PHYS 501: Mathematical Physics I

Fall 2014

Homework #1

(Due: October 6, 2014)

1. (a) Let $\mathbf{a}$ and $\mathbf{b}$ be any two vectors in a real linear vector space, and define $\mathbf{c} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a scalar. By requiring that $\mathbf{c}\cdot\mathbf{c} \geq \mathbf{0}$ for all λ, derive the Cauchy-Schwartz inequality

$$(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{b}) \geq (\mathbf{a}\cdot\mathbf{b})^2\,.$$

When does equality hold? (Use only the general properties of the inner product. Do *not* assume that it is possible to write $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.)

(b) Let A be any square matrix, and define

$$B = e^A \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!}\,.$$

Prove that an eigenvector of A with eigenvalue λ is an eigenvector of B, with eigenvalue e^λ.

2. Consider the 4-dimensional vector space of polynomials of degree less than or equal to 3, on the range $-1 \leq x \leq 1$, spanned by the basis set $\{1, x, x^2, x^3\}$. The inner product of two polynomials in this space is defined as

$$(f,g) = \int_{-1}^{1} |x| f(x) g(x)\,dx.$$

Use Gram–Schmidt orthogonalization to construct two orthonormal basis sets, as follows:

(i) start with the set as listed above and begin the procedure with the function 1, as in class (note that the weighting is different from the class example!).

(ii) rewrite the set as $\{x^2, x, 1, x^3\}$ and begin the orthogonalization procedure starting with x^2.

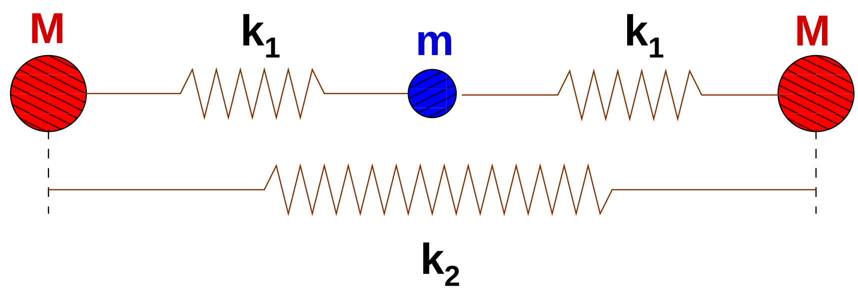
Write down the matrix representing the transformation from basis (i) to basis (ii), and demonstrate that it is orthogonal.

3. (a) Transform the matrix A and the vector $\mathbf{x}$ below into a coordinate system in which A is diagonal, with the diagonal elements *increasing* from top to bottom. Write down the transformation matrix, the diagonalized A, and the transformed $\mathbf{x}$.

$$A = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & i & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \\ -1 \end{pmatrix}$$

(b) A matrix B has real eigenvalues. Does it necessarily follow that B is hermitian?

4. Find the normal modes and normal frequencies for linear vibrations (i.e. vibrations in the horizontal direction, as drawn) of the (over)simplified "CO_2 molecule" modeled by the collection of masses and springs sketched below.



Describe qualitatively the appearance of each normal mode.