Ehrenfest Theorems

Robert Gilmore
Physics Department, Drexel University, Philadelphia, PA 19104

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Abstract

Ehrenfest's Theorems provide a bridge between quantum and classical mechanics. They relate time derivatives of expectation values to expectation values of appropriate operators. The expectation values are computed on quantum mechanical operators. The results assume a form "as close as possible" to the corresponding classical equations.

1 Introduction

The Ehrenfest Theorem, or Theorems, have the form

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A(x,t)}{\partial t}\rangle + \frac{i}{\hbar}\langle[H,A]\rangle \tag{1}$$

In this expression $H(x,p,t)$ is the hamiltonian that describes the system and $A(q,p,t)$ is some operator.

2 Proofs

The theorem is true both for pure states and for mixed states. Pure states are described by a wavefunction $\psi(x,t)$ (in the coordinate representation) and mixed states are described by a hermitian density operator $\rho(x,t)$ (again, in the coordinate representation). The wave function, its adjoint, and the density operator obey the following equations of motion:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t) \qquad -i\hbar\frac{\partial\psi(x,t)^\dagger}{\partial t} = \psi(x,t)^\dagger H \qquad i\hbar\frac{\partial\rho(x,t)}{\partial t} = [H,\rho] \tag{2}$$

2.1 Pure States

The expection value of an operator $A(x,t)$ in a pure state $\psi(x,t)$ is

$$\langle A\rangle = \int \psi^\dagger(x,t)A(x,t)\psi(x,t)dV \tag{3}$$

We take the time derivatives of both sides:

$$i\hbar\frac{d}{dt}\langle A\rangle = \int\left(i\hbar\frac{\partial\psi^\dagger}{\partial t}\right)A\psi\ dV + \int\psi^\dagger A\left(i\hbar\frac{\partial\psi}{\partial t}\right)dV + \int\psi^\dagger\left(i\hbar\frac{\partial A}{\partial t}\right)\psi\ dV \tag{4}$$

and use Eq.(2) for the evolution of $\psi(x,t)$ and its adjoint to find

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A(x,t)}{\partial t}\rangle + \frac{1}{i\hbar}\langle(-HA+AH)\rangle = \langle\frac{\partial A(x,t)}{\partial t}\rangle + \frac{i}{\hbar}\langle[H,A]\rangle \tag{5}$$

This is the Ehrenfest Theorem for pure states.

2.2 Mixed States

For mixed states, expectation values are constructed with the aid of the density operator $\rho(x,t)$ through $\langle A\rangle = \mathrm{Tr}\rho(x,t)A(x,t) = \mathrm{Tr}A(x,t)\rho(x,t)$. The equation of motion for the expectation value of A is

$$i\hbar\ \frac{d}{dt}\ \mathrm{Tr}\rho(x,t)A(x,t) = \mathrm{Tr}\left(i\hbar\frac{\partial\rho(x,t)}{\partial t}\right)A(x,t) + i\hbar\ \mathrm{Tr}\ \frac{\partial A(x,t)}{\partial t}\ \rho(x,t) \tag{6}$$

We use the equation of motion for $\rho(x,t)$ to replace its time derivative by $[H,\rho]$, open up the commutator, use the invariance of the trace under cyclic rotation of the operators, and divide by $i\hbar$ to obtain

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A(x,t)}{\partial t}\rangle + \frac{i}{\hbar}\langle[H,A]\rangle \tag{7}$$

Ehrenfest's Theorem holds in unchanged form for both pure and mixed states.

2.3 Heisenberg Representation

We have used the expression $\langle\psi(t)|A|\psi(t)\rangle$ for the expectation value of an operator A in the Schrödinger representation. In the Heisenberg representation is this written as $\langle\psi(0)U^{-1}(t)|A|U(t)\psi(0)\rangle = \langle\psi(0)|U^{-1}(t)AU(t)|\psi(0)\rangle$. In this representation the states are fixed and all the time dependence is expressed in terms of the operator. The unitary operator obeys the equation of motion $i\hbar\frac{\partial}{\partial t}U(t) = HU(t)$. The time derivative of the operator $A(t) = U^{-1}(t)A(0)U(t)$ is

$$\frac{d}{dt}\ A(t) = U^{-1}(t)\frac{\partial A}{\partial t}U(t) + U^{-1}(t)\left(\frac{i}{\hbar}HA - \frac{i}{\hbar}AH\right)U(t) = U^{-1}(t)\left(\frac{\partial A}{\partial t} + \frac{i}{\hbar}\left[H,A\right]\right)U(t) \tag{8}$$

The right hand side includes the explicit time dependence of the operator A through the partial derivative and the implicit time dependence through the hamiltonian dynamics. The expectation value is taken using the time independent wavefunctions

$$\frac{d}{dt}\langle\psi(0)|A(t)|\psi(0)\rangle = \langle\psi(0)|\frac{dA}{dt}|\psi(0)\rangle = \langle\psi(0)|U^{-1}(t)\left(\frac{\partial A}{\partial t} + \frac{i}{\hbar}\left[H,A\right]\right)U(t)|\psi(0)\rangle = \langle\left(\frac{\partial}{\partial t} + \frac{i}{\hbar}\left[H,\right]\right)A\rangle \tag{9}$$

On the left, since the wavefunctions are independent of time, the time derivative can be commuted out of the expectation value. On the right we find the standard expression for Ehrenfest's theorem. The same result is obtained in the Heisenberg representation using the density operator: $\rho(t) = U(t)\rho(0)U^{-1}(t)$.

