PHYS 502: Mathematical Physics II

Winter 2015, Homework #4

(Due February 25, 2015)

1. (a) Show explicitly from the series solutions that

$$\begin{aligned} J_{1/2}(x) &= A\,x^{-1/2}\sin x \\ J_{-1/2}(x) &= B\,x^{-1/2}\cos x\,. \end{aligned}$$

Hence, taking $A = B = 1$ and using the recurrence relations, write down expressions for $J_{3/2}(x)$ and $J_{5/2}(x)$.

(b) A function $f(x)$ is expressed as a Bessel series

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn} x),$$

where α_{mn} is the n-th root of J_m. Prove the Parseval relation

$$\int_0^1 [f(x)]^2 x\,dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2\,[J_{m+1}(\alpha_{mn})]^2\,.$$

2. The curved surface of a long cylinder of radius b is kept at a constant temperature $T = 0$. Initially the cylinder is at a uniform temperature $T_0 > 0$. Derive an expression for the temperature at the center of the cylinder at any time $t > 0$, and write down a simplified solution (not $T = 0$!) valid in the limit $t \gg b^2/\kappa$, where κ is the heat diffusion coefficient of the cylinder.

3. (a) Two hemispherical shells each of radius a are fitted together, insulated around their circle of contact, and kept at potentials $\pm V_0$, respectively. Find the potential $\Phi(r,\theta)$ inside the resulting sphere, where $\nabla^2\Phi = 0$ inside the sphere, $\Phi = \pm V_0$ on the two hemispheres, and the polar axis $\theta = 0$ is the axis of symmetry.

(b) Now suppose that the potentials of the hemispheres in part (a) oscillate in time, with $V(t) = \pm V_0 e^{-i\omega t}$. Find the *exterior* solution $\Phi(r,\theta,t)$ to the wave equation

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$$

for $r > a$, with $\Phi = \pm V(t)$ on the hemispheres, that describes an *outgoing* wave as $r \to \infty$.

[Recall that the asymptotic forms of the spherical Bessel functions $j_l(x)$ and $n_l(x)$ as $x \to \infty$ are $j_l(x) \sim \frac{1}{x}\cos\{x - \frac{\pi}{2}(l+1)\}$ and $n_l(x) \sim \frac{1}{x}\sin\{x - \frac{\pi}{2}(l+1)\}$.]

4. Each of the two 1S electrons in a helium atom may be described by a hydrogenic wave function

$$\psi(\mathbf{r}) = \left(\frac{8}{\pi a_0^3}\right)^{1/2} e^{-2r/a_0}$$

in the absence of the other electron. Here, $a_0 = \hbar^2/me^2$ is the Bohr radius. Use the expansion

$$\frac{1}{r_{12}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi}{2n+1} \left[Y_n^m(\theta_1, \phi_1)\right]^* Y_n^m(\theta_2, \phi_2) \begin{cases} \frac{r_1^n}{r_2^{n+1}}, & |\mathbf{r}_1| < |\mathbf{r}_2| \\ \frac{r_2^n}{r_1^{n+1}}, & |\mathbf{r}_1| > |\mathbf{r}_2| \end{cases}$$

to find the mutual electrostatic potential energy of the two electrons

$$U = \int \psi^*(\mathbf{r}_1)\psi^*(\mathbf{r}_2) \frac{e^2}{r_{12}} \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\, d^3r_1 d^3r_2,$$

where $r_i = |\mathbf{r}_i|$, $d^3r_i = r_i^2 dr_i\, \sin\theta_i d\theta_i\, d\phi_i$ and $r_{12} = |\mathbf{r}_1 - \mathbf{r}_1|$.