# Math Phys II HW 3

Vince Baker

February 9, 2015

**Abstract**

## 1 Problem 1

We investigate solutions of Laplace' equation in two dimensions.

$$\nabla^2 U = 0 \tag{1.1}$$
$$U(\rho,\phi) = R(\rho)\Phi(\phi) \tag{1.2}$$

Separating variables and dividing through by $R\Phi$:

$$\frac{1}{\rho}\frac{d}{d\rho}(rR^{'})\Phi + \frac{1}{\rho^2}\Phi^{''}R = 0 \tag{1.3}$$
$$\frac{\rho}{R}(rR^{'})^{'} + \frac{\Phi^{''}}{\Phi} = 0 \tag{1.4}$$

We use $\Phi(\phi) = e^{im\phi}$ as the angular solution, then:

$$\frac{\rho}{R}(rR^{'})^{'} = m^2 \tag{1.5}$$
$$\rho^2 R^{''} + \rho R^{'} - m^2 R = 0 \tag{1.6}$$

We write down the series solution to this problem around the regular singular point 0.

$$R = \sum c_i \rho^{\alpha+i} \tag{1.7}$$
$$R^{'} = \sum (\alpha+i)c_i\rho^{\alpha+i-1} \tag{1.8}$$
$$R^{''} = \sum (\alpha+i)(\alpha+i-1)c_i\rho^{\alpha+i-2} \tag{1.9}$$
$$\sum(\alpha+i)(\alpha+i-1)c_i\rho^{\alpha+i} + \sum(\alpha+i)c_i\rho^{\alpha+i} - m^2\sum c_i\rho^{\alpha+i} = 0 \tag{1.10}$$

Collecting the coefficients of the 0th term and equating them to 0:

$$[\alpha(\alpha-1) + \alpha - m^2] = 0 \tag{1.11}$$
$$\alpha = \pm m \tag{1.12}$$
$$R = \rho^{\pm m} \tag{1.13}$$

We can now write the general solution.

$$U(\rho, \phi) = \sum (a_m \rho^m + b_m \rho^{-m}) e^{im\phi} \tag{1.14}$$

With boundary condition $U(a, \phi) = U_0 cos^2\phi$, we now look for the coefficients. We write the boundary condition as $\frac{U_0}{2} + \frac{U_0 cos2\phi}{2}$ and expand the general $e^{im\phi}$ term into cosines and sines.

$$U(a, \phi) = \sum (a_m a^m + b_m a^{-m})(A_m cos\ m\phi + B_m sin\ m\phi) \tag{1.15}$$

It is clear from the form of the boundary condition that only the m=0 and m=2 coefficients will survive. Collecting the various coefficients:

$$U(a, \phi) = A_0 + A_2 cos\ 2\phi = \frac{U_0}{2} + \frac{U_0 cos2\phi}{2} \tag{1.16}$$

$$A_0 = A_2 = \frac{U_0}{2} \tag{1.17}$$

So the general solution is:

$$U(\rho, \phi) = \frac{U_0}{2} + \frac{U_0}{2}\rho^2 cos\ 2\phi \tag{1.18}$$

# 2 Problem 2

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{2.1}$$

We recognize the time part of the solution as $e^{i\omega t}$. The sptial solutions of Equation 2.1 are then the solutions to a Helmholtz equation in three dimensions.

$$u(r, \theta, \phi) = \sum j_\ell(kr) P_\ell^m(cos\theta) e^{im\phi} \tag{2.2}$$

The surface boundary conditions $\frac{\partial u}{\partial r} = 0,\ r = R_0$, require that $\frac{d}{dr} j_\ell(cR_0) = 0$. We calculate $\frac{d}{dr} j_\ell(cR_0)$ for $\ell = 0, 1, 2$.

$$j_0 = \frac{sinx}{x}, j_0^{'} = -\frac{-sinx}{x^2} + \frac{cosx}{x} = 0 \Rightarrow x = 4.49 \tag{2.3}$$

$$j_1 = \frac{sinx}{x^2} - \frac{cosx}{x}, j_1^{'} = x^2 sinx + 2xcosx - 2sinx = 0 \Rightarrow x = 2.08 \tag{2.4}$$

$$j_2 = \frac{3sinx}{x^3} - \frac{sinx}{x} - \frac{3cosx}{x^2}, j_2^{'} = -x^3 cosx + 4x^2 sinx + 9xcosx - 9sinx = 0 \Rightarrow x = 3.34 \tag{2.5}$$

We can now solve for k and find $\omega = kc$.

$$\omega_{10} = \frac{4.49}{R} c \tag{2.6}$$

$$\omega_{11} = \frac{2.08}{R} c \tag{2.7}$$

$$\omega_{12} = \frac{3.34}{R} c \tag{2.8}$$

# 3 Problem 3

$$\nabla^2 n + \lambda n = \frac{1}{\kappa}\frac{\partial n}{\partial t} \tag{3.1}$$

