DREXEL
UNIVERSITY

DEPARTMENT OF PHYSICS AND ATMOSPHERIC SCIENCE

Ph.D. Qualifying Exam **Classical Physics**

Friday, 20 September 1996 9 AM - 12 NOON

PRINT YOUR NAME ____________________________

EXAM CODE ______________________

PUT YOUR EXAM CODE, **NOT YOUR NAME,** ON EACH PIECE OF PAPER YOU HAND IN. (This allows us to grade each student only on the work presented.)

Do each problem or question on a separate sheet of paper. (This allows us to grade them simultaneously.)

Answer 5 of 7 short-answer questions in part I and 3 of 5 longer problems in part II, with at least one problem from Group A and one from Group B. By circling the numbers below, indicate which questions you have answered.

Short Problems	*Long Problems*
1. ________	A1. ________
2. ________	A2. ________
3. ________	A3. ________
4. ________	B1. ________
5. ________	B2. ________
6. ________	B3. ________
7. ________	

CLASSICAL PHYSICS

PART I: Short Problems (25%)

ANSWER 5 OF 7 QUESTIONS

Problem 1
A small hockey puck sits on top of a smooth mound of ice shaped like a hemisphere. If the puck is slightly disturbed, at what point does it fall off the mound?

Problem 2
A particle of mass m moves in a plane under the influence of an inverse square attractive force. Use plane polar coordinates (r, θ).

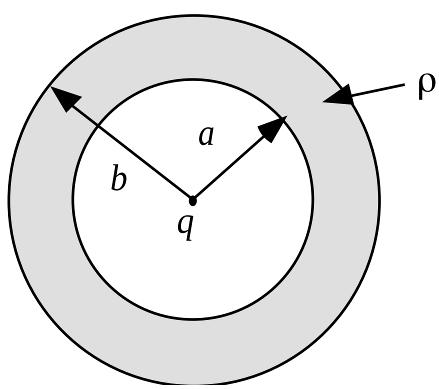
(a) Construct the Lagrangian for this one-particle system.

(b) Use the Lagrangian to obtain the equations of motion for r and θ.

(c) Explain why the equation of motion for θ tells us that the angular momentum is a constant of the motion.

Problem 3
You are given a set of polarizing elements with which to rotate the polarization of a linearly polarized beam of light by 60° with the requirement that the intensity is reduced by only 5%. How many polarizing elements are required and how should they be arranged?

Problem 4
A spherical region $a < r < b$ carries a charge per unit volume of $\rho = \dfrac{A}{r}$, where A is a constant. In addition, at the center there is a point charge q. Find the value of A so that the electric field in the region $a < r < b$ is a constant.



Problem 5

You wish to measure earth's magnetic field on the roof of Disque Hall.

(a) Give three techniques by which one could measure magnitude and direction of the earth's magnetic field. Describe each technique in reasonable detail emphasizing the physical principles involved.

(b) What is the magnitude and direction of the earth's magnetic field on the roof of Disque Hall?

Problem 6

A coil (dc resistance = 4Ω) and a capacitor are connected in parallel across a signal generator operating at a frequency f = 10 kHz and a peak voltage V_p = 10 mV as shown in Figure a. Calculate:

(a) the impedance of the coil;

(b) the impedance of the parallel circuit and the total current in the coil;

(c) the resistance, R' and inductance L' of the equivalent parallel circuit (Figure b).

