2001 PhD Qualifying Exam — Solutions

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Classical Physics

Short Answers

1. *"Gemini 4, piloted by Jim McDivitt and Ed White, reached orbit on June 3, 1965. In accordance with his flight plan, McDivitt immediately tried to maneuver close to the spent second stage of the launch rocket, which was drifting about 650 feet away.... He found that simply pointing the craft toward the target and firing his rear attitude thrusters did not help him overtake the spent stage.... Indeed, the distance between them actually increased, as if he were pressing a car's gas pedal in forward gear and the car was moving in reverse." (from "Docking in space", by R. Zimmerman, Invention and Technology, Fall 2001, p. 12). Explain. (You may approximate the orbits as circular.)*

   Assuming the spent stage and Gemini 4 are located 650 ft. apart in the same orbit, if McDivitt pointed himself towards the target and fired his thrusters accordingly, he would be increasing his tangential velocity. Since angular momentum is conserved, this would cause the radius of his orbit to increase, thus pushing him further from the target. To reach the target, McDivitt would have to reduce his tangential velocity, fall into a lower orbit, catch up to the spent stage, then increase his tangential velocity to return to the original orbit.

2. *In a famous, recent experiment (reported even in the New York Times), Lene Hau, of Harvard University, sent a pulse of light through a dilute gas (density $\approx 10^{13}$ atoms/cm^3) and measured its group velocity to be a mere 17 m/s (about the speed of a bicycle). Because the index of refraction of such dilute gas is virtually equal to unity, how can we understand such a remarkably small group velocity?*

   The group velocity is dependent on the rate of change of the index of refraction with respect to the wavenumber.
   $$v_g = c\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$
   If the frequency of the light matches the density of the gas, then $\frac{dn}{dk} \gg 0$, and the group velocity will be very small.

3. *Two cannons A and B on a level surface are going to shoot cannon balls that will collide. Cannon A points at an angle θ_A above the horizontal, aiming in the direction of B. Cannon B points back in the direction of A, with an angle θ_B above the horizontal. Ignoring all friction and air resistance, derive an equation for the conditions on the velocities v_A and v_B and the angles θ_A and θ_B such that the cannon balls collide in mid-air.*

   The position of the cannonball ejected from cannon A is
   $$x_A = v_A \cos\theta_A t$$
   $$y_A = v_A \sin\theta_A t - \frac{1}{2}gt^2$$

The position of the cannonball ejected from cannon B is

$$x_B = x_{B0} - v_B \cos\theta_B t$$
$$y_B = v_B \sin\theta_B t - \frac{1}{2}gt^2$$

We have placed cannon A at the origin, and cannon B is somewhere to the right of cannon A ($x_{B0} > 0$). For the two cannonballs to collide midair, $x_A(t_c) = x_B(t_c)$ and $y_A(t_c) = y_B(t_c)$, where t_c is the time of the collision. Evaluating the condition on y,

$$\begin{aligned} y_A(t_c) &= y_B(t_c) \\ v_A \sin\theta_A t_c - \frac{1}{2}gt_c^2 &= v_B \sin\theta_B t_c - \frac{1}{2}gt_c^2 \\ &\boxed{v_A \sin\theta_A = v_B \sin\theta_B} \end{aligned}$$

The condition on the x-coordinate shows

$$\begin{aligned} x_A(t_c) &= x_B(t_c) \\ v_A \cos\theta_A t_c &= x_{B0} - v_B \cos\theta_B t_c \\ t_c(v_A \cos\theta_A + v_B \cos\theta_B) &= x_{B0} \\ t_c &= \frac{x_{B0}}{v_A \cos\theta_A + v_B \cos\theta_B} \end{aligned} \tag{1}$$

Obviously, t_c must be greater than 0 (since the balls must be in the air at the time of the collision). Applying this condition to Eqn. 1 only tells us that $x_{B0} > 0$, which was an initial assumption. To find the upper limit on t_c, we solve for when the y position of both cannon balls equal 0.

$$\begin{aligned} 0 &= v_A \sin\theta_A t - \frac{1}{2}gt^2 \\ &= t(v_A \sin\theta_A - \frac{1}{2}gt) \\ \frac{1}{2}gt_{max} &= v_A \sin\theta_A \\ t_{max} &= 2\frac{v_A}{g}\sin\theta_A \end{aligned}$$

The same time is found when solving for the total flight time of cannonball B.

Applying this time limit to Eqn. 1, we find the second relationship between the velocities and launch angles of the two cannonballs.

$$\begin{aligned} t_c = \frac{x_{B0}}{v_A \cos\theta_A + v_B \cos\theta_B} &< 2\frac{v_A}{g}\sin\theta_A \\ &\boxed{x_{B0} < 2\frac{v_A}{g}\sin\theta_A(v_A \cos\theta_A + v_B \cos\theta_B)} \end{aligned}$$

4. *A thin, flat plate of dielectric with parallel faces has been manufactured by modern techniques so that its index of refraction is a function of distance horizontally along the surface as well as wavelength, i.e., $n = n(x, \lambda)$. Describe the behavior of a beam of white light that strikes the place at normal incidence (i.e. traveling in the z-direction).*

   Striking at normal incidence, the light will travel straight through the medium. The λ-dependence of the index of refraction will cause the travel time for each wavelength to be different, so if a pulse of light were sent into the dielectric, a "rainbow laser" would be emitted. The additional x-dependence of the index of refraction has no influence here, since the light is not changing its x-position.

5. *Which exerts more force on an umbrella: rain drops or hailstones of the same mass and size as the raindrops falling from the same height? Explain your answer.*

   The force exerted on the umbrella by each object is equal to the change in momentum divided by the change in time. The raindrop will result in an inelastic collision, so the total change in momentum will be mv. The time over which this change occurs is rather long, though. For hail, the collision will be elastic, so the total change in momentum is $2mv$, and the time span will be very short. Thus, the hailstone will exert a greater force on the umbrella.

6. *A solid copper box can shield a piece of equipment inside it from radio waves. If the box has holes in it, how large can the holes be and still allow the shielding to be generally effective?*

   Since it is a conductor, the copper box must remain an equipotential surface. This means that the variations of the electromagnetic field must be greater than or equal to the size of the holes in the box (so it still "appears" solid to the field). Therefore, the holes are allowed to be up to the smallest wavelength desired to be blocked from the equipment. Radio waves have a wavelength range of 1 mm to 100 km, so the largest the holes should be are slightly less than 1 mm.

7. *When a bowling ball is sent down a bowling lane, it first slides without rolling and then begins to roll. What determines when the rolling begins?*

   When the bowling ball's linear velocity slows down so that kinetic friction converts to static friction, the bowling ball will start to roll without slipping.

   The torque on the bowling ball when it begins to roll is equal to

$$\begin{aligned}
\tau &= I\alpha \\
r \times F &= \frac{2}{5}mr^2\alpha \\
\mu_s rmg &= \\
\alpha &= \frac{5}{2}\mu_s\frac{g}{r}
\end{aligned}$$

   Knowing the angular acceleration of the ball at the point of rolling, we can calculate the angular velocity of the ball, from which we can find the length of time since the ball first started sliding.

$$\begin{aligned}
\omega &= \omega_0 + \alpha t \\
\frac{v}{r} &= \frac{5}{2}\mu_s\frac{gt}{r} \\
v &= \frac{5}{2}\mu_s gt \\
t &= \frac{2}{5}\frac{v}{\mu_s g}
\end{aligned}$$

   The linear acceleration of the bowling ball is

$$\begin{aligned}
F &= ma \\
-\mu_s mg &= ma \\
a &= -\mu_s g
\end{aligned}$$

Using this acceleration and the time, we can find the velocity of the ball at the beginning of rolling.

$$\begin{aligned}
v &= v_0 + at \\
&= v_0 - \mu_s g \frac{2}{5} \frac{v}{\mu_s g} \\
&= v_0 - \frac{2}{5} v \\
\frac{7}{5} v &= v_0 \\
v &= \frac{5}{7} v_0
\end{aligned}$$

The bowling ball will begin to rotate when its tangential velocity is $\frac{5}{7}$ of its original velocity.

