BOSE-EINSTEIN CONDENSATION IN THE PRESENCE OF A UNIFORM FIELD AND A POINT-LIKE IMPURITY

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Abstract

The behavior of an ideal D-dimensional boson gas in the presence of a uniform gravitational field is analyzed. It is explicitly shown that, contrarily to an old standing folklore, the three-dimensional gas does not undergo Bose-Einstein condensation at finite temperature. On the other hand, Bose-Einstein condensation occurs at $T \neq 0$ for $D = 1, 2, 3$ if there is a point-like impurity at the bottom of the vessel containing the gas.

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I. INTRODUCTION

The response of quantum systems to the influence of external background fields is of utmost importance in a wide number of physical applications. As well, the role of disorder, i.e., the presence of impurities in condensed matter systems, is often crucial in the occurrence of remarkable physical effects. It is the aim of the present paper to investigate the behavior of an ideal boson gas in the presence of a uniform (i.e., constant and homogeneous) gravitational field and of extremely localized (actually point-like) impurities affecting the quantum dynamics of the bosonic particles.

It is well known since a long time [1,2] that an ideal three-dimensional boson gas in free space undergoes a phase transition called *Bose-Einstein condensation* (BEC), in which a finite fraction of its constituent molecules condenses in the single-particle ground state. Such a condensation differs from the usual condensation of a vapor into a liquid in that there is no phase separation. For this reason, BEC is commonly described as a phase transition in momentum space — the particles condense into the $|\mathbf{p} = \mathbf{0}\rangle$ state, which has a uniform spatial distribution. It is also well known [2] that such a phase transition is no longer possible, for free bosons, in one and two dimensions — although in both cases it does occur in the presence of a point-like attractive potential [3,4]. A long standing popular belief [1,5–10] is that if the particles of a 3D ideal boson gas were placed in a (uniform) gravitational field, then BEC would still occur, but in the condensation region there would be a spatial separation of the two phases, just as in a gas-liquid condensation.

In the present paper we study the exactly solvable quantum mechanical model of an ideal boson gas in $D = 1, 2, 3$ dimensions in the presence of a uniform gravitational field and of a point-like impurity formally described by a δ-function potential. In order to make the Hamiltonian bounded from below, so that the system may attain a state of thermodynamic equilibrium, we shall enclose the gas in a container with impenetrable walls. Concerning the mathematical description of a point-like impurity, it should be remarked that a δ-potential is generally ill-defined when $D > 1$, and some renormalization procedure is mandatory. Actually, the rigorous mathematical procedure to deal with point-like interactions involves the analysis of the self-adjoint extensions of the symmetric Hamiltonian operator [11]. In the present work, however, we prefer to follow a more informal approach [12] which is closer to the physical intuition, but reaches the same final result as the rigorous though more involved method of self-adjoint extensions [13]. To be specific, we formally treat the contact interaction as a D-dimensional δ-potential, then proceed to the renormalization procedure in physical terms, and finally obtain the so-called Krein's formula for the Green's function, from which it is possible to extract the energy spectrum of the single-particle Hamiltonian.

In Section II we prove that an ideal boson gas in the presence of a uniform gravitational field does not undergo BEC at finite temperature, except in the one-dimensional case. This implies, in particular, that in the three-dimensional case no phase separation occurs in the thermodynamic limit, at variance with the above quoted conventional wisdom. We also provide a rather general *sufficient* condition for the occurrence of BEC in a trapped ideal gas, which generalizes some results obtained by other authors [14–18] for power-law potentials. In Section III we show that the onset of BEC in a uniform gravitational field is made possible in $D = 2, 3$ if a point-like impurity (i.e., a δ-potential) is placed at the bottom of the vessel containing the gas. The reason is that the presence of the impurity entails the existence of a bound state, whose energy gap with respect to the continuous spectrum is what is needed for the ideal gas to undergo BEC. In Section IV we draw our conclusions, whereas some technical details are presented in two Appendices.

II. D-DIMENSIONAL BOSON GAS IN A UNIFORM FIELD

It is convenient to first analyze and discuss the impurity-free case, which turns out to exhibit, as we shall see below, rather surprising features. Thus, in this Section we shall study the quantum mechanical behavior of an ideal boson gas in the presence of a uniform gravitational field. The existence of a (single-particle) ground state is guaranteed by the presence of an impenetrable wall at the bottom of the vessel containing the gas. The single-particle Hamiltonian is given by

$$H_0^{(D)}(g) = \frac{\mathbf{p}^2}{2m} + mgx, \tag{1}$$

in which we have set

$$\mathbf{x} = (x_1, \ldots, x_D) \equiv (\mathbf{r}, x), \qquad \mathbf{p} = (p_1, \ldots, p_D) \equiv (\mathbf{k}, p). \tag{2}$$