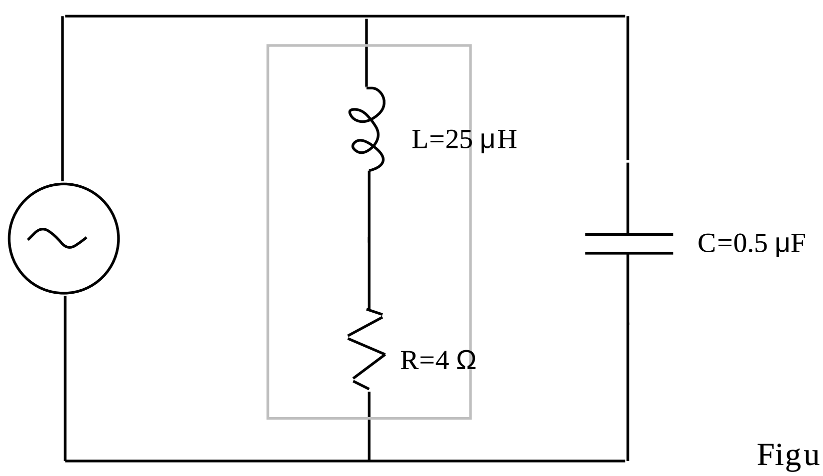


Figure a

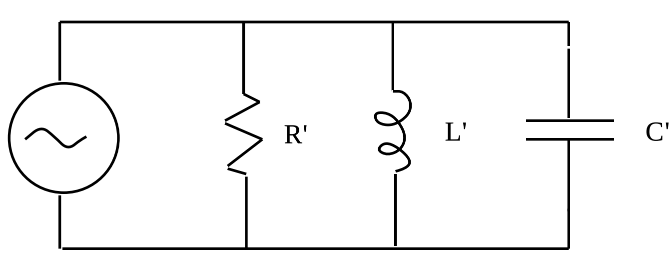
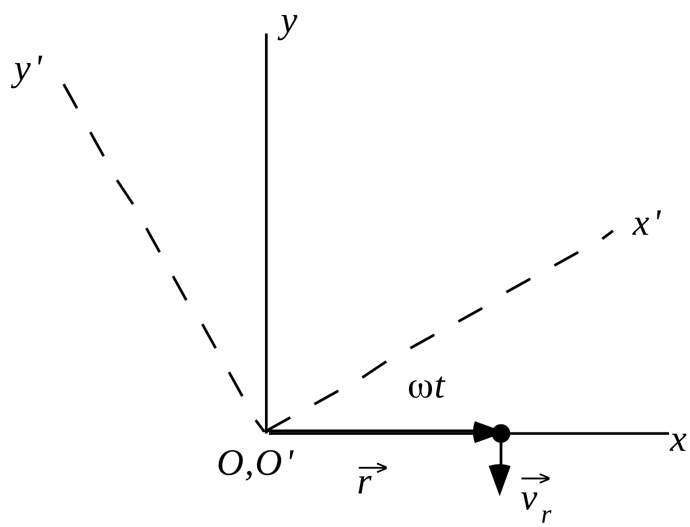


Figure b

Problem 7

The O-frame, with its z axis out of the page, is a fixed coordinate system. The O'-frame is a rotating system with x' and y' rotating with constant angular velocity ω with respect to the fixed system about the z = z' axis. An observer at O' in the rotating system sees a rock fixed on the x axis a distance r from O = O'. To the observer in the O'-frame, the rock moves in a circular orbit with speed $v_r = \omega r$. What force holds the rock in the circular orbit for the rotating observer?



PART II: Long Problems (75%)

ANSWER 3 OF 5 QUESTIONS, WITH AT LEAST ONE FROM GROUP A AND ONE FROM GROUP B.

Problem A1

A particle of unit mass (m = 1) is moving in an attractive central force potential (e.g., Newtonian gravity); the system conserves energy (dissipationless). The particle has total energy E and an angular momentum ℓ (which are both constants of the motion). The motion of the particle in the radial direction, r, is given by the energy conservation equation

$$E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{1}{r} + \frac{1}{2}\frac{\ell^2}{r^2} \quad .$$

This can be written in the form

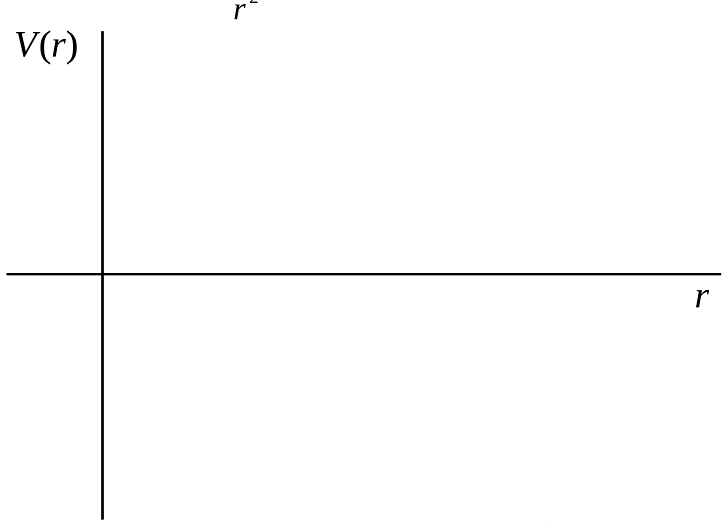
$$\left(\frac{dr}{dt}\right)^2 = 2E - V \ ,$$

where the effective potential is

$$V = -\frac{2}{r} + \frac{\ell^2}{r^2} \quad .$$

V is plotted in the diagram.

