DEPARTMENT OF PHYSICS

PhD Qualifying Exam **Modern Physics**

Friday, September 21, 2001 1 pm - 4 pm

PRINT YOUR NAME________________________

EXAM CODE__________________

1. PUT YOUR EXAM CODE, **NOT YOUR NAME,** ON EACH PIECE OF PAPER YOU HAND IN. (This allows us to grade each student only on the work presented.)

2. Do each problem or question on a separate sheet of paper…even the short ones. (This allows us to grade them simultaneously.)

Answer 5 of 7 short answer questions in part I and 3 of 5 longer problems in part II, with at least one problem from Group A and one from Group B. *Circle* the numbers below to indicate which questions you have answered—write nothing on the lines.

Short questions	*Long Problems*
1. _____	A1. _____
2. _____	A2. _____
3. _____	A3. _____
4. _____	B1. _____
5. _____	B2. _____
6. _____	
7. _____	

MODERN PHYSICS

PART I: Short answers (25%)

ANSWER 5 OF 7 QUESTIONS

1. What is the rule of thumb that enables you to decide if a state of N identical particles needs to be symmetrized or not?

2. It is believed that electron neutrinos can transform into muon neutrinos. Assume an electron neutrino is in an initial state

   $|\psi(t=0)\rangle = |e\rangle = \cos\theta\ |m_1\rangle\ -\ \sin\theta\ |m_2\rangle$

   in which we have the mass eigenestates, related to electron and muon states, as

   $|m_1\rangle = \cos\theta\ |e\rangle\ +\ \sin\theta\ |\mu\rangle$

   $|m_2\rangle = -\sin\theta\ |m_1\rangle\ +\ \cos\theta\ |\mu\rangle$

   and θ is the mixing angle. The masses are slightly different: $\delta m^2 = m_2^2 - m_1^2$ and we assume that $\delta m^2/m^2 \ll 1$. The energy of a neutrino of mass m_i is given by $E_i^2 = p^2c^2 + m_i^2c^4$. The initial electron neutrino travels a distance $c\,t$ and evolves into a state

   $|\psi(t)\rangle = \cos\theta \exp(-iE_1 t/\hbar)\ |m_1\rangle - \sin\theta \exp(-iE_2 t/\hbar)\ |m_2\rangle$.

   Find the probagbility for transformation from an electron to a muon neutrino state and show that it vanishes as mass difference vanishes.

3. The atom of sodium has one electron outside a closed shell. Thus, roughly, we can view this system as a one-electron atom. Provide convincing arguments in support of the following spectroscopic fact: In the absence of an applied magnetic field each s-states of sodium ($\ell = 0$) consists of a single line, whereas the p, d, f, etc. states are split into closely spaced, but readily resolvable doublets.

4. If a beam of spin-1/2 particles (e.g. silver atoms) passes through a Stern-Gerlach analyzer, one generally expects to observe two spots at the exit of the device. Suppose one constructed a double Stern-Gerlach system in which the beam passed simultaneous through two magnetic fields crossed at 90° to one another, rather than just one. How many spots one one expect to see at the exit of this device? (Just a brief answer is needed.)

5. What is the approximate rms velocity of a molecule of the air in this room? You may take its mass as 30 x m_p, where the mass of the proton is 1.67 x 10^{-27} kg, and Boltzmann's constant is 1.4 x 10^{-23} J/K.

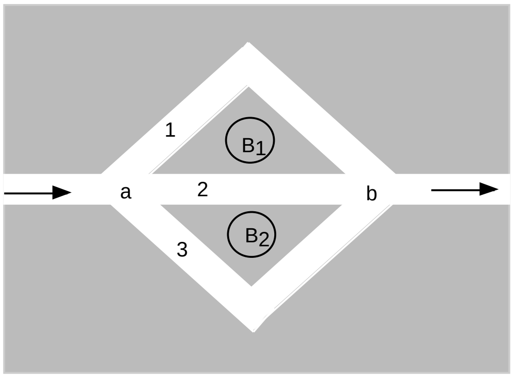
6. A tire has a gage pressure of 200 kPa at 25°C. If the atmospheric pressure is 100 kPa, find the gage pressure in the tire at a temperature of 50°C.

