and all m = 1, 2, ...

Lemma 2.21. – E_S is subadditive, $E_S(\sigma \otimes \rho) \leq E_S(\sigma) + E_S(\rho)$ for all $\sigma, \rho \in \mathcal{S}(\mathcal{H})$, where σ and ρ are states of quantum systems held by the same parties.

Proof: Let $\sigma = \sum_j \mu_j |\phi_j\rangle \langle \phi_j|$ and $\rho = \sum_k \eta_k |\varphi_k\rangle \langle \varphi_k|$ be the optimal decompositions of σ and ρ respectively for which the minima in Eq. (2.131) are attained. Then

$$\sum_{j,k} \eta_k \mu_j |\varphi_k\rangle \langle \varphi_k| \otimes |\phi_j\rangle \langle \phi_j| \tag{2.143}$$

is a decomposition of $\sigma \otimes \rho$, but not necessarily the one for which Eq. (2.131) becomes minimal.

It should be noted that out of the conditions for an entanglement measure in the strict sense of Ref. [86] E_S does not satisfy a continuity criterion. In particular, it is not weakly continuous in the sense of Eq. (2.4). This is why the uniqueness theorem for entanglement measures of pure states (see Ref. [86]) cannot be applied. Hence, E_S does not have to coincide with the von-Neumann entropy of one subsystem $S(\operatorname{tr}_A[|\psi\rangle\langle\psi|])$ for a bi-partite system in a pure state $|\psi\rangle\langle\psi|$, and indeed, it does not coincide in general.

2.3.2 Classification of Multi-Particle Entanglement

Although the Schmidt measure of a mixed state is defined via a minimization over all possible realizations of the state, it can be calculated exactly for a rather large class of states. This is mainly due to the fact that it is a coarse grained measure. All terms that appear in Eq. (2.131) are logarithms of natural numbers weighted with respective probabilities. This

quality of the Schmidt measure is both the strength and the weakness of this quantity. It allows, however, for a detailed classification of multi-particle entangled states. The subsequent investigations will be restricted to the multi-qubit case, where $\mathcal{H}=(\mathbb{C}^2)^{\otimes N}$. In this classification the Schmidt measure will also be furnished with an index indicating the respective split. For example, in a system consisting of parts A_1 , A_2 , and A_3 the Schmidt measure associated with the 3-split $A_1A_2A_3$ is written as $E_S^{A_1A_2A_3}$, and the one belonging to $(A_1A_2)A_3$ is denoted by $E_S^{(A_1A_2)A_3}$ (see Fig. 2.4).

For each *k*-split the Schmidt measure cannot increase on average in the course of a LOCC operation, as the Schmidt measure is an entanglement monotone with respect to every possible split of the system. This means in turn that once one knows that the Schmidt measure would become larger on average, one can be sure that the corresponding transformation cannot be implemented under LOCC.

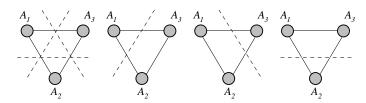


Figure 2.4: The 3-split $A_1A_2A_3$ and the splits $(A_2A_3)A_1$, $(A_1A_2)A_3$, and $(A_3A_1)A_2$.

Example 2.22. – Consider a three party W-state with

$$|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$$
 (2.144)

that has been used in Ref. [124]. $E_S^{A_1A_2A_3}(|\mathbf{W}\rangle\langle\mathbf{W}|) = \log_2(3)$, while the three party GHZ-state

$$|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$$
 (2.145)

obtains the value $E_S^{A_1A_2A_3}(|\text{GHZ}\rangle\langle\text{GHZ}|)=1$. Hence, in the 3-split E_S discriminates between the GHZ-state, the W-state and product states (value 0). The 3-tangle, proposed in Ref. [121], is also an entanglement monotone (see Ref. [124]), and it can distinguish the W from the GHZ state. However, it is defined only for three qubits, and it gives the same value for the W-state and for product states.

Other splits with k = 2 reveal further information and give rise to the full classification.

Example 2.23. – A two-party Bell state with state vector $|\phi^+\rangle_{A_1A_2}=(|00\rangle_{A_1A_2}+|11\rangle_{A_1A_2})/\sqrt{2}$ held by A_1 and A_2 with party A_3 in the state $|0\rangle\langle 0|_{A_3}$ cannot be transformed into $|\text{GHZ}\rangle\langle \text{GHZ}|$ of A_1 , A_2 , and A_3 under local operations and classical communication. This has not yet become obvious from the values of the Schmidt measure corresponding to the split $A_1A_2A_3$, as both states yield

$$E_S^{A_1 A_2 A_3}(|\phi^+\rangle \langle \phi^+|_{A_1 A_2} \otimes |0\rangle \langle 0|_{A_3}) = E_S^{A_1 A_2 A_3}(|\text{GHZ}\rangle \langle \text{GHZ}|) = 1.$$
 (2.146)

However, $E_S^{(A_1A_2)A_3}(|\phi^+\rangle\langle\phi^+|_{A_1A_2}\otimes|0\rangle\langle0|_{A_3})=0$, and

$$E_S^{(A_1 A_2)A_3}(|GHZ\rangle\langle GHZ|) = 1.$$
 (2.147)

Table 2.1: Values of the Schmidt measure E_S for some four qubit pure states (the four party GHZ-state $(|0000\rangle + |1111\rangle)/\sqrt{2}$, the generalized W-state [9] $(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)/2$, the cluster state [131, 132] $|\phi_4\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$, and the state $|\phi^+\rangle|\phi^+\rangle = (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)/2$).

	GHZ	W	$ \phi_4\rangle$	$ \phi^+\rangle \phi^+\rangle$
$\begin{array}{c} A_1 A_2 A_3 A_4 \\ A_1 A_2 (A_3 A_4) \\ (A_1 A_2) A_3 A_4 \\ (A_1 A_2) (A_3 A_4) \\ (A_1 A_3) (A_2 A_4) \\ (A_1 A_4) (A_2 A_3) \end{array}$	1 1 1 1 1 1	$ \begin{array}{c} 2 \\ \log_2 3 \\ \log_2 3 \\ 1 \\ 1 \end{array} $	$ \begin{array}{c c} & 1 & \\ & 1 & \\ & 1 & \\ & 2 & \\ & 2 & \\ \end{array} $	2 1 1 0 2
$(A_1A_2A_3)A_4$	1	1	1	1

And it is for this reason that the split $(A_1A_2)A_3$ indicates that

$$|\phi^{+}\rangle\langle\phi^{+}|_{A_{1}A_{2}}\otimes|0\rangle\langle0|_{A_{3}}\longrightarrow|\text{GHZ}\rangle\langle\text{GHZ}|$$
 (2.148)

under LOCC.

Further examples can be found in Table 2.1. It shows the values of the Schmidt measure with respect to all possible splits for some pure states of a four-partite system. In order to calculate the Schmidt measure of a mixed state a minimization over all possible decompositions of the state is required. The convex roof construction provides upper bounds of E_S : If $\sigma = \sum \eta_i |\psi_i\rangle \langle \psi_i|$ is any not necessarily optimal decomposition of a state $\sigma \in \mathcal{S}(\mathcal{H})$, then $\sum_i \eta_i E_S(|\psi_i\rangle \langle \psi_i|)$ is an upper bound of $E_S(\sigma)$. For many states E_S can however be fully evaluated.

Example 2.24. – Consider two parties A_1 and A_2 sharing two qubits in the Werner state [46]

$$\rho_W(\lambda) = \lambda |\psi^-\rangle \langle \psi^-| + (1 - \lambda) \mathbb{1}/4, \tag{2.149}$$

with $|\psi^-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}$, $0\leq\lambda\leq1$. As all pure states in the range of $\rho_W(\lambda)$ have Schmidt measure 0 or 1, one has to identify in any decomposition $\rho_W(\lambda)=\sum_i\eta_i|\psi_i\rangle\langle\psi_i|$ the terms with Schmidt measure 0 (product states) or 1 (entangled states). Hence, the Schmidt measure is given by $E_S(\rho_W(\lambda))=1-s$, where s is the weight of the separable state that can maximally be subtracted from $\rho_W(\lambda)$ while maintaining the semi-positivity of the state. As shown in [74] it follows that

$$E_S(|\psi\rangle\langle\psi|) = \begin{cases} \frac{3}{2}\lambda - \frac{1}{2}, & \text{for } 1/3 < \lambda \le 1, \\ 0, & \text{for } 0 \le \lambda \le 1/3. \end{cases}$$
 (2.150)

In other words, E_S is given by the weight of the inseparable state in the best separable approximation in the sense of Ref. [74]. The Schmidt rank of all states in the range of any state from $\mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is smaller or equal to 2, and hence, this statement holds for all mixed states of systems consisting of two qubits.

Corollary 2.25. – Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, and let $\lambda(\sigma)$ be the weight of the separable contribution of the best separable approximation σ_s in $\sigma = \lambda(\sigma)\sigma_s + (1 - \lambda(\sigma))\delta\sigma$ of a state $\sigma \in \mathcal{S}(\mathcal{H})$. Then λ is an

	$\rho_{\rm G}(\lambda)$	ρ_M	$\rho(\lambda,\mu)$
$ \begin{array}{c} A_1 A_2 A_3 \\ (A_1 A_2) A_3 \\ (A_1 A_3) A_2 \\ (A_2 A_3) A_1 \end{array} $	λ λ λ	1 2/3 2/3 2/3	$1 \\ 1 - \lambda \\ \lambda + \mu \\ 1 - \mu$

Table 2.2: The Schmidt measure E_S for some mixed quantum states $(\rho_G(\lambda), \rho_M, \text{ and } \rho(\lambda, \mu))$.

entanglement monotone.

For more than two parties different entanglement classes can be distinguished. The Schmidt measure is again defined for all possible k-splits, k=2,...,N. If the N-partite system is separable with respect to a particular k-split, the value of the corresponding Schmidt measure is 0: $E_S(\sigma)=0$ if and only if σ is fully separable. Hence, the Schmidt measure with respect to a certain split gives an account of the separability of the state.

For three systems, e.g., the classes of one-system bi-separable states, two-system bi-separable states, three-system bi-separable states, and fully separable states (see above) can be distinguished. For a state σ which is taken to be a one-qubit bi-separable state with respect to party A_1 , E_S takes the values

$$E_S^{A_1(A_2A_3)}(\sigma)=0, \text{ but } E_S^{A_3(A_1A_2)}(\sigma)>0 \text{ and } E_S^{A_2(A_3A_1)}(\sigma)>0. \tag{2.151}$$

The Schmidt measure, however, can reveal more structure, since the entanglement is also quantified.

Example 2.26. – By way of an example, let

$$\rho(\lambda,\mu) = \lambda |\phi^{+}\rangle \langle \phi^{+}|_{A_{1}A_{2}} \otimes |0\rangle \langle 0|_{A_{3}} + \mu |\phi^{+}\rangle \langle \phi^{+}|_{A_{2}A_{3}} \otimes |0\rangle \langle 0|_{A_{1}}$$

$$+ (1 - \lambda - \mu)|\phi^{+}\rangle \langle \phi^{+}|_{A_{3}A_{1}} \otimes |0\rangle \langle 0|_{A_{2}},$$
(2.152)

 $0 \le \lambda, \mu \le 1$. For $\lambda = \mu = 1/3$ this state reduces to the three-party molecule state

$$\rho_{M} = \frac{1}{3} (|\phi^{+}\rangle\langle\phi^{+}|_{A_{1}A_{2}} \otimes |0\rangle\langle0|_{A_{3}} + |\phi^{+}\rangle\langle\phi^{+}|_{A_{2}A_{3}} \otimes |0\rangle\langle0|_{A_{1}}
+ |\phi^{+}\rangle\langle\phi^{+}|_{A_{3}A_{1}} \otimes |0\rangle\langle0|_{A_{2}})$$
(2.153)

studied in Ref. [134]. The Schmidt measure $E_S^{A_1A_2A_3}(\rho_M)=1$ is equal to the Schmidt measure of a state where A_1 and A_2 hold a $|\phi^+\rangle\langle\phi^+|$ state and A_3 is in the state $|0\rangle\langle0|$, as the mere classical ignorance of which parties are actually holding the Bell state cannot increase the amount of entanglement. In Table 2.2 the values of the Schmidt measures of all splits of the states $\rho(\lambda,\mu)$, ρ_M , and

$$\rho_{G}(\lambda) = \lambda |GHZ\rangle\langle GHZ| + (1 - \lambda)|000\rangle\langle 000|, \qquad (2.154)$$

 $0 \le \lambda \le 1$, are shown.

2.3.3 Remarks on the Asymptotic Limit

The Schmidt measure is the basis for a rather detailed classification of mixed state entanglement and it is hoped to be giving useful information about the possibility to transform one state into another using LOCC operations. However, its relation to a number of other concepts of multi-particle entanglement is not yet fully investigated. For cluster states [131] the Schmidt measure coincides with the *persistency of entanglement* [131], and it would be interesting to establish the exact connection between the definition of the Schmidt measure and a – possibly refined – definition of persistency.

Another potentially rewarding direction for further research is the notorious problem of transformations of multi-partite pure state entanglement in the asymptotic limit. Much is known about *stochastic transformations* of single copies of three-qubit systems [124]. However, the question is not resolved whether every pure state of three qubits can be prepared in an asymptotically reversible way starting from copies of states taken from a finite set of pure states. To shed light on this issue, entanglement monotones like the Schmidt measure would be desirable, which are – unlike $E_{\it S}$ – continuous functionals.

Before being able to state this problem in a more formal way [119, 135, 107], the concept of asymptotical reducibility [119] needs to be introduced. A pure state $|\psi\rangle\langle\psi|$ is said to be asymptotically reducible to $|\phi\rangle\langle\phi|$ if the following condition holds: For all $\delta>0$ and all $\varepsilon>0$ there exist $n,m\in\mathbb{N}$ and an LOCC operation $\mathcal E$ taking inputs from $\mathcal S(\mathcal H^{\otimes m})$ and mapping them on $\mathcal S(\mathcal H^{\otimes n})$ with the property

$$\left| \frac{n}{m} - 1 \right| < \delta \quad \text{and} \quad F(\mathcal{E}(|\psi\rangle\langle\psi|^{\otimes m}), |\phi\rangle\langle\phi|^{\otimes n}) \ge 1 - \varepsilon.$$
 (2.155)

The statement that $|\psi\rangle\langle\psi|$ is asymptotically reducible to $|\phi\rangle\langle\phi|$ is abbreviated as

$$|\psi\rangle\langle\psi|\longrightarrow|\phi\rangle\langle\phi|$$
 under ALOCC. (2.156)

The yield in asymptotic reducibilities can also be non-integer. One writes in short

$$|\psi\rangle\langle\psi|^{\otimes x} \longrightarrow |\phi\rangle\langle\phi|^{\otimes y}$$
 under ALOCC (2.157)

with $x,y\geq 0$ if for all $\delta>0$ and all $\varepsilon>0$ there exist $n,m\in\mathbb{N}$ and an LOCC operation $\mathcal E$ with the property

$$\left| \frac{n}{m} - \frac{x}{y} \right| < \delta \quad \text{and} \quad F(\mathcal{E}(|\psi\rangle\langle\psi|^{\otimes m}), |\phi\rangle\langle\phi|^{\otimes n}) \ge 1 - \varepsilon.$$
 (2.158)

With this concept at hand, a reversible entanglement generating set (REGS) can be defined precisely: Let \mathcal{H} be the Hilbert space of a multi-partite quantum system. A REGS is a set $\{|\psi_1\rangle,...,|\psi_n\rangle\}$, $|\psi_i\rangle\in\mathcal{H}$, i=1,...,n, with the property that there are coefficients $x_i\geq 0$ such that

$$\bigotimes_{i=1}^{n} |\psi_{i}\rangle\langle\psi_{i}|^{\otimes x_{i}} \longrightarrow |\psi\rangle\langle\psi| \text{ under ALOCC and}$$

$$|\psi\rangle\langle\psi| \longrightarrow \bigotimes_{i=1}^{n} |\psi_{i}\rangle\langle\psi_{i}|^{\otimes x_{i}} \text{ under ALOCC}.$$
(2.159)

An MREGS [119, 135, 107] is a minimal REGS in the sense that it is a reversible entanglement generating set with the minimal number of elements.

As is well known, the set $\{|\phi^+\rangle\}$ is an MREGS for all systems with $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2$, that is, for bi-partite qubit systems. Every state of a two-qubit system can be prepared in an asymptotically reversible way from Bell states. Unfortunately, it is not known yet what

the MREGS is for systems consisting of three qubits A_1 , A_2 , and A_3 ? There is a strong indication that $\{|\phi^+\rangle_{A_1A_2}, |\phi^+\rangle_{A_2A_3}, |\phi^+\rangle_{A_1A_3}, |\text{GHZ}\rangle\}$ is not sufficient (see Refs. [107] and [109]). In order to see whether particular states can be prepared in a reversible way from certain ingredients in the asymptotic limit, it would be very useful to dispose of continuous entanglement monotones. The Schmidt measure alone is not appropriate to tackle this problem.

It can however be used to address the question of the minimal amount of resources needed to prepare a particular multi-partite pure state. To be more specific, one may ask how many Bell states are needed on average asymptotically to generate a certain final three qubit state. This procedure is typically expected to be irreversible. Three qubit GHZ-states can by no means be prepared irreversibly from two qubit Bell states, not even in the asymptotic limit [119]. Nevertheless, if one allows the protocol to be irreversible all pure states of three qubits can be achieved by using only Bell states as a resource. The Schmidt measure provides a lower bound for this minimal average number of Bell states. The subsequent proposition is concerned with tri-partite systems, but an analogous statement also holds for arbitrary N-party systems.

Proposition 2.27. – Consider a tripartite system with parts A_1 , A_2 , and A_3 , with $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Let $|\phi\rangle \in \mathcal{H}$. There exist numbers $x_1, x_2, x_3 \geq 0$ such that

$$|\phi^{+}\rangle\langle\phi^{+}|_{A_{1}A_{2}}^{\otimes x_{1}}\otimes|\phi^{+}\rangle\langle\phi^{+}|_{A_{1}A_{3}}^{\otimes x_{2}}\otimes|\phi^{+}\rangle\langle\phi^{+}|_{A_{2}A_{3}}^{\otimes x_{3}}\longrightarrow|\phi\rangle\langle\phi|$$
 under ALOCC (2.160)

and such that the sum $E_X(|\phi\rangle\langle\phi|) = x_1 + x_2 + x_3$ attains its infimum. Then

$$E_X(|\phi\rangle\langle\phi|) \ge E_S^{A_1 A_2 A_3}(|\phi\rangle\langle\phi|). \tag{2.161}$$

Proof: This statement is a consequence of the fact that $E_S^{A_1A_2A_3}$ is an entanglement monotone and of its additivity property with respect to pure states. For all $\varepsilon>0$ and all $\delta>0$ there exist $n,n_1,n_2,n_3\in\mathbb{N}$ and an LOCC operation $\mathcal E$ such that $|n_i/n-x_i|<\delta$ for i=1,2,3 and

$$F(\mathcal{E}(|\phi^{+}\rangle\langle\phi^{+}|_{A_{1}A_{2}}^{\otimes n_{1}}\otimes|\phi^{+}\rangle\langle\phi^{+}|_{A_{1}A_{3}}^{\otimes n_{2}}\otimes|\phi^{+}\rangle\langle\phi^{+}|_{A_{2}A_{3}}^{\otimes n_{3}}),|\phi\rangle\langle\phi|^{\otimes n})\geq 1-\varepsilon. \tag{2.162}$$

The statement of the proposition follows from the fact that

$$n_1 + n_2 + n_3 \ge E_S^{A_1 A_2 A_3}(|\phi\rangle\langle\phi|^{\otimes n}) = nE_S^{A_1 A_2 A_3}(|\phi\rangle\langle\phi|).$$
 (2.163)

Example 2.28. – How many Bell states are needed to prepare copies of W-states in the asymptotic limit? A lower bound can be deduced from Proposition 2.27, but it seems hard to tell whether this bound can be achieved. To approach this problem one can at least produce copies of W-states in two steps as shown below.

1. Two parties prepare copies of

$$\frac{1}{\sqrt{3}} \left(|0\rangle_{A_1} |0\rangle_{A_2} |1\rangle_{A'_2} + |0\rangle_{A_1} |1\rangle_{A_2} |0\rangle_{A'_2} + |1\rangle_{A_1} |0\rangle_{A_2} |0\rangle_{A'_2} \right), \tag{2.164}$$

where A_2 and A'_2 are held by the same party.

- 2. The state of A'_2 is teleported to A_3 .
- 3. The total amount of entanglement used per copy of the final state is given by $1/3 + \log_2(3) = 1.918$.

While this procedure is far from being optimal, it is possible to state that in this example

$$1.585 = \log_2(3) = E_S^{A_1 A_2 A_3}(|W\rangle\langle W|) \le E_X(|W\rangle\langle W|) \le 1/3 + \log_2(3) = 1.918.$$
 (2.165)

This example concludes the considerations of the quantification of quantum entanglement. To summarize, several good measures of entanglement have been proposed, and their properties have been studied. After all, the degree of entanglement gives information about the possibility to manipulate the resource entanglement. Entanglement cannot be created on average with local operations and classical communication. However, it is in general not sufficient to know the degree of entanglement present in the initial and in the final situation to find out whether a certain state transformation can – in principle – be implemented. For single copies of quantum systems more information is needed. This points towards the topic of the next chapter, in which criteria for the transformation of quantum states will be presented.

Chapter 3

Entanglement Transformations

3.1 Introduction

Given a machine that is designed in such a way that it can implement any LOCC operation on a bi-partite quantum system, what tasks may this machine accomplish? If one inserts a *single* copy of a bi-partite quantum system in an entangled state into this machine, one can be sure that no output state with a larger amount of entanglement can be produced with probability one: Local operations and classical communication alone cannot increase the amount of entanglement in a such a deterministic transformation, as quantified by any entanglement monotone. But what particular states *can* be prepared with LOCC operations starting from a given state?

The theory of entanglement transformations aims at answering questions of this type. Essentially, the issue is what state transformations can be done with a single copy of a bipartite quantum system using LOCC operations. On the one hand this analysis aims at a classification of states via the class of LOCC operations, on the other hand it provides answers to practical questions concerning the manipulation of the resource entanglement.

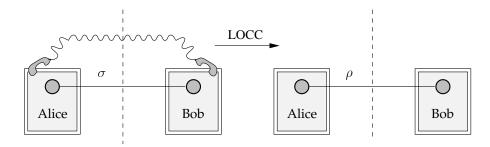


Figure 3.1: A schematic representation of a transformation of an initial state σ into the state ρ with the use of local quantum operations and classical communication (LOCC).

More formally, let σ and ρ be states taken from the state space $\mathcal{S}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the Hilbert space associated with a bipartite quantum system consisting of parts A

and B. If there exists a trace-preserving LOCC operation \mathcal{E} mapping σ on $\mathcal{E}(\sigma) = \rho$, one writes [42]

$$\sigma \longrightarrow \rho \ under \ LOCC.$$
 (3.1)

The statement that such a transformation is not possible within this class of operations is abbreviated as

$$\sigma \not\longrightarrow \rho \text{ under LOCC}.$$
 (3.2)

States ρ and σ are called *incommensurate* if both $\sigma \not\longrightarrow \rho$ and $\rho \not\longrightarrow \sigma$ under LOCC.

The question whether an initial state σ can be transformed into a particular final state ρ is different from the question of asymptotic reducibility. Under LOCC operations, it may well be possible to transform a large number of copies of σ into a large number of copies of ρ in a reversible way, while with a single copy at hand the corresponding transformation $\sigma \longrightarrow \rho$ under LOCC is not possible.

Much work has been done on entanglement transformations of pure states. In Ref. [42] a necessary and sufficient criterion for entanglement transformation has been presented. Ref. [43] concentrates on *probabilistic transformations*, and the maximal probability is calculated of the successful transformation from an initial pure entangled state into another final pure state. In Ref. [45] and [136] these results are generalized to the case where a pure state is mapped on an ensemble of pure states. *Approximate transformations* are considered in Refs. [44] and [137]. In aggregate, pure-state entanglement transformations are rather well-understood.

The structure of the chapter is as follows: First of all, two important results [42, 43, 45] for the transformation of pure states will be mentioned. This chapter however deals mainly with transformations of mixed states, about which much less is known. In the subsequent section transformation criteria for mixed states will be derived. The main emphasis of the chapter is on a new class of entanglement transformations proposed in Ref. [138] and developed for mixed states in Ref. [E1]: the class of so-called *entanglement-assisted local quantum operations*. Most of the results presented in this chapter have been published in Ref. [E1].

3.2 Entanglement Manipulation for Pure States

The theorem proved in Ref. [42] provides a necessary and sufficient condition for entanglement transformations. It links the problem of manipulating quantum systems in entangled states to the theory of *majorization*, a topic from linear algebra and matrix analysis. The criterion itself is given by a *majorization relation*. In this chapter it will be assumed that the dimensions $N = \dim[\mathcal{H}_A] = \dim[\mathcal{H}_B]$ of the Hilbert spaces \mathcal{H}_A and \mathcal{H}_B of the bi-partite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ are identical.

Theorem (Nielsen). – Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $\sigma, \rho \in \mathcal{S}(\mathcal{H})$ be pure states. Then $\sigma \longrightarrow \rho$ under LOCC if and only if

$$\sum_{i=1}^{k} \alpha_{i} \leq \sum_{i=1}^{k} \beta_{i} \text{ for all } k = 1, ..., N,$$
(3.3)

where $\alpha_1,...,\alpha_N$ and $\beta_1,...,\beta_N$ with $\alpha_1 \ge ... \ge \alpha_N$ and $\beta_1 \ge ... \ge \beta_N$ are the eigenvalues of $tr_A[\sigma]$ and $tr_A[\rho]$, respectively.¹

¹It should be noted that if an entanglement transformation is possible, it suffices to use one-local operations instead of the full class of LOCC operations. This fact has already been demonstrated in Ref. [139].

A list of the latter type will also be referred to as *ordered list*. The content of the N conditions on the eigenvalues of $\operatorname{tr}_A[\sigma]$ and $\operatorname{tr}_A[\rho]$ in Eq. (3.3) is typically abbreviated as

$$\operatorname{tr}_A[\sigma] \prec \operatorname{tr}_A[\rho].$$
 (3.4)

The symbol \prec in Eq. (3.4) is the (operator) majorization relation [140, 141, 100] and means that " $\operatorname{tr}_A[\rho]$ is more mixed than $\operatorname{tr}_A[\sigma]$ ". Such a majorization relation is defined for all Hermitian matrices. However, it does not lead directly to a criterion for entanglement transformations for mixed states. For states the N-th condition given by Eq. (3.3) is trivially satisfied due to the normalization constraint.

Example 3.1. – Take $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^3$, and label the basis vectors of \mathcal{H}_A as $|1\rangle, |2\rangle, |3\rangle$. Consider the states $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ with

$$|\psi\rangle = \sqrt{0.6}|11\rangle + \sqrt{0.3}|22\rangle + \sqrt{0.1}|33\rangle, \tag{3.5}$$

$$|\phi\rangle = \sqrt{0.15}|11\rangle + \sqrt{0.7}|22\rangle + \sqrt{0.15}|33\rangle.$$
 (3.6)

The transformation $|\psi\rangle\langle\psi|$ to $|\phi\rangle\langle\phi|$ is not possible with local operations and classical communication, that is, $|\psi\rangle\langle\psi| \not\longrightarrow |\phi\rangle\langle\phi|$ under LOCC, by virtue of the above theorem. The ordered list of eigenvalues of $\mathrm{tr}_A[|\psi\rangle\langle\psi|] = 0.6|1\rangle\langle1| + 0.3|2\rangle\langle2| + 0.1|3\rangle\langle3|$ is given by 0.6, 0.3, 0.1, and the corresponding ordered list for $\mathrm{tr}_A[|\phi\rangle\langle\phi|]$ is 0.7, 0.15, 0.15. As

$$0.6 \le 0.7$$
, but $0.6 + 0.3 > 0.7 + 0.15$, (3.7)

the majorization criterion of Nielsen's theorem is not satisfied.

The second important result is related to transformations from a pure state to a mixed state. In Ref. [43] an abstract criterion is given for the probabilistic transformation from one pure state to a set of other pure states. This result can be used to give necessary and sufficient conditions for the deterministic transformation from a pure to a mixed state, as the desired mixed state can always be generated by mapping the initial state to a known set of certain final pure states and mixing of the results. That is, the classical information about the outcomes is discarded. The explicit criterion is given in Ref. [45] and reads as follows.

