PHYS 502: Mathematical Physics II

Winter 2015, Homework #6

(Due March 17, 2015)

1. Repeat the brachistchrone problem discussed in class, but assume that the particle has initial speed v_0. That is, a particle is launched from a point $A = (0, 0)$ with speed v_0, and is constrained to move along a frictionless wire connecting A to a point $B = (x, y)$ lying a vertical distance y below A. The shape of the wire $y(x)$ is to be chosen to minimize the travel time
$$t = \int_A^B \frac{ds}{v},$$
where $v^2 = v_0^2 + 2gy$. Write down and solve the Euler-Lagrange equation for the motion, and show that the solution $y(x)$ coincides with the solution to the classical problem (zero initial speed) that starts at $y = -v_0^2/2g$ and passes through A.

2. The distance ds between two points (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ on the surface of a sphere of radius R is given by
$$ds^2 = R^2(d\theta^2 + \cos^2\theta\, d\phi^2)\,,$$
where θ is "colatitude," $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, with $\theta = 0$ on the equator, and ϕ is "longitude," $0 \le \phi < 2\pi$. A *geodesic* is defined as the shortest distance ($\int ds$) between any two given points on the surface.

   (a) Solve the Euler-Lagrange equations to show that a geodesic $\phi(\theta)$ through the point $\theta = \phi = 0$ satisfies
$$\tan\phi = \frac{\cos\theta_m \sin\theta}{\sqrt{\sin^2\theta_m - \sin^2\theta}}\,,$$
for any choice of θ_m.

   (b) By considering a general point (θ, ϕ) on the geodesic and the two particular points $(0, 0)$ and $(\theta_m, \frac{\pi}{2})$, show that the above equation describes a *great circle*—that is, the intersection of the surface of the sphere with a plane through the sphere's center. [Recall that the condition for three (3-D) vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to be coplanar is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.]

3. Fermat's principle of least time states that a ray of light between two given points travels along the path which it can traverse in the shortest time.

   (a) Consider a region in which the index of refraction varies linearly with height: $n = n_0(1 + \alpha z)$. Determine the light path between points P and P' at the same height $z = 0$, separated by some fixed horizontal distance d.

   (b) If $\alpha d \ll 1$, compute, to first order in αd, the angle θ to the horizontal at which light leaving P arrives at P'.

4. A *fixed* volume of incompressible fluid rotates at constant angular speed Ω inside a cylinder of radius R. The cylinder's axis is vertical. Use a variational method to find the shape of the water surface, by requiring that it minimize the total effective potential energy (i.e. including both gravitational and centrifugal terms) of the water in the frame rotating with angular speed ω.

5. (a) Use a variational method to estimate the lowest frequency mode of oscillation of a semi-circular drum head of radius R. Take the wave speed to be c.

For the Helmholtz equation

$$\nabla^2 u + k^2 u = 0,$$

the lowest eigenmode k_{min} satisfies

$$k_{min}^2 \;\leq\; K[u] \;\equiv\; \frac{\int (\nabla u)^2 d^2x}{\int u^2 d^2x}$$

for *any* choice of the function u. Use trial functions of the form

$$u(r,\theta) = r\,\left[1 - (r/R)^n\right]\,\cos\theta\,,$$

where $n > 0$ and $\theta = 0$ is the axis of symmetry, and vary n to minimize your frequency estimate.

(b) Compare your result to the analytical solution for the lowest frequency.