

# Statmech II HW5

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## 1 Problem 8.1

## 2 Problem 8.2

We find a general expression for the temperature:

$$\frac{1}{v} = \frac{g}{\lambda^3} f_{3/2}(z) \quad (2.1)$$

$$\lambda = \frac{h}{\sqrt{2\pi mkT}} \quad (2.2)$$

$$T = \frac{h^2}{2\pi mk} (gv f_{3/2}(z))^{-2/3} \quad (2.3)$$

At  $T_0$ ,  $z = 1$ , and we find (from E.14 in Pathria):

$$f_{3/2}(1) = (1 - \frac{1}{\sqrt{2}}) \zeta(\frac{3}{2}) = 0.7650 \quad (2.4)$$

We set  $t_f = \frac{e_f}{k}$  and use Pathria equation 8.1.24 for  $e_f$ .

$$T_f = \left( \frac{3}{4\pi gv} \right)^{2/3} \frac{h^2}{2mk} \quad (2.5)$$

$$\frac{T_f}{T_0} = \frac{1}{\pi} \left( \frac{4\pi}{3f_{3/2}(1)} \right)^{2/3} = 0.9889 \quad (2.6)$$

So the temperature at which  $\mu = 0$  is approximately the Fermi temperature.

### 3 Problem 8.3

Starting from the expression for pressure and using the recurrence relation for the Fermi integrals:

$$P = kT \frac{g}{\lambda^3} f_{5/2}(z) \quad (3.1)$$

$$P = kg \left( \frac{h}{\sqrt{2\pi mk}} \right)^3 T^{5/2} f_{5/2}(z) \quad (3.2)$$

$$\left( \frac{\partial P}{\partial T} \right)_P = kg \left( \frac{h}{\sqrt{2\pi mk}} \right)^3 \frac{5}{2} T^{\frac{3}{2}} f_{5/2}(z) + kg \left( \frac{h}{\sqrt{2\pi mk}} \right)^3 T^{5/2} \frac{1}{z} f_{3/2}(z) \left( \frac{\partial z}{\partial T} \right)_P \quad (3.3)$$

$$\frac{1}{z} \left( \frac{\partial z}{\partial T} \right)_P = - \frac{5}{2T} \frac{f_{5/2}(z)}{f_{3/2}(z)} \quad (3.4)$$

From equation 8.1.9 in Pathria we can prove the desired relation  $\gamma$ :

$$\frac{1}{z} \left( \frac{\partial z}{\partial T} \right)_v = - \frac{3}{2T} \frac{f_{3/2}(z)}{f_{1/2}(z)} \quad (3.5)$$

$$\gamma = \frac{C_P}{C_v} = \frac{5}{3} \frac{f_{5/2}(z) f_{1/2}(z)}{(f_{3/2}(z))^2} \quad (3.6)$$

Using eqns 8.1.30-8.1.32 of Pathria:

$$\gamma = \frac{5}{3} \frac{\frac{8}{15\sqrt{\pi}} (\ln z)^{5/2} (1 + \frac{5\pi^2}{8} (\ln z)^{-2} + \dots) \frac{2}{\sqrt{\pi}} (\ln z)^{1/2} (1 - \frac{\pi^2}{24} (\ln z)^{-2} + \dots)}{\left( \frac{4}{3\sqrt{\pi}} (1 + \frac{\pi^2}{8} (\ln z)^{-2} + \dots) \right)^2} \quad (3.7)$$

Keeping up to second-order terms:

$$\gamma = \frac{3}{5} \frac{5}{3} \frac{(1 + \frac{5\pi^2}{8} \ln z^{-2} - \frac{\pi^2}{24} \ln z^{-2} + \dots)}{(1 + 2\frac{\pi^2}{8} \ln z^{-2} + \dots)} \quad (3.8)$$

$$\gamma \simeq 1 + \frac{\pi^2}{3} (\ln z)^{-2} \quad (3.9)$$

Substituting  $\ln z = \frac{e_f}{kT}$  we prove  $\gamma \simeq 1 + \frac{\pi^2}{3} \left( \frac{kT}{e_f} \right)^2$ .

