PHYS 502: Mathematical Physics II

Winter 2015

Solutions to Homework #5

1. Assume that the solution is a function of $\mathbf{x} - \mathbf{x}'$ and take $\mathbf{x}' = 0$ for convenience. Then the Green's function satisfies
$$\nabla^2 G + k^2 G = \delta(\mathbf{x}).$$
For $\mathbf{x} \neq 0$, we have $\nabla^2 G + k^2 G = 0$ and G is a sum of terms of the form
$$\left[a_l j_l(kr) + b_l n_l(kr)\right] Y_l^m(\theta, \phi).$$
Since $j_0(x) = \sin x/x$ and $n_0(x) = -\cos x/x$, we obtain the solution representing an outgoing spherical wave at infinity ($G \sim e^{ikr}/r$) by adopting spherical symmetry ($l = m = 0$) and choosing $b_0 = ia_0$ (so $G = -ib_0 h_0^{(1)}(kr)$, where $h_0^{(1)} = j_0 + in_0$ is a Hankel function). Near $r = 0$,
$$G \;\sim\; b_0\, n_0(kr) \;\sim\; -\frac{b_0}{kr}.$$
Integrating the differential equation over an infinitesimal sphere centered on the origin, assuming G is continuous, and applying the divergence theorem to the $\nabla^2 G$ term as discussed in class, we find, near $r = 0$,
$$\begin{aligned} \frac{\partial G}{\partial r} &\sim \frac{1}{4\pi r^2} \\ \Rightarrow \qquad G &\sim -\frac{1}{4\pi r}. \end{aligned}$$
The two expressions for $G(r \to 0)$ are consistent if
$$b_0 = \frac{k}{4\pi}.$$
so
$$G \;=\; -\frac{e^{ikr}}{4\pi r} \;=\; -\frac{ikh_0^{(1)}(kr)}{4\pi}.$$

2. The Green's function is
$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x}' - \mathbf{x}|} + \frac{\beta}{4\pi|\mathbf{x}' - \mathbf{x}_1|},$$
where $\mathbf{x}_1 = \alpha\mathbf{x}$ is the image point.

(a) We apply the boundary condition $G(\mathbf{x}, \mathbf{x}') = 0$ when $r' \equiv |\mathbf{x}'| = a$ at the two points $\mathbf{x}_A = a\mathbf{x}/r$ and $\mathbf{x}_B = -a\mathbf{x}/r$ where the diameter through $\mathbf{x}$ intersects the surface of the sphere. When $\mathbf{x}' = \mathbf{x}_A$, we have $|\mathbf{x}' - \mathbf{x}| = a - r$, $|\mathbf{x}' - \mathbf{x}_1| = \alpha r - a$, so setting $G = 0$ implies
$$\frac{-1}{a - r} + \frac{\beta}{\alpha r - a} = 0,$$

or

$$\beta(a - r) = \alpha r - a.$$

Similarly, when $\mathbf{x}' = \mathbf{x}_B$, we have

$$\beta(a + r) = \alpha r + a.$$

The solutions to these two equations are easily seen to be

$$\beta = \frac{a}{r}, \qquad \alpha = \frac{a^2}{r^2}.$$

(b) The solution to $\nabla^2 u = 0$ with $u(a, \theta, \phi) = f(\theta, \phi)$ is then

$$u(r, \theta, \phi) = \int a^2 d\Omega' \, f(\theta', \phi') \left. \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial r'} \right|_{r'=a}.$$

Writing $\rho = |\mathbf{x}' - \mathbf{x}|$, $\rho_1 = |\mathbf{x}' - \mathbf{x}_1|$, and noting that

$$\nabla' \rho = \frac{\mathbf{x}' - \mathbf{x}}{\rho},$$

it follows that

$$\begin{aligned}
\frac{\partial}{\partial r'} \left(\frac{1}{\rho} \right)_{r'=a} &= -\frac{1}{\rho^2} \frac{\mathbf{x}'}{a} \cdot \nabla' \rho \\
&= \frac{a - r\cos\gamma}{\rho^3}, \\
\frac{\partial}{\partial r'} \left(\frac{1}{\rho_1} \right)_{r'=a} &= \frac{a - \alpha r \cos\gamma}{\beta^3 \rho^3},
\end{aligned}$$

where

$$\begin{aligned}
\rho^2 &= a^2 + r^2 - 2ar\cos\gamma \qquad \text{and} \\
\cos\gamma &= \mathbf{x}' \cdot \mathbf{x}/ar \\
&= \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos(\phi' - \phi).
\end{aligned}$$

Substituting in, we have

$$u(r, \theta, \phi) = -\frac{1}{4\pi} \int d\Omega' \, f(\theta', \phi') \left(\frac{a}{\rho} \right)^3 \left[1 - \left(\frac{r}{a} \right)^3 - \frac{r}{a} \left(1 - \frac{r}{a} \right) \cos\gamma \right].$$

(c) The series solution to the problem is

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} r^l Y_l^m(\theta, \phi),$$

where

$$a_{lm} a^l = \int d\Omega' \, f(\theta', \phi') \, Y_l^{m*}(\theta', \phi'),$$

so

$$u(r,\theta,\phi) = \sum_{l,m} \left(\frac{r}{a}\right)^l \int d\Omega' \, f(\theta',\phi') \, Y_l^{m*}(\theta',\phi') \, Y_l^m(\theta,\phi).$$