Long Answers

Spring, as normal

A block of mass M is attached to a spring (spring constant k) and slides along a frictionless table. Attached to the side of the block and extending below the table is a mass m attached to a rigid rod of length l. Find the Lagrangian and the equations of motion for the system, using the variables z and θ. Find the normal modes of small vibrations for the system.

The kinetic energy of the large mass M is

$$K = \frac{1}{2} M \dot{z}^2 \tag{2}$$

Its potential energy is due only to the spring:

$$U = \frac{1}{2} k z^2 \tag{3}$$

The x- and y-coordinates of the smaller mass m are

$$\begin{aligned}
x &= z + l \sin\theta \\
y &= -l \cos\theta
\end{aligned}$$

y is defined to be 0 at the surface of the table.

The time derivatives of these positions are

$$\begin{aligned}
\dot{x} &= \dot{z} + l \cos\theta \dot{\theta} \\
\dot{y} &= l \sin\theta \dot{\theta}
\end{aligned}$$

Thus, the kinetic energy of the small mass is

$$\begin{aligned}
K &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \\
&= \frac{1}{2} m \left((\dot{z} + l \cos\theta \dot{\theta})^2 + l^2 \sin^2\theta \dot{\theta}^2 \right) \\
&= \frac{1}{2} m (\dot{z}^2 + 2l \cos\theta \dot{z} \dot{\theta} + l^2 \cos^2\theta \dot{\theta}^2 + l^2 \sin^2\theta \dot{\theta}^2) \\
K &= \frac{1}{2} m (\dot{z}^2 + 2l \cos\theta \dot{z} \dot{\theta} + l^2 \dot{\theta}^2) \qquad (4)
\end{aligned}$$

Its potential energy is

$$U = mgy = -mgl \cos\theta \tag{5}$$

Combining Eqn. 2, 3, 4, and 5, we find the Lagrangian of the system to be

$$\begin{aligned}
\mathcal{L} &= K + U \\
&= \frac{1}{2}M\dot{z}^2 + \frac{1}{2}m\left(\dot{z}^2 + 2l\cos\theta\dot{z}\dot{\theta} + l^2\dot{\theta}^2\right) - \frac{1}{2}kz^2 + mgl\cos\theta
\end{aligned}$$

$$\boxed{\mathcal{L} = \frac{1}{2}(M+m)\dot{z}^2 + ml\cos\theta\dot{z}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}kz^2 + mgl\cos\theta}$$

Using the Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) = \frac{\partial\mathcal{L}}{\partial q}$, we can find the two equations of motion.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) &= \frac{\partial\mathcal{L}}{\partial z} \\
\frac{d}{dt}\left((M+m)\dot{z} + ml\cos\theta\dot{\theta}\right) &= -kz
\end{aligned}$$

$$\boxed{(M+m)\ddot{z} + ml\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) = -kz}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= \frac{\partial\mathcal{L}}{\partial\theta} \\
\frac{d}{dt}\left(ml\cos\theta\dot{z} + ml^2\dot{\theta}\right) &= -ml\sin\theta\dot{z}\dot{\theta} - mgl\sin\theta \\
ml\left(\cos\theta\ddot{z} - \sin\dot{\theta}\dot{z}\right) &=
\end{aligned}$$

$$\boxed{\cos\theta\ddot{z} + l\ddot{\theta} = -g\sin\theta}$$

To find the normal modes, we need to take the small angle approximation on the energies. When expanding the cosines, we want to keep all quadratic terms and lower. The new Lagrangian is then

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{z}^2 + ml\dot{z}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}kz^2 + mgl - \frac{1}{2}mgl\theta^2$$

The corresponding equations of motion are then

$$\begin{aligned}
\ddot{z} &= \frac{m}{M}g\theta - \frac{k}{M}z \\
\ddot{\theta} &= -\frac{(M+m)g}{Ml}\theta + \frac{k}{Ml}z
\end{aligned}$$

In matrix form,

$$\begin{pmatrix}\ddot{z}\\ \ddot{\theta}\end{pmatrix} = \begin{pmatrix}-\frac{k}{M} & \frac{mg}{M}\\ \frac{k}{Ml} & -\frac{(M+m)g}{Ml}\end{pmatrix}\begin{pmatrix}z\\ \theta\end{pmatrix} \tag{6}$$

We are searching for something of the form $\ddot{\phi} = \omega^2\phi$; the only non-zero solutions will be when $\left|\mathbf{L} - \omega^2\mathbf{I}\right| = 0$, where **L** is the coefficient matrix in Eqn. 6.

$$
\begin{aligned}
0 &= \begin{vmatrix} -\frac{k}{M} - \omega^2 & \frac{mg}{M} \\ \frac{k}{Ml} & -\frac{(M+m)g}{Ml} - \omega^2 \end{vmatrix} \\
&= \left(-\frac{k}{M} - \omega^2\right)\left(-\frac{(M+m)g}{Ml} - \omega^2\right) - \left(\frac{mg}{M}\right)\left(\frac{k}{Ml}\right) \\
&= \omega^4 + \left(\frac{k}{M} + \frac{(M+m)g}{Ml}\right)\omega^2 + \frac{kg}{Ml} \\
\omega^2 &= -\frac{1}{2}\left(\frac{k}{M} + \frac{(M+m)g}{Ml}\right) \pm \frac{1}{2}\sqrt{\left(\frac{k}{M} + \frac{(M+m)g}{Ml}\right)^2 - 4\frac{kg}{Ml}}
\end{aligned}
$$

$$
\boxed{\omega^2 = -\frac{1}{2}\left(\frac{k}{M} + \frac{(M+m)g}{Ml}\right) \pm \frac{1}{2}\sqrt{\frac{k^2}{M^2} + \frac{2k(m-M)g}{M^2 l} + \frac{(M+m)^2 g^2}{M^2 l^2}}}
$$

Pardon me, but your belt is slipping

A belt of negligible mass passes between two solid cylinders and is pulled to the right with a force P. It has acceleration a to the right. The top cylinder, cylinder A, of radius $R_A = 50$ mm, is free to move vertically, as well as rotate, in the bearing blocks that support its shaft; its mass is 6 kg. Cylinder B of radius $R_B = 120$ mm is held in its bearings so it only rotates; its mass is 24 kg. Coefficients of friction between the belt and the cylinders are unknown, but it is known that only rolling occurs, without any slippage of the cylinders on the belt. Neglect the friction in the bearings on the shafts of the cylinder.

a. *If the acceleration of the belt is $a = 1.5m/s^2$ to the right, determine the angular accelerations α_A and α_B of the cylinders A and B. Indicate the directions of these.*

   Angular acceleration is related to linear acceleration by the formula $\alpha = \frac{a}{r}$, so

$$
\begin{aligned}
\alpha_A &= \frac{a}{r_A} = \frac{1.5\text{m/s}^2}{50\text{mm}} = & \boxed{30\text{rad/s}^2 = \alpha_\text{A}} \\
\alpha_B &= \frac{a}{r_B} = \frac{1.5\text{m/s}^2}{120\text{mm}} = & \boxed{12.5\text{rad/s}^2 = \alpha_\text{B}}
\end{aligned}
$$

   α_A is counter-clockwise, while α_B is clockwise.

b. *What is the static frictional force between the belt and the cylinder B so there is no slippage? (Static friction provides the necessary torque.)*

