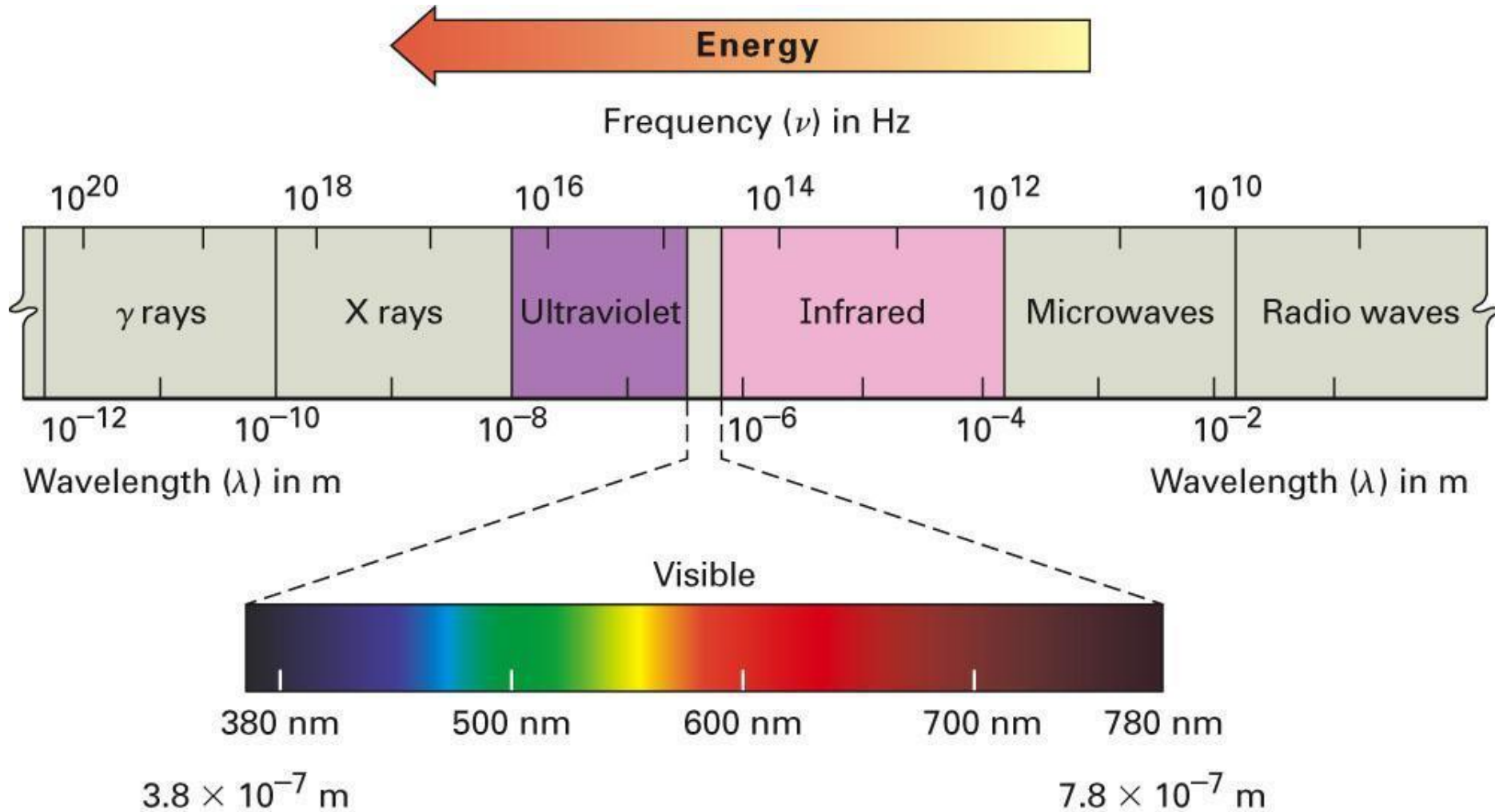


# **Electromagnetic Waves**

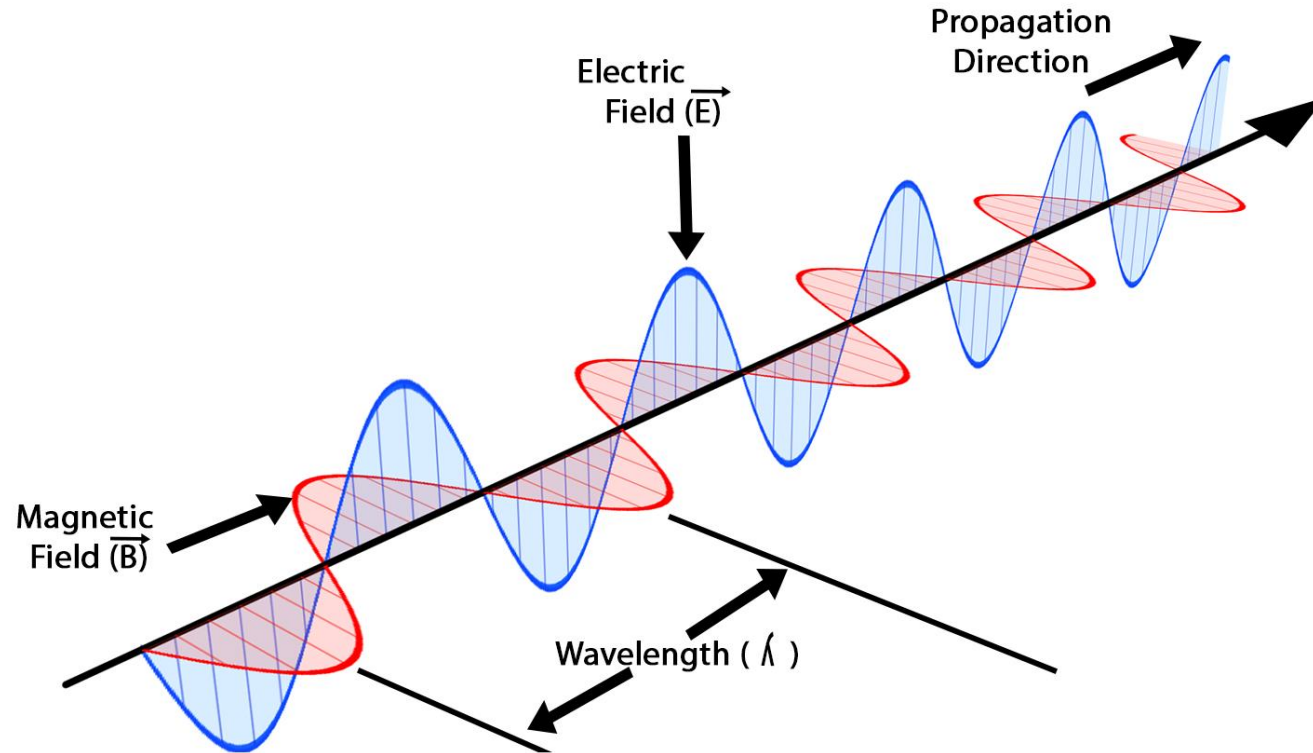
# Contents :

- ☐ Introduction
- ☐ Scalar and vector fields
- ☐ Gradient, divergence, and curl
- ☐ Stokes' and Green's (Gauss') theorems
- ☐ Concept of Displacement current
- ☐ Maxwell's equations
- ☐ Electromagnetic wave equations in free space and conducting media
- ☐ Skin depth and its applications

# Electromagnetic Waves :



# Electromagnetic Waves



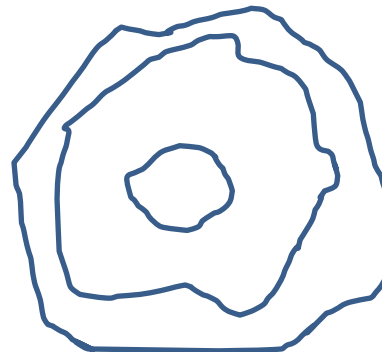
# Properties of Electromagnetic Waves:

- They are transverse in nature i.e. direction of propagation ( $\mathbf{k}$ ) is perpendicular to electric field ( $\mathbf{E}$ ) and magnetic field ( $\mathbf{B}$ ).  
 $(\mathbf{E} \perp \mathbf{B} \perp \mathbf{k})$
- EM waves have oscillating electric and magnetic field.
- They travel with fixed speed ( $3 \times 10^8$  m/s) in vacuum.
- They don't need medium to propagate.
- While travelling through medium, their speed is less than  $c$ .

**Scalar Fields :** like temperature, electric potential etc.

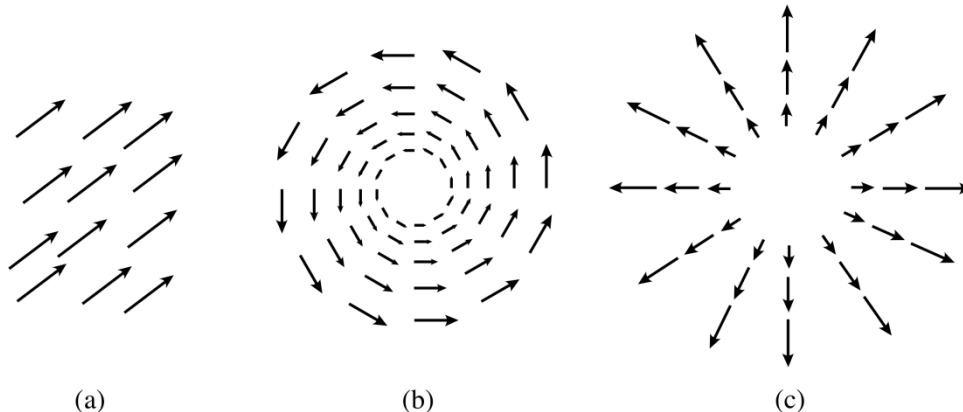
- Scalar fields are used to represent scalar quantities/functions in a region.
- These fields can be represented by contours which are imaginary surfaces drawn through all points where field has same value (called as equipotential surfaces).
- No two equipotential surfaces cut each other.