The gas is supposed to be enclosed in a rectangular box of sides $L_1, L_2, \ldots, L_D$, with its bottom fixed at the plane $x = 0$. Since we are interested in the thermodynamic limit, we can, without lack of generality, impose periodic boundary conditions in the $x_1, \ldots, x_{D-1}$ directions and Neumann boundary condition[1] at $x = 0$ and $x = L_D$, i.e.,

$$\psi(x_1, \ldots, x_j + L_j, \ldots, x_D) = \psi(x_1, \ldots, x_j, \ldots, x_D), \qquad j = 1, \ldots, D-1, \tag{3}$$

$$\partial_x \psi(\mathbf{r}, x = 0) = \partial_x \psi(\mathbf{r}, x = L_D) = 0, \tag{4}$$

and then take the limits $L_j \to \infty$, $j = 1, \ldots, D$. After these limits are taken, the eigenfunctions and eigenvalues of $H_0^{(D)}(g)$ read

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = \frac{\exp\{(i/\hbar)\,\mathbf{k}\cdot\mathbf{r}\}}{(2\pi\hbar)^{(D-1)/2}} \sqrt{-\frac{\kappa}{a'_n}}\,\frac{\mathrm{Ai}(\kappa x + a'_n)}{\mathrm{Ai}(a'_n)}, \tag{5}$$

$$E_{n,\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - E_g a'_n\,, \qquad n \in \mathbb{N},\ \mathbf{k} \in \mathbb{R}^{D-1}, \tag{6}$$

where $\mathrm{Ai}(x)$ is the Airy function [19], a'_n are the zeros of $\mathrm{Ai}'(x)$, and the parameters κ and E_g are defined as

[1]The reason why we impose Neumann boundary condition, instead of the seemingly more natural Dirichlet one, will be explained in Section III.

$$\kappa \equiv \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}, \qquad E_g \equiv \frac{mg}{\kappa} = \frac{\hbar^2\kappa^2}{2m}. \tag{7}$$

All the zeros of $\mathrm{Ai}'(x)$ are negative, hence the energy levels $E_{n,\mathbf{k}}$ are positive.

If $D > 1$ the spectrum is purely continuous and the corresponding improper eigenfunctions are normalized according to

$$\langle \psi_{n',\mathbf{k}'} | \psi_{n,\mathbf{k}} \rangle = \delta_{n,n'}\, \delta^{(D-1)}(\mathbf{k} - \mathbf{k}'). \tag{8}$$

On the other hand, in the one-dimensional case the spectrum is purely discrete, the normalized eigenfunctions and eigenvalues being respectively

$$\psi_n(x) = \sqrt{-\frac{\kappa}{a'_n}}\, \frac{\mathrm{Ai}(\kappa x + a'_n)}{\mathrm{Ai}(a'_n)}, \tag{9}$$

$$E_n = -E_g a'_n\,, \qquad n \in \mathbb{N}. \tag{10}$$

Let us first analyze in detail the Bose-Einstein condensation (BEC) for such a one-dimensional system. In the grand canonical ensemble the average number of particles N at temperature T and chemical potential μ reads

$$N = \sum_{n=1}^{\infty} \frac{1}{\exp\left[\beta(E_n - \mu)\right] - 1}, \tag{11}$$

where, as usual, $\beta = 1/k_B T$. The criterion for the occurrence of BEC is that the average population of the excited states remains finite as the chemical potential approaches the ground state energy from below, i.e.,

$$\lim_{\mu \uparrow E_1} N_{\mathrm{ex}} = \lim_{\mu \uparrow E_1} \sum_{n=2}^{\infty} \frac{1}{\exp\left[\beta(E_n - \mu)\right] - 1} < \infty. \tag{12}$$

Notice that the ground state population has been split off, that being the reason why the above sum begins at $n = 2$. The sequence of eigenvalues (10) is such that the above mentioned BEC criterion is satisfied. Consequently, Bose-Einstein condensation is expected to occur, although, in order to specify the critical temperature, it would be necessary to sum up the series, which, up to our knowledge, cannot be done analytically. Nonetheless, one can estimate the critical quantities using the asymptotic behavior of E_n for large n [19]:

$$E_n = -E_g a'_n \sim E_g \left[3\pi(4n-3)/8\right]^{2/3}, \qquad n \gg 1. \tag{13}$$

This corresponds to a density of states of the form

$$\rho(E) \approx \frac{dn}{dE} \sim \frac{1}{\pi}\, E_g^{-3/2}\, E^{1/2}, \qquad E \gg E_g\,. \tag{14}$$

Since $E_g \propto g^{2/3}$, as $g \to 0$ the energy spectrum becomes denser and denser and the ground state energy approaches zero. Thus, in a weak gravitational field it is reasonable to extrapolate in the continuum the density of states (14) down to $E = 0$. We can then approximate the series in Eq. (12) by an integral, and eventually obtain

$$N_{\rm ex} \sim \int_0^\infty \frac{dE}{\pi} \frac{E_g^{-3/2} E^{1/2}}{\exp\left[\beta(E-\mu)\right]-1} = 4\pi \left(\kappa\lambda_T\right)^{-3} g_{3/2}(e^{\beta\mu}), \tag{15}$$

where $\lambda_T \equiv h/\sqrt{2\pi m k_B T}$ is the thermal wavelength and $g_s(x) \equiv \sum_{n=1}^{\infty} n^{-s}\, x^n$ is the Bose-Einstein function [1]. To obtain the critical temperature, we take the limit $\mu \to 0$ in Eq. (15) and equate $N_{\rm ex}$ to the total number of particles in the gas; solving for T then yields the approximate critical temperature

$$T_{\rm c} \sim \frac{E_g}{k_B} (4\pi)^{1/3} \left(\frac{N}{g_{3/2}(1)}\right)^{2/3}. \tag{16}$$