Theorem (Plenio, Jonathan, Vidal) [43, 45]. – Let $|\psi\rangle \in \mathcal{H}$ and let $\rho \in \mathcal{S}(\mathcal{H})$ be a mixed state. For any $|\phi\rangle \in \mathcal{H}$ let

$$E_k(|\phi\rangle) = \sum_{i=k}^{N} \alpha_i, \quad k = 1, 2, ..., N,$$
 (3.8)

where $\alpha_1, ..., \alpha_N$ is the ordered list of eigenvalues of $tr_A[|\phi\rangle\langle\phi|]$. Then

$$|\psi\rangle\langle\psi| \longrightarrow \rho \text{ under LOCC}$$
 (3.9)

if and only if there exists a decomposition $ho = \sum_i \mu_i |\phi_i\rangle \langle \phi_i|$ of ho in terms of pure states such that

$$\sum_{i} \mu_{i} E_{k}(|\phi_{i}\rangle) \le E_{k}(|\psi\rangle) \tag{3.10}$$

for all k = 1, 2, ..., N.

3.3 State Transformations for Mixed States

In any practical application, however, one would expect to always deal with entangled mixed states rather than with pure states. Unfortunately, much less is known about entanglement transformations of mixed states, and such a powerful tool as the criterion of Ref. [42] is missing in this case. The question whether a particular entanglement transformation from one mixed state into another mixed state is possible seems to be much more involved. As has been pointed out in the previous chapter, in quantum mechanical states both classical correlations and intrinsic quantum mechanical correlations may be present, which makes the structure of mixed-state entanglement a much more complex matter. A different aspect of the same problem is the well known fact that a representation of a mixed state in terms of pure states is not uniquely defined, and it is essentially this ambiguity that prohibits a straightforward application of the techniques of the pure-state case. Due to these difficulties one can hardly hope for as convenient tools as are available for pure states.

In this subsection transformation criteria for mixed states will be developed. A still rather simple setting is one where the projections in the spectral decompositions of the initial state can be locally distinguished. Examples of such states will be given in Chapter 5. In this case Alice may design a projective measurement discriminating between the respective components. A particular strategy for manipulating such a quantum state can easily be identified: she performs a projective measurement yielding a pure state, transforms this pure state to an appropriate other pure state, and finally discards the classical information of the outcome of the measurement in order to realize a certain mixed state. Based on this strategy, one can introduce the following lemma providing both necessary and sufficient conditions for the transformation of states of this type into other such states.

Lemma 3.2. – Consider the states of rank two

$$\sigma = \lambda |\psi\rangle\langle\psi| + (1-\lambda)|\eta\rangle\langle\eta|_A \otimes |\xi\rangle\langle\xi|_B, \tag{3.11}$$

$$\rho = \lambda |\phi\rangle\langle\phi| + (1-\lambda)|\varphi\rangle\langle\varphi|_A \otimes |\nu\rangle\langle\nu|_B, \tag{3.12}$$

 $\lambda \in (0,1)$, where $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ are pure entangled states satisfying $\langle\eta|_A(tr_B[|\psi\rangle\langle\psi|])|\eta\rangle_A=0$, $\langle\varphi|_A(tr_B[|\phi\rangle\langle\phi|])|\varphi\rangle_A=0$, $\langle\xi|_B(tr_B[|\psi\rangle\langle\psi|])|\xi\rangle_B=0$, $\langle\nu|_B(tr_B[|\phi\rangle\langle\phi|])|\nu\rangle_B=0$. Then $\sigma \longrightarrow \rho$ under LOCC if and only if $|\psi\rangle\langle\psi| \longrightarrow |\phi\rangle\langle\phi|$ under LOCC.

Proof: σ can be transformed into ρ by applying LOCC if $|\psi\rangle\langle\psi| \longrightarrow |\phi\rangle\langle\phi|$ under LOCC: Alice performs a local selective and projective measurement in \mathcal{H}_A distinguishing between $|\psi\rangle\langle\psi|$ and $|\eta\rangle\langle\eta|_A\otimes|\xi\rangle\langle\xi|_B$. If the final state of this selective measurement is $|\psi\rangle\langle\psi|$, she applies an appropriate LOCC operation to obtain the state $|\phi\rangle\langle\phi|$. In the other case she transforms $|\eta\rangle\langle\eta|_A\otimes|\xi\rangle\langle\xi|_B$ into $|\varphi\rangle\langle\varphi|_A\otimes|\nu\rangle\langle\nu|_B$ by using LOCC operations. In a last step she disregards the classical information about the outcomes of the projective measurement to achieve ρ . Conversely, let \mathcal{E} be the LOCC operation realizing $\mathcal{E}(\sigma) = \rho$. Then $\mathcal{E}(|\eta\rangle\langle\eta|_A\otimes|\xi\rangle\langle\xi|_B) = |\varphi\rangle\langle\varphi|_A\otimes|\nu\rangle\langle\nu|_B$, as $|\varphi\rangle_A\otimes|\nu\rangle_B$ is the only product vector in range $[\rho]$. By linearity of \mathcal{E} is follows that $\mathcal{E}(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|$. \square

In a sense the states in Lemma 3.2 can be regarded as essentially pure states: the parties can go back and forth from the initial mixed state to one of the pure states in the spectral decomposition by implementing measurements and discarding information. Finding criteria for the transformation of states for which this procedure is not possible is a more challenging task. Such states for which the projections of the spectral decomposition cannot

be locally distinguished will be called *genuinely mixed states*. The subsequent lemma gives a necessary condition in order for an entanglement transformation to be possible. The class of mixed states considered in Lemma 3.3 is actually rather artificial, and it presents a particular case in which an explicit necessary condition can be given. Propositions 3.4 - 3.6 will be more general statements; however, in the context of later considerations Lemma 3.3 will turn out to be very useful.

Lemma 3.3. – Let σ and ρ be mixed states of rank two of the form

$$\sigma = \lambda |\psi\rangle\langle\psi| + (1-\lambda)|\eta\rangle\langle\eta|_A \otimes |\xi\rangle\langle\xi|_B, \tag{3.13}$$

$$\rho = \mu |\phi\rangle\langle\phi| + (1-\mu)|\eta\rangle\langle\eta|_A \otimes |\xi\rangle\langle\xi|_B, \tag{3.14}$$

where $\langle \eta |_A (tr_B[|\phi\rangle\langle\phi|]) | \eta \rangle_A = \langle \eta |_B (tr_A[|\phi\rangle\langle\phi|]) | \eta \rangle_B = 0$, and $\mu = \lambda tr[\chi]$, $\chi = \Pi |\psi\rangle\langle\psi|\Pi$. The projector Π is given by

$$\Pi = \mathbb{1} - |\eta\rangle\langle\eta|_A \otimes |\xi\rangle\langle\xi|_B. \tag{3.15}$$

 $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ are entangled pure states. Then

$$\sigma \longrightarrow \rho \text{ under LOCC } \Longrightarrow \frac{tr_A[\chi]}{tr[\chi]} \prec tr_A[|\phi\rangle\langle\phi|].$$
 (3.16)

Proof: Let $\sigma \longrightarrow \rho$ under LOCC. The set of LOCC operations is a subset of the set of separable operations. Therefore, one may consider a separable operation \mathcal{E} ,

$$\mathcal{E}(\sigma) = \sum_{i=1}^{K} (A_i \otimes B_i) \sigma(A_i \otimes B_i)^{\dagger}, \tag{3.17}$$

such that $\mathcal{E}(\sigma) = \rho$. The Kraus-operators A_i , B_i , i=1,...,K, act in \mathcal{H}_A and \mathcal{H}_B , respectively, and satisfy $\sum_{i=1}^K A_i^\dagger A_i = \mathbbm{1}_A$, $\sum_{i=1}^K B_i^\dagger B_i = \mathbbm{1}_B$. For each i the image of σ under this map must be element of the range of ρ , that is

$$(A_i \otimes B_i)\sigma(A_i \otimes B_i)^{\dagger} \in \text{range}[\rho].$$
 (3.18)

There is only a single product vector included in the range of ρ , which then amounts to a best separable approximation in the sense of [74]. As $\mathcal{E}(|\eta\rangle\langle\eta|_A\otimes|\xi\rangle\langle\xi|_B)=|\eta\rangle\langle\eta|_A\otimes|\xi\rangle\langle\xi|_B$, it follows that

$$\mathcal{E}(\sigma) = \lambda \mathcal{E}(|\psi\rangle\langle\psi|) + (1-\lambda)|\eta\rangle\langle\eta|_A \otimes |\xi\rangle\langle\xi|_B, \tag{3.19}$$

and therefore, the state $|\psi\rangle\langle\psi|$ must be mapped on

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \nu|\phi\rangle\langle\phi| + (1-\nu)|\eta\rangle\langle\eta|_A \otimes |\xi\rangle\langle\xi|_B, \tag{3.20}$$

where $\nu = \mu/\lambda$. Thus,

$$\Pi(A_i \otimes B_i)|\psi\rangle = \Pi(A_i \otimes B_i)\Pi|\psi\rangle \tag{3.21}$$

for i = 1, ..., K. Hence, the following chain holds:

$$\nu = \operatorname{tr}\left[\prod \sum_{i=1}^{K} (A_i \otimes B_i) |\psi\rangle \langle \psi | (A_i \otimes B_i)^{\dagger} \Pi\right]$$

$$= \operatorname{tr}\left[\sum_{i=1}^{K} \prod (A_i \otimes B_i) \chi (A_i \otimes B_i)^{\dagger} \Pi\right]$$

$$\leq \operatorname{tr}[\chi]. \tag{3.22}$$

But $tr[\chi] = \nu$ by definition, and therefore,

$$\chi/\text{tr}[\chi] \longrightarrow |\phi\rangle\langle\phi|$$
 (3.24)

under LOCC, which in turn implies, according to Nielsen's theorem, that

$$\operatorname{tr}_{A}[\chi]/\operatorname{tr}[\chi] \prec \operatorname{tr}_{A}[|\phi\rangle\langle\phi|].$$
 (3.25)

This is the statement of the lemma.

This lemma provides a convenient tool for the considerations of the next section. In particular, it will be used to show that also in the mixed state domain entanglement-assisted operations are more powerful than ordinary LOCC operations. The general problem of finding necessary and sufficient criteria for the transformation of general mixed states of full rank into other mixed states is extremely difficult. Ref. [142] partially addresses this problem and gives some hints, related to the observation that the Schmidt number should not increase on average under LOCC operations. The *Schmidt number* – which is different from the Schmidt measure proposed in the previous chapter – can be conceived as a useful generalization of the Schmidt rank of pure states to the mixed-state domain. It seems likely that any simple generalization of Nielsen's theorem will not be appropriate. Even the connection of this problem to majorization theory is not obvious at all. The following three statements are necessary criteria, but unfortunately, their practical implications are rather limited, as it is not known how to find the optimal decompositions of the involved mixed states in terms of pure states.

Proposition 3.4. – Let $\sigma, \rho \in \mathcal{S}(\mathcal{H})$ be states satisfying $\sigma \longrightarrow \rho$ under LOCC. Then for any LOCC operation \mathcal{E} for which $\mathcal{E}(\sigma) = \rho$ and any decomposition $\sigma = \sum_{i=1}^{n} \mu_i |\psi_i\rangle \langle \psi_i|$ of σ in terms of pure states

$$range[\mathcal{E}(|\psi_i\rangle\langle\psi_i|)] \subset range[\rho]$$
 (3.26)

holds for all i = 1, ..., n.

Proposition 3.5. – Let $\sigma, \rho \in \mathcal{S}(\mathcal{H})$ be states satisfying $\sigma \longrightarrow \rho$ under LOCC. Then for any decomposition $\sigma = \sum_{i=1}^n \mu_i |\psi_i\rangle \langle \psi_i|$ of σ in terms of pure states there exists a decomposition $\rho = \sum_{i=1}^n \mu_i \rho_i$ of ρ and for each i = 1, 2, ..., n a decomposition $\rho_i = \sum_j \eta_{i,j} |\phi_{i,j}\rangle \langle \phi_{i,j}|$ of ρ_i in terms of pure states such that

$$\sum_{j} \eta_{i,j} E_k(|\phi_{i,j}\rangle) \le E_k(|\psi_i\rangle) \quad \text{for all } k = 1, 2, ..., N \text{ and all } i = 1, 2, ..., n.$$
 (3.27)

Proof: Let \mathcal{E} be the LOCC operation realizing $\mathcal{E}(\sigma) = \rho$, then $\rho_i = \mathcal{E}(|\psi_i\rangle\langle\psi_i|)$ satisfy $\sum_i \mu_i \rho_i = \rho$. The proof of the statement then follows from the theorem of Ref. [45]. \square

Proposition 3.6. – Let $\sigma, \rho \in \mathcal{S}(\mathcal{H})$ be states satisfying $\sigma \longrightarrow \rho$ under LOCC. Let $|\psi\rangle \in \mathcal{H}$ and let $\sigma = \sum_{i=1}^{n} \mu_i |\psi_i\rangle \langle \psi_i|$ be a decomposition of σ in terms of pure states satisfying

$$\sum_{i=1}^{n} \mu_i E_k(|\psi_i\rangle) \le E_k(|\psi\rangle) \tag{3.28}$$

for all k=1,2,...,N. Then there exists a decomposition $\rho=\sum_{j=1}^m \eta_j |\phi_j\rangle\langle\phi_j|$ of ρ in terms of pure states such that

$$\sum_{j=1}^{m} \eta_j E_k(|\phi_j\rangle) \le E_k(|\psi\rangle) \tag{3.29}$$

for k = 1, ..., N.

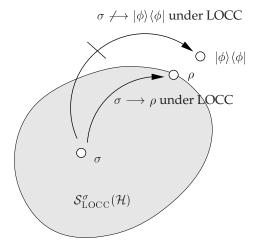


Figure 3.2: Although the state σ cannot be transformed into $|\phi\rangle\langle\phi|$ by use of LOCC operations, the similar state ρ can be reached. ρ maximizes the fidelity with respect to $|\phi\rangle\langle\phi|$ in the set $\mathcal{S}^{\sigma}_{\text{LOCC}}(\mathcal{H})$.

Sometimes it is not of primary interest to find out whether a particular final state can exactly be reached when performing a LOCC operation. Instead, if a transformation from a certain initial state σ to a final state $|\phi\rangle\langle\phi|$ is not possible, the objective is to prepare a mixed state ρ that is closest to the desired pure state as measured by the fidelity. Such transformations are called *approximate transformations* [44]. In general the task is to increase the fidelity with respect to $|\phi\rangle\langle\phi|$,

$$F(\sigma, |\phi\rangle\langle\phi|) = \langle\phi|\sigma|\phi\rangle,\tag{3.30}$$

to the maximal value which can be attained by applying LOCC operations on the initial mixed state σ . This maximal value will hereafter be denoted as

$$F_{\text{LOCC}}(\sigma, |\phi\rangle\langle\phi|) = \max_{\rho \in \mathcal{S}_{\text{LOCC}}^{\sigma}(\mathcal{H})} \langle\phi|\rho|\phi\rangle. \tag{3.31}$$

 $\mathcal{S}^{\sigma}_{\mathrm{LOCC}}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ is in this equation the set of states ρ for which $\sigma \longrightarrow \rho$ under LOCC. This set is a convex subset of $\mathcal{S}(\mathcal{H})$: If ρ_1 is accessible from σ under LOCC and the same is true for another state ρ_2 , then also the convex combination $\lambda \rho_1 + (1-\lambda)\rho_2$ can be achieved with the use of such transformations for all $\lambda \in [0,1]$. This set is also a compact set. As the fidelity with respect to a fixed pure state is a linear functional, the max in Eq. (3.31) is justified because the corresponding maximum is actually attained.

3.4 Entanglement-Assisted Transformations

As has been pointed out in the introduction, entanglement is often viewed as the essential resource for many tasks of quantum information processing. Typically, the entanglement is used up in the course of the implementation of a certain protocol. So it comes as a surprise that "the mere presence of entanglement" can be an advantage when one intends to transform an initial state into a particular final state with the use of local quantum operations with classical communication. It has been demonstrated in Ref. [45] that there are indeed target states which cannot be reached by LOCC starting from a particular initial state, but with the assistance of a distributed pair of auxiliary quantum systems in a particular known state, even though these auxiliary quantum systems are left in exactly the same state and remain finally completely uncorrelated to the quantum system of interest. This phenomenon is quite remarkable as the entanglement which serves as a "catalyst" for the otherwise forbidden "reaction" is not consumed.

The basis of the example given in Ref. [45] is the above majorization criterion of Ref. [42]. Let σ and ρ be states of a bi-partite quantum system with Hilbert space \mathcal{H} . The notation

$$\sigma \longrightarrow \rho$$
 under ELOCC (3.32)

will be used if there exists a Hilbert space K of another bi-partite system and a pure state $\omega \in \mathcal{S}(\mathcal{H})$ such that

$$\sigma \otimes \omega \longrightarrow \rho \otimes \omega$$
 under LOCC. (3.33)

 \mathcal{H} and \mathcal{K} are Hilbert spaces corresponding to bi-partite quantum systems, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{K} = \mathcal{K}_A \otimes \mathcal{K}_B$, respectively, such that local operations of Alice, say, are operations in $\mathcal{H}_A \otimes \mathcal{K}_A$ (see Figs. 3.3 and 3.4). The phenomenon of "catalysis" of entanglement transformations might serve as a basis for applications in cryptography. For first steps in this direction see Refs. [137] and [143].

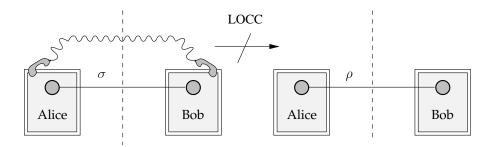


Figure 3.3: If σ and ρ are entangled states of a bi-partite quantum system, it may well be that $\sigma \not\longrightarrow \rho$ under LOCC, even if ρ is less entangled than σ with respect to some entanglement monotone.

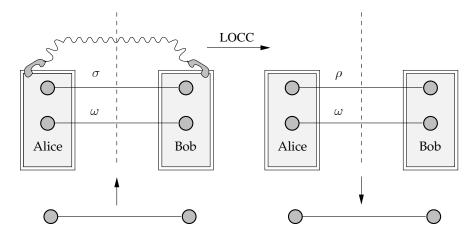


Figure 3.4: There are, however, cases for which $\sigma \not\longrightarrow \rho$ under LOCC, but $\sigma \otimes \omega \longrightarrow \rho \otimes \omega$ under LOCC for an appropriately chosen state ω of another bi-partite quantum system. That is, the auxiliary system in the state ω is borrowed by Alice and Bob, such that they hold a composite system in the state $\sigma \otimes \omega$. This state can then be transformed into $\rho \otimes \omega$ under LOCC operations. The auxiliary system can in a last step be returned. The state ω remains unchanged in the course of the protocol, and the auxiliary system is finally fully uncorrelated to the original quantum system.

3.4.1 Pure State Case

For pure states one can use the strong tool that is provided by Nielsen's theorem. On applying the majorization criterion one can look numerically for examples for catalysis of entanglement manipulation.

Example (Jonathan and Plenio). – In this example [138] the initial pure state $|\psi\rangle\langle\psi|$ and the final state $|\phi\rangle\langle\phi|$ are taken from the state space corresponding to $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, $\mathcal{H}_A=\mathcal{H}_B=\mathbb{C}^4$. The basis is denoted as $\{|1\rangle,|2\rangle,|3\rangle,|4\rangle\}$. The state vectors can be represented in the Schmidt decomposition as

$$|\psi\rangle = \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle + \sqrt{0.1}|33\rangle,$$
 (3.34)

$$|\phi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle.$$
 (3.35)

It follows from Nielsen's theorem [42] that $|\psi\rangle\langle\psi| \not\longrightarrow |\phi\rangle\langle\phi|$, as the majorization relation is not satisfied, $\mathrm{tr}_A[|\psi\rangle\langle\psi|] \not\prec \mathrm{tr}_A[|\phi\rangle\langle\phi|]$. However, surprisingly, if one appends the catalyst state

$$\omega = \left(\sqrt{0.6}|44\rangle + \sqrt{0.4}|55\rangle\right)\left(\sqrt{0.6}\langle 44| + \sqrt{0.4}\langle 55|\right),\tag{3.36}$$

then

$$\operatorname{tr}_{A}[|\psi\rangle\langle\psi|\otimes\omega] \prec \operatorname{tr}_{A}[|\phi\rangle\langle\phi|\otimes\omega].$$
 (3.37)

In other words, $|\psi\rangle\langle\psi|$ can be transformed into $|\phi\rangle\langle\phi|$ by means of ELOCC operations.

The general problem of *catalysis for pure states* remains unresolved. The issue is basically a topic from matrix analysis and majorization theory. Let M_1 and M_2 be Hermitian matrices with $M_1^2 = M_1$, $M_2^2 = M_2$, satisfying $M_1 \not\prec M_2$. What are the (necessary and sufficient) conditions on M_1 and M_2 for the existence of a Hermitian matrix C, $C^2 = C$, such that

$$M_1 \otimes C \prec M_1 \otimes C.$$
 (3.38)

Translated into quantum mechanical terms this question reads: Let σ and ρ be pure states on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. What are the (necessary and sufficient) conditions on σ and ρ for the existence of a Hilbert space $\mathcal{K} = \mathcal{K}_A \otimes \mathcal{K}_B$ and a pure state $\omega \in \mathcal{S}(\mathcal{K})$ such that

$$\operatorname{tr}_{A}[\sigma] \otimes \operatorname{tr}_{A}[\omega] \prec \operatorname{tr}_{A}[\rho] \otimes \operatorname{tr}_{A}[\omega]$$
? (3.39)

In the absence of a complete answer to this question, practically relevant necessary conditions are already helpful. A step in this direction could be the subsequent statement²:

Proposition 3.7. – *Let* $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}$. *Then*

$$|\psi\rangle\langle\psi| \longrightarrow |\phi\rangle\langle\phi| \quad under \; ELOCC \implies \sum_{i=1}^{N} \alpha_i^{\xi} \le \sum_{i=1}^{N} \beta_i^{\xi}$$
 (3.40)

for all $\xi > 0$, where $\alpha_1, ..., \alpha_N$ and $\beta_1, ..., \beta_N$ are the eigenvalues of $tr_A[|\psi\rangle\langle\psi|]$ and $tr_A[|\phi\rangle\langle\phi|]$, respectively.

Proof: Assume that $|\psi\rangle\langle\psi| \longrightarrow |\phi\rangle\langle\phi|$ under ELOCC. Then there exists a $\mathcal{K} = \mathcal{K}_A \otimes \mathcal{K}_B$ and a pure state $\omega \in \mathcal{S}(\mathcal{K})$ such that $|\psi\rangle\langle\psi|\otimes\omega \longrightarrow |\phi\rangle\langle\phi|\otimes\omega$ under LOCC and hence,

$$\operatorname{tr}_{A}[|\psi\rangle\langle\psi|\otimes\omega] \prec \operatorname{tr}_{A}[|\phi\rangle\langle\phi|\otimes\omega].$$
 (3.41)

Let $\gamma_1,...,\gamma_M$, $M=\dim[\mathcal{K}_B]$, be the eigenvalues of $\operatorname{tr}_A[\omega]$, such that $\alpha_i\gamma_j$ and $\beta_i\gamma_j$, i=1,...,N, j=1,...,M are the respective eigenvalues of $\operatorname{tr}_A[|\psi\rangle\langle\psi|\otimes\omega]$ and $\operatorname{tr}_A[|\phi\rangle\langle\phi|\otimes\omega]$. So,

$$\sum_{i=1}^{N} \sum_{j=1}^{M} g(\alpha_i \gamma_j) \le \sum_{i=1}^{N} \sum_{j=1}^{M} g(\beta_i \gamma_j)$$
 (3.42)

for any convex function $g:\mathbb{R}\longrightarrow\mathbb{R}$ [140, 100]. In particular, take the function f defined as $f(x)=x^{\xi},\,\xi>0.$ As

$$f(\beta_i \gamma_j) = f(\beta_i) f(\gamma_j) \tag{3.43}$$

for all i = 1, ..., N, j = 1, ..., M, the statement of Eq. (3.40) holds.

This proposition provides pairs of states for which one can tell that the corresponding entanglement-assisted transformation is not possible. Accordingly, one can rule out a rather large class of pairs of pure states which are otherwise eligible for enhancement of entanglement manipulations through additional catalyst quantum systems.

Example 3.8. – Consider the initial state $|\psi\rangle\langle\psi|$ and the final state $|\phi\rangle\langle\phi|$ with

$$|\psi\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.2 - \varepsilon}|22\rangle + \sqrt{0.2 + \varepsilon}|33\rangle + \sqrt{0.1}|44\rangle,$$
 (3.44)

$$|\phi\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.2}|22\rangle + \sqrt{0.2}|33\rangle + \sqrt{0.1}|44\rangle,$$
 (3.45)

 $\varepsilon \in (0,0.1)$. As follows from Nielsen's theorem $|\psi\rangle\langle\psi| \not\longrightarrow |\phi\rangle\langle\phi|$ under LOCC for all values of ε . In addition, one cannot gain from the possibility of using ELOCC operations, as can be

²After completion of the work on this proposition the author found out that in unpublished lecture notes on majorization theory statements can be found which are similar in spirit [144].

inferred from Proposition 3.7: take as an appropriate function $f: \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = x^2$. Since

$$0.5^{2} + (0.2 - \varepsilon)^{2} + (0.2 + \varepsilon)^{2} + 0.1^{2} > 0.5^{2} + 2 \times 0.2^{2} + 0.1^{2}$$
(3.46)

for all $\varepsilon \in (0, 0.1)$, also $|\psi\rangle\langle\psi| \not\longrightarrow |\phi\rangle\langle\phi|$ under ELOCC for all $\varepsilon \in (0, 0.1)$.

Example 3.9. – It has been shown that $|\psi\rangle\langle\psi| \not\longrightarrow |\phi\rangle\langle\phi|$ under LOCC, but $|\psi\rangle\langle\psi| \longrightarrow |\phi\rangle\langle\phi|$ under ELOCC, where

$$|\psi\rangle = \sqrt{0.4}|11\rangle + \sqrt{0.4}|22\rangle + \sqrt{0.1}|33\rangle + \sqrt{0.1}|44\rangle,$$
 (3.47)

$$|\phi\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.25}|22\rangle + \sqrt{0.25}|33\rangle. \tag{3.48}$$

In order to investigate to what degree one can change the coefficients without changing the structure of the example, take the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = x^2$. Then the criterion of Proposition 3.7 is satisfied, since

$$2 \times 0.4^2 + 2 \times 0.1^2 = 0.34 < 0.5^2 + 2 \times 0.25^2 = 0.375.$$
 (3.49)

The slightly modified states

$$|\psi\rangle = \sqrt{0.43}|11\rangle + \sqrt{0.43}|22\rangle + \sqrt{0.07}|33\rangle + \sqrt{0.07}|44\rangle,$$
 (3.50)

$$|\phi\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.25}|22\rangle + \sqrt{0.25}|33\rangle.$$
 (3.51)

are nevertheless not appropriate for an entanglement-assisted transformation. From Proposition 3.7 one can infer that the transformation from $|\psi\rangle\langle\psi|$ to $|\phi\rangle\langle\phi|$ under ELOCC is certainly not possible.

3.4.2 Mixed-State Catalysis of Entanglement Manipulation

It is not clear a priori whether in the mixed state domain the set of tasks that can be accomplished with entanglement-assisted local operations is strictly larger than the set of tasks which may be performed with mere LOCC. The possibilities of manipulating entanglement are often much different for pure and for mixed states. For example, while (pure) singlets can be produced from non-maximally entangled pure states by individual measurements, this is not true for mixed states, not even if they are sufficiently close to being singlets [145].

In an abstract form, the class of ELOCC operations can be easily explored in terms of separable operations. The class of LOCC operations is contained in the set of separable operations, which can be represented using Kraus operators as explained in the first chapter. Certainly, the statement that a certain operation is a ELOCC operation implies particular restrictions on the Kraus operators:

Proposition 3.10. – Let $\sigma, \rho \in \mathcal{S}(\mathcal{H})$ be states for which

$$\sigma \longrightarrow \rho$$
 under ELOCC. (3.52)

Then there exist a Hilbert space K, a state vector $|\tilde{\psi}\rangle \in K$ and Kraus operators $A_1,...,A_K:\mathcal{H}_A \to \mathcal{H}_A$ and $B_1,...,B_K:\mathcal{H}_B \to \mathcal{H}_B$ satisfying $\sum_{k=1}^K A_k^\dagger A_k = \mathbbm{1}_A$ and $\sum_{k=1}^K B_k^\dagger B_k = \mathbbm{1}_B$, such that for every decomposition $\sigma = \sum_{i=1}^n \mu_i |\psi_i\rangle \langle \psi_i|$ of σ in terms of pure states

$$(A_k \otimes B_k)|\psi_i\rangle \otimes |\tilde{\psi}\rangle \propto |\phi_i^{(k)}\rangle \otimes |\tilde{\psi}\rangle \tag{3.53}$$

holds for all k = 1, ..., K, i = 1, ..., n, where $|\phi_i^{(k)}\rangle \in \mathcal{H}_A$ are appropriate state vectors.