Ex : equipotential surface :



**Vector Fields :** quantities with magnitude and direction like electric field, force, velocity etc.

- Vector fields are used to represent vector quantities/functions in a region.
- These fields are represented by flux or field lines drawn in such a way that tangent at any point of the line gives direction of vector field at that point.
- Lines representing vector fields can not cross each other because that would give non-unique value at that point.



**Del operator ( $\vec{\nabla}$ )** : Not a scalar or vector but operator.

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

It doesn't have meaning until it acts (not multiply) up on a function.

It is an instruction to differentiate what follows.

There are 3 ways  $\vec{\nabla}$  can act :

1. On a scalar function  $T$  :  $\vec{\nabla}T$  (**gradient**)
2. On a vector function ( $\vec{v}$ ) by dot product :  $\vec{\nabla} \cdot \vec{v}$  (**divergence**)
3. On a vector function ( $\vec{v}$ ) by cross product :  $\vec{\nabla} \times \vec{v}$  (**curl**)



## Gradient $\vec{\nabla}T$ :

Suppose scalar quantity (Let us say temperature)  $T$  is function of  $(x,y,z)$ . Theorem on partial derivatives states

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz \quad (1)$$

Physically this tell us how  $T$  changes when three variables are changed by infinitesimal amounts  $dx$ ,  $dy$ ,  $dz$ .

Equation (1) can be written as dot product

$$\begin{aligned} dT &= \left( \left(\frac{\partial T}{\partial x}\right) \hat{x} + \left(\frac{\partial T}{\partial y}\right) \hat{y} + \left(\frac{\partial T}{\partial z}\right) \hat{z} \right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) \\ &= (\vec{\nabla}T) \cdot (d\vec{l}) \end{aligned} \quad (2)$$

Where  $\vec{\nabla}T = \left(\frac{\partial T}{\partial x}\right) \hat{x} + \left(\frac{\partial T}{\partial y}\right) \hat{y} + \left(\frac{\partial T}{\partial z}\right) \hat{z}$  is gradient of  $T$ . It is vector quantity with three components.

**Gradient  $\vec{\nabla}T$  interpretation :** Gradient has magnitude as well as direction.

$$dT = (\vec{\nabla}T) \cdot (\vec{dl}) = |\vec{\nabla}T| |\vec{dl}| \cos \theta$$

Where  $\theta$  is angle between  $\vec{\nabla}T$  and  $\vec{dl}$ .

If magnitude of  $|\vec{dl}|$  is fixed and  $\theta$  is varied,  $dT$  is maximum when  $\theta = 0$  ( $\cos \theta = 1$ ). It means for fixed distance  $|\vec{dl}|$ ,  $dT$  is maximum when you move in direction of  $\vec{\nabla}T$ .

or

Gradient  $\vec{\nabla}T$  points in direction of maximum increase of function  $T$ .

and

Magnitude  $|\vec{\nabla}T|$  gives the slope along this maximal direction.

## Gradient $\vec{\nabla}T = 0$ meaning :

If  $\vec{\nabla}T = 0$  at  $(x, y, z)$  then  $dT = 0$  for small displacements about the point  $(x, y, z)$ . This is then **stationary point** of the function  $T(x, y, z)$ . It could be a maximum, a minimum or a shoulder.

If you want to locate extrema of a function of three variables, set its gradient equal to 0.

## Numerical on gradient :

The height of certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where  $y$  is distance (in miles) in north and  $x$  the distance in east.

- a) Where is the top of hill located?
- b) How high is the hill?
- c) How steep is slope (in feet per mile) at point 1 mile north and 1 mile east?

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

$$\vec{\nabla} h(x, y, z) = \left( \frac{\partial h}{\partial x} \right) \hat{x} + \left( \frac{\partial h}{\partial y} \right) \hat{y}$$

$$\vec{\nabla} h(x, y, z) = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$$

**(a) Where is the top of hill located ?**

Remember to find maxima, minima, you put gradient = 0

$$2y - 6x - 18 = 0$$

$$2x - 8y + 28 = 0$$

Solving these  $x = -2$ ,  $y = 3$  (location of top of hill)

**(b) How high is the hill?**

Putting  $x = -2$  and  $y = 3$ ,

$$h = 10(-12 - 12 - 36 + 36 + 84 + 12) = 720 \text{ ft}$$

**(c) How steep is slope (in feet per mile) at point 1 mile north and 1 mile east?**

Remember Magnitude  $|\vec{\nabla}T|$  gives the slope.

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

$$\vec{\nabla}h(x, y, z) = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$$

Putting  $x = 1$  and  $y = 1$

$$\vec{\nabla}h(x, y, z) = 10(2 - 6 - 18)\hat{x} + 10(2 - 8 + 28)\hat{y}$$

$$\vec{\nabla}h(x, y, z) = -220\hat{x} + 220\hat{y}$$

$$|\vec{\nabla}h| = 220\sqrt{2}$$

## Divergence $\vec{\nabla} \cdot \vec{v}$ :

From definition of  $\vec{\nabla}$ , divergence will be

$$\begin{aligned}\vec{\nabla} \cdot \vec{v} &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{x} + v_y \hat{y} + v_z \hat{z}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\end{aligned}$$

Divergence of vector function is a scalar quantity.

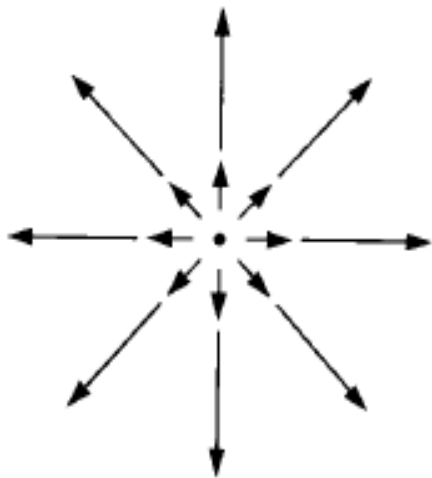
If  $\vec{\nabla} \cdot \vec{v} = 0$ , then it is called solenoidal field.

Ex: Calculate divergence of function  $\vec{v} = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

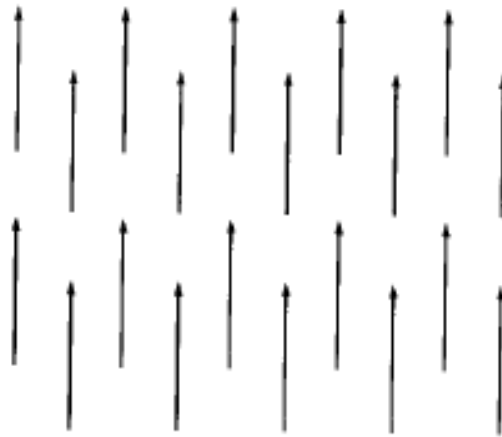
## Geometrical Interpretation of divergence ( $\vec{\nabla} \cdot \vec{v}$ ):

$\vec{\nabla} \cdot \vec{v}$  is a measure of how much the vector  $\vec{v}$  spreads out (diverges) from the point in question.

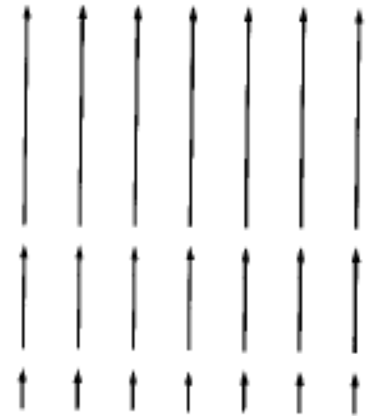
A point of positive divergence is a “source” and a point of negative divergence is a “sink” or “drain”.



Positive  
divergence

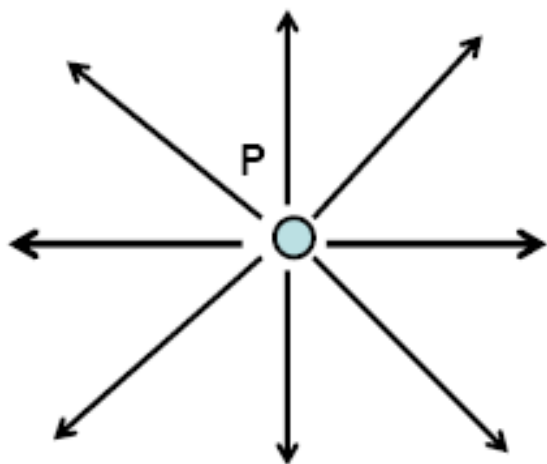


Zero  
divergence

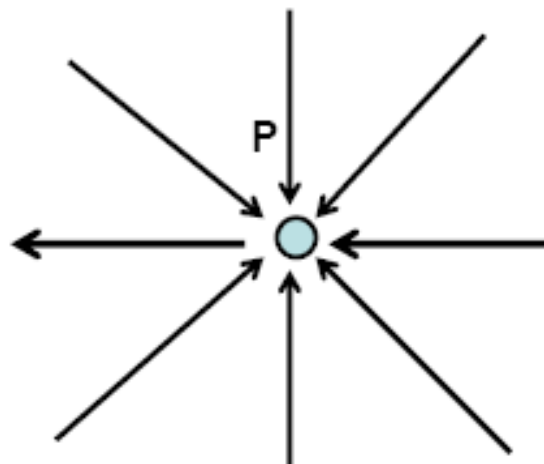


Positive  
divergence

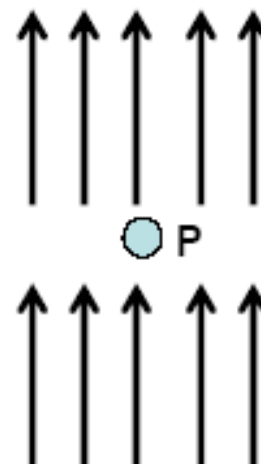




Positive  
Divergence



Negative  
Divergence



Zero  
Divergence

## The curl ( $\vec{\nabla} \times \vec{v}$ ):

From definition of  $\vec{\nabla}$ , curl will be

$$\begin{aligned}\vec{\nabla} \times \vec{v} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{x} \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{y} \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{z} \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)\end{aligned}$$