$\frac{1}{r^2}$ term dominates as $r \to 0$



$-\frac{1}{r}$ term dominates as $r \to \infty$

(a) Show that

$$\frac{d^2r}{dt^2} = -\frac{1}{2}\frac{dV}{dr} \quad .$$

(b) A circular orbit has r = constant. Use the equation derived in (a) to calculate the radius of the circular orbit.

(c) Locate the circular orbit on the potential diagram. Is this a stable or an unstable orbit?

Another particle of unit mass moves under a *different* central force. This particle has constant angular momentum L and total energy E'. The effective potential $V_{eff}(r)$ in this case takes the form

$$V_{eff}(r) = 1 - \frac{2}{r} + \frac{L^2}{r^2} - \frac{2L^2}{r^3}$$

so that $V_{eff}(r) \to 1$ as $r \to \infty$. We then have

$$\left(\frac{dr}{dt}\right)^2 = E' - V_{eff}$$

(d) Sketch the effective potential $V_{eff}(r)$.

(e) Derive a relationship for $\frac{d^2r}{dt^2}$ analogous to the one you found in part (a).

(f) Use the equation derived in (e) to calculate the radii of circular orbits in this potential.

(g) State under what conditions there are

(i) two distinct circular orbits;

(ii) one circular orbit (calculate the radius in this case);

(iii) no circular orbits.

(h) For the case in which there are two distinct circular orbits,

(i) locate r_+ (outer) and r_- (inner) on the potential diagram;

(ii) state whether the orbits at r_+ and r_- are stable or unstable;

(iii) explain what happens to a particle at r_- that receives a small inward perturbation.

Problem A2

A moving particle makes a perfectly elastic collision with a second particle, initially at rest, along their line of centers.

(a) Find the ratio of the masses which makes the kinetic energy transferred to the second particle a maximum.

(b) If the ratio does not satisfy this condition, show that the amount of energy transferred can be increased by inserting a third particle between the first two.

(c) For optimal energy transfer, show that the mass of the third particle should be the geometric mean of the other two.

Problem A3

Consider a particle of mass M.

(a) Write down the equation of motion for the particle if it is subject to the following forces:

(i) a linear restoring force of magnitude kx, where k is a constant and x is the magnitude of the displacement from the equilibrium position of the particle;

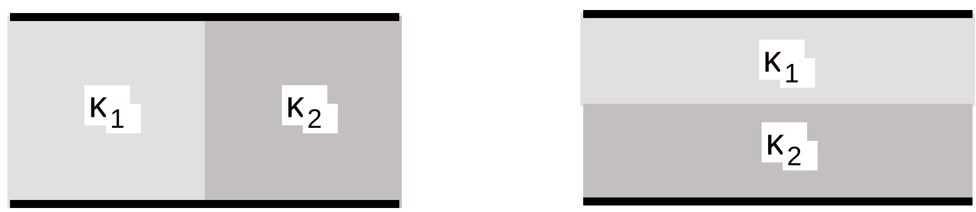
(ii) a frictional force of magnitude $bv(t)$, where b is a constant damping parameter and $v(t)$ is the particle velocity time t;

(iii) an external force $F(t) = F_o\left(1 - e^{-\alpha t}\right)$ where F_o and α are constants.

(b) Solve the equation of motion obtained in part (a) to obtain the particle position $x(t)$ as a function of time. Use $k = 4m\alpha^2$ and $b = m\alpha$. Sketch $x(t)$ vs. t.

Problem B1

Find the ratio of the two capacitances of the two capacitors shown. The capacitors were identical before the dielectrics were placed symmetrically between the plates as shown. There are no air gaps in the capacitors.



