PHYS 501: Mathematical Physics I

Fall 2014

Homework #3

(Due: October 24, 2014)

1. (a) Sketch the image in the w-plane of the region $\text{Re}(z) > A$, under the mapping $w = 1/z$. Indicate on your sketch the images of (i) lines of constant $\text{Re}(z)$ and (ii) lines of constant $\text{Im}(z)$. Be sure to distinguish between the cases $A > 0$ and $A < 0$.

   (b) Repeat part (a) for the mapping $w = e^z$.

   (c) Locate the branch points and draw three distinct sets of branch cuts of the function

   $$w(z) = \sqrt{z(z^2 + 1)}\,.$$

2. (a) Find *all* Laurent or Taylor expansions of the function

   $$f(z) = \frac{z}{z^2 - 1}$$

   about the point $z = 2$, i.e. expand the function as a series of the form

   $$f(z) = \sum_{n=-\infty}^{\infty} c_n \, (z - 2)^n.$$

   Note that there are several different regions in which different expansions apply.

   (b) Fill in the blanks, where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of the analytic function $w(z)$:

   (i) $u(x, y) = e^x \cos y, \quad v(x, y) = ?, \quad w(z) = ?$

   (ii) $u(x, y) = ?, \quad v(x, y) = y(3x^2 - y^2 - 1), \quad w(z) = ?$

   (iii) $u(x, y) = ?, \quad v(x, y) = ?, \quad w(z) = \tan^{-1} z.$

3. Evaluate the following integrals, all taken counterclockwise around the circle $|z| = 1.5$

   (a) $\oint \frac{\cos z}{z} \, dz$ (b) $\oint \frac{\tan z}{z} \, dz$ (c) $\oint \frac{2z^2 + 3z - 1}{z - 1 + i} \, dz$ (d) $\oint \frac{dz}{(z - 2)^2}$

4. Evaluate the following integrals using the residue theorem:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3}$ (b) $\displaystyle\int_{0}^{\infty} \frac{x \sin x \, dx}{x^2 + 1}$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)}$ (d) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 e^x \, dx}{1 + e^{2x}}$,

   where a and b are nonzero and $a \neq b$. In each case, sketch the contour you choose and clearly quote all theorems used in the derivation of your results.