Electromagnetic Theory II HW6

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8.4

The general wave equation for a cylindrical waveguide is:

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \tag{0.1}$$
$$\psi = E_z \ \textbf{(TM)} \ \textbf{,} B_z \ \textbf{(TE)} \tag{0.2}$$

The solutions that are regular at $\rho = 0$ are:

$$\psi = A_m J_m(\gamma\rho)e^{\pm im\phi} \tag{0.3}$$

The boundary conditions on the conductor surface $\rho = R$ will be different for TE and TM modes. For TM modes the E field vanishes at the surface, so the eigenvalues are the zeros of $J_m(\gamma R)$. For TE modes the normal derivative of the magnetic field vanishes at the surface, so the eigenvalues are the zeros of $J'_m(\gamma R)$. Therefore the mode frequencies are:

$$\omega_{m,n} = \frac{Z_m(n)}{\sqrt{\mu\epsilon}R} \ \textbf{(TM)} \tag{0.4}$$
$$\omega_{m,n} = \frac{Z'_m(n)}{\sqrt{\mu\epsilon}R} \ \textbf{(TE)} \tag{0.5}$$

Where we have defined $Z_m(n)$ as the Nth zero of J_m and $Z'_m(n)$ as the Nth zero of J'_m. The first few roots are:

$$Z_0 = 2.41, 5.52, 8.65 \tag{0.6}$$
$$Z_1 = 3.83, 7.02, 10.17 \tag{0.7}$$
$$Z_2 = 5.14, 8.41, 11.62 \tag{0.8}$$
$$Z'_0 = 3.83, 7.02, 10.17 \tag{0.9}$$
$$Z'_1 = 1.84, 5.33, 8.54 \tag{0.10}$$
$$Z'_2 = 3.05, 6.71, 9.97 \tag{0.11}$$

The lowest root is $Z_1'(1)$, so the the dominant mode is TE_{11}. Listing the dominant mode and the next four higher modes:

Mode	Frequency	Ratio
TE_{11}	1.84	1
TM_{01}	2.41	1.31
TE_{21}	3.05	1.66
TE_{01}, TM_{11}	3.83	2.08
TM_{21}	5.14	2.79

b) The attenuation coefficients are found from (Jackson 8.57):

$$\beta_\lambda = -\frac{1}{2P}\frac{dP}{dz} \tag{0.12}$$

The power and power loss for TE and TM modes are:

$$P_{TM} = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\epsilon\int_A \psi^*\psi\ da \tag{0.13}$$

$$\frac{dP}{dz}_{TM} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2\oint_C \frac{1}{\mu^2\omega_\lambda^2}|\frac{\partial\psi}{\partial n}|^2\ d\ell \tag{0.14}$$

$$P_{TE} = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\mu\int_A \psi^*\psi\ da \tag{0.15}$$

$$\frac{dP}{dz}_{TE} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2\oint_C \frac{1}{\mu\epsilon\omega_\lambda^2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)|\mathbf{n}\times\nabla_t\psi|^2 + |\frac{\partial\psi}{\partial n}|^2\ d\ell \tag{0.16}$$

We can evaluate the power expressions for the TM modes using the orthogonality of the Bessel functions:

$$\int_0^1 xJ_m\left(xZ_m(n)\right)J_m\left(xZ_m(n)\right)\ dx = \frac{1}{2}\left[J_{m+1}\left(xZ_m(n)\right)\right]^2 \tag{0.17}$$

For the TM modes, integrating from $\rho = 0$ to R with $u \equiv Z_m(n)/R$ making the substitution $\rho' = \rho/R$ we find:

$$P_{TM} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}2\pi\int_0^1 R^2\rho'\left[J_m(\rho' Z_m(n))\right]^2\ d\rho' \tag{0.18}$$

$$P_{TM} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\pi R^2\left[J_{m+1}(Z_m(n))\right]^2 \tag{0.19}$$

The normal derivative is just $-\frac{d\psi}{d\rho}$, so we can calculate the power loss:

$$\frac{d\psi}{d\rho} = -\gamma J_m'\left(Z_m(n)\right)e^{im\phi} \tag{0.20}$$

$$\frac{dP}{dz} = -\frac{1}{2\sigma\delta}\frac{\epsilon}{\mu}(2\pi R)[J_m'(Z_m(n))]^2 \tag{0.21}$$

$$\frac{dP}{dz} = -\frac{1}{2\sigma\delta}\frac{\epsilon}{\mu}(2\pi R)[J_{m+1}(Z_m(n))]^2 \tag{0.22}$$

Where in the last step we have used a recursion relation for Bessel functions. We can now calculate the attenuation coefficient:

$$\beta_\lambda = \frac{1}{2\sigma\delta}\left(\frac{\epsilon}{\mu}\right)^{3/2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\frac{2\pi R}{\pi R^2} \tag{0.23}$$

$$\beta_\lambda = \frac{1}{\sigma\delta}\left(\frac{\epsilon}{\mu}\right)^{3/2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\frac{1}{R} \tag{0.24}$$

For the TE modes we are working with the zeros of the derivatives of the Bessel functions. Using an identity:

$$\int_0^1 x\,[J_m(ax)]^2\ dx = \frac{1}{2}\left([J_m'(a)]^2 + (1-m^2/a^2)[J_m(a)]^2\right) \tag{0.25}$$

We calculate the power in the TE modes:

$$P_{TE} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\pi R^2\left(1-\frac{m^2}{Z_m'(n)^2}\right)\left[J_m(Z_m'(n))\right]^2 \tag{0.26}$$

The TE expressions for dP/dz includes both a normal derivative term and transverse gradient term. The normal derivative term can be evaluated as above:

$$\oint_C |\frac{\partial\psi}{\partial n}|^2\ d\ell = 2\pi R[J_m(Z_m'(n))]^2 \tag{0.27}$$

