

Statmech II HW3

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November 1, 2015

1 Problem 1

We prove the relation:

$$\ln w(i) = n_i \ln \left(\frac{g_i}{n_i} - a \right) - \frac{g_i}{a} \ln \left(1 - a \frac{n_i}{g_i} \right) \quad (1.1)$$

Where $a = -1$ for Bose-Einstein systems, $a = 1$ for Fermi-Dirac systems, and $a = 0$ for classical Maxwell-Boltzmann systems. We assume that $n_i \gg 1$ and $g_i \gg 1$.

For Bose-Einstein systems:

$$w(i) = \frac{(n_i + g_i - 1)!}{(n_i)!(g_i - 1)!} \quad (1.2)$$

$$\ln w(i) = \ln \left(\frac{(n_i + g_i - 1)!}{(n_i)!(g_i - 1)!} \right) \quad (1.3)$$

$$\ln w(i) = (n_i + g_i - 1) \ln (n_i + g_i - 1) - n_i \ln n_i - (g_i - 1) \ln (g_i - 1) \quad (1.4)$$

$$\ln w(i) = n_i \ln \left(\frac{n_i + g_i - 1}{n_i} \right) + (g_i - 1) \ln \left(\frac{n_i}{g_i - 1} + 1 \right) \quad (1.5)$$

$$\ln w(i) = n_i \ln \left(\frac{g_i}{n_i} + 1 \right) + g_i \ln \left(\frac{n_i}{g_i} + 1 \right) \quad (1.6)$$

Where in 1.5 we have used $n_i \gg 1$ and $g_i \gg 1$ to simplify the expression.

For Fermi-Dirac systems:

$$w(i) = \frac{(g_i)!}{(n_i)!(g_i - n_i)!} \quad (1.7)$$

$$\ln w(i) = \ln \left(\frac{(g_i)!}{(n_i)!(g_i - n_i)!} \right) \quad (1.8)$$

$$\ln w(i) = g_i \ln g_i - n_i \ln n_i - (g_i - n_i) \ln (g_i - n_i) \quad (1.9)$$

$$\ln w(i) = n_i (\ln (g_i - n_i) - \ln n_i) + g_i \left(\ln \frac{g_i}{g_i - n_i} \right) \quad (1.10)$$

$$\ln w(i) = n_i \ln \left(\frac{g_i}{n_i} - 1 \right) - g_i \ln \left(1 - \frac{n_i}{g_i} \right) \quad (1.11)$$

For Maxwell-Boltzmann systems:

$$w(i) = \frac{(g_i)^{n_i}}{(n_i)!} \quad (1.12)$$

$$\ln w(i) = n_i \ln g_i - n_i \ln n_i + n_i \quad (1.13)$$

Taking the limit of $-\frac{g_i}{a} \ln(1 - a\frac{n_i}{g_i})$ as $a \rightarrow 0$ using L'Hopital's rule:

$$\lim_{a \rightarrow 0} 0 - \frac{g_i}{a} \ln(1 - a\frac{n_i}{g_i}) = n_i \quad (1.14)$$

$$\ln w(i) = n_i \ln g_i - n_i \ln n_i - \frac{g_i}{a} \ln(1 - a\frac{n_i}{g_i}) \quad (a = 0) \quad (1.15)$$

2 Problem 2

We prove the maximization result using Lagrange multipliers:

$$\delta \ln W_i - \left(\alpha \sum \delta n_i + \beta \sum \epsilon_i \delta n_i \right) = \sum \left(\ln \left(\frac{g_i}{n_i} - a \right) - \alpha - \beta \epsilon_i \right) \delta n_i \quad (2.1)$$

We take the derivative with respect to n_i of $\ln w_i$ (equation 1.1).

$$\frac{\partial \ln w_i}{\partial n_i} = \ln \left(\frac{g_i}{n_i} - a \right) - \frac{g_i}{n_i} \frac{1}{g_i/n_i - a} + \frac{1}{1 - an_i/g_i} \quad (2.2)$$

$$\frac{\partial \ln w_i}{\partial n_i} = \ln \left(\frac{g_i}{n_i} - a \right) \quad (2.3)$$

With $W_i = \prod w_i$ and using the logarithm addition rules this proves 2.1.

3 Problem 3

Starting from the grand canonical equivalent of the microcanonical ensemble expression for number of microstates, with $g_i = 1$ and replacing n_i^* with

$\langle n_e \rangle$:

$$\ln W = \sum_e \left(\langle n_e \rangle \ln \left(\frac{1}{\langle n_e \rangle} - a \right) - \frac{1}{a} \ln (1 - a \langle n_e \rangle) \right) \quad (3.1)$$

$$S = k \ln W = k \sum_e \left(\langle n_e \rangle \ln \left(\frac{1}{\langle n_e \rangle} - a \right) - \frac{1}{a} \ln (1 - a \langle n_e \rangle) \right) \quad (3.2)$$

$$S = k \sum_e \langle n_e \rangle \ln \left(\frac{1 - a \langle n_e \rangle}{\langle n_e \rangle} \right) - \frac{1}{a} \ln (1 - a \langle n_e \rangle) \quad (3.3)$$

$$S = k \sum_e \left(\langle n_e \rangle - \frac{1}{a} \right) \ln (1 - a \langle n_e \rangle) - \langle n_e \rangle \ln \langle n_e \rangle \quad (3.4)$$

So we have proved the relation with $a = -1$ for Bosons and $a = 1$ for Fermions.

For Fermions there are only two probabilities:

$$\rho_0 = 1 - \langle n_e \rangle \quad (3.5)$$

$$\rho_1 = \langle n_e \rangle \quad (3.6)$$

So the probability form of the entropy is:

$$S = -k \sum_e ((1 - \langle n_e \rangle) \ln (1 - \langle n_e \rangle) + \langle n_e \rangle \ln \langle n_e \rangle) \quad (3.7)$$

$$S = k \sum_e ((\langle n_e \rangle - 1) \ln (1 - \langle n_e \rangle) - \langle n_e \rangle \ln \langle n_e \rangle) \quad (3.8)$$

For Bosons, the probability is:

$$\rho_e = \frac{\langle n_e \rangle^n}{(\langle n_e \rangle + 1)^{n+1}} \quad (3.9)$$

We can write the entropy equation terms as the expectation value of $\ln \rho_e$.

$$\sum_n \rho_e \ln \rho_e = \langle \ln \rho_e \rangle \quad (3.10)$$

$$S = -k \sum_e \langle \ln \langle \rho_e \rangle \rangle \quad (3.11)$$

$$S = -k \sum_e \langle n \ln \langle n_e \rangle - (n + 1) \ln (\langle n_e \rangle + 1) \rangle \quad (3.12)$$

$$S = k \sum_e (\langle n_e \rangle + 1) \ln (\langle n_e \rangle + 1) - \langle n_e \rangle \ln \langle n_e \rangle \quad (3.13)$$