   The torque on the cylinder is equal to the cross product of the radius and the frictional force. Torque is also defined as the product of the angular acceleration and the moment of inertia of the cylinder. Setting these equal to each other, we can solve for the frictional force.

$$
\begin{aligned}
\tau = \alpha_B I &= r_B \times f_B \\
f_B = \frac{1}{2} M_B r_B \alpha_B &= \frac{1}{2} a M_B \qquad (7)
\end{aligned}
$$

   For cylinder B, where $a = 1.5\text{m/s}^2$ and $M_B = 24\text{kg}$, the frictional force is 18 N.

c. *What is the static frictional force between the belt and the cylinder A so there is no slippage? (Static friction provides the necessary torque.)*

   Using Eqn. 7 with the values for cylinder A ($M_A = 6\text{kg}$), the frictional force between the belt and cylinder A is 4.5 N.

d. *What is the magnitude of the required force P to perform this task?*

The force P on the end of the belt must be equal to the sum of the two frictional forces between itself and the cylinders, or 22.5 N.

e. *If the maximum acceleration of the belt is $a_{max} = 2.0m/s^2$ before any slipping occurs, find the coefficient of static friction μ_s between the cylinder B and the belt.*

The frictional force is equal to the product of the coefficient of friction and the normal force. For cylinder B, the normal force responsible for the friction is the weight of cylinder A, so

$$f_B = \mu_s M_A g$$
$$\mu_s = \frac{f_B}{M_A g} = \frac{a_{max} M_B}{2 M_A g}$$

The coefficient of static friction between cylinder B and the belt must be 0.408.

f. *What is the maximum angular acceleration of the cylinder A for $a_{max} = 2.0\ m/s^2$?*

Using the same relation as in part (a), the maximum angular acceleration of cylinder A is 40 rad/s^2.

Unexpected support

A battery of emf $\varepsilon = 24\ V$ and internal resistance $r = 0.45\Omega$ is connected in series with a variable resistance R', two conducting vertical springs each of spring constant $k = 19.62\ N/m$, and a linear conductor of length $l = 120\ cm$, mass $m = 80\ g$ and resistance $R = 2.55\Omega$ as shown. Neglect the mass and the resistance of the springs. (The internal resistance is not drawn.)

a. *What is the extension (Δy) in each spring due to the weight of the conductor?*

With the switch S open, the only force on the springs is the weight of the conductor. Balancing these three forces, we have

$$2F_s = mg$$
$$2kx =$$
$$x = \frac{mg}{2k} = \frac{(80\text{ g})(9.8\text{ m/s}^2)}{2(19.62\text{ N/m})}$$
$$\boxed{\Delta y = 0.01998\text{ m}}$$

b. *Switch S is closed, and a uniform magnetic field of 0.436 T is applied to reduce the elastic force on the spring to zero. Find the direction of the applied magnetic field and the magnitude of current needed to reduce the elastic force on the spring to zero. Also determine the value of the variable resistor R'.*

The current in the circuit is in the counter-clockwise direction, so the current through the conductor is flowing from left to right. For the magnetic force to counteract gravity (so up), by the right-hand rule the applied magnetic field must be into the page. To perfectly cancel out the gravitational force,

$$F_g = F_B$$
$$mg = Il \times B$$
$$I = \frac{mg}{lB} = \frac{(80\text{ g})(9.8\text{ m/s}^2)}{(120\text{ cm})(0.436\text{ T})}$$
$$\boxed{I = 1.498\text{ A}}$$

Applying Kirchoff's voltage law to the circuit,

$$
\begin{aligned}
0 &= \varepsilon - IR - IR' - Ir \\
IR' &= \varepsilon - I(R + r) \\
R' &= \frac{\varepsilon}{I} - (R + r) = \frac{24 \text{ V}}{1.498 \text{ A}} - (2.55\Omega + 0.45\Omega) \\
&\boxed{R' = 13.02\Omega}
\end{aligned}
$$

c. *The magnitude and direction of the applied magnetic field remain unchanged as above. The value of R' is adjusted to $\frac{7}{3}\Omega$. What is the length of the spring if the unstretched length for each spring is 60 cm?*

If $R' = \frac{7}{3}\Omega$, then the current will be (using Kirchoff's law)

$$
\begin{aligned}
0 &= \varepsilon - I(R + R' + r) \\
I &= \frac{\varepsilon}{R + R' + r} = \frac{24 \text{ V}}{2.55\Omega + \frac{7}{3}\Omega + 0.45\Omega} = 4.5 \text{ A}
\end{aligned}
$$

The force on the conductor is then

$$F = Il \times B = (4.5 \text{ A})(120 \text{ cm})(0.436 \text{ T}) = 2.354 \text{ N up}$$

The total force on both springs is then

$$\sum F = F_B - F_g = 2.354 \text{ N} - (80 \text{ g})(9.8 \text{ m/s}^2) = 1.570 \text{ N up}$$

Each spring has half the total force, so $F_s = 0.7852$ N up.

$$
\begin{aligned}
F_s &= kx \\
x &= \frac{F_s}{k} = \frac{0.7852 \text{ N}}{19.62 \text{ N/m}} = 0.0400 \text{ m}
\end{aligned}
$$

If the unstretched length of one of the springs is 60 cm, then the total length now $L = 60 \text{ cm} - 0.0400 \text{ m} = 0.5599$ m.

d. *Magnetic field stays the same as in part (b) with $B = 0.436$ T and the variable resistance is set at 7Ω. Switch S is kept open. A capacitor $C = 50\mu F$ is connected across the circuit. Clock is reset to $t = 0$ and at this instant the switch S is closed. Find the initial current through the resistor R, the initial extension or compression in the spring and the charge on the capacitor C.*

Initially, $Q = 0$ on the capacitor, so there is no voltage drop across it. This makes the current through the linear conductor $I_3 = 0$. The spring extension will be caused just by gravity, so $\Delta y = 0.01998$ m.

As time passes, the capacitor will charge exponentially. Writing out the three equations from Kirchoff's laws, we know that

$$0 = \varepsilon - I_1 r - I_1 R' - \frac{Q}{C} \tag{8}$$

$$0 = -\frac{Q}{C} + I_3 R \tag{9}$$

$$I_1 = I_2 + I_3 \tag{10}$$

where I_1 is the current through the battery and the variable resistor, I_2 is the current through the capacitor, and I_3 is the current through the linear conductor.

From Eqn. 9, we see that $Q = I_3 RC$. Substituting this and Eqn. 10 into Eqn. 8, we find that

$$I_3 = \frac{\varepsilon - I_2(r + R')}{r + R' + R}$$

Substituting this expression for I_3 back into that for Q, we see that

$$Q = \frac{RC(\varepsilon - I_2(r + R'))}{r + R' + R}$$
$$q = \frac{RC\varepsilon}{r + R' + R} - \frac{RC(r + R')}{r + R' + R}\frac{dq}{dt}$$

Isolating the q's from the t,

$$dt = \frac{-\frac{RC(r+R')}{r+R'+R}}{q - \frac{RC\varepsilon}{r+R'+R}} dq$$

Integrating both sides ($0 < t < t$ and $0 < q < Q$), we find that

$$t = -\frac{RC(r + R')}{r + R' + R} \ln\left(\frac{Q - \frac{RC\varepsilon}{r+R'+R}}{-\frac{RC\varepsilon}{r+R'+R}}\right)$$

Solving for Q, we find that

$$Q = \frac{RC\varepsilon}{r + R' + R}\left(1 - e^{-\frac{r+R'+R}{RC(r+R')}t}\right)$$
$$Q = 3.06 \times 10^{-4}\mathrm{C}\left(1 - e^{-10527.70\mathrm{s}^{-1}t}\right)$$

So, at $t = \infty$, $Q = 3.06 \times 10^{-4}$C, $I_3 = 2.4$A, and $\Delta y = 0.0120$m.