Proof: Let \mathcal{E} be the separable operation realizing $\mathcal{E}(\sigma \otimes |\tilde{\psi}\rangle\langle\tilde{\psi}|) = \sum_{k=1}^K (A_k \otimes B_k)(\rho \otimes |\tilde{\psi}\rangle\langle\tilde{\psi}|)(A_k \otimes B_k)^\dagger = \rho$, and let $\sigma = \sum_{i=1}^n \mu_i |\psi_i\rangle\langle\psi_i|$ be a decomposition of σ in terms of pure states. Then $(A_k \otimes B_k)|\psi_i\rangle \otimes |\tilde{\psi}\rangle$ must be element of the range of $\rho \otimes |\tilde{\psi}\rangle\langle\tilde{\psi}|$ for every k = 1, ..., K and every i = 1, ..., n. This is condition Eq. (3.53).

These conditions are rather restrictive, but from this statement alone it cannot be judged whether entanglement-assisted operations are possible for mixed states. This is because Proposition 3.10 does not present a constructive way of actually finding the Kraus operators which satisfy Eq. (3.53). Nevertheless, it can be shown that examples of mixed-state catalysis of entanglement transformations can indeed be found. The major result of this subsection is an example of incommensurate genuinely mixed states such that with the use of some appropriately chosen catalyst state, the initial state can be converted into the final state while fully retaining the catalyst state. That is, there exist mixed states σ , ρ such that $\sigma \longrightarrow \rho$ under ELOCC but not $\sigma \longrightarrow \rho$ under LOCC. To show that the transformation $\sigma \longrightarrow \rho$ is actually not possible by means of LOCC operations, Lemma 3.3 will be used.

Example 3.11. – Consider the two one-parameter sets of states of rank two, which are density operators on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^5$ with basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle\}$. For a $\lambda \in (0, 1)$ let

$$\sigma = \lambda |\psi\rangle\langle\psi| + (1-\lambda)|55\rangle\langle55|, \tag{3.54}$$

$$\rho = \mu |\phi\rangle\langle\phi| + (1-\mu)|55\rangle\langle55|, \tag{3.55}$$

with $\mu = 0.95\lambda$ and

$$|\psi\rangle = \sqrt{0.38}|11\rangle + \sqrt{0.38}|22\rangle + \sqrt{0.095}|33\rangle + \sqrt{0.095}|44\rangle + \sqrt{0.05}|55\rangle,$$
 (3.56)

$$|\phi\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.25}|22\rangle + \sqrt{0.25}|33\rangle,$$
 (3.57)

The initial state σ is indeed genuinely mixed for all $\lambda \in (0,1)$, as the components of the spectral decomposition cannot be locally distinguished. These states are included in the sets of states considered in Lemma 3.3, and this lemma can be applied. $\chi/{\rm tr}[\chi] = |\eta\rangle\langle\eta|$, where

$$|\varphi\rangle = \sqrt{0.4}|11\rangle + \sqrt{0.4}|22\rangle + \sqrt{0.1}|33\rangle + \sqrt{0.1}|44\rangle \tag{3.58}$$

as in Ref. [45]. Hence,

$$\frac{\operatorname{tr}_{A}[\chi]}{\operatorname{tr}[\chi]} \not\prec \operatorname{tr}_{A}[|\phi\rangle\langle\phi|],\tag{3.59}$$

and therefore,

$$\sigma \not\longrightarrow \rho \text{ under LOCC}$$
 (3.60)

for all values of $\lambda \in (0,1)$. However, it can be shown that $\sigma \longrightarrow \rho$ under ELOCC by designing a protocol that accomplishes this task:

1. Alice performs a local projective measurement in system A associated with Kraus operators $E_1 = A_1 \otimes \mathbb{1}_B$, $E_2 = A_1 \otimes \mathbb{1}_B$, where

$$A_1 = \sum_{i=1}^{4} |ii\rangle\langle ii|, \ A_2 = |55\rangle\langle 55|$$
 (3.61)

satisfying $A_1^{\dagger}A_1 + A_2^{\dagger}A_2 = \mathbb{1}_A$. (A similar decomposition will be used in Chapter 5 in Eq. (5.15).)

- 2. If the outcome corresponds to A_2 , she does not implement any further operations.
- 3. In the alternative case the final state is the pure state $|\varphi\rangle\langle\varphi|$ given by Eq. (3.58). As in Ref. [45] this state can be transformed into $|\phi\rangle\langle\phi|$ by the help of the pure catalyst state

$$\omega = (\sqrt{0.4}|66\rangle + \sqrt{0.6}|77\rangle)(\sqrt{0.4}\langle 66| + \sqrt{0.6}\langle 77|). \tag{3.62}$$

This follows from the fact that

$$\operatorname{tr}_{A}[|\varphi\rangle\langle\varphi|\otimes\omega] \prec \operatorname{tr}_{A}[|\phi\rangle\langle\phi|\otimes\omega],$$
 (3.63)

as the criterion of Nielsen's theorem is satisfied.

4. Finally, the classical information about the outcomes in the first step is discarded in order to achieve ρ .

Hence, it turns out that the initial state σ cannot be transformed into ρ if – except for the classical communication to coordinate the steps – only local quantum operations on the quantum system itself can be performed. In the presence of an additional quantum system in the pure state ω this task may be feasible.

3.4.3 Increasing the Proportion of a Pure State in a Mixture

In the mixed-state regime entanglement-assisted LOCC operations are more powerful than mere local operations with classical communication. This has an implication on the efficiency of attempts to increase the quota of an entangled state $|\phi\rangle\langle\phi|$ in a mixed state σ by applying a trace-preserving operation. When the task is to maximize the fidelity

$$F(\sigma, |\phi\rangle\langle\phi|) = \langle\phi|\sigma|\phi\rangle \tag{3.64}$$

of an initial state with respect to a particular entangled pure state, protocols which make appropriate use of catalyst states can indeed be more efficient than protocols in which such additional quantum systems are not available. This result is quite amazing since in this case a "number" is increased (the efficiency of the protocol) without consuming the additional resource at all (the entanglement of the catalyst state).

Proposition 3.12. – There exist mixed states σ and pure states $|\phi\rangle\langle\phi|$ with the property that the maximal average attainable value of the fidelity under ELOCC

$$F_{\text{ELOCC}}(\sigma, |\phi\rangle\langle\phi|) = \max_{\rho \in \mathcal{S}_{\text{ELOCC}}^{\sigma}(\mathcal{H})} \langle\phi|\rho|\phi\rangle$$
 (3.65)

is strictly larger than the maximal attainable fidelity under LOCC,

$$F_{\text{LOCC}}(\sigma, |\phi\rangle\langle\phi|) = \max_{\rho \in \mathcal{S}_{\text{LOCC}}^{\sigma}(\mathcal{H})} \langle\phi|\rho|\phi\rangle. \tag{3.66}$$

The set $S_{\text{ELOCC}}^{\sigma}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ *consists of the states* ρ *for which* $\sigma \longrightarrow \rho$ *under ELOCC.*

Proof: This statement can be proven by taking an initial state σ of the form specified in Eq. (3.54), $\sigma = \lambda |\psi\rangle\langle\psi| + (1-\lambda)|55\rangle\langle55|$, $\lambda \in [0,1]$, with

$$|\psi\rangle = \varepsilon(\sqrt{0.4}|11\rangle + \sqrt{0.4}|22\rangle + \sqrt{0.1}|33\rangle + \sqrt{0.1}|44\rangle) + \sqrt{1 - \varepsilon^2}|55\rangle, \tag{3.67}$$

and one may choose

$$|\phi\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.25}|22\rangle + \sqrt{0.25}|33\rangle$$
 (3.68)

as in Eq. (3.57). As the components of the initial state σ are not locally distinguishable for $\varepsilon < 1$ and since the achievable fidelity can be no better than the sum of both largest possible fidelities of each contribution,

$$F_{\text{LOCC}}(\sigma, |\phi\rangle\langle\phi|) \leq (1 - \lambda)F_{\text{LOCC}}(|55\rangle\langle55|, |\phi\rangle\langle\phi|) + \lambda F_{\text{LOCC}}(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|).$$
(3.69)

The not necessarily pure separable state ρ which maximizes the fidelity $F(\rho,|\phi\rangle\langle\phi|)=\langle\phi|\rho|\phi\rangle$ is given by $|11\rangle\langle11|$ as follows from Lemma 3.13, and for any mixed separable state this fidelity takes a smaller value. Therefore,

$$F_{\text{LOCC}}(|55\rangle\langle 55|, |\phi\rangle\langle \phi|) = 1/2. \tag{3.70}$$

Finally, from

$$F_{\text{ELOCC}}(\sigma, |\phi\rangle\langle\phi|) \ge \lambda \varepsilon^2 + (1 - \lambda \varepsilon^2)/2$$
 (3.71)

and due to the continuity of $F_{\rm LOCC}(|\psi\rangle\langle\psi|,|\phi\rangle\langle\phi|)$ taken as a function of ε one can conclude that there exists a $\tilde{\varepsilon} \in (0,1)$ such that the inequality

$$F_{\text{ELOCC}}(\sigma, |\phi\rangle\langle\phi|) > F_{\text{LOCC}}(\sigma, |\phi\rangle\langle\phi|)$$
 (3.72)

certainly holds for all $\varepsilon \in (\tilde{\varepsilon},1]$. For all $\varepsilon < 1$ the initial state is also genuinely mixed. \square

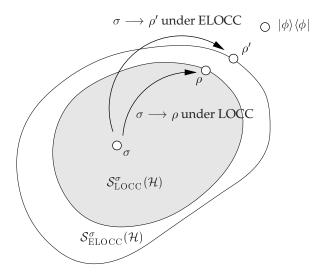


Figure 3.5: As in Fig. 3.2 the state σ cannot be transformed into the desired final state $|\phi\rangle\langle\phi|$ under LOCC operations. The closest state to $|\phi\rangle\langle\phi|$ in the set $\mathcal{S}^{\sigma}_{\mathrm{LOCC}}(\mathcal{H})$ as measured by the fidelity is given by ρ . In the set $\mathcal{S}^{\sigma}_{\mathrm{ELOCC}}(\mathcal{H})$ even the state ρ' is accessible which is closer to $|\phi\rangle\langle\phi|$. Consequently, the approximate transformation is more efficient under ELOCC operations than under LOCC operations.

Lemma 3.13. – Let $|\phi\rangle \in \mathcal{H}$, and let $|\psi\rangle \in \mathcal{H}$, $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, be a product state vector. Let $\beta_1, ..., \beta_N$ be the ordered list of eigenvalues of $tr_A[|\phi\rangle\langle\phi|]$. Then $F_{LOCC}(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = \beta_1$ holds.

Proof: Under LOCC operations any separable state is accessible starting from the product state $|\psi\rangle\langle\psi|$. All separable states can be written in the form $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle\langle\psi_i|$ are pure product states and $p_1,p_2,...$ is a probability distribution. Under these states the fidelity $\langle\phi|\rho|\phi\rangle$ is maximal if $\rho=|11\rangle\langle11|$, and $F_{\rm LOCC}(|\psi\rangle\langle\psi|,|\phi\rangle\langle\phi|)=\beta_1$ holds.

It should be noted that the set $\mathcal{S}^{\sigma}_{\mathrm{ELOCC}}(\mathcal{H})$ is not automatically a convex set. If both $\sigma \longrightarrow \rho_1$ and $\sigma \longrightarrow \rho_2$ under ELOCC, then of course any convex combination $\lambda \rho_1 + (1-\lambda)\rho_2$, $\lambda \in [0,1]$, can be achieved through mixing. But it may not be possible to assume the additional states ω_1 and ω_2 that are needed to render the respective entanglement manipulation possible to be identical. In general no state ω can be found such that

$$\sigma \otimes \omega \longrightarrow (\lambda \rho_1 + (1 - \lambda)\rho_2) \otimes \omega$$
 (3.73)

under LOCC. $S_{\text{ELOCC}}^{\sigma}(\mathcal{H})$ is nevertheless compact.

Example 3.14. – The transformation investigated in Proposition 3.12 can be explored further, as $F_{\rm LOCC}(|\psi\rangle\langle\psi|,|\phi\rangle\langle\phi|)$ can be calculated exactly. Following Ref. [44] the quantities defined in Eq. (3.8) are needed in order to evaluate the optimal achievable fidelity; they are given by

$$E_{1}(|\psi\rangle) = 1, E_{2}(|\psi\rangle) = 1 - 0.4\varepsilon^{2}, E_{3}(|\psi\rangle) = 1 - 0.8\varepsilon^{2}, E_{4}(|\psi\rangle) = 1 - 0.9\varepsilon^{2},$$

$$E_{5}(|\psi\rangle) = 1 - \varepsilon^{2};$$

$$E_{1}(|\phi\rangle) = 1, E_{2}(|\phi\rangle) = 0.5, E_{3}(|\phi\rangle) = 0.25, E_{4}(|\phi\rangle) = E_{5}(|\phi\rangle) = 0.$$
(3.74)

According to Ref. [44] the state which maximizes the fidelity with respect to $|\phi\rangle\langle\phi|$ among all states ρ for which $|\psi\rangle\langle\psi| \longrightarrow \rho$ under LOCC is a pure state. This optimal pure state will be referred to as $|\xi\rangle\langle\xi|$. Now let $\varepsilon > \sqrt{15}/4$, otherwise $F_{\rm LOCC}(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = 1$. In this case $|\xi\rangle\langle\xi|$ can be evaluated as

$$|\xi\rangle = \varepsilon \sqrt{\frac{8}{15}} |11\rangle + \varepsilon \sqrt{\frac{4}{15}} |22\rangle + \sqrt{1 - \frac{4}{5}\varepsilon^2} |33\rangle,$$
 (3.75)

such that

$$F_{\text{LOCC}}(|\psi\rangle\langle\psi|,|\phi\rangle\langle\phi|) = |\langle\xi|\phi\rangle|^2 = \left(5 + 8\varepsilon^2 + 4\sqrt{3}\sqrt{5 - 4\varepsilon^2}\varepsilon\right)/20.$$
 (3.76)

Fig. 3.6 shows an upper bound on the maximally attainable fidelity under LOCC operations, $F_{\rm LOCC}(\sigma, |\phi\rangle\langle\phi|)$, and a lower bound on $F_{\rm ELOCC}(\sigma, |\phi\rangle\langle\phi|)$ for $\lambda = 1/2$.

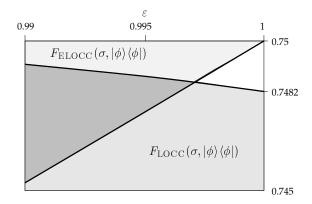


Figure 3.6: This diagram depicts an upper bound on $F_{\text{LOCC}}(\sigma, |\phi\rangle\langle\phi|)$ and a lower bound on $F_{\text{ELOCC}}(\sigma, |\phi\rangle\langle\phi|)$ as functions of ε . The parameter λ is taken to be 1/2. If $\varepsilon = 1$, $F_{\text{ELOCC}}(\sigma, |\phi\rangle\langle\phi|) \geq 3/4$ and $F_{\text{LOCC}}(\sigma, |\phi\rangle\langle\phi|) \leq (5 + 8 + 4\sqrt{3})/20$.

3.4.4 Purification Procedures

The results of the two previous subsections indicate that the class of ELOCC operations is more powerful than LOCC operations not only on the subset of the boundary of $\mathcal{S}(\mathcal{H})$ comprising the pure states, but also in the interior of the set $\mathcal{S}(\mathcal{H})$. This fact suggests that the use of supplementary catalyst states opens up possibilities to enhance general purification procedures. However, ELOCC operations cannot improve the efficiency in the subsequent practically motivated problem:

Clearly, any mixed state cannot be mapped on a maximally entangled state in a deterministic transformation. However, it may be transformed into a maximally entangled state with a certain probability p>0. Transformations in which the target state has to be reached with a probability p>0 are called *probabilistic transformations* [43, 119] or stochastic transformations, abbreviated as SLOCC operations. The process of transforming mixed states into maximally entangled states under SLOCC operations is referred to as *purification* – but the reader should be warned that the term "purification" has at least four different meanings³.

³A) A SLOCC transformation from a mixed state to a maximally entangled state, see, e.g., Refs. [145, 146]. B) Entanglement distillation [31, 32, 64] as described in Chapter 2. C) The reconstruction of a pure state from many copies of mixed states which are a convex combination of the same pure state and the maximally mixed state (this kind of purification aims at "reversing" the process of decoherence), see Refs. [147, 148]. D) A pure state on a larger Hilbert space which is consistent with a given mixed state on a smaller Hilbert space. More general, the purification of a positive linear form is a lift to a pure linear form of a larger algebra.

This subsection concentrates on a particular physically relevant class of mixed states, and a slightly more general problem than that of purification is addressed. This considered class of states includes the set of states consisting of a convex combination of some pure state and the maximally mixed state in the corresponding state space. It is of major interest in connection to practical applications.

Proposition 3.15. – *Consider the class of states*

$$\sigma = \lambda |\psi\rangle\langle\psi| + (1 - \lambda)\chi \tag{3.77}$$

with the property that there exists a $\lambda_0 \in (0,1)$ such that σ is a separable state and so that every state with a larger weight of $|\psi\rangle\langle\psi|$ is entangled. Furthermore, it is assumed that $\langle\psi|\chi|\psi\rangle=0$. Then the fidelity $F(\sigma,|\psi\rangle\langle\psi|)$ of σ with respect to $|\psi\rangle\langle\psi|$ cannot be increased under ELOCC with a fixed operation on the convex set $\{\sigma|\sigma=\lambda|\psi\rangle\langle\psi|+(1-\lambda)\chi \text{ for all }\lambda\in[0,1]\}$ with a non-vanishing probability.

The same class of states as in Proposition 3.15 has been investigated in Ref. [145] in the case that no catalyst state is available. It has been shown that for this class of states, the proportion of $|\psi\rangle\langle\psi|$ cannot be increased with SLOCC operations. Proposition 3.15 shows that an analogous statement is also true of probabilistic ELOCC operations: any catalyst state cannot increase the efficiency of this procedure.

Before continuing with the proof, the last statement of Proposition 3.15 might need some further explanation: for the task of purification one specifies a certain LOCC operation, which is optimized with respect to the states $|\psi\rangle\langle\psi|$ and χ of the convex combination Eq. (3.77). Under such conditions it is then not possible to increase the proportion of $|\psi\rangle\langle\psi|$, not even if one performs a selective generalized measurement which corresponds to a non-trace preserving completely positive map. In a more general treatment, one could, however, include the possibility of modifying the operation on the particular initial state from the above set, such that the operation itself becomes dependent on the parameter λ .

Proof: Let $\sigma \in \mathcal{S}(\mathcal{H})$ be such a state, and let $\omega \in \mathcal{S}(\mathcal{K})$ be an appropriate catalyst state. The above transformation then amounts to a map

$$\sigma \otimes \omega \longmapsto \rho \otimes \omega = \frac{\sum\limits_{i=1}^{K} (A_i \otimes B_i)(\sigma \otimes \omega)(A_i \otimes B_i)^{\dagger}}{\operatorname{tr} \left[\sum\limits_{i=1}^{K} (A_i \otimes B_i)(\sigma \otimes \omega)(A_i \otimes B_i)^{\dagger}\right]}.$$
 (3.78)

Since the map does not have to be trace-preserving, A_i and B_i corresponding to this separable operation satisfy

$$\sum_{i=1}^{K} A_i^{\dagger} A_i \le \mathbb{1}_A, \ \sum_{i=1}^{K} B_i^{\dagger} B_i \le \mathbb{1}_B, \tag{3.79}$$

but not necessarily $\sum_{i=1}^K A_i^{\dagger} A_i = \mathbb{1}_A$ and $\sum_{i=1}^K B_i^{\dagger} B_i = \mathbb{1}_B$. The operation defined by Eq. (3.78) is separable in the sense that A_i and B_i act only in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{K}_A)$ and $\mathcal{S}(\mathcal{H}_B \otimes \mathcal{K}_B)$, respectively. The quantity to be considered is the fidelity

$$F(\rho, |\psi\rangle\langle\psi|) = \langle\psi|\rho|\psi\rangle \tag{3.80}$$

of the normalized final state ρ with respect to $|\psi\rangle\langle\psi|$. This fidelity decreases under convex combination of states, and therefore it suffices to consider the smaller class of

transformations which includes a single Kraus operator $A_i \otimes B_i$ for some i = 1, ..., K. Hence, it is no restriction on generality to investigate the map

$$\sigma \otimes \omega \longmapsto \rho \otimes \omega = \frac{(A_i \otimes B_i)(\sigma \otimes \omega)(A_i \otimes B_i)^{\dagger}}{\operatorname{tr}\left[(A_i \otimes B_i)(\sigma \otimes \omega)(A_i \otimes B_i)^{\dagger}\right]}.$$
 (3.81)

Regarded as a function of λ the fidelity in Eq. (3.80) is given by

$$F(\lambda) = \operatorname{tr}_{\mathcal{K}} \left(\lambda \langle \psi | \left[(A_i \otimes B_i)(|\psi\rangle \langle \psi | \otimes \omega)(A_i \otimes B_i)^{\dagger} \right] |\psi\rangle \right.$$

$$\left. + \left. (1 - \lambda) \langle \psi | \left[(A_i \otimes B_i)(\chi \otimes \omega)(A_i \otimes B_i)^{\dagger} \right] |\psi\rangle \right) / \xi(\lambda) ,$$

where the normalization constant $\xi(\lambda)$ is

$$\xi(\lambda) = \operatorname{tr}\left[(A_i \otimes B_i)((\lambda|\psi\rangle\langle\psi| + (1-\lambda)\tilde{\sigma}) \otimes \omega)(A_i \otimes B_i)^{\dagger} \right] .$$

The second derivative of F with respect to λ can be calculated as

$$\frac{dF^{2}(\lambda)}{d^{2}\lambda} = \frac{\operatorname{tr}_{\mathcal{K}}\langle\psi|\left[(A_{i}\otimes B_{i})((\chi-|\psi\rangle\langle\psi|)\otimes\omega)(A_{i}\otimes B_{i})^{\dagger}\right]|\psi\rangle\,\xi'(\lambda)}{\xi(\lambda)^{3}}.$$
(3.82)

In particular, it is of the form

$$\frac{dF^2(\lambda)}{d^2\lambda} = \frac{\Xi}{\xi(\lambda)^3} \tag{3.83}$$

with a number Ξ independent of λ , and it is safe to argue that the sign of the second derivative of the function

$$f(\lambda) = F(\lambda) - \lambda \tag{3.84}$$

is constant for all $\lambda \in (0,1)$ (as in the case of local operations without a catalyst state [145]). Therefore, the function f must be either convex or concave or linear over the whole interval. At $\lambda=0$, $f(0)\geq 0$ as $f(\lambda)\geq -\lambda$ for all $\lambda\in (0,1)$, and $f(1)\leq 0$. $f(\lambda_0)\leq 0$ follows from the fact that the map Eq. (3.78) cannot transform the state pertaining to λ_0 to an entangled state. Hence,

$$f(\lambda) \le 0 \text{ for all } \lambda \in [\lambda_0, 1),$$
 (3.85)

i.e., the proportion of $|\psi\rangle\langle\psi|$ can only decrease.

3.4.5 Entanglement-Assisted Small Transformations

A transformation from one pure state $|\psi\rangle\langle\psi|$ to another pure state $|\phi\rangle\langle\phi|$ is obviously not always possible under LOCC operations, even if the overlap $|\langle\psi|\phi\rangle|^2$ is arbitrarily close to 1 and the two states are therefore arbitrarily similar as measured by the fidelity. Instead, for a given $|\psi\rangle\langle\psi|$ and for any δ there exist pure states $|\phi\rangle\langle\phi|$ with $|\langle\psi|\phi\rangle|^2>1-\delta$ which cannot be accessed by these operations. One might suspect that if such a "small transformation" is not possible, entanglement-assisted operations might under certain circumstances provide the tools to perform the task, because the resources required would be small in any case.

However, this intuitive approach does not generate a valid statement as will be shown in Proposition 3.16: For a $|\psi\rangle\langle\psi|$ there is a $\delta>0$ such that for all pure states $|\phi\rangle\langle\phi|$ with $|\langle\psi|\phi\rangle|^2>1-\delta$ the presence of catalyst states does not imply an advantage at all, and a transformation under ELOCC is only possible if it can already be realized with LOCC operations. In the mixed state regime things are again different, see Fig. 3.7. Quite surprisingly,

it has become apparent that pure states behave differently from mixed states as far as small transformations are concerned.

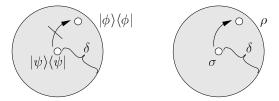


Figure 3.7: A schematic representation of entanglement-assisted small transformations in the pure state case and for mixed states. For a given pure state $\sigma = |\psi\rangle\langle\psi|$ there exists a $\delta>0$ such that for all pure states $\rho = |\phi\rangle\langle\phi|$ with $|\langle\psi|\phi\rangle|^2>1-\delta$ no transformation $\sigma\longrightarrow\rho$ is possible under ELOCC. This is represented in the picture on the left hand side. The other figure depicts a possible transformation in the mixed state case where there are states σ such that for every $\delta>0$ there exist other mixed states with $F(\sigma,\rho)>1-\delta$ and $\sigma\longrightarrow\rho$ under ELOCC.

Proposition 3.16. – For all $|\psi\rangle \in \mathcal{H}$ and all $|\tilde{\psi}\rangle \in \mathcal{K}$ there exists a $\delta > 0$ such that

 $|\psi\rangle\langle\psi|\not\longrightarrow|\phi\rangle\langle\phi|\ under\ LOCC\ \implies\ |\psi\rangle\langle\psi|\otimes|\tilde{\psi}\rangle\langle\tilde{\psi}|\not\longrightarrow|\phi\rangle\langle\phi|\otimes|\tilde{\psi}\rangle\langle\tilde{\psi}|\ under\ LOCC$ for all $|\phi\rangle\in\mathcal{H}$ with $|\langle\psi|\phi\rangle|^2>1-\delta$.

Proof: Let

$$\alpha_1, ..., \alpha_N, \quad N = \dim[\mathcal{H}_A] \tag{3.86}$$

be the ordered lists of eigenvalues of $\operatorname{tr}_A[|\psi\rangle\langle\psi|]$, and let $\gamma_1,...,\gamma_M$, $M=\dim[\mathcal{K}_A]$, be the corresponding list of the pure catalyst state. Let $\varepsilon>0$ and call an ε -list a list

$$\beta_1, ..., \beta_N, \quad 1 \ge \beta_1 \ge ... \ge \beta_N \ge 0$$
 (3.87)

that has the property $|\beta_i - \alpha_i| < \varepsilon$ for all i = 1, ..., N. There exists an $\varepsilon_0 > 0$ such that for all ε -lists $\beta_1, ..., \beta_N$ the statement that $\alpha_i \gamma_j > \alpha_k \gamma_l$ for some i, k = 1, ..., N and j, l = 1, ..., M implies that

$$\beta_i \gamma_i > \beta_k \gamma_l. \tag{3.88}$$

Additionally, there exists a $\delta > 0$ such that for each $|\phi\rangle \in \mathcal{H}$ with

$$|\langle \psi | \phi \rangle|^2 > 1 - \delta, \tag{3.89}$$

the ordered eigenvalues of $\operatorname{tr}_A[|\phi\rangle\langle\phi|]$ form an ε_0 -list – and hence, for such states it is not possible that $\beta_i\gamma_j<\beta_k\gamma_l$ and $\alpha_i\gamma_j>\alpha_k\gamma_l$. It follows that for all such $|\phi\rangle\in\mathcal{H}$ with $|\langle\psi|\phi\rangle|^2>1-\delta$ the majorization relation

$$\operatorname{tr}_{A}[|\psi\rangle\langle\psi|\otimes\omega] \not\prec \operatorname{tr}_{A}[|\phi\rangle\langle\phi|\otimes\omega]$$
 (3.90)

holds if

$$\operatorname{tr}_{A}[|\psi\rangle\langle\psi|] \not\prec \operatorname{tr}_{A}[|\phi\rangle\langle\phi|].$$
 (3.91)

Finally, this implies the statement of Proposition 3.16.

A corresponding statement does not hold for mixed states. The generalized fidelity of two states σ and ρ can, e.g., be taken to be [141, 98]

$$F(\sigma, \rho) = \left(\text{tr} \left[(\sqrt{\sigma} \rho \sqrt{\sigma})^{1/2} \right] \right)^2. \tag{3.92}$$

The statement for mixed states can then be formulated as follows.

Proposition 3.17. – There exist states $\sigma \in \mathcal{S}(\mathcal{H})$ such that for every $\delta > 0$ there exist states $\rho \in \mathcal{S}(\mathcal{H})$ with the property that

$$F(\sigma, \rho) > 1 - \delta \tag{3.93}$$

and $\sigma \not\longrightarrow \rho$ under LOCC, but $\sigma \longrightarrow \rho$ under ELOCC.

Such states can, e.g., be constructed using the class of states defined in Eq. (3.54), Eq. (3.56), and Eq. (3.57). For any given $\delta>0$ there is a sufficiently small $\lambda>0$ such that the fidelity satisfies $F(\sigma,\rho)>1-\delta$. Hence, quite surprisingly, in the case of entanglement manipulations from an initial pure state to a close pure state entanglement-assisted operations do not add any power to LOCC operations. Put differently, there is no catalysis for sufficiently close pure states. Yet, for mixed states there can be catalysis for such close states.

