Quantum III HW1

Vince Baker

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1 Problem 1

Using the classic perturbation theory:

$$i\hbar \dot{c_1} = e^{\frac{i}{\hbar}\omega_{12}t} V_{21} c_2 \tag{1.1}$$
$$i\hbar \dot{c_2} = e^{-\frac{i}{\hbar}\omega_{12}t} V_{12} c_1 \tag{1.2}$$

We insert the initial values of c_1, c_2 on the right hand side and find the first-order solution.

$$i\hbar \dot{c_1} = 0,\ c_1 = 1 \tag{1.3}$$
$$i\hbar \dot{c_2} = e^{\frac{i}{\hbar}\omega_{21}t} \lambda \cos \omega t \tag{1.4}$$
$$c_2 = \frac{-i}{\hbar} \int \lambda \cos{(\omega t)} e^{-\frac{i}{\hbar}\omega_{12}t}\ dt \tag{1.5}$$
$$c_2 = \frac{-i\lambda}{\hbar} \int e^{i(\omega_{12}+\omega)t} + e^{i(\omega_{12}-\omega)t}\ dt \tag{1.6}$$
$$c_2 = \frac{-\lambda}{2\hbar} \left(\frac{1}{\omega_{12}+\omega} (e^{i(\omega_{12}+\omega)t} - 1) + \frac{1}{\omega_{12}-\omega} (e^{i(\omega_{12}-\omega)t} - 1) \right) \tag{1.7}$$

If $E_2 - E_1$ is close to $\hbar\omega$ then the second term will become large, so this approximation is not completely valid since we assume a small perturbation. However, this approximation does capture the qualitative transition probabilities for emission and absorption under a harmonic perturbation.

2 Problem 2

Using $\delta(x - ct) = \frac{1}{c}\delta(\frac{x}{c} - t)$ we can write the first-order Dyson series term as:

$$c_2^1 = \frac{-iA}{\hbar c} \int_0^t dt' \left\langle 2|\hat{x}\right\rangle \delta(\frac{x}{c} - t') \left\langle \hat{x}|1\right\rangle \tag{2.1}$$
$$c_2^1 = \frac{-iA}{\hbar c} \int_{-\infty}^{\infty} dx\ \psi_2^{\dagger} \psi_1 e^{i\omega_{12}(\frac{x}{c})} \tag{2.2}$$

Using the integral form of the exponential $\delta(x-x_0) = \int e^{ik(x-x_0)}dk$ we can view the exponential as an expansion in harmonic terms. However our result shows that only the term with $k = \omega_{12}$ contributes to the transition probability.

3 Problem 3

We have two coupled differential equations:

$$i\hbar\dot{c_1} = \gamma e^{i(\omega+\omega_{12})t}c_2 \tag{3.1}$$

$$i\hbar\dot{c_2} = \gamma e^{-i(\omega-\omega_{21})t}c_1 \tag{3.2}$$

Rearranging 3.2 for c_1, we take the time derivative $\dot{c_1}$ and substitute into 3.1:

$$-\frac{\hbar^2}{\gamma}e^{i(\omega-\omega_{21})t}(i(\omega-\omega_{21}\dot{c_2}+\ddot{c_2})) = \gamma e^{i(\omega+\omega_{12})t}c_2 \tag{3.3}$$

$$\ddot{c_2} + i(\omega-\omega_{21})\dot{c_2} + \frac{\gamma^2}{\hbar^2} = 0 \tag{3.4}$$

Since the coefficients are all constants the solution will be a linear combination of exponentials, and we can find the roots of the quadratic equation.