In this representation it is possible to obtain more powerful results than the Ehrenfest Theorem. For example, expectation values of time derivatives of two or more operators are easily expressed:

$$\langle A\frac{dB}{dt}\rangle = \langle A\left(\frac{\partial B}{\partial t} + \frac{i}{\hbar}\left[H,B\right]\right)\rangle \tag{10}$$

$$\langle\frac{dA}{dt}\frac{dB}{dt}\rangle = \langle\left(\frac{\partial A}{\partial t} + \frac{i}{\hbar}\left[H,A\right]\right)\left(\frac{\partial B}{\partial t} + \frac{i}{\hbar}\left[H,B\right]\right)\rangle \tag{11}$$

Higher time derivatives involve mixtures of higher partial derivatives and multiple commutators:

$$\frac{d^2}{dt^2}\langle A\rangle = \langle\psi(0)|\frac{d^2}{dt^2}A(t)|\psi(0)\rangle = \langle\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t}+\left[\frac{i}{\hbar}H,A\right]\right)\rangle + \langle\left[\frac{i}{\hbar}H,\frac{\partial A}{\partial t}+\left[\frac{i}{\hbar}H,A\right]\right]\rangle \tag{12}$$

2.4 Classical — Quantum Comparison

In classical mechanics the time evolution of an operator $A(q(t),p(t);t)$ is

$$\frac{dA(q,p;t)}{dt} = \frac{\partial A}{\partial t} + \sum_i\left(\frac{\partial A}{\partial q_i}\frac{dq_i}{dt}+\frac{\partial A}{\partial p_i}\frac{dp_i}{dt}\right) \tag{13}$$

This consists of the sum of two contributions. One comes from the *explicit* time dependence of $A(q,p;t)$ through the term $\partial A/\partial t$. The value of A also depends *implicitly* on time through the dependence of the coordinates $q_i(t)$ and momenta $p_i(t)$ on time. These terms are contained within the large bracket. We use Hamilton's equations of motion $dq_i/dt = \partial H/\partial p_i$ and $dp_i/dt = -\partial H/\partial q_i$ to write this evolution equation in the suggestive form

$$\frac{dA(q,p;t)}{dt} = \frac{\partial A}{\partial t} + \sum_i\left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \to \frac{\partial A}{\partial t}+\{A,H\} \tag{14}$$

The curly brackets { } are the standard representation for the Poisson bracket:

$$\{A,B\} = \sum_i\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}\right) \tag{15}$$

By comparing the Classical result Eq. (14) with the Quantum result Eq. (7) we can infer that a useful quantization condition, allowing immediate, algorithmic passage from classical to quantum mechanics, is the identification between Poisson brackets and commutator brackets:

$$\{A,B\} \leftrightarrow \frac{1}{i\hbar}[A,B] \tag{16}$$

This correspondence is the *raison d'etre* for Goldstein's beautiful book. Applying this *quantization algorithm* to the coordinate and momentum operators gives:

$$[q_i,p_j] = i\hbar\{q_i,p_j\} = i\hbar\delta_{ij} \tag{17}$$

This algorithm shows clearly that the canonical coordinates are no longer commuting variables. We can choose to express either p in terms of q as $p_j \to \frac{\hbar}{i}\frac{\partial}{\partial q_j}$ or q in terms of p as $q_j \to -\frac{\hbar}{i}\frac{\partial}{\partial p_j}$. In finite-dimensional spaces we must choose the $n\times n$ matrix representatives of q_i and p_j so that they satisfy these commutation relations: $[q_i,p_j] = i\hbar I_n$.

3 Applications

This theorem is applied to a number of important operators. These include the position and momentum operators $\mathbf{x}$ and $\mathbf{p}$ for a single particle. These operators have no explicit time dependence, so that $\partial\mathbf{x}/\partial t = \partial\mathbf{p}/\partial t = 0$. For these operators, and for a hamiltonian with the simple form $H(\mathbf{x},\mathbf{p},t) = \frac{\mathbf{p}\cdot\mathbf{p}}{2m}+V(\mathbf{x},t)$, the important commutators are

$$[H,\mathbf{x}] = \frac{\hbar}{i}\frac{\mathbf{p}}{m} \qquad\qquad [H,\mathbf{p}] = -\frac{\hbar}{i}\nabla V(\mathbf{x},t) \tag{18}$$

For more general hamiltonians that depend on N coordinates q_i and momenta p_j which are canonically conjugate (this means $[q_r, p_s] = -\frac{\hbar}{i}\delta_{rs}$) the important commutators are

$$[H(q,p,t), q_r] = +\frac{\hbar}{i}\frac{\partial H}{\partial p_r} \qquad\qquad [H(q,p,t), p_s] = -\frac{\hbar}{i}\frac{\partial H}{\partial q_s} \tag{19}$$

The second relation is obtained by expressing p_s in the coordinate representation as $p_s = \frac{\hbar}{i}\frac{\partial}{\partial q_s}$, while the first is obtained by expressing q_r in the momentum representation as $q_r = -\frac{\hbar}{i}\frac{\partial}{\partial p_r}$.

3.1 Momentum and Velocity: $\mathbf{A} = \mathbf{x}$

The the equation of motion for the centroid of $\mathbf{x}$ is

$$\frac{d}{dt}\langle \mathbf{x}\rangle = \langle \frac{\partial \mathbf{x}}{\partial t}\rangle + \frac{i}{\hbar}\langle [H, \mathbf{x}]\rangle = \langle \frac{\mathbf{p}}{m}\rangle \tag{20}$$

3.2 Newton's Second Law: $\mathbf{A} = \mathbf{p}$

Similarly,

$$\frac{d}{dt}\langle \mathbf{p}\rangle = \frac{i}{\hbar}\langle [H, \mathbf{p}]\rangle = \langle -\nabla V(\mathbf{x})\rangle = \langle \mathbf{F}(\mathbf{x})\rangle \tag{21}$$

3.3 Acceleration: Second Derivative

The second derivative of the expectation value of the position operator is

$$m\frac{d^2}{dt^2}\langle \mathbf{x}\rangle = \langle \left[\frac{i}{\hbar}H, \left[\frac{i}{\hbar}H, m\mathbf{x}\right]\right]\rangle = \langle \left[\frac{i}{\hbar}H, \mathbf{p}\right]\rangle = \langle -\nabla V\rangle = \langle \mathbf{F}\rangle \tag{22}$$

3.4 Generalized Harmonic Motion

The hamiltonain that describes simple harmonic motion in one dimension is $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$. The second order equation of motion for $\langle x\rangle$ is

$$\frac{d^2}{dt^2}\langle x\rangle = \langle \left[\frac{i}{\hbar}\ H, \left[\frac{i}{\hbar}\ H, x\right]\right]\rangle = \langle \left[\frac{i}{\hbar}\ H, \frac{p}{m}\right]\rangle = \langle -\frac{k}{m}\ x\rangle \tag{23}$$