For the transverse gradient term the cross-product with $\mathbf{n}$ picks out only the azimuthal part of the gradient:

$$\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} - \frac{1}{\rho}imJ_m(Z_m'(n)) \tag{0.28}$$

Therefore the gradient term is:

$$\oint_C |\mathbf{n}\times\nabla_t\psi|^2\ d\ell = 2\pi R\,(m/R)^2\,[J_m(Z_m'(n))]^2 \tag{0.29}$$

$$\tag{0.30}$$

We now have the complete expression for power loss for TE modes:

$$\frac{dP}{dz}_{TE} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2[J_m(Z_m'(n))]^2\left(\frac{m^2}{R^2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)+1\right) \tag{0.31}$$

We can now calculate the attenuation coefficient:

$$\beta_\lambda = \frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{1}{R}\left(\frac{m^2}{R^2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)+1\right)\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{-1/2} \tag{0.32}$$

8.6

We have solved the cylindrical waveguide in the previous problem, the cylindrical cavity frequencies come from the modified expression for γ in a cavity:

$$\gamma^2 = \mu\epsilon\omega^2 - \left(\frac{p\pi}{d}\right)^2 \tag{0.33}$$

$$\omega_{\lambda p}^2 = \frac{1}{\mu\epsilon}\left\{\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2\right\} \tag{0.34}$$

Where $p = 0, 1, 2..$ for TM modes (cosine solutions in z) and $p = 1, 2, 3...$ for TE modes (sine solutions in z). Writing the modes in terms of the zeros of the Bessel functions and their derivatives, and pulling out a factor of $1/R^2$:

$$\omega_{m,n,p} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{\left(Z_m(n)^2 + \left(\frac{p\pi R}{d}\right)^2\right)} \textbf{ (TM)} \tag{0.35}$$

$$\omega_{m,n,p} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{\left(Z_m'(n)^2 + \left(\frac{p\pi R}{d}\right)^2\right)} \textbf{ (TE)} \tag{0.36}$$

$$\tag{0.37}$$

The mode plots (figure 1) show that the frequency depends on the R/L ratio if $p \neq 0$. Each Bessel function root has a family of p-modes.
b) The lowest mode with $R/L = 2/3$ is the TM_{01} mode. We find the Q factor from:

$$Q = \omega\frac{U}{P_{loss}} \tag{0.38}$$

For the TM_{01} mode with $p = 0$, $U = \frac{L}{2}\int_A |\psi|^2\ da$ from Jackson 8.92. The power loss is (from Jackson 8.94 taking $\xi = 1$):

$$P_{loss} = \frac{\epsilon}{2\sigma\delta}\left(1 + CL/2A\right)^{-1}\int_A |\psi|^2\ da \tag{0.39}$$

Taking the ratio and using the equation for skin depth $\delta = \sqrt{2/\omega\mu\sigma}$:

$$Q = \frac{L}{\delta}\left(1 + L/R\right)^{-1} \tag{0.40}$$

For the given dimensions:

$$Q = \frac{.012}{\delta} \tag{0.41}$$

Assuming the interior of the cavity is free space, the fundamental frequency is about $\frac{2.41c}{.02} = 3.61e10$ or about 5.75 GHz. Pasternack's online skin depth calculator provides a value of $0.860\mu m$ at this frequency. Therefore the Q of this cavity is about 14,000.

9.3

Since the problem is axially symmetric but not spherically symmetric we expect the dipole term to dominate. The electric dipole moment is give by:

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\ d^3x' \tag{0.42}$$

We have previously solved the dipole potential of charged hemispheres:

$$\Phi = \frac{3}{2}VR^2\frac{1}{r^3}\cos(\theta) \tag{0.43}$$

The potential from a dipole is:

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \tag{0.44}$$

So the dipole moment of the hemispheres is:

$$\mathbf{p} = 6\pi\epsilon_0 VR^2\hat{z} \tag{0.45}$$

Ignoring the near-field component, we can directly calculate the **H** and **E** fields (Jackson 9.19).

$$\mathbf{H} = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r} \tag{0.46}$$

$$\mathbf{H} = -\frac{3ck^2\epsilon_0}{2}VR^2\frac{e^{ikr}}{r}\sin(\theta)\hat{\phi} \tag{0.47}$$

$$\mathbf{E} = Z_0\mathbf{H}\times\mathbf{n} \tag{0.48}$$

$$\mathbf{E} = -\sqrt{\frac{\epsilon_0^2\mu_0}{\epsilon_0^2\mu_0}}\frac{3}{2}Vk^2R^2\frac{e^{ikr}}{r}\sin(\theta)\hat{\theta} \tag{0.49}$$

$$\mathbf{E} = -\frac{3}{2}Vk^2R^2\frac{e^{ikr}}{r}\sin(\theta)\hat{\theta} \tag{0.50}$$

The power radiated per unit solid angle is (Jackson 9.23):

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2(\theta) \tag{0.51}$$

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{32\pi^2}k^4\left(36\pi^2\epsilon_0^2V^2R^4\right)\sin^2(\theta) \tag{0.52}$$

$$\frac{dP}{d\Omega} = \frac{9}{8}c^2Z_0k^4\epsilon_0^2V^2R^4\sin^2(\theta) \tag{0.53}$$

The total radiated power is (Jackson 9.24):

$$P = \frac{c^2Z_0k^4}{12\pi}|\mathbf{p}|^2 \tag{0.54}$$

$$P = \frac{3\pi}{Z_0}k^4V^2R^4 \tag{0.55}$$

Figure 1: TE and TM modes