PHYS 501: Mathematical Physics I

Fall 2014

Homework #1 solutions

1. (a) For $c = a + \lambda b$, if λ is *real*, we have

$$c^2 = c \cdot c = a^2 + 2\lambda a \cdot b + \lambda^2 b^2.$$

The condition for $c^2 \geq 0$ for all a and b is that the discriminant of the above equation should be ≤ 0, so

$$(2a \cdot b)^2 \leq 4a^2 b^2,$$

or

$$a^2 b^2 \geq (a \cdot b)^2.$$

Equality occurs when $a = \alpha b$.

If instead we allow λ to be *complex*, $\lambda = \lambda_r + i\lambda_i$, we have

$$c \cdot c = a^2 + 2\lambda_i a \cdot b + (\lambda_r^2 + \lambda_i^2)\, b^2.$$

We can find the minimum value of $c \cdot c$ by looking at partial derivatives with respect to λ_r and λ_i:

$$\begin{aligned} \frac{\partial(c \cdot c)}{\partial \lambda_r} &= 2a \cdot b + 2\lambda_r b^2 = 0 \Rightarrow \lambda_r = -\frac{a \cdot b}{b^2} \\ \frac{\partial(c \cdot c)}{\partial \lambda_i} &= 2\lambda_i b^2 = 0 \Rightarrow \lambda_i = 0. \end{aligned}$$

Hence the minimum value of $c \cdot c$ is $a^2 - (a \cdot b)^2/b^2$. Requiring this to be non-negative leads to the desired result.

(b) If u is an eigenvector of A, with $Au = \lambda u$, then u is an eigenvector of A^n, with $A^n u = \lambda^n u$. Hence, if

$$B = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

then

$$Bu = \sum_{n=0}^{\infty} \frac{A^n}{n!} u = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} u = e^\lambda u,$$

so u is an eigenvector of e^A, with eigenvalue e^λ.

2. (i) Starting from the basis set $\{a_i, i = 1, \ldots, 4\} = \{1, x, x^2, x^3\}$, with inner product $(f, g) = \int_{-1}^{1} |x| f(x) g(x)\, dx$, and noting that

$$(1,1) = \int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = 1,$$

$$(1,x) = \int_{-1}^{1} |x| x\, dx = 0,$$

$$(1,x^2) = \int_{-1}^{1} |x| x^2\, dx = 2\int_0^1 x^3\, dx = \tfrac{1}{2},$$

$$(1, x^3) = \int_{-1}^{1} |x| x^3 \, dx = 0,$$

and similarly,

$$(x, x) = \tfrac{1}{2}, \quad (x, x^2) = 0, \quad (x, x^3) = \tfrac{1}{3},$$
$$(x^2, x^2) = \tfrac{1}{3}, \quad (x^2, x^3) = 0, \quad (x^3, x^3) = \tfrac{1}{4},$$

we define the new orthonormal basis set $\{e_i\}$ as follows:

$$e'_1 = a_1 = 1$$
$$e_1 = e'_1/|e'_1| = 1$$

$$e'_2 = a_2 - (a_2, e_1)e_1 = x$$
$$e_2 = e'_2/|e'_2| = \sqrt{2}x$$

$$e'_3 = a_3 - (a_3, e_1)e_1 - (a_3, e_2)e_2 = x^2 - \tfrac{1}{2}$$
$$(e'_3, e'_3) = 2\int_0^1 x(x^4 - x^2 + \tfrac{1}{4})\, dx = \tfrac{1}{12}, \text{ so } e_3 = e'_3/|e'_3| = 2\sqrt{3}(x^2 - \tfrac{1}{2})$$

$$e'_4 = a_4 - (a_4, e_1)e_1 - (a_4, e_2)e_2 - (a_4, e_3)e_3 = x^3 - \tfrac{2}{3}x$$
$$(e'_4, e'_4) = 2\int_0^1 x^3(x^4 - \tfrac{4}{3}x^2 + \tfrac{4}{9})\, dx = \tfrac{1}{36}, \text{ so } e_4 = e'_4/|e'_4| = 6x^3 - 4x.$$

Thus the orthonormal "e" basis set is $\{1, \sqrt{2}x, 2\sqrt{3}(x^2 - \frac{1}{2}), 6x^3 - 4x\}$.

