Ph.D. Qualifying Exam Solutions
2003

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Classical

Part I: Short Answers (25%)

Problem 1

A wound up yo-yo is attached to a spring scale. The weight of the yo-yo is W, the weight of the string can be neglected.

a) *If the yo-yo is allowed to unwind downward, would the scale indicate a reading less than, equal to, or greater than W?*

The tension of the string on one side of the yo-yo will produce a torque on the yo-yo, resulting in an angular acceleration. Assuming no slipping, there will be an associated linear acceleration downwards, with the result that the scale reads less than W.

b) *After reaching the lowest point, the yo-yo moves upward. Would the scale now indicate a reading less than, equal to, or greater than W?*

The opposite effect is going to occur here, with the yo-yo accelerating upwards and the scale reading greater than W.

Problem 2

A particle moves in a circular orbit or radius r_0 under the influence of an attractive central force of magnitude $F(r) = kr^{-n}$, where k is a constant. Derive a condition on F near radius $r = r_0$ for the orbit to be stable.

For a particle in orbit, the orbit will be stable when two conditions are fulfilled, namely that the first derivative of the effective potential be zero, and the second derivative be greater than zero (i.e. the particle is located at the bottom of a potential well.) To find this, consider the energy of the system

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

Our effective potential is the angular momentum term and attractive central force, and with a little rearranging gives us

$$U(r) = V(r) + \frac{L^2}{2mr^2}$$

Considering $F(r) = -\nabla V(r)$, we already have our first derivative as the force, and so, along with finding the second derivative we have

$$\frac{dU}{dr} = F + \frac{-2L^2}{2mr_0^3} = kr_0^{-n} + \frac{-2L^2}{2mr_0^3} = 0$$

$$kr_0^{3-n} = \frac{L^2}{m}$$

$$\frac{d^2U}{dr^2} = -nkr_0^{-n-1} + \frac{6L^2}{2mr_0^4} > 0$$

$$r_0^{3-n}k < \frac{3L^2}{mn}$$

$$\cancel{r_0^{3-n}k} < \frac{3}{n}\cancel{kr_0^{3-n}}$$

$$\boxed{n < 3}$$

Problem 3

Astronomers have recently discovered that a very massive black hole in the Perseus galaxy cluster is whistling at a practically pure note. Specifically, it is emitting sound waves (pressure-density waves) in a thin gas in the cluster. The wavelength is $\lambda = 30,000$ light years and the period is 10,000,000 years.

a) *What is the speed of these waves?*

$$v = \frac{\lambda}{T}$$

$$\boxed{v = 9 \times 10^5 \text{ m/s}}$$

b) *What is the frequency?*

$$f = 1/T = \boxed{3.17 \times 10^{-15} \text{ Hz}}$$

c) *How many octaves below the concert scale (above) is this frequency?*

Realize that a musical octave is a frequency logarithmic scale in base 2. If a signal has a frequency of 440 Hz, successive octaves below would have frequencies of 220 Hz, 110 Hz, and so on. Thus we can measure a ratio of octaves by

$$\frac{f_1}{f_2} = 2^n$$

where n is the difference in octaves. Obviously, we need to pick a reference frequency in the presented musical spectrum. By picking a frequency in the middle, we can pick the best estimate for the number of octaves.

$$\frac{600 \text{ Hz}}{3.17 * 10^{-15} \text{ Hz}} = 2^n \qquad n = 57$$

This frequency is 57 octaves lower than the concert scale.

d) *What is the pitch of these waves?*

Using the same formula,

$$\frac{x}{3.17 * 10^{-15}} = 2^{57} \qquad x = 456.8\text{Hz}$$

which corresponds to an offtune A approaching a $\text{B}^\flat$.

Problem 4

An athlete can throw a javelin 60 m from a standing position. If he can run 100 m at a constant velocity in 10 s, how far could he hope to throw the javelin from a running start? (Neglect air resistance.)
According to upper sources, the problem is flawed, as it was intended to act as a momentum problem, where the man converts all of his momentum into that of the javelin. Without knowing the mass of the athlete, we cannot calculate the additional momentum of the javelin.

Problem 5

An aluminum sphere of radius $r = 10.0$ cm is subjected to monochromatic ultraviolet radiation of wavelength $\lambda = 2900\mathring{A}$ for a long time. What is the charge on the sphere?
The work function of Al is $\phi = 4.19$ eV, corresponding to a wavelength of $\lambda_0 = 2960\mathring{A}$.

$$\begin{aligned} h &= 6.625 * 10^{-27} \text{ erg sec} \\ c &= 3 \times 10^{10} \text{ cm/sec} \\ hc &= 12.41 \times 10^{-5} \text{ eV cm} \\ 1\mathring{A} &= 10^{-8} \text{ cm} \end{aligned}$$

The radiation ionizes electrons from the surface through the photoelectric effect. This results in an induced charge on the sphere, thereby creating an extra potential barrier and effectively increasing the work function. The system will be in equilibrium when the maximum kinetic energy of ejected electrons is equal to the electric potential energy of an escaping electron.

$$\begin{aligned} \text{KE}_{\text{max}} &= E_{rad} - \phi \\ &= \frac{hc}{\lambda} - \phi = 0.089 \text{ eV} \\ \text{U}_{electric} &= \frac{kQq}{R} = \text{KE}_{max} \\ &\boxed{Q = 9.88 \times 10^{-13} \text{ C}} \end{aligned}$$

Problem 6

As a lecture demonstration, a short cylindrical bar magnet is dropped down a vertical aluminum pipe a few meters long. It takes several seconds for the magnet to emerge at the bottom. An otherwise identical piece of unmagnetized metal makes the trip in a fraction of a second. Provide a convincing *argument to explain why the magnet falls more slowly.*

As the magnet falls, the magnetic field in the stationary coordinate system changes. This change in the magnetic flux induces Eddy currents in the conductive pipe. Below the magnet, the induced magnetic field from these induced currents opposes the magnet's, pushing the magnet upwards. Likewise, the net induced magnetic field above the magnet is in the same direction as the magnet's magnetic field, exerting an additional force up on the magnet. Together, these forces cancel some of the downward gravitational force, causing the net acceleration of the magnet to be less than that of the magnetized object.