The expectation value of x (also of p) obeys the harmonic equation:

$$\left(\frac{d^2}{dt^2} + \frac{k}{m}\right)\langle x\rangle = 0 \tag{24}$$

More generally, harmonic motion involving many coupled degrees of freedom, as in a lattice or a molecule under the small amplitude approximation, is governed by the hamiltonian

$$H = \frac{1}{2}(M^{-1})_{rs}p_r p_s + \frac{1}{2}K_{rs}q_r q_s \tag{25}$$

Here M is the mass matrix (usually taken as diagonal and nonsingular) and K describes the connectivity of the degrees of freedom. It may be singular, and when it is there is a Goldstone mode. The coordinates and momenta are canonical: $[p_r, q_s] = (\hbar/i)\delta_{rs}$. The equation of motion for the expectation value q_r is

$$\frac{d^2}{dt^2}\langle q_r\rangle = \langle\left[\frac{i}{\hbar}\,H,\left[\frac{i}{\hbar}\,H,q_r\right]\right]\rangle = \langle\left[\frac{i}{\hbar}\,H,(M^{-1})_{rs}p_s\right]\rangle = \langle -(M^{-1})_{rs}K_{st}q_t\rangle = -(M^{-1})_{rs}K_{st}\langle q_t\rangle \tag{26}$$

This expression and easily be disentangled and expressed in a form standard in classical physics:

$$\left(M_{rs}\frac{d^2}{dt^2}+K_{rs}\right)\langle q_s\rangle = 0 \tag{27}$$

In this form it is clear that the classical and quantum normal modes 'are the same', behave the same way, obey the same generalized eigenvalue equation, and have the same eigenstructure.

3.5 Orbital Angular Momentum and Torque: $\mathbf{A} = \mathbf{r}\times\mathbf{p}$

The expectation value of the orbital angular momentum is equal to the torque on a body. In quantum form this is

$$\frac{d}{dt}\langle\mathbf{r}\times\mathbf{p}\rangle = \frac{i}{\hbar}\langle[H,\mathbf{r}\times\mathbf{p}]\rangle = \frac{i}{\hbar}\langle[H,\mathbf{r}]\times\mathbf{p}\rangle + \frac{i}{\hbar}\langle\mathbf{r}\times[H,\mathbf{p}]\rangle = \langle\frac{\mathbf{p}}{m}\times\mathbf{p}\rangle + \langle\mathbf{r}\times(-\nabla V)\rangle = \langle\mathbf{r}\times\mathbf{F}(\mathbf{x})\rangle \tag{28}$$

3.6 Angular Momentum and Precession

First we treat spin precession. Then we treat the general angular momentum case.

3.6.1 Spin Angular Momentum: $\mathbf{A} = \mathbf{S}$

The magnetic moment μ of a point particle with charge q and spin $\mathbf{S}$ is $\mu = \frac{q}{mc}\mathbf{S}$. The interaction of the moment with an external magnetic field is described by the hamiltonian $H = -\mu\cdot\mathbf{B}$. For the simplest fermions (spin $\frac{1}{2}$) the spin is $\mathbf{S} = \frac{\hbar}{2}\sigma$, where the Pauli spin matrices satisfy the commutation relations $[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k$. The standard representation for these matrices is

$$\sigma_1 = \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \tag{29}$$

The operators $\mathbf{S}$ satisfy $[S_i,S_j] = i\hbar\epsilon_{ijk}S_k$. The time evolution of $\mathbf{S}$ is

$$\frac{d}{dt}\langle\mathbf{S}\rangle = \frac{i}{\hbar}\langle[-\mu\cdot\mathbf{B},\mathbf{S}]\rangle = \frac{q}{mc}\langle\mathbf{S}\times\mathbf{B}\rangle = \langle\mu\rangle\times\mathbf{B} \tag{30}$$

3.6.2 Arbitrary Angular Momentum: $\mathbf{A} = \mathbf{J}$

In this case $\mu = \gamma\mathbf{J}$ and the angular momentum operators satisfy the commutation relations $[J_i,J_j] = i\hbar\epsilon_{ijk}J_k$. The equations of motion for $\langle\mathbf{J}\rangle$ are

$$\frac{d}{dt}\langle\mathbf{J}\rangle = \frac{i}{\hbar}\langle[-\gamma\mathbf{J}\cdot\mathbf{B},\mathbf{J}]\rangle = \langle\gamma\mathbf{J}\times\mathbf{B}\rangle = \langle\mu\rangle\times\mathbf{B} \tag{31}$$

3.7 Lorentz Force: $\mathbf{A} = m\mathbf{v} = \mathbf{\Pi} = \mathbf{p} - \frac{q}{c}\mathbf{A}$

In the presence of an electromagnetic field that can be described in terms of vector and scalar potentials $\mathbf{A}(\mathbf{x},t)$ and $\Phi(\mathbf{x},t)$ by

$$\mathbf{B}(\mathbf{x},t) = \nabla\times\mathbf{A}(\mathbf{x},t) \qquad\qquad \mathbf{E}(\mathbf{x},t) = -\nabla\Phi(\mathbf{x},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} \tag{32}$$

the following relation exists between the Newtonian momentum $m\mathbf{v}$ and the canonical momentum $\mathbf{p}$. For a particle of charge q

$$\mathbf{p} = \partial L(\mathbf{x},\dot{\mathbf{x}},t)/\partial\dot{\mathbf{x}} = m\mathbf{v} + \frac{q}{c}\mathbf{A}(\mathbf{x},t) \tag{33}$$

The hamiltonian is $H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{x},t)\right)^2 + q\Phi(\mathbf{x},t)$. The time evolution for the expectation value of the Newtonian momentum is

$$\frac{d}{dt}\langle m\mathbf{v}\rangle = \langle\frac{\partial\Pi}{\partial t}\rangle + \frac{i}{\hbar}\langle\left[\frac{\Pi^2}{2m} + q\Phi, \Pi\right]\rangle \tag{34}$$

where $\Pi = p - \frac{q}{c}\mathbf{A}(\mathbf{x},t) = m\mathbf{v}$. The time derivative of Π is $-\frac{q}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}$. The second simple contribution to the commutator is $[q\Phi,\Pi] = -\frac{\hbar}{i}(q\nabla\Phi)$. The contribution $\left[\Pi^2,\Pi\right]$ is not simple, since the operators Π_j do not commute. Rather, $[\Pi_i,\Pi_j] = -\frac{q}{c}\frac{\hbar}{i}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) = -\frac{q}{c}\frac{\hbar}{i}\epsilon_{ijk}B_k$. The result is

$$\frac{1}{2m}\left[\Pi^2,\Pi\right] = \frac{q}{c}\frac{\hbar}{i}\,\frac{1}{2}(\mathbf{v}\times\mathbf{B} - \mathbf{B}\times\mathbf{v}) \tag{35}$$

where we have used $\Pi/m = \mathbf{v}$. The net result is

$$\frac{d}{dt}\langle m\mathbf{v}\rangle = -\langle\frac{q}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}\rangle - \langle(q\nabla\Phi)\rangle + \langle\frac{q}{c}\,\frac{1}{2}(\mathbf{v}\times\mathbf{B} - \mathbf{B}\times\mathbf{v})\rangle \tag{36}$$

