PHYS 501: Mathematical Physics I

Fall 2014

Homework #4
Solutions

1. The desired integral is
$$I = \int_{-1}^{1} \frac{dx}{(a^2+x^2)\sqrt{1-x^2}} \,.$$
We can't simply choose a contour that runs from $z = -1$ to $z = 1$ along the top of the branch cut, loops around $z = 1$ and returns along the bottom of the cut, because the integrand is not analytic along the entire length of the cut, so the conditions for application of the residue theorem are not satisfied. Instead, we choose a contour that runs from $z = -1$ to $z = 1$ along the top of the cut, and from $z = 1$ to $z = R$ along the positive real axis, then circles the entire complex plane at radius $|z| = R$ (contour C_R, say), returns from $z = R$ to $z = 1$ along the positive real axis, runs from $z = 1$ to $z = -1$ along the bottom of the cut, and finally closes by looping around $z = -1$ on a circle C_ϵ of radius ϵ. Then

   - the integrand is only mildly singular at $z = -1$, so the integral around $C_\epsilon \sim \epsilon^{1/2} \to 0$ as $\epsilon \to 0$,
   - the integral around $C_R \sim R^{-2} \to 0$ as $R \to \infty$,
   - the integral from $z = -1$ to $z = 1$ is I,
   - the presence of the branch cut means that there is a sign change as we cross the cut—specifically, for $z = iy$, $\sqrt{1-z^2} = +\sqrt{1+y^2}$ for $y > 0$ and $-\sqrt{1+y^2}$ for $y < 0$, so the integral from $z = 1$ to $z = -1$ is $-(-I) = I$,
   - the two integrals between $z = 1$ and $z = R$ cancel,
   - the contour encloses poles at $z = \pm ia$; taking into account the sign change across the cut, the residue at each is $1/2ia\sqrt{1+a^2}$.

   Thus we have, by the residue theorem,
$$2I = 2\pi i \left(\frac{2}{2ia\sqrt{1+a^2}}\right) \,,$$
or
$$I = \frac{\pi}{a\sqrt{1+a^2}} \,.$$

2. (a) We seek a series solution of the ODE $(1-x^2)y'' - xy' + n^2 y = 0$ in the form
$$y(x) \;=\; x^k \sum_{m=0}^{\infty} c_m\, x^m \;=\; \sum_{m=0}^{\infty} c_m\, x^{m+k} \,,$$

where $c_0 \neq 0$. Substituting the sum into the differential equation yields

$$\begin{aligned}\sum_{m=0}^{\infty} (m+k)(m+k-1)\, c_m\, x^{m+k-2}& \\ -\sum_{m=0}^{\infty} (m+k)(m+k-1)\, c_m\, x^{m+k}& \\ -\sum_{m=0}^{\infty} (m+k)\, c_m\, x^{m+k}& \\ +\sum_{m=0}^{\infty} n^2\, c_m\, x^{m+k} &= 0\,,\end{aligned}$$

or, collecting terms

$$\begin{aligned}\sum_{m=-2}^{\infty} & (m+k+2)(m+k+1)\, c_{m+2}\, x^{m+k} \\ -\sum_{m=0}^{\infty} & \Big\{ (m+k)(m+k-1)\, c_m\, x^{m+k} \\ & + (m+k)\, c_m\, x^{m+k} \\ & - n^2\, c_m\, x^{m+k} \Big\} = 0\,.\end{aligned}$$

The leading term (x^{k-2}, from $m = -2$ in the first sum) gives the indicial equation

$$k(k-1) = 0\,,$$

so $k = 0$ or 1. For $k = 0$ the next term $[(k+1)k\, c_1]$ is automatically zero, so there is no constraint on c_1. For $k = 1$, we must have $c_1 = 0$. The remaining terms imply

$$(m+k+2)(m+k+1)\, c_{m+2} \;=\; [(m+k)^2 - n^2]\, c_m\,,$$

connecting even to even and odd to odd terms. Obviously, the odd terms in the $k = 0$ case, starting with $c_1 x$, give the same sequence as the even terms in the $k = 1$ case, starting with $c_0 x$. Accordingly, we can consider the odd and even series separately. Both are regular at $x = 0$.

Since

$$c_{m+2} \;=\; \frac{(m+k)^2 - n^2}{(m+k+2)(m+k+1)}\, c_m\,,$$

we see that $\lim_{n\to\infty} c_{m+2}/c_m = 1$, and the ratio test shows that each series has radius of convergence 1; in fact, both converge for $|x| = 1$ (see Arfken & Weber, §5.2). Both series diverge for $|x| > 1$ unless n is an integer, in which case the series terminate at $m = n - k$. (The solution in this case is the Chebyshev polynomial T_n.)