$$c_2 = C_A e^{i\alpha_+ t} + C_B e^{i\alpha_- t} \tag{3.5}$$

$$\alpha_\pm = \frac{1}{2}\left(\omega-\omega_{21} \pm \sqrt{(\omega-\omega_{21})^2 - 4(\gamma^2/\hbar^2)}\right) \tag{3.6}$$

$$c_2(0) = 0,\ C_A = -C_B \tag{3.7}$$

$$c_1(0) = \frac{i\hbar}{\gamma}\dot{c_2}(0) = 1 \tag{3.8}$$

$$c_2 = \frac{i\gamma e^{i\frac{\omega-\omega_{21}}{2}t}}{\hbar\sqrt{(\omega-\omega_{21})^2 + 4(\gamma^2/\hbar^2)}}\sin\left(\frac{\sqrt{(\omega-\omega_{21})^2+(\gamma^2/\hbar^2)}}{2}t\right) \tag{3.9}$$

$$|c_2|^2 = \frac{\gamma^2}{\hbar^2((\omega-\omega_{21})^2+4(\gamma^2/\hbar^2))}\sin^2\left(\frac{\sqrt{(\omega-\omega_{21})^2+(\gamma^2/\hbar^2)}}{2}t\right) \tag{3.10}$$

$$|c_1|^2 = 1 - |c_2|^2 \tag{3.11}$$

We can also examine the problem using perturbation theory. Working up to second order:

$$c_2^0 = 0 \tag{3.12}$$

$$c_2^{(1)} = \frac{i}{\hbar}\int_0^t dt'\gamma e^{-i(\omega-\omega_{21})t'} \tag{3.13}$$

$$c_2^{(1)} = \frac{\gamma}{\hbar}\frac{ie^{-i(\omega-\omega_{21})t/2}}{2(\omega-\omega_{21})}(\sin{(\omega-\omega_{21})t/2}) \tag{3.14}$$

$$c_2^{(2)} = (\frac{-i}{\hbar})^2\int_0^t dt'\gamma e^{-i(\omega-\omega_{21})t'}\int_0^{t'} dt''\gamma e^{i(\omega-\omega_{21})t} \tag{3.15}$$

$$c_2^{(2)} = \frac{\gamma^2}{\hbar^2}\frac{i}{\omega-\omega_{21}}\left(t-\frac{i}{\omega-\omega_{21}}(e^{-i(\omega-\omega_{21})t}-1)\right) \tag{3.16}$$

$$c_2^{(2)} = \frac{\gamma^2}{\hbar^2}\frac{i}{\omega-\omega_{21}}\left(t-\frac{e^{-i(\omega-\omega_{21})t/2}}{2(\omega-\omega_{21})}(\sin{(\omega-\omega_{21})t/2})\right) \tag{3.17}$$

$$c_2^{(2)} = \frac{\gamma^2}{\hbar^2}\frac{i}{\omega-\omega_{21}}t-\frac{\gamma^2}{\hbar^2}\frac{ie^{-i(\omega-\omega_{21})t/2}}{2(\omega-\omega_{21})^2}\sin{(\omega-\omega_{21})t/2)} \tag{3.18}$$

When $\omega \neq \omega_{21}$ the $\frac{\gamma^2}{\hbar^2}\frac{i}{\omega-\omega_{21}}t$ term will dominate for large t, since the other terms are oscillatory. This asymptotic behvior is not indicated in the exact solution. The probability wil lincrease as t^2.
When $\omega \approx \omega_{21}$ the dominant term will be:

$$c_2 \approx \frac{\gamma^2}{\hbar^2}\frac{ie^{-i(\omega-\omega_{21})t/2}}{2(\omega-\omega_{21})^2}\sin{(\omega-\omega_{21})t/2} \tag{3.19}$$

$$|c_2|^2 = \frac{\gamma^4}{\hbar^4}\frac{1}{4(\omega-\omega_{21})^4}\sin^2{(\omega-\omega_{21})t/2} \tag{3.20}$$

Which has similar behavior to the exact result, although the amplitude and frequency are different.