Rearranging the terms slightly, we find

$$\frac{d}{dt}\langle m\mathbf{v}\rangle = q\left(\langle -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}\rangle + \frac{1}{c}\,\langle\frac{1}{2}(\mathbf{v}\times\mathbf{B} - \mathbf{B}\times\mathbf{v})\rangle\right) = q\left(\langle\mathbf{E}\rangle + \frac{1}{c}\,\frac{1}{2}\langle(\mathbf{v}\times\mathbf{B} - \mathbf{B}\times\mathbf{v})\rangle\right) \tag{37}$$

3.8 Hamilton's Equations

For the general hamiltonian $H(q,p,t)$ depending on N coordinates and their conjugate momenta, the equations of motion for the expectation values $\langle q_i\rangle, \langle p_j\rangle$ ($\partial q_i/\partial t = \partial p_j/\partial t = 0$) are as close as can be expected to their classical counterparts:

$$\frac{d\langle q_i\rangle}{dt} = \langle+\frac{\partial H}{\partial p_i}\rangle \qquad\qquad \frac{d\langle p_j\rangle}{dt} = \langle-\frac{\partial H}{\partial q_j}\rangle \tag{38}$$

3.9 The Virial: $A = \mathbf{x}\cdot\mathbf{p}$

The virial operator is $\mathbf{x}\cdot\mathbf{p}$. The time evolution of its expectation value is

$$\frac{d}{dt}\langle\mathbf{x}\cdot\mathbf{p}\rangle = \frac{i}{\hbar}\langle[H,\mathbf{x}\cdot\mathbf{p}]\rangle = \frac{i}{\hbar}\langle[H,\mathbf{x}]\cdot\mathbf{p} + \mathbf{x}\cdot[H,\mathbf{p}]\rangle = \langle\frac{\mathbf{p}}{m}\cdot\mathbf{p} + \mathbf{x}\cdot\mathbf{F}\rangle = \langle 2K + \mathbf{x}\cdot\mathbf{F}\rangle \tag{39}$$

where K is the kinetic energy operator. In a bound state $\langle \mathbf{x} \cdot \mathbf{p} \rangle$ is bounded, so that its time averaged value is zero

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \langle \mathbf{x} \cdot \mathbf{p} \rangle = \lim_{T\to\infty} \frac{\text{finite} - \text{finite}}{T} \to 0 \tag{40}$$

As a consequence, the time averaged value of $\langle 2K + \mathbf{x} \cdot \mathbf{F} \rangle$ also vanishes. This can be expressed as

$$\langle \langle 2K + \mathbf{x} \cdot \mathbf{F} \rangle \rangle_T = 0 \tag{41}$$

Double averages occur in this expression. The inner average is a quantum mechanical average. The outer average is a long time average. For a homogeneous potential $V(\lambda \mathbf{x}) = \lambda^n V(\mathbf{x})$ (for the Coulomb potential $n = -1$), an Euler theorem about homogeneous functions ($\mathbf{x} \cdot \nabla V = nV$) allows us to write

$$\langle \langle 2K - nV \rangle \rangle_T = 0 \tag{42}$$

If the localized state is also stationary (i.e., a bound eigenstate) the inner average is time independent so the outer long time average can be removed, and we find

$$\langle 2K \rangle = n \langle V \rangle \tag{43}$$

For the Coulomb potential ($n = -1$) this result tells us that the mean value of the potential energy is twice that of the kinetic energy and of opposite sign. For the harmonic potential ($n = 2$) the average kinetic energy is equal to the average potential energy (equipartition theorem).

3.10 Isotropic Oscillator and Quadrupole Tensor: $A = \mathbf{Q}_{ij}$

The three-dimensional isotropic harmonic oscillator has its three resonance frequencies in $1:1:1$ resonance and is described by the rotationally invariant hamiltonian $H = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + \frac{1}{2} k \mathbf{x} \cdot \mathbf{x} = \frac{1}{2}\hbar\omega(a_i^\dagger a_i + a_i a_i^\dagger)$.

3.10.1 A Conserved Quantity

In addition to the orbital angular momentum $\mathbf{L}$ there is another set of constants of motion. These are the components of the nine operators

$$\mathbf{Q}_{ij} = \frac{1}{2m} p_i p_j + \frac{k}{2} x_i x_j = \frac{1}{2}\hbar\omega(a_i^\dagger a_j + a_j^\dagger a_i) = \mathbf{Q}_{ji} \qquad \langle \mathbf{Q}_{ij} \rangle = Q_{ij} \tag{44}$$

The trace of $\mathbf{Q}$ is the hamiltonian H. The three sets of operators $H, \mathbf{L}, \mathbf{Q} - \frac{1}{3}\delta_{ij} H$ are irreducible tensor operators of degree $l = 0, 1, 2$ and of dimension $1, 3$, and 5, respectively. These nine operators span the Lie algebra $\mathfrak{u}(3)$ and the eight operators $\mathbf{L}$ and $\mathbf{Q} - \frac{1}{3}\delta_{ij} H$ span $\mathfrak{su}(3)$.

The expectation values of the quadrupole moment operators share many of the same properties as the moment of inertia tensor:

- The nine expectation values $Q_{ij} = \langle \mathbf{Q}_{ij} \rangle$ can be considered as the elements of a three by three matrix.
- The matrix is symmetric.
- The Q_{ij} are constants of motion.

- The real symmetric tensor Q_{ij} depends on the initial conditions. The eigenvalues of Q_{ij} are the principal moments of inertia and the eigenvectors are the principal axes of the averaged motion.

