Statmech II HW4

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October 29, 2015

1 Problem 6.2

We derive the expressions for $\langle n_e^2 \rangle - \langle n_e \rangle^2$ starting from the probabilities.

For Bose-Einstein statistics:

$$\langle n_e^2 \rangle = \sum_n^\infty n^2 \rho_e(n) \tag{1.1}$$

$$\langle n_e^2 \rangle = \sum_n^\infty n^2 \frac{\langle n_e \rangle^n}{(\langle n_e \rangle + 1)^{n+1}} \tag{1.2}$$

$$\tag{1.3}$$

We use the relation $\sum_n^\infty n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ and pull out a factor of $1/(\langle n_e \rangle + 1)$.

$$\langle n_e^2 \rangle = \frac{1}{\langle n_e \rangle + 1} \left(\frac{\frac{\langle n_e \rangle}{\langle n_e \rangle + 1}(1 + \frac{\langle n_e \rangle}{\langle n_e \rangle + 1})}{(1 - \frac{\langle n_e \rangle}{\langle n_e \rangle + 1})^3} \right) \tag{1.4}$$

$$\langle n_e^2 \rangle = \frac{\langle n_e \rangle}{(\langle n_e + 1 \rangle)^2} \frac{2 \langle n_e \rangle + 1}{\langle n_e \rangle + 1} (\langle n_e \rangle + 1)^3 \tag{1.5}$$

$$\langle n_e^2 \rangle = 2 \langle n_e \rangle^2 + \langle n_e \rangle \tag{1.6}$$

$$\sigma^2 = \langle n_e \rangle^2 + \langle n_e \rangle \tag{1.7}$$

We also prove the differential relation:

$$kT \frac{\partial \langle n_e \rangle}{\partial \mu} = kT \frac{\partial}{\partial \mu} \left(\frac{1}{z^{-1} e^{\beta e} - 1} \right) \tag{1.8}$$

$$kT \frac{\partial \langle n_e \rangle}{\partial \mu} = \left(\frac{1}{(z^{-1} e^{\beta e} - 1)^2} + \frac{1}{z^{-1} e^{\beta e} - 1} \right) \tag{1.9}$$

$$kT \frac{\partial \langle n_e \rangle}{\partial \mu} = \langle n_e \rangle^2 + \langle n_e \rangle = \sigma^2 \tag{1.10}$$

For Fermi-Dirac statistics:

$$\rho_e(n) = (1 - \langle n_e \rangle, \langle n_e \rangle) \tag{1.11}$$

$$\langle n_e^2 \rangle = \frac{0^2 * (1 - \langle n_e \rangle)}{1} + \frac{1^2 * \langle n_e \rangle}{1} = \langle n_e \rangle \tag{1.12}$$

$$\sigma^2 = \langle n_e \rangle - \langle n_e \rangle^2 \tag{1.13}$$

We also prove the differential relation:

$$kT\frac{\partial \langle n_e \rangle}{\partial \mu} = kT\frac{\partial}{\partial \mu}\left(\frac{1}{z^{-1}e^{\beta e} + 1}\right) \tag{1.14}$$

$$kT\frac{\partial \langle n_e \rangle}{\partial \mu} = \left(\frac{1}{z^{-1}e^{\beta e} + 1} - \frac{1}{(z^{-1}e^{\beta e} + 1)^2}\right) \tag{1.15}$$

$$kT\frac{\partial \langle n_e \rangle}{\partial \mu} = \langle n_e \rangle - \langle n_e \rangle^2 = \sigma^2 \tag{1.16}$$

For Maxwell-Boltzmann statistics:

$$\langle n_e^2 \rangle = \sum n^2 \frac{\langle n_e \rangle^n}{n!} e^{-\langle n_e \rangle} \tag{1.17}$$

$$\langle n_e^2 \rangle = e^{-\langle n_e \rangle} \sum (n - 1 + 1) \frac{\langle n_e \rangle^n}{(n-1)!} \tag{1.18}$$

$$\langle n_e^2 \rangle = e^{-\langle n_e \rangle} \left(\sum \frac{\langle n_e \rangle^n}{(n-2)!} + \sum \frac{\langle n_e \rangle^n}{(n-1)!}\right) \tag{1.19}$$

$$\langle n_e^2 \rangle = e^{-\langle n_e \rangle} \left(\langle n_e \rangle^2 e^{\langle n_e \rangle} + \langle n_e \rangle e^{\langle n_e \rangle}\right) \tag{1.20}$$

$$\langle n_e^2 \rangle = \langle n_e \rangle^2 + \langle n_e \rangle \tag{1.21}$$

$$\sigma^2 = \langle n_e \rangle \tag{1.22}$$

We also prove the differential relation:

$$\sigma^2 = \langle n_e \rangle \tag{1.23}$$

$$kT\frac{\partial \langle n_e \rangle}{\partial \mu} = kT\frac{\partial}{\partial \mu}\left(ze^{-\beta e}\right) \tag{1.24}$$

$$kT\frac{\partial \langle n_e \rangle}{\partial \mu} = \langle n_e \rangle \tag{1.25}$$