Doing the wave with a group

Maxwell's wave equation for the electric field in a non-magnetic medium has the well-known form

$$\nabla_2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2\epsilon_0}\frac{\partial^2 P}{\partial t^2} \tag{11}$$

with the conventional meaning of the symbols. If the medium is a dilute dielectric with a single resonance, it may be modeled as a distribution of N positive and N negative charges per unit volume. The positive charges are immobile and each negative charge is tied to a positive charge by an elastic force with resonance frequency ω_0. Thus, the equation of motion for each negative charge is

$$\left(\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right) R = -\frac{q}{m}E \tag{12}$$

where m is the mass, γ a phenomenological damping constant, R the displacement of the negative charge from the corresponding positive charge, and q is the absolute value of the charge; E is the electric field at the location of the charge. The macroscopic induced polarization in the medium is given by

$$P = -qNR$$

at the location of the charge under consideration.

a. *We want to solve the coupled equations 11 and 12 at steady state for the case of plane polarized electric and polarization waves, propagating in the positive direction of the z-axis, i.e.*

$$E(z,t) = \hat{i} E_{st} e^{i(kz-\omega t)} + c.c. \tag{13a}$$

$$P(z,t) = \hat{i} P_{st} e^{i(kz-\omega t)} + c.c. \tag{13b}$$

Find the dispersion relation $k = k(\omega)$ so that the above waves (13a, 13b) satisfy Eqns. 11 and 12.

Substituting in the expressions for E and P into Eqns. 11 and 12, we will be able to solve for k. Start by first taking the Laplacian of E.

$$\begin{aligned} \nabla^2 E &= \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} \\ &= \frac{\partial}{\partial z}\left(E_{st} ik e^{i(kz-\omega t)} \hat{i}\right) \\ \nabla^2 E &= -E_{st} k^2 e^{i(kz-\omega t)} \hat{i} \end{aligned}$$

We also need the second time derivative of E.

$$\begin{aligned} \frac{\partial^2 E}{\partial t^2} &= \frac{\partial}{\partial t}\left(E_{st}(-i\omega) e^{i(kz-\omega t)} \hat{i}\right) \\ \frac{\partial^2 E}{\partial t^2} &= -\omega^2 E_{st} e^{i(kz-\omega t)} \hat{i} \end{aligned}$$

The second time derivative of P is

$$\begin{aligned} \frac{\partial^2 P}{\partial t^2} &= \frac{\partial}{\partial t}\left(P_{st}(-i\omega) e^{i(kz-\omega t)} \hat{i}\right) \\ \frac{\partial^2 P}{\partial t^2} &= -\omega^2 P_{st} e^{i(kz-\omega t)} \hat{i} \end{aligned}$$

Inserting these expressions into Eqn. 11, we find that

$$\left(\frac{\omega^2}{c^2} - k^2\right) E_{st} = -\frac{\omega^2}{c^2 \epsilon_0} P_{st} \tag{14}$$

Likewise, when substituting these expressions into Eqn. 12, we find that

$$\begin{aligned} (-\omega^2 - i\omega\gamma + \omega_0^2) P_{st} &= \frac{q^2 N}{m} E_{st} \\ E_{st} &= \frac{m}{q^2 N}(-\omega^2 - i\omega\gamma + \omega_0^2) P_{st} \end{aligned}$$

Inserting this back into Eqn. 14 and solving for k, we find that

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{q^2 N}{m\epsilon_0(\omega^2 + i\omega\gamma - \omega_0^2)}\right)}$$

b. *Under the generally valid assumption $\gamma \ll \omega_0, \omega$ and in the frequency range $|\omega - \omega_0| \approx \gamma$, calculate the phase and group velocity of the electromagnetic wave in the medium.*

Under the stated assumption, our expression for k^2 becomes

$$k^2 \approx \frac{\omega^2}{c^2}\left(1 - \frac{q^2 N}{m\epsilon_0\left(\gamma(\omega + \omega_0) + i\omega\gamma\right)}\right)$$

In the given frequency range,

$$k^2 \approx \frac{\omega^2}{c^2}\left(1 - \frac{q^2 N}{m\epsilon_0(\omega + \omega_0 + i\omega)}\right) \tag{15}$$

The phase velocity is then

$$\begin{aligned} v_p &= \frac{\omega}{k} \\ &= \frac{c\omega}{\omega\sqrt{1 - \frac{q^2N}{m\epsilon_0(\omega+\omega_0+i\omega)}}} \\ &\boxed{v_p = c\left(1 - \frac{q^2 N}{m\epsilon_0(\omega+\omega_0+i\omega)}\right)^{-1/2}} \end{aligned}$$

Going back to Eqn. 15, we can rearrange it to make it slightly easier to manage.

$$0 = \omega^3 + \frac{\omega_0}{1+i}\omega^2 - c^2k^2\omega - \frac{q^2N + c^2k^2m\epsilon_0\omega_0}{m\epsilon_0(1+i)}$$

Since the group velocity is defined as $v_g = \frac{\partial\omega}{\partial k}$, we can take the implicit derivative of the above equation to solve for v_g.

$$\begin{aligned} 0 &= 2m\epsilon_0\omega_0\omega\frac{\partial\omega}{\partial k} + 3m\epsilon_0(1+i)\omega^2\frac{\partial\omega}{\partial k} - 2c^2km\epsilon_0\omega_0 - c^2m\epsilon_0(1+i)\left(k^2\frac{\partial\omega}{\partial k} + 2k\omega\right) \\ &= 2\omega_0\omega v_p + 3(1+i)\omega^2 v_p - 2c^2k\omega_0 - c^2(1+i)(k^2v_p + 2k\omega) \\ &\boxed{v_p = \frac{2c^2k\omega_0 + 2c^2k\omega(1+i)}{2\omega_0\omega + 3(1+i)\omega^2 - c^2k^2(1+i)}} \end{aligned}$$

Hint for part b: for a dilute medium (e.g. a gas) the index of refraction is very close to unity.

Charging an egg

a. *How does charge distribute itself on and within an isolated conductor?*

Charge distributes itself so as to minimize the potential energy of the system. For a charged conductor, this translates to no electric field on the inside of the isolated conductor. All charge is found on the surface.

b. *How and why does such a distribution occur?*

If the conductor is initially in a state such that there is an electric field within, the electric field will create a force on the electrons, moving them around until there is no longer an electric field.

c. *A certain conductor has the shape of an ellipsoid described by the equation*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

where $a < b < c$. What is the ratio of the maximum to minimum charge densities on the conductor's surface?

The maximum charge density will be at the ends of the ellipsoid ($z = \pm c$), while the minimum charge density will be at the sides ($x = \pm a$). The surface of the conductor is a surface of equipotential (to minimize the potential energy).

Take the surface of the ellipsoid to be the small volume between two ellipsoids of nearly the same size. The total charge in the ellipsoid Q is

$$Q = \rho V = \rho \left(\frac{4}{3}\pi(a+\delta a)(b+\delta b)(c+\delta c) - \frac{4}{3}\pi abc \right) \tag{16}$$

We can relate all the additional expansions by the similarity

$$\frac{\delta a}{a} = \frac{\delta b}{b} = \frac{\delta c}{c} = \frac{\delta d}{d}$$

where δd is the surface thickness and d is the distance from the center of the ellipsoid to the plane tangent to a point on the surface. Using this similarity to rewrite Eqn. 16,

$$\begin{aligned}
Q &= \rho \frac{4}{3}\pi abc \left(\left(1 + \frac{\delta d}{d} \right)^3 - 1 \right) \\
&= 4\pi \frac{abc}{d} \rho \delta d \\
&= \frac{4\pi abc}{d} \sigma \\
\sigma &= \frac{Qd}{4\pi abc}
\end{aligned}$$

Therefore, $\boxed{\frac{\sigma_c}{\sigma_a} = \frac{c}{a}}$.

