PHYS 502: Mathematical Physics II

Winter 2015

Solutions to Homework #1

1. (a) Let $\{\mathcal{R}_j\}$ be a random real sequence with $\mathcal{R}_j$ uniformly distributed on [-1,1]. Then $\langle \mathcal{R}_j \rangle = 0$, $\langle {\mathcal{R}_j}^2 \rangle = \frac{1}{3}$, and $\langle \mathcal{R}_j \mathcal{R}_{j'} \rangle = \frac{1}{3}\delta_{jj'}$. Now let

$$r_k = \sum_{j=0}^{N-1} \mathcal{R}_j \, e^{2\pi i jk/N} \, ,$$

so

$$|r_k|^2 = r_k r_k^* = \sum_{j=0}^{N-1} \sum_{j'=0}^{N-1} \mathcal{R}_j \mathcal{R}_{j'} \, e^{2\pi i (j-j')k/N} \, .$$

Hence

$$\langle |r_k|^2 \rangle = \sum_{j=0}^{N-1} \sum_{j'=0}^{N-1} \langle \mathcal{R}_j \mathcal{R}_{j'} \rangle \, e^{2\pi i (j-j')k/N} = \sum_{j=0}^{N-1} \tfrac{1}{3} = \tfrac{1}{3} N,$$

and

$$\langle P_k \rangle = \langle |r_k|^2 + |r_{N-k}|^2 \rangle = \tfrac{2}{3} N.$$

The variance is

$$\sigma_k^2 = \langle P_k^2 \rangle - \langle P_k \rangle^2.$$

The computation of $\langle P_k^2 \rangle$ involves calculating $\langle |r_k|^4 \rangle, \langle |r_{N-k}|^4 \rangle$, and $2\langle |r_k|^2 |r_{N-k}|^2 \rangle$, which contain terms of the form

$$\langle |r_k|^4 \rangle = \sum_{j=0}^{N-1} \sum_{j'=0}^{N-1} \sum_{j''=0}^{N-1} \sum_{j'''=0}^{N-1} \langle \mathcal{R}_j \mathcal{R}_{j'} \mathcal{R}_{j''} \mathcal{R}_{j'''} \rangle \, e^{2\pi i (j-j')k/N} e^{2\pi i (j''-j''')k/N}.$$

The expectation value $\langle \mathcal{R}_j \mathcal{R}_{j'} \mathcal{R}_{j''} \mathcal{R}_{j'''} \rangle$ is zero unless the indices are all equal (N possibilities) or consist of two pairs of equal values (e.g. $j = j', j'' = j'''$; total of $3N^2$ possibilities). We neglect the former, assuming $N \gg 1$, and find $\langle |r_k|^4 \rangle = \langle |r_{N-k}|^4 \rangle = \frac{1}{3}N^2$, $\langle |r_k|^2 |r_{N-k}|^2 \rangle = \frac{1}{9}N^2$. Hence $\langle P_k^2 \rangle = \frac{8}{9}N^2$ and

$$\sigma_k^2 = \tfrac{4}{9} N^2 = \langle P_k \rangle^2.$$

Hence $\sigma_k = \langle P_k \rangle$, independent of N.

(b), (c) See numerical solutions.

2. See numerical solutions.