(ii) Starting instead from the permuted basis set $\{b_i, i = 1, \ldots, 4\} = \{x^2, x, 1, x^3\}$, and following the same procedure as above, we obtain

$$f'_1 = b_1 = x^2$$
$$f_1 = f'_1/|f'_1| = \sqrt{3}x^2$$

$$f'_2 = b_2 - (b_2, f_1)f_1 = x$$
$$f_2 = f'_2/|f'_2| = \sqrt{2}x = e_2$$

$$f'_3 = b_3 - (b_3, f_1)f_1 - (b_3, f_2)f_2 = 1 - \tfrac{3}{2}x^2$$
$$(f'_3, f'_3) = 2\int_0^1 x(1 - 3x^2 + \tfrac{9}{4}x^4)\, dx = \tfrac{1}{4}, \text{ so } f_3 = f'_3/|f'_3| = 2 - 3x^2$$

$$f'_4 = b_4 - (b_4, f_1)f_1 - (b_4, f_2)f_2 - (b_4, f_3)f_3 = x^3 - \tfrac{2}{3}x = e'_4$$
$$f_4 = 6x^3 - 4x = e_4.$$

Hence the new orthonormal "f" basis set is $\{\sqrt{3}x^2, \sqrt{2}x, 2 - 3x^2, 6x^3 - 4x\}$.

The transformation matrix γ that takes us from the e basis to the f basis is defined by

$$f_j = \sum_i \gamma_{ij}\, e_i\,.$$

Writing

$$\begin{aligned} f_1 &= \tfrac{1}{2}e_3 + \tfrac{\sqrt{3}}{2}e_1, \quad f_2 = e_2, \\ f_3 &= -\tfrac{\sqrt{3}}{2}e_3 + \tfrac{1}{2}e_1, \quad f_4 = e_4, \end{aligned}$$

we find

$$\gamma = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is obviously orthogonal (representing a reflection and a "60° rotation" in the 1–3 plane).

3. (a) We wish to diagonalize the matrix

$$A = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & i & 1 \end{pmatrix}$$

By inspection, we see that the matrix $A - \lambda I$ breaks up into three pieces: a 2×2 matrix (top left), a 1×1 matrix (center), and another 2×2 matrix (bottom right), so the secular equation $|A - \lambda I| = 0$ becomes

$$(\lambda^2 - 1)(3 - \lambda)(\,[1 - \lambda]^2 - 1) = 0,$$

the solutions to which are $\lambda = \pm 1, 3, 0,$ and 2. The corresponding eigenvectors are

$$\begin{aligned} \lambda = \pm 1: \quad u_{\pm 1} \quad &= \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \lambda = 3: \quad u_3 \quad &= \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ \lambda = 0, 2: \quad u_{0,2} \quad &= \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \mp i \end{pmatrix}. \end{aligned}$$

Hence, the diagonalized matrix (with eigenvalues in ascending order) is

$$A' = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

and the diagonalizing transformation is

$$\gamma = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ -i & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -i & 0 & i & 0 \end{pmatrix}$$

The transformed x is

$$x' = \gamma^{-1}x = \gamma^{\dagger}x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i \\ 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ i \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 1 \\ i \\ \sqrt{2}i \end{pmatrix}$$

(b) No. For example, replace the i in the first column of A by $2i$.

