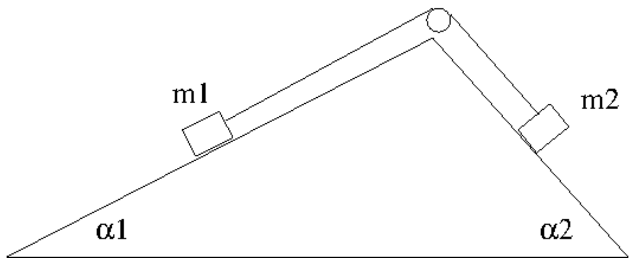
PH.D. QUALIFYING EXAM 2004

MATT THIESSE

Classical

Problem 1. *Find the relation between α_1 and α_2 for the static condition shown below (no friction):*



Balancing forces,

$$T = m_1 g \sin\alpha_1 \qquad T = m_2 g \sin\alpha_2$$
$$m_1 g \sin\alpha_1 = m_2 g \sin\alpha_2$$
$$\boxed{\frac{m_1}{m_2} = \frac{\sin\alpha_2}{\sin\alpha_1}}$$

Problem 2. *A 50g bullet travels at 500 m/s toward a hanging wooden block of mass 4.95 kg, and becomes embedded. The block hangs from a 2m length of rope. What is the maximum angle, θ, reached by the pendulum? What fraction of the initial kinetic energy is lost due to the collision?*

Conservation of momentum,

$$mv = (m+M)V$$

Conservation of energy,

$$\frac{1}{2}(m+M)V^2 = (m+M)gh$$

Combining equations and noting that the block rises a distance $h = L - L\cos\theta$,

$$\frac{m^2v^2}{(m+M)^2} = 2gL(1-\cos\theta)$$

so,

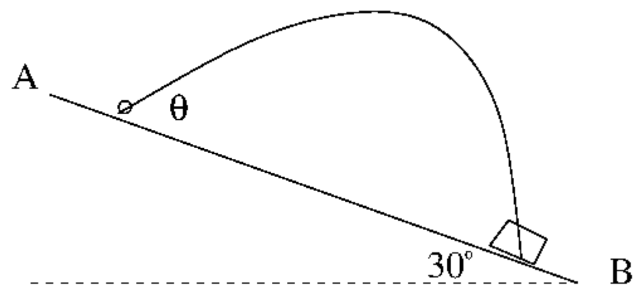
$$\boxed{\theta = 68.76^\circ}$$

The fraction of kinetic energy lost is

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}(m+M)V^2}{\frac{1}{2}mv^2} = 0.01$$

So, 99% of the bullet's initial kinetic energy is lost in the collision.

Problem 3. *In a demonstration experiment, a ball is shot from A with an initial speed v at an angle θ with an inclined plane AB. At the same instant, a cup is released from A and slides frictionlessly down the plane which is at an angle of 30° with the horizontal. If θ is chosen properly, the ball will always land in the cup regardless of the value of v. What is that proper value of θ?*



This is easier than it looks. When we rotate our coordinates, gravity pulls in a new direction such that there is an acceleration $a = g\sin 30$ along our new x axis. So, the x position of the ball and cup are

$$x_{cup} = \frac{1}{2} g \sin 30 t^2 \qquad x_{ball} = v_i \cos\theta t + \frac{1}{2} g \sin 30 t^2$$

The ball and cup must always have the same x position so we get simply

$$v_i \cos\theta t = 0$$

so

$$\boxed{\theta = \frac{\pi}{2}}$$

Problem 4. *When an unpolarized electromagnetic wave is incident on a medium at an incident angle $\theta_i = 55°$, the reflected rays are completely polarized. What is the refractive index of the medium?*

