PHYS 502: Mathematical Physics II

Winter 2015

Solutions to Homework #6

1. We wish to minimize the travel time

$$t = \int_A^B \frac{ds}{v},$$

where $ds = \left(1 + y'^2\right)^{1/2}$ and $v^2 = v_0^2 + 2gy$. The integrand, $f(y, y', x) = \left(1 + y'^2\right)^{1/2} \left(v_0^2 + 2gy\right)^{-1/2}$, is formally independent of x, so the Euler-Lagrange equation gives

$$f - y' \frac{\partial f}{\partial y'} = \text{constant},$$

so

$$\left(1 + y'^2\right) \left(v_0^2 + 2gy\right) = C.$$

We solve this equation by setting $y' = \cot\theta$, so $1 + y'^2 = \text{cosec}^2\theta$ and

$$\begin{aligned} v_0^2 + 2gy &= C\sin^2\theta, \\ y &= \frac{1}{2g}\left(C\sin^2\theta - v_0^2\right) \\ &= \frac{C}{4g}(1 - \cos 2\theta) - \frac{v_0^2}{2g}. \end{aligned}$$

We can solve for x by writing

$$\frac{dx}{d\theta} = \frac{dy}{d\theta} \Big/ \frac{dy}{dx} = \frac{C}{2g}(1 - \cos 2\theta)$$

so

$$x = \frac{C}{4g}(2\theta - \sin 2\theta).$$

Writing $\phi = 2\theta$ we recover the standard brachistochrone solution discussed in class, except that y is offset by an amount $v_0^2/2g$, the height needed to account for the initial speed v_0.

2. (a) The line element on the surface of a sphere of radius R is

$$ds^2 = R^2(d\theta^2 + \cos^2\theta\, d\phi^2),$$

for "latitude" $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and "longitude" $0 \le \phi < 2\pi$. The geodesic equation is

$$\delta \int_A^B ds \equiv \delta \int_A^B Rf(\phi, \phi', \theta)\, d\theta = 0,$$

where $\phi(\theta)$ is the desired path from A to B and $f = \left[1 + \cos^2\theta\,(\phi')^2\right]^{1/2}$. The Euler-Lagrange equation for this problem is

$$\frac{\partial f}{\partial \phi'} = \text{constant} = \alpha,$$

so

$$\begin{aligned}
\cos^4\theta\left(\phi'\right)^2 &= \alpha^2\left[1+\cos^2\theta\left(\phi'\right)^2\right], \\
\frac{d\phi}{d\theta} &= \frac{\alpha}{\cos\theta\sqrt{\cos^2\theta-\alpha^2}} \\
\phi &= \int_0^\theta \frac{\alpha\, d\theta}{\cos\theta\sqrt{\cos^2\theta-\alpha^2}},
\end{aligned}$$

where we have chosen $\phi = 0$ when $\theta = 0$. Clearly $\cos^2\theta < \alpha^2$, and we write $\alpha = \cos\theta_{max}$. We can evaluate the integral by setting $s = \sin\theta$, so

$$\begin{aligned}
\phi &= \int_0^s \frac{\alpha\, ds}{(1-s^2)\sqrt{1-\alpha^2-s^2}} \\
&= \tan^{-1}\frac{\alpha s}{\sqrt{1-\alpha^2-s^2}} \\
\text{so } \tan\phi &= \frac{\cos\theta_{max}\sin\theta}{\sqrt{\sin^2\theta_{max}-\sin^2\theta}}\,. \qquad (1)
\end{aligned}$$

(b) To prove that this represents a great circle, we must prove that it is coplanar with the origin (the center of the sphere). Consider the following three points on the curve

(a) $\theta = 0, \phi = 0$: $(1, 0, 0)$,

(b) $\theta = \theta_{max}, \phi = \frac{\pi}{2}$: $(0, \cos\theta_{max}, \sin\theta_{max})$,

(c) arbitrary (θ, ϕ): $(x, y, z) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$.

The condition for these three points to be coplanar with the origin is

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{max} & \sin\theta_{max} \\ x & y & z \end{vmatrix} = 0\,,$$

which implies

$$z\cos\theta_{max} - y\sin\theta_{max} = 0$$

so

$$\sin\theta\cos\theta_{max} = \cos\theta\sin\phi\sin\theta_{max}$$

or

$$\sin\phi = \tan\theta\cot\theta_{max}\,,$$

which is easily shown to be equivalent to equation (1) above.

3. The light travel time from $P = (0, 0)$ to $P' = (d, 0)$ is

$$t = \int_0^d \frac{ds}{v}\,,$$

where $z(x)$ is the path, $ds^2 = dx^2 + dz^2$, and

$$v = \frac{c}{n} = \frac{c}{n_0(1+\alpha z)}.$$

Thus

$$t = \frac{n_0}{c} \int_0^d dx\, (1 + z'^2)^{1/2}\, (1 + \alpha z).$$

(a) We seek a path such that $\delta t = 0$. The integrand is independent of x, so the Euler-Lagrange equation implies

$$(1 + \alpha z)(1 + z'^2)^{1/2} - z'^2(1 + z'^2)^{-1/2}(1 + \alpha z) = A,$$

where A is a constant, so

$$\begin{aligned}
(1 + z'^2)^{-1/2}(1 + \alpha z) &= A \\
\left(\frac{dz}{dx}\right)^2 &= A^{-2}(1 + \alpha z)^2 - 1 \\
\Rightarrow \qquad \frac{A dz}{\sqrt{(1 + \alpha z)^2 - A^2}} &= dx.
\end{aligned}$$