4 Problem 4

At time $t = 0$ the system is in state $\alpha\,|1\rangle + \beta\,|2\rangle$. The zeroth-order term will be $c_1^0 = \alpha, c_2^0 = \beta$. Referencing equation 1.4 for the first-order terms we find:

$$c_1 = \alpha - \frac{\beta}{2\hbar}\left(\frac{1}{\omega_{21}+\omega}(e^{i(\omega_{21}+\omega)t}-1)+\frac{1}{\omega_{21}-\omega}(e^{i(\omega_{21}-\omega)t}-1)\right) \tag{4.1}$$

$$c_2 = \beta - \frac{\alpha}{2\hbar}\left(\frac{1}{\omega_{12}+\omega}(e^{i(\omega_{12}+\omega)t}-1)+\frac{1}{\omega_{12}-\omega}(e^{i(\omega_{12}-\omega)t}-1)\right) \tag{4.2}$$

With $\hbar\omega \approx \hbar(E_2 - E_1)$ we can discard the terms where the denominator is not close to 0. We then have:

$$c_1 = \alpha - \frac{\beta}{2\hbar}\left(\frac{1}{\omega - \omega_{12}}(e^{i(\omega-\omega_{21})t} - 1)\right) \tag{4.3}$$

$$c_1 = \alpha - i\frac{\beta}{\hbar}\frac{e^{i(\omega-\omega_{21})t/2}}{\omega - \omega_{12}}(\sin{(\omega - \omega_{21})t/2})) \tag{4.4}$$

$$c_2 = \beta - \frac{\alpha}{2\hbar}\left(\frac{1}{\omega_{12} - \omega}(e^{i(\omega_{12}-\omega)t} - 1)\right) \tag{4.5}$$

$$c_2 = \beta - i\frac{\alpha}{\hbar}\frac{e^{i(\omega_{12}-\omega)t/2}}{\omega_{12} - \omega}(\sin{(\omega_{12} - \omega)t/2})) \tag{4.6}$$

Generally, if the system starts in a mixed state, the probabilities will involve cross terms. However, the transition probabilities from a pure state are much simpler.

$$|c_1^2|_{\alpha=0} = \frac{\beta^2}{\hbar^2}\frac{\sin^2{(\omega - \omega_{21})t/2)}}{(\omega - \omega_{12})^2} \tag{4.7}$$

$$|c_2^2|_{\beta=0} = \frac{\alpha^2}{\hbar^2}\frac{\sin^2{(\omega_{12} - \omega)t/2)}}{(\omega_{12} - \omega)^2} \tag{4.8}$$

5 Problem 5

a) Starting from the Schrodinger equation we derive the time evolution of the density operator.

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{5.1}$$

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}\left|\psi\right\rangle\left\langle\psi\right| + \left|\psi\right\rangle\frac{\partial}{\partial t}\left\langle\psi\right| \tag{5.2}$$

$$i\hbar\frac{\partial\rho}{\partial t} = H\left|\psi\right\rangle\left\langle\psi\right| - \left|\psi\right\rangle\left\langle\psi\right|H^\dagger \tag{5.3}$$

$$i\hbar\frac{\partial\rho}{\partial t} = [H, \rho] \tag{5.4}$$

b) Since $\langle\rho\rangle = \langle\psi|\psi\rangle\langle\psi|\psi\rangle = 1$, the time derivative of ρ is 0.
c) Making use of 5.4:

$$i\hbar\frac{\partial\rho}{\partial t} = H\left|\psi\right\rangle\left\langle\psi\right| - \left|\psi\right\rangle\left\langle\psi\right|H^\dagger \tag{5.5}$$

$$\langle\frac{\partial\rho}{\partial t}\rangle = \frac{i}{\hbar}\left(\left\langle\psi\right|\rho H\left|\psi\right\rangle - \left\langle\psi\right|H\rho\left|\psi\right\rangle\right) \tag{5.6}$$

$$\langle\frac{\partial\rho}{\partial t}\rangle = \frac{i}{\hbar}\left(E_\psi\left\langle\psi\right|\rho\left|\psi\right\rangle - \left\langle\psi\right|H\left|\psi\right\rangle\left\langle\psi\right|\left|\psi\right\rangle\right) \tag{5.7}$$

$$\langle\frac{\partial\rho}{\partial t}\rangle = \frac{i}{\hbar}(E_\psi - E_\psi) = 0 \tag{5.8}$$