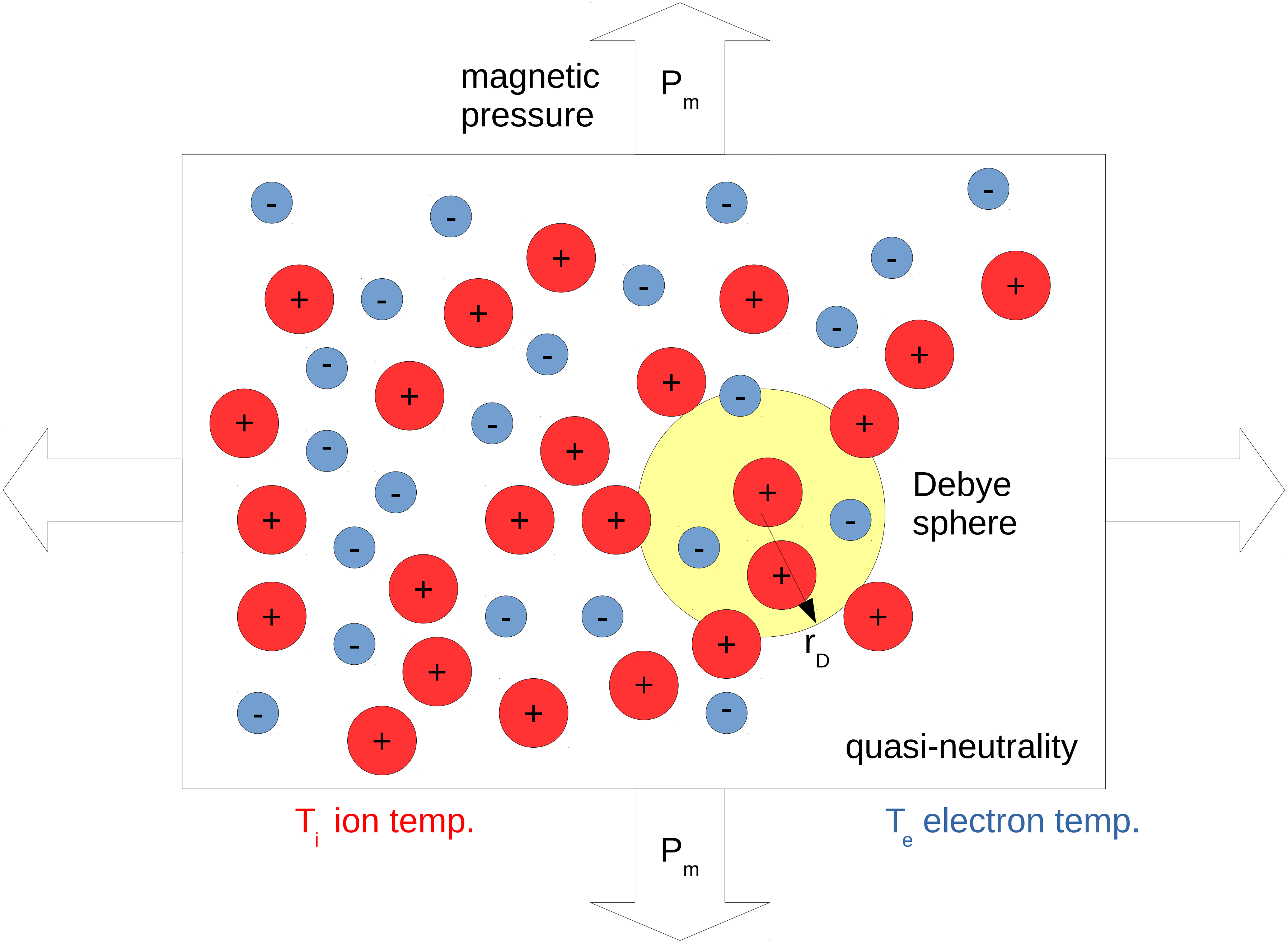
Magnetohydrodynamics (MHD)



T_i ion temp.

T_e electron temp.

