EXPANSIVITY ON JULIA SETS

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Abstract. The expansivity of a rational map on its Julia set is closely related to an invariant quantity known as the asymptotic $\overline{E}^\infty$ energy. Using metric graphs embedded in Julia sets, we outline methods developed for bounding and calculating $\overline{E}^\infty$ for quadratic and cubic post-critically finite hyperbolic rational maps and demonstrate this quantity's relation to expansivity. Upper bounds are calculated by constructing a map forcing efficient lifts, while lower bounds are calculated through assigning a Markov partition to invariant graphs.

1. Introduction

Fix some post-critically finite (PCF) hyperbolic rational map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with Julia set $\mathcal{J}$. Nekrashevych showed how to approximate Julia sets using sequences of combinatorial spaces (e.g., graphs G_n) so that the inverse limit as n approaches infinity is $\mathcal{J}(f)$ [Nek14]. Later, D. Thurston developed machinery using elastic graphs to study PCF hyperbolic maps [Thu16, Thu19, Thu20]. In doing so, a family of $\overline{E}^p$ energies of rational maps associated with these graphs was defined. Data is currently incomplete in understanding these energies. Park showed in his thesis that $p = 1$ regulates the topology of the Julia set [Par21]. D. Thurston showed that $p = 2$ determines whether a certain branched self-cover is equivalent to a rational map it has also been shown that $p = \infty$ controls whether f is expanding on $\mathcal{J}(f)$ [DDT22, Thu20, Thu16].

We continue the study of E^∞. In particular, we would like to study it as an expansion measurement of f on its Julia set from the graph G_0. We define this "expansivity energy" a map $\varphi_0^n : G_n \to G_0$:

$$E^\infty(\varphi_0^n) = \sup_{z\in G_n} |(\varphi_0^n)'(z)|,$$

where the supremum is taken over all points $z \in G_n$. We also consider the energy over the homotopy class of φ_0^n defined for G_n in the following manner where φ_0^n is a map between metric graphs:

$$E^\infty[\varphi_0^n] := \inf_{\psi_0^n\in[\varphi_0^n]} E^\infty(\psi_0^n)$$

A more concise method of calculating $E^\infty[\varphi_0^n]$ was shown by Bestvina and White:

Lemma 1 (Bestvina-White [Bes11, Prop. 2.1])**.** *For any $\varphi_0^n : G_n \to G_0$, we have*

$$E^\infty[\varphi_0^n] = \sup_{C\subseteq G_n} \frac{\ell_0(\varphi_0^n(C))}{\ell_n(C)}$$

where the supremum ranges over all curves $C \subseteq G_n$. Moreover, the supremum can be taken as the maximum over all curves in G_n that run over each edge at most once in each direction.

If $\varphi_0^n : G_n \to G_0$ is homotopic to an embedding, and $\lim_{n\to\infty} G_n$ is homeomorphic to $J(f)$, then it is natural to assume this quantity relates to some property of the Julia set. We study this idea using the *asymptotic* ∞-energy, $\overline{E}^\infty[f, \varphi_0^n]$.

Definition 2. The *asymptotic Lipschitz energy* $\overline{E}^\infty[f, \varphi_0^n]$ is

$$\overline{E}^\infty[f, \varphi_0^n] = \lim_{n\to\infty} (E^\infty[\varphi_0^n])^{\frac{1}{n}}$$

Recall that, for a given PCF function f, $(E^\infty(\varphi_0^n, f))^{1/n} \geq (E^\infty[\varphi_0^n, f])^{1/n} \geq \overline{E}^\infty(f)$, where $\varphi_0^n : G_n \to G_0$ is a map induced by the deformation retract $\hat{\mathbb{C}} \setminus f^{-n}(P(f)) \to G_0$ which commutes, up to homotopy, with the embedding $\varepsilon_n : G_n \to \hat{\mathbb{C}} \setminus (f^{-1})^{\circ n}(P(f))$, the inclusion map $i_0^n : \hat{\mathbb{C}} \setminus (f^{-1})^{\circ n}(P(f)) \to \hat{\mathbb{C}} \setminus P(f)$ and the deformation retract $\pi_0 : \hat{\mathbb{C}} \setminus P(f) \to G_0$, i.e. $\varphi_0^n \sim \pi_0 \circ i_0^n \circ \varepsilon_n$. Here, G_0 is a spine, and is obtained through the deformation retract π_0. The method of constructing these graphs G_n through pullback is outlined in [DDT22].