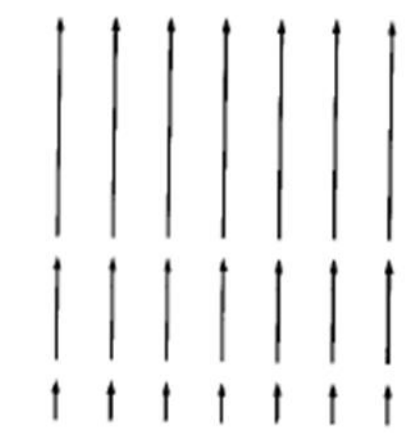
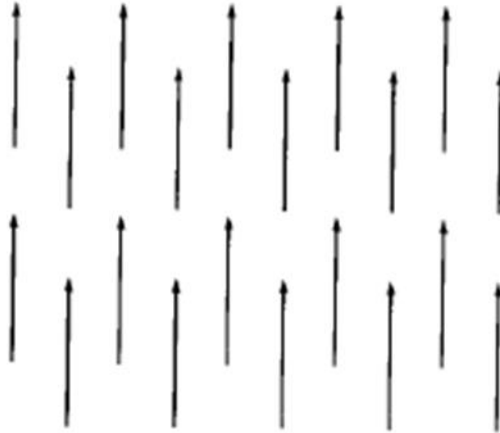
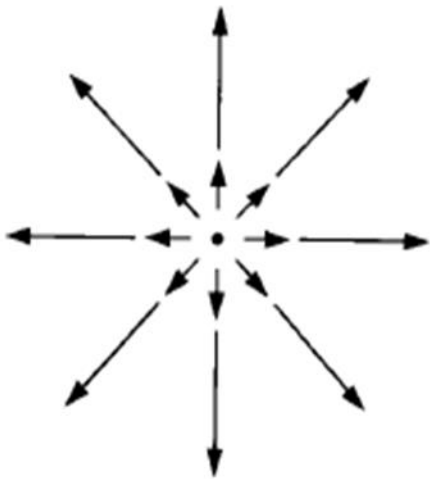
Curl of vector function  $\vec{v}$  is, like any cross product, a vector.

If  $\vec{\nabla} \times \vec{v} = \mathbf{0}$ , then it is called irrotational field.

# Geometrical interpretation of the curl ( $\vec{\nabla} \times \vec{v}$ ):

$\vec{\nabla} \times \vec{v}$  is a measure of how much the vector curls/swirls around the point in question.

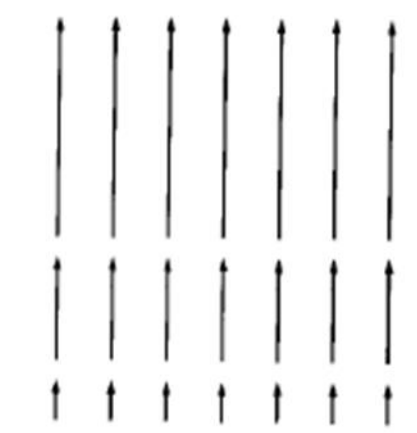
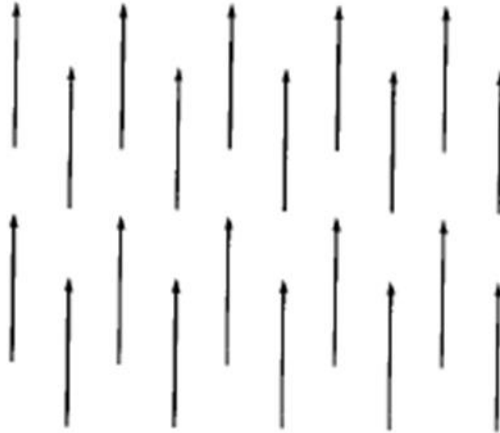
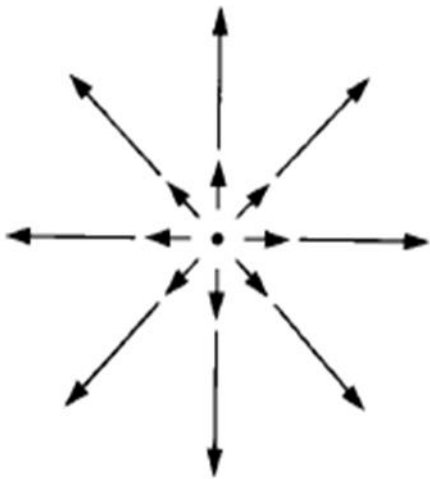
What is curl of these functions?



# Geometrical interpretation of the curl ( $\vec{\nabla} \times \vec{v}$ ):

$\vec{\nabla} \times \vec{v}$  is a measure of how much the vector curls/swirls around the point in question.

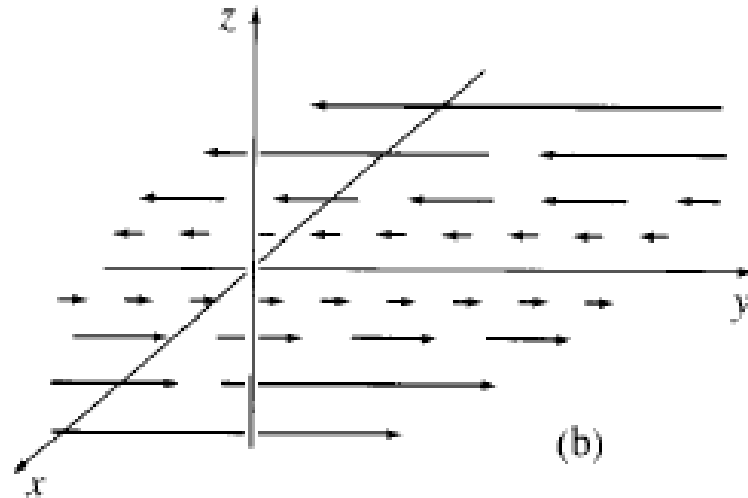
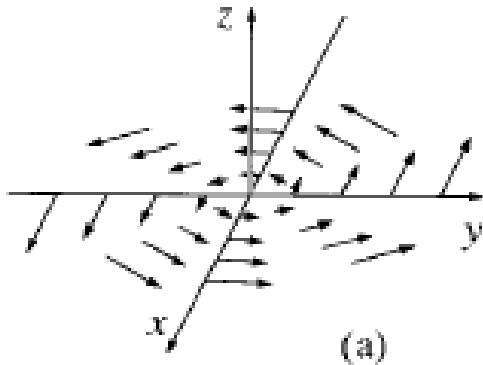
What is curl of these functions?



**Zero**

## Geometrical interpretation of the curl ( $\vec{\nabla} \times \vec{v}$ ):

$\vec{\nabla} \times \vec{v}$  is a measure of how much the vector curls/swirls around the point in question.



These functions have curls pointing in z-direction (given by right hand rule : Curl your fingers in direction of swirl, then thumb gives direction of curl.)

**Calculate curl of the function  $\vec{v} = -y \hat{x} + x \hat{y}$ .**

# Fundamental Theorem for divergences (Green's theorem or Gauss's theorem):

This theorem states that :

$$\int_V (\vec{\nabla} \cdot \vec{v}) d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

(V represents volume and  $\vec{v}$  represents a vector)

$d\tau$  is integration over volume V ( $dx dy dz$ ).

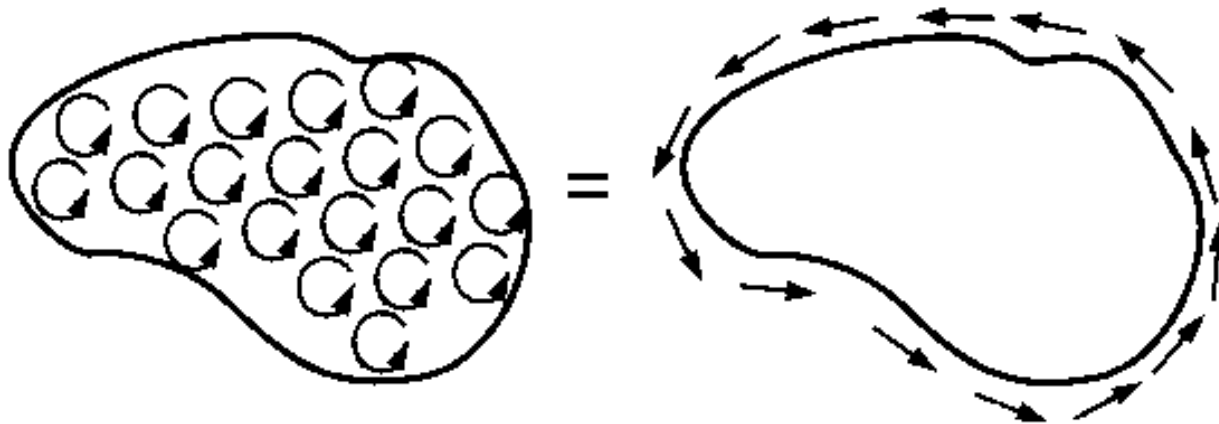
In words, it says that the integral of a divergence over a volume V is equal to the value of the function at the boundary (here, surface S that bounds volume).