Debye Shielding

$$r_D = \frac{T_e}{q_e^2 n_e}$$

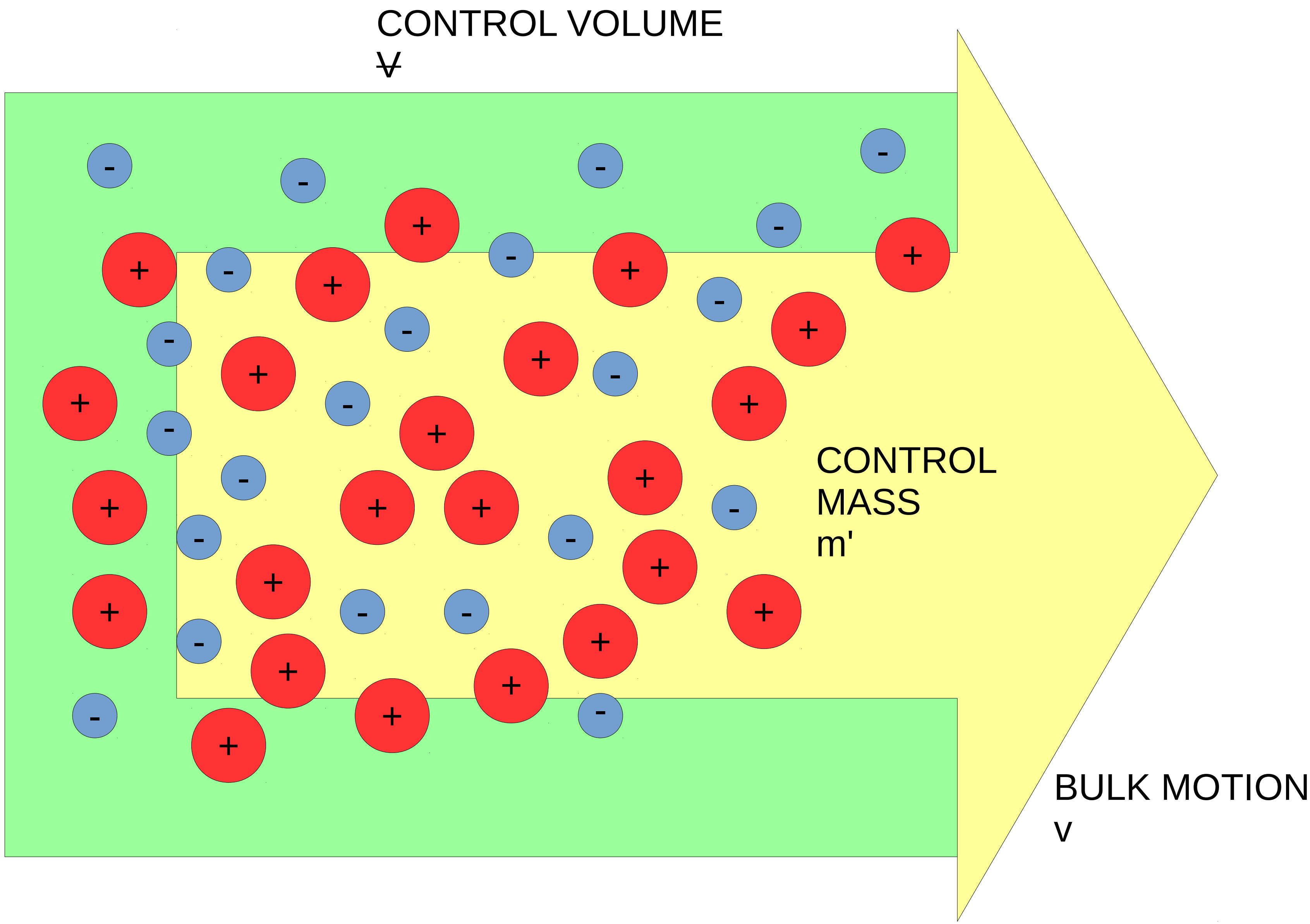
Debye radius – limit of quasi-neutrality

boundary between small-scale collisional effects and collective fluid motion

If control volume has $R \gg r_D$ possible to neglect self electric field balance of electrons and ions cancels Coulomb forces