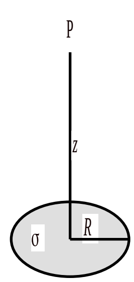
Problem B2

The potential on the axis of a disk of radius R and uniform surface charge density σ is given by

$$\phi(r,0) = \frac{\sigma}{2\varepsilon_o}\left[\sqrt{z^2 + R^2} - z\right]$$

Use this result, together with the fact that $P_n(\cos 0^\circ) = 1$ (all orders of Legendre polynomials have a value of 1 for $\theta = 0^\circ$) to calculate the first three terms in the expression for the potential

$$\phi(r,\theta) = \sum_n (A_n r^n + B_n r^{-(n+1)}) P_n(\cos\theta)$$



Problem B3

A Fabry-Perot interferometer can be constructed with a plane-parallel plate of thickness ℓ and index of refraction n immersed in a medium of index of refraction n'. Here, we will set n' =1. Suppose a plane wave is incident on the plate at an angle θ' with respect to the normal. An infinite number of reflections occur at the interfaces and this determines the reflection and transmission characteristics of the interferometer as a function of the vacuum wavelength λ.

The phase delay between two partial waves that differ by one round trip is

$$\delta = \frac{4\pi n\ell\cos\theta}{\lambda}$$

where θ is the internal angle of incidence. The complex amplitude of the incident wave is A_i. Then

$$\begin{aligned} &B_1 = rA_i \\ &B_2 = tt'r'A_i e^{i\delta} \\ &etc. \end{aligned}$$

where r is the reflection coefficient and t the transmission coefficient for waves incident from air towards the medium and r' and t' are analogous terms but for waves incident from the medium towards air. The complex amplitude of the total reflected wave is

$$\begin{aligned} &A_r = B_1 + B_2 + ... \\ &A_r = \left\{ r + tt'r'e^{i\delta}\left(1 + r'^2 e^{i\delta} + r'^4 e^{2i\delta} + ...\right)\right\} A_i \end{aligned}$$

(a) Summing the series $\left(note: 1 + a + a^2 + \ldots = \frac{1}{1-a} \right)$ while recognizing (1) the difference in phase shifts of the reflected waves such that $r' = -r$ and (2) conservation of energy in lossless media such that $r^2 + tt' = 1$ and defining the reflectance R and transmittance T as

$$R = r^2 = r'^2$$
$$T = tt'$$

show that the fraction of reflected intensity with respect to the incident intensity is

$$\frac{I_r}{I_i} = \frac{A_r A_r^*}{A_i A_i^*} = \frac{4R\sin^2\left(\frac{\delta}{2}\right)}{(1-R)^2 + 4R\sin^2\left(\frac{\delta}{2}\right)} \quad .$$

(b) Find A_t and $\frac{I_t}{I_i}$.

(c) Sketch $\frac{I_t}{I_i}$ vs. δ and $\frac{I_t}{I_i}$ vs. λ.

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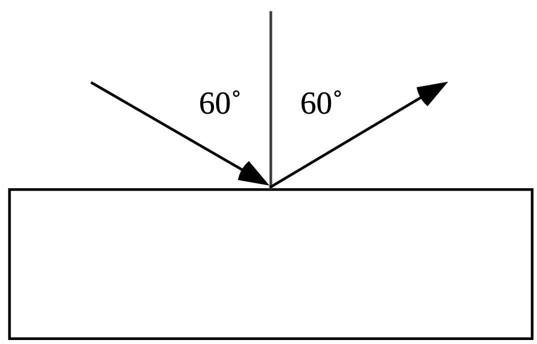
MODERN PHYSICS

PART I: Short Problems (25%)

ANSWER 5 OF 7 QUESTIONS

Problem 1

An electron beam is incident on the (100) surface of a hypothetical crystalline material at an angle of 60˚ with respect to the surface normal as shown in the figure. The material possesses a simple cubic crystal structure and the interlayer spacing is 2 Å. The electron beam energy is adjustable from 0 to 500 eV. As a first approximation, we neglect any change of energy when the electron enters the material. Estimate the energies for which maxima will be observed in the specular reflected beam ($\theta_{out} = \theta_{in}$) due to diffraction effects.



Problem 2

Bose-Einstein Condensation (BEC) has been observed by several groups.

a) Explain briefly what BEC is.

b) In one experiment, "Rubidium atoms, trapped magnetically, were cooled to below 1 μK by lasers and then cooled by evaporation."

Explain in 2-3 sentences for each of the following items:

- laser cooling
- magnetic traps
- cooling by evaporation

c) It was observed that, because the trap was shorter in one dimension, the mean momentum was greater in that dimension. Now, for a classical (non-flowing) gas, the momentum distribution is isotropic. Why is the BE condensate different?

d) Is the experimental observation of BEC worth a Nobel Prize? Defend your answer.