Below $T_{\rm c}$ the fraction of particles occupying the ground state is given by

$$\frac{N_0}{N} = 1 - \frac{N_{\rm ex}}{N} = 1 - \left(\frac{T}{T_{\rm c}}\right)^{3/2}. \tag{17}$$

The reasoning which led us to the conclusion that a one-dimensional ideal boson gas in a uniform gravitational field displays BEC can be easily generalized to higher dimensions and other types of potential. This is the content of the following theorem.

Theorem 1 *Suppose the single-particle energy spectrum of an ideal boson gas satisfies the following conditions: (i) there is a gap between the fundamental and the first excited energy levels, i.e.,* $E_1 - E_0 = \Delta > 0$*; (ii) the single-particle partition function is finite, i.e.,* $Z \equiv \sum_{n=0}^{\infty} d_n \exp(-\beta E_n) < \infty$*,* d_n *being the finite degeneracy of the* n*-th eigenvalue of the single-particle Hamiltonian. Then this gas displays Bose-Einstein condensation at finite temperature.*

Proof. If $\mu < E_0$, the number of particles in the excited states is bounded from above by

$$N_{\rm ex} = \sum_{n=1}^{\infty} \frac{d_n \exp[-\beta(E_n-\mu)]}{1-\exp[-\beta(E_n-\mu)]} \leq \frac{\exp(\beta\mu)}{1-\exp[-\beta(E_1-\mu)]} \sum_{n=1}^{\infty} d_n \exp(-\beta E_n). \tag{18}$$

Therefore

$$\lim_{\mu\to E_0} N_{\rm ex} \leq \frac{\exp(\beta E_0)}{1-\exp(-\beta\Delta)} \left[Z - d_0 \exp(-\beta E_0)\right] < \infty, \tag{19}$$

since, by hypothesis, Z and d_0 are finite and $\Delta > 0$. *Q.E.D.*

We notice that the above statement may be generalized to some cases in which part of the spectrum is continuous or there are infinitely degenerate energy levels. This is done under the suitable introduction of the density of particles in the excited states and of the single-particle partition function per unit volume. Some explicit examples of this generalization are discussed in Ref. [4] and in Section III of the present paper.

There are many papers that discuss the problem of Bose-Einstein condensation of an ideal gas confined in a power-law potential [14–18], mainly using some kind of semiclassical approximation. In particular, they predict that a one-dimensional gas displays BEC iff the power-law potential is *less* confining than the parabolic one, i.e., $V(x) \propto x^\eta$, $\eta < 2$. Theorem 1 shows that this condition is too strong: BEC occurs for any positive η. It should be clear that the reason of such a discrepancy is not the semiclassical approximation *per se*, but the substitution of the discrete spectrum by a smooth density of states, which may miss some relevant features of the energy spectrum.

Let us return to the problem of an ideal boson gas in a uniform gravitational field. We shall now consider the two- and three-dimensional cases. Due to the translation invariance along the transverse direction(s), the proper quantity to be discussed is the number of particles per unit area $n^{(D)} \equiv \lim_{L_j\to\infty} N/L_1 \cdots L_{D-1}$. The density of particles in the excited states is then given by

$$\begin{aligned} n_{\rm ex}^{(D)} &= \sum_{j=1}^{\infty} \int \frac{d^{D-1}k}{(2\pi\hbar)^{D-1}} \left\{ \exp\left[\beta\left(\frac{\mathbf{k}^2}{2m} - E_g a_j' - \mu\right)\right] - 1 \right\}^{-1} \\ &= \lambda_T^{1-D} \sum_{j=1}^{\infty} g_{(D-1)/2}\left[\exp \beta(E_g a_j' + \mu)\right], \qquad \mu < -E_g a_1' \,. \end{aligned} \tag{20}$$

The integral in Eq. (20) is well defined for arbitrary $D > 1$ due to the condition $\mu < -E_g a_1'$. Now, since $\lim_{x\to 1} g_s(x) = \infty$ if $s \le 1$, the first term of the series on the r.h.s. of Eq. (20) diverges for $D \le 3$ as $\mu \to -E_g a_1'$. Therefore, a two- or three-dimensional ideal boson gas in a uniform gravitational field *does not* display Bose-Einstein condensation at $T \neq 0$.

Some remarks are in order here:

(a) At first sight, Eq. (20) seems to imply absence of BEC in $D = 1$ too. It should be noted, however, that in one dimension there is no integration over transverse momenta. Hence, in order to remove the contribution of the ground state from the sum over states in Eq. (20), one has to begin it at $j = 2$. Then $n_{\rm ex}^{(1)}$ $(= N_{\rm ex})$ has a finite limit as $\mu \to -E_g a_1'$.