According to Snell's law,

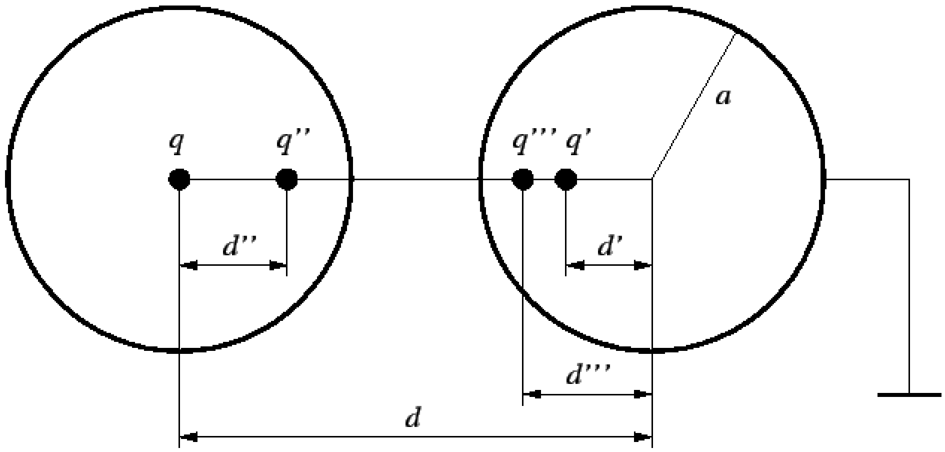
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

In order for the wave to be polarized, the transmitted wave must be perpendicular to the reflected wave. So, since the reflected wave exits the surface at θ_1, the transmitted wave must enter at $90-\theta_1$. We get

$$n_1 \sin 55 = n_2 \sin(90 - 55)$$

If we assume $n_1 = 1$, n_2 becomes $\boxed{1.428}$

Problem 5. *Two conducting spheres of radius R have centers separated by a distance d and are used as a capacitor. What is the capacitance?*



Use the method of images. If we place a test charge, q, at the center of the left sphere, it is now an equipotential but the right sphere is no longer equipotential. So, we must put a charge, q' in the right sphere a distance d' from the center of the right sphere. Now, the right sphere is equipotential but the left is disrupted. We continue placing charges in either sphere until each approaches equipotential. To calculate the values and positions of each of these charges we require the potential on the surface of either sphere to be zero. On the left side of the right sphere and the right side of the right sphere, the potential is, respectively

$$\frac{q}{d-R} + \frac{q'}{R-d'} = 0 \qquad \frac{q}{d+R} + \frac{q'}{d'+R} = 0$$

Solving these two equations gives $q' = -\frac{R}{d}q$ and $d' = \frac{R^2}{d}$. Continue doing this for each successive charge on each sphere. After lots of work you will get the total charge on the left sphere to be

$$q_{tot} = q\left(1 + \frac{R^2}{d^2 - R^2} + \frac{R^4}{d^4 - 3d^2R^2 + R^4} + \dots\right)$$

which is the same as the total charge on the right sphere in the limit as the number of added test charges goes to infinity. For some reason, the only charge that contributes to the potential is q, so the potential is simply

$$V = \frac{q}{4\pi\epsilon_0 R}$$

and the capacitance is

$$\boxed{C = 4\pi\epsilon_0 R\left(1 + \frac{R^2}{d^2 - R^2} + \frac{R^4}{d^4 - 3d^2R^2 + R^4} + \dots\right)}$$

If you have questions, see `http://www.iue.tuwien.ac.at/phd/wasshuber/node77.html`.