Problem 3

Isospin is a way to represent protons (P) and neutrons (N) in terms of two component objects. Define the *isospin operator* in terms of the Pauli matrices

$$\hat{t}_i = \frac{1}{2}\hat{\sigma}_i$$

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) What are the simultaneous eigenstates of $\hat{t}_2$ and $\hat{t}_3$ and the corresponding eigenvalues?

(b) Define a *charge operator* such that $\hat{Q}_P|P\rangle = 1|P\rangle$ and $\hat{Q}_N|N\rangle = 0$

(c) In beta decay, neutrons decay into protons and protons decay into neutrons. What should the dependence of the *beta transition operator* on isospin be?

(d) What is the connection of isospin with spin?

Problem 4

Each of the following statements is either true or false. Evaluate each in these terms and give a brief reason for each of your answers. (Note: H is the Hamiltonian and L is the angular momentum operator.)

(a) A wavefunction can be found that is a simultaneous eigenfunction of H, L^2, and L_y.

(b) A wavefunction that is a simultaneous eigenfunction of H, L_x, and L_y cannot be constructed.

(c) Exact energy eigenfunctions are obtainable for the case of three identical fermions interacting with an external potential well, with not with each other.

(d) $L_+ \, Y_{lm}(\theta,\phi)$ is an eigenfunction of L^2.

Problem 5

Consider a lithium atom (Z = 3).

(a) What is the ground state configuration of a lithium atom?

(b) Give a rough estimate of the energy of the 2p state of lithium.

Problem 6

A proton in a hydrogen atom is replaced by a positron. The resulting bound system is an exotic atom called *positronium*.

(a) If, loosely speaking, we use the diameter of the atom as a measure of its size, what is the ratio of the size of an ordinary hydrogen atom to that of the positronium?

(b) How does the energy spectrum of positronium compare with that of a hydrogen atom in a simple case when one considers only a Coulomb interaction and no relativistic effects?

(c) Explain why the spin-orbit splitting (or fine structure splitting) in positronium is of the same order of magnitude as the hyperfine splitting.

Problem 7

Four spin-$\frac{1}{2}$ particles are placed at the corners of a square. The interaction Hamiltonian is

$$H = -\sum_{nn} \sigma_{iz}\sigma_{jz}$$

where nn means nearest-neighbors.

(a) Compute the energy level spectrum including degeneracies.

(b) Compute the partition function at temperature *T*.

(c) Compute the mean energy at temperature *T*.

(d) Compute the specific heat at temperature *T*.

PART II: Long Problems (75%)

ANSWER 3 OF 5 QUESTIONS, WITH AT LEAST ONE FROM GROUP A AND ONE FROM GROUP B.

Problem A1

A very crude model of a nucleus is obtained by filling "orbitals" in a central potential with neutrons and protons. Assume that the spherically symmetric central potential is

$$V(r) = -V_o + \frac{1}{2} M\omega^2 r^2$$

(a) Find the spectrum of one particle in such a potential well.

(b) Neutrons and protons possess spin which results in an interaction with the angular momentum via the spin-orbit interaction which can be taken as

$$V_{ls}(r) = V_{so}\vec{l} \cdot \vec{s} \qquad \text{where } V_{so} < 0 .$$

Modify the spectrum obtained above to accommodate V_{ls}.

(c) Assuming no interaction between protons and neutrons, fill in the levels with neutrons and protons and predict *nuclear magic numbers.*

Problem A2

Consider a spinless particle of charge q and mass m moving in the presence of a static magnetic field B directed along the z axis and described by the vector potential

$$\vec{A}(r) = \frac{1}{2}(\vec{B} \times \vec{r}) \ .$$

The Hamiltonian for this particle is

$$H = \frac{1}{2m}\left[\vec{P} - q\vec{A}(r)\right]^2 \equiv \frac{1}{2} mV^2$$

(a) Derive explicit expressions for the three components V_x, V_y, and V_z of the velocity operator in terms of *P* and *r*.

(b) Calculate their commutation relations.

(c) Derive the explicit form of the Heisenberg equations for the velocity components.

(d) Deduce form these equations the behavior of the particle trajectory and construct an expression for the so-called cyclotron frequency ω_C.

