Electromagnetic Theory II HW1

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a) We start with the general solution for the wave equation and the provided source function:

$$\Psi(\mathbf{x}, t) = \int d^3x' \, \frac{\{f(x', t')\}_{ret}}{|x - x'|} \tag{1.1}$$

$$f(x', t') = \delta(x')\delta(y')\delta(t') \tag{1.2}$$

We can write this out in components, defining $\rho^2 = x^2 + y^2$:

$$\Psi(\mathbf{x}, t) = \int d^3x' \, \frac{\delta(x')\delta(y')\delta(t - \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}/c)}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{1.3}$$

$$\Psi(\mathbf{x}, t) = \int dz' \, \frac{\delta(t - \sqrt{\rho^2 + (z - z')^2}/c)}{\sqrt{\rho^2 + (z - z')^2}} \tag{1.4}$$

To evaluate the delta function of a function, we use:

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{1.5}$$

Where the x_i are the roots of $g(x)$. In this case $g(z') = t - \sqrt{\rho^2 + (z - z')^2}/c$.

$$g'(z') = -\frac{1}{2c} \frac{2(z - z')(-1)}{\sqrt{\rho^2 + (z - z')^2}} = \frac{z - z'}{c\sqrt{\rho^2 + (z - z')^2}} \tag{1.6}$$

We find the roots of $g(z')$:

$$t - \sqrt{\rho^2 + (z - z')^2}/c = 0 \tag{1.7}$$

$$(z - z')^2 = c^2t^2 - \rho^2 \tag{1.8}$$

$$z' = z \pm \sqrt{c^2t^2 - \rho^2} \tag{1.9}$$

Plugging in the two roots the delta function is written as:

$$\delta(t - \sqrt{\rho^2 + (z - z')^2}/c) = \frac{c^2 t}{\sqrt{c^2t^2 - \rho^2}}\{\delta(z' - z + \sqrt{c^2t^2 - \rho^2}) + \delta(z' - z - \sqrt{c^2t^2 - \rho^2})\} \tag{1.10}$$

We can now evaluate (4):

$$\Psi(\mathbf{x}, t) = \frac{c^2 t}{\sqrt{c^2t^2 - \rho^2}} \left\{ \frac{1}{\sqrt{\rho^2 + (z - z')^2}} \right\}_{z' = z \pm \sqrt{c^2t^2 - \rho^2}} \tag{1.11}$$

$$\Psi(\mathbf{x}, t) = \frac{c^2 t}{\sqrt{c^2t^2 - \rho^2}} \left\{ \frac{1}{ct} + \frac{1}{ct} \right\} \tag{1.12}$$

$$\Psi(\mathbf{x}, t) = \frac{2c}{\sqrt{c^2t^2 - \rho^2}} \tag{1.13}$$

The suggest answer is this function multipled by the unit step function of $(ct - \rho)$. Inspecting (9), we see that the delta function will only have real roots when $ct > \rho$. Therefore we can write the wave equation in the suggested form:

$$\Psi(\mathbf{x}, t) = \frac{2c}{\sqrt{c^2t^2 - \rho^2}} \Theta(ct - \rho) \tag{1.14}$$

b) We now extend our line charge to a sheet charge in the y-z plane, which will produce an effective one-dimensional source in the x direction. The formulation is similar to part A, we just remove the $\delta(y')$ from the source term.

$$\Psi(\mathbf{x}, t) = \int d^3x' \; \frac{\delta(x')\delta(t - \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}/c)}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{1.15}$$

$$\Psi(\mathbf{x}, t) = \int dy' \; dz' \; \frac{\delta(t - \sqrt{x^2 + (y - y')^2 + (z - z')^2}/c)}{\sqrt{x^2 + (y - y')^2 + (z - z')^2}} \tag{1.16}$$

We do the two-dimensional integral in polar coordinates, setting the observation point at $z = 0, y = 0$. We define $\rho'^2 = z'^2 + y'^2$. The ϕ integral is easy due to our choice of origin and is just 2π. We now have:

$$\Psi(\mathbf{x}, t) = 2\pi \int_0^\infty d\rho' \; \rho' \frac{\delta(t - \sqrt{x^2 + \rho'^2}/c)}{\sqrt{x^2 + \rho'^2}} \tag{1.17}$$

We again find the derivative of the argument of the delta function:

$$\frac{d}{d\rho'}(t - \sqrt{x^2 + \rho'^2}/c) = -\frac{\rho'}{c\sqrt{x^2 + \rho'^2}} \tag{1.18}$$

The roots of the argument are at $\rho' = \pm\sqrt{c^2t^2 - x^2}$. They roots will only be real when $ct > |x|$. We also note that the negative root will not be used since the integral starts at 0. We can then write the delta function:

$$\delta(t - \sqrt{x^2 + \rho'^2}/c) = c^2 t \frac{\delta(\rho' - \sqrt{c^2t^2 - x^2})}{\sqrt{c^2t^2 - x^2}} \tag{1.19}$$

With the correct delta function expression our wave function is now:

$$\Psi(\mathbf{x}, t) = 2\pi \frac{c^2 t}{\sqrt{c^2t^2 - x^2}} \frac{1}{\sqrt{x^2 + c^2t^2 - x^2}} \sqrt{c^2t^2 - x^2} \tag{1.20}$$

$$\Psi(\mathbf{x}, t) = 2\pi c \tag{1.21}$$

We enforce the restriction from the real roots of the delta function using a unit step function of $ct - |x|$ as in part A.

$$\Psi(\mathbf{x}, t) = 2\pi c \Theta(ct - |x|) \tag{1.22}$$