Math Phys II HW 2

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Abstract

1 Problem 1

We seek solutions of the Kortweg-deVries equation:

$$\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi}{\partial x}+\frac{\partial^3\psi}{\partial x^3}=0 \tag{1.1}$$

We look for solutions $\psi(\xi)$, with $\xi=x-ct$. To write 1.1 in terms of ξ, we calculate the partial derivatives:

$$\frac{\partial\psi}{\partial t}=-c\frac{d\psi(\xi)}{d\xi}$$
$$\frac{\partial\psi}{\partial x}=\frac{d\psi(\xi)}{d\xi}$$
$$\frac{\partial^3\psi}{\partial x^3}=\frac{d^3\psi(\xi)}{d\xi^3}$$

We can now write 1.1 in terms of ξ:

$$-c\frac{d\psi}{d\xi}+\psi\frac{d\psi}{d\xi}+\frac{d^3\psi}{d\xi^3}=0 \tag{1.2}$$

This simplifies to:

$$(\psi-c)\frac{d\psi}{d\xi}+\frac{d^3\psi}{d\xi^3}=0 \tag{1.3}$$

We can integrate 1.3 to find:

$$\frac{d^2\psi}{d\xi^2}=c\psi-\frac{\psi^2}{2} \tag{1.4}$$

We then integrate again and multiply by $\frac{d\psi}{d\xi}$:

$$\frac{d\psi}{d\xi}=\int(c\psi-\frac{\psi^2}{2}) \tag{1.5}$$

$$(\frac{d\psi}{d\xi})^2=\frac{\psi^2}{2}(c-\frac{\psi}{3}) \tag{1.6}$$

$$\frac{d\psi}{d\xi}=\frac{\psi}{\sqrt{2}}(c-\frac{\psi}{3})^{\frac{1}{2}} \tag{1.7}$$

We can now integrate for ξ:

$$d\xi = \int \frac{d\psi}{\frac{\psi}{\sqrt{2}}(c - \frac{\psi}{3})^{\frac{1}{2}}} \tag{1.8}$$

And then rearrange to find ψ as a function of ξ and c.

2 Problem 2

The general form of a second-order linear PDE is:

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x\partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} \tag{2.1}$$

The characteristic equation, with solutions $\xi(x,y)$ and $\eta(x,y)$, is:

$$A(\frac{dy}{dx})^2 + 2B(\frac{dy}{dx}) + C = 0 \tag{2.2}$$

We wish to write Eq. 1 in terms of ξ and η. We differentiate $\psi(\xi,\eta)$:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x} \tag{2.3}$$

Now we calculate the other partials with respect to η and ξ.

$$\frac{\partial}{\partial x}(\frac{\partial\psi}{\partial\xi}) = \frac{\partial^2\psi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x} \tag{2.4}$$

$$\frac{\partial}{\partial x}(\frac{\partial\psi}{\partial\eta}) = \frac{\partial^2\psi}{\partial\eta^2}\frac{\partial\eta}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x} \tag{2.5}$$

We use 3,4 and 5 to calculate $\frac{\partial^2\psi}{\partial x^2}$.

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} &= \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial\xi}{\partial x}(\frac{\partial^2\psi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x}) \\ &+\frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial\eta}{\partial x}(\frac{\partial^2\psi}{\partial\eta^2}\frac{\partial\eta}{\partial x} + \frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}) \\ = \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} &+ \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2 + \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta} + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 + \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta}\end{aligned}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 + 2(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta}) \tag{2.6}$$

The calculation of $\frac{\partial^2\psi}{\partial y^2}$ is identical.

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\xi}{\partial y^2}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial y^2}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial y})^2 + \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial y})^2 + 2(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2\psi}{\partial\xi\partial\eta}) \tag{2.7}$$

We now take $\frac{\partial}{\partial y}$ of equation 1:

$$\begin{aligned}\frac{\partial^2\psi}{\partial x\partial y} &= \frac{\partial^2\xi}{\partial x\partial y}\frac{\partial\psi}{\partial\xi} + \frac{\partial^2\eta}{\partial x\partial y}\frac{\partial\psi}{\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}) \\ &+ \frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}) + \frac{\partial^2\psi}{\partial\xi\partial\eta}(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y})\end{aligned} \tag{2.8}$$

We can now write Eq. 1 in terms of ξ and η:

$$\begin{aligned}
A\{\frac{\partial^2\xi}{\partial x^2}\frac{\partial\psi}{\partial\xi}+\frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta}+\frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x})^2+\frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2+2(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2\psi}{\partial\xi\partial\eta})\}\\
+2B\{\frac{\partial^2\xi}{\partial x\partial y}\frac{\partial\psi}{\partial\xi}+\frac{\partial^2\eta}{\partial x\partial y}\frac{\partial\psi}{\partial\eta}+\frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y})\\
+\frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y})+\frac{\partial^2\psi}{\partial\xi\partial\eta}(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y})\}\\
+C\{\frac{\partial^2\xi}{\partial y^2}\frac{\partial\psi}{\partial\xi}+\frac{\partial^2\eta}{\partial y^2}\frac{\partial\psi}{\partial\eta}+\frac{\partial^2\psi}{\partial\xi^2}(\frac{\partial\xi}{\partial y})^2+\frac{\partial^2\psi}{\partial\eta^2}(\frac{\partial\eta}{\partial y})^2+2(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2\psi}{\partial\xi\partial\eta})\}
\end{aligned}\tag{2.9}$$

We now take a break to stop Eq. 9 from giving us a migraine brought on by eye strain.