We can then conclude that, finding upper bounds for $\overline{E}^\infty$ is the same as computing $(E^\infty[\varphi_0^n, f])^{1/n}$ for a given n-th pullback. With this in mind, we delineate an iterative procedure for obtaining such quantities, outlined in (2).

H. Dai, C. Davis, and D. Thurston developed methods during Summer 2022 using nonlinear inequalities to rigorously bound and calculate the asymptotic $\overline{E}^\infty$ for the seventeen cubic Belyi rational maps [BBL+00, DDT22]. These methods relied on what will shortly be defined as "efficiently lifting curves." Here, we introduce key terminology from Summer 2022's study:

A PCF rational map is *hyperbolic* if and only if every critical point attracts to an attracting cycle of type $C(d,p)$ under iteration. Each cycle of type $C(d,p) = \{z_1, \ldots, z_p\} \subset P(f)$ contains two pieces of data: the *period* $p = |C(d,p)|$, and $d = \prod_{i=1}^p \deg_f(z_i)$, the product of degrees with which elements $z_i \in C(d,p)$ map to their following term.

Lemma 3. *For f PCF with cycles of type $C(d,p)$, where d is the product of local degrees in the cycle and p is the number of points in the cycle, we have* $\max(d^{-1/p}) \leq \overline{E}^\infty$.

Lemma 4. *Suppose that there exists a map $\varphi_0^1 : G_1 \to G_0$ where $E^\infty[\varphi_0^1] = \alpha$. Then we have $\overline{E}^\infty \leq \alpha$.*

Definition 5. A curve $C \subset G_n$ *lifts efficiently* to G_{n+1} if there exists $C' \subset G_{n+1}$ such that $\varphi_n^{n+1}(C') = C$ and $\frac{\ell_n(C)}{\ell_{n+1}(C')} = \alpha$.

For all n, consider the set of curves C_n in G_n that lift efficiently to G_{n+1}. Denote by T_n be the set of curves in G_0 that is the image of C_n under φ_0^n.

Lemma 6. *If the intersection $\bigcap_{n\in\mathbb{N}} T_n$ is nonempty, then $\overline{E}^\infty = \alpha$.*

Utilizing these definitions and results, $\overline{E}^\infty$ was calculated for sixteen of seventeen cubic Belyi rational maps.

The remaining problem rational map in question is the following, #51 in the cubic census:

$$f_{3,51}(z) = \frac{4(z-1)^3}{27z}$$

Using methods from Summer 2022, it was found that $0.5 \leq \overline{E}^\infty \leq \phi^{-1}$ where ϕ is the golden ratio.

Due to their non-linear nature, these methods became difficult to implement in cases with $|P(f)| = 4$. Thus, new heuristics were necessary in computing $\overline{E}^\infty$ for these cases.

In studying cubic map #51, we developed novel, linear methods to bound and compute $\overline{E}^\infty$. Additionally, we have shown a concrete relation between $\overline{E}^\infty$ and the expansivity of a rational map. This allowed us to improve the #51 bound from $[0.5, 0.618]$ to $[0.553, 0.563]$. These linear methods, then, were easier to apply to $|P(f)| = 4$ cases. Hence, we have calculated $\overline{E}^\infty$ for three of seven quadratic $|P_f| = 4$ rational maps and in the Fall will be further investigating another method which seems to resolve three more cases.

We summarize these methods, applications, and results in the following sections.

2. Upper Bounds

We now present a heuristic process which was found to be very useful in computing $E^\infty[\phi_0^n, f^n]$ (which are upper bounds for $\overline{E}^\infty$).

Definition 7. A *separating curve* in G_n with respect to $\emptyset \neq K \subset P_f$ is a curve which separates the elements of K from the elements of $P_f \setminus K$. The set of all separating curves in G_n for a given subset K is denoted $\Psi_n(K)$ (in practice, the function f and the inclusion map in question φ_0^n will generally be implied from context). Note that, $\Psi_n(K) = \Psi_n(P_f \setminus K)$.

Separating curves are very useful in our construction a method to calculate upper bounds as it is not difficult to designate minimally stretched separating curves in G_0. With these curves, designated, we can exploit Lemma 1 and tweak only these relevant separating curves in pullback.