According to Prof. Goldberg, all we have to do is to assume all the charge collects at the point on each sphere which is closest to the other sphere. So, there is Q on the left sphere and $-Q$ on the right sphere. The charge is separated a distance of $d - 2R$ so the potential is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{d-2R}$$

So the capacitance is

$$C = \frac{Q}{V} = \frac{Q}{\frac{1}{4\pi\epsilon_0}\frac{Q}{d-2R}} = \boxed{4\pi\epsilon_0(d-2R)}$$

Problem 6. *A 10 μF capacitor is charged to 1000V. The capacitor is then discharged by connecting it to a 10 kΩ resistor. How long does it take for the capacitor to dissipate 2 J of energy in the resistor?*

The power delivered to the resistor is $P = \frac{dE}{dt}$ so the energy dissipated is

$$E = \int P dt = \int I^2 R dt$$

The discharging capacitor provides a current of $I = I_0 e^{-t/RC}$ so the energy integral becomes

$$\begin{aligned}
E &= \int_0^t I_0^2 e^{2t/RC} R dt \\
&= I_0^2 R e^{-2t/RC} \frac{1}{-2/RC} \\
\frac{-2E}{I_0^2 R^2 C} &= e^{-2t/RC} - 1 \\
\frac{-RC}{2} \ln\left(1 - \frac{2E}{I_0^2 R^2 C}\right) &= t \\
\Aboxed{t &= 0.0255 \text{ s}}
\end{aligned}$$

Problem 7. *A star of mass M radiates energy isotropically into space at a rate of L watts. A particle of mass m at distance r from the star presents an effective cross-section σ to the star's radiation.*

a) *Write down the condition for the star's radiation pressure to exceed the gravitational force on the particle.*

Clearly, the radiation pressure must be greater than $\frac{GMm}{r^2}$. The radiative force is given by $\frac{dp}{dt}$. The photon momentum is $p = \frac{E}{c}$. To find the energy, we start with the intensity,

$$I = \frac{L}{4\pi r^2}$$

which has units of power/area. So the power is

$$P = \frac{L}{4\pi r^2}\sigma$$

with units energy/time. Integrating the power from 0 to t we get an energy of

$$E = \frac{L\sigma t}{4\pi r^2}$$

Therefore, the momentum is

$$p = \frac{E}{c} = \frac{L\sigma t}{4\pi r^2 c}$$

and the force is

$$F = \frac{dp}{dt} = \frac{L\sigma}{4\pi r^2 c}$$

which must be greater than $\frac{GMm}{r^2}$.

b) *Estimate the maximum possible luminosity of a star of mass equal to that of the Sun. (Assume that the particles are protons, with mass $m_p = 1.7 \times 10^{-27} kg$ and cross section $\sigma_T = 6.7 \times 10^{-29} m^2$, and the mass of the Sun is $2.0 \times 10^{30} kg$.)*

$$L = \frac{4\pi c G M_S m_p}{\sigma_T} = \boxed{1.27 \times 10^{31}\ \mathrm{W}}$$

Problem A1. *Two masses m and $M = 2m$ are connected by a light inextensible string which passes without slipping over a uniform circular pulley of mass $2m$ and radius r. The pulley is free to rotate about a frictionless horizontal axis of rotation. Using the angular position of a point on the rim of the pulley as a generalized coordinate, write down the Lagrangian equation, and determine the acceleration of mass M.*

The relationship between the linear coordinate of the masses' positions and the angular coordinate of the pulley's rotation is $x = R\theta$ and the relationship between the velocities is therefore $\dot{x} = R\dot{\theta}$. So the kinetic energy of the system is

$$T = \frac{1}{2}(2m)R^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{2}(2m)R^2\right)\dot{\theta}^2$$
$$T = 2mR^2\dot{\theta}^2$$

and the potential energy is

$$V = -2mgR\theta + mgR\theta$$
$$V = -mgR\theta.$$

The Lagrangian is $\mathcal{L} = T - V = 2mR^2\dot{\theta}^2 + mgR\theta$. Then, from the Euler-Lagrange equation

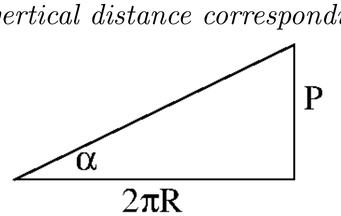
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

we can solve for the linear acceleration of the two masses.

