PHYS 502: Mathematical Physics II

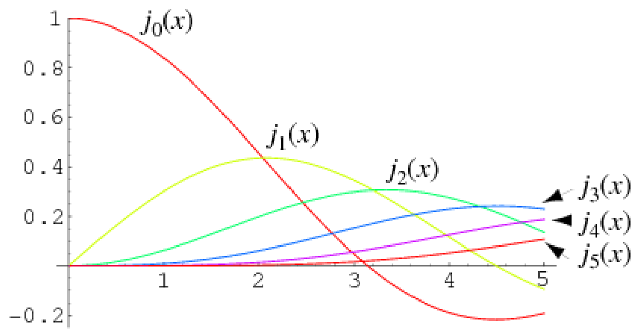
Winter 2014

Solutions to Homework #3

1. The solutions to the wave equation in a sphere are of the form

$$u(r,\theta,\phi) = j_l(kr)P_l^m(\cos\theta)e^{im\phi},$$

for integer l and m. The boundary condition $\partial u/\partial r = 0$ at $r = R$ requires $j_l'(kR) = 0$. As illustrated in the figure below, the three lowest allowed values of kR correspond, respectively, to the first zeros of j_1' and j_2', and the second zero of j_0'.



Since

$$j_0(x) = \frac{\sin x}{x},$$

we have

$$j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2},$$

so $j_0'(x) = 0 \to \tan x = x$, or $x = 4.49$. Similarly, since

$$\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, \\ j_2(x) &= \sin x\left(\frac{3}{x^2} - 1\right) - \frac{3\cos x}{x^2}, \end{aligned}$$

$j_1'(x) = 0$ for $x = 2.08$, $j_2'(x) = 0$ for $x = 3.34$. (Note that the first zero of j_3' is at $x = 4.52$.) Thus, the three lowest frequencies are $\omega = kc = 2.08c/R, 3.34c/R, 4.49c/R$.

2. The equation to be solved is

$$\nabla^2 n + \lambda n = \frac{1}{\kappa}\frac{\partial n}{\partial t},$$

where $\lambda, \kappa > 0$ and $n = 0$ on the surface. For assumed time dependence $n \sim e^{\alpha t}$, the equation becomes

$$\nabla^2 n + k^2 n = 0,$$

where $k^2 = \lambda - \alpha/\kappa$. The critical case has $\alpha = 0$, or $k^2 = \lambda$.

(a) For a sphere, the general solution is $n \sim j_l(kr)P_l^m(\cos\theta)\, e^{im\phi}$. The surface boundary condition is $j_l(kR) = 0$, and the minimum k corresponds to the first root of j_0, so $l = m = 0$. Since $j_0(x) \sim \sin x/x$, we find $kR = \pi$ and the critical radius is

$$R_0 = \frac{\pi}{k} = \frac{\pi}{\sqrt{\lambda}}\,.$$

Note that, in order to satisfy the boundary condition, increasing R has the effect of decreasing k and hence of increasing $\alpha = \kappa(\lambda - k^2)$. Thus the sphere is unstable for $R > R_0$.

(b) For a *hemisphere*, the extra boundary condition at $\theta = \pi/2$ means that the $l = 0$ mode is not a solution. We now require $P_l^m(\cos\theta) = 0$ at $\theta = \pi/2$ (where we have assumed that the z axis is the axis of symmetry of the hemisphere). The lowest-order P_l^m satisfying the boundary condition is $P_1^0 = \cos\theta$, so $l = 1$ and the radial boundary condition becomes $j_1(kR) = 0$. Since $j_1(x) \sim \sin x/x^2 - \cos x/x$, the first zero has $x = \tan x$, or $x = 1.43\pi = 4.49$. The critical ($\alpha = 0$) radius for this geometry then is

$$R_1 = \frac{1.43\pi}{k} = \frac{1.43\pi}{\sqrt{\lambda}} = 1.43 R_0\,.$$

(c) Now the system is spherical again, but the radius is $R_1 > R_0$ and the system is unstable. Writing $\beta = 1.43$, the boundary condition now implies

$$\begin{aligned} kR_1 = \left(\lambda - \frac{\alpha}{\kappa}\right)^{1/2} R_1 &= \pi \\ \Rightarrow \quad \alpha &= \kappa\lambda\left(1 - \beta^{-2}\right). \end{aligned}$$

The growth time scale therefore is

$$\tau = \alpha^{-1} = \left(\frac{\beta^2}{\beta^2 - 1}\right)\frac{1}{\kappa\lambda} = \frac{1.96}{\kappa\lambda}\,.$$

3. The temperature satisfies the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa\,\nabla^2 T.$$

Separating out the time dependence $T(\mathbf{x}, t) = \chi(\mathbf{x})e^{-\kappa k^2 t}$, we have

$$\nabla^2\chi + k^2\chi = 0,$$

with χ regular as $r = |\mathbf{x}| \to 0$ and $\chi = 0$ at $r = b$. The general solution is a sum of terms of the form

$$\chi(r, \phi) = J_m(kr)\, e^{im\phi},$$

where we have assumed that the solution is independent of z. The axisymmetric initial and boundary conditions imply that only the $m = 0$ term contributes, and the boundary condition at $r = b$ implies $J_0(kb) = 0$, so $k = k_n = \alpha_{0n}/b$, where α_{mn} is the n-th root of J_m. Thus the solution is

$$T(r, t) = \sum_n a_n\, J_0(k_n r)\, e^{-\kappa k_n^2 t}.$$

We determine the a_n by satisfying the initial conditions:

$$u(r,0) = T_0 = \sum_n a_n\, J_0\left(\frac{\alpha_{0n} r}{b}\right).$$

Inverting this Bessel series gives

$$a_n = \frac{2T_0}{b^2 J_1^2(\alpha_{0n})} \int_0^b J_0\left(\frac{\alpha_{0n} r}{b}\right) r dr.$$