Definition 8. A *σ-triplet* for a pair (G_n, G_0) is a metric graph Γ_n together with a map $\sigma_n : G_n \to \Gamma_n$, and a scaling of graphs Λ_n, satisfying the following properties:

(1) σ_n commutes with respect to the following diagram:

$$\begin{tikzcd} G_n \arrow[r, "\varphi_0^n"] \arrow[d, "\sigma_n"'] & G_0 \\ \Gamma_n \arrow[ur, "\Lambda_n"'] & \end{tikzcd}$$

where Λ_n is a map with constant derivative $\alpha_n < 1$, i.e., a scaling of graphs isomorphic to the identity.

(2) $|\sigma_n'(x)| \leq 1$ for all $x \in G_n$ and $\|\sigma_n'\|_\infty = 1$, we call maps that satisfy this condition *short maps*.

Note that, given we are able to find such a triplet, $|(\varphi_0^n)'(x)| = |\Lambda'(\sigma_n)\sigma_n'(x)| \leq \alpha_n$.

We now turn to a particular method of constructing such a map σ_n. Our technique involves prescribing an arbitrary metric on G_0 , and then preserving the length of a big enough choice of separating curves under the map σ_n such that Γ_n is determined (this will automatically imply relationships between the lengths of each edge in G_0). We then check for the additional constraint $|\sigma_n'| \leq 1$.

To do this, we chose as many curves as there are edges (modulo edges of the same length) in G_0. We must choose at most one separating curve around each subset $K \subset P_f$. This is because we want $|\sigma_n'| \leq 1$, and under our construction, the curves we choose realize $|\sigma_n'| = 1$, implying that all other curves must stretch by at most the amount that the selected curves are stretching. As two curves around the same K need not have the same length under an arbitrary metric, choosing more then one curve under any given K would contradict $|\sigma_n'| \leq 1$. Sometimes, more than one curve around K might have its length preserved under σ_n, and in this case choosing any such curves will yield the same σ-triplet.

Note that, the maximum number of edges of different lengths in G_0 for any PCF f is bounded above by $\frac{|P_f|(|P_f|-1)}{2}$ (for $|P_f| \leq 4$ we have an equality). However, the number of separating curves one can chose is $\frac{|\mathcal{P}(P_f)|}{2} - 1$. Thus, the number of separating curves for $|P_f| > 3$ is bigger than the number of edges one needs to solve for. This means that, beyond simply choosing the curves for any given number of subsets of $P(f)\setminus\{\infty\}$, we also have some freedom regarding the choice of subsets to consider lifts that realize the maximum stretch.

After obtaining a graph Γ_n, we then construct an eigenvalue equation which encapsulates the scaling Λ_n:

$$A\mathbf{x_0} = \mathbf{x_n} = \frac{\mathbf{1}}{\alpha_\mathbf{n}}\mathbf{x_0}$$

Here we denote $\mathbf{x_n}$ by the set of length of distinct edges on Γ_n. After this step, we have candidates for a metric in G_0, $\mathbf{x_0}$, and the the Lipschitz energy $E^\infty[\varphi_0^n]$, α_n. For Γ_n to be valid for some σ-triplet, we need to check our assumptions. First, we must check the metric x_0 and guarantee that the chosen separating curves in G_n are indeed the smallest ones for their respective subset $K \subset P(f)$. If this does not hold, the chosen curve will not realize the supremum in Lemma 1, contradicting $|\sigma_n'| \leq 1$. However, even if the resulting eigenvector respects our relationships between edges, we need to guarantee that there are no separating curves around other subsets of $P(f) \setminus \{\infty\}$ which stretch more than the chosen ones.

To do this, we need a way to guarantee that the attained σ_n indeed obeys $|\sigma_n \leq 1$. The way we do this is by finding an embedding of G_n into Γ_n (conversely G_0) which obeys $|\sigma_n| \leq 1$ (conversely $|\varphi_0^n| \leq \alpha_n$). To visualize if this is possible, we inflate Γ_n and draw G_n inside of it. If we can embed G_n in Γ_n without stretching it we have obtained a valid σ-triplet; we call such an embedding a *taut map*. Detailed examples of this procedure will be shown in the applications section.