Within $R < r_D$ must include all 2 particle Coulomb interactions

$$\vec{J} = \vec{J}\,' + \rho_Q \vec{v}$$



MHD Assumptions

one component fluid $\rho_Q = 0$

neglect displacement current $\frac{\partial D}{\partial t} = 0$

very high conductivity

scale much larger than Debye radius $R \gg r_D$

scale much larger than collisional scale

- thermal plasma $T_e = T_i = T$

Fluid Equations

continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Cauchy equation

$$\overbrace{\rho\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\vec{v}}^{\text{rate of change in momentum density}} = \overbrace{\vec{J}\times\vec{B} - \nabla P + F_{\nu} + \rho\vec{g}}^{\text{sum of body forces}}$$

convective derivative

Lorentz force

pressure gradient

viscosity

gravity

also need an equation of state, such as the ideal gas law

Maxwell's Equations

magnetic
transport
equation

$$\left.\begin{aligned} \nabla \times E + \frac{\partial B}{\partial t} &= 0 \\ \nabla \cdot D &= \rho_Q \\ \nabla \times H &= J + \frac{\partial D}{\partial t} \\ \nabla \cdot B &= 0 \end{aligned}\right\}$$

$$\boxed{\frac{\partial B}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\mu \sigma} \nabla^2 B}$$

$$\vec{J} \times \vec{B} = \boxed{\frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}} - \boxed{\nabla \left(\frac{1}{2} \frac{\vec{B}^2}{\mu} \right)}$$

magnetic tension

magnetic pressure

Magnetic Diffusion

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{\sigma \mu} \nabla^2 \vec{B}$$

transport eqn.
with no flow

magnetic
viscosity

$$\tau = \mu \sigma L^2$$

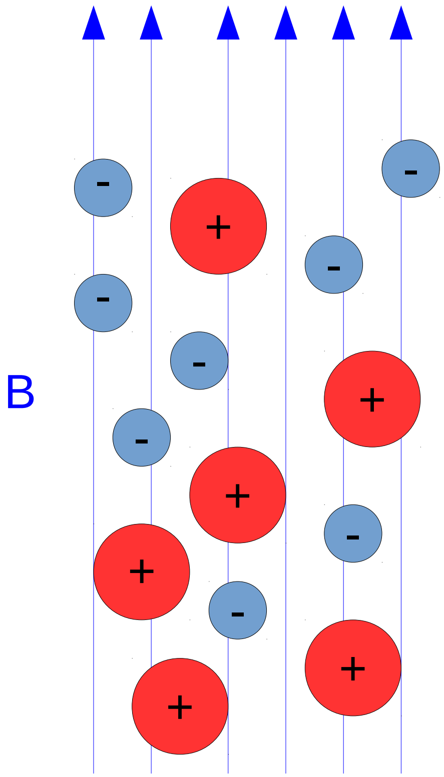
An initial field, B, diffuses in a decay time, τ
Characteristic length, L, found from problem geometry

Frozen Field Lines

$$\frac{\partial B}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

advection

transport equation for
$t \ll \tau$ or $\sigma \rightarrow \infty$



magnetic lines
"frozen" to the fluid

as fluid moves lines of magnetic force are "pulled" along

B is caused by current fluid motion changes current locally and results in shift of overall field

Dimensionless Numbers

$$Ha = B L \sqrt{\frac{\sigma}{\rho \nu}}$$

Hartmann number – ratio of Lorentz to viscous forces

$$Re = \frac{v L}{\nu}$$

Reynold's number – ratio of inertial to viscous forces

$$Re_m = B L \sigma \sqrt{\frac{\mu_0}{\rho}}$$

magnetic Reynold's number – ratio of advection to diffusion
can be written in same form as Re using phase velocity and magnetic viscosity

MHD requires $Re_m >> 1$

Characteristic length scales found from force balance

Alfven Velocity

characteristic velocity in Re_m related to v_A $\quad v = \frac{c}{n} \propto v_a$