## 4 8.4

a) We prove the following relations for the isothermal compressibility and adiabatic compressibility of an ideal Fermi gas:

$$\kappa_T = \frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \quad (4.1)$$

$$\kappa_S = \frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_S \quad (4.2)$$

Using the expressions for P and n and taking the derivatives with T held constant and with z held constant:

$$P = kT \frac{g}{\lambda^3} f_{5/2}(z) \quad (4.3)$$

$$\frac{\partial P}{\partial z} = kT \frac{g}{\lambda^3} \frac{1}{z} f_{3/2}(z) \quad (4.4)$$

$$\frac{\partial P}{\partial T} = \frac{5}{2} \frac{(2\pi m)^{3/2}}{gh^3} k^{5/2} T^{3/2} f_{5/2}(z) \quad (4.5)$$

$$n = \frac{g}{\lambda^3} f_{3/2}(z) \quad (4.6)$$

$$\frac{\partial n}{\partial z} = \frac{g}{\lambda^3} \frac{1}{z} f_{1/2}(z) \quad (4.7)$$

$$\frac{\partial n}{\partial T} = \frac{3}{2} \frac{(2\pi mk)^{3/2}}{gh^3} T^{1/2} f_{3/2}(z) \quad (4.8)$$

Writing the compressibility expressions as functions of n and using the appropriate derivatives, we show:

$$\kappa_T = \frac{N}{V} \left( \frac{\partial(\frac{N}{V})}{\partial P} \right)_T = \frac{1}{n} \left( \frac{\partial n}{\partial P} \right)_T \quad (4.9)$$

$$\kappa_T = \frac{1}{nkT} \frac{f_{1/2}(z)}{f_{3/2}(z)} \quad (4.10)$$

$$\kappa_S = \frac{N}{V} \left( \frac{\partial(\frac{N}{V})}{\partial P} \right)_S = \frac{1}{n} \left( \frac{\partial n}{\partial P} \right)_S \quad (4.11)$$

$$\kappa_S = \frac{3}{5nkT} \frac{f_{3/2}(z)}{f_{5/2}(z)} \quad (4.12)$$

Using the expansions of the Fermi integrals in powers of  $\ln z$  up to second

order (eqns 8.1.30-8.1.32 of Pathria):

$$\kappa_T \simeq \frac{1}{nkT} \frac{3}{2 \ln z} (1 - \frac{\pi^2}{24} (\ln z)^{-2} + \dots) / (1 + \frac{\pi^2}{8} (\ln z)^{-2} + \dots) \quad (4.13)$$

$$\kappa_T \simeq \frac{1}{nkT} \frac{3}{2 \ln z} (1 - \pi^2 (\frac{1}{24} + \frac{1}{8}) (\ln z)^{-2} + \dots) \quad (4.14)$$

$$\kappa_T \simeq \frac{3}{2nkT \ln z} (1 - \frac{\pi^2}{6} (\ln z)^{-2} + \dots) \quad (4.15)$$

We substitute Pathria 8.1.35 for  $\ln z$  and keep only the first-order part in the exponential term. Working up to second order:

$$\kappa_T \simeq \frac{3}{2nkT \frac{e_f}{kT} (1 - \frac{\pi^2}{12} (\frac{kT}{e_f})^2)} (1 - \frac{\pi^2}{6} (\frac{kT}{e_f})^2) \quad (4.16)$$

$$\kappa_T \simeq \frac{3}{2ne_f} (1 - \frac{\pi^2}{6} (\frac{kT}{e_f})^2) (1 + \frac{\pi^2}{12} (\frac{kT}{e_f})^2) \quad (4.17)$$

$$\kappa_T \simeq \frac{3}{2ne_f} \left( 1 - \frac{\pi^2}{12} (\frac{kT}{e_f})^2 \right) \quad (4.18)$$

We now follow the same procedure to calculate  $\kappa_S$ .