Separating the time and space parts, we find:

$$T^{'} - \kappa c T = 0 \tag{3.2}$$

With the time solution of the form $e^{\alpha t}$, then $\alpha = \kappa c$. We separate the space part:

$$\nabla^2\chi + k^2\chi = 0,\ k^2 \equiv \lambda - c \tag{3.3}$$

For spherically symmetric modes $\ell = m = 0$, and the solution is:

$$\chi(r) = j_0(kr) \tag{3.4}$$

Applying the boundary condition $n(R) = 0$:

$$j_0(kR) = \frac{sinx}{x} = 0 \tag{3.5}$$

The first zero will be at $\pi$, so we find $R_0 = \frac{\pi}{k}$. At the critical point $\alpha = 0, k^2 = \lambda$. So the critical radius $R_0 = \frac{\pi}{\sqrt{\lambda}}$.

Repeating for a hemisphere we apply axial symmetry (m=0) and find:

$$\chi(r,\theta) = j_\ell(kr)P_\ell^0(cos\theta) \tag{3.6}$$

We now have a boundary on the upper surface (r=R) and on the bottom of the hemisphere ($\theta = \frac{\pi}{2}$). For the bottom surface boundary, only $P_1^0 = cos\theta$ satisfies the the boundary condition so $\ell = 1$. We can now apply the radial boundary condition:

$$j_1(kR) = \frac{sinx}{x^2} - \frac{cosx}{x} = 0 \Rightarrow x = 4.49 \tag{3.7}$$

$$R_0 = \frac{4.49}{\sqrt{\lambda}} \tag{3.8}$$

We can now combine the two critical hemispheres into a single sphere of radius $\frac{4.49}{\lambda}$. The critical radius of the sphere is only $\frac{\pi}{\lambda}$, so the sphere is now unstable. The critical radius for the sphere is $\frac{\pi}{\sqrt{\lambda}}$, so we can express the new radius as $1.43R_0$. Using 3.4, we write:

$$k^2 = \lambda - \frac{\alpha}{\kappa} = \tag{3.9}$$

$$k_0 = \frac{\pi}{R_0},\ k_1 = \frac{\pi}{1.43R_0} = \frac{\sqrt{\lambda}}{1.43} \tag{3.10}$$

$$k_1^2 = \frac{\lambda}{1.43^2} = \lambda - \frac{\alpha}{\kappa} \tag{3.11}$$

$$\alpha = \kappa\lambda(0.511) \tag{3.12}$$

$$\tau = \frac{1}{\alpha} = \frac{1.957}{\kappa\lambda} \tag{3.13}$$

# 4 Problem 4

The general solution to the homogenous 2D Helholtz equation is:

$$u(r,\phi) = \sum_m a_m J_m(kr) e^{im\phi} \tag{4.1}$$

Applying the boundary condition, we get:

$$u(R,\phi) = \sum_m a_m J_m(kR) e^{im\phi} = f(\phi) \tag{4.2}$$

we recognize this as the Fourier expansion of $f(\phi)$, with the Fourier coeeficients $c_m = a_m J_m(kR)$. We can now combine the coeeficients into a general coefficient $B_m$ and solve for $B_m$.

$$B_m = \frac{1}{2\pi} \int_0^{2\pi} f(\phi^{'}) e^{im\phi^{'}} d\phi^{'} \tag{4.3}$$

$$a_m = \frac{1}{J_m(kR)} \frac{1}{2\pi} \int_0^{2\pi} f(\phi^{'}) e^{im\phi^{'}} d\phi^{'} \tag{4.4}$$

Substituting the $a_m$ back into 4.2 and rearranging:

$$u(r,\phi,\phi^{'}) = \frac{1}{J_m(kR)} \frac{1}{2\pi} \int_0^{2\pi} \{f(\phi^{'}) e^{im\phi^{'}} d\phi^{'}\} J_m(kr) e^{im\phi} \tag{4.5}$$

$$= \int_0^{2\pi} \sum_m \frac{J_m(kr)}{J_m(kR)} \frac{1}{2\pi} e^{im\phi} f(\phi^{'}) e^{im\phi^{'}} d\phi^{'} \tag{4.6}$$

$$= \int_0^{2\pi} K(r,\phi,\phi^{'} f(\phi^{'}) d\phi^{'} \tag{4.7}$$

$$K = \sum_m \frac{J_m(kr)}{J_m(kR)} \frac{1}{2\pi} e^{im\phi} e^{im\phi^{'}} \tag{4.8}$$

To solve for $f(\phi) = cos^2\phi$ we substitute into 4.2.

$$u(R,\phi) = \sum_m a_m J_m(kR) e^{im\phi} = cos^2\phi \tag{4.9}$$

Using the same approach as problem 1, we find that only the m=0 and m=2 coefficients survive and are equal to $\frac{1}{2J_0(kR)}$ and $\frac{1}{2J_2(kR)}$.

$$u(r,\phi) = \frac{1}{2J_0(kR)} J_0(kr) + \frac{1}{2J_2(kR)} J_2(kr) cos\ 2\phi \tag{4.10}$$