$$\frac{d}{dt}\left[4mR^2\dot{\theta}\right] = mgR$$
$$\ddot{\theta} = \frac{g}{4R}$$
$$R\ddot{\theta} = \boxed{\frac{g}{4} = a}$$

Problem A2. *Consider the motion of a bead of mass m which slides on a smooth fixed wire bent in the form of a helix having radius R and a constant inclination angle α relative to the horizontal. The helix is at the surface of the earth.*

(a) *Use cylindrical coordinates and write down the constraint equation for z and θ with the parameter P (the pitch, defined as the vertical distance corresponding to 2π rotation in θ)*



Clearly,

$$P = 2\pi R \tan\alpha$$

and for a general height, z,

$$z = R\theta \tan\alpha$$

so

$$\boxed{z = \frac{\theta P}{2\pi}}$$

(b) *Assuming that the central axis of the helix is vertical, find the time required for the bead to slide through a vertical distance d, in terms of d, R, and P, after it is released from rest.*

For the bead on an inclined plane, the acceleration along the wire is $a = g\sin\alpha$ so the distance traveled along the wire at any time t is

$$\ell = \frac{1}{2} g \sin\alpha t^2$$

so the vertical distance traveled d is

$$d = \ell \sin\alpha = \frac{1}{2} g \sin^2\alpha t^2$$

and, hence, the time required is

$$t^2 = \frac{2d}{g\sin^2\alpha}$$

From part (a), we can easily show that $\sin\alpha = \frac{P}{2\pi R}$, so the time is

$$\boxed{t = \sqrt{\frac{8\pi^2 d R^2}{g P^2}}}$$

Problem A3. *In an experiment, a abnormally large woman is dropped from rest and falls to the ground from a building. She encounters air resistance during her flight. The position-time data are shown,*

x (m)	t (s)
0.61	0.347
1.00	0.470
1.22	0.519
1.52	0.582
1.83	0.650

(a) *Try to fit the data points with a resistive force that is linear in velocity, i.e. of the form kv. Note what you believe to be the value of the resistive constant (k) that yields a "reasonably good" fit.*

The equation for the force exerted on the fat woman is

$$F = k\dot{x} - mg = ma$$

which leads to the differential equation

$$\frac{d^2x}{dt^2} - \frac{k}{m}\frac{dx}{dt} = g$$

We see that the homogeneous solution is clearly exponential

$$x_h(t) = c_1 e^{\frac{k}{m}t}$$

The particular solution is clearly a polynomial

$$x_p(t) = c_2 t + c_3$$

so the full solution is of the form

$$x(t) = c_1 e^{\frac{k}{m}t} + c_2 t + c_3$$

Plugging this into the differential equation reveals the value $c_2 = \frac{-mg}{k}$. The differential equation is constrained by $x(0) = 0$ and $x'(0) = 0$ so we can easily find the other two constants. The first constraint leads to $c_1 = -c_3$ and the second constraint leads to $c_1 = \frac{m^2 g}{k^2}$. So, the full solution to which we will fit the given data is

$$x(t) = \frac{m^2 g}{k^2} e^{\frac{k}{m}t} - \frac{mg}{k} t - \frac{m^2 g}{k^2}$$

Now, get out your TI-83 and plug in each pair of x-t values to solve for $\frac{k}{m}$. The five values you get are $\frac{k}{m} = 0.2858,\ -0.516,\ -0.464,\ -0.4637,\ -0.5877$ so the best guess which gives a "reasonably good" fit is the average of these values: $\boxed{\frac{k}{m} = -0.349}$.