The boundary of a volume is a closed surface, that of a surface is a closed line. But the boundary of a line is just two points.

## Fundamental Theorem for curls (Stoke's theorem):

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

It says that integral of curl over a region of surface S is equal to the value of function at the boundary (here perimeter of surface, P).



# Maxwell's first equation (Gauss's Law) :

Flux of  $\vec{E}$  through a surface S is measure of “number of field lines” passing through S

$$\varphi_E \equiv \int_S \vec{E} \cdot d\vec{a}$$

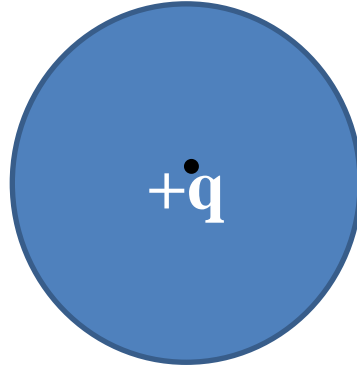
(The dot product picks out area  $d\vec{a}$  in direction of  $\vec{E}$ .)

This means total flux through a **closed surface** is measure of total charge inside. This is essence of **Gauss's Law**.



## Maxwell's first equation (Gauss's Law) :

If there is point charge at origin, flux  $E$  through a spherical radius of  $r$  is



$$\oint \vec{E} \cdot d\vec{a} = \int \left( \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$$
$$= \frac{q}{4\pi\epsilon_0} \int \sin\theta \, d\theta \, d\phi = \frac{q}{\epsilon_0}$$

For any closed surface, whatever its shape, would be pierced by same number of field lines. Hence, flux through any surface enclosing charge  $q$  is  $q/\epsilon_0$ .

## Maxwell's first equation (Gauss's Law) :

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \quad (1)$$

Applying divergence theorem  $\int_V (\vec{\nabla} \cdot \vec{v}) d\tau = \oint_S \vec{v} \cdot d\vec{a}$  on

L.H.S., eq. (1) can be written

$$\int_V (\vec{\nabla} \cdot \vec{E}) d\tau = \frac{q}{\epsilon_0} \Rightarrow \int_V (\vec{\nabla} \cdot \vec{E}) d\tau = \int_V \frac{\rho}{\epsilon_0} d\tau \quad \left( \because q = \int_V \rho d\tau \right)$$

***Hence,  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$***  (Maxwell's First Equation in differential form)

## Maxwell's Second equation:

Flux of  $\vec{B}$  through a surface S is measure of “number of field lines” passing through S

$$\varphi_B \equiv \int \vec{B} \cdot d\vec{a}$$

(The dot product picks out area  $d\vec{a}$  in direction of  $\vec{B}$ .) For a closed surface S,

$$\oint_S \vec{B} \cdot d\vec{a} = 0 \quad \text{(Because number of lines entering and leaving closed surface has to be same)}$$

$$\oint_S \vec{B} \cdot d\vec{a} = 0 \quad (2)$$

Applying divergence theorem  $\int_V (\vec{\nabla} \cdot \vec{v}) d\tau = \oint_S \vec{v} \cdot d\vec{a}$

$$\int_V (\vec{\nabla} \cdot \vec{B}) d\tau = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = 0 \quad (\text{Maxwell's Second Equation in differential form})$$

## Maxwell's third equation (Faraday's Law):

Faraday's law of electromagnetic induction says that the induced emf ( $\varepsilon$ ) is rate of change of magnetic flux ( $\varphi$ )

$$\varepsilon = -\frac{d\varphi}{dt}$$

$$\varepsilon = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a}$$

$$\varepsilon = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \quad (2)$$

## Maxwell's third equation (Faraday's Law):

$\varepsilon$  is actually potential or energy per unit charge. So,  $\varepsilon$  is work done in carrying a unit positive charge around a closed loop. So,

$$\varepsilon = \oint_P \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} \quad (3)$$

Using Stoke's theorem  $\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint_P \vec{E} \cdot d\vec{l}$

From eq. (2) and (3)  $\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$

$$\Rightarrow \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad \text{(Maxwell's Third Equation in differential form)}$$

## Maxwell's third equation (Faraday's Law):

$$\Rightarrow \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

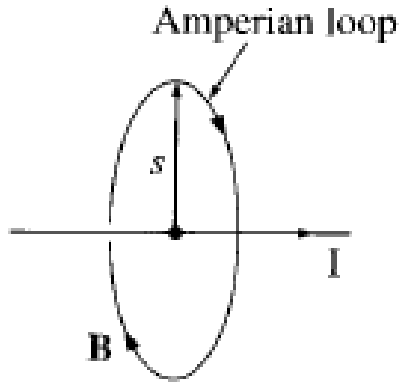
Taking divergence on both sides

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = - \frac{\partial (\vec{\nabla} \cdot \vec{B})}{\partial t}$$

L. H. S. of above eq. is zero because divergence of curl of any vector is zero and R. H. S. is zero from Maxwell's second equation ( $\vec{\nabla} \cdot \vec{B} = 0$ ).

**Hence, everything is okay so far.**

# Ampere's Circuital Law:



According to Ampere's circuital law, if  $d\vec{l}$  is perimeter of Amperian loop and  $I_{enc}$  is current enclosed by that loop

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \quad (4)$$

Using Stoke's theorem:  $\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$

And

$$I_{enc} = \int \vec{J} \cdot d\vec{a}$$

Hence, equation (4) becomes

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

Therefore,  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$  (Ampere's Circuital Law in differential form)



# Electrodynamics before Maxwell:

$$1. \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

(Gauss's Law)

$$2. \quad \vec{\nabla} \cdot \vec{B} = 0$$

(No Name)

$$3. \quad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

(Faraday's Law)

$$4. \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

(Ampere's Law)

## Equation of continuity:

Current density ( $\vec{J}$ ) : Defined as current per unit area (area being parallel to direction of flow)

$$I = \int_S \vec{J} \cdot d\vec{a}$$

The **charge per unit time leaving a volume** is

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{J}) d\tau \quad (\text{Using Gauss' divergence theorem})$$

Because charge is conserved, so whatever is flowing through surface must come at expense of what remains inside

$$\int_V (\vec{\nabla} \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_V \rho d\tau = - \int_V \left( \frac{\partial \rho}{\partial t} \right) d\tau$$

## Equation of continuity:

$$\int_{\underline{V}} (\vec{\nabla} \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_{\underline{V}} \rho d\tau = - \int_{\underline{V}} \left( \frac{\partial \rho}{\partial t} \right) d\tau$$

$\rho$  is the charge density (charge per unit volume).

(-ve sign because the outward flow decreases the charge left in volume  $V$ .)

$$\Rightarrow \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad \textbf{(Equation of continuity)}$$

When a steady current ( $I$ ) is flowing through a wire then its magnitude  $I$  must be same all along line; otherwise charge would be piling up somewhere. Because  $\frac{\partial \rho}{\partial t} = 0$ , hence  $\vec{\nabla} \cdot \vec{J} = 0$ .

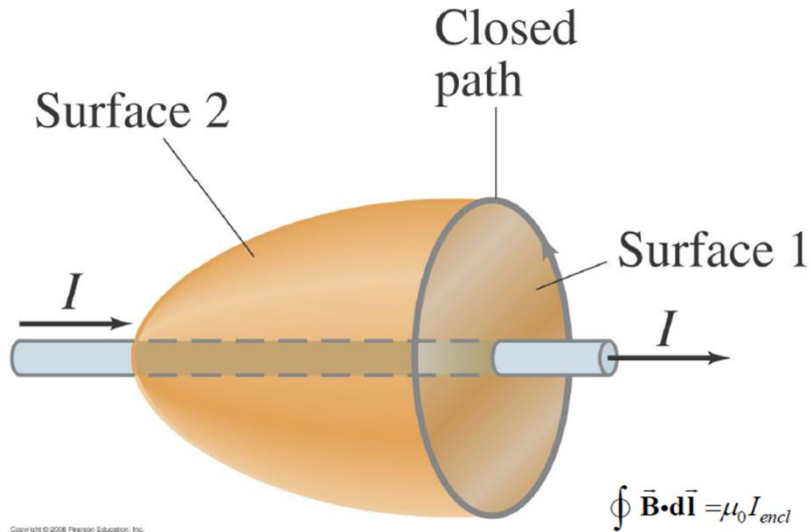
## Problem with Ampere's Circuital Law:

Ampere's circuital law states that  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Taking divergence on both sides  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J})$

As discussed before divergence of curl for any vector is zero , so, L.H.S. is zero but R. H.S. might not be zero necessarily. Actually. R.H.S. is zero only when a steady current is flowing (As discussed in continuity equation).

