Statmech II HW2

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1 Problem 1

We prove the relation:

$$\ln w(i) = n_i \ln\left(\frac{g_i}{n_i} - a\right) - \frac{g_i}{a} \ln\left(1 - a\frac{n_i}{g_i}\right) \tag{1.1}$$

Where $a = -1$ for Bose-Einstein systems, $a = 1$ for Fermi-Dirac systems, and $a = 0$ for classical Maxwell-Boltzmann systems. We assume that $n_i \gg 1$ and $g_i \gg 1$.
For Bose-Einstein systems:

$$w(i) = \frac{(n_i + g_i - 1)!}{(n_i)!(g_i - 1)!} \tag{1.2}$$

$$\ln w(i) = \ln\left(\frac{(n_i + g_i - 1)!}{(n_i)!(g_i - 1)!}\right) \tag{1.3}$$

$$\ln w(i) = (n_i + g_i - 1)\ln(n_i + g_i - 1) - n_i \ln n_i - (g_i - 1)\ln(g_i - 1) \tag{1.4}$$

$$\ln w(i) = n_i \ln\left(\frac{n_i + g_i - 1}{n_i}\right) + (g_i - 1)\ln\left(\frac{n_i}{g_i + 1} + 1\right) \tag{1.5}$$

$$\ln w(i) = n_i \ln\left(\frac{g_i}{n_i} + 1\right) + g_i \ln\left(\frac{n_i}{g_i} + 1\right) \tag{1.6}$$

Where in 1.5 we have used $n_i \gg 1$ and $g_i \gg 1$ to simplify the expression.
For Fermi-Dirac systems:

$$w(i) = \frac{(g_i)!}{(n_i)!(g_i - n_i)!} \tag{1.7}$$

$$\ln w(i) = \ln\left(\frac{(g_i)!}{(n_i)!(g_i - n_i)!}\right) \tag{1.8}$$

$$\ln w(i) = g_i \ln g_i - n_i \ln n_i - (g_i - n_i)\ln(n_i - g_i) \tag{1.9}$$

$$\ln w(i) = n_i (\ln(g_i - n_i) - \ln n_i) + g_i \left(\ln \frac{g_i}{g_i - n_i}\right) \tag{1.10}$$

$$\ln w(i) = n_i \ln\left(\frac{g_i}{n_i} - 1\right) - g_i \ln\left(1 - \frac{n_i}{g_i}\right) \tag{1.11}$$

For Maxwell-Boltazmann systems:

$$w(i) = \frac{(g_i)^{n_i}}{(n_i)!} \tag{1.12}$$

$$\ln w(i) = n_i \ln g_i - n_i \ln n_i + n_i \tag{1.13}$$

Taking the limit of $-\frac{g_i}{a} \ln(1 - a\frac{n_i}{g_i})$ as $a \to 0$ using L'Hopital's rule:

$$\lim_{\longrightarrow} 0 - \frac{g_i}{a} \ln(1 - a\frac{n_i}{g_i}) = n_i \tag{1.14}$$

$$\ln w(i) = n_i \ln g_i - n_i \ln n_i - \frac{g_i}{a} \ln(1 - a\frac{n_i}{g_i}) \; (a = 0) \tag{1.15}$$

2 Problem 2

We prove the maximization result using Lagrange multipliers:

$$\delta \ln W_i - \left(\alpha \sum \delta n_i + \beta \sum \epsilon_i \delta n_i\right) = \sum \left(\ln(\frac{g_i}{n_i} - a) - \alpha - \beta \epsilon_i\right) \delta n_i \tag{2.1}$$

We take the derivative with respect to n_i of $\ln w_i$ (equation 1.1).

$$\frac{\partial \ln w_i}{\partial n_i} = \ln(\frac{g_i}{n_i} - a) - \frac{g_i}{n_i} \frac{1}{g_i/n_i - a} + \frac{1}{1 - an_i/g_i} \tag{2.2}$$

$$\frac{\partial \ln w_i}{\partial n_i} = \ln(\frac{g_i}{n_i} - a) \tag{2.3}$$

With $W_i = \prod w_i$ and using the logarithm addition rules this proves 2.1.

3 Problem 3

Starting from the grand canonical equivalent of the microcanonical ensmble expression for number of microstates, with $g_i = 1$ and replacing n_i^* with

$< n_e >$:

$$\ln W = \sum_e \left(< n_e > \ln\left(\frac{1}{< n_e >} - a\right) - \frac{1}{a} \ln\left(1 - a < n_e >\right) \right) \tag{3.1}$$

$$S = k \ln W = k \sum_e \left(< n_e > \ln\left(\frac{1}{< n_e >} - a\right) - \frac{1}{a} \ln\left(1 - a < n_e >\right) \right) \tag{3.2}$$

$$S = k \sum_e < n_e > \ln\left(\frac{1 - a < n_e >}{< n_e >}\right) - \frac{1}{a} \ln\left(1 - a < n_e >\right) \tag{3.3}$$

$$S = k \sum_e \left(< n_e > - \frac{1}{a} \right) \ln\left(1 - a < n_e >\right) - < n_e > \ln < n_e > \tag{3.4}$$

So we have proved the relation with $a = -1$ for Bosons and $a = 1$ for Fermions.
For Fermions there are only two probabilities:

$$\rho_0 = 1 - < n_e > \tag{3.5}$$

$$\rho_1 = < n_e > \tag{3.6}$$

So the probability form of the entropy is:

$$S = -k \sum_e \left((1 - < n_e > \ln\left(1 - < n_e >\right) + < n_e > \ln < n_e > \right) \tag{3.7}$$

$$S = k \sum_e \left((< n_e > -1) \ln\left(1 - < n_e >\right) - < n_e > \ln < n_e > \right) \tag{3.8}$$

For Bosons, the probability is:

$$\rho_e = \frac{< n_e >^n}{(< n_e > +1)^{n+1}} \tag{3.9}$$

We can write the entropy equation terms as the expectation value of $\ln \rho_e$.

$$\sum_n \rho_e \ln \rho_e = < \ln \rho_e > \tag{3.10}$$

$$S = -k \sum_e \langle \ln \langle \rho_e \rangle \rangle \tag{3.11}$$

$$S = -k \sum_e \langle n \ln \langle n_e \rangle - (n+1) \ln\left(\langle n_e \rangle + 1\right) \rangle \tag{3.12}$$

$$S = k \sum_e (\langle n_e \rangle + 1) \ln\left(\langle n_e \rangle + 1\right) - \langle n_e \rangle \ln \langle n_e \rangle \tag{3.13}$$