3.5 Concluding Remarks

In this chapter entanglement transformations have been investigated. In particular, it has been the emphasis of this chapter to explore the power of entanglement-assisted manipulation of entangled quantum systems in mixed states. It became apparent that the class of ELOCC operations, despite their counterintuitive character, is superior to mere LOCC operations also in the interior of the state space, for which such strong tools as the majorization criterion of Ref. [42] are not available. It is the hope that these findings contribute significantly to the quest for a better understanding of mixed-state entanglement transformations.

There remain, however, numerous open questions. A general criterion for entanglement transformations to be possible in the mixed state domain would be very useful. The isomorphism between quantum operations and quantum states of bi-partite systems as presented in Ref. [65] (going back to Ref. [149]) could possibly point towards a resolution of the general problem. Also, the characterization of mixing and measurement in terms of majorization relations for the eigenvalues of the involved matrices could be useful [150] when approaching this problem.

There might also be a connection between entanglement-assisted local operations and bound entanglement. As has been mentioned in the first chapter bound entangled states cannot be transformed into free entangled states using LOCC operations. The main implication is as follows: Let $\sigma \in \mathcal{S}(\mathcal{H})$ with $\sigma^{T_A} \geq 0$ on a Hilbert space $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$ of a bi-partite $N \times N$ -system. One can safely say that any state ρ which satisfies $\sigma \longrightarrow \rho$ under LOCC is not distillable [41]. However, for stochastic ELOCC operations one may conjecture the validity of the following statement.

Conjecture 3.18. – There exist states $\sigma \in \mathcal{S}(\mathcal{H})$ and $\omega \in \mathcal{S}(\mathcal{K})$ (\mathcal{H} and \mathcal{K} are appropriate Hilbert spaces of bi-partite systems) satisfying

(i)
$$\sigma^{T_A} \geq 0$$
,

(ii) $\sigma \otimes \omega \longrightarrow \rho \otimes \omega$ under SLOCC, such that ρ is distillable, that is, such that there exists a $K \in \mathbb{N}$ and a state vector $|\psi\rangle$ taken from a 2×2 -dimensional subspace $\mathcal{C} \subset \mathcal{H}^{\otimes K}$ such that

$$\langle \psi | (\rho^{T_A})^{\otimes K} | \psi \rangle < 0. \tag{3.94}$$

This is not the same as activating bound entanglement in the sense of Ref. [82], as here the entanglement of the free entangled catalyst state is not used up in the course of the protocol. A priori it is not clear why such a transformation should not be possible under ELOCC operations. This would imply that bound entanglement could be "unlocked" without using up any free entanglement, that is, it could be transformed into a form of entanglement which can be distilled to singlet form.

Chapter 4

Non-Local Implementation of Joint Unitary Operations

4.1 Introduction

In order to better understand the properties and structure of quantum entanglement the typical class of quantum operations on bi-partite quantum systems has been studied in the previous chapter. In entanglement transformations both the initial state and the desired final state are known, and it is a matter of investigation to see if or to what degree the task of transforming the states into each other can be accomplished with a given physical instrument. But what if the initial and the final state are not known in advance? In particular, joint unitary operations have major practical implications. How can *joint unitary operations* at different nodes be implemented without knowing the involved states? And what resources are needed to perform such a task? In the words of Ref. [151], the problem is to characterize the non-local character of unitary operations. This non-locality is measured in terms of the used resources, namely, shared quantum systems in entangled states and the amount of necessary classical communication.

In mathematical terms, the problem of *non-local implementation* of joint unitary operations can be formulated as follows. Let $\mathcal H$ be the Hilbert space of a multi-partite quantum system (at least bi-partite). For any unitary operator $U:\mathcal H\longrightarrow\mathcal H$ there exists a Hilbert space $\mathcal K$ of another multi-partite quantum system, a state $\omega\in\mathcal S(\mathcal K)$ and a trace-preserving LOCC operation $\mathcal E$ such that

$$\mathcal{E}(\sigma \otimes \omega) = U \sigma U^{\dagger} \otimes \omega' \tag{4.1}$$

for all $\sigma \in \mathcal{S}(\mathcal{H})$, with $\omega' \in \mathcal{S}(\mathcal{K})$. Local quantum operation here means that the quantum operations are performed on those parts of \mathcal{H} and \mathcal{K} that correspond to quantum systems which are physically held by the same parties. The question now is: for a given U, what is the minimal amount of classical communication needed in order to implement \mathcal{E} , and what is the minimal entanglement of the state ω ?

This matter is not only of academic interest. From a practical point of view, the problem is essentially how to implement elementary *quantum gates* in a quantum computer which consists of several parts at remotely located positions. Such a so-called *distributed quantum computer* may have – once realized – several advantages compared to a quantum computer located at a single site. The actual construction of a full large scale quantum computer is hindered by daunting problems. The main obstacle to the experimental realization of such a quantum computer is the process of decoherence [55, 54]. Quantum systems in pure states get entangled with their environment and are degraded in purity, which makes the manipulation and storing of sufficiently many quantum systems [5, 152] extraordinarily difficult.

Leaving the fundamental problems aside, imperfections in applying the appropriate quantum operations cannot be avoided. In a quantum optical implementation, say, one would have to expect fluctuations in timing, length, and intensity of the applied laser pulses. It has been shown that under non-ideal conditions a distributed quantum computer may be superior to a non-distributed quantum computer for certain problems [22, 29].

In this chapter several protocols implementing gates that effect qubits at different nodes will be presented using only LOCC operations and previously shared entanglement. Optimality is measured by the consumption of the basic experimental resources of entanglement and classical communication between nodes. As the emphasis of the chapter is on rather practical issues, all considerations are restricted to the case where the quantum systems at different nodes are qubits. The results which are presented in this chapter can be found in published form in [E4] ¹.

4.2 Quantum Gates

Computation can be made reversible, both logically and thermodynamically [19]. Quantum computation amounts to the implementation of one or more than one unitary quantum operation. In essentially all proposals for quantum computation these unitary operations are built up of elementary quantum operations – or *quantum gates* [154, 18, 20, 4] – which are applied on one, two or more separated quantum systems (for an exception see Ref. [132]). Single qubit gates are unitary operations on \mathbb{C}^2 with basis $\{|0\rangle, |1\rangle\}$ and can be represented by unitary 2×2 matrices. Among the single qubit gates the *Hadamard gate* and the σ_z operation will be used subsequently. The Hadamard gate is defined by the map

$$|0\rangle \longmapsto (|0\rangle + |1\rangle)/\sqrt{2}, \qquad |1\rangle \longmapsto (|0\rangle - |1\rangle)/\sqrt{2}, \tag{4.2}$$

a σ_z operation is specified by

$$|0\rangle \longmapsto |0\rangle, \qquad |1\rangle \longmapsto -|1\rangle.$$
 (4.3)

The quantum *CNOT gate* is an operation on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and acts as

$$|e\rangle_A|f\rangle_B \longmapsto |e\rangle_A|e \oplus f\rangle_B,$$
 (4.4)

where $e, f \in \{0, 1\}$; \oplus denotes addition modulo two. In accordance with their roles in the transformation the first qubit is called *control qubit*, the second *target qubit*. For N qubits acting as nodes, N = 2, 3, ..., let

$$U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} \tag{4.5}$$

be the matrix representation of a unitary operator and let

$$\Lambda_n(U) | x_1, ..., x_{N-1}, y \rangle$$

$$= \begin{cases} u_{y0} | x_1, ..., x_{N-1}, 0 \rangle + u_{y1} | x_1, ..., x_{N-1}, 1 \rangle, & \text{if } x_1 = ... = x_{N-1} = 1, \\ | x_1, ..., x_{N-1}, y \rangle & \text{otherwise.} \end{cases}$$
(4.6)

¹Independently, similar results were published on the eprint server of the Los Alamos National Lab on the same day by D. Collins, N. Linden, and S. Popescu [151]. Also, the results presented in Ref. [153] – an expanded version of a plenary speech given at the 1998 International Conference on Group Theoretic Methods in Physics – imply a scheme for the non-local implementation of a CNOT gate the authors of Ref. [E4] were not aware of.

 $\Lambda_N(U)$ is an N node control-U gate [154]. In matrix form the gate $\Lambda_N(U)$ can be represented as

$$\begin{pmatrix}
1 & & & & & \\
& 1 & & & & \\
& & \dots & & & \\
& & u_{00} & u_{01} \\
& & u_{10} & u_{11}
\end{pmatrix},$$
(4.7)

where again, the basis states are lexicographically ordered. If

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.8}$$

and N=2, then such a control-U gate reduces to the quantum CNOT gate. For N=3 this particular kind of control-U gate is called *Toffoli gate*.

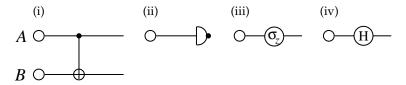


Figure 4.1: A schematic representation of (i) a quantum CNOT gate where A is the control qubit and B is the target. The CNOT maps $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |01\rangle$, $|10\rangle \mapsto |11\rangle$, and $|11\rangle \mapsto |10\rangle$. (ii) is the symbol of a projective selective measurement in the computational basis (with Kraus operators $E_1 = |0\rangle\langle 0|$ and $E_2 = |1\rangle\langle 1|$), (iii) stands for a σ_z gate, and (iv) depicts a Hadamard transformation.

4.3 Implementation of Two-Qubit Gates in Distributed Quantum Computation

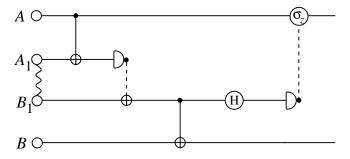


Figure 4.2: A representation of a quantum circuit to perform a quantum CNOT gate with the minimal use of resources. The qubits A and B are the qubits on which the CNOT gate is applied. In addition to A and B Alice and Bob hold the qubits A_1 and B_1 respectively, which are initially in a maximally entangled state. Each dashed line corresponds to a single bit of classical communication.

The first result is concerned with the optimal non-local implementation of a quantum CNOT gate. Optimal means in this context that the least possible resources are used up,

both as far as shared entanglement and classical communication are concerned.

Proposition 4.1. – A single shared ebit and one bit of classical communication in each direction are necessary and sufficient for the non-local implementation of a quantum CNOT gate.

Proof: (i) *Sufficiency*: This statement can be proved by constructing an explicit circuit which performs the CNOT using these resources, see Fig. 4.2. The CNOT is performed between the qubits A and B. Alice holds the qubits A and A_1 , and Bob holds the qubits B and B_1 . The state of A_1 and B_1 will be taken to be a maximally entangled state corresponding to $(|00\rangle_{A_1B_1}+|11\rangle_{A_1B_1})/\sqrt{2}$. The initial state of A is necessarily arbitrary, it can, however, without loss of generality be taken to be a pure state $\alpha|0\rangle_A+\beta|1\rangle_A$. The initial state of B is also arbitrary and is given by $\gamma|0\rangle_B+\delta|1\rangle_B$. First a local CNOT is performed with A as the control and A_1 as the target. After this the combined state² of A, A_1 and B_1 is

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle_{AA_1B_1} + \alpha|011\rangle_{AA_1B_1} + \beta|110\rangle_{AA_1B_1} + \beta|111\rangle_{AA_1B_1}). \tag{4.9}$$

Alice then performs a measurement on A_1 in the computational basis, and the line corresponding to this qubit terminates. The result of the measurement is one bit of information, which is communicated to Bob, and this communication is denoted by the dashed line. If the result is 0 Bob does nothing, and if the result is 1 Bob performs the not operation. At this point the combined state of A and B_1 is $\alpha|00\rangle_{AB_1}+\beta|11\rangle_{AB_1}$. That is, this procedure amounts effectively to the implementation of a CNOT between A and B_1 . Now particle B_1 contains the necessary information about the state of A. One may now perform a CNOT between B_1 and B. The combined state of A, B_1 and B is

$$\frac{1}{\sqrt{2}}(\alpha\gamma|000\rangle_{AB_1B} + \alpha\delta|001\rangle_{AB_1B} + \beta\delta|110\rangle_{AB_1B} + \beta\gamma|111\rangle_{AB_1B})$$
(4.10)

All that needs to be done is to remove B_1 from the state. This is done by performing a Hadamard transformation on B_1 , and then measuring B_1 in the computational basis, at which point the line denoting B_1 terminates. The result of the measurement (one bit) is communicated to Alice. If the result is 0 Alice does nothing, and if the result is 1 she performs a (state-independent) σ_z operation on particle A. This completes the non-local CNOT.

(ii) *Necessity*: The procedure consists of LOCC operations, and hence, all information which has been sent at the end of the operation must have been sent classically. If the target qubit is initialized in the state $|0\rangle$, then its final state will be $|0\rangle$ or $|1\rangle$ depending on the initial state of the control qubit being $|0\rangle$ or $|1\rangle$ respectively. Therefore, the final result of the gate in this case is the communication of one bit of information from Alice (holding the control qubit) to Bob (holding the target qubit). Consequently, in the non-local implementation, one bit of classical information must have been sent classically from Alice to Bob. In order to understand that one bit must also have been sent from Bob to Alice, note that in the basis $|\pm\rangle=(|0\rangle\pm|1\rangle)/\sqrt{2}$ the role of control and target in a CNOT gate are reversed, see Fig. 4.3. Assume that Alice's particle is prepared in the standard state $|+\rangle$ and Bob chooses to prepare his particle either in state $|+\rangle$ or $|-\rangle$. After the application of the CNOT gate, Alice will hold a quantum system which is either in the state $|+\rangle$ or $|-\rangle$ depending on the state Bob's particle has been prepared in. Consequently, one bit of information has been transmitted from

²In this chapter both Hermitian, positive trace-class operators with trace one and state vectors are called states.

Bob to Alice. As the implementation of the CNOT must be independent of the initial state, the procedure has to allow for one bit of communication in each direction. Thus the non-local implementation must involve, as a minimum, one bit of communication in both directions. A CNOT gate acting on the initial state $(|0\rangle_A + |1\rangle_A)|0\rangle_B$ leads to a maximally entangled state $(|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$, which is why at least one ebit is required.

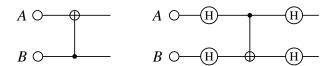


Figure 4.3: Two equivalent formulations of a quantum CNOT gate with qubit A as target and B as control qubit.

A control-U gate can be implemented using one shared ebit and one bit of classical communication in each direction. A control-U gate is defined as a gate that applies the identity on the target qubit if the control bit is in state $|0\rangle$ and it applies the unitary operator U to the target if the control qubit is in state $|1\rangle$. It is possible to use the same quantum circuit as in Fig. 4.2 except that the CNOT gate on Bob's side is replaced by a control-U gate.

In general a single application of a control-U gate cannot be employed to create one ebit from an initial product state of two qubits. Also, the amount of classical information that can be sent from Alice to Bob via a general control-U gate is less than or equal to one bit. In fact, it has been shown very recently that one can implement certain phase gates with a very small amount of initial entanglement [65]. In particular, this proposal provides the first example of a non-local implementation which makes use of a non-integer number of ebits. The basis for this approach is an isomorphism between certain quantum operations and quantum states going back to Ref. [149].

Lemma 4.2. – Two bits of classical communication in both directions and two shared ebits are sufficient for the non-local implementation of a general two-bit gate. This upper bound cannot be made smaller.

Proof: Any operation may be performed by teleporting Alice's state to Bob [24], at which point Bob may locally perform the operation and then teleport the resulting state back to Alice. This procedure requires two bits of communication in each direction and two shared ebits. This bound may be saturated: The *state swapper* mapping unitarily $|\psi\rangle_A\otimes|\phi\rangle_B$ on $|\phi\rangle_A\otimes|\psi\rangle_B$ – that is, applying a permutation operator – requires two bits of classical communication and two ebits of entanglement (see Fig. 4.4): By single use of a state swapper on two qubits two ebits of entanglement can be created. Accordingly, two initial ebits of entanglement are necessary in the local implementation. Using a state swapper of the above type enables Alice to send two bits of classical information to Bob (and vice versa). This can be done using a *dense coding protocol* [25]. It thus becomes apparent that two bits of classical information need to be sent in both directions.

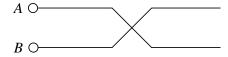


Figure 4.4: Schematic representation of the state swapper.

As any quantum gate array can be constructed from quantum CNOT gates and onequbit gates only [154], there is a realization of a state swapper on two qubits using only those gates. In can be shown that in such an implementation three CNOT gates would be necessary [109] (see Fig. 4.5). This procedure would, however, make use of three ebits of entanglement instead of two.

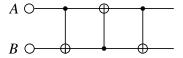


Figure 4.5: A state swapper represented in terms of CNOT gates.

4.4 Implementation of Multi-Qubit Gates

In this subsection, it will be presented how the implementation of gates can be generalized to certain multi-qubit gates, that is, to the realization of gates where more than two parties are involved. At first, the implementation of the Toffoli gate will be investigated. The generalization to multi-party gates will then become evident.

Proposition 4.3. – Two shared ebits and four bits of classical communication are necessary and sufficient for the local implementation of a non-local three-party quantum Toffoli gate.

Proof: (i) *Sufficiency*: Again, a certain circuit will be constructed to perform the task, see Fig. 4.6. Alice, Bob, and Charles initially hold the qubits labeled A, B, and C respectively such that $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. In addition to this, let Alice and Charles share a pair A_1 and C_1 of qubits in a maximally entangled state $|\phi^+\rangle = (|00\rangle_{A_1C_1} + |11\rangle_{A_1C_1})/\sqrt{2}$, and let Bob and Charles share another pair B_1 and C_2 in the same maximally entangled state. The initial state $|\psi\rangle$ of A, B, C, A_1 , B_1 , C_1 and C_2 is then given by

$$|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B \otimes |\varphi\rangle_C \otimes |\phi^+\rangle_{A_1C_1} \otimes |\phi^+\rangle_{B_1C_2}, \tag{4.11}$$

where

$$|\psi\rangle_A = \alpha |0\rangle_A + \beta |1\rangle_A, \ |\psi\rangle_B = \gamma |0\rangle_B + \delta |1\rangle_B, \ |\psi\rangle_C = \eta |0\rangle_C + \xi |1\rangle_C.$$
 (4.12)

The first step is a local quantum CNOT gate on A and A_1 with A as control. Then Alice measures particle A_1 and Charles performs a NOT operation on his particle C_1 if Alice finds 1 in the measurement and the identity if Alice finds 0. Qubit A_1 is then discarded, and Bob applies a local CNOT with B being the control and B_1 being the target. Then Bob measures particle B_1 and Charles performs a NOT operation on his

particle C_2 if Bob finds 1 and the identity if Bob finds 0. Qubit B_1 is then discarded. Now the state of the remaining qubits A, B, C, C_1 and C_2 is given by

$$(\alpha|00\rangle_{AC_1} + \beta|11\rangle_{AC_1}) \otimes (\gamma|00\rangle_{BC_2} + \delta|11\rangle_{BC_2}) \otimes |\psi\rangle_C \tag{4.13}$$

In a further step Charles locally applies a Toffoli with C_1 and C_2 being the control qubits. Charles then applies Hadamard gates to both C_1 and C_2 . He measures C_2 in the computational basis and applies σ_z or the identity $\mathbbm{1}$ to B if his result is 1 or 0, respectively. Finally he measures C_1 and applies σ_z or the identity to A if his result is 1 or 0. This completes the Toffoli gate. The total number of classical bits which have to be communicated is four, and two shared ebits of entanglement are consumed.

(ii) Necessity: Two ebits of entanglement being necessary is taken to mean that the pure initial state of the auxiliary quantum systems with Hilbert space $\mathcal K$ is a product of two maximally entangled states in $\mathbb C^2\otimes\mathbb C^2$ each. Assume first that $|\psi\rangle_A=|1\rangle_A$. Then the initial state

$$|\psi\rangle = |1\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)(\gamma|0\rangle_C + \delta|1\rangle_C) \tag{4.14}$$

is mapped on

$$|\phi\rangle = |1\rangle_A (\alpha\gamma|00\rangle_{BC} + \alpha\delta|01\rangle_{BC} + \beta\gamma|11\rangle_{BC} + \beta\delta|10\rangle_{BC}) \tag{4.15}$$

by application of the three-party quantum Toffoli gate. Hence, in this case, the quantum Toffoli gate amounts to a quantum CNOT between the qubits B and C. According to Proposition 4.1 it follows that a single classical bit of information has to be exchanged in both directions between Alice and Charles. Also, Alice and Charles have to effectively share one ebit of entanglement. That is, the initial state of the auxiliary qubits labeled A_1 and C_1 has to have the property that with local operations and classical communication between Alice, Bob, and Charles this state can be transformed into a maximally entangled state between Alice and Charles with unit probability. The same argument can be applied when initially $|\psi\rangle_B = |1\rangle_B$. Hence, four bits of classical information and two ebits are the necessary resources. 3

Again, these results can be generalized to three-party control-U operations. The local Toffoli gate merely needs to be replaced by a local three-party control-U. Also, on the basis of these findings, a statement about control-U operations involving N parties can be made with $N \geq 3$.

Proposition 4.4. – Two shared ebits and four bits of classical communication are sufficient for the local implementation of a non-local control-U gate of three parties.

³Note that while Alice and Charles on the one hand and Bob and Charles on the other hand have to share effectively a maximally entangled state before the appropriate LOCC operation is performed, the parties Alice, Bob, and Charles do not necessarily have to share the qubits A_1, B_1, C_1 , and C_2 in a state of the form $|\phi^+\rangle_{A_1C_1}\otimes |\phi^+\rangle_{B_1C_2}$ or unitarily equivalent via local unitary operations. It might well be that Bob has an additional qubit B_2 at hand; the initial state of A_1 , B_1 , B_2 , and C_2 could then be $|\phi^+\rangle_{A_1B_1}\otimes |\phi^+\rangle_{B_2C_1}$, as by *entanglement swapping* [155, 156] at Bob's site, Alice and Charles may obtain a maximally entangled state of two qubits.

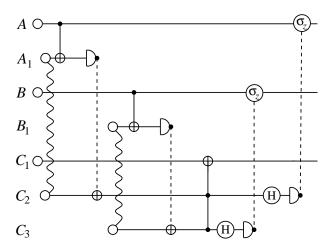


Figure 4.6: A representation of the optimal non-local implementation of a quantum Toffoli gate.

Proposition 4.5. – An N party control-U gate with $N \ge 3$ can be implemented using 2(N-1) bits of classical communication and N-1 shared ebits.

Proof: The N-1 control qubits are denoted by $A_1,...,A_{N-1}$, the control qubit is labeled T, $\mathcal{H}=(\mathbb{C}^2)^{\otimes N}$. Initially there are N-1 shared ebits: In addition to A_i , i=1,...,N-1, the i-th party has the qubit P_i at hand which is maximally entangled with a qubit Q_i held by the N-th party, see Fig. 4.7.

The first N-1 steps of the protocol are essentially analogous to the previous protocol. In the i-th step, i=1,...,N-1, a local quantum CNOT gate is applied on A_i and P_i with A_i as control. Then party i measures particle P_i (again a projective measurement in the computational basis). The target party, the N-th, performs a NOT operation on his ancillary qubit Q_i if the i-th party finds 1 and the identity if the i-th party finds 0 in the measurement. Qubit P_i is afterwards discarded. In the N-th step an N-party control-U gate is performed by the N-th party on $Q_1,...,Q_{N-1}$ and T, where T is the target qubit. Eventually, the target party performs Hadamard gates on each of the qubits $Q_1,...,Q_{N-1}$.

The last N-1 steps again involve measurements, each of them being of the following type: Q_i , i=1,...,N-1 is measured in the computational basis. If the outcome is 1, then σ_z is applied to A_i , if the outcome is 0 then no action is taken. Qubit Q_i may finally be discarded. Hence, the total required resources are 2(N-1) bits of classical information and N-1 initially shared ebits.

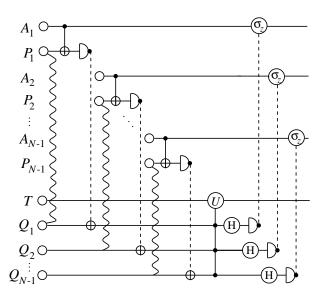


Figure 4.7: Representation of the protocol described in Proposition 4.5.

Quite surprisingly, a mere N-1 ebits are required in this protocol. Obviously, one could also construct a quantum gate array performing an N-party control-U gate by using two-party control-U gates and CNOT gates only, as described in Ref. [154]. In each step, the two-party gates would then have to be realized non-locally. In such a protocol a supply of $3 \times 2^{N-1} - 4$ ebits would be necessary. A more efficient teleportation-based protocol [157] in which the respective states of the qubits at different nodes are twice teleported would still use 2(N-1) ebits and 4(N-1) bits of classical information.

4.5 Concluding Remarks

In all of the above prototols the amount of classical bits that needed to be sent from Alice to Bob and from Bob to Alice was the same. In this sense there was a symmetry in the classical communication cost in the non-local implementation of a joint unitary operation. The same is true for the protocol presented in Ref. [65]. One may wonder whether this symmetry is a general property of such protocols. For a large class of gates it can be proved that this symmetry holds [E4] – in the sense that an implementation can be found for which the minimal amount of transmitted classical information is the same in both directions – but not for all possible unitary operations. Another open question is whether the achievable minimal (average) amount of entanglement that is necessary to implement a gate is always identical to the entanglement that can be maximally created by the gate. In all the above implementations this is the case, but it is not obvious whether it is always possible to find a non-local implementation with this property.

Chapter 5

Entanglement and Classical Information

5.1 Introduction

The amount of entanglement two parties share is related to the knowledge of the parties about the state of the quantum system. If a composite quantum system is in one of a set of possible entangled states, but the parties do not know in which one, it may happen that the entanglement is of no use. In principle, the parties could transform the initial state into an appropriate form, but in order to do so, they would have to know what particular protocol they should implement.

This chapter deals with a relationship between the loss of classical information about a quantum system and the concomitant loss of entanglement. For reasons of simplicity, a particularly clear way of losing classical information will be considered: the classical record of the identity of the quantum systems will be discarded. In the first section simple examples will be presented. In the second section the reflections will be generalized using group theoretical methods. In the last section a number of remarks on bounds for the loss of entanglement due to loss of classical information will be added, and the connection to the hashing inequality will be pointed out.

In order to explain the structure of the problem in a more formal way, assume that two parties, Alice and Bob, prepare n copies of pairs of qubits in a particular pure state. This state is taken to be an entangled state which may be used to implement a protocol for quantum communication. Let us also assume that the following misfortune happens: the parties forget which system is which, such that they do not know any more which qubit is entangled with which, without having manipulated the systems. They may have, for instance, written the relevant classical information on a sheet of paper that has been taken away. After all, there is no way of finding out what pairs of qubits are in the original state.

The important question is whether a system of this type could still be used for the original purpose. The two parties would like to recover as much of the entanglement as possible by operational means, and hence, the appropriate concept to quantify the entanglement is the entanglement of distillation with respect to LOCC operations D_{\leftrightarrow} [31, 64, 91]. The loss of classical information may essentially be regarded as a mixing procedure; this is why the degree of distillable entanglement will decrease. But *how much* distillable entanglement will be lost in the course of the loss of classical information? Is there a general upper bound for $\Delta D_{\leftrightarrow}$, the difference in distillable entanglement in the initial and the final situation?

And is there a connection to the amount of lost classical information ΔI , as quantified by an appropriate functional?

The relevance of these considerations is twofold: Firstly, from the perspective of a theory of entanglement this analysis is useful as the distillable entanglement D_{\leftrightarrow} after the permutation of the qubits can actually be evaluated. This means that these investigations provide one of the very few classes of mixed states for which this quantity can be computed. Secondly, the relevance in the context of channel capacities arises from the fact that the process of losing information can be regarded as the result of the transmission of qubits through a particular noisy quantum channel. The statement of the proposition can then be interpreted as a statement about the quantum capacity of the corresponding quantum channel. Most of the results of this chapter have been published in a shorter form in [E2].

5.2 Examples

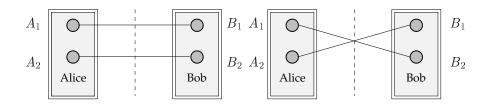


Figure 5.1: The situation in Example 5.1.

Example 5.1. – Consider the situation where Alice and Bob initially share two pairs of qubits each in a maximally entangled state, so that they share two ebits of entanglement. The parties then lose the information about the order of the quantum systems. This means that they do not know whether the two pairs are in the original order or have been permuted (see Fig. 5.1).

