Electromagnetic Theory I HW1

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1 Problem 1.1

a) Consider an element of excess charge inside the conductor. Enclose the excess charge in a Gaussian sphere of arbitrary radius such that the surface of the sphere remains inside the conductor. By Gauss' Law there will be an electric field normal to the surface of the sphere, porportional to the encolsed excess charge. However, any electric field inside a conductor will cause the free charges to move. Therefore, in equilibrium, any excess charge will be distributed on the surface of the conductor.

b) We have shown that there are no electric fields inside a conductor. First consider a solid conductor placed in an external field. Since there is no excess charge inside the surface integral of $\mathbf{E} \cdot d\mathbf{a}$ is equal to 0. On the surface of a conductor at equilibrium there must be no tangential component to the electric field, since any tangential component would cause the free charge carriers to move. With no tangential component to the electric field a line integral $\int \mathbf{E} \cdot d\mathbf{l}$ along the surface will identically be 0, and the entire surface is therefore an equipotential. We can see that these same boundary conditions hold for a conducting shell with the same outer surface as a solid conductor. Both the solid conductor and the shell will have the same surface charge distribution at equilibrium. Clearly this charge distribution must make the electric field inside the surface vanish, since the field inside a conductor is identically 0.

c) On the surface of a conductor at equilibrium there must be no tangential component to the electric field, since any tangential component would cause the free charge carriers to move. To calculate the normal electric field due to the surface charge distribution take a Gaussian pillbox around a differential area element da. Let the depth of the pillbox become arbitrarily small, so that the tangential components of the surface integral will be 0. The total charge enclosed in the pillbox is $\sigma\ da$. Gauss's law is then:

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0}\sigma da \tag{1.1}$$

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{n} \tag{1.2}$$

2 Problem 1.4

All three charge distributions are spherically symmetric, so the field will have no angular dependence. For all three distributions the field external to the sphere at distance R can be calculated from Gauss' law and the total charge Q in the sphere.

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0} \tag{2.1}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{\mathbf{n}} \tag{2.2}$$

The conducting sphere has no field inside the sphere, as it is a conductor. In the uniform sphere at a point of radius R we draw a Gaussian sphere that will enclose total charge $\frac{4}{3}\pi R^3\rho$, with $\rho = \frac{3Q}{4\pi a^3}$. The total enclosed charge is therefore $Q\frac{R^3}{a^3}$. Using Gauss' law:

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0}\frac{R^3}{a^3} \tag{2.3}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{R}{a^3}\hat{\mathbf{n}} \tag{2.4}$$

We note that at $R = a$ the internal and external fields are equal, so the field is continuous as expected.

For the exponential charge distribution, we first solve for the constant of porportionality to make the total charge Q.

$$\int Cr^n r^2\ dr\ \sin\theta\ d\theta\ d\phi = Q \tag{2.5}$$

$$C\frac{1}{n+3}a^{n+3} = \frac{Q}{4\pi} \tag{2.6}$$

$$C = \frac{Q}{4\pi}\frac{n+3}{a^{n+3}} \tag{2.7}$$

We can now calculate the field at a point at radius R inside the sphere from Gauss' law.

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \int_V \frac{Q}{4\pi\epsilon_0} \frac{n+3}{a^{n+3}} r^n \, dV \tag{2.8}$$

$$4\pi R^2 E = \frac{Q}{\epsilon_0} \int_0^R \frac{n+3}{a^{n+3}} r^{n+2} \tag{2.9}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{R^{n+1}}{a^{n+3}} \tag{2.10}$$

The fields for all three distributions are plotted below in units such that $\frac{Q}{4\pi\epsilon_0} = 1$.

Figure 1: Conducting sphere

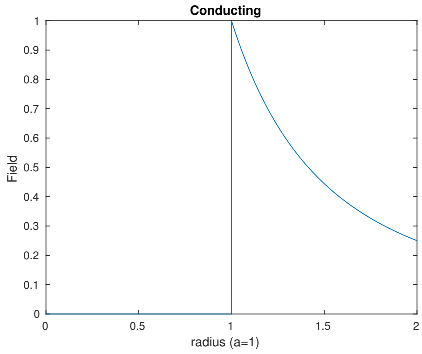

Figure 2: Sphere of uniform charge distribution

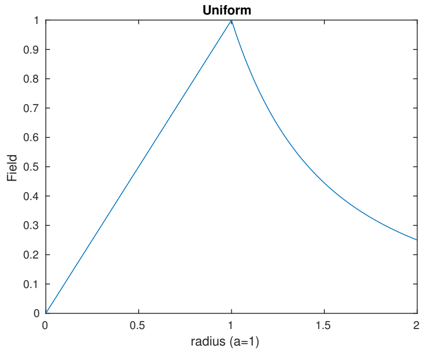

Figure 3: Sphere with exponential radial charge distribution