Computational Biophysics HW7

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1 3.9

The conventional solution in cylindrical coordinates takes the seperation constant k^2, resulting in $\sinh, \cosh$ solutions for the z equation. In this problem we have boundary conditions $\Phi(\rho, \phi, 0) = 0$ and $\Phi(\rho, \phi, L) = 0$. These cannot be satisfied by the hyperbolic functions. Instead, we take the separation constant to be $-k^2$ so that we get the sin solution for z which will satisfy the boundary conditions. We are then working with the modified Bessel functions.

We are looking for an interior solution, so we discard the Neumann terms. The ϕ solutions are the typical exponential combinations, with the usual integer restriction on m so they are single-valued. We can then write the general solution in cylindrical coordinates:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m(\frac{n\pi}{L}\rho) \sin\left(\frac{n\pi}{L}z\right) \left(A_{mn}e^{im\phi} + B_{mn}e^{-im\phi}\right) \tag{1.1}$$

Where we have removed the $n = 0$ term, since it will be zero due to the sine function.

We now solve for the coefficients using the other provided boundary condition:

$$\Phi(b, \phi, z) = V(\phi, z) \tag{1.2}$$

$$V(\phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m(\frac{n\pi}{L}b) \sin\left(\frac{n\pi}{L}z\right) \left(A_{mn}e^{im\phi} + B_{mn}e^{-im\phi}\right) \tag{1.3}$$

We multiply both sides by a $e^{-im'\phi} \sin \frac{n\pi}{L}z'$ and integrate, using the delta function relations:

$$\int_0^{2\pi} e^{i(m-m')\phi} \, d\phi = 2\pi\delta(m - m') \tag{1.4}$$

$$\int_0^L \sin(n\pi z/L) \sin(n'\pi z/L) \, dz = \frac{L}{2}\delta(n - n') \tag{1.5}$$

We then have:

$$\int_0^{2\pi}\int_0^L V(\phi,z)e^{-im\phi}\sin\left(\frac{n\pi}{L}z\right)\,d\phi\;dz = I_m(\frac{n\pi}{L}b)\pi L A_{mn} \tag{1.6}$$

$$A_{mn} = \left(I_m(\frac{n\pi}{L}b)\pi L\right)^{-1}\int_0^{2\pi}\int_0^L V(\phi,z)e^{-im\phi}\sin\left(\frac{n\pi}{L}z\right)\,d\phi\;dz \tag{1.7}$$

Following the same process, but multiplying by $e^{im'\phi}\sin\left(\frac{n\pi}{L}z'\right)$, we find the B_{mn}:

$$B_{mn} = \left(I_m(\frac{n\pi}{L}b)\pi L\right)^{-1}\int_0^{2\pi}\int_0^L V(\phi,z)e^{im\phi}\sin\left(\frac{n\pi}{L}z\right)\,d\phi\;dz \tag{1.8}$$

Inserting the coefficients into (1) we now have a complete expression for the potential inside the cylinder.

2 4.1

a) The charge distribution consists of 4 point charges, all at $r = a$ and $\theta = \pi/2$. The four ϕ angles are $0, \pm\pi/2, \pi$. We can write the charge distribution in terms of delta functions:

$$\rho = q\delta(r-a)\delta(\theta-\pi/2)\left\{\delta(\phi)+\delta(\phi-\pi/2)-\delta(\phi-\pi/2)-\delta(\phi-\pi)\right\} \tag{2.1}$$

The coefficients are then:

$$q_{\ell m} = \int Y^*_{\ell m}(\theta,\phi)r^\ell\rho(\mathbf{x})d^3x \tag{2.2}$$

$$q_{\ell m} = qa^\ell\left(Y^*_{\ell m}(0,0)+Y^*_{\ell m}(0,\pi/2)-Y^*_{\ell m}(0,-\pi/2)-Y^*_{\ell m}(0,\pi)\right) \tag{2.3}$$

We pull out the common θ part of the spherical harmonics:

$$q_{\ell m} = qa^\ell P_\ell(0)\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\left(e^{im0}+e^{im\pi/2}-e^{-im\pi/2}-e^{im\pi}\right) \tag{2.4}$$

When m is even the first and last ϕ terms will cancel, as will the second and third. Therefore $q_{\ell m} = 0$ for m even. When m is odd the ϕ terms reduce to $2 \mp 2i$ with the sign alternating. We can then write the coefficients as:

$$q_{\ell m} = 2qa^\ell P_\ell(0)\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\left(1-i^m\right)\ \text{(m odd)} \tag{2.5}$$

b) The charge density in b consists of three point charges, so we again write the charge density in terms of delta functions:

$$\rho = -2q(\delta(r))+q\delta(r)\delta(\theta)+q\delta(r)\delta(\theta-\pi) \tag{2.6}$$

$$\tag{2.7}$$

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