(b) *For that good fit, by comparing the difference between the potential energy lost and the kinetic energy gained to the total work done by friction to any point (all to be evaluated explicitly) during the fat woman's travel, show that the Work-Energy Theorem is confirmed.*

The Work-Energy Theorem states that

$$mg\Delta x = \frac{1}{2} m v(t)^2 - \int F_f \cdot dx$$

From the general solution, we can find the velocity

$$v(t) = x'(t) = \frac{mg}{k} e^{kt/m} - \frac{mg}{k}$$

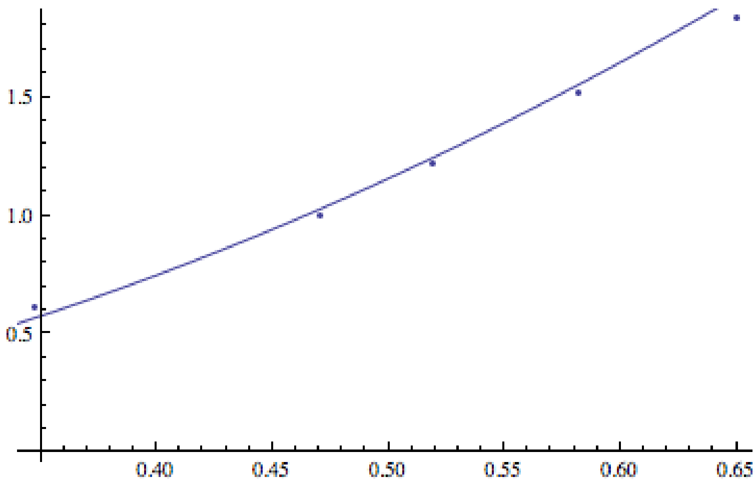
We can then plug this into the W-E Theorem:

$$\begin{aligned}
mg\Delta x &= \frac{1}{2} m v(t)^2 - \int k v^2 dt \\
&= \frac{1}{2} m \left(\frac{mg}{k} e^{kt/m} - \frac{mg}{k} \right)^2 - \int k \left(\frac{mg}{k} e^{kt/m} - \frac{mg}{k} \right)^2 dt \\
\Delta x &= \frac{1}{2} \frac{m^2}{k^2} g \left(e^{2kt/m} - 2e^{kt/m} + 1 \right) \Big|_0^{t_f} - \frac{mg}{k} \left(\frac{m}{2k} e^{2kt/m} - 2\frac{m}{k} e^{kt/m} + t \right) \Big|_0^{t_f} \\
&= \frac{m^2 g}{2k^2} \left(e^{2kt_f/m} - 2e^{kt_f/m} + 1 \right) - \frac{m^2 g}{2k^2} \left(e^{2kt_f/m} - 4e^{kt_f/m} + \frac{2k}{m} t_f + 3 \right) \\
\Delta x &= \frac{m^2 g}{k^2} \left(e^{kt_f/m} - \frac{kt_f}{m} - 1 \right)
\end{aligned}$$

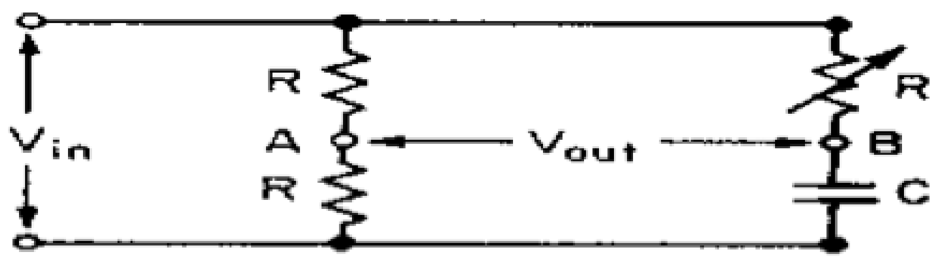
If we plug in the values, $g = 9.8 m/s^2$, $\frac{k}{m} = -0.349$, and $t_f = \{0.347, 0.470, 0.519, 0.582, 0.65\}$ we get the following values for Δx:

Time	Δx(fit)	Δx(actual)	% Difference
0.347	0.567	0.61	7.07
0.47	1.026	1.00	−2.56
0.519	1.244	1.22	−1.94
0.582	1.553	1.52	−2.16
0.65	1.922	1.83	−5.04

Furthermore, we can see the fit by plotting the fit function $x(t)$ versus time against the given data points. We can clearly see that the fit is a good one and the fact that the Work-Energy Theorem correctly predicts (approximately) the value of the position at any time during the fall.