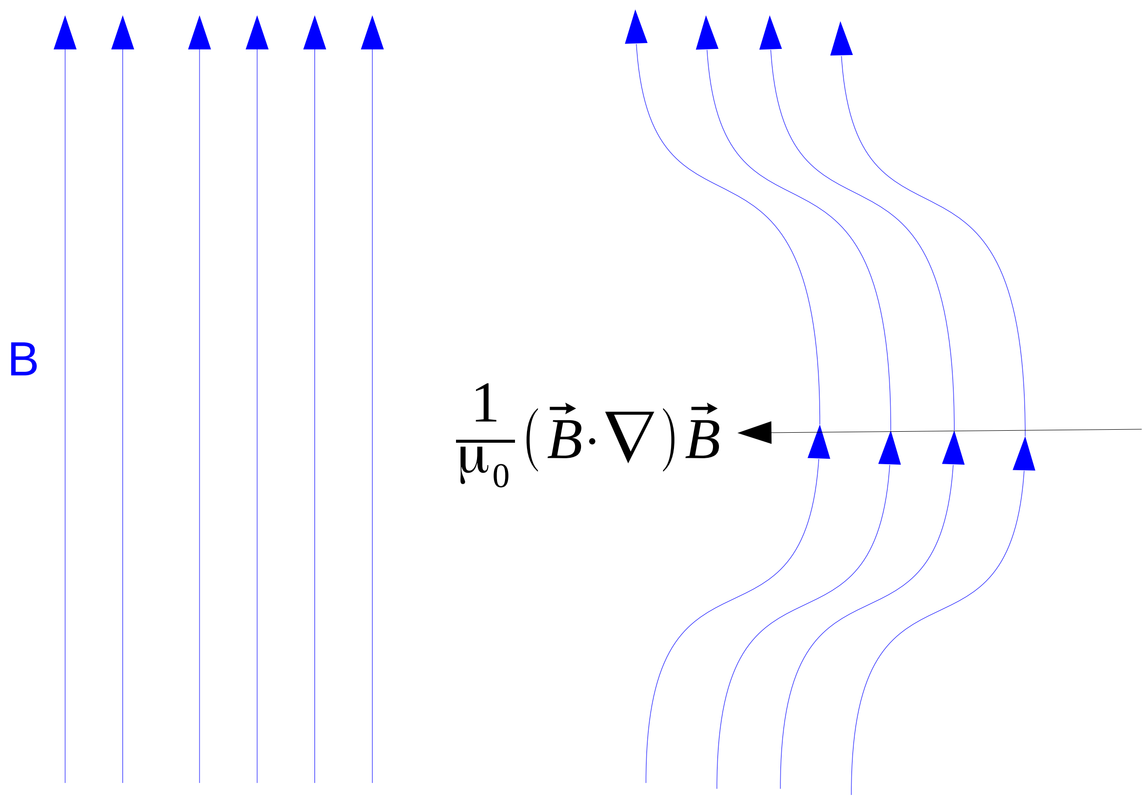
index of refraction in a single fluid plasma $\quad n^2 = 1 + \left(\frac{c}{v_A}\right)^2$

for $v_A << c$
$n \approx c$, $v \approx v_A$

$$\vec{v}_A = \frac{\vec{B}}{\sqrt{\rho \mu_0}}$$

Alfven velocity – analogous to speed of sound

Alfven Waves



If fluid is displaced magnetic tension establishes restoring force

Inertia causes overshoot and fluid oscillates with plasma frequency

$$\omega_p^2 = \frac{q_e^2 n_e}{\epsilon_0 m_e}$$

transverse wave travels along field lines with velocity v_A

Force Balance

Cauch eqn.
for static fluid

$$\frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B} \;-\; \nabla\left(\frac{1}{2}\frac{B^2}{\mu} \;+\; P\right) \;+\; F_\nu \;+\; \rho\vec{g} \;=\; 0$$

0 for 1-component
B field

total pressure

$$\vec{B}=\vec{B}_{\mathrm{ind}}+\vec{B}_{ext}$$

induced B field from current
external B field applied

References

(1)Jackson, J. *Classical Electrodynamics* John Wiley and Sons, New York, 1962.