(b) It is easy to see that the absence of BEC in a two- or three-dimensional ideal boson gas in a uniform gravitational field in the x-direction is due to the quantization of the motion in that direction. Thus, any potential V that depends only on x, and such that the one-dimensional Hamiltonian

$$H_x = \frac{p_x^2}{2m} + V(x) \tag{21}$$

has a discrete spectrum, will do the job of hindering BEC in $D = 2, 3$.

(c) There are claims in the Literature [1,5–10] that a three-dimensional ideal boson gas in a uniform field may undergo BEC at $T \neq 0$. This is an artifact of approximating the sum in Eq. (20) by an integral (remember that Eq. (20) holds true for $D > 1$). Indeed, using the density of states given by Eq. (14) we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} g_{(D-1)/2}\left[\exp \beta(E_g a_j' + \mu)\right] &\approx \frac{1}{\pi} E_g^{-3/2} \int_0^\infty dE\, E^{1/2} \sum_{n=1}^{\infty} \frac{e^{-n\beta(E-\mu)}}{n^{(D-1)/2}} \\ &= \frac{1}{\pi} (\beta E_g)^{-3/2}\, \Gamma(3/2) \sum_{n=1}^{\infty} \frac{e^{n\beta\mu}}{n^{(D+2)/2}} \\ &= 4\pi\, (\kappa\lambda_T)^{-3}\, g_{(D+2)/2}(e^{\beta\mu}). \end{aligned} \tag{22}$$

Inserting this result into Eq. (20), one would be led to the incorrect conclusion that BEC occurs at finite temperature in $D = 2$ and $D = 3$ in the presence of a uniform field, because $\lim_{\mu\to 0} g_{(D+2)/2}(e^{\beta\mu}) < \infty$ if $D > 0$.

(d) It should be clear by now that none of our conclusions so far depends crucially on the use of Neumann boundary condition. They would remain correct, at least qualitatively, had we used Dirichlet or Robin boundary condition instead.

III. D-DIMENSIONAL BOSON GAS INTERACTING WITH A POINT-LIKE IMPURITY AT THE BOTTOM OF THE CONTAINER

In this Section we finally come to the most interesting physical case in which, in addition to the gravitational field, there is a point-like impurity at the bottom of the vessel containing the gas. As we shall show here, such an impurity is enough to restore BEC in the three-dimensional case — and to allow its existence in the two-dimensional case, in which it is absent with or without the gravitational field. The single-particle Hamiltonian takes now the form

$$H^{(D)}(g, \lambda_D) = \frac{\mathbf{p}^2}{2m} + mgx + \lambda_D\, \delta^{(D)}(\mathbf{x}) \equiv H_0^{(D)}(g) + \lambda_D\, \delta^{(D)}(\mathbf{x}). \tag{23}$$

Our main task will be to show that the δ-potential creates a bound state in the two- and three-dimensional cases, thus paving the way for the occurrence of Bose-Einstein condensation, at variance with the impurity-free situation discussed in the previous Section.

Our basic tool to tackle this problem is the Green's function

$$G^{(D)}(z; \mathbf{x}, \mathbf{x}') = \left\langle \mathbf{x} \left| \left[H^{(D)}(g, \lambda_D) - z\right]^{-1} \right| \mathbf{x}' \right\rangle, \qquad z \in \mathbb{C}, \tag{24}$$

from which it is possible to extract the energy spectrum. A formal expression for $G^{(D)}(z; \mathbf{x}, \mathbf{x}')$ can be obtained by solving the Lippmann-Schwinger integral equation,

$$G^{(D)}(z; \mathbf{x}, \mathbf{x}') = G_0^{(D)}(z; \mathbf{x}, \mathbf{x}') - \int d^D y\, G_0^{(D)}(z; \mathbf{x}, \mathbf{y})\, V(\mathbf{y})\, G^{(D)}(z; \mathbf{y}, \mathbf{x}'), \tag{25}$$

where $G_0^{(D)}$ and $G^{(D)}$ are the Green's functions associated to $H_0^{(D)}$ and $H^{(D)} = H_0^{(D)} + V(\mathbf{x})$, respectively. For $V(\mathbf{x}) = \lambda_D\, \delta^{(D)}(\mathbf{x})$ the integral in Eq. (25) can be done trivially, resulting in

$$G^{(D)}(z; \mathbf{x}, \mathbf{x}') = G_0^{(D)}(z; \mathbf{x}, \mathbf{x}') - \lambda_D\, G_0^{(D)}(z; \mathbf{x}, \mathbf{0})\, G^{(D)}(z; \mathbf{0}, \mathbf{x}'). \tag{26}$$