4. The equations of motion are

$$\begin{aligned} \ddot{x}_1 &= -\frac{k_1}{M}(x_1 - x_2) - \frac{k_2}{M}(x_1 - x_3) \\ \ddot{x}_2 &= \frac{k_1}{m}(x_1 - x_2) - \frac{k_2}{m}(x_2 - x_3) \\ \ddot{x}_3 &= \frac{k_2}{M}(x_1 - x_3) + \frac{k_1}{M}(x_2 - x_3), \end{aligned}$$

where x_1, x_2, and x_3 are the displacements of the three masses. Looking for normal modes with time dependence $e^{i\omega t}$ (so $\ddot{x}_j \to -\omega^2 x_j$) leads to the secular equation

$$\begin{vmatrix} \frac{k_1+k_2}{M} - \omega^2 & -\frac{k_1}{M} & -\frac{k_2}{M} \\ -\frac{k_1}{m} & 2\frac{k_1}{m} - \omega^2 & -\frac{k_1}{m} \\ -\frac{k_2}{M} & -\frac{k_1}{M} & \frac{k_1+k_2}{M} - \omega^2 \end{vmatrix} = 0.$$

Rearranging, we obtain

$$\begin{vmatrix} k_1 + k_2 - M\omega^2 & -k_1 & -k_2 \\ -k_1 & 2k_1 - m\omega^2 & -k_1 \\ -k_2 & -k_1 & k_1 + k_2 - M\omega^2 \end{vmatrix} = 0,$$

then, subtracting row 3 from row 1,

$$(k_1 + 2k_2 - M\omega^2)\begin{vmatrix} 1 & 0 & -1 \\ -k_1 & 2k_1 - m\omega^2 & -k_1 \\ -k_2 & -k_1 & k_1 + k_2 - M\omega^2 \end{vmatrix} = 0,$$

and, adding column 1 to column 3,

$$(k_1 + 2k_2 - M\omega^2)\begin{vmatrix} 1 & 0 & 0 \\ -k_1 & 2k_1 - m\omega^2 & -2k_1 \\ -k_2 & -k_1 & k_1 - M\omega^2 \end{vmatrix} = 0.$$

Hence

$$(k_1 + 2k_2 - M\omega^2)\left[mM\omega^4 - k_1(m + 2M)\omega^2\right] = 0,$$

so

$$\omega^2 = \frac{k_1 + 2k_2}{M}, \quad 0, \quad \text{or} \quad k_1\left(\frac{1}{M} + \frac{2}{m}\right).$$

Substituting back into the equations of motion, we obtain the eigenvectors:

$$\omega^2 = 0: \qquad \begin{aligned} (k_1 + k_2)x_1 - k_1 x_2 - k_2 x_3 &= 0 \\ -k_1 x_1 + 2k_1 x_2 - k_1 x_3 &= 0 \\ -k_2 x_1 - k_1 x_2 + (k_1 + k_2)x_3 &= 0, \end{aligned}$$

so

$$x_1 = x_2 = x_3\,,$$

corresponding to uniform translation of the entire system. Similarly,

$$\omega^2 = \frac{k_1 + 2k_2}{M}: \qquad \begin{aligned} -k_2 x_1 - k_1 x_2 - k_2 x_3 &= 0 \\ -k_1 x_1 + \left[2k_1 - \tfrac{m}{M}(k_1 + 2k_2)\right] x_2 - k_1 x_3 &= 0 \\ -k_2 x_1 - k_1 x_2 - k_2 x_3 &= 0, \end{aligned}$$

so

$$x_2 = 0, \quad x_3 = -x_1$$

describing motion in which the left and right masses oscillate 180° out of phase about the central mass, which stays fixed. Finally,

$$\omega^2 = k_1\left(\frac{1}{M} + \frac{2}{m}\right): \qquad \begin{aligned} (k_2 - 2\tfrac{k_1 M}{m})x_1 - k_1 x_2 - k_2 x_3 &= 0 \\ -k_1 x_1 - \tfrac{m}{M}k_1 x_2 - k_1 x_3 &= 0 \\ -k_2 x_1 - k_1 x_2 + (k_2 - 2\tfrac{k_1 M}{m})x_3 &= 0, \end{aligned}$$

so

$$x_3 = x_1, \quad x_2 = -2\frac{M}{m}x_1\,,$$

where the outer masses oscillate in phase with one another and 180° out of phase with the central mass.