Quantum II HW3

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April 29, 2015

1 Problem 1

a. We expand the trace of both AB and BA:

$$trace(AB) = \sum_i \sum_j a_{ij} b_{ji} \tag{1.1}$$

$$trace(BA) = \sum_i \sum_j b_{ij} a_{ji} \tag{1.2}$$

But i and j are just index variables over the same range, so we can write 1.2 as:

$$trace(BA) = \sum_j \sum_i b_{ji} a_{ij} \tag{1.3}$$

This is a sum of the same terms as 1.1, so $trace(AB) = trace(BA)$.
b. We define the product matrix $BCD...Z \equiv P$. Then we have $trace(AP) = trace(PA)$ which we have proved in part a.
c. The proof in A holds only for the diagonal elements used in the trace (since they have the same row/column indices). The general elements of BC and CB will be different, so $trace(ABC) \neq trace(ACB)$.
d. The diagonal elements of $|u\rangle \langle v|$ are $u_i v_i$, so the trace is $\sum u_i v_i = \langle v|u\rangle$.

2 Problem 2

For each matrix, we check its eigenvalues and symmetry to see if it is a valid density matrix. If it is a density matrix we determine if it represents a pure state through the eigenvalues (one eigenvalue is 1, others are 0). If it does represent a pure state, we determine the pure state by finding the eigenvector corresponding to the eigenvalue that equals 1.

ρ_1 and ρ_4 have negative eigenvalues and are therefore not valid desnity matrices.
In ρ_3, collecting the three ineer products into one constant C, we have:

$$\rho_3 = \frac{C}{3} |u\rangle \langle v| \tag{2.1}$$

$$\rho_3 = \frac{C}{3} \langle v|u\rangle = 0 \tag{2.2}$$

So ρ_3 is not a valid density matrix.
ρ_5 is a valid density matrix but does not represent a pure state.
ρ_2 is a valid density matrix that represents the pure state $\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$.

3 Problem 3

With the two eignvalues 1,-1 the operator of the dynamical variable is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.
The states in vector form are:

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.1}$$

$$|-\rangle = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{3.2}$$

a. For the pure states $|\theta\rangle = \sqrt{\frac{1}{2}} \left(|+\rangle + e^{i\theta} |-\rangle\right)$ we find:

$$\langle\theta| M |\theta\rangle = \langle\theta| \sqrt{\frac{1}{2}} \begin{bmatrix} |+\rangle \\ -e^{i\theta} |-\rangle \end{bmatrix} \tag{3.3}$$

$$\langle\theta| M |\theta\rangle = \frac{1}{2} \left(\langle+|+\rangle - (e^{-i\theta} e^{i\theta}) \langle-|-\rangle\right) \tag{3.4}$$

$$\langle\theta| M |\theta\rangle = \frac{1}{2}(1 - 1) = 0 \tag{3.5}$$

For the mixed state $\rho = \frac{1}{2}(\left|+\right\rangle\left\langle+\right| + \left|-\right\rangle\left\langle-\right|)$, writing the inner product as a constant C, we find:

$$\rho = \frac{C}{2}(\left|+\right\rangle\left\langle-\right|) \tag{3.6}$$

$$\rho^{\dagger}M\rho = \rho^{\dagger}\frac{C}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \tag{3.7}$$

$$\rho^{\dagger}M\rho = \rho^{\dagger}\frac{C}{2}\begin{bmatrix}1\\0\end{bmatrix} \tag{3.8}$$

$$\rho^{\dagger}M\rho = \frac{C^2}{4}\left|+\right\rangle\left\langle-\right|\left|+\right\rangle = \frac{C^2}{4}\left|+\right\rangle\{\left\langle-|+\right\rangle\} = 0 \tag{3.9}$$

Since $\left|+\right\rangle, \left|-\right\rangle$ are orthogonal.
b. The pure and mixed states can be distinguished by studying their time evolution. The pure state evolution is just a phase factor, whereas the mixed state will undergo amplitude transitions.

4 Problem 4

To find the probability that σ_y is positive we first find its eigenstates. Solving the inidicial equation we find that the eigenvalues are $\pm\frac{\hbar}{2}$ as expected. We write the eigenvectors as $e_{-\hbar/2} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)$ and $e_{\hbar/2} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} i \\ -1 \end{smallmatrix}\right)$.
We project the state vector $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ on to the state with energy $\frac{\hbar}{2}$ by:

$$\frac{1}{\sqrt{2}}\left\langle\alpha\ \beta|i\ \ -1\right\rangle = \frac{\alpha i - \beta}{\sqrt{2}} \tag{4.1}$$

Since $\frac{\hbar}{2}$ is the only positive state the probahility that σ_y is positive is $\frac{\alpha^2+\beta^2}{2}$.

5 Problem 5

For the operator M the eigenvalues are $\{-\sqrt{2}, 0, \sqrt{2}\}$. The eigenvector correspong to e=0 is $e_2 = [-\frac{1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}]$. We find the conditional probability $P(M = 0|p_i) = \left\langle e_2\right| M \left|e_2\right\rangle$.

$$P(M = 0|\rho_1) = \frac{1}{4} \tag{5.1}$$

$$P(M = 0|\rho_2) = 0 \tag{5.2}$$

$$P(M = 0|\rho_3) = \frac{1}{2} \tag{5.3}$$

6 Problem 6

For $R = \left(\begin{smallmatrix} 6 & -2 \\ -2 & 9 \end{smallmatrix}\right)$ and an arbitrary state vector $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ we calcualte $\langle R^2 \rangle$ two ways.
a. We calculate $\langle R^2 \rangle$ directly.

$$R^2 = \begin{bmatrix} 40 & -20 \\ -30 & 35 \end{bmatrix} \tag{6.1}$$

$$\langle \Psi | R^2 | \Psi \rangle = 40a^2 - 60ab + 85b^2 \tag{6.2}$$

b. We find the eigenvalues and eigenvectors of R.

$$R = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} \tag{6.3}$$

$$(6 - \lambda)(9 - \lambda) - 4 = 0 \tag{6.4}$$

$$\lambda^2 - 15\lambda + 50 = 0 \tag{6.5}$$

$$\lambda = \{5, 10\} \tag{6.6}$$

$$e_5 : x_1 = 2x_2,\ e_5 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{6.7}$$

$$e_{10} : x_1 = -\frac{1}{2} x_2,\ e_1 0 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{6.8}$$

We can write $|a\ b\rangle$ as a linear combination of the two eigenvectors by calculating the projection of $|a\ b\rangle$ onto each one.

$$\langle a\ b | e_5 \rangle = \frac{2a + b}{\sqrt{5}} \tag{6.9}$$

$$\langle a\ b | e_{10} \rangle = \frac{a - 2b}{\sqrt{5}} \tag{6.10}$$

We can now evaluate $\langle R^2 \rangle$.

$$\langle R^2 \rangle = \langle a\ b | e_5 \rangle^2 5^2 + \langle a\ b | e_{10} \rangle^2 10^2 \tag{6.11}$$

$$\langle R^2 \rangle = 40a^2 - 60ab + 85b^2 \tag{6.12}$$

6.12 and 6.2 are the same result as expected.