If we now set $\mathbf{x} = \mathbf{0}$, we obtain an algebraic equation for $G^{(D)}(z; \mathbf{0}, \mathbf{x}')$. Solving that equation and inserting the result into Eq. (26), we end up with

$$G^{(D)}(z; \mathbf{x}, \mathbf{x}') = G_0^{(D)}(z; \mathbf{x}, \mathbf{x}') - \frac{G_0^{(D)}(z; \mathbf{x}, \mathbf{0})\, G_0^{(D)}(z; \mathbf{0}, \mathbf{x}')}{\frac{1}{\lambda_D} + G_0^{(D)}(z; \mathbf{0}, \mathbf{0})}\,. \tag{27}$$

As we shall see below, $G_0^{(D)}(z; \mathbf{0}, \mathbf{0})$ is formally divergent for $D \geq 2$, but one can still give a well defined meaning to Eq. (27) by renormalizing the coupling parameter λ_D. The resulting expression, which then makes sense also for $D = 2, 3$, is known as the Krein's formula [11] and encodes the one-parameter family of self-adjoint extensions of the symmetric Hamiltonian operator $H_0^{(D)}(g)$. This precisely corresponds to the mathematically rigorous description of the δ-potential.

To complete the construction of $G^{(D)}$ we still have to obtain the Green's function in the absence of the impurity. This is done in Appendix A, with the result

$$G_0^{(D)}(z; \mathbf{x}, \mathbf{x}') = -\frac{\pi\kappa}{E_g} \int \frac{d^{D-1}k}{(2\pi\hbar)^{D-1}} \, \exp\left\{\frac{i}{\hbar}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right\} \frac{u[\xi(x_<)]\, v[\xi(x_>)]}{\mathrm{Ai}'[\xi(0)]}\,, \tag{28}$$

where the functions $u(\xi)$ and $v(\xi)$ are defined in Eq. (A7), $\xi(x)$ is defined in Eq. (A4), and $x_<(x_>) = \min(\max)\{x, x'\}$. Setting $\mathbf{x} = \mathbf{x}' = \mathbf{0}$ in Eq. (28) we formally obtain

$$\begin{aligned} G_0^{(D)}(z; \mathbf{0}, \mathbf{0}) &= -\frac{\kappa}{E_g} \int \frac{d^{D-1}k}{(2\pi\hbar)^{D-1}} \frac{\mathrm{Ai}\left[(\mathbf{k}^2/2mE_g) - (z/E_g)\right]}{\mathrm{Ai}'\left[(\mathbf{k}^2/2mE_g) - (z/E_g)\right]} \\ &= -C_D \int_0^\infty dy\, y^{(D-3)/2} \frac{\mathrm{Ai}(y-\zeta)}{\mathrm{Ai}'(y-\zeta)}\,, \end{aligned} \tag{29}$$

where

$$C_D \equiv \frac{\kappa^D \,(4\pi)^{(1-D)/2}}{E_g\,\Gamma[(D-1)/2]}\,, \qquad \zeta \equiv \frac{z}{E_g}\,. \tag{30}$$

It follows from the asymptotic behavior of the Airy function $\mathrm{Ai}(x)$ for large x [19],

$$\mathrm{Ai}(x) \overset{x\to\infty}{\sim} \frac{1}{2\sqrt{\pi}x^{1/4}} \, \exp\left(-\frac{2}{3}\,x^{3/2}\right) \left[1 + O(x^{-3/2})\right]\,, \tag{31}$$

that the integral in Eq. (29) diverges in the UV region for $D \geq 2$, as anticipated. (The integral is finite in the IR for $D > 1$.)

Before we show how to make sense of Eq. (27) for $D = 2, 3$, let us discuss the one-dimensional case, which does not need renormalization. In this case, the energy spectrum can be obtained by solving[2]

$$\frac{1}{\lambda_1} + G_0^{(1)}(z; 0, 0) = 0, \tag{32}$$

or, more explicitly (see Appendix A),

$$\frac{1}{\lambda_1} - \frac{\kappa}{E_g} \frac{\mathrm{Ai}\,(-z/E_g)}{\mathrm{Ai}'\,(-z/E_g)} = 0. \tag{33}$$

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[2]One can easily check that the residue of $G_0^{(1)}(z; x, x')$ at $z = -E_g a_n'$ cancels against the residue of the second term on the r.h.s. of Eq. (27) at the same pole. Therefore, all the poles of $G^{(1)}(z; x, x')$ are given by the solutions to Eq. (32).

This equation is equivalent to the imposition of Robin boundary condition at the origin, i.e., $\psi'(0) + c\,\psi(0) = 0$. It interpolates between the Neumann boundary condition, for $\lambda_1 \to 0$, and the Dirichlet one, for $\lambda_1 \to \infty$. Any of these boundary conditions prevents the flow of particles across the origin, so any of them can be used to represent an impenetrable wall at the bottom of the container. Nevertheless, it is more convenient to impose Neumann boundary condition in the impurity-free case, because it is then possible to model an impurity at the bottom of the container by a δ-potential. This would not be possible had we imposed Dirichlet boundary condition instead. In any case, the energy spectrum obtained by solving Eq. (33) will be purely discrete and bounded from below. As a consequence, we can say that in the one-dimensional case the Bose-Einstein condensation actually occurs at the lowest discrete energy level, although the ground state energy itself as well as the critical quantities are shifted with respect to the previously discussed impurity-free case.