We can connect this to the Green's function solution as follows. Using the addition theorem for $r < a$, $r_1 > a$, $r' \approx a$, expand

$$\frac{1}{\rho} = \sum_{l,m} \frac{4\pi}{2l+1} Y_l^{m*}(\theta',\phi') \, Y_l^m(\theta,\phi) \frac{r^l}{(r')^{l+1}},$$

with a similar expression for $1/\rho_1$ (with the same θ and ϕ). The Green's function thus is

$$G = -\sum_{l,m} \frac{1}{2l+1} Y_l^{m*}(\theta',\phi') \, Y_l^m(\theta,\phi) \left[\frac{r^l}{(r')^{l+1}} - \beta \frac{(r')^l}{r_1^{l+1}} \right].$$

Hence

$$\begin{aligned} \left.\frac{\partial G}{\partial r'}\right|_{r'=a} &= -\sum_{l,m} \frac{1}{2l+1} Y_l^{m*}(\theta',\phi') \, Y_l^m(\theta,\phi) \left[-(l+1)\frac{r^l}{a^{l+2}} - l\frac{r^l}{a^{l+2}} \right] \\ &= \frac{1}{a^2} \sum_{l,m} \left(\frac{r}{a}\right)^l Y_l^{m*}(\theta',\phi') \, Y_l^m(\theta,\phi), \end{aligned}$$

in agreement with the series solution.

3. The Green's function for this problem is

$$G(\mathbf{x}-\mathbf{x}', t-t') = \begin{cases} 0 & (t < t') \\ -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} \, \delta(t - t' - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|) & (t > t'), \end{cases}$$

so

$$\phi(\mathbf{x},t) = -\frac{1}{4\pi} \int d^3\mathbf{x}' \int dt' \, \frac{\delta[\mathbf{x}' - \boldsymbol{\xi}(t')] \, \delta(t - t' - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}.$$

Clearly the only contribution to the integral occurs when $\mathbf{x}' = \boldsymbol{\xi}(t')$, $t - t' = |\mathbf{x}-\mathbf{x}'|/c$. To determine the contribution from that point, we note that the integral

$$I = \int\int\int\int dx \, dy \, dz \, dt \, \delta[f_1(x,y,z,t)] \, \delta[f_2(x,y,z,t)] \, \delta[f_3(x,y,z,t)] \, \delta[f_4(x,y,z,t)]$$

(temporarily dropping the primes for convenience) can be evaluated by transforming to f_1, f_2, f_3, f_4 as independent variables in the vicinity of $f_i = 0$, to obtain

$$\begin{aligned} I &= \int\int\int\int df_1 \, df_2 \, df_3 \, df_4 \left| \frac{\partial(x,y,z,t)}{\partial(f_1,f_2,f_3,f_4)} \right| \delta(f_1)\delta(f_2)\delta(f_3)\delta(f_4) \\ &= \left| \frac{\partial(x,y,z,t)}{\partial(f_1,f_2,f_3,f_4)} \right|_{f_i=0} \\ &= \left| \frac{\partial(f_1,f_2,f_3,f_4)}{\partial(x,y,z,t)} \right|^{-1}_{f_i=0}, \end{aligned}$$

where $J = \partial(x_1,x_2,x_3,x_4)/\partial(f_1,f_2,f_3,f_4)$ is the Jacobian matrix

$$J_{ij} = \frac{\partial x_i}{\partial f_j}.$$

Here (reinstating the primes),

$$\begin{aligned} f_1 &= x' - \xi_x(t') \\ f_2 &= y' - \xi_y(t') \\ f_3 &= z' - \xi_z(t') \\ f_4 &= t - t' - \frac{1}{c}|\mathbf{x} - \mathbf{x}'|\,, \end{aligned}$$

so

$$\frac{\partial(f_1, f_2, f_3, f_4)}{\partial(x', y', z', t')} = \begin{pmatrix} 1 & 0 & 0 & -\dot{\xi}_x(t') \\ 0 & 1 & 0 & -\dot{\xi}_y(t') \\ 0 & 0 & 1 & -\dot{\xi}_z(t') \\ \frac{x-x'}{c|\mathbf{x}-\mathbf{x}'|} & \frac{y-y'}{c|\mathbf{x}-\mathbf{x}'|} & \frac{z-z'}{c|\mathbf{x}-\mathbf{x}'|} & -1 \end{pmatrix},$$

(where we have used the fact that if $r = |\mathbf{x}|$, then $\nabla r = \mathbf{x}/r$), and hence

$$\left|\frac{\partial(f_1, f_2, f_3, f_4)}{\partial(x', y', z', t')}\right| = -1 - \frac{1}{c}\,\frac{\dot{\boldsymbol{\xi}}(t') \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,.$$

Doing the integrals, the delta functions now imply $\mathbf{x}' = \boldsymbol{\xi}(t')$, and

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi}\,\frac{1}{|\mathbf{x} - \boldsymbol{\xi}(t')| + \frac{1}{c}\dot{\boldsymbol{\xi}}(t') \cdot [\mathbf{x} - \boldsymbol{\xi}(t')]}\,,$$

where t' is the solution of the implicit equation

$$c(t - t') = |\mathbf{x} - \boldsymbol{\xi}(t')|.$$

The above expression for ϕ is the so-called *Lienard-Wiechert* potential.