

PHYS 502: Mathematical Physics II

Winter 2015

Solutions to Homework #2

1. (a) Writing $\xi = x - ct$ and seeking solutions $\psi(\xi)$, we have

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= \frac{d\psi}{d\xi} \frac{\partial \xi}{\partial t} = -c\psi'(\xi) \\ \frac{\partial \psi}{\partial x} &= \psi'(\xi) \\ \frac{\partial^3 \psi}{\partial x^3} &= \psi'''(\xi),\end{aligned}$$

so the equation becomes

$$(\psi - c)\psi' + \psi''' = 0.$$

Integrating once, we have

$$\frac{1}{2}\psi^2 - c\psi + \psi'' = 0$$

so

$$\psi'' = c\psi - \frac{1}{2}\psi^2.$$

- (b) Multiplying by ψ' and integrating again, we have

$$(\psi')^2 = c\psi^2 - \frac{1}{3}\psi^3$$

or

$$\psi' = \psi \left(c - \frac{1}{3}\psi \right)^{1/2}.$$

Hence, writing $u = \psi/3c$, we have

$$\begin{aligned}\xi &= \int \frac{d\psi}{\psi \left(c - \frac{1}{3}\psi \right)^{1/2}} \\ &= \frac{1}{\sqrt{c}} \int \frac{du}{u(1-u)^{1/2}} \\ &= \frac{1}{\sqrt{c}} \log \left(\frac{1 - \sqrt{1-u}}{1 + \sqrt{1-u}} \right).\end{aligned}$$

Inverting, and after some algebra, we find

$$\psi = \frac{3c}{\cosh^2 \sqrt{c}\xi/2},$$

which represents a non-dispersive, traveling nonlinear wave.

2. For the PDE

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} = 0,$$

the two solutions of the characteristic equation

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$

are

$$\begin{aligned}\xi(x, y) &= \text{constant}, \\ \eta(x, y) &= \text{constant}.\end{aligned}$$

Hence, along a characteristic,

$$\frac{dy}{dx} = -\frac{\partial\xi}{\partial x} / \frac{\partial\xi}{\partial y} = -\xi_x/\xi_y,$$

so ξ satisfies

$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0, \quad (1)$$

and similarly for η . We want to use ξ and η as coordinates and write the PDE in terms of them. We assume that the functions A, B , and C can always be written explicitly in terms of ξ and η (which is in principle true, but often difficult in practice!).

We start by expanding

$$\begin{aligned}\psi_x &= \psi_\xi\xi_x + \psi_\eta\eta_x, \\ \psi_{xx} &= (\psi_{\xi\xi}\xi_x + \psi_{\xi\eta}\eta_x)\xi_x + \psi_\xi\xi_{xx} + (\psi_{\xi\eta}\xi_x + \psi_{\eta\eta}\eta_x)\eta_x + \psi_\eta\eta_{xx} \\ &= \psi_{\xi\xi}\xi_x^2 + 2\psi_{\xi\eta}\xi_x\eta_x + \psi_{\eta\eta}\eta_x^2 + \psi_\xi\xi_{xx} + \psi_\eta\eta_{xx}.\end{aligned}$$

Similarly, we find

$$\begin{aligned}\psi_{yy} &= \psi_{\xi\xi}\xi_y^2 + 2\psi_{\xi\eta}\xi_y\eta_y + \psi_{\eta\eta}\eta_y^2 + \psi_\xi\xi_{yy} + \psi_\eta\eta_{yy}, \\ \psi_{xy} &= \psi_{\xi\xi}\xi_x\xi_y + \psi_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \psi_{\eta\eta}\eta_x\eta_y + \psi_\xi\xi_{xy} + \psi_\eta\eta_{xy}.\end{aligned}$$

Combining terms, the coefficients of $\psi_{\xi\xi}$ and $\psi_{\eta\eta}$ are, respectively, $A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2$ and $A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2$, which are both zero, by Equation (1), so

$$\begin{aligned}A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} &= 2[A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y]\psi_{\xi\eta} + D(\xi, \eta, \psi_\xi, \psi_\eta) \\ &= 0,\end{aligned}$$

where the function D involves only first derivatives of ψ (and in fact is linear in them). Dividing through by the coefficient of $\psi_{\xi\eta}$ brings the equation into the desired form.

3. (a) In this case, $A = 1, B = 0, C = -c(x)^2$, and the characteristic equation is

$$\left(\frac{dx}{dt}\right)^2 = c(x)^2,$$

the solutions to which are

$$t = \pm \int_{x_0}^x \frac{ds}{c(s)}.$$

For $c(x) = c_0(1 + |x|/a)^{-1}$, we find

$$c_0 t = \pm \int^x ds (1 + |s|/a) = \pm \left[x + \text{sign}(x) \frac{x^2}{2a} \right] + \text{constant}.$$