Modern Physics

Short Answers

1. *What is the rule of thumb that enables you to decide if a state of N identical particles needs to be symmetrized or not?*

   If they are bosons, the state needs to be symmetric. If they are fermions, the state needs to be anti-symmetric.

2. *It is believed that electron neutrinos can transform into muon neutrinos. Assume an electron neutrino is in an initial state*

   $$|\psi(t=0)\rangle = |e\rangle = \cos\theta|m_1\rangle - \sin\theta|m_2\rangle$$

   in which we have the mass eigenstates, related to electron and muon states, as

   $$\begin{aligned}
   |m_1\rangle &= \cos\theta|e\rangle + \sin\theta|\mu\rangle \\
   |m_2\rangle &= -\sin\theta|m_1\rangle + \cos\theta|\mu\rangle
   \end{aligned}$$

   and θ is the mixing angle. The masses are slightly different: $\delta m^2 = m_2^2 - m_1^2$ and we assume that $\frac{\delta m^2}{m^2} \ll 1$. The energy of a neutrino of mass m_i is given by $E_i^2 = p^2c^2 + m_i^2c^4$. The initial electron neutrino travels a distance ct and evolves into a state

   $$|\psi(t)\rangle = \cos\theta e^{-iE_1t/}|m_1\rangle - \sin\theta e^{-iE_2t/}|m_2\rangle$$

   Find the probability for transformation from an electron to a muon neutrino state and show that it vanishes as mass difference vanishes.

After a time t, the state will have evolved so that

$$|\psi(t)\rangle = \cos\theta e^{-iE_1 t/\hbar}|m_1\rangle - \sin\theta e^{-iE_2 t/\hbar}|m_2\rangle$$

Writing this in terms of the electron and muon states,

$$|\psi(t)\rangle = \left(\cos^2\theta e^{-iE_1 t/\hbar} + \sin^2\theta e^{-iE_2 t/\hbar}\right)|e\rangle + \sin\theta\cos\theta\left(e^{-iE_1 t/\hbar} - e^{-iE_2 t/\hbar}\right)|\mu\rangle$$

The probability of the muon transforming to a muon state is then

$$\begin{aligned} P_\mu &= \sin^2\theta\cos^2\theta\left(e^{-iE_1 t/\hbar} - e^{-iE_2 t/\hbar}\right)\left(e^{iE_1 t/\hbar} - e^{iE_2 t/\hbar}\right) \\ P_\mu &= 2\sin^2\theta\cos^2\theta(1 - \cos(E_2 - E_1)t) \end{aligned} \tag{17}$$

From the definition of the energy, we can write

$$\begin{aligned} (E_2 + E_1)(E_2 - E_1) &= p^2c^2 + m_2^2c^4 - p^2c^2 + m_1^2c^4 \\ &= (m_2^2 - m_1^2)c^4 \\ &= \delta m^2 c^4 \\ E_2 - E_1 &= \frac{\delta m^2 c^4}{E_2 + E_1} \end{aligned}$$

Substituting this into Eqn. 17, we see that

$$\boxed{P_\mu = 2\sin^2\theta\cos^2\theta\left(1 - \cos\frac{\delta m^2 c^4}{E_2 + E_1}\right)}$$

As can be seen, as $\delta m^2 \to 0, P_\mu \to 2\sin^2\theta\cos^2\theta(1 - \cos 0) = 0$.

3. *The atom of sodium has one electron outside a closed shell. Thus, roughly, we can view this system as a one-electron atom. Provide convincing arguments in support of the following spectroscopic fact: in the absence of an applied magnetic field each s-state of sodium ($l = 0$) consists of a single line, whereas the p, d, f, etc. states are split into closely spaced but readily resolvable doublets.*

   (For this question to make sense, you must assume two things:

   1. you are taking the spectrum of a gas of sodium atoms, so that you can see more than one transition line, and
   2. there is something present to excite the atoms (or there was something once), allowing transitions to occur.)

   Allowed transitions of this single electron must follow the electric dipole selection rule $\Delta l = \pm 1$. As such, when the electron is transitioning to the p, d, f etc. states, it can be coming from two states — this will result in two lines in a spectrum. However, when the electron is moving to/from the s-state, it can only be coming from / going to the next-higher p-state, so there will only be one line in the spectrum at this energy.

4. *If a beam of spin-1/2 particles (e.g. silver atoms) passes through a Stern-Gerlach analyzer, one generally expects to observe two spots at the exit of the device. Suppose one constructed a double Stern-Gerlach system in which the beam passed simultaneously through two magnetic fields crossed at 90° to one another, rather than just one. How many spots would one expect to see at the exit of this device? (Just a brief answer is needed.)*

Since the two magnetic fields are acting on the beam simultaneously, they will add together, acting like a single magnetic field on the particles. Thus, the beam will only split once, so there will be two points seen at the exit of the device.

5. *What is the approximate rms velocity of a molecule of the air in this room? You may take its mass as* $30m_p$*, where the mass of the proton is* 1.67×10^{-27} *kg, and Boltzmann's constant is* 1.4×10^{-23} *J/K.*

   A diatomic molecule has five degrees of freedom, so its average energy $E = \frac{5}{2}kT$. Setting this equal to its kinetic energy,

$$\frac{1}{2}mv^2 = \frac{5}{2}kT$$
$$v^2 = \frac{5kT}{m}$$
$$\boxed{v = \sqrt{\frac{5(1.4 \times 10^{-23}\ \text{J/K})(300\ \text{K})}{30(1.67 \times 10^{-27}\ \text{kg})}} = 647.43\ \text{m/s}}$$

6. *A tire has a gage pressure of 200 kPa at* 25° *C. If the atmospheric pressure is 100 kPa, find the gage pressure in the tire at a temperature of* 50° *C.*

   Assuming an ideal gas, $PV = NkT$, or $\frac{P}{T} = \frac{Nk}{V}$. Neither the number of particles nor the volume are changing, so the ratio of the pressure to the temperature is constant. The pressure of the gas inside the tire is equal to the sum of the atmospheric pressure and the gage pressure, as the gage measures the pressure of the gas relative to the air pressure. So

$$\frac{P_1}{T_1} = \frac{P_2}{T_2}$$
$$\frac{P_{atm} + P_{25}}{T_{25}} = \frac{P_{atm} + P_{50}}{T_{50}}$$
$$P_{50} = P_{atm} + 2P_{25} = 100\ \text{kPa} + 2(200\ \text{kPa})$$
$$\boxed{P_{50} = 500\ \text{kPa}}$$

7. *There are* N *systems* $A, B, C, ...$ *which are in equilibrium with the same heat bath. If these systems are nearly independent of each other so that they can be considered as a compound system* $A + B + C + ...$*, show that the partition function and the (Helmholtz) free energy of the compound system can be expressed as* $Z_{A+B+C+...} = Z_A Z_B Z_C ...$ *and* $F_{A+B+C+...} = F_A + F_B + F_C + ...$*, where* $Z_A, Z_B, ...$ *and* $F_A, F_B, ...$ *are the partition functions and free energies of the individual systems, respectively.*

   The partition function is equal to

$$Z = \sum_s e^{-\beta E_s}$$

   Each energy state of the entire system is equal to the sum of all the energy states of each smaller system, so $E_s = E_{As} + E_{Bs} + E_{Cs} + \dots$ and

$$Z = \sum_s e^{-\beta(E_{As} + E_{Bs} + E_{Cs} + \dots)}$$
$$= \sum_s e^{-\beta E_{As}} e^{-\beta E_{Bs}} e^{-\beta E_{Cs}} \dots$$
$$\boxed{Z = Z_A Z_B Z_C \dots}$$

   The Helmholtz free energy is defined as

$$F = kT \ln Z$$

With the above partition function, this becomes

$$
\begin{aligned}
F &= kT\ln(Z_A Z_B Z_C \dots) \\
&= kT(\ln Z_A + \ln Z_B + \ln Z_C + \dots) \\
&\boxed{F = F_A + F_B + F_C + \dots}
\end{aligned}
$$

Long Problems

Split personality

The device shown has been constructed on a nanoscale and charged particles such as electrons can enter from the left, split at a and go down any (or all!) of the three legs of this interferometer, so the amplitudes recombine at b and the flow can emerge on the right. A circular wire can support a magnetic field B_1 between legs 1 and 2. Another circular wire can support a magnetic field B_2 between legs 2 and 3. The magnetic field is zero in each of the individual legs.