Let

$$|\psi\rangle = (|00\rangle_{A_1B_1} + |11\rangle_{A_1B_1}) \otimes (|00\rangle_{A_2B_2} + |11\rangle_{A_2B_2})/2 \tag{5.1}$$

be the state vector of the qubits labeled A_1 , B_1 , A_2 , and B_2 in the original situation (see Fig. 6.1), belonging to Alice and Bob, respectively;

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \quad \text{with} \quad \mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}, \quad \mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2},$$
 (5.2)

 $\mathcal{H}_{A_1}=\mathcal{H}_{A_2}=\mathcal{H}_{B_1}=\mathcal{H}_{B_2}=\mathbb{C}^2.$ In the computational basis $|\psi\rangle$ is given by

$$|\psi\rangle = (|0000\rangle_{A_1B_1A_2B_2} + |0011\rangle_{A_1B_1A_2B_2} + |1100\rangle_{A_1B_1A_2B_2} + |1111\rangle_{A_1B_1A_2B_2})/2, \quad (5.3)$$

while in the permuted case the role of B_1 and B_2 is interchanged,

$$|\phi\rangle = (|0000\rangle_{A_1B_1A_2B_2} + |0110\rangle_{A_1B_1A_2B_2} + |1001\rangle_{A_1B_1A_2B_2} + |1111\rangle_{A_1B_1A_2B_2})/2.$$
 (5.4)

The index indicating which entry corresponds to which quantum system will be omitted in the remainder of this example. As a result of the loss of the record about the order of the particles, the composite quantum system is now described by the mixed state

$$\sigma = \frac{1}{2} \left(|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi| \right). \tag{5.5}$$

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The question that now arises is how much distillable entanglement the state σ holds, i.e., how much entanglement is still accessible to Alice and Bob. In order to approach this problem consider the spectral decomposition of σ ,

$$\sigma = \frac{1}{4} |\phi_1\rangle\langle\phi_1| + \frac{3}{4} |\phi_2\rangle\langle\phi_2|,\tag{5.6}$$

where

$$|\phi_1\rangle = \frac{1}{2}(|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle),$$
 (5.7)

$$|\phi_2\rangle = \frac{1}{\sqrt{12}} (2|0000\rangle + |0011\rangle + |0110\rangle + |1001\rangle + |1100\rangle + 2|1111\rangle).$$
 (5.8)

The structure of the problem will become more transparent if one represents the states in terms of the angular momentum eigenstates. In the basis of angular momentum eigenstates $|j,m\rangle$ with j=0,1, m=-1,0,1,

$$|1, -1\rangle = |00\rangle, \qquad |1, 1\rangle = |11\rangle, \tag{5.9}$$

$$|1,0\rangle = (|01\rangle + |10\rangle)/\sqrt{2}, \qquad |0,0\rangle = (|01\rangle - |10\rangle)/\sqrt{2},$$
 (5.10)

the eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ read

$$|\phi_1\rangle = |0,0\rangle|0,0\rangle, \tag{5.11}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{3}} (|1, -1\rangle|1, -1\rangle + |1, 0\rangle|1, 0\rangle + |1, 1\rangle|1, 1\rangle).$$
 (5.12)

An upper bound for the distillable entanglement is given by the relative entropy of entanglement $E_R(\sigma)$ of σ which in turn is smaller or equal to the relative entropy with respect to any separable state $\rho \in \mathcal{D}(\mathcal{H})$. Hence, the distillable entanglement $D_{\leftrightarrow}(\sigma)$ of σ is bounded by

$$D_{\leftrightarrow}(\sigma) \le S(\sigma||\rho) = \frac{3}{4}\log_2 3,$$
 (5.13)

where the separable state ρ is taken to be

$$\rho = \frac{1}{4} \sum_{j=0}^{1} \sum_{m=-j}^{j} |j, m\rangle |j, m\rangle \langle j, m| \langle j, m|.$$

$$(5.14)$$

Here, the relative entropy of σ with respect to ρ can be evaluated using the direct sum property of the functional (see Appendix A). Surprisingly, the upper bound given in Eq. (5.13) can be achieved. This task may be accomplished by means of the optimal distillation protocol. This distillation protocol involves one-way classical communication only.

1. Alice performs a projective measurement with the two projections

$$A_1 = |0,0\rangle\langle 0,0|, \ A_2 = \sum_{m=-1}^{1} |j=1,m\rangle\langle j=1,m|,$$
 (5.15)

while Bob remains inactive, i.e., $E_1 = A_1 \otimes \mathbb{1}_B$, $E_2 = A_2 \otimes \mathbb{1}_B$. In this measurement, two locally distinguishable subspaces are discriminated (compare also Eq. (3.61)).

- 2. With probability $p_1 = 1/4$ they obtain the normalized output state $|\phi_1\rangle\langle\phi_1|$, which is a product state that has no further use in the distillation protocol.
- 3. The other final pure state of the selective measurement they get with probability $p_2 = 3/4$ is $|\phi_2\rangle\langle\phi_2|$. The entanglement of this state amounts to $\log_2(3)$ ebits.
- 4. The average number of maximally entangled states that can be distilled from σ is given by

$$D_{\to}(\sigma) = D_{\leftrightarrow}(\sigma) = \frac{3}{4} \log_2(3) \approx 1.189.$$
 (5.16)

As this realizes the bound Eq. (5.13) it is the maximally possible value. It is worth noting that this value is greater than one. This means that less than one ebit of entanglement is erased due to the loss of the classical information about the order. The classical information can be taken to be the von Neumann entropy of the mixed state that Alice and Bob share afterwards, i.e.,

$$\Delta I = S(\sigma) = -(1/4)\log_2(1/4) - (3/4)\log_2(3/4) = \frac{3}{4}\log_2(3). \tag{5.17}$$

This choice is justified as – according to Schumacher's noiseless coding theorem [158] – the amount of classical information that can be encoded in the two non-orthogonal states with state vectors $|\psi\rangle$ and $|\phi\rangle$ associated with weights p=1/2 and 1-p=1/2 is given by $S(\sigma)=S((|\psi\rangle\langle\psi|+|\phi\rangle\langle\phi|)/2)$. This quantifies the classical uncertainty about the order of the particles. The change in distillable entanglement is from now on denoted by

$$\Delta D_{\leftrightarrow} = D_{\leftrightarrow}(|\psi\rangle\langle\psi|) - D_{\leftrightarrow}(\sigma). \tag{5.18}$$

From Eq. (5.16) and Eq. (5.17) it follows that

$$\frac{\Delta D_{\leftrightarrow}}{\Delta I} = 1. \tag{5.19}$$

This means that in this case the amount of lost classical information and the loss of distillable entanglement are identical. This is the desired result for this particular case connecting the loss of distillable with the loss of classical information.

Example 5.2. – The above scenario can be generalized to a situation where Alice and Bob initially do not hold quantum systems in maximally entangled states but in arbitrary pure states with a given degree of entanglement. This case is suitable for investigation because it leads to an operationally defined one-parameter class of states for which the distillable entanglement can be analytically computed. This class could provide a useful tool for investigating distillable entanglement. The class of states to be considered is according

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to the Schmidt decomposition given by $\sigma = (|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|)/2$ with

$$\begin{split} |\psi\rangle &= (\sqrt{\alpha}|00\rangle_{A_{1}B_{1}} + \sqrt{\beta}|11\rangle_{A_{1}B_{1}}) \otimes (\sqrt{\alpha}|00\rangle_{A_{2}B_{2}} + \sqrt{\beta}|11\rangle_{A_{2}B_{2}})/2 \\ &= \alpha|0000\rangle_{A_{1}B_{1}A_{2}B_{2}} + \sqrt{\alpha\beta}|0011\rangle_{A_{1}B_{1}A_{2}B_{2}} \\ &+ \sqrt{\alpha\beta}|1100\rangle_{A_{1}B_{1}A_{2}B_{2}} + \beta|1111\rangle_{A_{1}B_{1}A_{2}B_{2}}, \\ |\phi\rangle &= \alpha|0000\rangle_{A_{1}B_{1}A_{2}B_{2}} + \sqrt{\alpha\beta}|0110\rangle_{A_{1}B_{1}A_{2}B_{2}} \\ &+ \sqrt{\alpha\beta}|1001\rangle_{A_{1}B_{1}A_{2}B_{2}} + \beta|1111\rangle_{A_{1}B_{1}A_{2}B_{2}}, \end{split} \tag{5.20}$$

where $\alpha \in [0,1]$, $\beta = 1 - \alpha$. Again, the indices will be omitted subsequently. The spectral decomposition of σ is given by

$$\sigma = \alpha \beta |\phi_1\rangle \langle \phi_1| + (\alpha^2 + \alpha \beta + \beta^2) |\phi_2\rangle \langle \phi_2|, \tag{5.22}$$

where, written in the same basis as above,

$$|\phi_1\rangle = |0,0\rangle|0,0\rangle, \tag{5.23}$$

$$|\phi_2\rangle = \frac{\alpha|1,-1\rangle|1,-1\rangle + \beta|1,1\rangle|1,1\rangle + \sqrt{\alpha\beta}|1,0\rangle|1,0\rangle}{\sqrt{\alpha^2 + \beta^2 + \alpha\beta}}.$$
 (5.24)

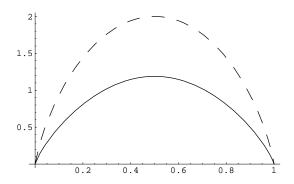


Figure 5.2: The distillable entanglement $D_{\leftrightarrow}(\sigma)$ of σ (solid line) and the entanglement of the initial pure state $D_{\leftrightarrow}(|\psi\rangle\langle\psi|)$ (dashed line) as functions of α in Example 5.2.

Now one can proceed in a similar fashion as before. The upper bound for distillable entanglement, given by the relative entropy with respect to the separable state

$$\rho = \alpha \beta (|0,0\rangle|0,0\rangle\langle0,0|\langle0,0|+|1,01,0\rangle\langle1,0|\langle1,0|) + \beta^{2}|1,1\rangle|1,1\rangle\langle1,1|\langle1,1|+\alpha^{2}|1,-1\rangle|1,-1\rangle\langle1,-1|\langle1,-1|,$$
(5.25)

can again be reached using the protocol of Example 5.1. It follows that the distillable entanglement is in this case given by

$$D_{\leftrightarrow}(\sigma) = E_R(\sigma) = (1 - \alpha\beta)\log(1 - \alpha\beta) - (\alpha^2\log(\alpha^2) + \beta^2\log_2(\beta^2) + \alpha\beta\log_2(\alpha\beta)),$$
 (5.26)

see Fig. 5.2. The entanglement of the initial pure state was given by the entropy of the reduced states of Alice or Bob, that is, by

$$D_{\leftrightarrow}(|\psi_1\rangle\langle\psi_1| = -2(\alpha\log_2(\alpha) + \beta\log_2(\beta)),\tag{5.27}$$

and since $S(\sigma) = -2(\alpha \log_2(\alpha) + \beta \log_2(\beta))$, again,

$$\frac{\Delta D_{\leftrightarrow}}{\Delta I} = 1 \tag{5.28}$$

for all $\alpha \in [0, 1]$. Of course, Eq. (5.26) reduces to $D_{\leftrightarrow}(\sigma) = (3/4) \log_2(3)$ for $\alpha = \beta = 1/2$.

Example 5.3. – Assume that Alice has merely a single qubit A at hand, while Bob possesses two qubits B_1 and B_2 ; $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$, $\mathcal{H}_A = \mathcal{H}_{B_1} = \mathcal{H}_{B_2} = \mathbb{C}^2$. Let initially, Alice and Bob share one pair of qubits in a maximally entangled state, Bob's second qubit is in the state with state vector $|0\rangle_{B_2}$. The state of A, B_1 , and B_2 after permutation can then be taken to be $\sigma = (|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)/2$, with

$$|\psi_1\rangle = (|000\rangle_{AB_1B_2} + |110\rangle_{AB_1B_2})/\sqrt{2},$$
 (5.29)

$$|\psi_2\rangle = (|000\rangle_{AB_1B_2} + |101\rangle_{AB_1B_2})/\sqrt{2}.$$
 (5.30)

Similarly as before, one may obtain a constructive method of distillation by investigating the spectral decomposition of σ which reads

$$\sigma = \frac{1}{4} |\phi_1\rangle \langle \phi_1| + \frac{3}{4} |\phi_2\rangle \langle \phi_2|, \tag{5.31}$$

where

$$|\phi_1\rangle = (|110\rangle - |111\rangle)/\sqrt{2},\tag{5.32}$$

$$|\phi_2\rangle = (2|000\rangle + |110\rangle + |111\rangle)/\sqrt{6}.$$
 (5.33)

In the basis

$$|\hat{0}\rangle = |00\rangle, \ |\hat{1}\rangle = |01\rangle, \ |\hat{2}\rangle = (|10\rangle - |11\rangle)/\sqrt{2}, \ |\hat{3}\rangle = (|10\rangle + |11\rangle)/\sqrt{2},$$
 (5.34)

satisfying $|\langle \hat{i}|\hat{j}\rangle|^2=\delta_{ij}$ for i,j=0,1,2,3, Eqs. (5.32) and (5.33) become

$$|\phi_1\rangle = |1\hat{2}\rangle, \tag{5.35}$$

$$|\phi_2\rangle = (\sqrt{2}|0\hat{0}\rangle + |1\hat{3}\rangle)/\sqrt{3}. \tag{5.36}$$

Hence, in this scenario

$$\Delta I = 2 - \frac{3}{4} \log_2(3), \quad \Delta D_{\leftrightarrow} = \frac{3}{4} \log_2(3) - \frac{1}{2}$$
 (5.37)

and therefore $\Delta D_{\leftrightarrow}/\Delta I < 1$. This means that the loss of classical information and the loss of distillable entanglement are not the same any more, but $\Delta D_{\leftrightarrow}$ is still bounded by above by the loss of classical information ΔI .

5.3 A General Result

In this section a generalized result will be proved. Alice and Bob hold n pairs of qubits in any pure state. Then the classical record about the order of the particles is lost. It will be shown that the distillable entanglement can be evaluated exactly for any state and any number of copies. The value of D_{\leftrightarrow} does not vanish in the limit of infinitely many copies. For any number of copies some entanglement can be recovered, and the distillable entanglement grows monotonically with the number of copies.

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The permutation of the quantum systems is incorporated by applying elements of the symmetric group S_n on the state. A $\pi \in S_n$ acts as

$$\pi |\psi\rangle_1 \otimes \dots \otimes |\psi\rangle_n = |\psi\rangle_{\pi(1)} \otimes \dots \otimes |\psi\rangle_{\pi(n)}. \tag{5.38}$$

Again, π denotes a permutation of degree n and at the same time the associated unitary. The subsequent proposition is the general statement in case Alice and Bob are initially sharing pairs of qubits in a maximally entangled state. First, the case will be considered where Alice and Bob hold an even number n of pairs of qubits, such that the Hilbert space of the composite system is given by

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B, \quad \mathcal{H}_A = \mathcal{H}_B = (\mathbb{C}^2)^{\otimes 2J},$$
 (5.39)

with an appropriate J. In Proposition 5.6 this restriction will be given up.

Proposition 5.4. – Let Alice and Bob share n=2J pairs of qubits each in the same maximally entangled state, such that the initial state of the composite system is given by $|\psi\rangle\langle\psi|^{\otimes n}$, $|\psi\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$. The associated Hilbert space is $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, where $\mathcal{H}_A=\mathcal{H}_B=(\mathbb{C}^2)^{\otimes n}$, J=1,2,... Both parties then lose all classical information about the order of their n particles, such that the state of the composite system becomes

$$\sigma = \sum_{\pi_A, \pi_B \in S_n} (\pi_A \otimes \pi_B) |\psi\rangle \langle \psi|^{\otimes n} (\pi_A \otimes \pi_B).$$
 (5.40)

The distillable entanglement of the state σ can then be calculated as

$$D_{\leftrightarrow}(\sigma) = \sum_{j=0}^{J} d_j^2 p_j \log(2j+1),$$
 (5.41)

where

$$d_j = \frac{2j+1}{2J+1} \binom{2J+1}{J-j} \tag{5.42}$$

and

$$p_j = (2j+1)/(2^{2J}d_j). (5.43)$$

The ratio between the change of distillable entanglement $\Delta D_{\leftrightarrow}$ and the amount of erased information $\Delta I = S(\sigma)$ for any J = 1, 2, ... obeys the inequality

$$\frac{\Delta D_{\leftrightarrow}}{\Delta I} \le 1,\tag{5.44}$$

with equality for J=1.

The following considerations will prepare the proof of this proposition. The Hilbert space corresponding to Alice's system, $\mathcal{H}_A=(\mathbb{C}^2)^{\otimes n}$, can be decomposed into a direct sum according to

$$\mathcal{H}_A = \bigoplus_j \mathcal{H}_A^{(j)} = \bigoplus_j \mathcal{M}_A^{(j)} \otimes \mathcal{K}_A^{(j)}.$$
 (5.45)

(see also Ref. [148]). Here, $\mathcal{K}_A^{(j)}$ is a multiplicity space. It carries a representation of the symmetric group S_n . Let $\pi_A \in S_n$, and denote the corresponding unitary on \mathcal{H}_A also by π_A . Then

$$\pi_A = \bigoplus_{i} \mathbb{1} \otimes \lambda_A^{(j)}(\pi_A), \tag{5.46}$$

where $\lambda_A^{(j)}$ is a suitable *unitary irreducible representation* of S_n . $\mathcal{M}_A^{(j)}$ carries the representation of SU(2). Let $U_A \in \mathrm{SU}(2)$, then the n-fold tensor product $U_A^{\otimes n}$ reads as

$$U_A^{\otimes n} = \bigoplus_j D_A^{(j)}(U_A) \otimes \mathbb{1}, \tag{5.47}$$

where $D_A^{(j)}$ is a spin-j irreducible representation of the group SU(2). For the relationship between the symmetric group S_n and SU(2) see, e.g., Refs. [159, 160]. The Hilbert space \mathcal{H}_B of Bob's quantum systems can be decomposed in a fully analogous fashion.

To be more specific, it is helpful to specify a particular basis for the sets $\mathcal{H}_A^{(j)} = \mathcal{M}_A^{(j)} \otimes \mathcal{K}_A^{(j)}$. Let – as in Ref. [147] –

$$|j, m, 1\rangle = |j, m\rangle \otimes \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)^{\otimes (J-j)}$$
 (5.48)

for j=1,...,J, where $|j,m\rangle$ is the state of 2j qubits with a fixed value of j and m with j-m qubits in $|0\rangle$. $|j,m,\alpha_j\rangle$ with $\alpha_j=1,...,d_j$ and m=-j,...,j are then constructed using permutation operators as

$$|j, m, \alpha_j\rangle = \sum_i \eta_i \pi_A^{(i)} |j, m, 1\rangle \tag{5.49}$$

(compare Eq. (5.46)), where $\eta_j \in [0, 1], \pi_A^{(i)} \in S_n, i = 1, 2, ...$, such that the set

$$\{|j, m, 1\rangle, \dots, |j, m, \alpha_j\rangle\} \tag{5.50}$$

forms an orthonormal set. The degeneracy (the dimension of the multiplicity space) is given by

$$d_j = \frac{2j+1}{2J+1} \binom{2J+1}{J-j}. \tag{5.51}$$

The state given by Eq. (5.40) can be further specified according to Lemma 5.5. It makes use of *Schur's second lemma* [159, 160]. If the matrices D(g) are the irreducible representations of a group G, and if

$$[A, D(g)] = 0 (5.52)$$

for all $g \in G$, then $A = \text{const} \times \mathbb{1}$.

Lemma 5.5. – Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B = (\mathbb{C}^2)^{\otimes 2J}$, n = 2J. $\sigma = |\phi\rangle\langle\phi|$, where

$$|\phi\rangle = \sum_{j=0}^{J} \sum_{m=-j}^{j} \sum_{\alpha_{j}=1}^{d_{j}} \frac{|j, m, \alpha_{j}\rangle|j, m, \alpha_{j}\rangle}{\sqrt{2^{n}}}.$$
 (5.53)

Then the state

$$\sigma = \sum_{\pi_A, \pi_B \in S_n} (\pi_A \otimes \pi_B) |\phi\rangle \langle \phi | (\pi_A \otimes \pi_B). \tag{5.54}$$

can be written as

$$\sigma = \sum_{j=0}^{d_j} \sum_{\alpha_j, \beta_j=1}^{d_j} p_j |\psi_j(\alpha_j, \beta_j)\rangle \langle \psi_j(\alpha_j, \beta_j)|,$$
 (5.55)

where

$$|\psi_j(\alpha_j, \beta_j)\rangle = \sum_{m=-j}^j \frac{|j, m, \alpha_j\rangle|j, m, \beta_j\rangle}{\sqrt{2j+1}}$$
(5.56)

5.3. A General Result 93

and

$$p_j = \frac{2j+1}{2^{2J}d_j}. (5.57)$$

Proof: Before applying the permutation operators, the reduced state of Alice is given by

$$\operatorname{tr}_{B}[|\phi\rangle\langle\phi|] = \sum_{j=0}^{J} \sum_{j'=0}^{J} \sum_{m=-j}^{j} \sum_{m'=-j'}^{j'} \sum_{\alpha_{j}=1}^{d_{j}} \sum_{\alpha'_{i'}=1}^{d_{j'}} \frac{|j,m,\alpha_{j}\rangle\langle j',m',\alpha'_{j'}|}{2^{n}}.$$
 (5.58)

After applying of the permutation operators the reduced state of Alice,

$$\operatorname{tr}_{B}[\sigma] = \sum_{\pi_{A} \in S_{n}} \pi_{A} \operatorname{tr}_{B}[|\phi\rangle\langle\phi|] \pi_{A}, \tag{5.59}$$

commutes with all $\pi_A \in S_n$ by construction. Hence, Schur's Lemma can be applied with respect to all spin-j irreducible representations, j=0,...,J. In accordance with Schur's Lemma $\operatorname{tr}_B[|\phi\rangle\langle\phi|]$ can be written in the form

$$\operatorname{tr}_{B}[\sigma] = \sum_{j=0}^{J} \sum_{m=-j}^{j} \sum_{m'=-j'}^{j'} \sum_{\alpha_{j}=1}^{d_{j}} p_{j}|j, m, \alpha_{j}\rangle\langle j, m', \alpha_{j}|,$$
 (5.60)

with $p_j = ((2j+1))2^{2J})/d_j$. Bob's reduced state $\operatorname{tr}_A[\sigma]$ is identical to $\operatorname{tr}_B[\sigma]$. The full final state σ can be identified by investigating the structure of $|\phi\rangle\langle\phi|$. Since

$$|\phi\rangle = \sum_{j=0}^{J} \sum_{m=-j}^{j} \sum_{\alpha_j=1}^{d_j} \frac{|j, m, \alpha_j\rangle|j, m, \alpha_j\rangle}{\sqrt{2^n}},$$
(5.61)

the final state σ is given by Eq. (5.55).

This partial result can be used to tackle the statement of the proposition:

Proof: (Proposition 5.4) It may initially be observed that $|\psi\rangle^{\otimes n}$ can be written in the form

$$|\psi\rangle^{\otimes n} = |\phi\rangle = \sum_{j=0}^{J} \sum_{m=-j}^{j} \sum_{\alpha_j=1}^{d_j} \frac{|j, m, \alpha_j\rangle|j, m, \alpha_j\rangle}{\sqrt{2^n}}$$
 (5.62)

as in Lemma 5.5. It is thus possible to apply Lemma 5.5 in order to find out the particular form of the state after applying the permutation operators: the form is given by Eqs. (5.55), (5.56), and (5.57). The next step is constructing the optimal entanglement distillation protocol. As before, the following distillation protocol is based on the fact that the subspaces of the state space corresponding to the above components of the underlying Hilbert space are locally distinguishable. Interestingly, this protocol is related to the algorithm proposed in Ref. [147] for the optimal purification of qubits.

- 1. Alice starts the protocol by implementing a local projective measurement. This measurement is designed in order to project the reduced state belonging to her system A on one of the orthogonal subspaces with fixed j and α_j for some j=0,...,J and $\alpha_j=1,...,d_j$.
- 2. In this measurement she will obtain a value of $\alpha_j=1,...,d_j$. If $\alpha_j\neq 1$, she further applies a local unitary operation U_{j,α_j}^A such that her reduced state is included in the subspace of the state space belonging to $\alpha_j=1$. In general, $|j,m,\alpha_j\rangle$ can be written in the form

$$|j, m, \alpha_j\rangle = \sum_i \eta_i(\pi_A^{(i)} \otimes \mathbb{1}_B)|j, m, 1\rangle,$$
 (5.63)

where $\eta_i \in [0,1]$. $\pi_A^{(i)} \in S_n$, i=1,2,..., are appropriate permutation operators acting in \mathcal{H}_A only. That is, $|j,m,\alpha_j\rangle$ is a linear superposition of $\pi_A^{(i)}|j,m,1\rangle$. It is for this reason that the task of this step can always be performed.

3. The reduced state σ_A of Alice is at this stage of the structure

$$\sigma_A = \omega_A \otimes \left(\frac{(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)}{2}\right)^{\otimes (J-j)}.$$
 (5.64)

The last J-j pairs of qubits in the singlet state are neither entangled with the other qubits on her side nor entangled with any of Bob's qubits. They will thus be of no further use in the remainder of the protocol.

- 4. Bob performs a local measurement projecting his reduced state on one of the subspaces associated with $\mathcal{H}_{k,\beta_k}^B$ for a k=0,1,...,J and a $\beta_k=1,2,...,d_k$. Due to the particular form of the initial state he will obtain the value k=j, but he may get a β_j different from α_j .
- 5. Equal to the precedent protocol Bob applies a local unitary operation U_{j,β_j}^B such that his reduced state is included in the subspace of the state space belonging to $\beta_i = 1$. The structure of the state is depicted in Fig. 5.3.
- 6. Alice and Bob attain with probability

$$d_j^2 p_j = \frac{(2j+1)d_j}{2^{2J}} \tag{5.65}$$

one of the pure states $|\psi_i\rangle\langle\psi_i|$, where

$$|\psi_j\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} |j, m, 1\rangle |j, m, 1\rangle.$$
 (5.66)

This state contains $\log_2(2j+1)$ ebits of entanglement.

7. The total average number of ebits achieved in this protocol is

$$\sum_{j=0}^{J} d_j^2 p_j S(\operatorname{tr}_A[|\psi_j\rangle \langle \psi_j|]) = \sum_{j} d_j^2 p_j \log(2j+1).$$
 (5.67)

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In order to show that the above protocol is actually optimal, the relative entropy functional of the state σ after permutation with respect to an appropriate separable state ρ will be calculated. The separable state ρ is taken to be $\rho = \sum_{j=0}^{J} p_j \rho_j$, where

$$\rho_{j} = \sum_{\alpha_{j},\beta_{j}=1}^{d_{j}} \sum_{m=-j}^{j} \frac{|j,m,\alpha_{j}\rangle\langle j,m,\alpha_{j}| \otimes |j,m,\beta_{j}\rangle\langle j,m,\beta_{j}|}{2j+1}.$$
 (5.68)

All subspaces associated with different values of j, m, α_j , and β_j are orthogonal. Therefore, one may use the direct sum property of the relative entropy to achieve

$$S(\sigma||\rho) = \sum_{j=0}^{J} d_j^2 p_j \log(2j+1).$$
 (5.69)

This is identical to the value given for the average number of maximally entangled states obtained when applying the above procedure. It is therefore also identical to the distillable entanglement $D_{\leftrightarrow}(\sigma)$ with respect to LOCC operations. As $\Delta I = S(\sigma) = -\sum_{j=0}^J p_j \log_2(p_j)$, it follows that $\Delta D_{\leftrightarrow}/\Delta I \leq 1$ for all n for this particular initial state.

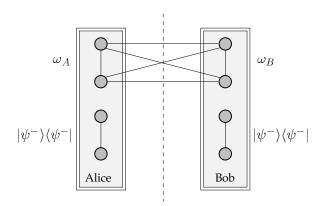


Figure 5.3: The structure of the state after step 5. in the above protocol. This figure corresponds to n=4, J=2, and j=1. Both Alice and Bob hold at this stage a pair of qubits in the singlet state $|\psi^-\rangle\langle\psi^-|$ with state vector $|\psi^-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}$, which can be discarded. ω_A and ω_B denote the reduced state of Alice and Bob, respectively, of the first two pairs of qubits.

Proposition 5.6. – Let Alice and Bob share n pairs of qubits each in the same pure state with a state vector $|\psi\rangle$. The associated Hilbert space is $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, where $\mathcal{H}_A=\mathcal{H}_B=(\mathbb{C}^2)^{\otimes n}$. Both parties then lose all classical information about the order of their n particles, such that the state of the composite system becomes

$$\sigma = \sum_{\pi_A, \pi_B \in S_n} (\pi_A \otimes \pi_B) |\psi\rangle\langle\psi|^{\otimes n} (\pi_A \otimes \pi_B).$$
 (5.70)

The distillable entanglement of the state σ can then be calculated exactly, and the ratio between the change of distillable entanglement $\Delta D_{\leftrightarrow}$ and the amount of erased information $\Delta I = S(\sigma)$ for any n = 1, 2, ... obeys the inequality

$$\frac{\Delta D_{\leftrightarrow}}{\Delta I} \le 1,\tag{5.71}$$

with equality for n=2.

