Statmech II HW6

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1 Problem 8.9

We calculate the second-order approximation to the specific heat of a Fermi gas using Sommerfeld's expansion of the Fermi integrals in powers of $\ln z$. To carry the whole solution to second-order we first extend the approximation of $\ln z$ derived from N/V, substituting ξ for $\ln z$:

$$\frac{N}{V} = \frac{h}{\lambda^3} f_{3/2}(z) \tag{1.1}$$

$$\frac{N}{V} \simeq \frac{4g}{3\sqrt{\pi}\lambda^3}(\xi)^{3/2}\left(1 + \frac{\pi^2}{8}\xi^{-2} - \frac{7\pi^4}{640}\xi^{-4}\right) \tag{1.2}$$

$$e_f = \left(\frac{3}{4\pi g}\right)^{2/3}\left(\frac{N}{V}\right)^{2/3}\frac{\hbar^2}{2m} \tag{1.3}$$

$$e_f \simeq \frac{\lambda^2}{\pi}\frac{\hbar^2}{2m}\xi\left(1 + \frac{\pi^2}{8}\xi^{-2} - \frac{7\pi^4}{640}\xi^{-4}\right)^{2/3} \tag{1.4}$$

Using a math package to help calculate the 2/3 power of the series we find:

$$e_f \simeq kT\ln z\left(1 + \frac{\pi^2}{8}(\ln z)^{-2} - \frac{\pi^4}{180}(\ln z)^{-4}\right) \tag{1.5}$$

We now approximate U/N:

$$\frac{U}{N} = \frac{3}{2}T\frac{f_{5/2}(z)}{f_{3/2}(z)} \tag{1.6}$$

$$f_{5/2}(z) \simeq \frac{8}{15\sqrt{\pi}}(\ln z)^{5/2}\left(1 + \frac{5\pi^2}{8}(\ln z)^{-2} - \frac{7\pi^4}{384}(\ln z)^{-4}\right) \tag{1.7}$$

$$\frac{U}{N} = \frac{3}{5}kT\ln z\left(1 + \frac{\pi^2}{2}(\ln z)^{-2} - \frac{11\pi^4}{120}(\ln z)^{-4}\right) \tag{1.8}$$

Using the provided result for $\mu = kT \ln z$:

$$\begin{aligned}\frac{U}{N} =& \frac{3}{5}e_f \left(1 + \frac{\pi^2}{2}(\frac{kT}{e_f})^2 - \frac{11\pi^4}{120}(\frac{kT}{e_f})^4\right) \\ & \times \left(1 - \frac{\pi^2}{12}(\frac{kT}{e_f})^2 - \frac{\pi^4}{80}(\frac{kT}{e_f})^4\right)\end{aligned} \tag{1.9}$$

$$\frac{U}{N} = \frac{3}{5}e_f \left(1 + \frac{5\pi^2}{12}(\frac{kT}{e_f})^2 - \frac{7\pi^4}{48}(\frac{kT}{e_f})^4\right) \tag{1.10}$$

From the temperature-dependent part we find:

$$\frac{C_V}{Nk} = \frac{\pi^2}{2}\frac{kT}{e_f} - \frac{7\pi^4}{20}\left(\frac{kT}{e_f}\right)^3 \tag{1.11}$$

2 Problem 8.10

We first find the density of energy states for a system of dimension n with $e = p^s$:

$$a(e) = \frac{V}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)}p^{n-1}dp \tag{2.1}$$

$$p = e^{1/s} \tag{2.2}$$

$$p^{n-1} = e^{(n-1)/s} \tag{2.3}$$

$$dp = \frac{1}{s}e^{\frac{1}{s}-1}de \tag{2.4}$$

$$a(e) = \frac{V}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)}e^{(n-1)/s}\frac{1}{s}e^{\frac{1}{s}-1}de \tag{2.5}$$

$$a(e) = \frac{V}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)s}e^{(n/s)-1}de \tag{2.6}$$

We can now do the usual integrals for P/kT and U/N.

$$\frac{P}{kT} = \frac{1}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)s}\int_0^\infty \ln\left(1 + ze^{-\beta e}\right)e^{(n/s)-1}de \tag{2.7}$$

$$\frac{P}{kT} = \frac{1}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)s}(kT)^{n/s}\int_0^\infty \ln\left(1 + ze^{-x}\right)x^{(n/s)-1}dx \tag{2.8}$$

$$\frac{P}{kT} = \frac{1}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)s}(kT)^{n/s}\Gamma(n/s)f_{n/s+1}(z) \tag{2.9}$$

$$\frac{N}{V} = \frac{1}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)s}\int_0^\infty \frac{e^{(n/s)-1}}{z^{-1}e^{\beta e}+1}de \tag{2.10}$$

$$\frac{N}{V} = \frac{1}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)s}(kT)^{n/s}\int_0^\infty \frac{x^{(n/s)-1}}{z^{-1}e^{x}+1}de \tag{2.11}$$

$$\frac{N}{V} = \frac{1}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)s}(kT)^{n/s}\Gamma(n/s)f_{n/s}(z) \tag{2.12}$$

Taking $U = kT^2 \left\{ \frac{\partial}{\partial T}(PV/kT) \right\}_{z,V}$ we have:

$$U = \frac{n}{s} V \frac{1}{h^n} \frac{2\pi^{n/2}}{\Gamma(n/2)s} \Gamma(n/s)(kT)^{n/s} f_{(n/s)+1}(z) \tag{2.13}$$

Comparing (13) and (9) we have shown that $U = \frac{n}{s}PV$ as required.
From (13) we have:

$$U = \frac{n}{s} NkT \frac{f_{(n/s)+1}(z)}{f_{n/s}(z)} \tag{2.14}$$

We will the derivative of z wrt T:

$$\frac{1}{z}\frac{\partial z}{\partial T} = -\frac{n}{s}\frac{1}{T}\frac{f_{n/s}(z)}{f_{(n/s)-1}(z)} \tag{2.15}$$

3 Problem 8.13