Problem 7

A taut metal wire is clamped between two fixed posts and a circuit loop is completed with some more loose wire, as sketched in the figure. If plucked transversely, the taut part of the wire oscillates, as expected. A horse-show magnet is now placed around the wire so that the direction of the magnetic field is roughly perpendicular to the direction of vibration.

a) *Will a current run through the wire? Why, or why not?*

Yes, an alternating current will be produced in the wire as the string vibrates perpendicular to the magnetic field.

b) *If the wire can oscillate for about one minute before the oscillations are damped out in absence of the magnet, will the length of the oscillations be affected by the magnetic field? Explain.*

The oscillations will be damped, with the current induced in the wire producing an opposing magnetic field, thereby slowing the velocity of the wire and causing it to dampen more quickly.

Part II: Long Answers (75%)

A1

Three particles, of masses m, $2m$, and m, respectively, are connected by two springs, each of spring constant k, and are free to move in one dimension without friction, as sketched below.
Determine the normal modes of oscillation of the system, and calculate their frequencies.
Here's one way to do it. Form a system of equations of sum of all forces on each mass

$$\begin{aligned}
k(x_2 - x_1) &= m\ddot{x}_1 \\
k(x_3 - x_2) - k(x_2 - x_1) &= 2m\ddot{x}_2 \\
-k(x_3 - x_2) &= m\ddot{x}_3
\end{aligned}$$

Making the ansatz $x_i = a_i \cos\omega t + \phi$ for each equation and rearranging, we get

$$\begin{aligned}
a_1(-k + m\omega^2) + a_2(k) &= 0 \\
a_1(k) + a_2(-k - k + 2m\omega^2) + a_3(k) &= 0 \\
a_2(k) + a_3(-k + m\omega^2) &= 0
\end{aligned}$$

There is a solution to this system of equations only when the determinant of this matrix is zero

$$\begin{vmatrix} m\omega^2 - k & k & 0 \\ k & 2m\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{vmatrix} = 0$$

which provides three solutions: $\omega^2 = k/m, \omega^2 = 2k/m, \omega^2 = 0$. With our eigenvalues, we can plug into our equations to find the eigenstates.

$$\begin{aligned}
\omega^2 = k/m : &\qquad a_2 = 0, \qquad a_1 = -a_3 \\
\omega^2 = 2k/m : &\qquad a_1 = -a_2, \qquad a_3 = -a_2 \\
\omega^2 = 0 : &\qquad a_1 = a_2 = a_3
\end{aligned}$$

And our corresponding modes are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Normalized, they are

$$\phi_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where $\omega_1^2 = 0$. Here, all the masses are moving together. The second mode is

$$\phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

where $\omega_2^2 = \frac{k}{m}$. Here, the center mass is stationary, and the outer two are oscillating in opposite directions. The third normal mode is

$$\phi_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

where $\omega_3^2 = \frac{2k}{m}$. The outer two masses are stationary here, with the center mass oscillating.

A2

*A belt of mass 0.8 kg passes between two solid cylinders and is pulled to the right by a constant force P. It has acceleration a to the right, as shown below. The top cylinder **A** (radius $R_A = 75$ mm, mass 8.0 kg, and moment of inertia $I_A = 2.25 \times 10^{-2}$ kg m^2) is free to move vertically, as well as rotate, in the bearing blocks that support its shaft. Cylinder **B** (radius $R_B = 150$ mm, mass 24.0 kg, and moment of inertia $I_B = 27 \times 10^{-2}$ kg m^2) is held in its bearings so it only rotates. Coefficients of friction between the belt and the cylinders are unknown, but it is known that only rolling (without any slipping between the belt and the cylinder) occurs. Neglect the friction in the bearings on the shafts of the cylinders.*

a) *Draw separate free-body force diagrams for the cylinders and the belt and show all the forces on each one of them.*

b) *If the acceleration of the belt is $a = 1.5$ m/s^2 to the right, determine the angular accelerations α_A and α_B of the cylinders A and B. Indicate the directions of these.*

$$\alpha = \frac{a_T}{r}$$
$$\alpha_A = 20 \text{ rad/s}^2 \odot$$
$$\alpha_B = 10 \text{ rad/s}^2 \otimes$$

c) *What is the static frictional force between the belt and cylinder **B** such that no slipping occurs?*

The torques from the frictional force and the angular acceleration must equal each other.

$$\tau = \alpha I$$
$$\tau = r \times F$$
$$F_F = \frac{\alpha I}{r}$$
$$F_{F,B} = 18 \text{ N} \qquad F_{F,A} = 6 \text{ N}$$

d) *What is the static frictional force between the belt and cylinder **A** such that no slipping occurs?*

See previous.

e) *What is the magnitude of the required force P to perform this task?*

The magnitude of P to accelerate the belt is just the sum of the frictional forces we found before using that acceleration. So P must equal 24 N.

f) *If the maximum acceleration of the belt is $a_{max} = 2.0$ m/s^2 before any slipping occurs, find the coefficient of static friction μ_S, between cylinder **B** and the belt.*

Assume the entire weight of the belt rests on the cylinder (not exactly physical, but there you are.) This means that the force on cylinder B is the weight of the belt and the weight of cylinder A. At acceleration $a = 2.0$ m/s^2, the force of friction from $F_F = \alpha_B I_B / r_B = a_T I_B / r_B^2 = 24$ N. Now $F_{F,B} = \mu F_N = \mu m g$. And $\mu = F_{F,B}/mg = 0.27$.

A3

Write down the Lagrangian of a system, undergoing small oscillations along a vertical axis, consisting of a mass M attached to a spring of mass m, length L, and a spring constant k. By solving the Lagrange's equation(s), show that the frequency of oscillation of mass M is given by

$$\omega = \left[\frac{3k}{3M+m}\right]^{1/2}$$

(Note: Assume that the amplitude of oscillations is much smaller than the length of the spring, and the speed of any part of the spring is linearly proportional to its distance from the rigid support.)