We collect the coefficients of all the derivates of ψ:

$$\begin{gathered}
\frac{\partial\psi}{\partial\xi}(A\frac{\partial^2\xi}{\partial x^2}+2B\frac{\partial^2\xi}{\partial x\partial y}+C\frac{\partial^2\xi}{\partial y^2})\\
\frac{\partial\psi}{\partial\eta}(A\frac{\partial^2\eta}{\partial x^2}+2B\frac{\partial^2\eta}{\partial x\partial y}+C\frac{\partial^2\eta}{\partial y^2})\\
\frac{\partial^2\psi}{\partial\xi^2}(A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2)\\
\frac{\partial^2\psi}{\partial\eta^2}(A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2)\\
\frac{\partial^2\psi}{\partial\xi\partial\eta}(2A(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x})+2B(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y})+2C(\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}))
\end{gathered}$$

After the break, we recognize that since $\xi(x,y)$ and $\eta(x,y)$ are solutions to Eq. 1:

$$A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2=0\tag{2.10}$$

$$A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2=0\tag{2.11}$$

3 Problem 3

We are solving the characteristic equation for:

$$\frac{\partial^2\psi}{\partial t^2}-c(x)^2\frac{\partial^2\psi}{\partial x^2}=0$$

With A=1, B=0, and $c=-c(x)^2$, the characteristic equation is:

$$(\frac{dx}{dt})^2-c(x)^2=0\tag{3.1}$$

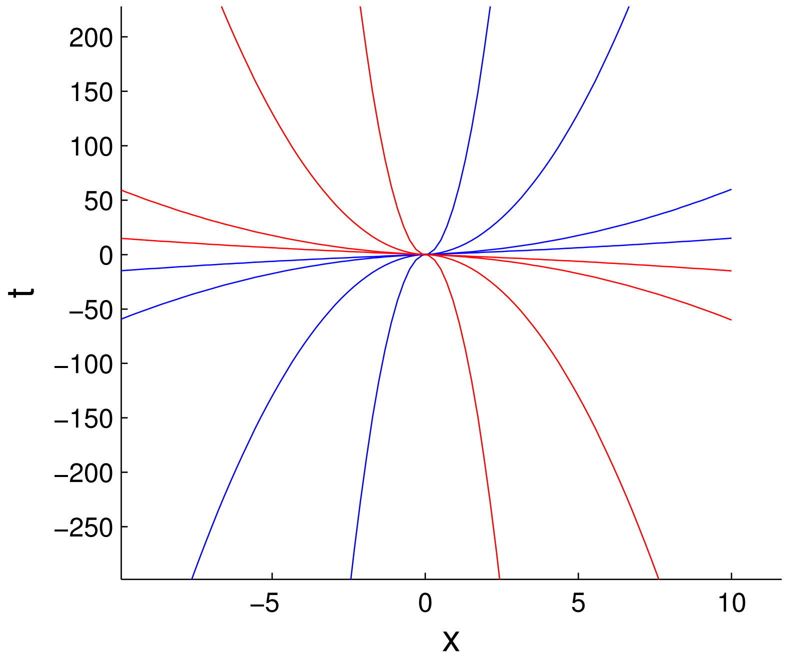
$$\frac{dx}{dt}=\pm c(x)\tag{3.2}$$

$$dt=\pm\frac{1}{c(x)}dx\tag{3.3}$$

With $c(x) = c_0(1 + \frac{|x|}{a})$ the characteristic curve can be written:

$$t = \pm\frac{1}{c_0}(x + sgn(x)\frac{x^2}{2a}) + C \tag{3.4}$$

Several characteristic curves are shown below for a-values 0.01, 0.1, 1 and 10. The positive curves are shown in blue, the negative curves in red.



We now look to find a solution given the initial conditions:

$$\psi(x, 0) = 0 \tag{3.5}$$

$$\frac{\partial\psi}{\partial t} |_{t=0}= e^{-|x|} \tag{3.6}$$

When $a = \infty$ the characteristic solutions become:

$$\xi = x + c_0 t \tag{3.7}$$

$$\eta = x - c_0 t \tag{3.8}$$

The solution can be written as a combination $\psi = f(\xi) + g(\eta)$. Using the initial condition $\psi(x, 0) = 0$ we see that $f(x) + g(x) = 0$, so that $g(x) = -f(x)$. We differentiate the combined solution with respect to t and use the second boundary condition:

$$-v\frac{df}{dt} + c_0\frac{dg}{dt} = e^{-|x|} \tag{3.9}$$

$$\int -2\frac{df}{dt} = \int \frac{1}{c_0}e^{-|x|} \tag{3.10}$$

$$f = \frac{-1}{2c_0}e^{-|x|} \tag{3.11}$$

We can now write the combined solution:

$$\psi(x,t) = \frac{1}{2c_0}(e^{-|x+vt|} - e^{-|x-vt|}) \tag{3.12}$$