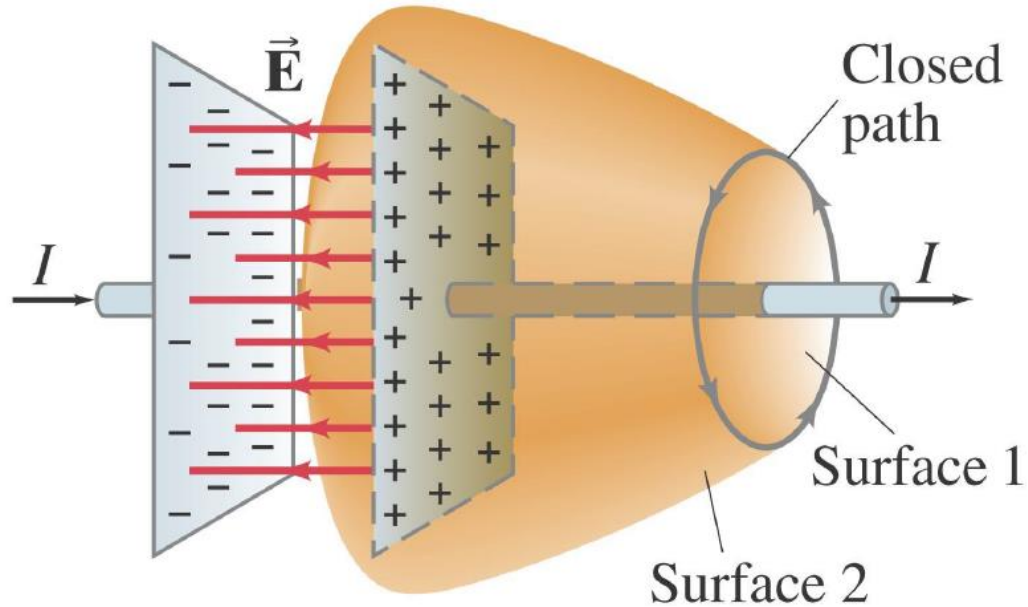
# Problem with Ampere's Circuital Law:



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

There could be two surfaces for which closed path is boundary and for both surfaces same amount of current pierces the surface. So, **Ampere's law works fine!!**

# Problem with Ampere's Circuital Law:



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

But when charge is piling up somewhere like in case of capacitor, **Ampere's circuital law fails**. Again for closed path as boundary, there are 2 surfaces shown in Fig. and for surface 1, current  $I$  pierces it but no current pierces surface 2 which is contradictory.

# How Maxwell fixed Ampere's Circuital Law:

Maxwell fixed it by purely theoretical arguments.

Ampere's circuital law states that  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Taking divergence on both sides  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J})$

L.H.S. is zero but R. H.S. might not be zero which is issue. R. H. S. can be rewritten using continuity equation :

$$\vec{\nabla} \cdot \vec{J} = - \frac{\partial \rho}{\partial t} \quad \text{Using Gauss Law : } \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{J} = - \frac{\partial}{\partial t} (\epsilon_0 \vec{\nabla} \cdot \vec{E}) = - \vec{\nabla} \cdot \left( \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

## How Maxwell fixed Ampere's Circuital Law:

$$\vec{\nabla} \cdot \vec{J} = - \frac{\partial}{\partial t} (\epsilon_0 \vec{\nabla} \cdot \vec{E}) = - \vec{\nabla} \cdot \left( \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \quad (5)$$

Ampere's circuital law states that  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Taking divergence on both sides  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J}) \quad (6)$

Problem was R. H. S. of equation (6) not being zero, but if we add negative of (5) in equation (6) R. H.S., then this too will be zero i.e.

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) &= \mu_0 \vec{\nabla} \cdot \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0 \\ \Rightarrow (\vec{\nabla} \times \vec{B}) &= \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \end{aligned}$$



# How Maxwell fixed Ampere's Circuital Law:

$$(\vec{\nabla} \times \vec{B}) = \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \quad \text{(Maxwell's fourth equation)}$$

Changing electric field produces magnetic field just as changing Magnetic field induces an electric field (Faraday's law)!!

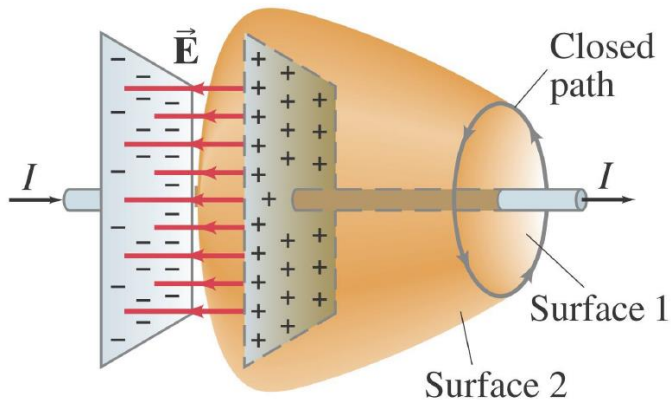
Maxwell called his extra term as displacement current

$$\vec{J}_D \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Now, Ampere's circuital law becomes

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left( \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$$

# How Maxwell fixed Ampere's Circuital Law:



Electric field between capacitor plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} \Rightarrow \frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A} \frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

Ampere's circuital law after Maxwell's change :

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left( \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$$

For surface 1,  $E = 0$  and  $I_{enc} = I$ . For surface 2,  $I_{enc} = 0$ , Hence

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \int \left( \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a} = \mu_0 \epsilon_0 \frac{I}{\epsilon_0} = \mu_0 I$$

Hence, we get same answer for either surface!!

# Maxwell's Equations :

1.  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$  (Gauss's Law)

2.  $\vec{\nabla} \cdot \vec{B} = 0$  (No Name)

3.  $\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$  (Faraday's Law)

4.  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$  (Ampere's Law with  
Maxwell's correction)

# EM wave equation in free space/vacuum :

In free space, charge density  $\rho = 0$ , current density  $\vec{J} = 0$ . Hence, Maxwell's equations are

1.  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$  becomes  $\vec{\nabla} \cdot \vec{E} = 0$

2.  $\vec{\nabla} \cdot \vec{B} = 0$  becomes  $\vec{\nabla} \cdot \vec{B} = 0$

3.  $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$  becomes  $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

4.  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$  becomes  $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Here  $\mu_0$  is permeability and  $\epsilon_0$  is permittivity of free space.

# EM wave equation in free space/vacuum :

Taking curl of Maxwell's third equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \left( \frac{\partial \vec{B}}{\partial t} \right)$$
$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

(because  $\vec{\nabla} \cdot \vec{E} = 0$  and  $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ )

$$\Rightarrow 0 - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left( \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E} \quad (7)$$

# EM wave equation in free space/vacuum :

Taking curl of Maxwell's fourth equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times (\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

(because  $\vec{\nabla} \cdot \vec{B} = 0$  and  $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ )

$$\Rightarrow 0 - \nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{B} \quad (8)$$

## EM wave equation in free space/vacuum :

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E} \quad (7) \quad \text{and} \quad \frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{B} \quad (8)$$

Equation of plane wave travelling in  $x$  direction is given by

$$y(x, t) = A \sin(kx - \omega t)$$

And

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad \left(v = \frac{\omega}{k}\right) \quad (9)$$

Comparing equation (9) with (7) and (8), speed of  $\vec{E}$  and  $\vec{B}$  will be given by  $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/sec}.$

# EM wave equation in free space/vacuum :

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E} \quad (7)$$

Solution of equation (7) will be

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{E}(\vec{r}, t) = (E_{0x} \hat{x} + E_{0y} \hat{y} + E_{0z} \hat{z}) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\begin{aligned} \vec{k} \cdot \vec{r} &= (k_x \hat{x} + k_y \hat{y} + k_z \hat{z}) \cdot (x \hat{x} + y \hat{y} + z \hat{z}) \\ &= (k_x x + k_y y + k_z z) \end{aligned}$$

or  $\vec{E}(\vec{r}, t) = (E_x \hat{x} + E_y \hat{y} + E_z \hat{z})$  where

$$E_x = E_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, E_y = E_{0y} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, E_z = E_{0z} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$



# EM wave equation in free space/vacuum :

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{B} \quad (8)$$

Solution of equation (8) will be

$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{B}(\vec{r}, t) = (B_{0x} \hat{x} + B_{0y} \hat{y} + B_{0z} \hat{z}) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\begin{aligned} \vec{k} \cdot \vec{r} &= (k_x \hat{x} + k_y \hat{y} + k_z \hat{z}) \cdot (x \hat{x} + y \hat{y} + z \hat{z}) \\ &= (k_x x + k_y y + k_z z) \end{aligned}$$

or  $\vec{B}(\vec{r}, t) = (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$  where

$$B_x = B_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, B_y = B_{0y} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, B_z = B_{0z} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

## EM wave equation in free space/vacuum :

$$\vec{E}(\vec{r}, t) = (E_x \hat{x} + E_y \hat{y} + E_z \hat{z})$$

$$E_x = E_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, E_y = E_{0y} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, E_z = E_{0z} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Putting these values in Maxwell's first equation in free space

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad (9)$$

$$\frac{\partial E_x}{\partial x} = ik_x E_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)} = i k_x E_x$$

$$\text{Similarly } \frac{\partial E_y}{\partial y} = ik_y E_y \quad \text{and} \quad \frac{\partial E_z}{\partial z} = ik_z E_z$$

$$\begin{aligned} \text{Hence equation (9) becomes} \quad ik_x E_x + ik_y E_y + ik_z E_z &= 0 \\ \Rightarrow i(\vec{k} \cdot \vec{E}) &= 0 \end{aligned}$$

**Dot product of two vectors is zero when they are perpendicular to each other. It means  $\vec{k} \perp \vec{E}$ .**

## EM wave equation in free space/vacuum :

$$\vec{B}(\vec{r}, t) = (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$B_x = B_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, B_y = B_{0y} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, B_z = B_{0z} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Putting these values in Maxwell's first equation in free space