For the block, its kinetic and potential energies are

$$K = \frac{1}{2}M\dot{z}^2$$
$$U = Mgz$$

The spring's kinetic energy is

$$K = \int_0^z \frac{1}{2}\dot{z}^2\frac{l^2}{z^2}\,dm$$

As a function of length, the mass of the spring is $m(l) = \frac{m}{z}l$ where l is the distance from the bottom of the table to a point along the spring, so $dm = \frac{m}{z}\,dl$. This allows us to evaluate the expression for its kinetic energy.

$$K = \frac{1}{2}\frac{\dot{z}^2}{z^2}\int_0^z l^2\frac{m}{z}\,dl$$
$$K = \frac{1}{6}m\dot{z}^2$$

Its potential energy is

$$U = -\frac{1}{2}k(z-L)^2 + \int_0^z gl\,dm$$
$$U = -\frac{1}{2}k(z-L)^2 + \frac{1}{2}mgz$$

Therefore, the Lagrangian of the system is

$$\mathcal{L} = K - U$$
$$\boxed{\mathcal{L} = \frac{1}{2}\dot{z}^2\left(M + \frac{1}{3}m\right) - gz\left(M + \frac{1}{2}m\right) + \frac{1}{2}k(z-L)^2}$$

The Euler-Lagrange equation with respect to z is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) = \frac{\partial\mathcal{L}}{\partial z}$$
$$\ddot{z}\left(M + \frac{1}{3}m\right) = -g\left(M + \frac{1}{2}m\right) + k(z-L)$$
$$\ddot{z} - \frac{k}{M+\frac{1}{3}m}z + \frac{kL + g(M+\frac{1}{2}m)}{M+\frac{1}{3}m} = 0$$

Therefore,

$$\omega^2 = \frac{k}{M + \frac{1}{3}m}$$

$$\boxed{\omega^2 = \frac{3k}{3M + m}}$$

B1

A long circular cylinder of radius R carries a magnetization

$$\vec{M} = Cr^3\hat{a}_\theta \qquad r \leq R$$

where C is a constant and r is the distance from the axis (of course, the magnetization is zero outside the cylinder). There are no free currents anywhere within our galaxy.

a) *Find the magnetic field $\vec{B}$ inside and outside the cylinder with the help of Ampere's law*

$$\oint_\gamma \vec{H} \cdot d\vec{l} = I_f^{encl}$$

where γ is a closed Amperian path of your choice.

First, a definition

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

So (no free currents), $\vec{H} = 0$. So $\vec{B} = \mu_0\vec{M}$. And

$$\vec{B} = \mu_0 Cr^3\hat{a_\theta} \qquad \text{inside}$$
$$\vec{B} = 0 \qquad \text{outside}$$

b) *Show that the boundary condition*

$$B_1^\perp = B_2^\perp$$

is satisfied at the interface between the inside of the cylinder and the rest of the world outside. The subscript 1 stands for 'inside' and 2 stands for 'outside'.

$$\vec{B_1} \cdot \hat{r} = B_1^\perp = 0$$
$$\vec{B_2} \cdot \hat{r} = 0$$
$$B_1^\perp = B_2^\perp$$

c) *Calculate the bound current density $\vec{J}_b$ in the region $r < R$, and the bound surface current density $\vec{K}_b$ at the surface of the cylinder (i.e., for $r = R$).*

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \left(\frac{1}{r}\cancelto{0}{\frac{\partial M_z}{\partial \theta}} - \cancelto{0}{\frac{\partial M_\theta}{\partial z}}\right)\hat{r} + \left(\cancelto{0}{\frac{\partial M_r}{\partial z}} - \cancelto{0}{\frac{\partial M_z}{\partial r}}\right)\hat{\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rM_\theta) - \cancelto{0}{\frac{1}{r}\frac{\partial M_r}{\partial \theta}}\right)\hat{z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r^4 C)\hat{z}$$

$$\boxed{\vec{J}_b = 4Cr^2\hat{z}}$$

$$\vec{K}_b = \vec{M} \times \hat{n} = \vec{M} \times \hat{r} = Cr^3\hat{\theta} \times \hat{r}$$

$$\boxed{\vec{K}_b = Cr^3\hat{z}}$$

d) *Calculate the magnetic field $\vec{B}$ inside the cylinder using the alternative form of Ampere's law*

$$\oint_\gamma \vec{B} \cdot d\vec{l} = \mu_0 I^{encl}$$

where γ is a closed Amperian path that you must select properly.

$$I^{enc} = \int \vec{J}_b \cdot d\vec{A}$$

$$\int \vec{B} \cdot d\vec{l} = 2\pi r B = \mu_0 \iint (4Cr^2)\hat{z} \cdot (r\, dr\, d\theta)\hat{z}$$

$$2\pi r B = 8\pi\mu_0 C \int_0^r r^3\, dr$$

$$B = \mu_0 C r^3$$

$$\boxed{\vec{B} = \mu_0 C r^3 \hat{\theta}}$$

with the direction following from the line integral dot product.