According to Feynman, the wavefunction $\Psi(b,t)$ is related to the wavefunction $\Psi(a,t-\Delta t)$ by $\Psi(b,t) = C\Psi(a,t-\Delta t)$ in which

$$C \sim \sum_{\textit{all paths}} e^{-\frac{i}{\hbar}S_{cl}(b,a\Delta t)}$$

and the action along a classical path from a to b in the time interval Δt is

$$S_{cl} = \int_{a,t-\Delta t}^{b,t} (\frac{m}{2}v^2 + \frac{e}{c}A\cdot v)dt$$

where A is the vector potential describing the magnetic field distribution. For our purposes, the only paths we need to consider are the classically allowed paths from a to b in the three legs of this interferometer. Assume the distance from a to b is L in leg 2, and $L\sqrt{2}$ in legs 1 and 3.

a. *Describe the simplest classically allowed path in each of the three legs (i.e., $a \to b$, neglect $a \to b \to a \to b$ and higher order paths).*

   The particles can travel from a along path 1 to b, from a along path 2 to b, or from a along path 3 to b.

b. *Show that the action along this path in leg 2 is $S_2 = \frac{mL^2}{2\Delta t} + \frac{e}{c}\int_2 A\cdot dx$ and evaluate the action along paths 1 and 3.*

$$
\begin{aligned}
S_2 &= \int_{0,0}^{L,t} \left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{e}{c}A\cdot\frac{dx}{dt}\right)dt \\
&= \frac{1}{2}m\int_0^L v dx + \frac{e}{c}\int_2 A\cdot dx \\
&= \frac{1}{2}mvL + \frac{e}{c}\int_2 A\cdot dx \\
&\boxed{S_2 = \frac{1}{2}m\frac{L^2}{\Delta t} + \frac{e}{c}\int_2 A\cdot dx}
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \int_0^{\sqrt{2}L} \left(\frac{1}{2}mv\frac{dx}{dt} + \frac{e}{c}A\cdot\frac{dx}{dt}\right) dt \\
&= \frac{1}{2}mv\sqrt{2}L + \frac{e}{c}\int_1 A\cdot dx \\
&= \frac{2mL^2}{2\Delta t} + \frac{e}{c}\int_1 A\cdot dx
\end{aligned}
$$

$$
\boxed{S_1 = \frac{mL^2}{\Delta t} + \frac{e}{c}\int_1 A\cdot dx}
$$

Likewise, $\boxed{S_3 = \frac{mL^2}{\Delta t} + \frac{e}{c}\int_3 A\cdot dx}$.

c. *Show that* $\Psi(b,t) = N\left(e^{iS_1/\hbar} + e^{iS_2/\hbar} + e^{iS_3/\hbar}\right)\Psi(a, t-\Delta t)$ *and therefore that*

$$
|\Psi(b,t)|^2 = N^2\left(3 + 2\sum_{i>j=1}^{3}\cos\left(\frac{S_i - S_j}{\hbar}\right)\right)|\Psi(a,t-\Delta t)|^2
$$

where N *is some normalization constant which you need not compute.*

$$
C \approx \sum e^{-\frac{i}{\hbar}S_{cl}(b,a,\Delta t)} = e^{-iS_1/\hbar} + e^{-iS_2/\hbar} + e^{-iS_3/\hbar}
$$

Therefore,

$$
\Psi(b,t) = C\Psi(a,t-\Delta t)
$$

$$
\boxed{\Psi(b,t) = N\left(e^{-iS_1/\hbar} + e^{-iS_2/\hbar} + e^{-iS_3/\hbar}\right)\Psi(a,t-\Delta t)}
$$

Squaring this,

$$
\begin{aligned}
|\Psi(b,t)|^2 &= \left[N\left(e^{iS_1/\hbar} + e^{iS_2/\hbar} + e^{iS_3/\hbar}\right)\Psi^*(a,t-\Delta t)\right]\left[N\left(e^{-iS_1/\hbar} + e^{-iS_2/\hbar} + e^{-iS_3/\hbar}\right)\Psi(a,t-\Delta t)\right] \\
&= N^2\left(3 + 2\cos\left(\frac{S_2 - S_1}{\hbar}\right) + 2\cos\left(\frac{S_3 - S_1}{\hbar}\right) + 2\cos\left(\frac{S_3 - S_2}{\hbar}\right)\right)|\Psi(a,t-\Delta t)|^2
\end{aligned}
$$

$$
\boxed{|\Psi(b,t)|^2 = N^2\left(3 + 2\sum_{i>j=1}^{3}\cos\left(\frac{S_i - S_j}{\hbar}\right)\right)|\Psi(a,t-\Delta t)|^2}
$$

d. *By recalling* $\oint A\cdot dS = \iint B\cdot dA = \Phi$, *where* Φ *is the magnetic flux, show that* $\cos\left(\frac{S_1 - S_2}{\hbar}\right) = \cos\left(\frac{mL^2}{2\hbar\Delta t} - \frac{e\Phi_1}{\hbar c}\right)$ *and then evaluate the other two cosine terms.*

$$
\begin{aligned}
S_2 - S_1 &= \frac{mL^2}{2\Delta t} + \frac{e}{c}\int_2 A\cdot dx - \frac{mL^2}{\Delta t} - \frac{e}{c}\int_1 A\cdot dx \\
&= -\frac{mL^2}{2\Delta t} + \frac{e}{c}\left(\int_2 A\cdot dx - \int_1 A\cdot dx\right) \\
&= -\frac{mL^2}{2\Delta t} + \frac{e}{c}\oint A_1\cdot ds \\
S_2 - S_1 &= -\frac{mL^2}{2\Delta t} + \frac{e}{c}\Phi_1
\end{aligned}
$$

Therefore,

$$
\boxed{\cos\left(\frac{S_2 - S_1}{\hbar}\right) = \cos\left(-\frac{mL^2}{2\hbar\Delta t} + \frac{e}{\hbar c}\Phi_1\right)}
$$

$$
\begin{aligned}
S_3 - S_2 &= \frac{mL^2}{\Delta t} + \frac{e}{c}\int_3 A\cdot dx - \frac{mL^2}{2\Delta t} - \frac{e}{c}\int_2 A\cdot dx \\
&= \frac{mL^2}{2\Delta t} + \frac{e}{c}\left(\int_3 A\cdot dx - \int_2 A\cdot dx\right) \\
&= \frac{mL^2}{2\Delta t} + \frac{e}{c}\Phi_2
\end{aligned}
$$

$$
\boxed{\cos\left(\frac{S_3 - S_2}{\hbar}\right) = \cos\left(\frac{mL^2}{2\hbar\Delta t} + \frac{e}{c\hbar}\Phi_2\right)}
$$

$$
\begin{aligned}
S_3 - S_1 &= \frac{mL^2}{\Delta t} + \frac{e}{c}\int_3 A\cdot dx - \frac{mL^2}{\Delta t} - \frac{e}{c}\int_1 A\cdot dx \\
&= \frac{e}{c}\left(\int_3 A\cdot dx - \int_1 A\cdot dx\right) \\
&= \frac{e}{c}(\Phi_1 + \Phi_2)
\end{aligned}
$$

$$
\boxed{\cos\left(\frac{S_3 - S_1}{\hbar}\right) = \cos\left(\frac{e}{c}(\Phi_1 + \Phi_2)\right)}
$$

Did you check with the operator?