$$\kappa_S \simeq \frac{3}{5nkT} \frac{5}{2 \ln z} (1 + \frac{\pi^2}{8} (\ln z)^{-2} + \dots) / (1 + \frac{5\pi^2}{8} (\ln z)^{-2} + \dots) \quad (4.19)$$

$$\kappa_S \simeq \frac{3}{5nkT} \frac{5}{2 \ln z} (1 + \pi^2 (\frac{1}{8} - \frac{5}{8}) (\ln z)^{-2} + \dots) \quad (4.20)$$

$$\kappa_S \simeq \frac{3}{2nkT \ln z} (1 - \frac{\pi^2}{2} (\ln z)^{-2} + \dots) \quad (4.21)$$

We substitute Pathria 8.1.35 for  $\ln z$  and keep only the first-order part in the exponential term. Working up to second order:

$$\kappa_S \simeq \frac{3}{2nkT \frac{e_f}{kT} (1 - \frac{\pi^2}{12} (\frac{kT}{e_f})^2)} (1 - \frac{\pi^2}{2} (\frac{kT}{e_f})^2) \quad (4.22)$$

$$\kappa_S \simeq \frac{3}{2ne_f} (1 - \frac{\pi^2}{2} (\frac{kT}{e_f})^2) (1 + \frac{\pi^2}{12} (\frac{kT}{e_f})^2) \quad (4.23)$$

$$\kappa_S \simeq \frac{3}{2ne_f} \left( 1 - \frac{5\pi^2}{12} (\frac{kT}{e_f})^2 \right) \quad (4.24)$$

b) Starting with the given relation for  $C_P - C_V$ :

$$C_P - C_V = TV\kappa_T \left( \frac{\partial P}{\partial T} \right)_V^2 \quad (4.25)$$

$$PV = \frac{2}{3}U \quad (4.26)$$

$$\left( \frac{\partial P}{\partial T} \right)_V = \frac{2}{3V} \left( \frac{\partial U}{\partial T} \right)_V = \frac{2}{3V} C_V \quad (4.27)$$

$$\frac{C_P - C_V}{C_V} = \frac{4}{9} \frac{1}{V} C_V \frac{1}{nk} \frac{f_{1/2}(z)}{f_{3/2}(z)} = \frac{4}{9} \frac{C_V}{Nk} \frac{f_{1/2}(z)}{f_{3/2}(z)} \quad (4.28)$$

Using  $C_V = \frac{\pi^2}{2} \frac{kT}{e_f}$  and again substituting the  $\ln z$  series expansion:

$$\frac{f_{1/2}}{f_{3/2}} = \frac{3}{2 \ln z} \left( 1 - \frac{\pi^2}{24} (\ln z)^{-2} \right) \left( 1 - \frac{\pi^2}{8} (\ln z)^{-2} \right) \quad (4.29)$$

$$\frac{f_{1/2}}{f_{3/2}} = \frac{3}{2} \frac{kT}{e_f} \left( 1 - \frac{\pi^2}{6} \left( \frac{kT}{e_f} \right)^2 \right) \quad (4.30)$$

$$\frac{C_P - C_V}{C_V} = \frac{4}{9} \frac{\pi^2}{2} \frac{kT}{e_f} \frac{3}{2} \frac{kT}{e_f} \left( 1 - \frac{\pi^2}{6} \left( \frac{kT}{e_f} \right)^2 \right) \quad (4.31)$$

$$\frac{C_P - C_V}{C_V} \simeq \frac{\pi^2}{3} \left( \frac{kT}{e_f} \right)^2 \quad (4.32)$$

c) Using the relation  $\gamma = \frac{\kappa_T}{\kappa_S}$ , keeping terms to second order, we find:

$$\gamma \simeq \left( 1 - \frac{\pi^2}{12} \left( \frac{kT}{e_f} \right)^2 \right) / \left( 1 - \frac{5\pi^2}{12} \left( \frac{kT}{e_f} \right)^2 \right) \quad (4.33)$$

$$\gamma \simeq \left( 1 - \frac{\pi^2}{12} \left( \frac{kT}{e_f} \right)^2 \right) \left( 1 + \frac{5\pi^2}{12} \left( \frac{kT}{e_f} \right)^2 \right) \quad (4.34)$$

$$\gamma \simeq 1 - \frac{\pi^2}{3} \left( \frac{kT}{e_f} \right)^2 \quad (4.35)$$

So we have verified the result in part A for  $\gamma$  at low temperatures.