B2

Two charges $+q$ and $-q$ are separated by a fixed distance d and rotate with a uniform angular velocity ω about an axis that passes mid-way between the two charges and which is perpendicular to the line joining the two charges. Calculate the time-averaged power radiated per unit solid angle $\left\langle \frac{dP}{d\Omega} \right\rangle$ and the total power $\langle P \rangle$ radiated by this system.
[Given: In the radiation zone the B-field due to the dipole of moment $\vec{p}$ is $B_{rad} = \frac{1}{c^2 r}[\ddot{\vec{p}}] \times \hat{n}$, where $\hat{n}$ is the unit vector in the direction of observation.]
The system's dipole moment $\vec{p}$ can be written as two oscillating dipoles along two axes perpendicular to the axis of rotation and each other. Taking the axis of rotation to be along $\hat{z}$, then

$$\vec{p} = qd(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y})$$

Taking the second time derivative of the dipole moment,

$$\begin{aligned}\ddot{\vec{p}} &= \frac{d^2}{dt^2}\left(qd\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right)\right)\\ &= -qd\omega^2\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right)\\ \ddot{\vec{p}} &= -\omega^2\vec{p}\end{aligned}$$

Therefore,

$$\vec{B} = -\frac{\omega^2}{c^2 r}\vec{p}\times\hat{n}$$

The radiated electric field is

$$\vec{E}_{rad} = \frac{1}{c^2 r}([\ddot{\vec{p}}]\times\hat{n})\times\hat{n} = \vec{B}\times\hat{n}$$

Knowing that $\hat{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$, we can now calculate the cross product of the dipole moment with the normal.

$$\vec{p}\times\hat{n} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \cos(\omega t) & \sin(\omega t) & 0\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\end{vmatrix}$$

$$\vec{p}\times\hat{n} = (\sin(\omega t)\cos\theta)\hat{x} + (\cos(\omega t)\cos\theta)\hat{y} + (\cos(\omega t)\sin\theta\sin\phi - \sin(\omega t)\sin\theta\cos\phi)\hat{z}$$

Therefore, the magnetic field is

$$\vec{B} = -\frac{\omega^2 qd}{c^2 r}\left[\sin(\omega t)\cos\theta\hat{x} + \cos(\omega t)\cos\theta\hat{y} + \sin\theta(\cos(\omega t)\sin\phi - \sin(\omega t)\cos\phi)\hat{z}\right]$$

Evaluating the cross product $\vec{B}\times\hat{n}$, we find that

$$\begin{aligned}\vec{E} = -\frac{\omega^2 qd}{c^2 r}&\big[(\cos(\omega t)\cos^2\theta - \sin^2\theta\sin\phi(\cos(\omega t)\sin\phi - \sin(\omega t)\cos\phi))\hat{x}\\ &+ (\sin(\omega t)\cos^2\theta - \sin^2\theta\cos\phi(\cos(\omega t)\sin\phi - \sin(\omega t)\cos\phi))\hat{y}\\ &+ \sin\theta\cos\theta(\sin(\omega t)\sin\phi - \cos(\omega t)\cos\phi)\hat{z}\big]\end{aligned}$$

The Poynting vector (after lots of algebra and simplification) is then

$$\begin{aligned}\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) = \frac{q^2 d^2\omega^4}{4\pi c^3 r^2}&\big\{-\sin^3\theta\cos\phi\sin^2(\phi - \omega t)\hat{x}\\ &+ \sin\theta\sin\phi\left[\cos^2\theta\cos(2\omega t) - \sin^2\theta\sin(\phi - \omega t)\right]\hat{y}\\ &+ \cos\theta\left[\cos^2\theta\cos(2\omega t) + \sin^2\theta\sin(\omega t - \phi)\sin(\omega t + \phi)\right]\hat{z}\big\}\end{aligned}$$

The power radiated per unit solid angle is then

$$\begin{aligned}\frac{dP}{d\Omega} = (\vec{S}\cdot\hat{n})r^2 = \frac{q^2 d^2\omega^4}{4\pi c^3}&\big[-\sin^4\theta\cos^2\phi\sin^2(\phi - \omega t) + \sin^2\theta\cos^2\theta\sin^2\phi\cos(2\omega t)\\ &- \sin^4\theta\sin^2\phi\sin(\phi - \omega t) + \cos^4\theta\cos(2\omega t) + \sin^2\theta\cos^2\theta\sin(\omega t - \phi)\sin(\omega t + \phi)\big]\end{aligned}$$

Observing that $\langle\sin^2(\phi - \omega t)\rangle = \frac{1}{2}$, $\langle\cos(2\omega t)\rangle = 0$, and $\langle\sin(\phi - \omega t)\rangle = 0$, we need only evaluate

$$\begin{aligned}\langle\sin(\omega t - \phi)\sin(\omega t + \phi)\rangle &= \langle\sin^2(\omega t)\cos^2\phi - \cos^2(\omega t)\sin^2\phi\rangle\\ &= \cos^2\phi\langle\sin^2(\omega t)\rangle - \sin^2\phi\langle\cos^2(\omega t)\rangle\\ &= \frac{1}{2}\cos^2\phi - \frac{1}{2}\sin^2\phi\\ &= \frac{1}{2}\cos(2\phi)\end{aligned}$$

Therefore,

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{q^2 d^2 \omega^4}{8\pi c^3} \sin^2\theta \left(\cos^2\theta \cos 2\phi - \sin^2\theta \cos^2\phi\right)}$$

The time-averaged power radiated is then

$$\begin{aligned}
\langle P \rangle &= \int \left\langle \frac{dP}{d\Omega} \right\rangle \sin\theta \, d\theta \, d\phi \\
&= \frac{q^2 d^2 \omega^4}{8\pi c^3} \int_0^{2\pi} d\phi \int_0^{\pi} (\sin^3\theta \cos^2\theta \cos 2\phi - \sin^5\theta \cos^2\phi) \, d\theta
\end{aligned}$$

$$\boxed{\langle P \rangle = -\frac{2q^2 d^2 \omega^4}{15 c^3}}$$

Modern

Part I: Short Answers (25%)

Problem 1

In two dimensions the parity operator Π is defined by the relation

$$\Pi\psi(x,y) = \psi(-x,-y)$$

where $\psi(x,y)$ is the wave function of the system of interest.

a) *What does this definition become if ψ is calculated in polar coordinates, i.e. what is $\Pi\psi(\rho,\varphi)$?*

$$\Pi\psi(\rho,\phi) = \psi(\rho,\phi+\pi)$$

b) *What is the parity of the wavefunction*

$$\psi(\rho,\varphi) = \frac{\cos\varphi}{\sqrt{\rho^2+A^2}}$$

where A is a constant?

$$\begin{aligned}\psi(\rho,\phi) &= \frac{\cos\phi}{\sqrt{\rho^2+A^2}}\\ &\to \frac{\cos\phi+\pi}{\sqrt{\rho^2+A^2}}\\ &= -\frac{\cos\phi}{\sqrt{\rho^2+A^2}}\\ &= -\psi(\rho,\phi)\end{aligned}$$

Parity of the wavefunction is -1.