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (10)$$

$$\frac{\partial B_x}{\partial x} = i k_x B_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)} = i k_x B_x$$

$$\text{Similarly } \frac{\partial B_y}{\partial y} = i k_y B_y \quad \text{and} \quad \frac{\partial B_z}{\partial z} = i k_z B_z$$

$$\begin{aligned} \text{Hence equation (10) becomes} \quad & i k_x B_x + i k_y B_y + i k_z B_z = 0 \\ & \Rightarrow i(\vec{k} \cdot \vec{B}) = 0 \end{aligned}$$

**It means  $\vec{k} \perp \vec{B}$ .**

## EM wave equation in free space/vacuum :

Maxwell's third equation : 
$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (11)$$

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \\ &= \hat{x} \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \hat{y} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + \hat{z} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \\ &= \hat{x} (ik_y E_z - ik_z E_y) - \hat{y} (ik_x E_z - ik_z E_x) + \hat{z} (ik_x E_y - ik_y E_x) \\ &= i (\vec{k} \times \vec{E}) \quad (12) \end{aligned}$$

$$(E_x = E_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, E_y = E_{0y} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, E_z = E_{0z} e^{i(\vec{k} \cdot \vec{r} - \omega t)})$$
$$\frac{\partial E_z}{\partial y} = ik_y E_{0z} e^{i(\vec{k} \cdot \vec{r} - \omega t)} = i k_y E_z \quad \text{and so on.....)}$$

## EM wave equation in free space/vacuum :

Maxwell's third equation : 
$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (11)$$

$$\vec{\nabla} \times \vec{E} = i (\vec{k} \times \vec{E})$$

And 
$$- \frac{\partial \vec{B}}{\partial t} = - \frac{\partial}{\partial t} \left( \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right) = i\omega \vec{B}$$

Putting value of L.H.S. and R. H.S. in equation (11)

$$i (\vec{k} \times \vec{E}) = i\omega \vec{B} \quad \text{or} \quad (\vec{k} \times \vec{E}) = \omega \vec{B} \quad (12)$$

**From equation (12),  $\vec{B} \perp \vec{E}$  and  $\vec{B} \perp \vec{k}$ . We already proved  $\vec{E} \perp \vec{k}$  which means  $\vec{B} \perp \vec{E} \perp \vec{k}$ .**

**Hence, EM waves are transverse in nature.**

# EM wave equation in free space/vacuum :

We just derived  $(\vec{k} \times \vec{E}) = \omega \vec{B}$

$$|\vec{k}| |\vec{E}| \sin 90^\circ = \omega |\vec{B}| \quad (13) \quad (\text{because } \vec{E} \perp \vec{k})$$

$$(\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \text{ and } \vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)})$$

$$|\vec{B}| = \sqrt{\vec{B} \vec{B}^*} = B_0 \quad \text{and similarly} \quad |\vec{E}| = \sqrt{\vec{E} \vec{E}^*} = E_0$$

So, equation (13) becomes

$$E_0 = \frac{\omega}{k} B_0 = c B_0 \quad (14) \quad \because \text{velocity } (c) = \frac{\omega}{k}$$

Hence, magnitude of electric field is  $c$  times magnitude of magnetic field. That is why direction of polarisation is denoted by electric field conventionally.

# EM wave equation in free space/vacuum :

Equation (14) states that  $E_0 = c B_0$

**Space impedance ( $Z_0$ )** is defined as

$$Z_0 = \left| \frac{E}{H} \right| = \mu_0 \frac{E_0}{B_0} \quad (B = \mu H)$$

$$= \mu_0 c \frac{B_0}{B_0} = \mu_0 c$$

$$= \mu_0 \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$= 376.7 \, \Omega$$

# EM wave equation in conducting medium :

From equation of continuity  $\vec{\nabla} \cdot \vec{J} = - \frac{\partial \rho}{\partial t}$

$$\Rightarrow \frac{\partial \rho}{\partial t} = - (\vec{\nabla} \cdot \sigma \vec{E})$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\sigma (\vec{\nabla} \cdot \vec{E})$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\sigma \frac{\rho}{\epsilon}$$

$$\Rightarrow \frac{\partial \rho}{\rho} = -\frac{\sigma}{\epsilon} dt$$

Integrating on both sides  $\ln(\rho) \Big|_{\rho_0}^{\rho} = -\frac{\sigma}{\epsilon} (t) \Big|_0^t$

$$\rho = \rho_0 e^{-\left(\frac{\sigma}{\epsilon}\right)t}$$



# EM wave equation in conducting medium :

$$\rho = \rho_0 e^{-\left(\frac{\sigma}{\epsilon}\right)t}$$

Characteristic time or charge relaxation time ( $\tau = \epsilon/\sigma$ ) is defined as time in which charge reduces to 1/e of its initial value.

Characteristic time is a measure of how good conductor is. Smaller it is, better conductor it is.

For ex. Characteristic time for Cu is  $4.5 \times 10^{-19} \text{ sec}$ .

**It means you can assume there is no charge inside the conductor since it immediately flows to the surface.**

# EM wave equation in conducting medium :

We just said that we can assume there is no charge inside the conductor so, one can say then  $\rho = 0$  inside a conductor.

Let us say that  $\mu$  is permeability and  $\epsilon$  is permittivity of this medium.

Maxwell's equations for conducting medium are :

$$1. \quad \vec{\nabla} \cdot \vec{E} = 0 \quad (\text{as } \rho = 0 \text{ inside a conductor})$$

$$2. \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$3. \quad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$4. \quad \vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t}$$

Here  $\mu$  is permeability and  $\epsilon$  is permittivity of conducting medium.

## EM wave equation in conducting medium :

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

Taking curl on both sides  $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = - \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = - \frac{\partial}{\partial t} \left( \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$\Rightarrow 0 - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = - \frac{\partial}{\partial t} \left( \mu\sigma\vec{E} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \right) \quad (\because \vec{\nabla} \cdot \vec{E} = 0 \text{ and } \vec{J} = \sigma\vec{E})$$

$$\Rightarrow \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad (13)$$

## EM wave equation in conducting medium :

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

Taking curl on both sides

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu (\vec{\nabla} \times \vec{J}) + \mu\epsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{B} = \mu (\vec{\nabla} \times \sigma \vec{E}) + \mu\epsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) \quad (\because \vec{J} = \sigma \vec{E})$$

$$\Rightarrow 0 - \nabla^2 \vec{B} = \left( -\mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \right) \quad (\because \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t})$$

$$\Rightarrow \nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \quad (14)$$

# EM wave equation in conducting medium :

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Since we have established EM waves are transverse in nature. So, if EM wave is travelling along z-axis, Electric field along y-axis then magnetic field will be along x-axis i.e.

$$\vec{E} = E_{0y} e^{i(kz - \omega t)} \hat{y} \qquad \vec{B} = B_{0x} e^{i(kz - \omega t)} \hat{x}$$

Putting value of  $\vec{E}$  in above equation (13) :

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (E_{0y} e^{i(kz - \omega t)}) = \mu\sigma \frac{\partial}{\partial t} (E_{0y} e^{i(kz - \omega t)}) + \mu\epsilon \frac{\partial^2}{\partial t^2} (E_{0y} e^{i(kz - \omega t)})$$

$$\Rightarrow (ik)^2 E_{0y} e^{i(kz - \omega t)} = \mu\sigma (-i\omega) E_{0y} e^{i(kz - \omega t)} + \mu\epsilon (-i\omega)^2 E_{0y} e^{i(kz - \omega t)}$$

# EM wave equation in conducting medium :

$$\Rightarrow (ik)^2 E_{0y} e^{i(kz - \omega t)} = \mu\sigma(-i\omega) E_{0y} e^{i(kz - \omega t)} + \mu\epsilon (-i\omega)^2 E_{0y} e^{i(kz - \omega t)}$$

$$\Rightarrow (ik)^2 = \mu\sigma(-i\omega) + \mu\epsilon (-i\omega)^2$$

$$\Rightarrow -k^2 = -i\mu\sigma\omega - \mu\epsilon \omega^2$$

$$\Rightarrow k^2 = i\mu\sigma\omega + \mu\epsilon \omega^2 \quad (15)$$

You can get equation (15) by putting value of  $\vec{B}$  in equation (14).  
Equation (15) also implies that  $k$  can be written as

$$k = k_+ + ik_- \quad \Rightarrow k^2 = k_+^2 - k_-^2 + 2ik_+k_- \quad (16)$$

Comparing eq. (15) and (16)

$$\mu\epsilon \omega^2 = k_+^2 - k_-^2$$

$$\mu\sigma\omega = 2k_+k_- \quad \Rightarrow k_- = \frac{\mu\sigma\omega}{2k_+}$$

# EM wave equation in conducting medium :

$$\mu\epsilon\omega^2 = k_+^2 - k_-^2 \quad (17)$$

$$\mu\sigma\omega = 2k_+k_- \quad \Rightarrow \quad k_- = \frac{\mu\sigma\omega}{2k_+} \quad (18)$$

Putting value of  $k_-$  from eq. (18) to (17)

$$\begin{aligned} \mu\epsilon\omega^2 &= k_+^2 - \left(\frac{\mu\sigma\omega}{2k_+}\right)^2 \\ \Rightarrow k_+^4 - \mu\epsilon\omega^2 k_+^2 - \left(\frac{\mu\sigma\omega}{2}\right)^2 &= 0 \end{aligned}$$

This is quadratic equation, solving it we will get

$$k_+^2 = \frac{\mu\epsilon\omega^2 \pm \sqrt{(\mu\epsilon\omega^2)^2 + (\mu\sigma\omega)^2}}{2}$$

$$\text{Or } k_+^2 = \frac{\mu\epsilon\omega^2}{2} \left[ 1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]$$