At the classical level it is a simple matter to show that the $\mathbf{Q}_{ij}$ are constants of motion:

$$\frac{d}{dt}\mathbf{Q}_{ij} = \frac{\dot{p}_i p_j}{2m} + \frac{k}{2}\dot{x}_i x_j + \frac{p_i \dot{p}_j}{2m} + \frac{k}{2}x_i \dot{x}_j = \frac{(-kx)_i p_j}{2m} + \frac{k}{2}\frac{p_i}{m}x_j + \frac{p_i(-kx)_j}{2m} + \frac{k}{2}x_i\frac{p_j}{m} = 0 \tag{45}$$

This result extends to the quantum level, for

$$\frac{dQ_{ij}}{dt} = \frac{d}{dt}\langle \mathbf{Q}_{ij}\rangle = \frac{i}{\hbar}\langle [H, \mathbf{Q}_{ij}]\rangle = \frac{i}{\hbar}\langle \frac{1}{2m}\left[p_r p_r, x_i x_j\right] + \frac{k}{2}\left[x_r x_r, p_i p_j\right]\rangle = 0 \tag{46}$$

3.10.2 Symmetry-Breaking

Nonquadratic perturbations to the hamiltonian break the $\mathfrak{u}(3)$ symmetry. As a consequence the principal axes will both rotate and change their magnitude. We consider a quartic perturbation of the form $H_{\text{pert}} = \alpha p^4 + \beta x^4$, where $p^4 = (\mathbf{p}\cdot\mathbf{p})^2$, and similarly for x^4. The expectation value Q_{ij} evolves according to

$$\frac{dQ_{ij}}{dt} = \frac{\alpha k}{2}\langle \{p^2, T_{ij}\}\rangle - \frac{\beta}{2m}\langle \{x^2, T_{ij}\}\rangle \tag{47}$$

where

$$T_{ij} = \frac{i}{\hbar}\left[p^2, x_i x_j\right] = -\frac{i}{\hbar}\left[x^2, p_i p_j\right] = \{x_i, p_j\} + \{x_j, p_i\} \tag{48}$$

and $\{A, B\} = AB + BA$. The relativistic "mass-velocity correction" introduces a correction to the nonrelativistic hamiltonian that is $-p^4/8m^3c^2$ to lowest order. In this case the correction term in H_{pert} has $\alpha = -1/8m^3c^2$ and $\beta = 0$.

3.11 Euler's Equations for Rigid body Motion

The hamiltonian for a rigid body with principle axis moments of inertia I_j is

$$H = \frac{\tilde{L}_1^2}{2I_1} + \frac{\tilde{L}_2^2}{2I_2} + \frac{\tilde{L}_3^2}{2I_3} + V(\mathbf{x}) \tag{49}$$

Here I_j are the moments of inertia along the principal axes of the rotating rigid body. The coordinates $\mathbf{x}$ and the angular momenta $\tilde{L}_j$ are measured with respect to the coordinate axis fixed in the body (rotating coordinate system). These angular momenta commute with the angular momenta in the laboratory frame, L_j (which satisfy $[L_i, L_j] = +i\epsilon_{ijk}L_k$), and satisfy the commutation relations among themselves $\left[\tilde{L}_i, \tilde{L}_j\right] = -i\epsilon_{ijk}\tilde{L}_k$. The two sets of angular momenta $\mathbf{L}$ and $\tilde{\mathbf{L}}$ span the Lie algebra $\mathfrak{so}(4)$ and are infinitesimal generators for the Lie group $SO(4)$. The equations of motion for the expectation values of the components of the angular momentum operator are

$$\begin{aligned}
\frac{d}{dt}\langle\tilde{L}_1\rangle &= \langle\frac{1}{2}\left\{\tilde{L}_2,\tilde{L}_3\right\}\rangle\left(\frac{1}{I_3}-\frac{1}{I_2}\right)+\langle(\mathbf{r}\times(-\nabla V))_1\rangle \\
\frac{d}{dt}\langle\tilde{L}_2\rangle &= \langle\frac{1}{2}\left\{\tilde{L}_3,\tilde{L}_1\right\}\rangle\left(\frac{1}{I_1}-\frac{1}{I_3}\right)+\langle(\mathbf{r}\times(-\nabla V))_2\rangle \\
\frac{d}{dt}\langle\tilde{L}_3\rangle &= \langle\frac{1}{2}\left\{\tilde{L}_1,\tilde{L}_2\right\}\rangle\left(\frac{1}{I_2}-\frac{1}{I_1}\right)+\langle(\mathbf{r}\times(-\nabla V))_3\rangle
\end{aligned} \tag{50}$$

The anticommutator $\{A,B\}=AB+BA$ projects out twice the hermitian part of the product of two hermitian operators A and B. If angular velocity operators $\omega_j=\tilde{\mathbf{L}}_J/I_J$ are introduced, these equations assume the standard form of the Euler rigid body equations of motion.

3.12 Runge-Lenz Vector

Laplace (1799) is usually credited with the discovery that in a gravitational potential the vector from the focus of elliptical motion to the perihelion is invariant. However, this observation goes back at least to Hermann (1710) and J. I. Bernoulli (1710). This vector was subsequently rediscovered by many others (including Hamilton). In the twentieth century it was again rediscovered by Runge (1919) and by Lenz (1924). It was later used by Pauli (1926) to construct a solution to the quantum hydrogen atom problem in the matrix mechanics formulation. This vector is sometimes referred to as the Laplace-Runge-Lenz-Pauli vector. It's name is usually shortened to Runge-Lenz vector.

3.12.1 Classical Description

The hamiltonian that describes a particle in an attractive Coulomb or gravitational potential is $H=\frac{\mathbf{p}\cdot\mathbf{p}}{2m}-K/r$, where $K=e^2$ or GMm and $r=|\mathbf{r}|$ and the central force is $-k\mathbf{r}/r^3$. Since the hamiltonian is rotationally invariant the orbital angular momentum $\mathbf{L}$ is a constant of motion: $d\mathbf{L}/dt=0$. There is another vector operator that is a constant of the motion. It is the Runge-Lenz vector

$$\mathbf{M}=\mathbf{v}\times\mathbf{L}-K\mathbf{r}/r \tag{51}$$

whose time derivative is zero:

$$\frac{d\mathbf{M}}{dt}=\frac{\mathbf{F}}{m}\times\mathbf{L}-K\frac{\dot{\mathbf{r}}(\mathbf{r}\cdot\mathbf{r})-\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}})}{r^3}=0 \tag{52}$$

This vector is orthogonal to $\mathbf{L}$ and it points in the direction of the elliptical orbit's perihelion. Its length is $\mathbf{M}\cdot\mathbf{M}=K^2\epsilon^2$, where $\epsilon=\left(1+\frac{2EL^2}{mK^2}\right)^{1/2}$ is the orbital eccentricity. The orbital trajectory is easily obtained by taking the dot product of $\mathbf{M}$ with $\mathbf{r}$:

$$r(\theta)=\frac{L^2/mK}{1+\epsilon\cos\theta} \tag{53}$$

The angle θ is measured from the perihelion. The period is $T=2\pi\sqrt{m/K}\left(\frac{K}{-2E}\right)^{3/2}$.