Problem 2

Consider a diatomic molecule (H_2, N_2, O_2) as a two point molecule with each particle having mass m connected by a massless rod of length r. Suppose the molecule rotates about an axis perpendicular to the rod through its midpoint. Using the Bohr-Sommerfeld quantization hypothesis, derive an expression for the allowed rotational kinetic energy states for this diatomic system.
Hint: Recall that the Bohr-Sommerfeld quantization hypothesis is $\oint pdq = nh$ for integer n.

$$\begin{aligned}\oint p_\theta d\theta &= nh\\ \int_0^{2\pi} L d\theta &= nh\\ 2\pi L &= nh\end{aligned}$$

Using the definitions of angular momentum and kinetic energy, where the kinetic energy is that for the entire system, we arrive at

$$\boxed{K = \frac{n^2\hbar^2}{mr^2}}$$

Problem 3

Suppose that the outermost electrons in the Carbon atom are both in the 2p orbit and produce a $^1D_2$ spectroscopic term (the lowest energy is $^3P_0$). What is the degeneracy of this term? In other words, how many different states produce a $^1D_2$ term?
Hint: How many different spin states m_S are allowed? How many different total angular momentum states m_J are allowed?
Think about this as a transition state problem. Our relevant rules are $\Delta L = \pm 1$ and $\Delta m = \pm 1, 0$. The atom is transitioning from an $L = 1$ state to an $L = 2$ state, so the possible m values go from $m = -1, 0, 1$ to $m = -2, -1, 0, 1, 2$. Each initial m value has three possible choices to transition to, so the total degeneracy of this transition is 9.

Problem 4

Consider an ideal free-electron gas (N non-interacting electrons). Let $\epsilon_1, \epsilon_2, ..., \epsilon_N$ denote the single-particle energies.

a) *What is the ground state energy of this system when $T \to 0$, where T denotes the absolute temperature?*

In a Fermi gas (large ensemble of fermions in gas form... obviously) at zero temperature, there can be no more than two particles in each energy state due to the Pauli exclusion principle. With energies $\epsilon_1, \epsilon_2, ..., \epsilon_N$, we have the total energy of the system summed up as

$$E = \sum_{i=1}^{N/2} 2\epsilon_i$$

b) *What do we mean by Fermi energy, E_F?*

The Fermi energy is the energy of the highest occupied quantum state in a system of fermions at $T = 0$ K.

c) *When $T \to 0$, the free-electron gas in a box has a density of states given by*

$$\rho(E) = \frac{3}{2} N \frac{E^{1/2}}{E_F^{3/2}} \qquad E \leq E_F$$

Sketch this function, and show with a second sketch how this function changes qualitatively when T is different from zero but still very small. Explain the physical reason for this behavior.

The problem describes the density of states for a particular energy level E. The number of combinations of particle energies that can produce the energy state with energy E increases as you increase the allowable energy states, and according to this equation acts as a square root dependence on the energy. As T increases, the states can acquire an extra 'kick' from random interactions, thereby skewing the graph more towards the right (increased probability densities per energy level).

Problem 5

Explain what is meant by the thermal de Broglie wavelength and give an expression for this.
The thermal de Broglie wavelength is the average de Broglie wavelength of gas particles in an ideal gas at a given temperature.

$$\lambda = h/p$$
$$E = 3/2kT = p^2/2m$$
$$p = \sqrt{3mkT}$$
$$\boxed{\lambda = \sqrt{\frac{h^2}{3mkT}}}$$

Problem 6

Physics principles can be used to understand problems associated with traffic jams. In a simple model, the probability of a car having speed in the range of $(v, v+dv)$ is assumed to be $\rho(v) = Avexp(-\frac{v}{v_0})dv$, where v_0 is a characteristic speed and A is a normalization constant. Find A and the average speed.

$$\sum p(v) = 1$$
$$\int_0^\infty p(v)\,dv = 1$$

Using integration by parts we get $\boxed{A = 1/v_0^2}$.

$$\langle v \rangle = \int_0^\infty vp(v)\,dv$$
$$\boxed{\langle v \rangle = 2v_0}$$

Problem 7

a) *You have heard that there is a finite probability of all the air molecules in your room being in the other half, leaving you breathless. How concerned are you about this? Why?*

Not concerned, as the amount of molecules is astronomical, so the probability of them being all in one half is staggeringly lower than that. Besides, air molecules move relatively quickly, so a breath would encompass a drawn out period, allowing that 'once in a universe' scenario to dismantle before you'd notice.

b) *Shrink the room to a few (say* $10 \sim 50$*) molecules, answer the same questions as in Part (a).*

Now the probability is much higher that all the molecules would be isolated at some point to one half of the room, but probably still somewhat low. If we only have these numbers of molecules to breathe, I think we have bigger problems than where they are in the room.

