PHYS 501: Mathematical Physics I

Fall 2014

Homework #5
(Due: December 1, 2014)

**** In all cases, turn in the program or script you have written—whether or not it works! ****

1. (a) Find the Fourier series $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$, for $-1 < x < 1$, for the sawtooth function

$$f(x) = \begin{cases} -1 - x & (-1 < x < 0) \\ 1 - x & (0 < x < 1)\,. \end{cases}$$

   (b) Plot the partial sums $S_N(x) = \sum_{n=1}^{N} b_n \sin(n\pi x)$ of the series for $0 \le x \le 1$, in steps of $\delta x = 0.0005$, and $N = 1$, 5, 10, 20, 50, 100, and 500. What is the maximum overshoot of the Fourier series relative to the original function in the $N = 500$ case, and at what value of x does it occur?

2. (a) Write a program to integrate the equations of motion for a particle moving in two dimensions under the influence of a central inverse-square force:

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{GM\mathbf{x}}{r^3}\,,$$

   where $r = |\mathbf{x}|$, using (i) the Midpoint method and (ii) second-order predictor–corrector, as described in class. In each case, set $GM = 1$ and take as initial conditions $x = 1, y = 0, v_x = 0, v_y = v_0 = 0.75$, where $\mathbf{x} = (x, y)$ and $\dot{\mathbf{x}} = (v_x, v_y)$. Your program should compute the trajectory from $t = 0$ to $t = 100$, using *fixed* time steps of size δt, to be defined below.

   (b) For each integration scheme and $\delta t = 0.04$, plot (i) the trajectory of the particle in the (x, y) plane and (ii) the time-dependence of the specific energy $E(t) = \frac{1}{2}v^2 - GM/r$.

   (c) For the Midpoint method, repeat your calculations with $\delta t = 0.02$, 0.01, 0.005, 0.0025, and 0.00125. In each case, calculate the overall energy error, $\max_t |E(t) - E_0|$, where $E_0 = E(0) = \frac{1}{2}v_0^2 - 1$ here. How does the error scale with δt?

   (d) Modify your program to use a *variable* time step $\delta t = 0.04\, r^{3/2}$, and repeat part (c).

3. Use a Runge-Kutta-4 scheme to integrate the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= -\sigma x + \sigma y \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= -bz + xy \end{aligned}$$

   for $(\sigma, b, r) = (10, 8/3, 30)$ and initial conditions $(x_0, y_0, z_0) = (5, 6, 10)$. Plot (i) the time variation of x, y, and z, on the same graph, and (ii) the solution in the (x, z) plane for $0 \le t \le 100$, with $\delta t = 0.01$. (This is the famous *Lorenz* system, which kick-started the science of nonlinear dynamics in the 1960s.)

4. Find the solution with the *smallest* value of $|y'(0)|$ which satisfies the second-order differential equation

$$y'' + y' + 50y^3 = 0$$

(where $' \equiv \mathrm{d}/\mathrm{d}x$), subject to the boundary conditions

$$y(0) = 2, \quad y(1) = -2\,.$$

Use the shooting method, starting at $x = 0$ with $y(0) = 2$ and integrating to $x = 1$ using Runge-Kutta-4 with $\Delta x = 0.01$. Iterate on $y'(0)$ until the boundary condition at $x = 1$ is satisfied to a relative accuracy of 1 part in 10^4.

Plot the solution $y(x)$ and give the value of $y'(0)$. Also plot a few representative intermediate iterates $y_n(x)$ to illustrate how the method converges to the solution.