Let us now discuss the two- and three-dimensional cases. In order to make sense of the denominator in Eq. (27), we first have to regularize $G_0^{(D)}(z;\mathbf{0},\mathbf{0})$. We shall do this by introducing a UV cutoff in Eq. (29), namely,

$$G_0^{(D)}(z;\mathbf{0},\mathbf{0}) \to G_0^{(D)}(\Lambda, z;\mathbf{0},\mathbf{0}) = -C_D \int_0^{\Lambda} dy\, y^{(D-3)/2}\, \frac{\mathrm{Ai}(y-\zeta)}{\mathrm{Ai}'(y-\zeta)}\,. \tag{34}$$

We now add to $G_0^{(D)}(\Lambda, z;\mathbf{0},\mathbf{0})$ the integral

$$I_D(\Lambda, z, \alpha) \equiv -C_D \int_0^{\Lambda} dy\, y^{(D-3)/2}\,(y+\alpha)^{-1/2}\,, \qquad \alpha > 0. \tag{35}$$

It follows from Eq. (31) that

$$\begin{aligned} \frac{\mathrm{Ai}(y-\zeta)}{\mathrm{Ai}'(y-\zeta)} &\overset{y\to\infty}{\sim} -(y-\zeta)^{-1/2} + O\left[(y-\zeta)^{-2}\right] \\ &\sim \; -y^{-1/2} + O\left(\zeta y^{-3/2}\right); \end{aligned} \tag{36}$$

hence, the integrand of $G_0^{(D)}(\Lambda, z;\mathbf{0},\mathbf{0}) + I_D(\Lambda, z, \alpha)$ behaves like $y^{(D-6)/2}$ for large y. This allows us to remove the UV regulator (i.e., to take the limit $\Lambda \to \infty$) for $D < 4$. At the same time, since we have added I_D to $G_0^{(D)}$, we must subtract it from λ_D^{-1} in order to keep the combination $\lambda_D^{-1} + G_0^{(D)}(z;\mathbf{0},\mathbf{0})$ unaltered. We may then define the renormalized coupling parameter λ_D^R as

$$\frac{1}{\lambda_D^R} = \lim_{\Lambda\to\infty}\left[\frac{1}{\lambda_D} - I_D(\Lambda, z, \alpha)\right], \tag{37}$$

where it is understood that λ_D depends on Λ in such a way that the limit exists. We then finally arrive at a meaningful expression for the Green's function $G^{(D)}(z;\mathbf{x},\mathbf{x}')$ for $D = 2, 3$, in which the denominator of Eq. (27) is replaced by the finite expression

$$\mathfrak{g}_D(\zeta, \alpha, \lambda_D^R) \equiv \frac{1}{\lambda_D^R} - C_D \int_0^{\infty} dy\, y^{(D-3)/2}\left[\frac{\mathrm{Ai}(y-\zeta)}{\mathrm{Ai}'(y-\zeta)} + (y+\alpha)^{-1/2}\right]. \tag{38}$$

It is possible to show (see Appendix B) that, for any finite value of λ_D^R, $\mathrm{g}_D(\zeta,\alpha,\lambda_D^R)$ has a single zero ζ_0 in the interval $-\infty < \zeta_0 < -a_1'$. In physical terms, this means the existence of a bound state with energy $E_0 = E_g\,\zeta_0$. The rest of the energy spectrum forms a continuum starting at $E = -E_g a_1'$. The presence of this gap in the energy spectrum is enough to guarantee the occurrence of BEC. The proof of this fact is similar to that of Theorem 1, the only difference being that what saturates in the limit $\mu \to E_0$ is not $N_{\rm ex}$, but $n_{\rm ex}^{(D)}$. Some examples of this phenomenon are discussed in detail in Ref. [4], where it is also shown how to obtain the critical quantities. Working in close analogy, one can obtain an estimate of the critical quantities in the present situation, taking Eq. (22) suitably into account. If the energy gap created by the impurity is much greater than the energy splitting due to the gravitational field, i.e., $\Delta \equiv -E_g a_1' - E_0 \gg -E_g a_2' + E_g a_1'$, one can obtain a good approximation to the critical temperature $T_{\rm c}$ by solving the equation

$$\lambda_{T_{\rm c}}^{D-1} n^{(D)} = 4\pi\,(\kappa\lambda_{T_{\rm c}})^{-3}\, g_{(D+2)/2}\left[\exp(-\Delta/k_B T_{\rm c})\right]. \tag{39}$$

It is worthwhile to stress that now, because the bound state energy E_0 is strictly below the continuum threshold $(-E_g a_1')$, we can safely use Eq. (22) to estimate the critical quantities in $D = 2, 3$.