Consider a spin-1/2 particle. Call its spin operator S, its orbital angular momentum L, and its state vector $|\psi\rangle$. The two functions $\psi_+(r)$ and $\psi_-(r)$ are defined by

$$
\psi_\pm(r) = \langle r, \pm|\psi\rangle
$$

Assume that

$$
\begin{aligned}
\psi_+(r) &= \sqrt{\frac{1}{14}}R(r)\left[2Y_0^0(\theta,\phi) - i\sqrt{3}Y_1^2(\theta,\phi)\right] \\
\psi_-(r) &= \frac{1}{2}R(r)\left[Y_0^0(\theta,\phi) + 2iY_1^1(\theta,\phi) + \sqrt{3}Y_1^{-1}(\theta,\phi)\right]
\end{aligned}
$$

a. *What condition must $R(r)$ satisfy for the state $|\psi\rangle$ to be normalized?*

For $|\psi\rangle$ to be normalized, we must have $1 = \langle\psi|\psi\rangle = \langle\psi|r\rangle\langle r|\psi\rangle$. Since this must be true for all values of $|\psi\rangle$, we can choose to run the calculation with either ψ_+ or ψ_-.

$$\begin{aligned}
1 &= \langle\psi|r\rangle\langle r|\psi\rangle \\
&= \int \psi_+^* \psi_+ r^2 \sin\theta dr d\theta d\phi \\
&= \frac{1}{14}\int_0^\infty r^2 R(r)^2 dr \int_0^{2\pi}\int_0^\pi \left(2Y_0^{0*} + i\sqrt{3}Y_2^{1*}\right)\left(2Y_0^0 - i\sqrt{3}Y_2^1\right)\sin\theta d\theta d\phi \\
14 &= \int_0^\infty r^2 R(r)^2 dr \int_0^{2\pi}\int_0^\pi \left(4|Y_0^0|^2 + 3|Y_2^1|^2 - 2i\sqrt{3}Y_0^{0*}Y_2^1 + 2i\sqrt{3}Y_0^0 Y_2^{1*}\right)\sin\theta d\theta d\phi
\end{aligned}$$

With $Y_0^0 = \sqrt{\frac{1}{4\pi}}$ and $Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$, the double integral works out so that

$$14 = 7\int_0^\infty r^2 R(r)^2 dr$$

$$\boxed{2 = \int_0^\infty r^2 R(r)^2 dr}$$

b. *S_z is measured with the particle in state $|\psi\rangle$. What results can be found and with what probabilities?*

It is (common) knowledge that $S_z|\psi\rangle = \hbar m_s|\psi\rangle$. For a particle of spin $\frac{1}{2}$, $s = \frac{1}{2}$, so $m_s = \pm\frac{1}{2}$. Therefore, the possible measured results will be $\pm\frac{\hbar}{2}$. There is no information provided about the probabilities of measuring S_z, since $\psi_{+,-}$ only carries details on the position of the particle. We cannot provide the probabilities of measuring these values.

c. *L_z is measured with the particle in state $|\psi\rangle$. What results can be found and with what probabilities?*

Again, it should be known that $L_z|\psi\rangle = \hbar m_l|\psi\rangle$. Given the two equations ψ_+ and ψ_-, we can see from the spherical harmonics that the possible values of m_l are $0, \pm1$. Assuming $|\psi\rangle$ is some linear combination of the two position wave functions ($|\psi\rangle = a\psi_+ + b\psi_-$), the probability of measuring 0 is

$$P_0 = \left|\frac{2a}{\sqrt{14}} + \frac{b}{2}\right|^2 = \frac{2a^2}{7} + \frac{b^2}{4} + \frac{2ab}{\sqrt{14}}$$

Likewise, the probability of measuring $\hbar$ is

$$P_1 = \left|-i\sqrt{\frac{3}{14}}a + ib\right|^2 = \frac{3a^2}{14} + b^2$$

The probability of measuring $-\hbar$ is

$$P_{-1} = \left|\frac{\sqrt{3}}{2}b\right|^2 = \frac{3b^2}{4}$$

d. *A measurement of L^2, with the particle in the state $|\psi\rangle$, has yielded $l = 1$. What normalized state describes the particle just after this measurement?*

A quick scan of the two possible wave functions reveals that $l = 1$ is only possible in the ψ_- state. l can equal either 0 or 2 in ψ_+.

Send in the spin

Consider a system composed of two spin-1/2 particles whose orbital variables are ignored. The Hamiltonian of the system is

$$H = \omega_1 S_{1z} + \omega_2 S_{2z}$$

where S_{1z} and S_{2z} are the projections of the spins S_1 and S_2 of the two particles onto the z-axis and ω_1 and ω_2 are real constants.

The initial state of the system at time $t = 0$ is $|\psi(0)\rangle = \sqrt{\frac{1}{3}}|+,-\rangle - \sqrt{\frac{2}{3}}|-,+\rangle$.

At time t, $S^2 = (S_1 + S_2)^2$ is measured. What results can be found, and with what probabilities?

The eigenstates of S^2 are $|m_{s1} m_{s2}\rangle$, where $m_s = 0, \pm 1$. We need to convert the given wavefunction into these eigenstates. Knowing that

$$|01\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$
$$|00\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

we can solve for $|+-\rangle$ and $|-+\rangle$. They are

$$|+-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$
$$|-+\rangle = -\frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right)$$

Substituting these into the wavefunction, we find that

$$|\psi\rangle = \frac{\sqrt{6} + 2\sqrt{3}}{6}|00\rangle - \frac{\sqrt{6} - 2\sqrt{3}}{6}|10\rangle$$

The eigenvalues for S^2 are $S^2|\psi\rangle = \hbar^2 s(s+1)|\psi\rangle$, where $s = 0, 1$. The two possible values that can be obtained from measurement are 0 and $2\hbar^2$. The probabilities of measuring each of these values is equal to the square of the coefficient of that state. Thus, the probability of measuring 0 is

$$P_0 = \left|\frac{\sqrt{6} + 2\sqrt{3}}{6}\right|^2 = \frac{3 + 2\sqrt{2}}{6}$$

The probability of measuring $2\hbar^2$ is

$$P_{2\hbar^2} = \left|\frac{\sqrt{6} - 2\sqrt{3}}{6}\right|^2 = \frac{3 - 2\sqrt{2}}{6}$$

So,

$$0 \text{ with } P = \frac{3 + 2\sqrt{2}}{6}$$
$$2\hbar^2 \text{ with } P = \frac{3 - 2\sqrt{2}}{6}$$

How long can this go on?

A polymer is a molecule composed of a long chain of identical molecular units, called monomers. Consider a long polymer, made of N rod-like monomers, each of length a attached end to end. One end of the polymer is held fixed while a constant force F is applied to the other end in the x-direction. The energy is $\epsilon_- = aF$ for monomers pointing in the negative x-direction and $\epsilon_+ = -aF$ for monomers pointing in the positive x-direction.

a. *Calculate, at temperature T, the average length of the polymer $L = N\langle l \rangle$, where $\langle l \rangle$ is the average projection of a monomer in the $+x$-direction, i.e., $\langle l \rangle = af_+ - af_-$ where f_+ and f_- are the probabilities of the monomer being in the $+x$- and $-x$-directions, respectively. What are the limiting values of L at $T = 0$ and $T = \infty$?*

The probability of a monomer being in the $+x$-direction is

$$f_+ = \frac{e^{-\beta\epsilon_+}}{e^{-\beta\epsilon_+} + e^{-\beta\epsilon_-}} = \frac{e^{a\beta F}}{e^{a\beta F} + e^{-a\beta F}} = \frac{1}{1 + e^{-2a\beta F}}$$

where $\beta = \frac{1}{kT}$.