(2)Davidson, P. *An Introduction to Magnetohydrodynamics* Cambridge University Press, New York, 2001.

(3)Fridman, A. *Plasma Chemistry* Cambridge University Press, New York, 2012.

(4)Caroll, B. Ostlie D. *An Introduction to Modern Astrophysics* Pearson, San Fransisco, 2007.

(5)Lorain, P. Lorrain, F. Houle, S. *Magneto-Fluid Dynamics* Springer, New York, 2006.

(6)wikipedia.org

Questions?

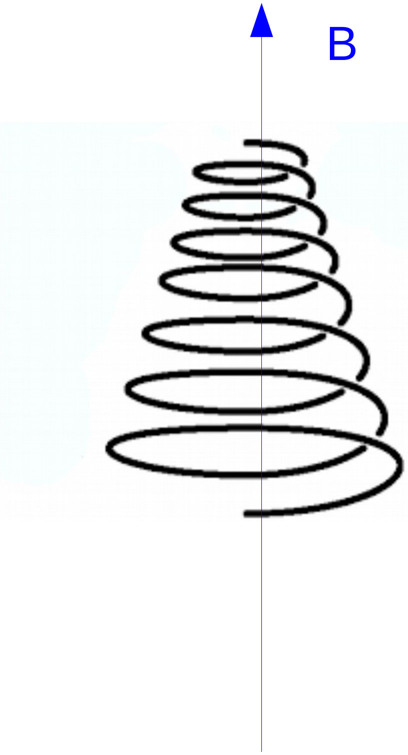
Supplemental Slides

Gyroradius

motion of light particles
around magnetic field
lines

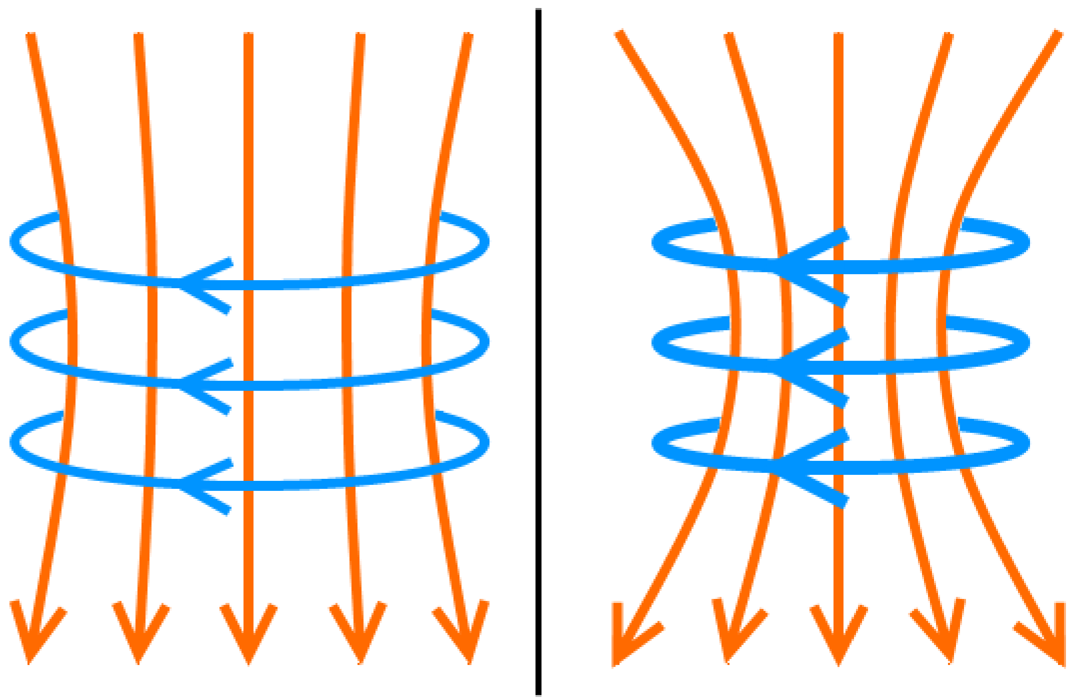
MHD requires scale
much larger than
gyroradius

$$r_g = \frac{v_\perp}{\omega_g}$$



$$\omega_g = \frac{q_e B}{m_e}$$

Pinch Effect



Plasma is contained by its own self-induced magnetic field.

