Electromagnetic Theory II HW4

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7.8

a) Inside each slab of linear, lossless, nondispersive media both the forward and backward waves will propagate without interaction or attenuation. They will only undergo a phase shift of kt_j ($-kt_j$ for the backward wave) where t_j is the thickness of slab j and $k = \frac{n_j \omega}{c}$. The transfer matrix is then:

$$T_j(n_j, t_j) = \begin{bmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{bmatrix} \tag{0.1}$$

Since $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ we can use Euler's formula to write this transfer matrix as:

$$T_j(n_j, t_j) = \begin{bmatrix} \cos(k_j t_j) + i \sin(k_j t_j) & 0 \\ 0 & \cos(k_j t_j) - i \sin(k_j t_j) \end{bmatrix} \tag{0.2}$$

$$T_j(n_j, t_j) = I \cos(k_j t_j) + i\sigma_3 \sin(k_j t_j) \tag{0.3}$$

b) Across a zero-width boundary between layers the phases of the forward and backward waves must be identical on both sides of the boundary. For normal incidence, with $\mu' = \mu$, the transmitted and reflected components of an incident wave (Jackson 7.39) can be wrtten:

$$E_{trans} = 2\left(1 + n'/n\right)^{-1} E_{incident} \tag{0.4}$$

$$E_{reflect} = \frac{n'/n - 1}{n'/n + 1} E_{incident} \tag{0.5}$$

We consider the backward-propagating wave in region 1 as the combination of the reflected part of the forward-propagating wave in region 1 and the transmitted part of the backward-propagating wave in region 2. Defining $\beta^+ \equiv \frac{1}{2}(n_1/n_2 + 1), \beta^- \equiv \frac{1}{2}(n_1/n_2 - 1)$:

$$E_- = E'_- \frac{1}{\beta^+} + E_+ \frac{\beta^-}{\beta^+} \tag{0.6}$$

$$E'_- = \beta^+ E_- - \beta^- E_+ \tag{0.7}$$

Defining $n \equiv n_1/n_2$ we have recovered the t_{21} and t_{22} transfer matrix elements. Across the two layers we must have $E_+ + E_- = E'_+ + E'_-$. Therefore must have $t_{11} = t_{22}$ and $t_{12} = t_{21}$. Using the Pauli matrix $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we can now write the transfer matrix as:

$$T_{i\to j} = I\frac{n+1}{2} - \sigma_1\frac{n-1}{2} \tag{0.8}$$

c) There will be no backward-propagating wave in the final semi-infinite region. Therefore the expression for total transmission and reflection by the stack is:

$$\begin{bmatrix} E'_+ \\ 0 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \end{bmatrix} \tag{0.9}$$

From the second row in this matrix equation it is clear that $E_- = -\frac{t_{21}}{t_{22}}E_+$. Plugging that result into the first row gives:

$$E'_+ = \left(t_{11} - \frac{t_{12}t_{21}}{t_{22}}\right)E_+ \tag{0.10}$$

$$E'_+ = \left(\frac{t_{11}t_{22} - t_{12}t_{21}}{t_{22}}\right)E_+ \tag{0.11}$$

$$E'_+ = \left(\frac{det(T)}{t_{22}}\right)E_+ \tag{0.12}$$

$$\tag{0.13}$$

7.12

a) Starting in the time domain with the charge/current equation, and using Ohm's law ($J = \sigma E$):

$$\nabla \cdot \mathbf{J}(\mathbf{x}, t) + \frac{\partial \rho(\mathbf{x}, t)}{\partial t} = 0 \tag{0.14}$$

$$\sigma\nabla \cdot \mathbf{E}(\mathbf{x}, t) + \frac{\partial \rho(\mathbf{x}, t)}{\partial t} = 0 \tag{0.15}$$

Tranforming to the Fourier domain, using the identity $F(dG/dx) = -i\omega F(G(x))$:

$$\sigma(\omega)\nabla \cdot \mathbf{E}(\mathbf{x}, \omega) - i\omega\rho(\mathbf{x}, \omega) = 0 \tag{0.16}$$

Since $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ we have:

$$(\sigma(\omega) - i\omega\epsilon_0)\,\rho(\mathbf{x}, \omega) = 0 \tag{0.17}$$

b) Plugging in the given representation for $\sigma(\omega)$:

$$\left(\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau} - i\omega\epsilon_0\right)\rho(\mathbf{x},\omega) = 0 \tag{0.18}$$

$$\left(\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau} - i\omega\epsilon_0\right) = 0 \tag{0.19}$$

$$\omega_p^2\tau = i\omega + \omega^2\tau \tag{0.20}$$

$$\omega = \frac{-i \pm \sqrt{4\tau^2\omega_p^2 - 1}}{2\tau} \tag{0.21}$$

In the approximation $\omega_p\tau \gg 1$ the roots are $-\frac{i}{2\tau} \pm \omega_p$. All other frequencies would require that $\rho = 0$ from eq. 5. Since ρ is only non-zero at two frequencies, it is straightforward to evaluate the inverse Fourier transform:

$$\rho(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\omega)e^{-i\omega t}\,d\omega \tag{0.22}$$

$$\rho(\mathbf{x},t) = \frac{1}{2\pi}\left(e^{-i(-\frac{i}{2\tau}+\omega_p)t} + e^{-i(-\frac{i}{2\tau}-\omega_p)t}\right) \tag{0.23}$$

$$\rho(\mathbf{x},t) = \frac{1}{2\pi}e^{-t/2\tau}\left(e^{-i\omega_p t} + e^{i\omega_p t}\right) \tag{0.24}$$

$$\rho(\mathbf{x},t) = \frac{1}{\pi}e^{-t/2\tau}\cos\left(\omega_p t\right) \tag{0.25}$$

This function form shows that the initial charge distribution at $t = 0$ will cause a damped oscillation at ω_p that will decay with time constant $1/2\tau$.