Statistical Mechanics II HW2

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April 23, 2015

1 Problem 1

We want to show the $p_i = \frac{1}{N}$ maximizes the total entropy. With $S(\{p_i\}) = p_i \ln p_i$, we maximize $\sum_{i=1}^{N} p_i \ln p_i$ subject to $\sum p_i = 1$. Using the method of Lagrange multipliers:

$$\sum_{i=1}^{N} p_i \ln p_i + \lambda \left(\sum_{i=1}^{N} p_i \right) = 0 \tag{1.1}$$

$$\frac{d}{dp_i} = p_i + 1 + \lambda = 0 \tag{1.2}$$

$$p_i = -1 - \lambda \tag{1.3}$$

$$\sum p_i - 1 = 0 \tag{1.4}$$

From 1.3 and 1.4 we find:

$$-N - N\lambda - 1 = 0 \tag{1.5}$$

$$\lambda = -\frac{1}{N} - 1 \tag{1.6}$$

Substituting back into 1.2:

$$p_i + 1 - \frac{1}{N} - 1 = 0 \tag{1.7}$$

$$p_i = \frac{1}{N} \tag{1.8}$$

2 Problem 2

Solve $Ax^2 + By^2$ subject to $ax + by - c = 0$.

$$g(x,y) = Ax^2 + By^2 + \lambda(ax + by - c) = 0 \tag{2.1}$$

$$\frac{\partial g}{\partial x} = 2Ax + \lambda a = 0,\ x = -\frac{\lambda}{2}\frac{a}{A} \tag{2.2}$$

$$\frac{\partial g}{\partial y} = 2By + \lambda b = 0,\ y = -\frac{\lambda}{2}\frac{b}{B} \tag{2.3}$$

$$-\frac{\lambda}{2}\left(\frac{a^2}{A} + \frac{b^2}{B}\right) - c = 0 \tag{2.4}$$

$$x = \frac{c\frac{a}{A}}{\frac{a^2}{A} + \frac{b^2}{B}} \tag{2.5}$$

$$y = \frac{c\frac{b}{B}}{\frac{a^2}{A} + \frac{b^2}{B}} \tag{2.6}$$

3 Problem 3

We find the probabilities that maximize total entropy constrained by an expectation relation. The expectation relation $< N >= \frac{2}{7}$ creates the constraint $p_1 + 2p_2 = \frac{2}{7}$.

$$g(p_i) = \sum_i p_i \ln(p_i) + \lambda_1(p_0 + p_1 + p_2 - 1) + \lambda_2(p_1 + 2p_2 - \frac{2}{7}) = 0 \tag{3.1}$$

$$\frac{\partial g}{\partial p_0} = \ln p_0 + 1 + \lambda_1 = 0,\ p_0 = e^{-(1+\lambda_1)} \tag{3.2}$$

$$\frac{\partial g}{\partial p_1} = \ln p_1 + 1 + \lambda_1 + \lambda_2 = 0,\ p_1 = e^{-(1+\lambda_1+\lambda_2)} \tag{3.3}$$

$$\frac{\partial g}{\partial p_2} = \ln p_2 + 1 + \lambda_1 + 2\lambda_2 = 0,\ p_2 = e^{-(1+\lambda_1+2\lambda_2)} \tag{3.4}$$

$$p_0 + p_1 + p_2 - 1 = 0 \tag{3.5}$$

$$p_1 + 2p_2 - \frac{2}{7} = 0 \tag{3.6}$$

4 Problem 4

Because S is an extensive parameter and using the multiplicative property of Ω, when we double the size of the system:

$$S = f(\Omega) \tag{4.1}$$
$$2S = f(\Omega^2) \tag{4.2}$$
$$2S - S = S \tag{4.3}$$
$$f(\Omega^2) - f(\Omega) = f(\Omega) \tag{4.4}$$
$$f(\Omega^2) = 2f(\Omega) \tag{4.5}$$

Therefore f must have the functional form $k \ln \Omega$.

5 Problem 5

We start with x=1, where $\ln x = x - 1$. Examining the derivatives:

$$\frac{d}{dx} \ln x = \frac{1}{x} \tag{5.1}$$
$$\frac{d}{dx}(x - 1) = 1 \tag{5.2}$$

We can clearly see that x-1 grows faster than $\ln x$ for $x > 1$. For $x < 1$ we see that $\frac{1}{x} > 1$ so that $\ln x$ will decrease faster than x - 1 as $x \to 0$.

6 Problem 6

We can write x as $2^{\log_2 x}$. Taking the natural logarithm of both sides:

$$\ln x = \ln 2^{\log_2 x} = \log_2 x \ln 2 \tag{6.1}$$
$$\log_2 x = \frac{\ln x}{\ln 2} \tag{6.2}$$