2 Problem 6.3

If the possible numer of particles in a level n_e is restricted to values $\leq \ell$, then we can write the grand partition function:

$$Z = \prod_e \sum_{n_e=0}^{\ell} (ze^{-\beta e})^{n_e} \tag{2.1}$$

Using the expression for the first ℓ terms of a power series we can write this:

$$Z = \prod_e \left(\frac{1-(ze^{-\beta e})^{\ell+1}}{1-ze^{-\beta e}} \right) \tag{2.2}$$

$$\Omega = \ln Z = \sum_e \ln \left(\frac{1-(ze^{-\beta e})^{\ell+1}}{1-ze^{-\beta e}} \right) \tag{2.3}$$

$$\langle n_e \rangle = -\frac{1}{\beta} \left(\frac{\partial \Omega}{e} \right) \tag{2.4}$$

$$\langle n_e \rangle = -\frac{1}{\beta} \left(\frac{1}{1+(ze^{-\beta e})^{\ell+1}} \beta(\ell+1)(ze^{-\beta e})^{\ell+1} - \frac{1}{1-ze^{-\beta e}} \beta z e^{-\beta e} \right) \tag{2.5}$$

$$\langle n_e \rangle = \frac{ze^{-\beta e}}{1-ze^{-\beta e}} - \frac{(\ell+1)(ze^{-\beta e})^{\ell+1}}{1-(ze^{-\beta e})^{\ell+1}} \tag{2.6}$$

$$\langle n_e \rangle = \frac{1}{z^{-1}e^{\beta e}-1} - \frac{\ell+1}{(z^{-1}e^{\beta e})^{\ell+1}-1} \tag{2.7}$$

$$\tag{2.8}$$

For Fermi-Dirac statistics $\ell = 1$, and we recover the expected result:

$$\langle n_e \rangle = \frac{1}{z^{-1}e^{\beta e}-1} - \frac{2}{(z^{-1}e^{\beta e})^2-1} \tag{2.9}$$

$$u \equiv z^{-1}e^{\beta e} \tag{2.10}$$

$$\frac{1}{u-1} - \frac{2}{(u-1)(u+1)} = \frac{u-1}{(u-1)(u+1)} = \frac{1}{u+1} \tag{2.11}$$

$$\langle n_e \rangle = \frac{1}{z^{-1}e^{\beta e}+1} \tag{2.12}$$

For Bose-Einstein statistics $\ell = \infty$. The right-hand term goes to 0, so we are left with $\langle n_e \rangle = \frac{1}{z^{-1}e^{\beta e}-1}$ as expected.

3 Problem 9.5

a) We calculate the parametric equation of state for the grand parition function $Z = (1+z)^V(1+z^{\alpha V})$:

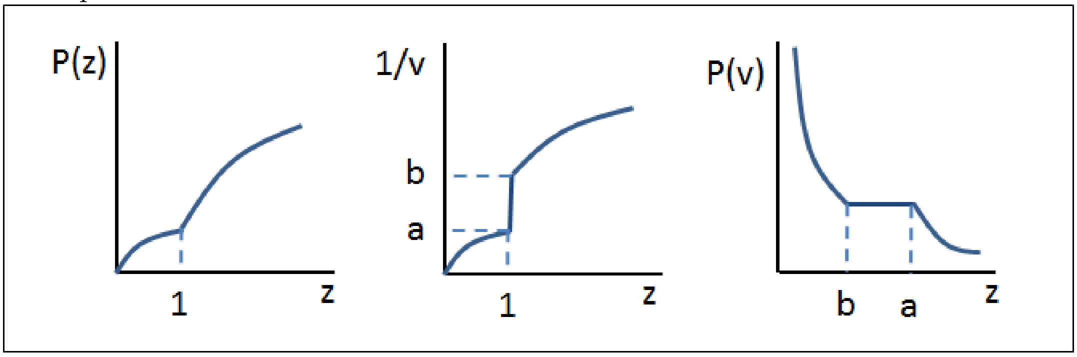
$$\frac{P}{kT} = V^{-1}\ln Z \tag{3.1}$$

$$\frac{P}{kT} = \ln(1+z) + \frac{1}{V}\ln(1+z^{\alpha V}) \tag{3.2}$$

$$\frac{1}{v} = V^{-1}z\frac{\partial}{\partial z}\ln Z \tag{3.3}$$

$$\frac{1}{v} = \left(\frac{z}{1+z} + \frac{\alpha}{z^{-\alpha V}+1}\right) \tag{3.4}$$

The graphs below show the discontinuities at $z = 1$ and the resulting first-order phase transition.



For $z < 1$ as $V \to \infty$ we have $\frac{1}{v} = \frac{z}{1+z}$.
For $z > 1$ as $V \to \infty$ we have $\frac{1}{v} = \frac{z}{1+z} + \alpha$.

b) For fixed V the roots in the complex plane for $Re(z) > 0$ are $z = (-1)^{\pm\frac{1}{\alpha V}}$. As $V \to \infty$ the roots converge to $z = 1 + i0$.

c) In the "gas" phase $z > 1$ the equation of state as $V \to \infty$ is:

$$\frac{P}{kT} = \ln(1+z) + \alpha\ln z \tag{3.5}$$

$$\frac{1}{v} = \frac{z}{z+1} + \alpha \tag{3.6}$$

Where in 3.6 we have taken the limit $V \to \infty$ before taking the derivative. These equations are continuous across $z = 1$ and so do not exhibit a phase transition.