# EM wave equation in conducting medium :

$$\text{Or } k_+^2 = \frac{\mu\epsilon\omega^2}{2} \left[ 1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]$$

On physical grounds, -ve sign is not acceptable, because  $k_+^2$  will be negative. From previous equations :

$$\mu\epsilon\omega^2 = k_+^2 - k_-^2$$

$$\mu\sigma\omega = 2k_+k_- \quad \Rightarrow \quad k_+ = \frac{\mu\sigma\omega}{2k_-}$$

Solving this time for  $k_-$  as before done for  $k_+$ , one will get

$$\text{Or } k_-^2 = \frac{\mu\epsilon\omega^2}{2} \left[ -1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]$$

Again, on physical grounds, -ve sign ( $\pm$  between brackets) is not acceptable, because  $k_-^2$  will be negative



# EM wave equation in conducting medium :

$$k_{\pm}^2 = \frac{\mu\epsilon\omega^2}{2} \left[ \pm 1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]$$

$$\Rightarrow k_+ = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ 1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]^{1/2} \quad \text{and} \quad k_- = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ -1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]^{1/2}$$

Both  $k_+$  as well as  $k_-$  are frequency ( $\omega$ ) dependent. We had defined wave vector ( $k$ ), electric field ( $\vec{E}$ ) and magnetic field ( $\vec{B}$ ) as

$$k = k_+ + ik_- \quad \text{and} \quad \vec{E} = E_{0y} e^{i(kz - \omega t)} \hat{y} \quad \text{and} \quad \vec{B} = B_{0x} e^{i(kz - \omega t)} \hat{x}$$

Electric field ( $\vec{E}$ ) and magnetic field ( $\vec{B}$ ) can be now written as :

$$\vec{E} = E_{0y} e^{i((k_+ + ik_-)z - \omega t)} \hat{y} \qquad \vec{B} = B_{0x} e^{i(((k_+ + ik_-)z - \omega t)} \hat{x}$$

$$\text{or} \quad \vec{E} = E_{0y} \underbrace{e^{-k_- z}}_{\text{Damping term}} e^{i(k_+ z - \omega t)} \hat{y} \qquad \vec{B} = B_{0x} \underbrace{e^{-k_- z}}_{\text{Damping term}} e^{i(k_+ z - \omega t)} \hat{x}$$

# EM wave equation in conducting medium :

$$\vec{E} = E_{0y} \underbrace{e^{-k_- z}} \underbrace{e^{i(k_+ z - \omega t)}}_{\text{Damping term}} \hat{y} \quad \vec{B} = B_{0x} \underbrace{e^{-k_- z}} \underbrace{e^{i(k_+ z - \omega t)}}_{\text{Damping term}} \hat{x}$$

**Skin depth ( $\delta$ )** is defined as the distance at which  $\vec{E}$  and  $\vec{B}$  are reduced to  $1/e$  of its initial value.

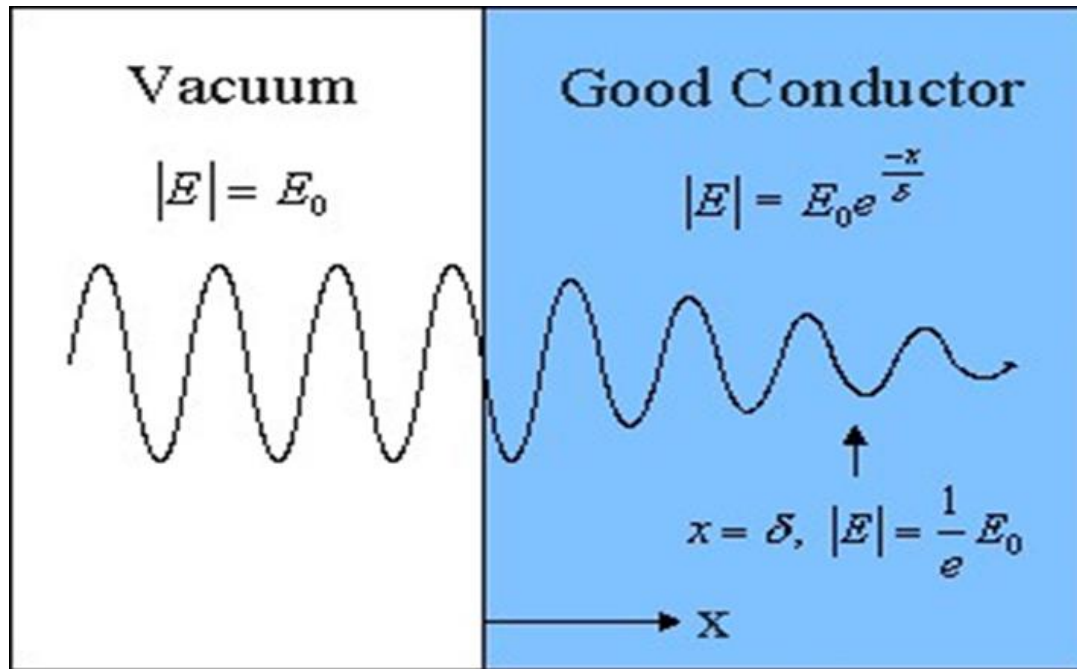
When  $z = \frac{1}{k_-}$ ,  $|\vec{E}| = \frac{E_{0y}}{e}$  and  $|\vec{B}| = \frac{B_{0x}}{e}$ , this value of  $z$  will be skin depth ( $\delta$ ).

Real part of wave vector  $k_+$  determines wavelength ( $\lambda$ ) and propagation speed ( $v$ ).

$$\lambda = \frac{2\pi}{k_+} \quad \text{and} \quad v = \frac{\omega}{k_+}$$

# EM wave equation in conducting medium :

Schematic representation of skin depth (Wave is travelling in x-direction):



# EM wave equation in conducting medium :

For poor conductor ( $\sigma \ll \epsilon\omega$  or  $\frac{\sigma}{\epsilon\omega} \ll 1$ ) :

$$k_+ = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ 1 + \sqrt{1 + \left( \frac{\sigma}{\epsilon\omega} \right)^2} \right]^{1/2}$$

Since  $\frac{\sigma}{\epsilon\omega} \ll 1$ , so it can be neglected

$$\Rightarrow k_+ \cong \omega \sqrt{\frac{\mu\epsilon}{2}} [1 + \sqrt{1}]^{1/2}$$

$$\Rightarrow k_+ \cong \omega \sqrt{\frac{\mu\epsilon}{2}} [2]^{1/2}$$

$$\Rightarrow k_+ \cong \omega \sqrt{\mu\epsilon}$$

# EM wave equation in conducting medium :

For poor conductor ( $\sigma \ll \epsilon\omega$  or  $\frac{\sigma}{\epsilon\omega} \ll 1$ ) :

$$k_- = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ -1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]^{1/2}$$

Using binomial theorem when  $x < 1$ ,  $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

$$\Rightarrow k_- = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ -1 + \left( 1 + \frac{1}{2} \left(\frac{\sigma}{\epsilon\omega}\right)^2 + \dots \right) \right]^{1/2}$$

$$\Rightarrow k_- = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ \frac{1}{2} \left(\frac{\sigma}{\epsilon\omega}\right)^2 \right]^{1/2}$$

$$\Rightarrow k_- = \omega \sqrt{\frac{\mu\epsilon}{2}} \frac{1}{\sqrt{2}} \left(\frac{\sigma}{\epsilon\omega}\right)$$

$$\Rightarrow k_- = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$$

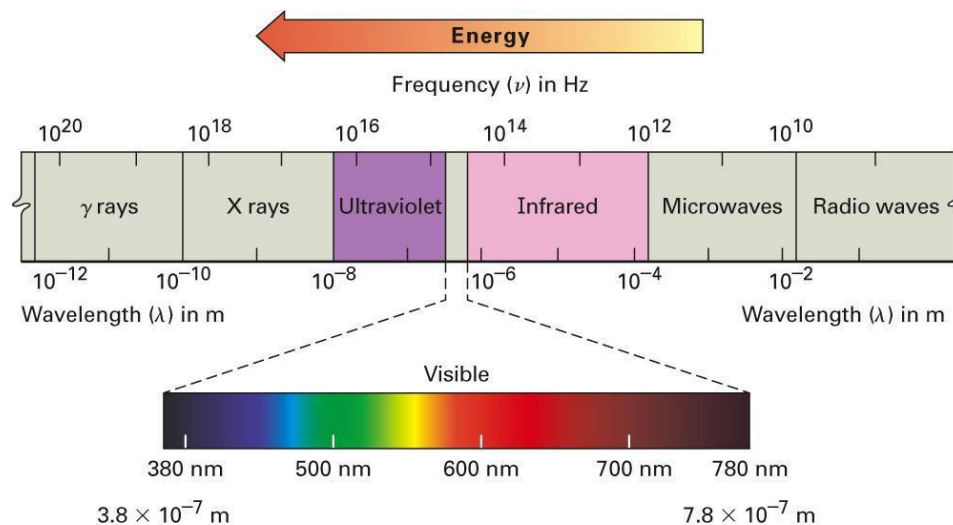
# EM wave equation in conducting medium :

For poor conductor ( $\sigma \ll \epsilon\omega$  or  $\frac{\sigma}{\epsilon\omega} \ll 1$ ) :

$$\Rightarrow k_- = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$$

$$\text{Skin depth } (\delta) = \frac{1}{k_-} = \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu}}$$

*Notice that skin depth ( $\delta$ ) is independent of frequency ( $\omega$ ) for a poor conductor. That means out of spectrum shown below whatever is incident wave, the penetration into medium will be same as long as  $\sigma \ll \epsilon\omega$  is true.*