Problem B1.



The above circuit contains several resistors of fixed resistance, R, a capacitor of known capacitance, C, and a tunable resistor, R'. At some initial time, $t = 0$, an induced alternating Voltage is applied to the system: $V_{in} = V_0 e^{i\omega t}$

(a) *What is the current passing through point, A, as a function of time? (Use only the real component)*

$$\begin{aligned} 0 &= V_{in} - I_1 R - I_1 R \\ 2I_1 R &= V_{in} \\ I_1 &= \frac{1}{2R} V_0 e^{i\omega t} \\ \Aboxed{\mathrm{Re}[I_1] &= \frac{V_0}{2R} \cos \omega t} \end{aligned}$$

(b) *What is the potential at A with respect to D as a function of time? (What is the real component?)*

$$V_A = I_1 R = \frac{V_0}{2} e^{i\omega t}$$
$$\boxed{\mathrm{Re}[V_A] = \frac{V_0}{2}\cos\omega t}$$

(c) *What is the charge on the capacitor as a function of time? (What is the real component?)*
We know that $Q = CV$ so the charge on the capacitor is given as

$$Q = C\left(V_0 e^{i\omega t} - IR'\right) = CV_0 e^{i\omega t} - C\frac{dQ}{dt}R'$$

and we are left with a differential equation

$$\frac{dQ}{dt} + \frac{Q}{R'C} = \frac{V_0}{R'}e^{i\omega t}$$

The solution is clearly of the form

$$Q(t) = Ae^{-t/R'C} + Be^{i\omega t}$$

By plugging this back into the differential equation, we find

$$B = \frac{V_0 C}{i\omega R'C + 1}$$

Also, by assuming $Q(0) = 0$, we see that $A = -B$. So the full solution is

$$Q(t) = \frac{V_0 C}{1 + i\omega R'C}\left(e^{i\omega t} - e^{-t/R'C}\right)$$

Let's now separate the real and imaginary parts: first the coefficient,

$$\frac{V_0 C}{1 + i\omega R'C}\,\frac{1 - i\omega R'C}{1 - i\omega R'C} = \frac{V_0 C - V_0 C^2 i\omega R'}{1 + \omega^2 R'^2 C^2}$$

then the exponential,

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

So the real part of the charge is

$$\boxed{Q(t) = \frac{V_0 C}{1 + \omega^2 R'^2 C^2}\left(\cos\omega t - e^{-t/R'C}\right)}$$

(d) *What is the current through B as a function of time? (What is the real component?)*
Since $I(t) = \frac{dQ(t)}{dt}$, the current is simply

$$\boxed{I(t) = \frac{V_0 C}{1 + \omega^2 R'^2 C^2}\left(-\omega\sin\omega t + \frac{e^{-t/R'C}}{R'C}\right)}$$

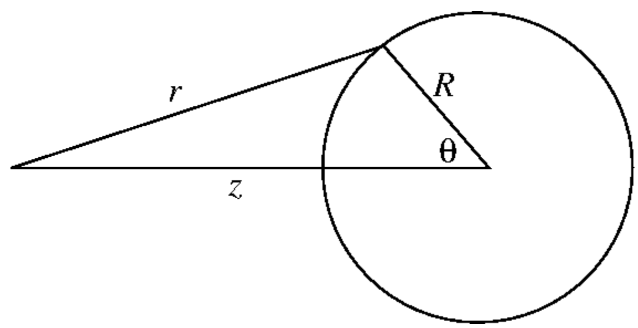
(e) *For an arbitrary value of R', what is the potential difference between A and B? (No need to compute the real part explicitly)*
The potential between A and B is

$$\boxed{V_{out} = \frac{-V_0}{2}\cos\omega t - \frac{V_0 C R'\omega}{1 + \omega^2 R'^2 C^2}\sin\omega t + \frac{V_0}{1 + \omega^2 R'^2 C^2}e^{-t/R'C}}$$

(f) *What is the purpose for this circuit?*
The output voltage $V_{out} = V_{AB}$ is the integral of the input voltage V_{in}. The circuit is an integrator.