3.12.2 Quantum Description

The corresponding quantum mechanical operator must be expressed in terms of the canonical momentum and must be hermitian. The obvious generalization is not hermitian, for $(\mathbf{p}\times\mathbf{L})^\dagger = -\mathbf{L}\times\mathbf{p}$. An appropriate hermitian operator is constructed by taking half the anticommutator of these operators:

$$\mathbf{M} = \frac{\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}}{2m} - K\frac{\mathbf{r}}{r} \longrightarrow \frac{\mathbf{r}(\mathbf{p}\cdot\mathbf{p}) - (\mathbf{r}\cdot\mathbf{p})\mathbf{p} - (\hbar/\mathrm{i})\mathbf{p}}{m} - K\frac{\mathbf{r}}{r} \tag{54}$$

The components of the orbital angular momentum operator $\mathbf{L}$ and those of the Runge-Lenz vector $\mathbf{M}$ satisfy the commutation relations:

$$\begin{aligned} [L_i, L_j] &= i\hbar\epsilon_{ijk}L_k \\ [L_i, M_j] &= i\hbar\epsilon_{ijk}M_k \\ [M_i, M_j] &= \left(-\frac{2H}{m}\right) i\hbar\epsilon_{ijk}L_k \end{aligned} \tag{55}$$

On the manifold of states with fixed energy ($H \to E$) these operators close under commutation. For bound states they span the compact Lie algebra $\mathfrak{so}(4)$ and for scattering states they span the noncompact algebra $\mathfrak{so}(3,1)$. Since $\mathbf{M}$ is (classically) a constant of motion, $[H, \mathbf{M}] = 0$. Since $\mathbf{M}$ carries no explicit time dependence the time derivative of its expectation value is zero:

$$\frac{d}{dt}\langle\mathbf{M}\rangle = \langle\frac{\partial\mathbf{M}}{\partial t}\rangle + \frac{i}{\hbar}\langle[H,\mathbf{M}]\rangle = 0 \tag{56}$$

3.13 Non Coulombic Perturbations

When the potential is not perfectly coulombic the Runge Lenz vector is no longer a constant of motion. If the perturbation from the $1/r$ potential is sufficiently small, the orbits can still usefully be approximated as ellipses. In such cases the eccentricity may vary with time and the perihelion could also vary with time.

3.13.1 Classical Description

In the presence of a spherically symmetric perturbation on an attracting Coulomb potential, the hamiltonian and the force are

$$\begin{aligned} H &= \frac{p^2}{2m} - \frac{K}{r} + V(r) \\ \mathbf{F} &= -K\frac{\mathbf{r}}{r^3} - \frac{dV}{dr}\frac{\mathbf{r}}{r} \end{aligned} \tag{57}$$

We will assume that $\langle V(r)\rangle \ll \langle K/r\rangle$ and $\langle\frac{1}{r}\frac{dV}{dr}\rangle \ll \langle K/r^2\rangle$. The orbital angular momentum $\mathbf{L}$ remains a constant of the motion but the Runge-Lenz vector is no longer a constant of the motion. It has neither constant length nor orientation but remains orthogonal to $\mathbf{L}$. The length is

$$M^2 = \frac{2EL^2}{m} + K^2 - \frac{2L^2}{m}V(r) = K^2\epsilon^2 - \frac{2L^2}{m}V(r) \tag{58}$$

The vector precesses in the direction of the perihelion with a vector angular velocity ω defined by

$$\mathbf{M} \times \dot{\mathbf{M}} = M^2\omega = \frac{1}{m}\left(-\frac{dV}{dr}\right)\left(K - \frac{L^2}{mr}\right)\mathbf{L} \tag{59}$$

As a result,

$$\omega = \frac{\frac{1}{m}\left(-\frac{dV}{dr}\right)\left(K - \frac{L^2}{mr}\right)}{K^2\epsilon^2 - \frac{2L^2}{m}V(r)}\,\mathbf{L} \tag{60}$$

The most important class of perturbations is the attractive $1/r^2$ potential, for which $V(r) = -\frac{\beta}{r^2}$ and $-\nabla V(r) = -\frac{2\beta}{r^3}\frac{\mathbf{r}}{r}$ ($\beta > 0$). Under such a perturbation the orbital trajectory has a form similar to that of the unperturbed trajectory:

$$r(\theta) = \frac{\mu^2 L^2/mK}{1 + \epsilon' \cos(\mu\theta)} \tag{61}$$

where

$$\mu = \left(1 - \frac{2m\beta}{L^2}\right)^{1/2} \qquad \epsilon' = \left(1 + \frac{2\mu^2 L^2 E}{mK^2}\right)^{1/2} \tag{62}$$

The angular advance $\delta\theta$ per period, measured in radians, is determined from

$$\mu(2\pi + \delta\theta) = 2\pi \Rightarrow \delta\theta = 2\pi(\mu^{-1} - 1) \simeq 2\pi\left(\frac{m\beta}{L^2}\right) \tag{63}$$

and the rate of precession is $\delta\theta/T = 2\pi m\beta/L^2T \simeq \beta\sqrt{m}(-2E)^{3/2}/KL^2$.