Proof: One can set according to the Schmidt decomposition

$$|\psi\rangle = \sqrt{\alpha}|00\rangle + \sqrt{\beta}|11\rangle,\tag{5.72}$$

with $\alpha \in [0,1]$, $\beta = 1-\alpha$. Assume that n=2J with a J=1,2,... It is then possible to pursue the same argument as in the preceding case, and the same protocol is optimal for distillation with respect to separable operations. The state σ after permutation will be found to be

$$\sigma = \sum_{j=0}^{J} p_j |\psi_j(\alpha_j, \beta_j)\rangle \langle \psi_j(\alpha_j, \beta_j)|$$
 (5.73)

with

$$p_j = \sum_{m=-j}^{j} \frac{\alpha^{(J-m)} \beta^{(J+m)}}{d_j}.$$
 (5.74)

The unnormalized state vectors $|\psi_j(\alpha_j,\beta_j)\rangle\langle\psi_j(\alpha_j,\beta_j)|$ are defined as

$$|\psi_j(\alpha_j, \beta_j)\rangle = \sum_{m=-j}^j \alpha^{j-m} \beta^{j+m} |j, m, \alpha_j\rangle |j, m, \beta_j\rangle.$$
 (5.75)

The distillable entanglement is given by

$$D_{\leftrightarrow}(\sigma) = \sum_{j=0}^{J} d_j^2 p_j S(\operatorname{tr}_A[|\psi_j(1,1)\rangle\langle\psi_j(1,1)|])$$
(5.76)

and $\Delta I = S(\sigma)$. In addition to this, there are initially

$$D_{\leftrightarrow}(|\phi\rangle\langle\phi|^{\otimes n}) = -n(\alpha\log_2(\alpha) + \beta\log_2(\beta)) \tag{5.77}$$

ebits of entanglement, so that it may be concluded that $\Delta D_{\leftrightarrow}/\Delta I \leq 1$ for all n holds for this case as well. The case that n=2J+1 with a J=1,2,... can be analyzed in an analogous fashion, and again, one obtains $\Delta D_{\leftrightarrow}/\Delta I \leq 1$. In the particular case that n=2 it follows that $\Delta D_{\leftrightarrow}=\Delta I$.

This proposition provides a general statement in the case that the lost classical information is the information about the order of several qubits. If a certain number of bits of classical information disappears, not more than the same amount of distillable entanglement is lost – as measured in ebits. Again, it should be emphasized that this setting can be interpreted in terms of channel capacities [90, 108].

5.4 Concluding Remarks

Motivated by these findings one may conjecture that the inequality $\Delta D_{\leftrightarrow}/\Delta I \leq 1$ holds in general, also if the loss of classical information is not due to the loss of a classical record about the identity of a number of qubits.

Conjecture 5.7. – Let $\sigma_1, ..., \sigma_n \in \mathcal{S}(\mathcal{H})$ be pure states, each one of which is assigned a classical probability $p_1, ..., p_n$. Let

$$\Delta D_{\leftrightarrow} = \sum_{i=1}^{n} p_i D_{\leftrightarrow}(\sigma_i) - D_{\leftrightarrow} \left(\sum_{i=1}^{n} p_i \sigma_i \right), \tag{5.78}$$

$$\Delta I = S(\sum_{i=1}^{n} p_i \sigma_i). \tag{5.79}$$

Then the change in distillable entanglement $\Delta D_{\leftrightarrow}$ and the loss of classical information ΔI obey the inequality

$$\frac{\Delta D_{\leftrightarrow}}{\Delta I} \le 1. \tag{5.80}$$

The problem can be considerably simplified if not the distillable entanglement D_{\leftrightarrow} but the relative entropy of entanglement E_R is taken to be the figure of merit instead of the distillable entanglement (see also Ref. [161] and Ref. [109]).

Proposition 5.8. – Let $\sigma_1, ..., \sigma_n \in \mathcal{S}(\mathcal{H})$ be pure states, each one of which is assigned a classical probability $p_1, ..., p_n$. Let

$$\Delta E_R = \sum_{i=1}^n p_i E_R(\sigma_i) - E_R\left(\sum_{i=1}^n p_i \sigma_i\right), \tag{5.81}$$

$$\Delta I = S(\sum_{i=1}^{n} p_i \sigma_i). \tag{5.82}$$

Then the change in relative entropy of entanglement ΔE_R and the loss of classical information ΔI obey the inequality

$$\frac{\Delta E_R}{\Delta I} \le 1. \tag{5.83}$$

Proof: The functional $f: \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \longrightarrow \mathbb{R}^+$ defined as $f(\sigma||\rho) = S(\sigma||\rho) + S(\sigma)$ is linear in its first argument. Let $\omega \in \mathcal{D}(\mathcal{H})$ be a state satisfying

$$S(\sum_{i=1}^{n} p_i \sigma_i || \omega) = \min_{\rho \in \mathcal{D}(\mathcal{H})} S(\sum_{i=1}^{n} p_i \sigma_i || \rho),$$
(5.84)

then

$$f(\sum_{i=1}^{n} p_i \sigma_i || \omega) = E_R(\sum_{i=1}^{n} p_i \sigma_i) + S(\sum_{i=1}^{n} p_i \sigma_i) = \sum_{i=1}^{n} p_i S(\sigma_i || \omega)$$

$$\geq \sum_{i=1}^{n} p_i E_R(\sigma_i), \qquad (5.85)$$

and therefore
$$\sum_{i=1}^{n} p_i E_R(\sigma_i) - E_R(\sum_{i=1}^{n} p_i \sigma_i) \leq S(\sum_{i=1}^{n} p_i \sigma_i)$$
.

The statement of Conjecture 5.7 is intimately related to another inequality which is highly relevant in the context of the *quantum capacity of a quantum channel*. In Ref. [90] Conjecture 5.7 has been discussed with reference to Ref. [E2]. It has therein been demonstrated that if the *hashing inequality*

$$D_{\to}(\rho) \ge I_B(\rho) \tag{5.86}$$

holds for all states $\rho \in \mathcal{S}(\mathcal{H})$, then Conjecture 5.7 is true as well. In inequality (5.86), $D_{\rightarrow}(\rho)$ denotes the distillable entanglement of ρ with respect to one-local operations with classical communication from Alice to Bob, and $I_B(\rho)$ is the *coherent information* of ρ with respect to Bob, that is,

$$I_B(\rho) = S(\operatorname{tr}_B[\rho]) - S(\rho). \tag{5.87}$$

It is in fact one of the major issues of quantum information theory to prove the validity of the hashing inequality. With this inequality at hand, the quantum channel capacity problem would essentially be solved. This astonishing fact has been pointed out in Ref. [90], which presents an elegant unifying approach to several quantum channel capacity problems. However, the hashing inequality is already known to be true for a rather large class of states. These are the pure states, mixtures of two Bell states, and all states for which the distillable entanglement is equal to the entanglement of formation [90, 109].

Chapter 6

Quantum Information and Game Theory

6.1 Introduction

Game theory is the theory of conflict between rationally acting opponents. The subject of game theory is the mathematical analysis of situations in which several parties are interested only in winning and make decisions according to their personal interest, situations that involve contest, rivalry, or struggle. The fundamentals of game theory were laid by Émile Borel, and, first and foremost, by the mathematician John von Neumann and the economist Oskar Morgenstern, who realized that several problems in economic behavior resemble in structure the mathematical notions of "games of strategy". Since the appearance of their seminal work (Ref. [162]) the theory has developed rapidly and has found manifold applications in the social sciences, biology, or economics. Of particular interest to the theory are *games of incomplete information* in which the parties may choose their plan of action with complete knowledge of the situation on rational grounds, but without knowing what decision the other parties would actually take¹. This chapter is devoted to the idea of identifying strategic moves in the sense of game theory with quantum operations as introduced in Refs. [E6] and [164]. This approach appears to be fruitful in several ways:

Firstly, some recently proposed applications of quantum information theory can already be conceived as competitive situations in which several parties – or players – interact with more or less opposed motives. Quantum cloning has been formulated as a game between two players [165]. Eavesdropping in quantum cryptography [16, 166] can equally be regarded as a game between the eavesdropper and the sender, and there are similarities of the extended form of quantum versions of games and quantum algorithms [21, 3].

Secondly, a generalization of the theory of decisions into the domain of quantum probabilities seems interesting, because the roots of game theory are partly in classical probability theory. In this context the question to be addressed would be what solutions could be attained if superpositions of strategies are allowed for.

¹Jacob Brownowski, a later co-worker of von Neumann, recalls a conversation with von Neumann during a taxi ride in which he explained his understanding of a game: "I naturally said to him, since I am an enthusiastic chess player, 'You mean, the theory of games like chess.' 'No, no,' he (von Neumann) said. 'Chess is not a game. Chess is a well-defined form of computation. You might not be able to work out the answers, but in theory there must be a solution, a right procedure in any position. Now real games,' he said, 'are not like that at all. Real life is not like that. Real life consists of bluffing, of little tactics of deception, of asking yourself what is the other man going to think I mean to do. And this is what games are about in my theory'." Cited after Ref. [163].

Thirdly, while game theory rarely deals with the transmission of information explicitly, it should nevertheless be noted that the practical implementation of any (classical) game inevitably involves the exchange of information by classical means. Bearing this in mind, it becomes legitimate to ask what happens if the carriers of information are taken to be quantum systems. ²

6.2 Game Theory

To start off, a few concepts will be summarized which are essential for the later considerations. As for a comprehensive general introduction to game theory the reader may be referred to Refs. [181, 182], which give an exhaustive overview. Ref. [183] is a non-technical introduction to game theory, and a text furnished with biographical remarks on John von Neumann can be found in Ref. [163].

Consider the following situation: two players, Alice and Bob, simultaneously place a penny on a table. The pennies can only show heads or tails up. If the pennies match, that is, if both heads or both tails are up, Alice is allowed to keep both pennies, if they do not match, Bob gets both pennies. The pay-off of each player for different outcomes is displayed in Fig. 6.1. Each player tries to maximize his or her final pay-off. What are they best advised to do?

		Bob		
		Heads	Tails	
Alice	Heads	(1,-1)	(-1,1)	
	Tails	(-1,1)	(1,-1)	
	1 4110	. , ,		

Figure 6.1: The pay-off table in the so-called *Matching Pennies game*. The first entry refers to Alice's pay-off, the second to Bob's.

This very simple example demonstrates the typical structure of a game. A *game* involves several *players* – at least two – who implement one out of several possible strategies, that is, make a decision. Many aspects of game theory can already be understood when only two players are involved. For reasons of simplicity, the following investigation will be restricted to the case of a *two-player game*. A *strategy* is a complete plan of action. Depending on the choices of both players they achieve a certain *pay-off*. More formally, a (strategic-form) two-player game $\Gamma = (\{A, B\}, S_A, S_B, u_A, u_B)$ is fully defined by the *set of players*, the *sets of strategies* S_A and S_B , the *utility functions* u_A and u_B defined on $S_A \times S_B$ specifying the payoff for each player, and additional rules of the game consistent with the set of strategies.

²Game theoretical settings in the quantum domain have first been considered in Refs. [E6] and [164]. In Ref. [164] an elegant quantum analogue of the Penny Flip game has been introduced, and the connection to quantum algorithms has been sketched. Some popular articles about the issues raised in Refs. [E6] and [164] can be found in Refs. [167, 168, 169, 170, 171]. In the meantime a series of other articles has appeared on investigations of such quantum games: in Ref. [172] (see also [173]) an alternative quantization scheme has been proposed. Multi-player extensions have been introduced and investigated in Refs. [174, 175, 176, 177]. Ref. [178] deals with the so-called Monty Hall problem. Evolutionary games have been considered in Ref. [179]. The connection between quantum algorithms and game theory has been further developed in Ref. [180], in which has been shown that a quantum algorithm for an oracle problem may be understood as a quantum strategy for a player in a certain game. This chapter deals with the results presented in Refs. [E5] and [E6].

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The *pay-off* provides a quantitative characterization of their individual preferences. Both players are assumed to want to maximize their respective pay-off, yet they must pick their choice without knowing the other player's decision.

The most important solution concept for a game of this type is the *Nash equilibrium*. A pair of strategies (s_A, s_B) is said to be a (Nash) equilibrium in *pure strategies* if

$$u_A(s_A, s_B) \geq u_A(s_A', s_B), \tag{6.1}$$

$$u_B(s_A, s_B) \geq u_B(s_A, s_B') \tag{6.2}$$

for all strategies $s'_A \in S_A$ and $s'_B \in S_B$. Hence, in an equilibrium no player can gain by unilaterally deviating from this equilibrium. Given that the other player will stick to the strategy corresponding to the equilibrium, the best result is achieved by also playing the equilibrium solution. Finding the equilibria of a game essentially equals "solving" a game.

An important class of games is the class of zero-sum games. A zero-sum game is defined in that the pay-offs of both players sum up to zero for all possible pairs of strategies of the players, that is, that $u_A(s_A, s_B) = -u_B(s_A, s_B)$ for all possible pairs (s_A, s_B) , $s_A \in S_A$, $s_B \in S_B$ of strategies. In a zero-sum game the interests of the players are diametrically opposed, and the gain of one player is the loss of the other player. Although not representing a typical competitive situation, this class of games provides a suitable object of analysis due to the strong tools of solution that are available. The most relevant property of such zero-sum games is summarized in von-Neumann's min-max theorem [162].

In the min-max theorem a value v is assigned to every two-person zero-sum game with finite sets of strategies, which is the average pay-off that one player can expect to win from the other player as long as both players act rationally. There exists at least one Nash equilibrium, while in the case of several equilibria the pay-off will always be represented by the same value v. A particular strategy that ensures this return v to each player is called min-max strategy: A player then adopts a pessimistic point of view and maximizes the minimal pay-off he or she may obtain when implementing this strategy. The equilibrium in min-max strategies is a Nash equilibrium.

			Bob	
	Н	eads (p=1)	(p=1/2)	(p=0)
	Heads $(p=1)$	(1,-1)	(0, 0)	(-1,1)
Alice	(p=1/2)	(0, 0)	(0, 0)	(0, 0)
	(p=0)	(-1,1)	(0, 0)	(1,-1)

Figure 6.2: The pay-off matrix of the *Matching Pennies game* including the (particular) mixed strategy in which a player chooses heads and tails with probability 1/2.

So what would be the min-max strategy in the simple Matching Pennies game as described above? Obviously, there is no pair of pure strategies which is a Nash equilibrium. If Alice plays heads, Bob is better off with tails, if she chooses tails, he is best advised to play heads. However, the players can do more than just choosing heads or tails: they can randomly take heads or tails with certain probabilities (which also corresponds to how such a game would be played in reality).

Such a strategy in which a player specifies a certain classical probability distribution on the set S_A or S_B of pure strategies, respectively, is called a *mixed strategy*. In such mixed strategies, the min-max strategy can easily be identified: both players are best advised to play heads and tails with probability 1/2, as displayed in Fig. 6.2. On average, both players achieve the pay-off 0 which is the value v of this zero-sum game.

As said before, non-zero-sum games are far more typical. Probably the most well-known non-zero-sum game is the so-called *Prisoners' Dilemma* [184]. In this game Alice and Bob have the choice between "cooperation" and "defection". Being well aware of the consequences of their decisions the players obtain a certain pay-off according to their respective strategies.

		Bob	
		С	D
Alice	С	(3,3)	(0,5)
	D	(5,0)	(1,1)

Figure 6.3: The pay-off matrix in the Prisoners' Dilemma game. The first entry refers to Alice's pay-off, the second to Bob's. If both players cooperate, they both get 3 units pay-off. If Bob defects and Alice happens to cooperate, he obtains 5 units, while Alice is in the unfortunate situation in which she does not receive any pay-off at all. Bob faces the same situation if he chooses to cooperate while Alice prefers to defect. If both defect, they equally get 1 unit pay-off.

Fig. 6.3 indicates the pay-off of Alice and Bob.³ As Alice is better off with defection regardless of Bob's choice, she will defect. The game being symmetric, the same argument applies to Bob. The players face a *dilemma* because rational reasoning makes them defect, although they would both benefit from mutual cooperation.

The strategy of "defection" is called *dominant strategy*: it is favorable regardless what strategy the other party picks. ("defection", "defection") is an equilibrium in dominant strategies. This equilibrium in dominant strategies is the unique Nash equilibrium, and this uniqueness also holds in mixed strategies. It is worth noting that if the players agree upon playing the game N times, they will both opt for "defection" N times 4 .

A pair of strategies is called *Pareto-optimal* if there is no outcome in which both players do simultaneously better. From this perspective the Dilemma lies in the fact that the unique Nash equilibrium is far from being Pareto-optimal. The importance of the Prisoners' Dilemma stems from the fact that it models a dilemma that may arise in many situations involving conflicting interests.

 $^{^3}$ For the purposes of this chapter, particular values have been chosen for A_{CC} , A_{CD} , A_{DC} , A_{DD} , B_{CC} , B_{CD} , B_{DC} , and B_{DD} . From a game theoretical viewpoint, any positive numbers satisfying the symmetry conditions $A_{CC} = B_{CC}$, $A_{DD} = B_{DD}$, $A_{CD} = B_{DC}$, $A_{DC} = B_{CD}$ and the inequalities $A_{DC} > A_{CC} > A_{DD} > A_{CD}$ and $A_{CC} \ge (A_{CD} + A_{DC})/2$ define a (strong) Prisoners' Dilemma.

 $^{^4}$ This statement may be deduced by backward-induction: in the last round – the N-th – the players will definitely not cooperate. In the previous round they would only choose cooperation if this behavior were rewarded in the N-th round. As any agreements concerning the choice of the N-th round would not be met, the players will stick to defection in the N-1-th round, and so on until the first round. This reasoning does not apply to the case where the number of rounds in such an iterated game is not known to the players in advance. In 1980 a computer tournament was devised by Robert Axelrod [185], and specialists of the field were asked to participate in a Prisoners' Dilemma tournament with a varying number of rounds. A very simple strategy submitted by Anatol Rapoport won: in the first round cooperation is played, and from then on the strategy of the other player of the previous round is copied (tit-for-tat).

6.3 Quantum Games, Strategies, and Equilibria

This subsection deals with a "quantum version" of a game. Any quantum system which can be manipulated by two parties or more and for which the utility of the moves can be quantified in an appropriate manner may be considered a quantum game. In the following definition, the physical system which serves as the underlying setup of the game is included, as the word "quantum" already points to the physical carrier of information. The quantum games proposed in Refs. [E6], [164], and [186], and later in Refs. [172, 174, 175, 176, 177, 178, 180] can be cast into this form; also, the quantum cloning device as described in [165] can be said to be a quantum game in this sense.

A two-player quantum game $\Gamma = (\mathcal{H}, \rho, S_A, S_B, P_A, P_B)$ is completely specified by

- (i) the underlying Hilbert space \mathcal{H} of the physical system,
- (ii) the initial state $\rho \in \mathcal{S}(\mathcal{H})$,
- (iii) the sets S_A and S_B of permissible quantum strategies of the two players, and
- (iv) the *utility functionals* P_A and P_B , which specify the utility for each player.

A *quantum strategy* $\mathcal{E}_A \in S_A$, $\mathcal{E}_B \in S_B$ is a quantum operation, that is, a completely positive trace-preserving map $\mathcal{E}_A, \mathcal{E}_B : \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H})$. The definition of a quantum game also includes certain implicit rules such as the order of the implementation of the respective quantum strategies.

As in the case of ordinary games, a quantum game is called *zero-sum game* if the expected pay-offs sum up to zero for all pairs of strategies, that is, if

$$P_A(\mathcal{E}_A, \mathcal{E}_B) = -P_B(\mathcal{E}_A, \mathcal{E}_B) \tag{6.3}$$

for all $\mathcal{E}_A \in S_A$, $\mathcal{E}_B \in S_B$. Otherwise, it is called a *non-zero-sum game*. Note that it is not required that a set of allowed strategies for a player forms a closed set. Two quantum strategies of Alice \mathcal{E}_A and \mathcal{E}_A' will be called *equivalent*, if

$$P_A(\mathcal{E}_A, \mathcal{E}_B) = P_A(\mathcal{E}_A', \mathcal{E}_B)$$
 and $P_B(\mathcal{E}_A, \mathcal{E}_B) = P_A(\mathcal{E}_A', \mathcal{E}_B)$ (6.4)

for all possible \mathcal{E}_B . That is, if \mathcal{E}_A and \mathcal{E}'_A yield the same expected pay-off for both players for all allowed strategies of Bob. Strategies \mathcal{E}_B and \mathcal{E}'_B of Bob will be identified accordingly.

A solution concept provides advice to the players with respect to the action they are best advised to take. As before, a quantum strategy of Alice \mathcal{E}_A is called a *dominant strategy* if

$$P_A(\mathcal{E}_A, \mathcal{E}_B') > P_A(\mathcal{E}_A', \mathcal{E}_B') \tag{6.5}$$

for all $\mathcal{E}'_A \in S_A$, $\mathcal{E}'_B \in S_B$. Analogously, one can define a dominant strategy for Bob. A pair $(\mathcal{E}_A, \mathcal{E}_B)$ is said to be an *equilibrium in dominant strategies* if \mathcal{E}_A and \mathcal{E}_B are the players' respective dominant strategies. A combination of strategies $(\mathcal{E}_A, \mathcal{E}_B)$ is called a *Nash equilibrium* if

$$P_A(\mathcal{E}_A, \mathcal{E}_B) \geq P_A(\mathcal{E}_A', \mathcal{E}_B),$$
 (6.6)

$$P_B(\mathcal{E}_A, \mathcal{E}_B) \geq P_B(\mathcal{E}_A, \mathcal{E}_B')$$
 (6.7)

for all $\mathcal{E}'_A \in S_A$, $\mathcal{E}'_B \in S_B$. Again, a pair of strategies $(\mathcal{E}_A, \mathcal{E}_B)$ is called *Pareto optimal*, if it is not possible to increase one player's pay-off without lessening the pay-off of the other player.

6.4 Two-Qubit Quantum Games

In this section specific games will be investigated where the *classical version of the game* is faithfully entailed in the quantum game. In a quantum version of a binary choice game, two qubits are prepared by an *arbiter* in a particular initial state, and are then sent to the two players who dispose of physical instruments in order to manipulate their qubits appropriately. In a last step the qubits are sent back to the arbiter who performs a measurement to evaluate the pay-off. ⁵

For such a bi-partite quantum game the system of interest is a quantum system with underlying Hilbert space $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, $\mathcal{H}_A=\mathcal{H}_B=\mathbb{C}^2$, and associated state space $\mathcal{S}(\mathcal{H})$. The quantum strategies of Alice and Bob \mathcal{E}_A and \mathcal{E}_B are local trace-preserving quantum operations acting in \mathcal{H}_A and \mathcal{H}_B respectively. In other words, Alice and Bob are restricted to implementing their respective quantum strategy \mathcal{E}_A and \mathcal{E}_B on their qubit only. In this step they may choose any quantum strategy that is included in the sets of strategies S_A and S_B . They are both aware of the sets S_A and S_B , but they do not know which particular quantum strategy the other party will actually implement. As the application of both quantum strategies amounts to a map $\mathcal{E}_A\otimes\mathcal{E}_B:\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})$, the system will after execution of the moves be in the state

$$\sigma = (\mathcal{E}_A \otimes \mathcal{E}_B)(\rho). \tag{6.8}$$

The quantum strategies $\mathcal{E}_A \otimes \mathbb{1}_B$ and $\mathbb{1}_A \otimes \mathcal{E}_B$ are identified with \mathcal{E}_A and \mathcal{E}_B , respectively. Particular attention will be paid to unitary operations, which are associated with unitary operators U_A and U_B , written as $\mathcal{E}_A \simeq U_A$ and $\mathcal{E}_B \simeq U_B$. In this case the final state σ is given by

$$\sigma = (U_A \otimes U_B)\rho(U_A \otimes U_B)^{\dagger}. \tag{6.9}$$

If not otherwise specified both the sets of strategies of Alice and Bob and the pay-off functionals are taken to be identical, that is,

$$S_A = S_B = S$$
 and $P_A = P_B = P$, (6.10)

such that both parties face the same situation.

6.4.1 General Setup

Let ρ be a maximally entangled state on $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2$. In order to be consistent with Ref. [E6] let $\rho=|\psi_{CC}\rangle\langle\psi_{CC}|$ with

$$|\psi_{CC}\rangle = (|00\rangle + i|11\rangle)/\sqrt{2}.\tag{6.11}$$

Any other maximally entangled state on $\mathbb{C}^2 \otimes \mathbb{C}^2$ would also be appropriate. The quantum game $\Gamma = (\mathbb{C}^2 \otimes \mathbb{C}^2, \rho, S, S, P, P)$ can be played in the following way: The two qubits are forwarded to the arbiter who performs a projective selective measurement on the final state σ with Kraus operators π_{CC} , π_{CD} , π_{DC} , and π_{DD} , where

$$\pi_{CC} = |\psi_{CC}\rangle\langle\psi_{CC}|, \quad |\psi_{CC}\rangle = (|00\rangle + i|11\rangle)/\sqrt{2},$$
(6.12)

$$\pi_{CD} = |\psi_{CD}\rangle\langle\psi_{CD}|, \quad |\psi_{CD}\rangle = (|01\rangle - i|10\rangle)/\sqrt{2}, \tag{6.13}$$

$$\pi_{DC} = |\psi_{DC}\rangle\langle\psi_{DC}|, \quad |\psi_{DC}\rangle = (|10\rangle - i|01\rangle)/\sqrt{2},$$
(6.14)

$$\pi_{DD} = |\psi_{DD}\rangle\langle\psi_{DD}|, \quad |\psi_{DD}\rangle = (|11\rangle + i|00\rangle)/\sqrt{2}. \tag{6.15}$$

⁵By classical means, a two player binary choice game may be played as follows: An arbiter takes two coins and forwards one each to the players. The players then receive their coin with heads up and may keep it as it is (the first pure strategy) or turn it upside down so that tails is up (the second strategy). Both players then return the coins to the arbiter who calculates the players' final pay-off corresponding to the combination of strategies he obtains from the players. Here, the coins serve as the physical carrier of information in the game.

The Kraus operators correspond to a projective measurement associated with the basis of \mathcal{H} consisting of $|\psi_{CC}\rangle$ and three orthonormal state vectors. On the one hand, due to this choice the system will be in a maximally entangled state when it comes to implementing the quantum operations. On the other hand it is guaranteed that if both players prefer to implement the identity operation, that is, "do nothing", then the detector will click in the channel with label CC with certainty.

According to the outcome of the measurement, a pay-off of A_{CC} , A_{CD} , A_{DC} , or A_{DD} is given to Alice, Bob receives B_{CC} , B_{CD} , B_{DC} , or B_{DD} . The utility functionals, also referred to as expected pay-off of Alice and Bob, read

$$P_{A}(\mathcal{E}_{A}, \mathcal{E}_{B}) = A_{CC} \operatorname{tr}[\pi_{CC}\sigma] + A_{CD} \operatorname{tr}[\pi_{CD}\sigma] + A_{DC} \operatorname{tr}[\pi_{DC}\sigma] + A_{DD} \operatorname{tr}[\pi_{DD}\sigma],$$

$$P_{B}(\mathcal{E}_{A}, \mathcal{E}_{B}) = B_{CC} \operatorname{tr}[\pi_{CC}\sigma] + B_{CD} \operatorname{tr}[\pi_{CD}\sigma] + B_{DC} \operatorname{tr}[\pi_{DC}\sigma] + B_{DD} \operatorname{tr}[\pi_{DD}\sigma].$$

$$(6.16)$$

The Kraus operators are chosen in such a way that the classical game is fully entailed in the quantum game: The *classical strategies* cooperation and defection are associated with particular unitary operations,

$$C \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
 (6.18)

C does not change the state at all, D implements a "spin-flip". If both parties stick to these classical strategies, Eq. (6.16) and Eq. (6.17) guarantee that the expected pay-off is exactly the pay-off of the corresponding classical game defined by the numbers A_{CC} , A_{CD} , A_{DC} , A_{DD} , B_{CC} , B_{CD} , B_{DC} , and B_{DD} .

To give an example, if Alice plays C and Bob chooses D, the state σ after implementation of the strategies is given by

$$\sigma = (C \otimes D)(\rho) = |\psi_{CD}\rangle\langle\psi_{CD}|,\tag{6.19}$$

such that Alice obtains A_{CD} units and Bob B_{CD} units pay-off. In this way the specificities of strategic moves in the quantum domain can be adequately studied. The players may make use of additional degrees of freedom which are not available by randomization of the classical strategies, but they can also stick to mere classical strategies. This scheme can be applied to any two player binary choice game and is canonical to a high extent.

6.4.2 Prisoners' Dilemma

For the Prisoners' Dilemma, the values in the table in Eqs. (6.16) and (6.17) are given by (see Fig. 6.3),

$$A_{CC} = B_{CC} = 3, \ A_{DD} = B_{DD} = 1,$$
 (6.20)

$$A_{CD} = B_{DC} = 0, \ A_{DC} = B_{CD} = 5.$$
 (6.21)

In all of the following sets of allowed strategies S the classical options (to defect and to cooperate) are included. Several interesting sets of strategies and solution concepts will now be studied. The first three subsections involve local unitary operations only, while in the last subsection other quantum operations will be considered as well.