We can evaluate the integral using the recurrence relation $xJ_0(x) = [xJ_1(x)]'$, to find

$$\begin{aligned}
\int_0^b J_0\left(\frac{\alpha_{0n} r}{b}\right) r dr &= \frac{b^2}{\alpha_{0n}^2} \int_0^{\alpha_{0n}} sJ_0(s)\, ds \\
&= \frac{b^2}{\alpha_{0n}^2} \int_0^{\alpha_{0n}} [sJ_1(s)]'\, ds \\
&= \frac{b^2}{\alpha_{0n}} J_1(\alpha_{0n}),
\end{aligned}$$

resulting in

$$a_n = \frac{2T_0}{\alpha_{0n} J_1(\alpha_{0n})}.$$

The central temperature is

$$\begin{aligned}
T(0,t) &= \sum_n a_n\, e^{-\kappa k_n^2 t} \\
&\approx a_1\, e^{-\kappa k_1^2 t} = \frac{2T_0}{\alpha_{01} J_1(\alpha_{01})}\, e^{-\kappa \alpha_{01}^2 t/b^2},
\end{aligned}$$

where the leading term dominates the sum if

$$\kappa t(\alpha_{02}^2 - \alpha_{01}^2)/b^2 \gg 1,$$

or (since $\alpha_{01} = 2.40$, $\alpha_{02} = 5.52$)

$$t \gg \frac{b^2}{24.7\kappa}.$$

4. (a) The general regular solution (in polar coordinates) to the 2-D Helmholtz equation is

$$u(r,\theta) = \sum_m J_m(kr)\,(a_m \cos m\theta + b_m \sin m\theta).$$

The boundary condition $u(R,\theta) = f(\theta)$ implies

$$\begin{aligned}
\sum_m J_m(kR)\,(a_m \cos m\theta + b_m \sin m\theta) &= f(\theta) \\
&= \sum_m (A_m \cos m\theta + B_m \sin m\theta),
\end{aligned}$$

where

$$\begin{aligned}
A_m &= \frac{1}{\pi} \int_0^{2\pi} f(\theta') \cos m\theta'\, d\theta' \qquad (\times\tfrac{1}{2} \text{ for } m=0) \\
B_m &= \frac{1}{\pi} \int_0^{2\pi} f(\theta') \sin m\theta'\, d\theta' \qquad (m > 0).
\end{aligned}$$

Hence

$$a_m = \frac{A_m}{J_m(kR)}, \qquad b_m = \frac{B_m}{J_m(kR)},$$

and so

$$u(r,\theta) = \int_0^{2\pi} K(r,\theta,\theta') f(\theta')\, d\theta',$$

where

$$\begin{aligned} K(r,\theta,\theta') &= = \sum_m \frac{J_m(kr)}{\pi J_m(kR)} \left(\cos m\theta \cos m\theta' + \sin m\theta \sin m\theta'\right) \\ &= \frac{1}{\pi} \sum_m \frac{J_m(kr)}{J_m(kR)} \cos m(\theta - \theta') \end{aligned}$$

(again with an extra factor of $\frac{1}{2}$ in the $m = 0$ term).

(b) For $f(\theta) = \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we have $A_0 = A_2 = \frac{1}{2}$, and all the other As and Bs are zero. Hence

$$u(r,\theta) = \tfrac{1}{2}\left[\frac{J_0(kr)}{J_0(kR)} + \frac{J_2(kr)}{J_2(kR)} \cos 2\theta\right].$$

5. Bessel's equation is

$$x^2 y + xy + (x^2 - m^2) y = 0\,.$$

Seeking a series solution of the form $y(x) = x^\alpha \sum_{n=0}^{\infty} c_n x^n$ and substituting into the equation we find, as usual

$$\begin{aligned} \alpha &= \pm m\,, \\ c_1 &= 0\,, \\ c_n &= \frac{-c_{n-2}}{(n+\alpha)^2 - m^2}\,. \end{aligned}$$

(i) For $m = \frac{1}{2}, \alpha = \frac{1}{2}$, we have $(n+\alpha)^2 - m^2 = (n+1)n$, so

$$\begin{aligned} c_{2k} &= \frac{(-1)^k c_0}{(2k+1)!}\,, \\ J_{\frac{1}{2}}(x) &= c_0 x^{\frac{1}{2}} (1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots) \\ &= c_0 x^{-\frac{1}{2}} \sin x\,. \end{aligned}$$

(ii) For $m = \frac{1}{2}, \alpha = -\frac{1}{2}$, we have $(n+\alpha)^2 - m^2 = n(n-1)$, so

$$\begin{aligned} c_{2k} &= \frac{(-1)^k c_0}{2k!}\,, \\ J_{\frac{1}{2}}(x) &= c_0 x^{-\frac{1}{2}} (1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots) \\ &= c_0 x^{-\frac{1}{2}} \cos x\,. \end{aligned}$$

Using the recurrence relation $J_{m+1}(x) = (2m/x) J_m(x) - J_{m-1}(x)$ (and setting each $c_0 = 1$ for simplicity), we find

$$\begin{aligned} J_{\frac{3}{2}}(x) &= x^{-1} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) &= x^{-\frac{3}{2}} (\sin x - x \cos x) \\ J_{\frac{5}{2}}(x) &= 3x^{-1} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) &= x^{-\frac{5}{2}} (3 \sin x - 3x \cos x - x^2 \sin x)\,. \end{aligned}$$