Statmech II HW1

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1 Problem 1

We write the wavefunctions corresponding to the two polarizations as $(1, 0), (0, 1)$. In this basis we can write the density matrix as:

$$\rho = \begin{bmatrix} f & 0 \\ 0 & 1-f \end{bmatrix} \tag{1.1}$$

2 Problem 2

Since the $|n\rangle$ are a complete orthonormal basis, each diagonal element of $\rho\hat{A}$ is:

$$(\rho\hat{A})_n = e^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!} \langle n| \hat{A} |n\rangle \tag{2.1}$$

a) For $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, each term contains a dot product $\langle n|n-1\rangle = 0$ because the eigenvectors are orthogonal, so $\langle \hat{a} \rangle = 0$.
For $\langle a^\dagger a \rangle$ the diagonal terms are:

$$`(\rho a^\dagger a)_n = e^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!} \langle n| n |n\rangle \tag{2.2}$$

$$(\rho a^\dagger a)_n = e^{-\langle n \rangle} \frac{\langle n \rangle^n}{(n-1)!} \tag{2.3}$$

We and pull out one $\langle n \rangle$ from the numerator and use the series definition of e:

$$Tr(\rho a^\dagger a) = \langle n \rangle e^{-\langle n \rangle} \sum_{j=0}^{\infty} \frac{\langle n \rangle^{j-1}}{(j-1)!} \tag{2.4}$$

$$Tr(\rho a^\dagger a) = \langle n \rangle e^{-\langle n \rangle} e^{\langle n \rangle} \tag{2.5}$$

$$Tr(\rho a^\dagger a) = \langle n \rangle \tag{2.6}$$

b) To find $\langle (a^\dagger a)^2 \rangle$ we expand the exponential in its series:

$$(\rho a^\dagger a)_n = e^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!} \langle n | \, n^2 \, | n \rangle \tag{2.7}$$
$$(\rho (a^\dagger a)^2)_n = e^{-\langle n \rangle} \frac{n \langle n \rangle^n}{(n-1)!} \tag{2.8}$$
$$Tr((a^\dagger a)^2) = \langle n \rangle \, e^{-\langle n \rangle} \sum_{j=0}^{\infty} \frac{j \langle n \rangle^{j-1}}{(j-1)!} \tag{2.9}$$
$$Tr((a^\dagger a)^2) = \langle n \rangle \left(\sum_{i=0}^{\infty} \frac{\langle n \rangle^i}{i!} \right)^{-1} \sum_{j=0}^{\infty} \frac{j \langle n \rangle^{j-1}}{(j-1)!} \tag{2.10}$$
$$Tr((a^\dagger a)^2) = \frac{\langle n \rangle}{\langle n \rangle} \left(\sum_{i=0}^{\infty} \frac{i \langle n \rangle^{i-1}}{(i-1)!} \right)^{-1} \sum_{j=0}^{\infty} \frac{j \langle n \rangle^{j-1}}{(j-1)!} \tag{2.11}$$
$$Tr((a^\dagger a)^2) = 1 \tag{2.12}$$

So we find that the variance is $1 - \langle n \rangle^2$.

3 Problem 3

a) The Hamiltonian for a electron with magnetic moment μ_B in a magnetic field B along the z axis is:

$$H = -\mu_B \mathbf{S} \cdot \mathbf{B} \tag{3.1}$$
$$H = -\mu_B B \sigma_z \tag{3.2}$$

The eigenvalues of σ_z are $\pm \frac{\hbar}{2}$ in the basis where σ_z is diagonalized. Defining $\kappa \equiv \mu_b B \frac{\hbar}{2}$ we find the probabilities of the two states:

$$\rho_\uparrow = \frac{e^{\beta\kappa}}{e^{-\beta\kappa} + e^{\beta\kappa}} = \frac{1}{1 + e^{-2\beta\kappa}} \tag{3.3}$$
$$\rho_\downarrow = \frac{e^{-\beta\kappa}}{e^{-\beta\kappa} + e^{\beta\kappa}} = \frac{1}{1 + e^{2\beta\kappa}} \tag{3.4}$$

Knowing that the eigenvectors of σ_z are $(1,0)$ and $(0,1)$ we can directly write the density matrix:

$$\rho = \begin{bmatrix} \frac{1}{1+e^{-2\beta\kappa}} & 0 \\ 0 & \frac{1}{1+e^{2\beta\kappa}} \end{bmatrix} \tag{3.5}$$

b) We compute $\langle \sigma_z \rangle$ as $Tr(\rho\sigma_z)$:

$$\langle \sigma_z \rangle = Tr(\rho\sigma_z) \tag{3.6}$$

$$\langle \sigma_z \rangle = Tr\left(\begin{bmatrix} \frac{1}{1+e^{-2\beta\kappa}} & 0 \\ 0 & \frac{1}{1+e^{2\beta\kappa}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \tag{3.7}$$

$$\langle \sigma_z \rangle = \frac{e^{\beta\kappa} - e^{-\beta\kappa}}{e^{\beta\kappa} + e^{-\beta\kappa}} \tag{3.8}$$