For constant axial current a radial magnetic pressure is created inwards.

$$\langle P \rangle = \frac{I^2}{2\pi R^2 c^2} \qquad P(r) = 2\langle P \rangle \left(1 - \left(\frac{r}{R}\right)^2\right)$$

assuming steady-state quasi-static behavior

MHD Waves

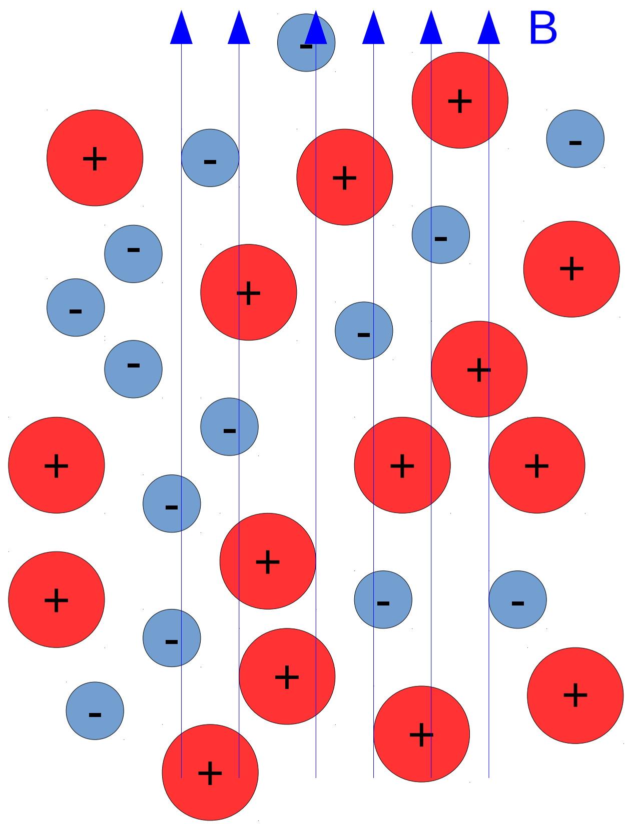
assuming equilibrium values
with some small perturbation

$$\left.\begin{aligned}\vec{B}&=\vec{B}_0+\vec{B}_1(\vec{x},t)\\ \rho&=\rho_0+\rho_1(\vec{x},t)\\ \vec{v}&=\vec{v}_1(\vec{x},t)\end{aligned}\right\}\quad -\omega^2\vec{v}_1+(v_s^2+v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k}+\vec{v}_A\cdot\vec{k}[(\vec{v}_A\cdot\vec{k})\vec{v}_1-(\vec{v}_A\cdot\vec{v}_1)\vec{k}-(\vec{k}\cdot\vec{v}_1)v_A]=0$$

allows ordinary longitudinal waves parallel to v_A $\quad (k^2 v_A^2-\omega^2)\vec{v}_1+[(\frac{v_s}{v_A})^2-1]k^2(\vec{v}_A\cdot\vec{v}_1)\vec{v}_A=0$

also allows transverse waves perpendicular to v_A $\quad \vec{v}_A\cdot\vec{v}_1=0$

Two Fluid Model



T_i ion temp. T_e electron temp.

quasi-neutrality

Similar to single fluid model, however electrons and ions treated as two separate fluids occupying same volume.
ions and electrons have different pressures, temperatures, and energies.

$$\vec{J} = q_e n_e (\vec{v}_e - \vec{v}_i)$$

$$\frac{\partial B}{\partial t} = \nabla \times (\vec{v}_e \times \vec{B})$$

field lines frozen in electron gas