Stat mech II HW7

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December 3, 2015

1 Problem 7.6

For $T > T_c$ we will take $\frac{\partial}{\partial \ln z}\frac{\partial \ln z}{\partial T}$. We drop the argument z of the Bose integrals for compactness.

$$\frac{C_V}{Nk} = \frac{15}{4}\frac{g_{5/2}}{g_{3/2}} - \frac{9}{4}\frac{g_{3/2}}{g_{1/2}} \tag{1.1}$$

$$\frac{1}{Nk}\frac{\partial C_v}{\partial \ln z} = \frac{15}{4}\left(g_{3/2}\frac{1}{g_{3/2}} - g_{5/2}\frac{g_{1/2}}{g_{3/2}^2}\right) - \frac{9}{4}\left(g_{1/2}\frac{1}{g_{1/2}} - g_{3/2}\frac{g_{-1/2}}{g_{1/2}^2}\right) \tag{1.2}$$

$$\frac{1}{Nk}\frac{\partial C_v}{\partial \ln z} = \frac{3}{2} - \frac{15}{4}\frac{g_{5/2}g_{1/2}}{g_{3/2}^2} + \frac{9}{4}\frac{g_{3/2}g_{-1/2}}{g_{1/2}^2} \tag{1.3}$$

Using the relation $\frac{1}{z}\left(\frac{\partial z}{\partial T}\right) = \frac{\partial \ln z}{\partial T} = \frac{3}{2T}\frac{g_{3/2}}{g_{1/2}}$ we have:

$$\frac{1}{Nk}\left(\frac{\partial C_V}{\partial T}\right) = \frac{1}{T}\left(\frac{45}{8}\frac{g_{5/2}}{g_{3/2}} - \frac{9}{4}\frac{g_{3/2}}{g_{1/2}} - \frac{27}{8}\frac{g_{3/2}^2 g_{-1/2}}{g_{1/2}^3}\right) \tag{1.4}$$

For $T < T_c$ we use the relation:

$$\frac{C_V}{Nk} = \frac{15}{4}\zeta\left(\frac{5}{2}\right)\frac{v}{\lambda^3} \tag{1.5}$$

With $\frac{1}{\lambda} \propto T^{3/2}$ we can take the derivative with respect to T directly. Adding a factor of $\frac{1}{T}$ we can keep the expression in terms of λ:

$$\frac{C_V}{Nk} = \frac{45}{8}\zeta\left(\frac{5}{2}\right)\frac{v}{T\lambda^3} \tag{1.6}$$

We have thus proved the required results.
We now examine the discontinuity at $T = T_c$. We take the difference of our two expressions as $T \to T_c$ from above and below. Using D.9 to make the approximation $(g_v(z) = \zeta(z)$

2 Problem 7.10

We start from the expression for particle density in a Bose system, and the Hamiltonian for an ideal gas in a uniform gravitational field:

$$N = \sum_e \frac{1}{z^{-1}e^{\beta e} - 1} \tag{2.1}$$

$$H = \frac{p^2}{2m} + mgz \tag{2.2}$$

To convert the sum to an integral we need to calculate the density of energy states $a(e)\ de$.

3 Problem 7.13

The number of particles with energy e in a Bose gas is:

$$\sum_e \frac{1}{z^{-1}e^{\beta e} - 1} \tag{3.1}$$

We convert this to an integral in the usual manner, except the gas is now confined to a 2D space. So the space integral evaluates to A rather than V, and we divide by h^2 for the per-particle area. Our 2D expression for the density of energy states is therefore:

$$a(e)\ de = \frac{2\pi m A}{h^2} e\ de \tag{3.2}$$

And our integral expression is:

$$\frac{N}{A} = \frac{2\pi m}{h^2} \int_0^\infty \frac{e\ de}{z^{-1}e^{\beta e} - 1} + \frac{1}{A}\frac{z}{1-z} \tag{3.3}$$

Where we have removed the singularity at $e = 0$ from the integral. The term on the left is the number of excited particles N_e while the right hand term is the number of particles in the ground state N_0. The solution to the integral is $g_1(z)$.
The critical temperature is defined by the number of particles that can be accomodated in the excited states in the limit as $z \to 1$, where $g_1(z) \to \zeta(1)$. Although $\zeta(1)$ is an undetermined form, the limits from the left and right are $\pm\infty$. This implies that the system can accomodate any number of paritcles in the excited states and shows that the system will not undergo Bose-Einstein condensation for any finite temperature.