(b) We again seek a series solution of the form

$$y(x) \;=\; \sum_{m=0}^{\infty} c_m\, x^{m+k}\,.$$

Because the differential equation $4x^2y'' + (1-p^2)y = 0$ is homogeneous, substituting this series into the equation implies that

$$[4(m+k)(m+k-1) + (1-p^2)]c_m = 0$$

for all m. Since $c_0 \neq 0$, we obtain

$$4k(k-1) + 1 - p^2 = 0\,,$$

so

$$k = \tfrac{1}{2}(1 \pm p)\,.$$

For $m > 0$, we find

$$4m(m \pm p)c_m = 0\,,$$

so $c_m = 0$ unless $p = \mp m$, in which case $m + k = \frac{1}{2}(1 \mp p)$, that is, the non-vanishing term is just the other power-law solution. Thus the two solutions are

$$y(x) \;=\; x^{\frac{1}{2}(1\pm p)}\,,$$

and these are easily shown to be independent by computing their Wronskian.

3. The first solution of

$$y'' - 2xy' = 0$$

is $y_1(x) = 1$. The Wronskian development gives, for the second solution

$$y_2(x) = y_1(x) \int^x e^{-\int^{x_2} P(x_1)\,dx_1}\,dx_2\,,$$

where $P(x) = -2x$ here. Thus

$$y_2(x) = \int^x e^{x_2^2}\,dx_2 = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)}\,,$$

where C is a constant. Near $x = 0$, $y_2 \sim C + x$.

4. (a) We can write $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, where

$$2\pi c_n = \int_{-\pi}^{\pi} f(x)e^{-inx}dx = \int_{-\pi}^{a} P(x)e^{-inx}dx + \int_{a}^{\pi} Q(x)e^{-inx}dx\,.$$

(It is convenient to work with the exponential form of the series. The result applies equally well to the trigonometric form.) Assuming that P' and Q' exist (which is certainly the case if P and Q are polynomials), integration by parts gives

$$\begin{aligned} 2\pi c_n &= \left[\frac{P(x)}{-in}e^{-inx}\right]_{-\pi}^{a} + \int_{-\pi}^{a} \frac{P'(x)}{in}e^{-inx}dx \\ &\quad + \left[\frac{Q(x)}{-in}e^{-inx}\right]_{a}^{\pi} + \int_{a}^{\pi} \frac{Q'(x)}{in}e^{-inx}dx \\ &= \frac{e^{-ina}}{in}[Q(a) - P(a)] + \frac{1}{in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx\,, \end{aligned}$$

where we have used the fact that $P(-\pi) = Q(\pi)$, by periodicity. If f is discontinuous at $x = a$, then the first term is nonzero and $c_n \sim 1/n$. Otherwise, the first term is zero, and similar arguments applied to f' show that c_n goes to zero at least as fast as $1/n^2$.

(b) From part (a), we expect that the function (f, say) is continuous, but that f' is discontinuous. We also note that f is periodic with period π, odd about $x = 0$ and, in $0 < x < \pi$, is symmetric about $x = \pi/2$. In addition,

$$f'' = -\sin x + \sin 3x - \sin 5x + \dots ,$$

which is reminiscent of a delta function—recall that we can write (subject to the usual caveats)

$$\delta(x) = \sum_{-\infty}^{\infty} \cos nx \, .$$

Heuristically, this suggests that we look for a solution of the form

$$f''(x) = A\left[\delta(x - \tfrac{\pi}{2}) - \delta(x + \tfrac{\pi}{2})\right].$$

Substitution into the usual formulae for the trigonometric Fourier series shows that the coefficient of $\sin nx$ is zero for n even, and $2A(-1)^m/\pi$, for n odd, with $n = 2m + 1$, so we must take $A = -\pi/2$. Integrating the series gives

$$f'(x) = \begin{cases} -\frac{\pi}{2}B & (-\pi < x < -\frac{\pi}{2}), \\ -\frac{\pi}{2}(B-1) & (-\frac{\pi}{2} < x < \frac{\pi}{2}), \\ -\frac{\pi}{2}B & (\frac{\pi}{2} < x < -\pi), \end{cases}$$

where B is an arbitrary constant. Integrating again gives

$$f(x) = C + \begin{cases} -\frac{\pi}{2}B(x+\pi) & (-\pi < x < -\frac{\pi}{2}), \\ -\frac{\pi^2}{4}B - \frac{\pi}{2}(B-1)(x + \frac{\pi}{2}) & (-\frac{\pi}{2} < x < \frac{\pi}{2}), \\ -\frac{\pi^2}{4}(3B-2) - \frac{\pi}{2}B(x - \frac{\pi}{2}) & (\frac{\pi}{2} < x < -\pi), \end{cases}$$

where C is another constant. The condition that $f(-\pi) = 0$ implies $C = 0$. Setting $f(0) = 0$ gives $B = \frac{1}{2}$. Thus we are left with

$$f(x) = \begin{cases} -\frac{\pi}{4}(x+\pi) & (-\pi < x < -\frac{\pi}{2}), \\ \frac{\pi}{4}x & (-\frac{\pi}{2} < x < \frac{\pi}{2}), \\ \frac{\pi}{4}(\pi - x) & (\frac{\pi}{2} < x < -\pi). \end{cases}$$

Note that the integration of the series for f'' is somewhat questionable mathematically, so it is always a good idea to check that the Fourier series of f really is what we expect. It is!