The Addition Theorem and Coulomb Integrals

The addition theorem for spherical harmonics (see H&R p. 340) allows us to write a spherical harmonic expansion for the inverse separation between two points $\mathbf{r}_1 = (r_1, \theta_1, \phi_1)$ and $\mathbf{r}_2 = (r_2, \theta_2, \phi_2)$:

$$\frac{1}{r_{12}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \left[Y_l^m(\theta_1, \phi_1)\right]^* Y_l^m(\theta_2, \phi_2) \begin{cases} \frac{r_1^l}{r_2^{l+1}}, & |\mathbf{r}_1| < |\mathbf{r}_2| \\ \frac{r_2^l}{r_1^{l+1}}, & |\mathbf{r}_1| > |\mathbf{r}_2| \end{cases}$$

This expansion is particularly useful for calculating Coulomb integrals involving the gravitational or electrostatic potential, since the angular part of the integral becomes easy to evaluate. More generally, the properties of the spherical harmonics can be used to facilitate integrals of the form

$$I = \int\int d^3r_1\, d^3r_2\, \frac{f(r_1, \theta_1, \phi_1)\, f(r_2, \theta_2, \phi_2)}{r_{12}}.$$

As a simple example, let's compute the electrostatic self-potential energy of a uniformly charged sphere of charge density ρ and radius R. In this case $f = \rho$, constant and the total energy is

$$U = \tfrac{1}{2} \int\int d^3r_1\, d^3r_2\, \frac{k\rho^2}{r_{12}},$$

where the leading factor of $\frac{1}{2}$ corrects for double counting. The method rests on the fact that, when we expand r_{12}^{-1} all the angular integrals reduce to the form

$$\int d\Omega\; Y_l^m(\theta, \phi)\, g(\theta, \phi),$$

where g is the angular part of f. In general, this integral represents a decomposition of g into spherical harmonics. In this case, g is constant and only the constant spherical harmonic $Y_0^0 = 1/\sqrt{4\pi}$ survives. Hence

$$\int d\Omega\; Y_l^m(\theta, \phi) \;=\; 4\pi Y_0^0\, \delta_{l0}\delta_{m0} \;=\; \sqrt{4\pi}\, \delta_{l0}\delta_{m0},$$

and the sums collapse to a single term.

Once the angular part is taken care of, the radial integrals are straightforward:

$$\begin{aligned}
U &= \tfrac{1}{2}k\rho^2 \int r_1^2\, dr_1 \int r_2^2\, dr_2\, \frac{4\pi}{1} \left(\sqrt{4\pi}\right)^2 \times \begin{cases} \frac{1}{r_1}, & r_2 < r_1 \\ \frac{1}{r_2}, & r_2 > r_1 \end{cases} \\
&= 8\pi^2 k\rho^2 \int_0^R r_1^2\, dr_1 \left\{ \int_0^{r_1} r_2^2\, dr_2\, \frac{1}{r_1} + \int_{r_1}^R r_2^2\, dr_2\, \frac{1}{r_2} \right\} \\
&= 8\pi^2 k\rho^2 \int_0^R r_1^2\, dr_1 \left\{ \tfrac{1}{3} r_1^3 + \tfrac{1}{2}(R^2 - r_1^2) \right\} \\
&= 8\pi^2 k\rho^2 \left\{ -\frac{R^5}{30} + \frac{R^5}{6} \right\} \\
&= 8\pi^2 k \left(\frac{3Q}{4\pi R^3} \right)^2 \frac{2R^5}{15} \\
&= \frac{3kQ^2}{5R}.
\end{aligned}$$

where Q is the total charge.