PHYS 502: Mathematical Physics II

Winter 2015, Homework #3

(Due February 9, 2015)

1. (a) On performing separation of variables of Laplace's equation

$$\nabla^2 u = 0$$

in plane polar coordinates, with

$$u(\rho, \phi) = R(\rho)\Phi(\phi),$$

show that the radial function $R(\rho)$ corresponding to angular dependence $\Phi(\phi) = e^{im\phi}$ satisfies the ODE

$$\rho^2 R'' + \rho R' - m^2 R = 0,$$

and that this equation has solutions $R = \rho^{\pm m}$.

(b) Hence write down the general solution to Laplace's equation in polar coordinates.

(c) Find the solution $u(\rho, \theta)$ of Laplace's equation inside a circle of radius a, where u is regular inside the circle and satisfies the boundary conditions

$$u(a, \phi) = U \cos^2 \phi\,.$$

2. Find the three lowest-frequency modes of oscillation of acoustic waves in a hollow sphere of radius R. Assume a boundary condition $\partial u/\partial r = 0$ at $r = R$, where u obeys the differential equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\,.$$

3. The neutron density n inside a spherical sample of fissionable material obeys the equation

$$\nabla^2 n + \lambda n = \frac{1}{\kappa}\frac{\partial n}{\partial t}\,,$$

where $\lambda > 0$, $\kappa > 0$, and $n = 0$ on the surface of the sample.

(a) Suppose the sample is spherical, of radius R. By seeking spherically symmetric modes with time dependence $e^{\alpha t}$, find the critical radius R_0 such that n is unstable and *increases* exponentially with time for $R > R_0$.

(b) Now suppose the sample is a hemisphere, again of radius R. Repeat part (a), for axially symmetric modes.

(c) Two hemispheres of the material, each just barely stable as in part (b), are brought together to form a sphere. This sphere is unstable, with

$$n \sim e^{t/\tau}\,.$$

Find the time constant τ of the resulting explosion.

4. (a) Consider the *homogeneous* two-dimensional Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

where $u(r,\theta)$ is finite inside the circle $r = R$ and satisfies the *inhomogeneous* boundary condition

$$u(R,\theta) = f(\theta)\,,$$

where f is some given function. By writing down the general separable solution to the homogeneous equation, show that the solution may be written in the form

$$u(r,\theta) = \int_0^{2\pi} K(r,\theta,\theta')\, f(\theta')\; d\theta'\,.$$

and determine the function K.

(b) Solve the above equation for $f(\theta) = \cos^2\theta$.