# EM wave equation in conducting medium :

For good conductor ( $\sigma \gg \epsilon\omega$  or  $\frac{\sigma}{\epsilon\omega} \gg 1$ ) :

$$k_+ = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ 1 + \sqrt{1 + \left( \frac{\sigma}{\epsilon\omega} \right)^2} \right]^{1/2}$$

Since  $\frac{\sigma}{\epsilon\omega} \gg 1$ , so we can ignore 1 in comparison to  $\frac{\sigma}{\epsilon\omega}$  and  $k_+$  will be

$$k_+ \cong \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ \left( \frac{\sigma}{\epsilon\omega} \right) \right]^{1/2}$$

$$\Rightarrow k_+ \cong \sqrt{\frac{\sigma\omega\mu}{2}}$$

# EM wave equation in conducting medium :

For good conductor ( $\sigma \gg \epsilon\omega$  or  $\frac{\sigma}{\epsilon\omega} \gg 1$ ) :

$$k_- = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ -1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right]^{1/2}$$

Since  $\frac{\sigma}{\epsilon\omega} \gg 1$ , so we can ignore 1 in comparison to  $\frac{\sigma}{\epsilon\omega}$  and  $k_-$  will be

$$k_- \cong \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ \left(\frac{\sigma}{\epsilon\omega}\right) \right]^{1/2}$$

$$\Rightarrow k_- \cong \sqrt{\frac{\sigma\omega\mu}{2}}$$

*we just derived*  $k_+ \cong \sqrt{\frac{\sigma\omega\mu}{2}}$

Hence, for good conductor,  $k_+ = k_-$



# EM wave equation in conducting medium :

**For good conductor ( $\sigma \gg \epsilon\omega$  or  $\frac{\sigma}{\epsilon\omega} \gg 1$ ) :**

$$k_- \cong \sqrt{\frac{\sigma\omega\mu}{2}}$$

$$\text{Skin depth } (\delta) = \frac{1}{k_-} = \sqrt{\frac{2}{\sigma\omega\mu}}$$

*Notice that skin depth ( $\delta$ ) is dependent on frequency ( $\omega$ ) for a good conductor. Higher is frequency ( $\omega$ ), less will be skin depth ( $\delta$ ).*

For copper, with  $\mu \approx \mu_0$  and  $\sigma = 5.8 \times 10^7 \text{ S/m}$  at a frequency of 60 Hz,  $\delta \approx 9 \text{ mm}$ ;

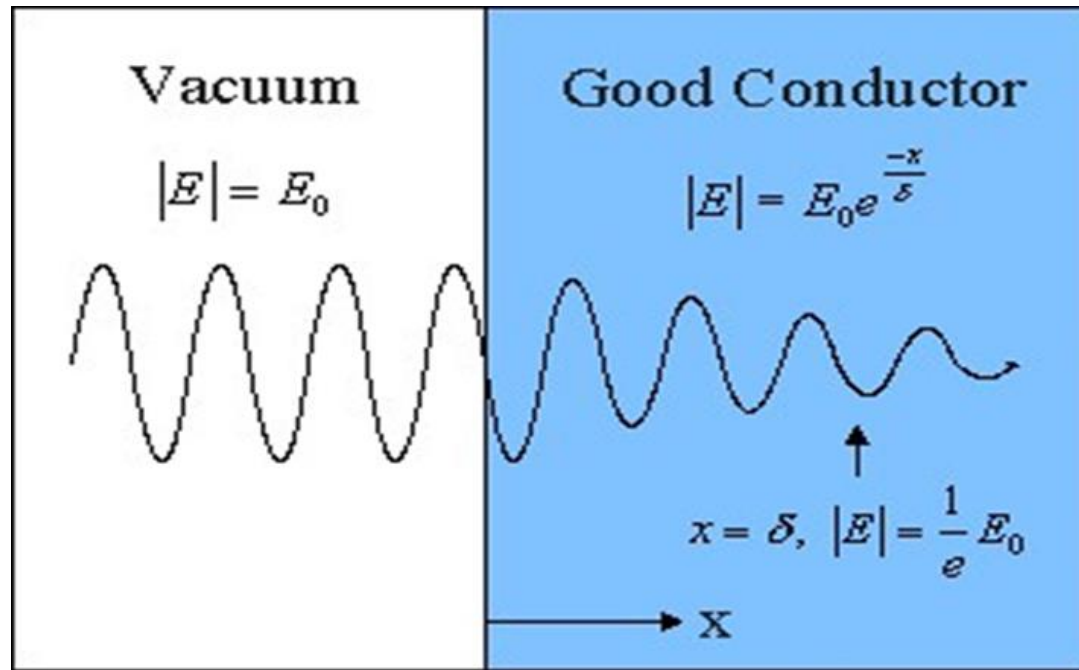
at 1 MHz,  $\delta \approx 6.6 \times 10^{-5} \text{ m}$

and at 30,000 MHz (radar wavelength of 1 cm),  $\delta \approx 3.8 \times 10^{-7} \text{ m}$ .

***We see also why a conductor can act to ‘shield’ a region from electromagnetic waves.***

# EM wave equation in conducting medium :

Schematic representation of skin depth :



For ex. Silver has conductivity  $\sigma = 6.30 \times 10^7 S/m$  and permittivity  $\epsilon_0 \approx \epsilon \approx 8.85 \times 10^{-12} F/m$ . For frequency  $10^{10}$  Hz, condition of good conductor is satisfied, skin depth is approx.  $0.6 \mu m$ .

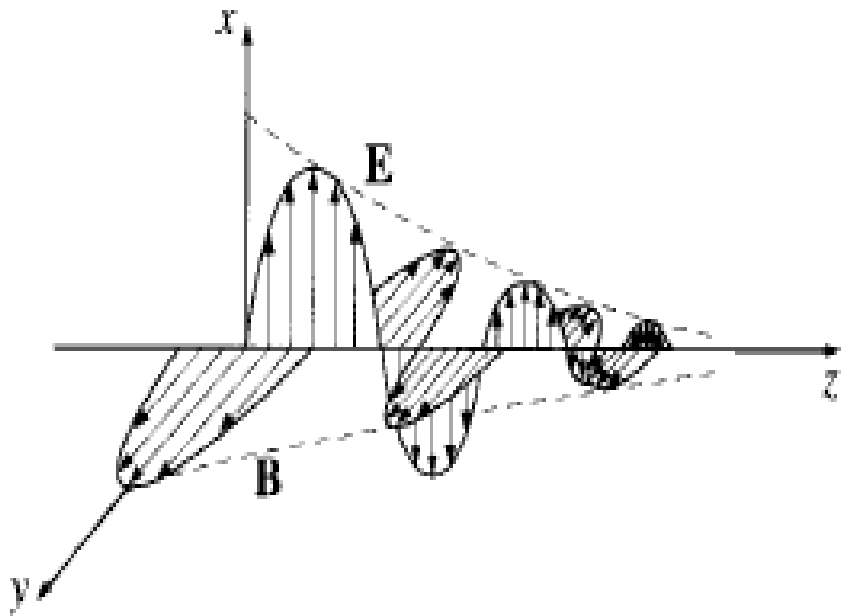
Wavelength of radiation of frequency  $10^{10}$  Hz is about 3 cm; but in silver wavelength is  $\lambda = \frac{2\pi}{k_+} = \frac{2\pi}{k_-} = 2\pi\delta \approx 4\mu m$ . (for good conductor  $k_+ = k_-$ )

# EM wave equation in conducting medium :

**For good conductor ( $\sigma \gg \epsilon\omega$  or  $\frac{\sigma}{\epsilon\omega} \gg 1$ ) :**

The phase difference between electric field and magnetic field in a good conductor is given by

$$\tan \phi = \frac{k_-}{k_+} = 1 \quad (\text{for good conductor } k_+ = k_-)$$
$$\Rightarrow \phi = 45^\circ$$



A schematic representation of *phase difference* and *exponential decaying amplitude* in a *good conductor* for a wave travelling in  $z$ -direction, electric field oscillating along  $x$ -axis and magnetic field oscillating along  $y$ -axis.

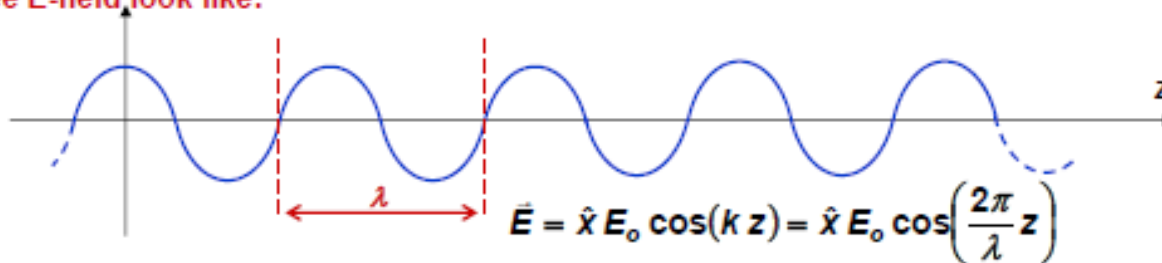
# A pictorial representation of travelling wave :

## Sinusoidal Solutions of Electromagnetic Wave Equation - V

Consider the plane wave:

$$\vec{E} = \hat{x} E_o \cos(\omega t - k z) \quad \vec{H} = \hat{y} \frac{E_o}{\eta_o} \cos(\omega t - k z)$$

If a person takes a snapshot of the wave in space at any time, say at  $t = 0$ , he will see E-field look like:



If a person sits at one location, say  $z = 0$ , he will see an oscillating E-field in time that looks like:

