PHYS 502: Mathematical Physics II

Winter 2014, Homework #2

(Due January 31, 2014)

1. The Kortweg-de Vries (KdV) equation

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0$$

is a well-known two-dimensional nonlinear partial differential equation. In general, the solutions are *dispersive*, meaning that, in a solution of the form $e^{ikx - i\omega t}$, the frequency ω depends on the wavenumber k. However, non-dispersive *soliton* solutions also exist.

(a) Let $\xi = x - ct$ and seek a traveling wave solution $\psi(\xi)$. Rewrite the above equation in terms of ξ and show that it implies

$$(\psi - c)\frac{d\psi}{d\xi} + \frac{d^3\psi}{d\xi^3} \quad = \quad 0,$$

and integrate to find

$$\frac{d^2\psi}{d\xi^2} \quad = \quad c\psi - \tfrac{1}{2}\psi^2.$$

(b) Hence show that

$$\left(\frac{d\psi}{d\xi}\right)^2 = c\psi^2 - \tfrac{1}{3}\psi^3,$$

and solve this equation to find the solution $\psi(\xi)$.

2. A second-order linear partial differential equation in two dimensions (x, y) has the form

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x \partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} = 0.$$

As discussed in class, the characteristic equation for this system is

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0.$$

Assume that the system is hyperbolic, and denote the two solution families (corresponding to the two roots of the above quadratic equation) of this ODE by

$$\begin{aligned} \xi(x,y) &= \text{constant}, \\ \eta(x,y) &= \text{constant}. \end{aligned}$$

Transform the PDE to the (ξ, η) coordinate system, and show that it takes the form

$$\frac{\partial^2\psi}{\partial\xi\partial\eta} = \cdots$$

where the (rather ugly) right-hand side depends only on known functions and first derivatives of ψ.

3. (a) Write down and solve the characteristic equation for the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} \; - \; c(x)^2 \, \frac{\partial^2 \psi}{\partial x^2} = 0.$$

where the signal speed is $c(x) = c_0 \, (1 + |x|/a)^{-1}$. Sketch some representative characteristic curves.

(b) For the case $c =$ constant $(a \to \infty)$ find, using the method of characteristics, the solution to the equation satisfying the initial conditions

$$\psi(x, 0) = 0, \qquad \left.\frac{\partial \psi}{\partial t}\right|_{t=0} \; = \; e^{-|x|},$$

for $x > 0, t > 0$.

4. A uniform cube of side L initially is at temperature $T = 0$. At time $t = 0$ the cube is immersed in a heat bath of temperature $T_0 > 0$. The temperature within the cube obeys the diffusion equation

$$\nabla^2 T = \frac{1}{\kappa} \frac{\partial T}{\partial t} \, .$$

Write down a general expression for the temperature in the cube, fit it to the boundary and initial conditions, and hence derive a formula (in the form of an infinite sum) for the temperature at any point within the cube at any subsequent time.

5. (a) A particle of mass m is contained in a cylinder of radius R and height H. The particle is described by a wavefunction $\psi(\rho, \phi, z)$ satisfying

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E \psi,$$

where ρ, ϕ, z are cylindrical polar coordinates and $\psi = 0$ on the surface of the cylinder $(\rho = R, z = 0, H)$. Find the ground-state energy of the system, and write down an explicit expression for the (unnormalized) lowest-energy wavefunction.

(b) Repeat part (a), but now for a particle moving in *two* dimensions, within a semicircular region of radius R, again with $\psi = 0$ on the boundary of the region.