We expect the solution to be symmetric about $x = d/2$, and the form of the left-hand integrand suggests that we try

$$\begin{aligned}
1 + \alpha z &= A \cosh \frac{x - d/2}{Q} \\
\Rightarrow \qquad \alpha dz &= \frac{A}{Q} \sinh \frac{x - d/2}{Q}\, dx
\end{aligned}$$

so

$$\frac{A dz}{\sqrt{(1 + \alpha z)^2 - A^2}} = \frac{A dx}{\alpha Q}\,,$$

so $Q = A/\alpha$ and the solution is

$$1 + \alpha z = A \cosh \frac{\alpha(x - d/2)}{A}.$$

We obtain A by requiring $z = 0$ when $x = 0, d$, so

$$A \cosh \frac{\alpha d}{2A} = 1.$$

Writing $\epsilon = \alpha d$ (dimensionless, and small in part b), we can recast this equation to read

$$\eta = \tfrac{1}{2}\epsilon \cosh \eta,$$

where $A = \epsilon/2\eta$. It is easily shown that this equation always has a solution for sufficiently small ϵ ($< 4/e$, approximately). For $\epsilon \ll 1$, the solution is $\eta \approx \frac{1}{2}\epsilon$, so $A = 1 + O(\epsilon)$.

(b) Now assume $\epsilon \ll 1$. The angle between the light path and the x-axis at $x = 0, d$ is θ, where

$$\tan \theta = \left.\frac{dz}{dx}\right|_{x=0} = \sinh \frac{\alpha d}{2A} = \sinh \frac{\epsilon}{2A}.$$

Thus for small ϵ we have, to first order,

$$\theta \approx \tfrac{1}{2}\epsilon = \tfrac{1}{2}\alpha d.$$

4. In the frame rotating with angular speed Ω, the water is at rest and the effective potential (per unit mass) is

$$\phi^{\text{eff}}(r,z) = gz - \tfrac{1}{2}\Omega^2 r^2,$$

where r is radial distance from the rotation axis and z is height (along the axis). We wish to minimize the total energy

$$\begin{aligned} E &= \int_0^R 2\pi r\, dr \int_0^{z(r)} ds\,(gs - \tfrac{1}{2}\Omega^2 r^2) \\ &= \int_0^R 2\pi r\, dr\,(\tfrac{1}{2}gz^2 - \tfrac{1}{2}\Omega^2 r^2 z), \end{aligned}$$

where $z(r)$ now represents the water surface, subject to the constraint

$$V = \int_0^R 2\pi r z\, dr \;=\; \text{constant}.$$

We accomplish this by performing an unconstrained minimization of

$$E + \lambda V = \int_0^R f(z, z', r) dr,$$

where

$$f(z,z',r) = \tfrac{1}{2}grz^2 - \tfrac{1}{2}\Omega^2 r^3 z + \lambda r z.$$

The Euler-Lagrange equation gives

$$\frac{\partial f}{\partial z} = 0 \quad\Rightarrow\quad gz = \tfrac{1}{2}\Omega^2 r^2 - \lambda,$$

so the surface is a parabola.

5. (a) The trial function

$$u(r,\theta) = r\,\left[1 - (r/R)^n\right]\,\cos\theta\,,$$

satisfies the boundary conditions on the edges of the drum head ($r = R, \theta = \pm\frac{\pi}{2}$) and has no interior nodes, so it plausibly has a shape similar to the fundamental mode of the Helmholtz equation

$$\nabla^2 u + k^2 u = 0.$$

We know that the lowest eigenmode k_{min} satisfies

$$k_{min}^2 \le K[u] \equiv \frac{\int (\nabla u)^2 d^2x}{\int u^2 d^2x}.$$

Using

$$(\nabla u)^2 = \left[1 - (n+1)\left(\frac{r}{R}\right)^n\right]^2 \cos^2\theta + \left[1 - \left(\frac{r}{R}\right)^n\right]^2 \sin^2\theta$$

and setting $x = r/R$, we evaluate this expression to obtain

$$\begin{aligned}
K[u] &= \frac{1}{R^2} \frac{\int_0^1 x dx \left\{[1-(n+1)x^n]^2 + (1-x^n)^2\right\}}{\int_0^1 x dx \, x^2 (1-x^n)^2} \\
&= \frac{1}{R^2} \frac{\int_0^1 \left\{2x - 2(n+2)x^{n+1} + [(n+1)^2+1]x^{2n+1}\right\} dx}{\int_0^1 (x^3 - 2x^{n+3} + x^{2n+3}) dx} \\
&= \frac{1}{R^2} \frac{-1 + \frac{(n+1)^2+1}{2(n+1)}}{\frac{1}{4} - \frac{2}{n+4} + \frac{1}{2n+4}} \\
&= \frac{1}{R^2} \frac{2(n+2)(n+4)}{n+1}.
\end{aligned}$$

This expression is minimized when n satisfies

$$n^2 + 2n - 2 = 0 \quad \Rightarrow \quad n = -1 + \sqrt{3} = 0.732,$$

so $K_{min} = 14.9/R^2$ and

$$k_{min} \leq \frac{3.86}{R}.$$

(b) The analytic solution is the sum of modes of the form

$$u = J_m(kr) \cos m\theta.$$

We choose $m = 1$ to satisfy the boundary conditions at $\theta = \pm\frac{\pi}{2}$, so

$$u = J_1(kr) \cos\theta.$$

The boundary condition at $r = R$ implies

$$k_{min} = \frac{\alpha_{11}}{R} = \frac{3.83}{R}.$$