In the language of the previous question, we have

$$\xi, \eta = x + \text{sign}(x) \frac{x^2}{2a} \pm c_0 t.$$

(b) For $a \rightarrow \infty$, we have $c(x) = c_0$, and the characteristics are simply given by $x \pm c_0 t = \text{constant}$. As discussed in class, the solution is $\psi(x, t) = f(\xi) + g(\eta)$, where $\xi = x + c_0 t, \eta = x - c_0 t$. Applying the initial conditions at $t = 0$, we have

$$\begin{aligned} f(x) + g(x) &= 0, \\ c_0 f'(x) - c_0 g'(x) &= e^{-|x|}, \end{aligned}$$

so

$$\begin{aligned} -g'(x) = f'(x) &= e^{-|x|}/2c_0, \\ -g(x) = f(x) &= \frac{1}{2c_0} \int^x e^{-|s|} ds = -\text{sign}(x) e^{-|x|}/2c_0 + \text{constant}, \end{aligned}$$

and hence

$$\psi(x, t) = f(x + c_0 t) - f(x - c_0 t) = \frac{1}{2c_0} \int_{x-c_0 t}^{x+c_0 t} e^{-|s|} ds.$$

4. It is most convenient to work in terms of $T' = T - T_0$, so

$$\nabla^2 T' = \frac{1}{\kappa} \frac{\partial T'}{\partial t},$$

with $T' = -T_0$ initially inside the cube and $T' = 0$ on the surface. As usual, we separate out the time dependence $e^{-\alpha \kappa t}$, so the spatial part of the solution $\chi(x, y, z)$ satisfies

$$\nabla^2 \chi + \alpha \chi = 0.$$

Separating in x, y , and z , we find that, to satisfy the boundary conditions at $x, y, z = 0$, χ must be a sum of terms of the form

$$\chi \sim \sin ax \sin by \sin cz.$$

Applying the boundary conditions at $x, y, z = L$ gives

$$a = \frac{k\pi}{L}, \quad b = \frac{l\pi}{L}, \quad c = \frac{m\pi}{L},$$

and

$$\alpha = \alpha_{klm} = a^2 + b^2 + c^2 = \frac{\pi^2}{L^2} (k^2 + l^2 + m^2).$$

Thus the general solution satisfying the differential equation and the boundary conditions is

$$T = T_0 + \sum_{k,l,m} a_{klm} \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) e^{-\alpha_{klm} \kappa t}.$$

We determine the coefficients a_{klm} by enforcing the initial condition, $T = 0$, or $T' = -T_0$, so

$$\begin{aligned} a_{klm} &= \frac{8}{L^3} \int_0^L dx \int_0^L dy \int_0^L dz (-T_0) \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \\ &= -\frac{8T_0}{L^3} \left[\frac{L}{k\pi} \{1 - (-1)^k\} \right] \left[\frac{L}{l\pi} \{1 - (-1)^l\} \right] \left[\frac{L}{m\pi} \{1 - (-1)^m\} \right] \\ &= \begin{cases} -\frac{64T_0}{klm\pi^3} & (k, l, m \text{ all odd}) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned}$$

and hence

$$T(x, y, z, t) = T_0 \left[1 - \frac{64}{\pi^3} \sum_{\substack{k, l, m \\ \text{odd}}} \frac{1}{klm} \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) e^{-\alpha_{klm}\kappa t} \right].$$

5. (a) Schrödinger's equation is

$$(\nabla^2 + k^2)\psi = 0,$$

where $k^2 = 2mE/\hbar^2$. The boundary conditions are that $\psi = 0$ on all surfaces of a cylinder of radius R and height H . Take the axis of the cylinder to have $r = 0$ in cylindrical polar coordinates, and the flat faces to lie at $z = 0$ and $z = H$. The general form of the solution is a sum of terms of the form

$$\psi \sim J_m(\beta r) e^{im\phi} \sin lz,$$

where $\beta^2 + l^2 = k^2$ and the $\sin lz$ term is chosen to satisfy the boundary condition at $z = 0$. The boundary condition at $z = H$ then implies $lH = n\pi$, for integral n . The boundary condition at $r = R$ is $J_m(\beta R) = 0$, so $\beta R = \alpha_{mq}$, the q -th root of J_m . Hence

$$E_{mqn} = \frac{\hbar^2 k_{mqn}^2}{2m} = \frac{\hbar^2}{2m} [\beta^2 + l^2] = \frac{\hbar^2}{2m} \left[\left(\frac{\alpha_{mq}}{R} \right)^2 + \left(\frac{n\pi}{H} \right)^2 \right]$$

for integral m, q , and n . Clearly the minimum energy corresponds to $m = 0, q = 1, n = 1$, so

$$E_{min} = \frac{\hbar^2}{2m} \left[\left(\frac{\alpha_{01}}{R} \right)^2 + \left(\frac{\pi}{H} \right)^2 \right].$$

Here, $\alpha_{01} = 2.405$. The corresponding (unnormalized) wavefunction is

$$\psi \sim J_0\left(\frac{\alpha_{01} r}{R}\right) \sin\left(\frac{\pi z}{H}\right)$$

(b) In two dimensions, similar reasoning to that in part (a) leads to the conclusion that the wavefunction must have the form

$$\psi \sim J_m(kr) e^{im\theta}.$$

The boundary condition $\psi = 0$ at $r = R$ implies $J_m(kR) = 0$. The boundary condition at $\theta = 0, \pi$ implies that the appropriate $\sim e^{im\theta}$ term is actually $\sin m\theta$, where m is a positive integer. The minimum k , and hence E , occurs at the lowest nonzero root of J_m for $m > 0$, corresponding to the first root of J_1 , $\alpha_{11} = 3.83$. Hence the ground-state solution (again unnormalized) has

$$\psi \sim J_1\left(\frac{\alpha_{11} r}{R}\right) \sin \theta, \quad E = \frac{\hbar^2}{2m} \left(\frac{\alpha_{11}}{R} \right)^2.$$