7. There are N systems A, B, C, ···· , which are in equilibrium with the same heat bath. If these systems are nearly independent of each other so that they can be considered as a compound system A + B + C + ····, show that the partition function and the (Helmholtz) free energy of the compound system can be expressed as: $Z_{A+B+C+\cdots} = Z_A \cdot Z_B \cdot Z_C \cdots$ and $F_{A+B+C+\cdots} = F_A + F_B + F_C + \cdots$, where Z_A, Z_B ··· and F_A, F_B ···are the partition functions and free energies of the individual systems, respectively.

PART II: Long problems (75%)

ANSWER 3 OF 5 QUESTIONS, WITH AT LEAST ONE FROM GROUP A AND ONE FROM GROUP B.

A1. Split personality



The device shown has been constructed on a nanoscale and charged particles such as electrons can enter from the left, split at *a* and go down andy (or all!) of the three legs of this interferometer, so the amplitudes recombine at *b* and the flow can emerge on the right. A circular wire can support a magnetic field B_1 between legs 1 and 2. Another circular wire can support a magnetic filed B_2 between legs 2 and 3. The magnetic field is **zero** in each of the individual legs.

According to Feynman, the wavefunction Ψ(b,t) is related to the wavefunction Ψ(a,t-Δt) by Ψ(b,t) = C Ψ(a,t-Δt) in which

$$C \sim \sum_{all\ paths} e^{\frac{i}{\hbar} S_{cl}(b,a\Delta t)}$$

and the action along a classical path from a to b in the time interval Δt is

$$S_{cl} = \int_{a,t-\Delta t}^{b,t} \left(\frac{m}{2} v^2 + \frac{e}{c} \vec{A} \cdot \vec{v}\right) dt$$

where A is the vector potential describing the magnetic field distribution.
For our purposes the only paths we need to consider are the classically allowed paths from a to b in the three legs of this interferometer. Assume the distance from a to b is L in leg 2, and $L\sqrt{2}$ in legs 1 and 3.

1. Describe the simplest classically allowed path in each of the three legs (i.e., a → b, neglect a → b → a → b and higher order paths).
2. Show that the action along this path in leg 2 is $S_2 = \frac{mL^2}{2\Delta t} + \frac{e}{c} \int_2 \vec{A} \cdot d\vec{x}$

   and evaluate the action along paths 1 and 3.

3. Show that $\Psi(b,t) = N(e^{iS_1/\hbar} + e^{iS_2/\hbar} + e^{iS_3/\hbar})\Psi(a,t-\Delta t)$
   and therefore that

   $$|\Psi(b,t)|^2 = N^2 \left(3 + 2\sum_{i>j=1}^{3} \cos\left(\frac{S_i - S_j}{\hbar}\right)\right) |\Psi(a,t-\Delta t)|^2$$

   where N is some normalization constant which you need not compute.

7. By recalling $\oint \vec{A} \cdot d\vec{S} = \iint \vec{B} \cdot d\vec{A} = \Phi$ where Φ is the magnetic flux, show that

$$\cos\left(\frac{S_1 - S_2}{\hbar}\right) = \cos\left(\frac{mL^2}{2\hbar\Delta t} - \frac{e\Phi_1}{\hbar c}\right)$$ and then evaluate the other two cosine terms.

A2. Did you check with the operator?