Example 6.1. – One-parameter set of strategies. – The first set of strategies $S^{(6.1)}$ involves quantum operations \mathcal{E}_A and \mathcal{E}_B which are local rotations with a single parameter. The matrix representation of the corresponding unitary operators is taken to be

$$U(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$
 (6.22)

with $\theta \in [0, \pi]$. Therefore, selecting strategies \mathcal{E}_A and \mathcal{E}_B amounts in this simple case to choosing two angles θ_A and θ_B . The classical pure strategies of defection and cooperation can be implemented as $C \simeq U(0)$, $D \simeq U(\pi)$. An analysis of the expected pay-offs P_A and P_B ,

$$P_{A}(\theta_{A}, \theta_{B}) = 3|\cos(\theta_{A}/2)\cos(\theta_{B}/2)|^{2} + 5|\cos(\theta_{B}/2)\sin(\theta_{A}/2)|^{2} + |\sin(\theta_{A}/2)\sin(\theta_{B}/2)|^{2},$$

$$P_{B}(\theta_{A}, \theta_{B}) = 3|\cos(\theta_{A}/2)\cos(\theta_{B}/2)|^{2} + 5|\sin(\theta_{B}/2)\cos(\theta_{A}/2)|^{2} + |\sin(\theta_{A}/2)\sin(\theta_{B}/2)|^{2},$$
(6.23)

shows that this game is nothing else but the classical Prisoners' Dilemma game. The payoff functions are identical to the analogous functions in a Prisoners' Dilemma with mixed, that is, randomized, strategies, where cooperation is chosen with the classical probability $p = \cos^2(\theta/2)$. The inequalities

$$P_A(D, \mathcal{E}_B) \geq P_A(\mathcal{E}_A, \mathcal{E}_B),$$
 (6.25)

$$P_B(\mathcal{E}_A, D) \geq P_B(\mathcal{E}_A, \mathcal{E}_B)$$
 (6.26)

hold for all \mathcal{E}_A , $\mathcal{E}_B \in S^{(6.1)}$, and therefore, (D,D) is an equilibrium in dominant strategies, and thus the unique Nash equilibrium. As explained in Section 6.2, this equilibrium is far from being efficient, because $P_A(D,D) = P_B(D,D) = 1$ instead of the Pareto optimal payoff which would be 3.

Example 6.2. – Two-parameter set of strategies. – A more general set of strategies is the following two-parameter set $S^{(6.2)}$. The matrix representation of operators corresponding to quantum strategies from this set is given by

$$U(\theta, \phi) = \begin{pmatrix} e^{i\phi} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi} \cos(\theta/2) \end{pmatrix}$$
(6.27)

with $\theta \in [0, \pi]$ and $\phi \in [0, \pi/2]$. Selecting a strategy \mathcal{E}_A , \mathcal{E}_B then means choosing appropriate angles θ_A , ϕ_A and θ_B , ϕ_B . The classical strategies of defection and cooperation are also included in the set of possible strategies, as

$$C \simeq U(0,0)$$
 and $D \simeq U(\pi,0)$. (6.28)

The expected pay-off for Alice explicitly reads

$$P_{A}(\theta_{A}, \phi_{A}, \theta_{B}, \theta_{B}) = 3 \left| \cos(\phi_{A} + \phi_{B}) \cos(\theta_{A}/2) \cos(\theta_{B}/2) \right|^{2}$$

$$+ 5 \left| \sin(\phi_{A}) \cos(\theta_{A}/2) \sin(\theta_{B}/2) - \cos(\phi_{B}) \cos(\theta_{B}/2) \sin(\theta_{A}/2) \right|^{2}$$

$$+ \left| \sin(\phi_{A} + \phi_{B}) \cos(\theta_{A}/2) \cos(\theta_{B}/2) + \sin(\theta_{A}/2) \sin(\theta_{B}/2) \right|^{2}$$

$$+ \left| \sin(\phi_{A} + \phi_{B}) \cos(\theta_{A}/2) \cos(\theta_{B}/2) + \sin(\theta_{A}/2) \sin(\theta_{B}/2) \right|^{2}$$

and the expected pay-off of Bob is given by

$$P_{A}(\theta_{A}, \phi_{A}, \theta_{B}, \theta_{B}) = 3 \left| \cos(\phi_{A} + \phi_{B}) \cos(\theta_{A}/2) \cos(\theta_{B}/2) \right|^{2}$$

$$+ 5 \left| \sin(\phi_{B}) \cos(\theta_{B}/2) \sin(\theta_{A}/2) - \cos(\phi_{A}) \cos(\theta_{A}/2) \sin(\theta_{B}/2) \right|^{2}$$

$$+ \left| \sin(\phi_{A} + \phi_{B}) \cos(\theta_{A}/2) \cos(\theta_{B}/2) + \sin(\theta_{A}/2) \sin(\theta_{B}/2) \right|^{2}.$$
(6.30)

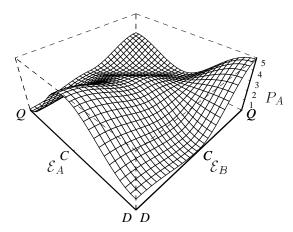


Figure 6.4: Alice's pay-off in the Prisoners' Dilemma game with the set of strategies being $S^{(6.2)}$. In this plot, a certain parameterization has been chosen such that the strategies \mathcal{E}_A and \mathcal{E}_B each depend on a single parameter $t \in [-1,1]$: $\mathcal{E}_A \simeq U(t\pi,0)$ for $t \in [0,1]$ and $\mathcal{E}_A \simeq U(0,-t\pi/2)$ for $t \in [-1,0)$, and analogously for Bob. Defection D corresponds to the value t=1, cooperation C to t=0, and Q is represented by t=-1.

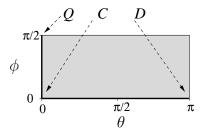


Figure 6.5: This figure shows the image of the parametric curve chosen in the previous figure.

It becomes evident that the previous Nash equilibrium (D,D) of $S^{(6.1)}$ is no longer an equilibrium solution, as both players may benefit from deviating from the strategy D. At the same time a new Nash equilibrium has emerged which will be given by (Q,Q). This strategy is not accessible in a classical game with mixed strategies. The strategy Q corresponds to a matrix

$$Q \simeq U(0, \pi/2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{6.31}$$

For all $\theta_A \in [0, \pi]$ and $\phi_B \in [0, \pi/2]$

$$P_A(\mathcal{E}_A, Q) = \cos^2(\theta_A/2) \left(3\sin^2(\phi_A) + \cos^2(\phi_A)\right) \le 3.$$
 (6.32)

In particular,

$$P_A(\mathcal{E}_A, Q) \le P_A(Q, Q) = 3 \tag{6.33}$$

$$P_B(Q, \mathcal{E}_B) \le P_B(Q, Q) = 3 \tag{6.34}$$

for all and all $\mathcal{E}_A \in S_A$ and all $\mathcal{E}_B \in S_B$, such that no player can gain from unilaterally deviating from Q. This Nash equilibrium is unique and serves as the only acceptable solution of the game.

The astonishing fact is that $P_A(Q,Q) = P_B(Q,Q) = 3$ (instead of 1) so that the Pareto optimum is realized. No player could gain without lessening the other player's expected pay-off. In this sense one can say that the Dilemma of the original game has fully disappeared. In the classical game only mutual cooperation is Pareto optimal, but this pair of strategies does not correspond to a Nash equilibrium. If the players may resort to quantum strategies, they can escape the Dilemma.

Example 6.3. – General unitary operations. – One can generalize the previous setting to the case where Alice and Bob can implement operations \mathcal{E}_A and \mathcal{E}_B taken from $S^{(6.3)}$, where $S^{(6.3)}$ is the set of general local unitary operations. It might be suspected that the solution becomes more efficient the larger the sets of allowed operations are. But on the contrary, the previous Pareto optimal unique Nash equilibrium (Q,Q) ceases to be an equilibrium solution if the set is enlarged: ⁶ For any strategy $\mathcal{E}_B \in S^{(6.3)}$ there exists an optimal answer $\mathcal{E}_A \in S^{(6.3)}$ resulting in

$$(\mathcal{E}_A \otimes \mathcal{E}_B)(\rho) = |\psi_{DC}\rangle\langle\psi_{DC}|,\tag{6.35}$$

with ρ given in Eq. (6.55). That is, for any strategy of Bob \mathcal{E}_B there is a strategy \mathcal{E}_A of Alice such that

$$P_A(\mathcal{E}_A, \mathcal{E}_B) = 5 \text{ and } P_B(\mathcal{E}_A, \mathcal{E}_B) = 0:$$
 (6.36)

Take

$$\mathcal{E}_A \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathcal{E}_B \simeq \begin{pmatrix} -ib & a \\ -d & -ic \end{pmatrix},$$
 (6.37)

where a,b,c,d are appropriate complex numbers. Given that Bob plays the strategy \mathcal{E}_B associated with a particular Nash equilibrium $(\mathcal{E}_A,\mathcal{E}_B)$, Alice can always apply the *optimal answer* \mathcal{E}_A to achieve the maximal possible pay-off. However, the resulting pair of quantum strategies cannot be an equilibrium since again, the game being symmetric, Bob can improve his pay-off by changing his strategy to his optimal answer \mathcal{E}_B' . Hence, there is no pair $(\mathcal{E}_A,\mathcal{E}_B)$ of pure strategies with the property that the players can only lose from unilaterally deviating from this pair of strategies.

Yet, there remain Nash equilibria in mixed strategies which are much more efficient than the classical outcome of the equilibrium in dominant strategies $P_A(D,D) = P_B(D,D) = 1$. In a mixed strategy of Alice, say, she selects a particular quantum strategy \mathcal{E}_A (which is then conceived as pure strategy) from the set of strategies \mathcal{E}_A with a certain classical probability. That is, mixed strategies of Alice and Bob are associated with maps of the form

$$\rho \longmapsto \sigma = \sum_{i,j=1}^{n} p_A^{(i)} p_B^{(j)} (U_A^{(i)} \otimes U_B^{(j)}) \rho (U_A^{(i)} \otimes U_B^{(j)})^{\dagger}, \tag{6.38}$$

 $p_A^{(i)}, p_B^{(i)} \in [0, 1], i, j = 1, 2, ..., n$, with

$$\sum_{i=1}^{n} p_A^{(i)} = \sum_{j=1}^{n} p_B^{(j)} = 1.$$
 (6.39)

⁶This has already been reported in Ref. [E6] and in greater detail in Ref. [187]. In fact, it is an open question whether there exist quantum games with (i) a unique equilibrium in *pure* strategies which is more efficient than the equilibrium of the corresponding classical game and (ii) in which the full group of local unitary operations is available to all players. In this context, the somewhat ambiguous term solution means either a unique Nash equilibrium or an equilibrium which is clearly distinguished from all the other equilibria. For example, refinement concepts like *perfect*, *proper*, *and persistent equilibria* [181] or the *focal point effect* [182] may be employed to eliminate all but one Nash equilibria. Multi-player games like the elegant scheme proposed in Ref. [174] (see also Ref. [175]) may point towards a resolution of this issue, although all equilibria of the game of Ref. [174] are fully symmetric, and the above refinement concepts do not lead to a particular distinguished equilibrium. However, it is the subject of the remainder of this example and Example 6.3 to show that in mixed strategies there still exist efficient solutions of the quantum game.

 $U_A^{(i)}$ and $U_B^{(j)}$ are local unitary operators corresponding to pure strategies $\mathcal{E}_A^{(i)}$ and $\mathcal{E}_B^{(j)}$. The map given by Eq. (6.38) acts in \mathcal{H}_A and \mathcal{H}_B as a doubly stochastic map, that is,

as a completely positive unital map [188]. As a result, the final reduced states $\operatorname{tr}_B[\sigma]$ and $\operatorname{tr}_A[\sigma]$ must be more mixed than the reduced initial states $\operatorname{tr}_B[\rho]$ and $\operatorname{tr}_A[\rho]$ in the sense of majorization theory [140]. The initial state ρ is a maximally entangled state, and therefore, the reduced states of Alice and Bob are initially maximally mixed. It follows that all accessible states after application of a mixed strategy of Alice and Bob are locally identical to the maximally mixed state $\mathbb{1}_A/2$ and $\mathbb{1}_B/2$, respectively.

For example, the following construction yields an equilibrium in mixed quantum strategies: Allow Alice to choose from two strategies $\mathcal{E}_A^{(1)}$ and $\mathcal{E}_A^{(2)}$ with probabilities $p_A^{(1)}=1/2$ and $p_A^{(2)}=1/2$, while Bob may take $\mathcal{E}_B^{(1)}$ or $\mathcal{E}_B^{(2)}$ with according probabilities, where

$$\mathcal{E}_A^{(1)} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{E}_A^{(2)} \simeq \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$
 (6.40)

$$\mathcal{E}_B^{(1)} \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{E}_B^{(2)} \simeq \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$
 (6.41)

The quantum strategies of Eq. (6.40) and Eq. (6.41) are mutually optimal answers and have the property

$$P_A(\mathcal{E}_A^{(i)}, \mathcal{E}_B^{(i)}) = 0, \qquad P_B(\mathcal{E}_A^{(i)}, \mathcal{E}_B^{(i)}) = 5,$$
 (6.42)

$$P_A(\mathcal{E}_A^{(i)}, \mathcal{E}_B^{(i)}) = 0, \qquad P_B(\mathcal{E}_A^{(i)}, \mathcal{E}_B^{(i)}) = 5,$$

$$P_A(\mathcal{E}_A^{(i)}, \mathcal{E}_B^{(3-i)}) = 5, \qquad P_B(\mathcal{E}_A^{(i)}, \mathcal{E}_B^{(3-i)}) = 0,$$
(6.42)

for i = 1, 2. Due to the particular constraints of Eq. (6.42) and Eq. (6.43) no other mixed strategy will entail a better pay-off for Bob than the above mixed strategy, given that Alice sticks to the equilibrium strategy.

This will become apparent in the following argument: Let Alice use this particular mixed quantum strategy as above and let Bob use any mixed quantum strategy

$$\mathcal{E}_B^{(1)}, \dots, \mathcal{E}_B^{(n)} \tag{6.44}$$

together with $p_A^{(1)},...,p_A^{(n)}$. The final state σ after application of the strategies is given by the convex combination

$$\sigma = \sum_{i=1,2} \sum_{j=1}^{n} p_A^{(i)} p_B^{(j)} (\mathcal{E}_A^{(i)} \otimes \mathcal{E}_B^{(j)})(\rho), \tag{6.45}$$

This convex combination can only lead to a smaller expected pay-off for Bob than the optimal pure strategy $\mathcal{E}_{R}^{(k)}$ in Eq. (6.44), $k \in \{1, ..., n\}$. Such optimal pure strategies are given by $\mathcal{E}_B^{(1)}$ and $\mathcal{E}_B^{(2)}$ as in Eq. (6.41) and lead to an expected pay-off for Bob of $P_B(\mathcal{E}_A,\mathcal{E}_B)=2.5$. There are no pure strategies which achieve a larger expected pay-off. While both pure strategies $\mathcal{E}_B^{(1)}$ and $\mathcal{E}_B^{(2)}$ do not correspond to an equilibrium, the mixed strategy where $\mathcal{E}_B^{(1)}$ and $\mathcal{E}_B^{(2)}$ are chosen with $p_B^{(1)}=1/2$ and $p_B^{(2)}=1/2$ actually does. Nash equilibria consist of pairs of mutually optimal answers, and only for this choice of Bob the original mixed quantum of $P_B^{(1)}$ and $P_B^{(2)}$ are chosen with $P_B^{(1)}$ and $P_B^{(2)}$ and $P_B^{(2)}$ are chosen with $P_B^{(1)}$ and $P_B^{(2)}$ and $P_B^{(2)}$ are chosen with $P_B^{(1)}$ and $P_B^{(2)}$ are chosen with $P_B^{(2)}$ and $P_B^{($ tum strategy of Alice is *her* optimal choice. The game being symmetric, the same argument applies also to her.

However, this Nash equilibrium is not the only one. There exist other four-tuples of matrices than the ones presented in Eq. (6.40) and Eq. (6.41) that satisfy Eq. (6.42) and Eq. (6.43). Such matrices can be found by appropriately rotating the matrices of Eq. (6.40) and Eq. (6.41). As this means that there is more than one equilibrium, it is not obvious which Nash equilibrium the players will realize. It is at first not even evident whether a Nash equilibrium will be played at all. But the game theoretical concept of the focal point effect [182, 181] helps to resolve this issue.

In order to explore the general structure of any Nash equilibrium in mixed strategies, let

$$U_A^{(1)}, ..., U_A^{(n)} (6.46)$$

together with $p_A^{(1)},...,p_A^{(n)}$ specify the mixed strategy pertinent to a Nash equilibrium of Alice. Then there is a mixed strategy $U_B^{(1)},...,U_B^{(n)},\,p_B^{(1)},...,p_B^{(n)}$ rewarding Bob with the best achievable pay-off, given that Alice plays this mixed strategy. Yet, the pair of mixed strategies associated with

$$QU_{A}^{(1)}Q^{\dagger},...,QU_{A}^{(n)}Q^{\dagger},\quad QU_{B}^{(1)}Q^{\dagger},...,QU_{B}^{(n)}Q^{\dagger} \tag{6.47}$$

with $p_A^{(1)},...,p_A^{(n)}$, $p_B^{(1)},...,p_B^{(n)}$ represents another Nash equilibrium. This equilibrium leads to the same expected pay-off for both players and is fully symmetric to the previous one. Doubly applying Q as $QQU_A^{(1)}Q^\dagger Q^\dagger,...,QQU_A^{(n)}Q^\dagger Q^\dagger$ results again in a situation with equivalent strategies as the original ones. For a given Nash equilibrium as in Eq. (6.46) the one specified by Eq. (6.47) will be called dual equilibrium.

There is a single Nash equilibrium (R,R) which is identical with its dual equilibrium: it is the simple map

$$\rho \longmapsto \sigma = 1/4. \tag{6.48}$$

There exist probabilities $p_A^{(1)},...,p_A^{(n)}$ and unitary operators $U_A^{(1)},...,U_A^{(n)}$ such that

$$\sum_{i} p_A^{(i)}(U_A^{(i)} \otimes \mathbb{1}_B) \rho(U_A^{(i)} \otimes \mathbb{1}_B)^{\dagger} = \mathbb{1}/4.$$
 (6.49)

[140]. If Alice has already selected $\mathcal{E}_A = R$, the application of $\mathcal{E}_B = R$ will not change the state of the quantum system any more.

Assume that Eq. (6.46) and Eq. (6.47) are associated with equivalent quantum strategies. This means that they have to produce the same expected pay-off for all quantum strategies \mathcal{E}_B of Bob. If Alice and Bob apply $\mathcal{E}_A \otimes \mathcal{E}_B$ they get an expected pay-off according to Eq. (6.16) and Eq. (6.17); if Alice after implementation of \mathcal{E}_A manipulates the quantum system by applying the local unitary operator $Q \otimes \mathbb{1}_B$, they obtain

$$P'_{A}(\mathcal{E}_{A}, \mathcal{E}_{B}) = A_{DD} \operatorname{tr}[\pi_{CC}\sigma] + A_{DC} \operatorname{tr}[\pi_{CD}\sigma] + A_{CD} \operatorname{tr}[\pi_{DC}\sigma] + A_{CC} \operatorname{tr}[\pi_{DD}\sigma],$$

$$P'_{B}(\mathcal{E}_{A}, \mathcal{E}_{B}) = B_{DD} \operatorname{tr}[\pi_{CC}\sigma] + B_{DC} \operatorname{tr}[\pi_{CD}\sigma] + B_{CD} \operatorname{tr}[\pi_{DC}\sigma] + B_{CC} \operatorname{tr}[\pi_{DD}\sigma].$$

$$(6.50)$$

The only \mathcal{E}_A with the property that $P_A'(\mathcal{E}_A, \mathcal{E}_B) = P_A(\mathcal{E}_A, \mathcal{E}_B)$ and $P_B'(\mathcal{E}_A, \mathcal{E}_B) = P_B(\mathcal{E}_A, \mathcal{E}_B)$ for all \mathcal{E}_B is the map given by Eq. (6.48).

In principle, any Nash equilibrium may become a *self-fulfilling prophecy* if the particular Nash equilibrium is expected by both players. It has been pointed out that in a game with more than one equilibrium, anything that attracts the players' attention towards one of the equilibria may make them expect and therefore realize it [182]. The corresponding *focal equilibrium* [181] is the one which is distinguished from the other Nash equilibria. There is indeed one Nash equilibrium which is different from all the others: it is the one that is equivalent to its dual equilibrium, the simple mapping of the initial state on the maximally mixed state. For all other expected pay-offs both players are ambivalent between (at least) two symmetric equilibria. The expected pay-off in this focal equilibrium,

$$P_A(R,R) = P_B(R,R) = 2.25,$$
 (6.52)

is not fully Pareto optimal, but it is again much more efficient than the classically achievable outcome of 1. Of course, in the classical game, too, both players could play C and D

with probabilities p=1/2 yielding the same expected pay-off of 2.25. However, this pair of mixed strategies would be no equilibrium solution, as any player could benefit from simply choosing the dominant strategy D.

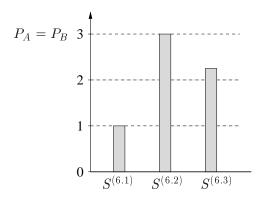


Figure 6.6: The expected pay-off P_A and P_B in the distinguished Nash equilibria in the sets of quantum strategies $S^{(6.1)}$, $S^{(6.2)}$, and $S^{(6.3)}$.

This study shows that the efficiency of the equilibrium the players can reach in this game depends on the actions the players may take. One feature, however, is present in both of the sets $S^{(6.2)}$ and $S^{(6.3)}$: both players can increase their expected pay-offs drastically by resorting to quantum strategies. $S^{(6.2)}$ is equivalent with the situation in the classical game with mixed strategies. Fig. 6.6 is a schematic representation of the achievable expected pay-off in the three sets of allowed strategies

$$S^{(6.1)} \subset S^{(6.2)} \subset S^{(6.3)}.$$
 (6.53)

In $S^{(6.2)}$ the unique Nash equilibrium is Pareto optimal and hence maximally efficient, in other sets the solution is less efficient.

In the subsequent last two examples for the quantum version of the Prisoners' Dilemma a different setting is studied: the case where the parties face an unfair situation. In Example 6.4 one party only is restricted to classical strategies, in Example 6.5 again such an asymmetric setting is investigated, but with an initial state with varying entanglement.

Example 6.4. – Classical versus quantum strategies. – An investigation of an unfair situation is particularly interesting in the case of the two-parameter strategic space $S^{(6.2)}$ introduced in Example 6.2. What happens if both parties do not have access to the same strategic space? Alice may use a quantum strategy, i.e., her strategic space is still $S^{(6.2)}$, while Bob is restricted to apply only classical pure strategies C or D or classically mixed strategies. In this case Alice is well advised to play

$$M \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1\\ -1 & -i \end{pmatrix}, \tag{6.54}$$

which corresponds to $\phi_A = \pi/2$ and $\theta_A = \pi/2$. This strategy will ensure that she gets $P_A(M, \mathcal{E}_B) = 3$ as pay-off, for all allowed mixed strategies of Bob; he may implement C and D with arbitrary probabilities p_B and $1 - p_B$, respectively, with $p_B \in [0, 1]$. Hence if in an unfair game Alice can be sure that Bob plays a mixed classical strategy, she may always

choose *M* as her preferred strategy in an iterated game. This certainly out-performs tit-fortat, but one must keep in mind that the assumed asymmetry is essential for this argument.

Example 6.5. – Advantange in an unfair game dependent on the entanglement of the initial state. – In the above considerations the initial state of the game was fixed, $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = (|00\rangle + i|00\rangle)/\sqrt{2}$. In this example the initial state will be varied. Depending on an entanglement parameter $\gamma, \gamma \in [0, \pi/2]$, $|\psi\rangle$ will be taken to be

$$|\psi\rangle = J|00\rangle, \quad J = \exp(i\gamma D \otimes D).$$
 (6.55)

The case $\gamma=\pi/2$ corresponds to the above case with a maximally entangled initial state, for $\gamma=0$ the state ρ is a product state. The entanglement of the pure state ρ for the values $\gamma\in[0,\pi/2]$ is given by $E(\rho)=-\sin^2(\gamma)\log_2(\sin^2(\gamma))-\cos^2(\gamma)\log_2(\cos^2(\gamma))$. Accordingly, the Kraus operators of the measurement are changed to

$$\pi_{CC} = |\psi_{CC}\rangle\langle\psi_{CC}|, \quad |\psi_{CC}\rangle = (C \otimes C)J|00\rangle,$$
(6.56)

$$\pi_{CD} = |\psi_{CD}\rangle\langle\psi_{CD}|, \quad |\psi_{CD}\rangle = (C \otimes D)J|00\rangle,$$
(6.57)

$$\pi_{DC} = |\psi_{DC}\rangle\langle\psi_{DC}|, \quad |\psi_{DC}\rangle = (D \otimes C)J|00\rangle,$$
(6.58)

$$\pi_{DD} = |\psi_{DD}\rangle\langle\psi_{DD}|, \quad |\psi_{DD}\rangle = (D \otimes D)J|00\rangle,$$
 (6.59)

such that again, the classical strategies cooperate and defect correspond to ${\cal C}$ and ${\cal D}$ with the matrix representations

$$C \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
 (6.60)

How does the comparative advantage of Alice depend on the parameter γ in a game in which Alice may implement any strategy from $S^{(6.2)}$, whereas Bob chooses C or D with a certain classical probability?

For any value of $\gamma \in [0, 1]$, the minimal expected pay-off m Alice can always attain by choosing an appropriate strategy \mathcal{E}_A is given by

$$m = \max_{\mathcal{E}_A \in S^{(6.2)}} \min_{\mathcal{E}_B: \mathcal{E}_B = p_B C + (1 - p_B)D} P_A(\mathcal{E}_A, \mathcal{E}_B). \tag{6.61}$$

Definitely, Alice will not settle for anything less than this quantity. Considering m a function of the entanglement parameter γ it is clear that m(0)=1 (since in this case the dominant strategy D is the optimal choice) while for maximal entanglement one finds $m(\pi/2)=3$ which is achieved by playing M. Fig. 6.7 shows m as a function of the entanglement parameter γ . As a matter of fact, m is a monotone increasing function of γ , and the maximal advantage is only accessible for maximal entanglement. Furthermore, Alice should deviate from the strategy D if and only if the degree of entanglement exceeds a certain threshold value

$$\Gamma = \arcsin(1/\sqrt{5}) = 0.4636.$$
 (6.62)

At the threshold she should discontinuously change her strategy from D to Q.

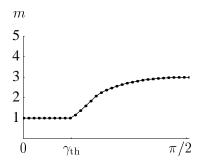


Figure 6.7: The minimal pay-off Alice can always attain on average as a function of γ in the unfair game of Example 6.5.

6.4.3 A Game With Two Equilibria

In the (classical) Prisoners' Dilemma the Dilemma of the players consists of the fact that the unique Nash equilibrium is not Pareto-optimal. Indeed, an unambiguous solution can be specified consisting of this Nash equilibrium, however, the solution is not efficient and hence not satisfactory to the players. In the so-called *Chicken game*⁷ [181, 163] the players face a different Dilemma: the classical game has multiple Nash equilibria. The situation of the players in the Chicken game,

$$A_{CC} = B_{CC} = 6, \ A_{CD} = B_{DC} = 8,$$
 (6.63)

$$A_{DC} = B_{CD} = 2, \ A_{DD} = B_{DD} = 0,$$
 (6.64)

can be described by the matrix of Fig. 6.8.

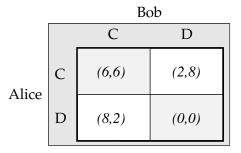


Figure 6.8: The pay-off matrix of the Chicken game.

This game has two Nash equilibria in pure strategies, namely (C,D) and (D,C), and it is not clear how to anticipate what the players' decision would be. In addition to the two Nash equilibria in pure strategies there is an equilibrium in mixed strategies, yielding an expected pay-off 4 [181]. In this equilibrium in mixed strategies both players choose each pure strategy with probability 1/2. In order to investigate the new features of the game if superpositions of classical strategies are allowed for, three set of strategies are briefly discussed:

⁷The name "Chicken" is inspired by a game mentioned in the 1955 movie called "Rebel Without a Cause".

Example 6.6. – One-parameter set of strategies. – The first set of strategies is again the set $S^{(6.1)}$ of one-dimensional rotations. Strategies \mathcal{E}_A and \mathcal{E}_B are associated with local unitary operators

$$U(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$
 (6.65)

with $\theta \in [0, \pi]$,

$$C \simeq U(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D \simeq U(\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
 (6.66)

Then as before, the quantum game yields the same expected pay-off as the classical game in randomized strategies. This means that still two Nash equilibria in pure strategies are present.