The relativistic "mass velocity correction" to the hamiltonian has the form $-p^4/8m^3c^2$. This perturbation is conveniently replaced by $-\left(\frac{p^2}{2m}\right)^2/2mc^2$, which is further simplified to $-\frac{1}{2mc^2}\left(E + \frac{K}{r}\right)^2$. When this is opened up there is a slight perturbation to the energy of $\frac{E^2}{2mc^2}$ and to the coupling constant, $K \to K(1 + E/mc^2)$. There is also a $1/r^2$ contribution to the potential, with $\beta = K^2/2mc^2$. This results in a precession of the perihelion that amounts to

$$\Delta\phi_{\text{S.R.}} = \frac{\pi GM}{a(1-\epsilon^2)} \qquad \text{radians/revolution} \tag{64}$$

for a planet with semimajor axis a and eccentricity ϵ in orbit around the sun, of mass M. This advance is due entirely to Special Relativistic effects. The advance in the perihelion due to General Relativistic effects is six times larger:

$$\Delta\phi_{\text{G.R.}} = 6\Delta\phi_{\text{S.R.}} \tag{65}$$

For Mercury in orbit around the sun these advances are small: 7 seconds/century according to Special Relativity and 43 seconds/century according to General Relativity.

3.13.2 Quantum Description

That $\mathbf{M}$ is no longer a constant of motion under the perturbation is shown by

$$\frac{d}{dt}\langle\mathbf{M}\rangle = \langle\frac{i}{\hbar}[H,\mathbf{M}]\rangle = \langle\frac{i}{\hbar}[V(r),\mathbf{M}]\rangle = -\frac{V'}{2mr}\{\mathbf{r}\times\mathbf{L} - \mathbf{L}\times\mathbf{r}\} = \langle -\frac{V'}{mr}\{\mathbf{r}(\mathbf{r}\cdot\mathbf{p}) + \mathbf{r}(\hbar/i) - (\mathbf{r}\cdot\mathbf{r})\mathbf{p}\}\rangle \tag{66}$$

The precession rate can be estimated from

$$\langle M^2\omega\rangle = \frac{1}{2}\langle \mathbf{M}\times\dot{\mathbf{M}} - \dot{\mathbf{M}}\times\mathbf{M}\rangle = \frac{i}{\hbar}\langle \mathbf{M}\times[H,\mathbf{M}] - [H,\mathbf{M}]\times\mathbf{M}\rangle \tag{67}$$

where

$$\mathbf{M} = \frac{\mathbf{r}(\mathbf{p}\cdot\mathbf{p}) - (\mathbf{r}\cdot\mathbf{p})\mathbf{p} - (\hbar/\mathrm{i})\mathbf{p}}{m} - K\frac{\mathbf{r}}{r} \tag{68}$$

where $\frac{i}{\hbar}[H,\mathbf{M}]$ is computed in Eq. (66). The result is:

(69)

4 Multilevel Systems

There are many instances in which it is desirable to provide an intuitive classical representation for the dynamics of a multilevel system. The oldest and best known case is the semiclassical representation of a precessing spin. It's motion is represented by the precession of the spin's Bloch vector. The Bloch vector itself is the expectation value of the three spin operators, themselves proportional to the Pauli spin matrices. The rotation of a "classical" three-vector provides physicists with a concrete representation of the dynamics at both the classical and quantum mechanical levels.

At the quantum level, the dynamics of an n-level system is described by the analogs of the Pauli spin operators. These analogs are $n\times n$ traceless hermitian matrices. There are n^2-1 linearly independent matrices that span this space: $X_i,\ i=1,2,...,n^2-1$. It is useful to choose their properties as follows (A. J. MacFarlane, A. Sudbury and P. H. Weisz, On Gell-Mann's λ-Matrices, d- and f-Tensors, Octets, and Parameterizations of $SU(3)$, Commun. Math. Phys. **11**, 77-90 (1968)):

$$\begin{array}{rcl} (X_i,X_j) &=& \mathrm{Tr}\ X_iX_j \ = \ 2\delta_{ij} \\ X_iX_j &=& \frac{2}{n}I_n\delta_{ij} + (d_{ijk}+if_{ijk})X_k \\ [X_i,X_j] &=& 2if_{ijk}X_k \\ \{X_i,X_j\} &=& 2d_{ijk}X_k + \frac{4}{n}I_n\delta_{ij} \end{array} \tag{70}$$

In the expressions above the third rank tensors f and d are completely antisymmetric and completely symmetric, respectively. For $n=2$ these conditions define the properties of the Pauli spin matrices and $f_{ijk}=\epsilon_{ijk}$, and for $n=3$ these conditions define the properties of the Gell-Mann $SU(3)$ matrices $\lambda_i,\ i=1,2,...,8$.

The quantum dynamics of an n level system can be described by a hamiltonian that is linear in the hermitian infinitesimal generators of the Lie group $U(3)$. If the hamiltonian is taken traceless by an appropriate resetting of the energy, it can be expressed as a linear superposition of the n^2-1 hermitian generators X_i of $SU(n)$. Before doing this, it is useful to define n^2-1 rescaled operators by $S_i=\frac{\hbar}{2}X_i$. The hamiltonian can be written

$$H = -\gamma B_iS_i \tag{71}$$

where the n^2-1 coefficients B_i must be real to preserve hermiticity and may be time dependent.

The dynamics is usefully described in terms of the expectation values $M_i=\langle S_i\rangle$. The time evolution of the expectation values M_i are

$$\frac{d}{dt}M_i = \frac{i}{\hbar}\langle[-B_j S_j, S_i]\rangle = -\gamma f_{ijk} B_j M_k \tag{72}$$

This is a generalization of Bloch dynamics from 2- to n-level systems. The expectation values $M_i = \langle S_i\rangle = \langle \frac{\hbar}{2} X_i\rangle$ obey linear equations of motion that preserves length: $\frac{d}{dt}M_i M_i = 0$.

Losses can be included by introducing damping terms:

$$\frac{d}{dt}M_i = -\gamma f_{ijk} B_j M_k - \frac{1}{T_i}(M_i - M_i(\infty)) \tag{73}$$

Here the dissipation terms are represented as temperatures, one for each degree of freedom. The $n^2 - 1$ terms T_i are not necessarily independent.

5 Systems Described by a Lie Algebra

A number of quantum systems are described by a hamiltonian that is linear in the generators of a Lie group. These include collections of N interacting harmonic oscillators; fields containing a finite number of modes; hamiltonians that are bilinear in the coordinate and momentum operators; systems containing only a finite number of levels of interest. In short, many quantum systems fall into this class, or can be approximated by models in this class.

Define X_i to be the generators of the Lie group ($i = 1, 2, ..., D$, where D is the dimension of the group/algebra), choose the X_i to be hermitian. This can be done if the group is compact. The structure of the Lie algebra is encoded in the structure constants, defined by $[X_i, X_j] = i{c_{ij}}^k X_k$. The structure constants are antisymmetric in the covariant indices i and j. It is not possible to discuss antisymmetrization in all three indices since k is a contravariant index. If it is possible to lower this index, then the symmetry properties of the index triple (ijk) can be investigated.