Part II: Long Answers (75%)

A1

The initial state of the hydrogen atom is described by the state vector

$$|\psi(0)\rangle = \frac{1}{\sqrt{6}}\left(|1,0,0\rangle - i\sqrt{2}|2,0,0\rangle + \sqrt{3}|3,2,-2\rangle\right)$$

where $|n,l,m\rangle$ is a hydrogen atom eigenstate.

a) *What is the state vector of the atom at time $t>0$?*

$$|\psi(t)\rangle = \frac{1}{\sqrt{6}}\left[e^{-iE_1t/\hbar}|100\rangle - i\sqrt{2}e^{-iE_2t/\hbar}|200\rangle + \sqrt{3}e^{-iE_3t/\hbar}|32-2\rangle\right]$$

with $E_n = E_1/n^2$ for hydrogen.

b) *What is the expectation value of the energy at time t?*

$$\begin{aligned}
\langle E\rangle &= p_1E_1 + p_2E_2 + p_3E_3 \\
&= p_1E_1 + p_2\left(\frac{E_1}{4}\right) + p_3\left(\frac{E_1}{9}\right) \\
&= \frac{1}{6}E_1 + \frac{2}{6}\left(\frac{E_1}{4}\right) + \frac{3}{6}\left(\frac{E_1}{9}\right) \\
&\boxed{\langle E\rangle = \frac{11}{36}E_1}
\end{aligned}$$

c) *What are the expectation values of the operators L^2 and L_z?*

$$L^2|\psi(t)\rangle = l(l+1)\hbar^2|\psi(t)\rangle \qquad L_z|\psi(t)\rangle = \hbar m|\psi(t)\rangle$$

where l and m are the quantum numbers of the state on which the operators are acting.

$$\begin{aligned}
&\langle\psi^*|L^2|\psi\rangle = \\
&\quad= \frac{1}{6}\left[\left[\langle 1,0,0| + i\sqrt{2}\langle 2,0,0| + \sqrt{3}\langle 3,2,-2|\right] l(l+1)\hbar^2\left[\cancel{|1,0,0\rangle} - \cancel{i\sqrt{2}|2,0,0\rangle} + \sqrt{3}|3,2,-2\rangle\right]\right] \\
&\quad= \frac{1}{6}\left[\sqrt{3}\langle 3,2,-2|\right]\left[2(3)\hbar^2\sqrt{3}|3,2,-2\rangle\right] \\
&\boxed{\langle L^2\rangle = 3\hbar^2}
\end{aligned}$$

$$\begin{aligned}
&\langle\psi^*|L_z|\psi\rangle = \\
&\quad= \frac{1}{6}\left[\left[\langle 1,0,0| + i\sqrt{2}\langle 2,0,0| + \sqrt{3}\langle 3,2,-2|\right]\hbar m\left[\cancel{|1,0,0\rangle} - \cancel{i\sqrt{2}|2,0,0\rangle} + \sqrt{3}|3,2,-2\rangle\right]\right] \\
&\quad= \frac{1}{6}\left[\sqrt{3}\langle 3,2,-2|\right]\left[\hbar(-2)\sqrt{3}|3,2,-2\rangle\right] \\
&\boxed{\langle L_z\rangle = -\hbar}
\end{aligned}$$

A2

A quantum mechanical harmonic oscillator is prepared in the initial state

$$|\psi(0)\rangle = |1\rangle - i\sqrt{2}|2\rangle$$

a) *Check if the state is properly normalized. If not, supply the required normalization constant.*

The sum of the coefficients squared must be equal to 1. Therefore, $[1^2 + (-i\sqrt{2} * i\sqrt{2})] * C^2 = 1$. So our normalization constant C must be equal to $\boxed{\frac{1}{\sqrt{3}}}$.

b) *Calculate the normalized state vector $\psi(t)$ at the arbitrary time $t > 0$.*

$$\begin{aligned}
E_n &= (n + \frac{1}{2})\hbar\omega \\
|\psi(t)\rangle &= \frac{1}{\sqrt{3}}\left[|1\rangle e^{-iE_1 t/\hbar} - i\sqrt{2}|2\rangle e^{-iE_2 t/\hbar}\right] \\
&= \frac{1}{\sqrt{3}}\left[|1\rangle e^{-\frac{3}{2}i\omega t} - i\sqrt{2}|2\rangle e^{-\frac{5}{2}i\omega t}\right]
\end{aligned}$$

c) *Calculate the expectation value $\langle P\rangle_t \equiv \langle\psi)(t)|P|\psi(t)\rangle$ of the momentum operator at time t. Interpret the motion of the oscillator.*

Remembering that $p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right)$, where $a^\dagger\psi_n = (\sqrt{n+1})\psi_{n+1}$ and $a\psi_n = (\sqrt{n})\psi_{n-1}$, we can see that

$$\begin{aligned}
\langle P\rangle &= \langle\psi(t)|P|\psi(t)\rangle \\
&= \frac{1}{3}i\sqrt{\frac{m\omega\hbar}{2}}\left(e^{i3\omega t/2}\langle 1| + i\sqrt{2}e^{i5\omega t/2}\langle 2|\right)\left(a^\dagger - a\right)\left(e^{-i3\omega t/2}|1\rangle - i\sqrt{2}e^{-i5\omega t/2}|2\rangle\right) \\
&= \frac{1}{3}i\sqrt{\frac{m\omega\hbar}{2}}\left(e^{i3\omega t/2}\langle 1| + i\sqrt{2}e^{i5\omega t/2}\langle 2|\right)\left(e^{-i3\omega t/2}\left(a^{dagger} - a\right)|1\rangle - i\sqrt{2}e^{-i5\omega t/2}\left(a^\dagger - a\right)|2\rangle\right) \\
&= \frac{1}{3}i\sqrt{\frac{m\omega\hbar}{2}}\left(e^{i3\omega t/2}\langle 1| + i\sqrt{2}e^{i5\omega t/2}\langle 2|\right)\left(e^{-i3\omega t/2}\left(\sqrt{2}|2\rangle - |0\rangle\right) - i\sqrt{2}e^{-i5\omega t/2}\left(\sqrt{3}|3\rangle - \sqrt{2}|1\rangle\right)\right) \\
&= \frac{1}{3}i\sqrt{\frac{m\omega\hbar}{2}}\Big(-\langle 1|0\rangle + \sqrt{2}\langle 1|2\rangle + i2e^{-i\omega t}\langle 1|1\rangle - i\sqrt{6}e^{-i\omega t}\langle 1|3\rangle - i\sqrt{2}e^{i\omega t}\langle 2|0\rangle + i2e^{i\omega t}\langle 2|2\rangle \\
&\qquad -2\sqrt{2}\langle 2|1\rangle + 2\sqrt{3}\langle 2|3\rangle\Big) \\
&= \frac{1}{3}i\sqrt{\frac{m\omega\hbar}{2}}\left(i2e^{-i\omega t} + i2e^{i\omega t}\right) \\
&= -\frac{2}{3}\sqrt{\frac{m\omega\hbar}{2}}2\cos\omega t
\end{aligned}$$

$$\boxed{\langle P\rangle = -\frac{4}{3}\sqrt{\frac{m\omega\hbar}{2}}\cos\omega t}$$

d) *Calculate the expectation value $\langle H \rangle_t \equiv \langle \psi(t)|H|\psi(t)\rangle$ of the energy of the oscillator at the time t.*