Problem B2. *Charge density $\sigma(\theta) = k\cos\theta$, where k is a constant, is glued over the surface of a spherical shell of radius a. Find the resulting electric potential inside and outside the sphere. Remember that the boundary conditions on the electric field at the surface assures continuity of the potential Φ at $r = a$ and that $\left(\frac{\partial\Phi_{out}}{\partial r} - \frac{\partial\Phi_{in}}{\partial r}\right)_{r=a} = -\frac{\sigma(\theta)}{\varepsilon_0}$, where ε_0 is the electrical permittivity of the vacuum.*

There are two ways to do this problem, neither of which are easier to do than the other. The first way is the most direct.



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{r}da$$

The distance r from any point on the shell to a point z away from the center of the shell is $r^2 = R^2 + z^2 - 2Rz\cos\theta$ and we are doing the integral over the entire surface so the equation we must solve is

$$V = \frac{k}{4\pi\epsilon_0}\int \frac{\cos\theta R^2 \sin\theta d\theta d\phi}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}}$$

The ϕ integral is easy, we just get a factor of 2π. The θ integral is hard. I don't know how to do it other than guess and check (or, of course, Mathematica) but the answer is

$$V = \frac{k}{4\pi\epsilon_0}\left(\frac{\sqrt{R^2 + z^2 - 2Rz\cos\theta}(R^2 + z^2 + Rz\cos\theta)}{3R^2z^2}\right)\Bigg|_0^\pi$$

$$V = \frac{kR^2}{2\epsilon_0}\left(\frac{\sqrt{(R+z)^2}(R^2 - Rz + z^2) - \sqrt{(R-z)^2}(R^2 + Rz + z^2)}{3R^2z^2}\right)$$

There are clearly two solutions, for $z < R$ and for $z > R$. So the potential becomes

$$V(z) = \begin{cases} \dfrac{kz}{3\epsilon_0} & z < R \\ \dfrac{kR^3}{3\epsilon_0 z^2} & z > R \end{cases}$$

The other way to solve this problem is by using the separable solution to Laplace's equation with azimuthal symmetry. The full solution is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

For the interior region of the shell, we only consider the first term as the potential since the second term blows up at the origin.

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

For the exterior region, only the second term contributes since the first term blows up at infinity.

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

To solve further for the coefficients, the first condition is that the potential must be continuous at the boundary of the shell, so

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

in particular,

$$B_l = A_l R^{2l+1}$$

The other condition is that the first derivative of the potential is discontinuous at the boundary according to

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)_{r=R} = -\frac{\sigma(\theta)}{\epsilon_0}$$

So,

$$\sum_{l=0}^{\infty} -(l+1)A_l \frac{R^{2l+1}}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{k\cos\theta}{\epsilon_0}$$

$$\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{k\cos\theta}{\epsilon_0}$$

There is only one l for which this equation holds, $l = 1$, since $P_1(\cos\theta) = \cos\theta$. So, the coefficient A_1 is

$$A_1 = \frac{k}{3\epsilon_0}$$

and B_1 is

$$B_1 = \frac{kR^3}{3\epsilon_0}$$

Therefore the potential due to the surface charge distribution is

$$\boxed{V(r,\theta) = \begin{cases} \dfrac{kr\cos\theta}{3\epsilon_0} & r < R \\ \dfrac{kR^3\cos\theta}{3\epsilon_0 r^2} & r > R \end{cases}}$$

This is more or less consistent with the direct integration method where $z = r\cos\theta$.