The process can be expressed in the following steps:

(1) Choose some G_0 and pull it back to G_n;
(2) Choose as many subsets $K \subset P(f) \setminus \{\infty\}$ as there are edges of different lengths in G_0;
(3) Now make a choice of the separating curves in each $\Psi_n(K)$ which will realize the maximum stretch, and record the implicit inequalities established between lengths of edges. Alternatively, we can start with an assumption regarding the inequalities between lengths of edges good enough so that we can distinguish separating curves sufficiently for us to choose the one with maximum lift;
(4) Construct Γ_n by forcing the length of it's shortest separating curves (which are determined without any metrics) around the initially chosen subsets K to be the same as the shortest curves determined in G_n;
(5) Obtain a candidate for the metric in G_0 and for α_n by constructing an eigenvalue equation which satisfies the constrains in Λ_n;
(6) Check if eigenvector agrees with our initial choice of curves;
    (a) If the eigenvector contradicts our assumptions, go back to (3) and choose different curves;
    (b) If the eigenvector agrees, check if a taut map can be constructed;
        (i) If it can't, go back to (2) and choose different subsets of $P(f) \setminus \{\infty\}$;

(ii) If it can, you've found a valid σ -triplet.

If the process was followed through exhaustion and a proper σ-triplet hasn't been achieved, then one must go back to (1) and choose a different initial spine G_0. Up to now, we haven't encountered any PCF rational map for which no spine is incompatible with this method.

3. Lower Bounds

In a very rigorous sense, $\overline{E}^{\infty}$ is related to a quantity shortly to be defined as the *expansivity* of a rational map on its Julia set. In particular, our methods used to acquire lower bounds for the former quantity utilize the latter. Thus, we first define this related terminology.

Definition 9. Fix a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with Julia set $\mathcal{J}$. Consider the dynamical system given by $f \colon \mathcal{J} \to \mathcal{J}$. For a given metric d on $\mathcal{J}$, expansion factor $\lambda > 1$, and $\epsilon > 0$, we say f is (λ, ϵ)-*expansive* on its Julia set with respect to d if, for every $z_1, z_2 \in \mathcal{J}$ with $|z_1 - z_2| < \epsilon$, we have

$$|f(z_1) - f(z_2)| \geq \lambda |z_1 - z_2|.$$

Definition 10. We say that f is λ-*expansive* on its Julia set if it is (λ, ϵ)-expansive for some $\epsilon > 0$.

Definition 11. Define the *expansivity of* f on its Julia set as

$$\overline{\lambda}(f) := \sup \{\lambda \mid \text{there is a metric } d \text{ on } \mathcal{J} \text{ so } f \text{ is } \lambda\text{-expansive wrt } d.\}$$

A second tool in bounding $\overline{E}^{\infty}$ comes from invariant graphs on the Julia set of a rational map, which will now be defined.

Definition 12. A *graph on* $\mathcal{J}(f)$ is a map $\phi : \Gamma \to \mathcal{J}(f)$ where Γ is a graph.

Definition 13. A graph Γ on $\mathcal{J}(f)$ is f-*invariant* if there is a map $g : \Gamma \to \Gamma$ such that $f \circ \phi = \phi \circ g$.

We utilize f-invariant, often denoted simply as *invariant*, graphs as a means to understand dynamics on a rational map's Julia set through studying a smaller subset of this space. A graph on $\mathcal{J}(f)$ can be visualized as being embedded in $\mathcal{J}(f)$. Hence, this invariant graph can be thought of as a sub-dynamical system of $f : \mathcal{J} \to \mathcal{J}$.

We now return to the context of pulling a spine back from G_0 to G_n. For each invariant graph (indexed by $k \in \mathbb{N}$) defined by induced map $g_k^n : G_0 \subset G_n \to G_0$, we assign a Markov partition matrix A_k^n (note n is not an exponential but an index for Markov matrices) where the basis for the matrix are the colored edges, and entry a_{ij} equals the number of times the ith edge maps to the jth edge. The following eigenvalue problem then allows one to treat the nth root of this Perron-Frobenius (PF) eigenvalue of this matrix, $\lambda_n(g_k^n)^{1/n}$, as a uniform expansion factor of G_0. Where $\mathbf{x_0}$ is the n-dimensional column vector of edge lengths in G_0, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & \vdots \\ a_{31} & a_{32} & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix} \mathbf{x_0} = \lambda_{\mathbf{n}}^{\mathbf{n}}(\mathbf{g_k^n})\mathbf{x_0}.$$

This nth root of this PF eigenvalue will then provide an effective upper bound of $\overline{\lambda}(f)$.

Following from these definitions are two lemmas for which we will save the proof for our research paper.