We close this section with a somewhat technical remark. Aside from being positive, the parameter α in Eq. (38) is arbitrary, and has to be fixed by some renormalization prescription. One possibility is the so called Bergmann-Manuel-Tarrach [20] renormalization prescription, in which the bound state energy E_0 labels the one-parameter family of self-adjoint extensions of the symmetric Hamiltonian $H_0^{(D)}(g)$. Then Eq. (38) becomes equivalent to the pair of equations

$$\mathrm{g}_D(\zeta,\zeta_0)|_{\rm BMT} = C_D \int_0^\infty dy\, y^{(D-3)/2}\left[\frac{\mathrm{Ai}(y-\zeta_0)}{\mathrm{Ai}'(y-\zeta_0)} - \frac{\mathrm{Ai}(y-\zeta)}{\mathrm{Ai}'(y-\zeta)}\right], \tag{40}$$

$$\frac{1}{\lambda_D^R(\alpha)} = C_D \int_0^\infty dy\, y^{(D-3)/2}\left[\frac{\mathrm{Ai}(y-\zeta_0)}{\mathrm{Ai}'(y-\zeta_0)} + (y+\alpha)^{-1/2}\right], \tag{41}$$

where $\zeta_0 = E_0/E_g < -a_1'$. The parameter $\alpha > 0$ is thus the subtraction point at which the "running" coupling parameter λ_D^R is defined.

IV. CONCLUSIONS

In this paper we have explicitly solved the quantum dynamics and studied the thermodynamic equilibrium of an ideal D-dimensional boson gas in the presence of a uniform gravitational field and a point-like impurity at the bottom of the vessel containing the gas. For convenience, in the present analysis we have imposed Neumann boundary condition at the bottom of the container, but our results might be generalized to Dirichlet or Robin boundary conditions without any substantial modification in the physical behavior. In the impurity-free case it has been shown that Bose-Einstein condensation at finite temperature is possible only in the one-dimensional case and an estimate of the critical temperature in this case has been obtained. It has also been elucidated why the conventional wisdom that

BEC (with a phase separation) might occur in the three-dimensional case does actually fail: the reason eventually lies in the illegitimate use of a continuous approximation to the density of states in the computation of the average number of particles in the excited states.

On the other hand, it has been proved that the presence of a point-like impurity is enough to allow BEC at $T \neq 0$ also in two and three dimensions. The reason is that the impurity creates a bound state in the single-particle spectrum, where particles can now accumulate. It should also be emphasized that a δ-potential in the presence of a uniform field is always attractive in two and three dimensions, irrespective of the sign of the renormalized coupling parameter.

The main interest in the study of the present model is in its exact solvability. Nonetheless, it is evident that the key physical features here exhibited will persist even if more realistic impurity potentials are used. The situation is less clear if one considers an interacting boson gas (for the general definition of BEC, applicable to this case, see Ref. [21]). It is reasonable to assume that our results still hold if the mean field interaction between the particles in the gas is smaller than (i) the energy splitting due to the gravitational field, and (ii) the energy gap created by the impurity (if the latter is present). This condition, however, is likely to be violated as more and more particles accumulate in the lowest energy level, until the interaction between the particles cannot be neglected anymore. What happens then awaits further investigation.

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APPENDIX A:

The Green's function $G_0^{(D)}(z;\mathbf{x},\mathbf{x}')$ satisfies the partial differential equation

$$\left[H_0^{(D)}(g) - z\right] G_0^{(D)}(z;\mathbf{x},\mathbf{x}') = \delta^{(D)}(\mathbf{x}-\mathbf{x}'). \tag{A1}$$

We can reduce Eq. (A1) to an ordinary differential equation by Fourier transforming in the transverse coordinates:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\mathbf{k}^2}{2m} + mgx - z\right)\mathcal{G}(z,\mathbf{k};x,x') = \delta(x-x'); \tag{A2}$$

the Green's function $G_0^{(D)}$ will then be given by[3]

$$G_0^{(D)}(z;\mathbf{x},\mathbf{x}') = \int \frac{d^{D-1}k}{(2\pi\hbar)^{D-1}}\, \exp\left\{\frac{i}{\hbar}\,\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right\}\mathcal{G}(z,\mathbf{k};x,x'). \tag{A3}$$

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[3]In the one-dimensional case we have instead $G_0^{(1)}(z;x,x') = \mathcal{G}(z,\mathbf{k}=\mathbf{0};x,x')$.

Upon the change of variable

$$\xi = \kappa x + E_g^{-1}\left(\frac{\mathbf{k}^2}{2m} - z\right), \tag{A4}$$

Eq. (A2) becomes

$$\left(\frac{\partial^2}{\partial \xi^2} - \xi\right)\mathcal{G}(\xi,\xi') = -\frac{\kappa}{E_g}\,\delta(\xi - \xi'). \tag{A5}$$