Consider a spin 1/2 particle. Call its spin operator $\vec{S}$, its orbital angular momentum $\vec{L}$, and its state vector $|\psi\rangle$. The two functions $\psi_+(\vec{r})$ and $\psi_-(\vec{r})$ are defined by

$$\psi_\pm(\vec{r}) = \langle \vec{r}, \pm|\psi\rangle .$$

Assume that

$$\psi_+(\vec{r}) = \sqrt{\frac{1}{14}}\, R(r)\left[2Y_0^0(\theta,\varphi) - i\sqrt{3}\,Y_2^1(\theta,\varphi)\right]$$

$$\psi_-(\vec{r}) = \frac{1}{2} R(r)\left[Y_0^0(\theta,\varphi) + 2iY_1^1(\theta,\varphi) + \sqrt{3}\,Y_1^{-1}(\theta,\varphi)\right],$$

where r,θ,φ are the spherical coordinates of the particle and $R(r)$ is a given function of r.

a)What condition must R(r) satisfy for the state $|\psi\rangle$ to be normalized?
b)S_z is measured with the particle in the state $|\psi\rangle$. What results can be found and with what probabilities?
c) L_z is measured with the particle in the state $|\psi\rangle$. What results can be found and with what probabilities?
d) A measurement of L^2, with the particle in the state $|\psi\rangle$, has yielded $\ell = 1$. What normalized state describes the particle just after this measurement?

A3. **Send in the spin**

Consider a system composed of two spin 1/2 particles whose orbital variables are ignored. The Hamiltonian of the system is

$$H = \omega_1 S_{1z} + \omega_2 S_{2z}$$

where S_{1z} and S_{2z} are the projections of the spins $\vec{S}_1$ and $\vec{S}_2$ of the two particles onto the z axis, and ω_1 and ω_2 are real constants.

The initial state of the system at time $t = 0$ is $|\psi(0)\rangle = \sqrt{\frac{1}{3}}\,|+,-\rangle - \sqrt{\frac{2}{3}}\,|-,+\rangle$

At time t, $S^2 = \left(\vec{S}_1 + \vec{S}_2\right)^2$ is measured. What results can be found, and with what probabilities?

B1. **How long can this go on?**

A polymer is a molecule composed of a long chain of identical molecular units, called monomers. Consider a long polymer, made of N rod-like monomers, each of length *a* attached end to end. One end of the polymer is held fixed while a constant force F is applied to the other end in the x direction. Each monomer can freely point along either in positive (+) or negative (-) direction. The energy is $\varepsilon_- = aF$ for monomers pointing in the negative x direction and $\varepsilon_+ = -aF$ for monomers pointing in the positive x direction.

(a) Calculate, at temperature T, the average length of the polymer, $L = N\langle l\rangle$ where $\langle l\rangle$ is the average projection of a monomer in the +x direction, i.e., $\langle l\rangle = a f_+ - a f_-$ where f_+ and f_- are the probabilities of the monomer being in the +x and –x directions, respectively. What are the limiting values of L at T = 0 and T = ∞ ?

(a) Calculate the thermal expansivity ($\alpha_l = (1/L)\,\partial L/\partial T$) of the polymer at temperature T and show that it is negative (as it is for rubber, which can also be very crudely represented by such a polymer.)

(b) What is the average internal energy of the polymer at temperature T?

B2. **Bathing the oscillator**

Consider a three-dimensional isotropic harmonic oscillator whose energy levels are given by

$$E_{n_1,n_2,n_3} = \hbar\omega(n_1 + n_2 + n_3 + \frac{3}{2})$$

where each of n_1, n_2, n_3 can be natural integers, 0, 1, 2, 3, et cetera.

a) Find the degeneracies of the levels of $\frac{7}{2}\hbar\omega$ and . $\frac{9}{2}\hbar\omega$

b) Given that the system is in thermal equilibrium with a heat bath at a temperature T, show that the $\frac{9}{2}\hbar\omega$ level is more populated than the $\frac{7}{2}\hbar\omega$ level if $k_B T$ is larger than $\dfrac{\hbar\omega}{\ln(\frac{5}{3})}$.

c) For a general energy level $(m + \frac{1}{2})\hbar\omega$, find the degeneracy and express it in terms of m.

d) If m is greater than m', above what temperature T, is the $(m + \frac{1}{2})\hbar\omega$ level more populated than the $(m' + \frac{1}{2})\hbar\omega$ level?