Problem A3

The wavefunction describing the ground state of a hydrogen atom has the form

$$\Psi_{1,0,0}(r) = Ce^{-r/a_o}$$

where a_o is the Bohr radius.

(a) Estimate the probability that the electron will be found inside the proton for a hydrogen atom in the ground state.

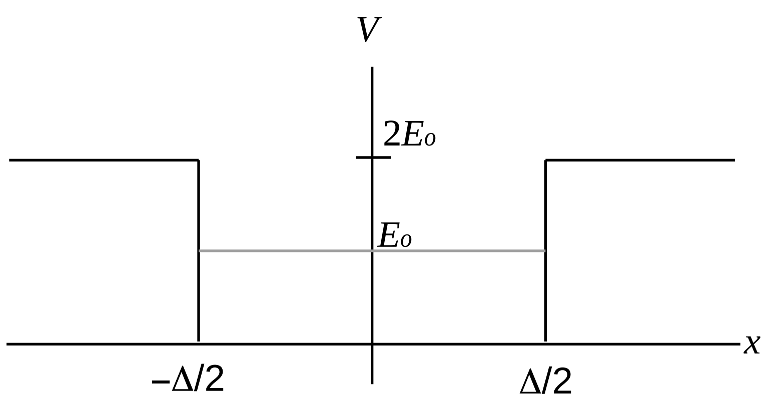
(b) Show that the electron is most likely to be found at a distance $r = a_o$ from the proton if the hydrogen atom is in the ground state.

Problem A4

A particle of mass m and energy E_o is trapped between two potential barriers of height $V_o = 2\ E_o$ separated by a distance Δ as shown. Assume that the particle inside the trap has the stationary state wavefunction

$$\phi_{inside} = A\cos(kx)$$

where $k = (2mE_o)^{1/2}$. The walls are symmetrical about $x = 0$.



(a) Find the lowest possible value for E_o.

(b) An electron is inside such a trap with Δ = 1 Å. What is the lowest value of E_o for the electron in the trap?

(c) What is the probability that the electron of part (b) is within the trap?

Problem B1

A fully degenerate non-relativistic Fermi gas has density of states

$$n(p) = \frac{8\pi p^2}{h^3} \qquad\qquad 0 \le p \le p_o$$

$$n(p) = 0 \qquad\qquad \text{otherwise}$$

where p is momentum and p_o is determined by the normalization

$$N = \int_0^{p_o} n(p)\,dp$$

where N is the total number density.

(a) Determine the equation of state of the gas relating the pressure P_{deg} to the number density N and the temperature T.

(b) By comparing the expressions for the pressure in the fully degenerate and fully nondegenerate (ideal gas) cases, estimate the density at which a sample of fully-ionized hydrogen gas at temperature T becomes significantly degenerate. Assume that only electrons contribute to P_{deg}.

Problem B2

Consider a set of N harmonic oscillators.

(a) Find the entropy of a set of N harmonic oscillators of frequency ω as a function of the total number of quanta, n, distributed among these oscillators.

(b) Let U denote the total energy $n\hbar\omega$ of the oscillators. Express the entropy σ as a function of U and N.

(c) Show that the total energy at temperature T is

$$U = \frac{N\hbar\omega}{e^{\hbar\omega/kT} - 1} \quad .$$

Note: The Stirling approximation is

$$\log N! \quad N \log N - N \; .$$

Problem B3

A 1.4 solar mass star (1.4 M_{solar} where $M_{\text{solar}} = 1.991 \times 10^{30}$ kg) collapses to a cold spherical neutron star of radius R and volume $V = \frac{4\pi}{3} R^3$ (neglect general relativity). The degenerate neutrons are non-relativistic (neglect special relativity).

(a) How many neutrons are present?

(b) For a three-dimensional degenerate Fermi gas consisting of N spin-$\frac{1}{2}$ particles occupying a volume V, show that the Fermi energy (ε_F) and the total kinetic energy (T) are given by

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$T = \frac{3}{5} N\varepsilon_F$$

(c) Show that the potential energy of the neutron star is estimated by

$$V = -k\frac{GM^2}{R} \quad .$$

If the neutron star has uniform density, show that $k = \frac{3}{5}$.

(d) Estimate the internal kinetic energy of the neutron star in terms of G, M, and R.

(e) Estimate R in km.