Likewise, the probability of a monomer being in the $-x$-direction is

$$f_- = \frac{e^{-2a\beta F}}{1 + e^{-2a\beta F}}$$

Therefore, the average projection of a monomer in the $+x$-direction is

$$\langle l \rangle = af_+ - af_-$$
$$\langle l \rangle = \frac{a\left(1 - e^{-2a\beta F}\right)}{1 + e^{-2a\beta F}}$$

The average length of the polymer at temperature T is then

$$L = N\langle l \rangle$$
$$\boxed{L = \frac{Na\left(1 - e^{-2a\beta F}\right)}{1 + e^{-2a\beta F}}}$$

At $T = 0$, $\beta \to \infty$, so $L = Na$. As $T \to \infty$, $\beta \to 0$, so $L \to 0$.

b. *Calculate the thermal expansivity ($\alpha_l = \frac{1}{L}\frac{\partial L}{\partial T}$) of the polymer at temperature T and show that it is negative (as it is for rubber, which can also be very crudely represented by such a polymer).*

$$\begin{aligned}
\alpha_l &= \frac{1}{L}\frac{\partial L}{\partial T} \\
&= \frac{Na}{L}\frac{\left(1 + e^{-2a\beta F}\right)\left(2aF\frac{\partial\beta}{\partial T}e^{-2a\beta F}\right) - \left(1 - e^{-2a\beta F}\right)\left(-2aF\frac{\partial\beta}{\partial T}e^{-2a\beta F}\right)}{\left(1 + e^{-2a\beta F}\right)^2} \\
&= \frac{Na}{L}\frac{\left(1 + e^{-2a\beta F}\right)\left(-\frac{2aF}{kT^2}e^{-2a\beta F}\right) - \left(1 - e^{-2a\beta F}\right)\left(\frac{2aF}{kT^2}e^{-2a\beta F}\right)}{\left(1 + e^{-2a\beta F}\right)^2}
\end{aligned}$$

$$\boxed{\alpha_l = -\frac{4Na^2Fe^{-2a\beta F}}{L\left(1 + e^{-2a\beta F}\right)^2}}$$

This quantity is less than 0, since $N, F, L > 0$.

c. *What is the average internal energy of the polymer at temperature T?*

Average internal energy can be defined as

$$
\begin{aligned}
U &= kT^2 \frac{\partial}{\partial T} \ln Z \\
&= kT^2 \frac{\partial}{\partial T} \ln \left(e^{a\beta F} + e^{-a\beta F}\right) \\
&= \frac{kT^2 \left(-\frac{aF}{kT^2} e^{a\beta F} + \frac{aF}{kT^2} e^{-a\beta F}\right)}{e^{-a\beta F} + e^{-a\beta F}} \\
&\boxed{U = \frac{aF\left(e^{-a\beta F} - e^{a\beta F}\right)}{e^{-a\beta F} + e^{a\beta F}}}
\end{aligned}
$$

Bathing the oscillator

Consider a three-dimensional isotropic harmonic oscillator whose energy levels are given by

$$E_{n_1,n_2,n_3} = \hbar\omega \left(n_1 + n_2 + n_3 + \frac{3}{2}\right)$$

where each of n_1, n_2, n_3 can be natural integers 0, 1, 2, 3, etc.

a. *Find the degeneracies of the levels of $\frac{7}{2}\hbar\omega$ and $\frac{9}{2}\hbar\omega$.*

If the energy is equal to $\frac{7}{2}\hbar\omega$, then the sum of the three integers must be

$$
\begin{aligned}
\frac{7}{2}\hbar\omega &= \hbar\omega(n_1 + n_2 + n_3 + \frac{3}{2}) \\
2 &= n_1 + n_2 + n_3
\end{aligned}
$$

All the possible combinations to get a sum of 2 are

n_1	n_2	n_3
2	0	0
0	2	0
0	0	2
1	1	0
1	0	1
0	1	1

Therefore, the degeneracy of this energy level is 6.

Repeating the same analysis on the energy level $\frac{9}{2}\hbar\omega$, we find that the sum of the three integers must be equal to 3, so the possible combinations are

n_1	n_2	n_3
3	0	0
0	3	0
0	0	3
2	1	0
2	0	1
1	2	0
1	0	2
0	2	1
0	1	2
1	1	1

The degeneracy for this level must therefore be 10.

b. *Given that the system is in thermal equilibrium with a heat bath at a temperature T, show that the $\frac{9}{2}\omega$ level is more populated than the $\frac{7}{2}\omega$ level if k_BT is larger than $\frac{\omega}{\ln\frac{5}{3}}$.*

The population of the i^{th} energy level $N_i = Np_i$, where p_i is the probability of being in the i^{th} level. $p_i = \frac{g_i e^{-\beta E_i}}{Z}$, where g_i is the degeneracy of that level. Thus,

$$\frac{N_{9/2}}{N_{7/2}} = \frac{g_{9/2}}{g_{7/2}}\frac{e^{-\beta E_{9/2}}}{e^{-\beta E_{7/2}}} = \frac{5}{3}\frac{e^{-\frac{9}{2}\beta\hbar\omega}}{e^{-\frac{7}{2}\beta\hbar\omega}}$$

If the higher energy level is more populated, then this fraction will be greater than 1.

$$\begin{aligned}
\frac{5}{3}\frac{e^{-\frac{9}{2}\beta\hbar\omega}}{e^{-\frac{7}{2}\beta\hbar\omega}} &> 1 \\
e^{-\beta\hbar\omega} &> \frac{3}{5} \\
-\beta\hbar\omega &> \ln\frac{3}{5} \\
\beta &> \frac{\ln\frac{5}{3}}{\hbar\omega} \\
\boxed{kT > \frac{\hbar\omega}{\ln\frac{5}{3}}}
\end{aligned}$$

c. *For a general energy level $\left(m+\frac{1}{2}\right)\omega$, find the degeneracy and express it in terms of m.*

For $m = 2$, it can be easily found that the degeneracy is 3. From part a, the degeneracy for $m = 3$ is 6, and the degeneracy for $m = 4$ is 10. Therefore, the degeneracy for m must be $1+2+3+\cdots+m = \boxed{\frac{1}{2}m(m+1)}$.

d. *If m is greater than m', above what temperature T is the $\left(m+\frac{1}{2}\right)\omega$ level more populated than the $\left(m'+\frac{1}{2}\right)\omega$ level?*

Repeating the same analysis as in part b,

$$\frac{N_m}{N_{m'}} = \frac{g_m}{g_{m'}}\frac{e^{-\beta(m+\frac{1}{2})\hbar\omega}}{e^{-\beta(m'+\frac{1}{2})\hbar\omega}}$$

Again, for an inverted population, this fraction must be larger than 1, so

$$\begin{aligned}
\frac{g_m}{g_{m'}}\frac{e^{-\beta(m+\frac{1}{2})\hbar\omega}}{e^{-\beta(m'+\frac{1}{2})\hbar\omega}} &> 1 \\
e^{-\beta\hbar\omega(m-m')} &> \frac{m'(m'+1)}{m(m+1)} \\
-\beta\hbar\omega(m-m') &> \ln\frac{m'(m'+1)}{m(m+1)} \\
\beta &> \frac{1}{\hbar\omega(m-m')}\ln\frac{m(m+1)}{m'(m'+1)} \\
\Aboxed{kT &> \frac{\hbar\omega(m-m')}{\ln\left(\frac{m(m+1)}{m'(m'+1)}\right)}}
\end{aligned}$$