Hint: $|n\rangle$ is an eigenvector of the number operator, and $P = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)$

Knowing that $H = \left(a^\dagger a - \frac{1}{2}\right)\hbar\omega$, we find that

$$\begin{aligned}
\langle H\rangle &= \langle\psi(t)|H|\psi(t)\rangle \\
&= \frac{1}{3}\hbar\omega\left(e^{i3\omega t/2}\langle 1| + i\sqrt{2}e^{i5\omega t/2}\langle 2|\right)\left(e^{-3i\omega t/2}a^\dagger a|1\rangle - \frac{1}{2}e^{-3i\omega t/2}|1\rangle - i\sqrt{2}e^{-5i\omega t/2}a^\dagger a|2\rangle + i\frac{\sqrt{2}}{2}e^{-5i\omega t/2}|2\rangle\right) \\
&= \frac{1}{3}\hbar\omega\left(e^{i3\omega t/2}\langle 1| + i\sqrt{2}e^{i5\omega t/2}\langle 2|\right)\left(e^{-3i\omega t/2}a^\dagger|0\rangle - \frac{1}{2}e^{-3i\omega t/2}|1\rangle - 2ie^{-5i\omega t/2}a^\dagger|1\rangle + \frac{\sqrt{2}}{2}ie^{-5i\omega t/2}|2\rangle\right) \\
&= \frac{1}{3}\hbar\omega\left(e^{i3\omega t/2}\langle 1| + i\sqrt{2}e^{i5\omega t/2}\langle 2|\right)\left(e^{-3i\omega t/2}|1\rangle - \frac{1}{2}e^{-3i\omega t/2}|1\rangle - 2\sqrt{2}ie^{-5i\omega t/2}|2\rangle + \frac{\sqrt{2}}{2}ie^{-5i\omega t/2}|2\rangle\right) \\
&= \frac{1}{3}\hbar\omega\left(e^{i3\omega t/2}\langle 1| + i\sqrt{2}e^{i5\omega t/2}\langle 2|\right)\left(\frac{1}{2}e^{-3i\omega t/2}|1\rangle - \frac{3\sqrt{2}}{2}ie^{-5i\omega t/2}|2\rangle\right) \\
&= \frac{1}{3}\hbar\omega\left(\frac{1}{2}\langle 1|1\rangle - \frac{3\sqrt{2}}{2}ie^{-i\omega t}\langle 1|2\rangle + \frac{\sqrt{2}}{2}ie^{i\omega t}\langle 2|1\rangle + 3\langle 2|2\rangle\right)
\end{aligned}$$

$$\boxed{\langle H\rangle = \frac{7}{6}\hbar\omega}$$

A3

Consider two non-interacting particles, both of mass m, in a one-dimensional box with infinite walls at $x = 0$ and $x = L$.

a) *If the particles are distinguishable, the states can be designated $|n_1, n_2\rangle$ where n_1, n_2 are integers. What is the degeneracy (number of states with same energy) of the first excited state? (You must show in detail how you computed this number. Simply guessing the answer without justification will earn zero credit.)*

Two particles in the first and second excited states can have two forms: $|1,2\rangle$ and $|2,1\rangle$ with the same energy. Thus there is a one-fold degeneracy, or the degeneracy is 2.

b) *If the particles are electrons, write the properly symmetrized wavefunctions (including the spin part) for the ground and first excited states. Indicate the spin component of the wavefunctions using χ_+ and χ_-, where $\chi_+(1)$ means that particle 1 is in a spin-up eigenstate.*

First, let's look at our possible spin states. The particles can be up-up, down-down, up-down, down-up (remember, one is excited and one is ground state, so Pauli exclusion principle is not violated). These 4 spin states can be written as

$$\begin{aligned}
\chi_+(1)\chi_-(2) \pm \chi_-(1)\chi_+(2) \\
\chi_+(1)\chi_+(2) \\
\chi_-(1)\chi_-(2)
\end{aligned}$$

And our one ground state and 2 possible excited states (electrons are indistinguishable) are

$$|1,1\rangle$$
$$\frac{1}{\sqrt{2}}\left(|1,2\rangle - |2,1\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|1,2\rangle + |2,1\rangle\right)$$

Now we need to combine the spin states with the energy states, ensuring they remain antisymmetric. The resulting symmetrized wavefunctions of the ground and possible excited states are

$$|1,1\rangle\left[\chi_+(1)\chi_-(2) - \chi_-(1)\chi_+(2)\right]$$
$$\frac{1}{\sqrt{2}}\left(|1,2\rangle - |2,1\rangle\right)\left[\chi_+(1)\chi_-(2) + \chi_-(1)\chi_+(2)\right]$$
$$\frac{1}{\sqrt{2}}\left(|1,2\rangle + |2,1\rangle\right)\left[\chi_+(1)\chi_-(2) - \chi_-(1)\chi_+(2)\right]$$
$$\frac{1}{\sqrt{2}}\left(|1,2\rangle - |2,1\rangle\right)\left[\chi_+(1)\chi_+(2)\right]$$
$$\frac{1}{\sqrt{2}}\left(|1,2\rangle - |2,1\rangle\right)\left[\chi_-(1)\chi_-(2)\right]$$

B1

A monatomic ideal gas with spin $\frac{1}{2}$ atoms in a magnetic field B has, in addition to its kinetic energy, a magnetic energy of $\pm\mu B$ per atom, where μ is the magnetic dipole moment. It is assumed that the gas is so dilute that the interaction of magnetic dipole moments may be neglected.