When $\xi \neq \xi'$, Eq. (A5) reduces to the Airy differential equation. Its solution must satisfy Neumann boundary condition at $x = 0$, i.e., $\partial_\xi \mathcal{G}(\xi,\xi')|_{x=0} = 0$, and it must vanish at infinity, $\lim_{\xi\to\infty} \mathcal{G}(\xi,\xi') = 0$. Thus,

$$\mathcal{G}(\xi,\xi') = C_1\, u(\xi)\,\theta(\xi' - \xi) + C_2\, v(\xi)\,\theta(\xi - \xi'), \tag{A6}$$

where $\theta(x)$ is the Heaviside step function and

$$u(\xi) \equiv \mathrm{Bi}'(\xi_0)\,\mathrm{Ai}(\xi) - \mathrm{Ai}'(\xi_0)\,\mathrm{Bi}(\xi), \qquad v(\xi) \equiv \mathrm{Ai}(\xi), \tag{A7}$$

with $\xi_0 \equiv \xi(x = 0)$. To fix the constants C_1 and C_2, one imposes continuity of $\mathcal{G}(\xi,\xi')$ at $\xi = \xi'$,

$$\mathcal{G}(\xi' + 0, \xi') = \mathcal{G}(\xi' - 0, \xi'), \tag{A8}$$

and a jump in $\partial_\xi \mathcal{G}(\xi,\xi')$ at the same point,

$$\partial_\xi \mathcal{G}(\xi' + 0, \xi') - \partial_\xi \mathcal{G}(\xi' - 0, \xi') = -\frac{\kappa}{E_g}, \tag{A9}$$

obtained by integrating Eq. (A5) from $\xi' - \epsilon$ to $\xi' + \epsilon$ and letting $\epsilon \downarrow 0$. Applying conditions (A8) and (A9) to the solution (A6), and using the fact that the Wronskian of $u(\xi)$ and $v(\xi)$ is given by

$$W\{u(\xi), v(\xi)\} = -\mathrm{Ai}'(\xi_0)\,W\{\mathrm{Bi}(\xi), \mathrm{Ai}(\xi)\} = \frac{1}{\pi}\,\mathrm{Ai}'(\xi_0), \tag{A10}$$

we finally obtain

$$\mathcal{G}(\xi,\xi') = -\frac{\pi\kappa\, u(\xi_<)\, v(\xi_>)}{E_g\,\mathrm{Ai}'(\xi_0)}, \tag{A11}$$

where $\xi_<(\xi_>) = \min\,(\max)\{\xi,\xi'\}$. Substituting (A11) into Eq. (A3) gives us the desired integral representation of Eq. (28) for $G_0^{(D)}(z;\mathbf{x},\mathbf{x}')$.

APPENDIX B:

Here we show that $\mathsf{g}_D(\zeta,\alpha,\lambda_D^R)$ has one (and only one) zero in the interval $-\infty<\zeta<-a_1'$. Indeed, for ζ large and negative we may use the first line of Eq. (36) to evaluate the integral in Eq. (38), obtaining

$$\mathsf{g}_2(\zeta,\alpha,\lambda_2^R) \overset{\zeta\to-\infty}{\sim} \frac{1}{\lambda_2^R} - C_2 \ln\left(-\frac{\zeta}{\alpha}\right), \tag{B1}$$

$$\mathsf{g}_3(\zeta,\alpha,\lambda_3^R) \overset{\zeta\to-\infty}{\sim} \frac{1}{\lambda_3^R} - 2\,C_3\left(\sqrt{-\zeta}-\sqrt{\alpha}\right). \tag{B2}$$

In both cases, $\lim_{\zeta\to-\infty}\mathsf{g}_D(\zeta,\alpha,\lambda_D^R)=-\infty$. On the other hand, the integral in Eq. (38) becomes divergent at the origin for $D\le 3$ if $\zeta\uparrow -a_1'$, as

$$\frac{\mathrm{Ai}(y+a_1')}{\mathrm{Ai}'(y+a_1')} \overset{y\to 0}{\sim} \frac{\mathrm{Ai}(a_1')}{\mathrm{Ai}''(a_1')\,y} = \frac{1}{a_1' y}\,. \tag{B3}$$

(The last equality is a consequence of Airy differential equation.) Since $a_1'<0$, it follows that $\lim_{\zeta\uparrow -a_1'}\mathsf{g}_D(\zeta,\alpha,\lambda_D^R)=+\infty$ $(D=2,3)$. By continuity, we may conclude that $\mathsf{g}_D(\zeta,\alpha,\lambda_D^R)$ vanishes at least once in the interval $-\infty<\zeta<-a_1'$. To show that it vanishes only once, it suffices to prove that $\mathsf{g}_D(\zeta,\alpha,\lambda_D^R)$ is a monotonically increasing function of ζ in that interval. This follows from the identity

$$\begin{aligned}\frac{\partial}{\partial\zeta}\,\mathsf{g}_D(\zeta,\alpha,\lambda_D^R) &= E_g\,\frac{\partial}{\partial z}\left[\frac{1}{\lambda_D}+G_0^{(D)}(z;\mathbf{0},\mathbf{0})\right]\\ &= E_g\left\langle \mathbf{0}\left|\left[H_0^{(D)}(g)-z\right]^{-2}\right|\mathbf{0}\right\rangle.\end{aligned} \tag{B4}$$

It shows that $\partial_\zeta\mathsf{g}_D(\zeta,\alpha,\lambda_D^R)>0$ if z is real and does not belong to the spectrum of $H_0^{(D)}(g)$. This occurs, as we have seen in Section II, for $z<-E_g a_1'$, or $\zeta<-a_1'$.

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