The contravariant index k can be lowered if it is possible to construct a nonsingular metric tensor $g_{ij} = g_{ji}$. The only data available with which to try to construct a metric tensor are the structure constants. In order to get something with two covariant indices from a tensor with two covariant indices and one controvariant index we use two of the latter type tensors, and perform a double contraction. The object $\sum_r \sum_s {c_{ir}}^r {c_{js}}^s$ loses information, as it depends on only D quantities $v_i = \sum_r {c_{ir}}^r$. A cross-contraction does the job: $g_{ij} = \sum_r \sum_s {c_{ir}}^s {c_{js}}^r$. If this is nonsingular, so that an inverse g^{ij} can also be computed, then it is possible to raise and lower indices without restraint. In particular, it is possible to show that $c_{ijk} = \sum_r {c_{ij}}^r g_{rk}$ is totally antisymmetric: $c_{ijk} = c_{jki} = c_{kij} = -c_{kji} = -c_{jik} = -c_{ikj}$. It is also possible to show that the scalar operator $\sum_i \sum_j g^{ij} X_i X_j$ commutes with all operators X_k. Such a scalar is called a Casimir invariant.

Write the hamiltonian as $H = \hbar\Omega_i X_i$ and introduce the expectation values $M_i = \langle X_i\rangle$ as the appropriate order parameters of the system. The dynamics of the order parameters are determined as usual

$$\frac{d}{dt}M_j = \frac{i}{\hbar}\langle[\hbar\Omega_i X_i, X_j]\rangle = -c_{ijk}\Omega_i M_k = +c_{jik}\Omega_i M_k = \mathbf{\Omega}\times\mathbf{M} \tag{74}$$

The cross product above, "$\times$", is a simple way to write the result in terms of the structure constants.

This result can be represented elegantly in terms of the *regular representation* of the Lie algebra, defined by

$$[Y, X_i] = Reg(Y)_i{}^j X_j \Rightarrow \dot{M} = Reg(\Omega)M \tag{75}$$

The $D \times D$ matrix (D is the dimension of the Lie algebra) $Reg(*)$ is antisymmetric. Loss terms can be put in by hand as before. The take the form of a diagonal matrix multiplying terms $(M_i - M_i(\infty))$, where $M_i(\infty)$ are the long term equilibrium values, usually externally imposed.

6 Systems Described by Two Lie Algebras

It is often useful to model two interacting quantum systems A and B by operators in two Lie algebras. A very useful example is often used in laser physics, where one quantum system consists of N two-level atoms, each described by spin operators $\sigma_z^{(i)}, \sigma_\pm^{(i)}$, and the other quantum system consists of a single mode of the electromagnetic field, the mode which is amplified to maser/laser action. This field is represented by the number operator $a^\dagger a$ and the creation and annihilation operators $a^\dagger$ and a.

In the general case we choose operators X_i, $i = 1, 2, \cdots, D_A$ to model the properties of system A and Y_α, $\alpha, 1, 2 \cdots, D_B$ to model properties of system B. These operators have commutation relations

$$[X_i, X_j] = ic_{ijk}^{(A)} X_k \qquad [Y_\alpha, Y_\beta] = ic_{\alpha\beta\gamma}^{(B)} Y_\gamma \qquad [X_i, Y_\beta] = 0 \tag{76}$$

We take the hamiltonian as linear in the operators of the two subsystems A and B (usually just a superposition of diagonal operators). The interaction terms is taken as simple as possible: bilinear - linear in the operators of each of the systems:

$$H = A_i X_i + B_\alpha Y_\alpha + M_{j\beta} X_j Y_\beta \tag{77}$$

The Ehrenfest equations of motion derived from this hamiltonian are:

$$\begin{aligned} \frac{d}{dt}\langle X_j \rangle &= -\frac{1}{\hbar} c_{ijk}^{(A)} \langle A_i X_k + M_{i\alpha} X_k Y_\alpha \rangle \\ \frac{d}{dt}\langle Y_\beta \rangle &= -\frac{1}{\hbar} c_{\alpha\beta\gamma}^{(B)} \langle B_\alpha Y_\gamma + M_{i\alpha} X_i Y_\gamma \rangle \end{aligned} \tag{78}$$

These equations cannot be solved without knowing about the expectation values of the bilinear terms $\langle X_i Y_\beta \rangle$. An Ehrenfest result for these terms can be evaluated. It contains terms of the form $\langle X_i X_j Y_\alpha \rangle$ and $\langle X_i Y_\alpha Y_\beta$. The equations of motion for these terms can be sought. They involve expectation values of products of four operators. Then $4 \to 5$ and $5 \to 6$, and so on up the ladder. This insanity gives an infinite set of equations that are not closed.

Alternatively, we can cheat and assume as some point that the products of $n_1 + n_2$ operators can be expressed as products of n_1 operators multiplied by products of n_2 operators: for example, products involving n_1 Xs and n_2 Ys factor into two products, one involving only the X variables, the other involving only the Y variables. This truncates the hierarchy of dynamical equations.

For example, if we assume this factorization can be effected in Eq. (78) above, then we find the pair of nonlinear coupled evolution equations:

$$\begin{aligned} \frac{d}{dt}\langle X_j \rangle &= -\frac{1}{\hbar} c_{ijk}^{(A)} \left(A_i \langle X_k \rangle + M_{i\alpha} \langle X_k \rangle \langle Y_\alpha \rangle \right) \\ \frac{d}{dt}\langle Y_\beta \rangle &= -\frac{1}{\hbar} c_{\alpha\beta\gamma}^{(B)} \left(B_\alpha \langle Y_\gamma \rangle + M_{i\alpha} \langle X_i \rangle \langle Y_\gamma \rangle \right) \end{aligned} \tag{79}$$

While these truncated equations are a set of $D_A + D_B$ coupled ordinary nonlinear dynamical equations to represent the evolution of the interacting systems A and B and are not easy to solve, they are a lot simpler than the next level in this hierarchy of rapidly increasing dimensional dynamical systems. These equations alone exhibit fascinating evolution possibilities, as witnessed, for example, by the complexity of the Lorenz equations, which have only two bilinear coupling terms.

7 Conclusion