a) *Derive an expression for the canonical partition function of N such atoms.*

$$Z_1 = \sum_i e^{-E_i\beta} = e^{-\mu\beta B} + e^{\mu\beta B}$$
$$Z_N = Z_1^N/N!$$
$$\boxed{Z_N = \frac{1}{N!}\left(e^{-\mu\beta B} + e^{\mu\beta B}\right)^N}$$

as the atoms are indistinguishable.

b) *Derive expressions for the Helmholtz free energy and the internal energy of this N-atom system from the partition function.*

$$F_N = -kT\ln Z_N$$
$$\boxed{F_N = -kTN\ln(2\cosh\mu\beta B) - \ln N + 1}$$

using Sterling's approximation ($\ln N! \simeq N\ln N - N$). As for the internal energy

$$U = -\frac{1}{Z_N}\frac{\partial Z_N}{\partial\beta} = -\mu BN\tanh\mu\beta B$$

c) *Calculate heat capacity C_V.*

$$C_V = \left.\frac{\partial U}{\partial T}\right|_V = \mu B N \operatorname{sech}^2\left(\frac{\mu B}{kT}\right) * \frac{\mu B}{k}\left(-\frac{1}{T^2}\right)$$

$$\boxed{C_V = \left(\frac{\mu B}{T}\right)^2 \frac{N}{k}\operatorname{sech}^2\left(\frac{\mu B}{kT}\right)}$$

B2

A 1-dimensional chain has N ($\gg 1$) elements of length a, and the angle between adjacent elements can only be $0°$ or $180°$. The joints can turn freely and the two ends of the chain are fixed at a distance L. (The elements in the drawing are displaced in the y-direction for clarity).

a) *Show that the number of arrangements that can give an overall length of $L = 2ma$, with $m > 0$, is*

$$\omega = \frac{N!}{(\frac{N}{2}+m)!(\frac{N}{2}-m)!}$$

Knowing that $\omega = \frac{N!}{(N_+)!(N_-)!}$,

$$\begin{aligned} L = 2ma &= a(n_+ - n_-) \\ 2m &= n_+ - n_- \qquad n_+ + n_- = N \\ 2m &= 2n_+ - N \\ n_+ &= \frac{N+2m}{2} \qquad n_- = \frac{N-2m}{2} \end{aligned}$$

Therefore, $\boxed{\omega = \frac{N!}{(\frac{N}{2}+m)!(\frac{N}{2}-m)!}}$

b) *Show that the entropy of the system is $S \approx k_B\left(N\ln 2 - \frac{L^2}{2Na^2}\right)$ under the condition of $N \gg 1$ and $L \ll Na$ (i.e., $m \ll N$).*

Starting with the definition

$$\begin{aligned} S = k\ln\omega &= k\ln\frac{N!}{(\frac{N}{2}+m)!(\frac{N}{2}-m)!} \\ &= k\left[\ln N! - \ln(N/2+m)! - \ln(N/2-m)!\right] \end{aligned}$$

Using Sterling's approximation ($\ln N! \approx N \ln N - N$),

$$
\begin{aligned}
S &\approx k\left[N\ln N - \cancel{N} - (\frac{N}{2}+m)\ln\left(\frac{N}{2}+m\right) + \cancel{\frac{N}{2}} + \cancel{m} - \left(\frac{N}{2}-m\right)\ln\left(\frac{N}{2}-m\right) + \cancel{\frac{N}{2}} - \cancel{m}\right] \\
&\approx k\left\{N\ln N - \left(\frac{N}{2}+m\right)\left[\ln\frac{N}{2} + \ln\left(1+\frac{2m}{N}\right)\right] - \left(\frac{N}{2}-m\right)\left[\ln\frac{N}{2} + \ln\left(1-\frac{2m}{N}\right)\right]\right\} \\
&\approx k\left\{N\ln N - \left(\frac{N}{2}+m\right)\left[\ln N - \ln 2 + \frac{2m}{N} - \frac{2m^2}{N^2}\right] - \left(\frac{N}{2}-m\right)\left[\ln N - \ln 2 - \frac{2m}{N} - \frac{2m^2}{N^2}\right]\right\} \\
&\approx k\left(\cancel{N\ln N} - \cancel{\frac{N}{2}\ln N} + \frac{N}{2}\ln 2 - \cancel{m} + \frac{m^2}{N} - \cancel{m\ln N} + \cancel{m\ln 2} - \frac{2m^2}{N} + \cancel{\frac{2m^3}{N^2}} - \cancel{\frac{N}{2}\ln N} + \frac{N}{2}\ln 2 + \cancel{m}\right. \\
&\qquad \left. + \frac{m^2}{N} + \cancel{m\ln N} - \cancel{m\ln 2} - \frac{2m^2}{N} - \cancel{\frac{2m^3}{N^2}}\right) \\
&\approx k\left(N\ln 2 - \frac{2m^2}{N}\right)
\end{aligned}
$$

$$\boxed{S \approx k\left(N\ln 2 - \frac{L^2}{2Na^2}\right)}$$

c) *Using the thermodynamic relation for change in internal energy $dU = TdS + fdL$ find the force f that is required to maintain the length L at a fixed temperature T under the same condition of $N \gg 1$ and $L \ll Na$.*

Hint: Useful formulas: $\ln n! \cong n\ln n - n$ *for* $n \gg 1$

$\ln 1 + x = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... + (-1)^n\frac{1}{n}x^n + ...$ for $-1 < x < 1$

$$
\begin{aligned}
\frac{dU}{dL} &= T\frac{dS}{dL} + f \\
f &= \frac{dU}{dL} - T\frac{dS}{dL}
\end{aligned}
$$

$$\frac{dU}{dL} = 0 \qquad \frac{dS}{dL} = \frac{-2kL}{2Na^2}$$

$$\boxed{f = \frac{TkL}{Na^2}}$$