Example 6.7. – Two-parameter set of strategies. – The players can actually take advantage of an additional degree of freedom which is not accessible in the classical game. If they may apply unitary operations from $S^{(6.2)}$ of the type

$$U(\theta, \phi) = \begin{pmatrix} e^{i\phi} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi} \cos(\theta/2) \end{pmatrix}$$
(6.67)

with $\theta \in [0,\pi]$ and $\phi \in [0,\pi/2]$ the situation is quite different than with $S^{(6.1)}$. (C,D) and (C,D) with $C \simeq U(0,0)$ and $D \simeq U(\pi,0)$ are no longer equilibrium solutions. E.g., given that $\mathcal{E}_A = D$ the pair of strategies (D,Q) with $Q \simeq U(0,\pi/2)$ yields a better expected payoff for Bob than (D,C), that is to say $P_B(D,Q) = 8$, $P_B(D,C) = 2$. In fact (Q,Q) is now the unique Nash equilibrium with $P_A(Q,Q) = P_B(Q,Q) = 6$, which follows from an investigation of the actual expected pay-offs of Alice and Bob analogous to Eq. (6.29). This solution is not only the unique acceptable solution of the game, but it is also an equilibrium that is Pareto optimal. This contrasts very much with the situation in the classical game, where the two equilibria were not that efficient.

Example 6.8. – General unitary operations. – As in the considerations concerning the Prisoners' Dilemma game, more than one Nash equilibrium is present, if both players can take quantum strategies from the set $S^{(6.3)}$, and all Nash equilibria emerge at least in pairs as above. The focal equilibrium is given by (R,R), resulting in a pay-off of $P_A(R,R) = P_B(R,R) = 4$, which is the same as the mixed strategy of the classical game.

6.5 Concluding Remarks

In this chapter the idea of implementing quantum operations as strategic moves in a game has been explored. First, the notion of a game has been introduced where strategic moves are identified with quantum operations. In detail, a certain model has been investigated which could be conceived as a generalization into the quantum domain of a two player binary choice game. As a toy model for more complex scenarios quantum games have been studied in which the efficiency of the equilibria that are attainable when using quantum strategies could be contrasted with the efficiency of solutions in the corresponding classical game.

The nature of a game is determined by the rules of the game. In particular, the appropriate solution concept depends on the available strategic moves. Obviously, a player cannot make a meaningful choice without knowing the options at his or her disposal. So it comes to no surprise that also the actual achievable pay-off in such a game depends on the set of allowed strategies. Roughly speaking, one can say that the possibility of utilizing strategies

which are not feasible in the analogous classical game implicates a significant advantage. In the models studied in detail two kinds of "dilemmas" were "resolved":

- (i) On the one hand there are quantum games with an efficient unambiguous solution, while in the classical analogue only an inefficient equilibrium can be identified. By taking advantage of appropriate quantum strategies much more efficient equilibria could be reached. In certain sets of strategies even a maximally efficient solution the Pareto optimum was attainable.
- (ii) On the other hand, there exist quantum games with a unique solution with a classical equivalent which offers two Nash equilibria of the same quality.

Also,

(iii) the performance of classical versus quantum strategies has been compared. It has been found [E6] (see also [164]) that the allowed quantum strategies outperform the classical strategies by far.

These investigations show that new features emerge in a game in which strategies are quantum operations applied on quantum mechanical carriers of information. In this chapter it has been the main emphasis to examine how such quantum generalizations are different from their classical equivalent. This work can be extended in many ways. In Refs. [174, 176, 175], e.g., multi-player generalizations of quantum games have been studied. Moreover, first steps in the analysis of evolutionary games have been taken [179]. *Iterated quantum games* have, however, not yet been studied in detail; it is not even fully clear how to introduce the notion of such an iterated game. It is the actual hope – at least according to the conception of the author of this thesis – that these investigations lead to a better understanding of competitive structures in a game theoretical sense in applications of quantum information theory. It seems that finding such more pragmatic applications is the real challenge in the next step of decision theoretical investigations in the quantum domain.

Chapter 7

Summary and Outlook

This thesis was concerned with quantum entanglement. In the center of interest was the resource character of entanglement in applications of quantum information theory. It was the intention of this thesis to clarify the theoretical possibilities in entanglement manipulation. The subsequent list summarizes the major contributions of this thesis to the academic debate on this subject:

Chapter 2. Quantification of Quantum Entanglement.

This chapter studied proper measures of quantum entanglement. New measures were introduced, their properties were investigated, and they were compared to already known measures. In particular, a fully additive entanglement monotone was proposed, and in a numerical investigation the value for the regularized relative entropy of entanglement was evaluated – that is, the average degree of entanglement of infinitely many copies of a certain state with high symmetry. The last section of the chapter dealt with multi-particle quantum systems. For such systems a general measure of entanglement was defined, and the implications were investigated.

Chapter 3. Entanglement Transformations.

The emphasis in this chapter was on transformation criteria for mixed states. The underlying question was: what tasks can in principle be performed by using only local operations and classical communication when starting from a given mixed state. Abstract criteria were presented for the possibility of a transformation from a single copy of a mixed state to another mixed state. A particular example was studied in more detail, which turned out to be useful in the context of entanglement-assisted transformations: In entanglement-assisted transformations an auxiliary bi-partite quantum system in an entangled state is borrowed, the desired quantum operation is performed locally, and then the auxiliary system is returned while leaving it in exactly the same state. It was shown in this chapter that such operations are more powerful than ordinary LOCC also in the mixed state-domain. Several aspects were addressed, such as the possibility to increase the proportion of a certain pure state in a mixture by using entanglement-assisted transformations, or small transformations.

Chapter 4. Non-Local Implementation of Joint Unitary Operations.

This chapter focused on the realization of quantum gates on remotely located nodes. In order to enable the implementation of such gates, both the exchange of classical information and initial entanglement are necessary. Protocols for a non-local implementation were presented for several quantum gates: these are the quantum CNOT gate, the two-party control-U gate, the state swapper, the three-party Toffoli gate, the three-party control-U gate, and an the N-party control-U gate.

Chapter 5. Entanglement and Classical Information.

The amount of usable entanglement is related to how mich is known about the state of the system. In this chapter a relationship between the loss of distillable entanglement and the loss of classical information was established, in case that the loss of information is due to the loss of information about the identity of the quantum systems. After presenting simple examples, the chapter focused on proving that the change in distillable entanglement is bounded by the loss of classical information for arbitrarily many copies of pairs of qubits in any pure state. The proof made use of group theoretical methods. Finally, a more general result of a relation between entanglement and classical information was derived when the relative entropy of entanglement is taken as the measure of entanglement.

Chapter 6. Quantum Information and Game Theory.

Chapter 6 concentrated on the connection between quantum information theory and the theory of games. The main idea of this chapter was to associate game theoretical strategies with quantum operations. A general framework of the notion of a quantum game was introduced. Simple toy models were investigated in detail in order to clarify which new features may emerge in such a situation compared to a classical game.

Needless to say, none of the research topics that have been addressed in this thesis have exhaustively been represented. The field of quantum information theory in general, and the theory of quantum entanglement in particular, is a young field of research, and many interesting questions are still waiting to be resolved. Several issues studied in detail lead directly to challenging new research topics, especially those related to mixed-state entanglement of bi-partite systems and pure-state entanglement in the multi-particle domain. At the root of the open questions in multi-particle entanglement is the MREGS-problem. In the absence of a general framework of investigation it seems appropriate to concentrate on practically motivated examples in the near future.

Symmetries can help to greatly simplify many problems in entanglement theory and quantum information theory while preserving the characteristic features of the original problem. The idea of systematically exploring symmetries is not yet fully exhausted. For example, in the context of the asymptotic limit of many copies of a state one can expect to yield useful results in further research. Another problem amenable for the approach of utilizing symmetries is the problem of evaluating quantum channel capacities.

Finally, a possibly fruitful line of thought could be the idea of applying the mathematical tools of the theory of entanglement to problems which are typically conceived not to be part of quantum information theory. An example could be the investigation of the entanglement between a quantum system and its environment in the context of decoherence, dissipation, and environment-induced selection [55, 54]. A promising model is the so-called *quantum Browian motion model*. In this model one considers the dynamics of a quantum harmonic oscillator that is linearly coupled to a heat bath. This heat bath likewise consists of harmonic oscillators, which are initially in a state corresponding to the canonical distribution associated with a certain temperature. This model – also known as Caldeira-Leggett-model [189] – is a frequent starting point of inquiries of the "emergence" of classical properties of quantum systems, partly because the model is analytically solvable [190]. As for Gaussian states the typical questions related to entanglement are no more difficult to answer than those related to small finite-dimensional systems, this model might be well-suited for a scrutiny of the entanglement with the tools of quantum information theory.

Appendix A: The von Neumann Entropy and the Relative Entropy Functional

The notion of quantum entropy is of major importance in quantum information theory. Originally, the entropy of the state of a physical system is a concept from phenomenological thermodynamics and statistical mechanics – the respective notions have been shaped mainly by R. Clausius, L. Boltzmann, and J.W. Gibbs in the 19th century. Later, entropy was applied and extended to an information theoretical context by C.E. Shannon [191, 101]. Quite naturally, in quantum information theory quantum entropy became one of the most fundamental concepts. In this appendix the major properties of the von Neumann entropy and the relative entropy for quantum states will be summarized.

The *Shannon entropy* [191, 101] is a quantity associated with a probability distribution. It is defined as

$$H(p_1, ..., p_n) = -\sum_{i=1}^n p_i \log_2(p_i).$$
(7.1)

Since $p_1, ..., p_n$ is a probability distribution, it is required that $0 \le p_i \le 1$ for i = 1, ..., n and $\sum_{i=1}^{n} p_i = 1$. The Shannon entropy expresses the average information one gains when learning about the value of a random variable X which takes the value x_i with the respective probability p_i .

The intuition behind the Shannon entropy can be explained as follows: Consider a source that produces strings $x^{(1)}, x^{(2)}, ...$, where the symbols $x^{(i)}$ are taken from the set $\{x_1, ..., x_n\}$ for i=1,2,.... Each $x^{(i)}$ is a realization of the random variable X with the above probability distribution. So one may ask: what is the minimal number of bits that are needed in order to store the information produced by the source, in the sense that the produced string can later be recovered? The answer to this question is provided by *Shannon's noiseless coding theorem* [191]: the minimal number of bits per source symbol is given by $H(p_1,...,p_n)$, the Shannon entropy of the probability distribution associated with the source.

The "quantum analogue" of the Shannon entropy is the von Neumann entropy [192]. The von Neumann entropy is a measure for the degree of "mixedness of a quantum state". It vanishes for pure states, and for maximally mixed states of the form $\rho=\mathbb{1}/\dim[\mathcal{H}]$ it attains its maximal value, where \mathcal{H} is the underlying Hilbert space. The state space of a quantum system is no Choquet simplex, and thus, a generic mixed state ρ has many decompositions of the type $\rho=\sum_i p_i |\psi_i\rangle\langle\psi_i|$. As in such a decomposition the projections are not necessarily orthogonal, a different Shannon information $H=-\sum_i p_i \log_2 p_i$ belongs to

each probability distribution $p_1, p_2, ...$, each one associated with a different "degree of mixing". The von Neumann entropy of a given state ρ is then taken to be the minimal value of such a Shannon entropy of a decomposition of ρ . This infimum corresponds to the case where the projections in the above decomposition are pairwise orthogonal (and therefore could be conceived as disjoint events). In this spirit the von Neumann entropy $S(\rho)$ of a state ρ is defined as [192, 104, 193]

$$S(\rho) = -\text{tr}[\rho \log \rho]. \tag{7.2}$$

If $\rho = \sum_{i=1}^{N} p_i |\psi_i\rangle\langle\psi_i|$ is the spectral decomposition of ρ , then

$$S(\rho) = -\sum_{i=1}^{N} p_i \log_2(p_i). \tag{7.3}$$

Among the important properties are the following: Let σ , ρ and ρ_i , σ_i , i=1,2,..., be states taken from $\mathcal{S}(\mathcal{H})$. For a composite system with Hilbert space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ let $\rho^{(1)}$, $\sigma^{(1)} \in \mathcal{S}(\mathcal{H}^{(1)})$ and $\rho^{(2)}$, $\sigma^{(2)} \in \mathcal{S}(\mathcal{H}^{(2)})$.

- Positivity: $S(\rho) \ge 0$.
- Symmetry: If $\sigma = U\rho U^{\dagger}$ for all unitary operators U, then $S(\sigma) = S(\rho)$.
- Concavity: $S(\lambda \rho_1 + (1 \lambda)\rho_2) \ge \lambda S(\rho_1) + (1 \lambda)S(\rho_2)$ for any $\lambda \in [0, 1]$.
- Additivity: $S(\rho^{(1)} \otimes \rho^{(2)}) = S(\rho^{(1)}) + S(\rho^{(2)})$.
- Let $\rho^{(1)} = \operatorname{tr}_2[\rho]$ and $\rho^{(2)} = \operatorname{tr}_1[\rho]$ be the reduced states of ρ . Then $S(\rho) \leq S(\rho^{(1)}) + S(\rho^{(2)})$.
- Lower semi-continuity: If $\lim_{n \to \infty} \|\rho_n \rho\| = 0$, then

$$S(\rho) \le \lim_{n \to \infty} \inf S(\rho_n).$$
 (7.4)

Here and in the following $\|.\|$ denotes the trace norm.

■ Monotonicity property: $S(\mathcal{E}(\rho)) \geq S(\rho)$ for any completely positive unital (that is, doubly stochastic) map \mathcal{E} .

There is a close connection between the concept of von Neumann entropy and the theory of majorization. As stated in Chapter 3 a state ρ is called *more mixed than* σ , if there exists a doubly stochastic map $\mathcal E$ such that $\rho=\mathcal E(\sigma)$. This more mixed relation – abbreviated as \prec – implies a partial order on the state space. In terms of the eigenvalues of the states this relation can be expressed as follows: Let $p_1,...,p_N$ and $q_1,...,q_N$ be the lists of eigenvalues of states ρ and σ , respectively, satisfying $1 \geq p_1 \geq ... \geq p_N \geq 0$ and $1 \geq q_1 \geq ... \geq q_N \geq 0$. $\rho \prec \sigma$ is equivalent with the statement that

$$\sum_{i=1}^{k} p_i \le \sum_{i=1}^{k} q_i \text{ for all } k = 1, ..., N.$$
 (7.5)

A connection between the relation \prec and the von-Neumann entropy is that

$$\rho \prec \sigma$$
 implies that $S(\rho) \leq S(\sigma)$. (7.6)

Such a statement holds also under more general circumstances, in that $\rho \prec \sigma$ implies that $tr[f(\rho)] \leq tr[f(\sigma)]$ for any convex function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$. For quantum systems with

Hilbert space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2$ and states $\sigma,\rho\in\mathcal{S}(\mathcal{H})$ the relation $\rho\prec\sigma$ is equivalent with $S(\rho)\leq S(\sigma)$.

The von Neumann entropy can be regarded as a special case of another functional, the *relative entropy functional*. In its simplest version it is defined for two states ρ and σ as [104, 193, 194, 195, 196]

$$S(\sigma \| \rho) = \operatorname{tr}[\sigma(\log_2 \sigma - \log \rho)]. \tag{7.7}$$

This relative entropy of σ with respect to ρ gives a measure of how different σ is from ρ in the sense of statistical distinguishability [35]. If σ and ρ are identical, the relative entropy vanishes, the larger the value of relative entropy is, the more information can – roughly speaking – be obtained from a measurement discriminating between σ and ρ . The expression given by Eq. (7.7) has to be interpreted in a similar way as above. Let $\rho = \sum_{i=1}^N p_i |\psi_i\rangle \langle \psi_i|$ and $\sigma = \sum_{j=1}^N q_j |\phi_j\rangle \langle \phi_j|$ be the two spectral decompositions of ρ and σ , respectively. Then

$$S(\sigma||\rho) = \sum_{i,j=1}^{N} (q_j \log_2 q_j - q_j \log_2 p_i) |\langle \psi_i | \phi_j \rangle|^2.$$
 (7.8)

Among the fundamental properties of the relative entropy are the following:

- Positivity: $S(\sigma||\rho) \ge 0$.
- Nilpotence property: $S(\rho || \rho) = 0$.
- Joint convexity:

$$S(\lambda \sigma_1 + (1 - \lambda)\sigma_2 || \lambda \rho_1 + (1 - \lambda)\rho_2) \le \lambda S(\sigma_1 || \rho_1) + (1 - \lambda)S(\sigma_2 || \rho_2) \tag{7.9}$$

for any $\lambda \in [0,1]$.

- Additivity: $S(\sigma^{(1)} \otimes \sigma^{(2)} || \rho^{(1)} \otimes \rho^{(2)}) = S(\sigma^{(1)} || \rho^{(1)}) + S(\sigma^{(2)} || \rho^{(2)}).$
- Lower semi-continuity: If $\lim_{n \to \infty} \|\sigma_n \sigma\| = 0$ and $\lim_{n \to \infty} \|\rho_n \rho\| = 0$, then $S(\sigma\|\rho) \leq \lim_{n \to \infty} \inf S(\sigma_n\|\rho_n)$. If there exists a positive number λ satisfying $\sigma_n \leq \lambda \rho_n$, then $\lim_{n \to \infty} S(\sigma_n\|\rho_n) = S(\sigma\|\rho)$.
- Monotonicity property: For any completely positive unital map \mathcal{E} (that is, for any doubly stochastic map) $S(\mathcal{E}(\sigma)||\mathcal{E}(\rho)) \leq S(\sigma||\rho)$.
- Direct sum property: For all $\lambda \in [0, 1]$

$$S(\lambda \sigma_1 + (1 - \lambda)\sigma_2 \|\lambda \rho_1 + (1 - \lambda)\rho_2) = S(\sigma_1 \|\rho_1) + S(\sigma_2 \|\rho_2), \tag{7.10}$$

if $\sigma_1 \sigma_2 = \rho_1 \rho_2 = \sigma_1 \rho_2 = \sigma_2 \rho_1 = 0$.

 \blacksquare Invariance property: for every unitary U

$$S(U\sigma U^{\dagger}||U\rho U^{\dagger}) = S(\sigma||\rho). \tag{7.11}$$

It should be noted that the relative entropy functional is no metric. In particular, it is not invariant under interchange of its arguments, i.e., in general $S(\sigma \| \rho) \neq S(\rho \| \sigma)$ for two states ρ and σ .

Appendix B: Numerical Evaluation of the Optimal PPT States

In Chapter 2 the quantities $B_R(\sigma^{\otimes n}||\rho_n)$ have been evaluated, n=1,2,... . The states ρ_n are given by

$$\rho_n = \sum_{k=0}^n \frac{p_k}{\binom{n}{k}} \sum_{\pi \in S_n} (\pi \otimes \pi) \left(\sigma_a^{\otimes k} \sigma_s^{\otimes (n-k)} \right) (\pi \otimes \pi), \tag{7.12}$$

where $p_0, ..., p_n$ is a probability distribution. The first seven probability distributions can be evaluated as

$$n=1$$
: $p_0 = \frac{1}{2}, \ p_1 = \frac{1}{2}$ (7.13)

$$n=2$$
: $p_0=\frac{2}{3}$, $p_1=0$, $p_2=\frac{1}{3}$, (7.14)

$$n=3$$
: $p_0 = \frac{4}{5}$, $p_1 = 0$, $p_2 = 0$, $p_3 = \frac{1}{5}$, (7.15)

$$n=4$$
: $p_0=\frac{3}{8}$, $p_1=\frac{1}{2}$, $p_2=0$, $p_3=0$, $p_4=\frac{1}{8}$, (7.16)

$$n = 5$$
: $p_0 = \frac{33}{106}$, $p_1 = \frac{45}{106}$, $p_2 = \frac{10}{53}$, $p_3 = 0$, $p_4 = 0$, $p_5 = \frac{4}{53}$, (7.17)

$$n = 6$$
: $p_0 = \frac{23}{87}$, $p_1 = \frac{30}{87}$, $p_2 = \frac{30}{87}$, $p_3 = 0$, $p_4 = 0$, $p_5 = 0$, $p_6 = \frac{4}{87}$, (7.18)

$$n=7$$
: $p_0=\frac{7}{24}$, $p_1=\frac{7}{72}$, $p_2=\frac{7}{12}$, $p_3=0$,

$$p_4 = 0, \ p_5 = 0, \ p_6 = 0, \ p_7 = \frac{1}{36}.$$
 (7.19)

For example,

$$\rho_1 = \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_s, \quad \rho_2 = \frac{1}{3}\sigma_a^{\otimes 2} + \frac{2}{3}\sigma_s^{\otimes 2}, \quad \rho_3 = \frac{1}{5}\sigma_a^{\otimes 3} + \frac{4}{5}\sigma_s^{\otimes 3}. \tag{7.20}$$

Appendix C: The Trace Norm Measure

In this appendix a further measure of entanglement will be proposed which involves the trace norm. This measure is defined in exactly the same way as the modified relative entropy of entanglement E_M , except that the relative entropy functional is replaced by the trace norm difference. In the notation of Chapter 2 this measure E_T is defined as

$$E_T(\sigma) = \min_{\rho \in \mathcal{D}_{\sigma}(\mathcal{H})} \|\sigma - \rho\|. \tag{7.21}$$

The subsequent proposition shows that E_T has all the properties of an entanglement monotone. It does not – however – coincide with $S(\text{tr}[|\psi\rangle\langle\psi|])$ for pure states $|\psi\rangle\langle\psi|$.

Proposition C.1. – E_T is an entanglement monotone.

Proof: Clearly, $E_T(\rho)=0$ for a separable state ρ . The set $\mathcal{D}_{\sigma}(\mathcal{H})$ is a compact and convex set for each $\sigma\in\mathcal{S}(\mathcal{H})$, and hence, E_T is also a convex functional by virtue of the triangle inequality. The remaining task is to show that in a local generalized measurement associated with Kraus operators $A_1,...,A_K$ the value of E_T may only decrease on average, that is, $\sum_{i=1}^K p_i E_T(\sigma_i) \leq E_T(\sigma)$ for all states σ , where $p_i = \operatorname{tr}[A_i \sigma A_i^{\dagger}]$ and $\sigma_i = A_i \sigma A_i^{\dagger}/p_i$ (see Chapter 2). The first ingredient to the proof is the fact that $\operatorname{tr}[A_i \rho A_i^{\dagger}] = \operatorname{tr}[A_i \sigma A_i^{\dagger}] = p_i$ for all $\rho \in \mathcal{D}_{\sigma}(\mathcal{H})$, and hence,

$$\sum_{i=1}^{K} p_{i} E_{T}(\sigma_{i}) = \sum_{i=1}^{K} p_{i} \min_{\rho_{i} \in \mathcal{D}_{\sigma_{i}}(\mathcal{H})} \|A_{i} \sigma A_{i}^{\dagger} / p_{i} - \rho_{i}\|$$

$$\leq \sum_{i=1}^{K} p_{i} \min_{\rho \in \mathcal{D}_{\sigma}(\mathcal{H})} \frac{\|A_{i} \sigma A_{i}^{\dagger} - A_{i} \rho A_{i}^{\dagger}\|}{p_{i}}$$

$$= \min_{\rho \in \mathcal{D}_{\sigma}(\mathcal{H})} \sum_{i=1}^{K} \|A_{i} (\sigma - \rho) A_{i}^{\dagger}\|. \tag{7.22}$$

The second ingredient is the statement of Lemma C.2,

$$\sum_{i=1}^{K} p_i E_T(\sigma_i) \leq \min_{\rho \in \mathcal{D}_{\sigma}(\mathcal{H})} \sum_{i=1}^{K} \|A_i^{\dagger} A_i | \sigma - \rho \| \|$$

$$= \min_{\rho \in \mathcal{D}_{\sigma}(\mathcal{H})} \|\sigma - \rho\| = E_T(\sigma). \tag{7.23}$$

Hence, E_T is an entanglement monotone.

It should be noted that the weaker condition $E_T(\mathcal{E}(\sigma)) \leq E_T(\sigma)$ for all LOCC \mathcal{E} and all states σ follows immediately from the fact that the trace norm fulfils

$$\|\mathcal{E}(\sigma) - \mathcal{E}(\rho)\| \le \|\sigma - \rho\| \tag{7.24}$$

for all quantum operations \mathcal{E} and all states σ, ρ [117].

Lemma C.2. – Let A, B be $n \times n$ matrices, and assume that $B = B^{\dagger}$. Then

$$||ABA^{\dagger}|| \le ||A^{\dagger}A|B||| \tag{7.25}$$

holds.

Proof: The trace norm is a unitarily invariant norm, and ABA^{\dagger} is a normal matrix.¹ Hence [100],

$$||A(BA^{\dagger})|| \le ||(BA^{\dagger})A||.$$
 (7.26)

Eq. (7.25) is a consequence of

$$\|(BA^{\dagger})A\| = \operatorname{tr}[(A^{\dagger}AB^{\dagger}BA^{\dagger}A)^{(1/2)}] = \operatorname{tr}[(A^{\dagger}A|B|^2A^{\dagger}A)^{(1/2)}] = \|A^{\dagger}A|B|\|. \quad (7.27)$$

¹An $n \times n$ -matrix M is called *normal*, if $MM^{\dagger} = M^{\dagger}M$. A norm $\|.\|$ on $n \times n$ -matrices is said to be *unitarily invariant*, if $\|UMV\| = \|M\|$ for all unitary U, V [100].

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Notations

\mathcal{H} $\mathcal{S}(\mathcal{H})$ $\mathcal{D}(\mathcal{H})$ $\mathcal{P}(\mathcal{H})$ $\mathcal{D}_{\sigma}(\mathcal{H})$ $\mathcal{P}_{\sigma}(\mathcal{H})$	Hilbert space State space associated with a Hilbert space \mathcal{H} Subset of $\mathcal{S}(\mathcal{H})$ of separable states Subset of $\mathcal{S}(\mathcal{H})$ of PPT states Subset of $\mathcal{D}(\mathcal{H})$ of separable states which are locally identical to σ Subset of $\mathcal{P}(\mathcal{H})$ of PPT states which are locally identical to σ
$\begin{array}{l} \text{dim} \\ \text{range} \\ \text{tr}_A \\ . \\ \otimes \\ \oplus \\ \prec \\ \mathcal{E} \\ E_1,, E_K \\ A_1,, A_K \\ A_1(A_2A_3) \end{array}$	Dimension of a vector space Range of a linear operator Partial trace with respect to system A Trace norm Tensor product Direct sum Majorization relation Completely positive linear map Kraus operators Kraus operators of a local quantum operation in system A A particular split of a system consisting of parts A_1 , A_2 , and A_3
B_{R} B_{M}^{∞} D_{C} D_{\leftrightarrow} D_{\rightarrow} E_{C} E_{F} E_{G} E_{F} E_{R} E_{N} E_{μ} E_{M} E_{S} E_{M} E_{S}	Relative entropy of entanglement with respect to PPT states Regularized relative entropy of entanglement with respect to PPT states Modified relative entropy of entanglement with respect to PPT states Distillable entanglement with respect to the class of operations C Distillable entanglement with respect to LOCC operations Distillable entanglement with respect to one-local operations Concurrence Entanglement of formation General distance measure based on the relative entropy Regularized entanglement of formation Relative entropy of entanglement Negativity Entanglement monotone defined in Proposition 2.6 Modified relative entropy of entanglement Schmidt measure
$E_S^{A_1(A_2A_3)}$ E_T E_X	Schmidt measure with respect to the split $A_1(A_2A_3)$ Trace norm measure Minimal asymptotic preparation cost of a multi-particle state

$ \begin{array}{l} (\sigma_n)_{n\in\mathbb{N}} \\ \rho^{T_A} \end{array} $ $ \rho^{\otimes n} \\ \sigma_a \\ \sigma_s \\ \rho_W \\ \psi^+\rangle, \psi^-\rangle, \phi^+\rangle, \phi^-\rangle \\ GHZ\rangle \\ W\rangle \\ \rho_M $	Series of states Partial transpose of ρ with respect to a certain basis of system A n -fold tensor product of a state ρ Antisymmetric state of a bi-partite quantum system Symmetric state of a bi-partite quantum system Werner state State vectors of the four Bell states State vector of the GHZ state of three qubits State vector of the W state of three qubits Molecule state
$F(\sigma, \rho)$ $H(p_1,, p_n)$ $S(\rho)$ $S(\sigma \rho)$	Fidelity of σ with respect to ρ Shannon entropy of the probability distribution $p_1,,p_n$ von Neumann entropy of ρ Relative entropy functional of σ with respect to ρ
$\sigma ightarrow ho$ under LOCC $\sigma ightarrow ho \ \ { m under ELOCC}$ $\Delta D_{\leftrightarrow} \ \ \Delta E_R \ \ \Delta I$	The state σ can be transformed into ρ under local operations with classical communication. The state σ can be transformed into ρ under entanglement-assisted local operations with classical communication. Change in distillable entanglement. Change in relative entropy of entanglement. Change in classical information
S_n π $SU(2)$ $D^{(j)}$	Symmetric group of degree n Permutation and corresponding unitary Special unitary group of degree 2 Spin- j irreducible representation of $SU(2)$
S_A P_A u_A $\Gamma = (\{A, B\}, S_A, S_B, u_A, u_B)$ $\Gamma = (\mathcal{H}, \rho, S_A, S_B, P_A, P_B)$	Set of quantum strategies of Alice Alice's utility functional Alice's utitily function (Classical) game Quantum game

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