Lemma 14. *If $f : \mathcal{J} \to \mathcal{J}$ has an f^n-invariant graph G_0 with induced map $g : G_0 \subset G_n \to G_0$ with Markov partition matrix A_k^n, then $\lambda(f) \leq (\lambda_n(g_k^n))^{1/n}$.*

Utilizing this inequality between the PF eigenvalue and expansivity of f on its Julia set, one can then calculate a lower bound for $\overline{E}^\infty[\pi, \phi]$ using expanding invariant graphs on Julia sets.

Lemma 15. *Let $\pi, \phi : G_1 \rightrightarrows G_0$ be a virtual endomorphism with $\overline{E}^\infty[\pi, \phi] < 1$. Then,*

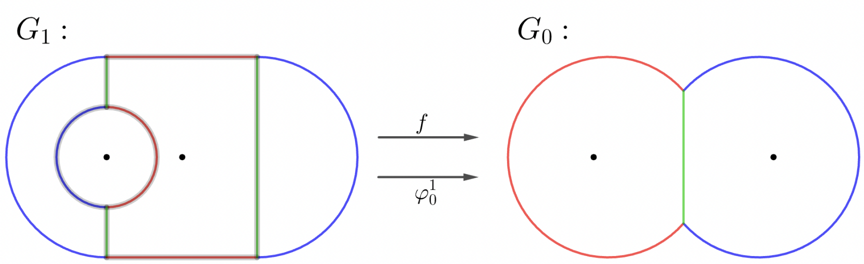
$$\overline{E}^\infty[\pi, \phi] \geq \frac{1}{\overline{\lambda}(f)}.$$

Furthermore, if the conditions for Lemma 14 are met, then for all k,

$$\overline{E}^\infty[\pi, \phi] \geq \frac{1}{(\lambda_n(g_k^n))^{1/n}}.$$

Example. We now provide an example of finding the a better lower bound for $f_{3,51}$ than is given by Lemma 3. Consider the following rational map with given G_0 pulled back to G_1:

$$f_{3,51}(z) = \frac{4(z-1)^3}{z}$$



Note there are multiple subgraphs of G_1 which are homotopic invariant graph candidates. Choose the candidate boldened in the above figure with following rules for its induced map:

- The red edge in G_0 is sent to the blue edge immediately surrounding $0 \in P(f)$.
- The green edge in G_0 is sent to the red edge separating $0, 1 \in P(f)$.

- The blue edge in G_1 is sent to a linear combination of three red and two green edges immediately surrounding $1 \in P(f)$.

Using this induced map, indexed $k = 1$, we construct Markov partition A_1^1 matrix with basis (r, g, b) and its corresponding eigenvalue problem:

$$A_1^1 \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \lambda_1(g_1^1) \begin{bmatrix} r \\ g \\ b \end{bmatrix}$$

Finding the Perron-Frobenius eigenvalue of this system amounts of solving the polynomial equation:

$$\lambda_1^3 - 2\lambda_1 - 3 = 0$$

We can then solve and find $\lambda_1(g_1^1) \approx 1.893$. Utilizing Lemmas 14 and 15, we then have

$$.\frac{1}{1.893} \approx 0.528 \leq \overline{E}^{\infty},$$

which is better than the Lemma 3 bound $0.5 \leq \overline{E}^{\infty}$.

Fix the nth pullback, G_n, of G_0. There exists a finite amount of f^n-invariant graphs with which one can apply Lemmas 14 and 15. For many choices of G_0, increasing n will give rise to an exhaustive, growing list of invariant candidates $g_k^n : G_0 \subset G_n \to G_0$. Note that for particular rational maps, many choices of G_0 will lead to no invariant graphs in any pullback. For a particular example of this phenomenon, see quadratic census map $f_{2,6.2}$.

The optimal lower bound for $\overline{E}^{\infty}$ in G_n is realized by the invariant graph $G_0 \subset G_n$ such that its Markov partition matrix A_i^n realizes $\lambda_n(g_i^n) = \min(\lambda_n(g_k^n))$ over all k.

Currently, there is no known algorithm to find such $g_i^n : G_0 \subset G_n \to G_0$. However, in all but three cases of our study, considering the few invariant graphs in G_1 is enough to find some invariant graph with Markov matrix A_i^1 such that $\frac{1}{\lambda_1(g_i^1)} = \overline{E}^{\infty}$.

4. Applications

As instructive examples, we will outline the process of finding (often equal) upper and lower bounds for $\overline{E}^{\infty}$. Recall the following tier system categorizing hyperbolic rational maps based on curves' efficient lifting in graphs on $\mathcal{J}(f)$:

(1) Blue: The entire spine G_0 has an efficient lift to a subgraph in G_1.
(2) Green: Every curve lifts efficiently in every preimage. Notably, this is not equivalent to Blue.
(3) Yellow: There exists a curve which lifts efficiently in every preimage. This is close to the minimum structure needed determine the $\overline{E}^{\infty}$ energy using our techniques.
(4) Red: Current methods presented in the last section are not sufficient to determine $\overline{E}^{\infty}$.

Among the examples outlined are quadratic census maps #6.1 (Blue), #7 (Green), and #3.2 (Yellow). Additionally, we will outline this process for cubic census map #51 (Red), for which $\overline{E}^\infty = \overline{\alpha}$ remains an open question.

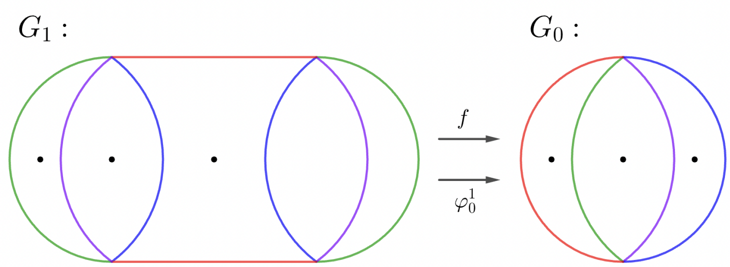
Example 1: Quadratic Map #6.1 (Blue)
Consider the following PCF hyperbolic rational map along with its corresponding critical portrait:

$$f(z) = \frac{\phi^{-2}}{(z-1)^2}$$

$$\begin{tikzcd}
2 \arrow[dr] & \phi \arrow[dr] & & & \\
& \phi^{-2} \arrow[r] & 1 \arrow[r, "2"] & \infty \arrow[r, "2"] & 0 \arrow[lll, bend left]
\end{tikzcd}$$

Note $P(f) = \{0, \phi^{-2}, 1, \infty\}$. Additionally, see that the critical portrait contains a cycle of type $C(4,4)$. From previous results, it is immediately clear that $2^{-1/2} \leq \overline{E}^\infty < 1$.

We must choose a spine G_0. For quadratic census map #6.1, we select the "eye spine" for G_0. We assign an arbitrary metric (r_0, g_0, p_0, b_0) to the edges of this graph, not yet requiring any edge length ordering assumptions. We then pull back to G_1.



First, we will improve the upper bound of $\overline{E}^\infty$ using the methods outlined in *(2)*.

In constructing $\sigma_1 : G_1 \to \Gamma_1$, we consider the shortest separating curves around subsets of $P(f) \setminus \{\infty\}$ in G_1. In order to find these curves, we must assume some *key inequalities* on the metric assigned to G_0.

In particular, finding the shortest curve around 1 in G_1 requires finding $\min\{g_0, p_0, b_0\}$. Before finding shortest curves in G_1, one should take inventory of all relevant edge length key inequalities in G_1, outlined for this choice of G_0 in the table below.

TABLE 1. Quadratic Census #6.1 Key Inequalities

$P(f)$ **Subset**	0	ϕ^{-2}	1	$0, \phi^{-2}$	$\phi^{-2}, 1$	$0, 1$	$0, \phi^{-2}, 1$
Key Inequalities	None	None	g_0, p_0, b_0	None	g_0, p_0, b_0	N/A	g_0, p_0, b_0

Fortunately, quadratic map #6.1 with the eye spine provides us with only one key inequality to consider. We first will assume $g_0 \leq p_0 \leq b_0$ in G_0. We are now able to choose the precise shortest curve around any subset of $P(f)$ in G_1. If the corresponding linear system which follows' PF eigenvector contradicts this choice of metric, we will reject this initial assumption and restart the process of finding $\sigma_1 : G_1 \to \Gamma_1$ accordingly.

In constructing $\sigma_1 : G_1 \to \Gamma_1$, we are searching for a metric (r_1, g_1, p_1, b_1) on Γ_1 forcing curves in G_0 to realize efficient lifts. Symbolically, we force

$$\frac{1}{\alpha_1}\begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} r_0 \\ g_0 \\ p_0 \\ b_0 \end{bmatrix}$$

In creating a well-determined linear system, we must choose four of the seven shortest curves in G_1 to equal to the corresponding curves in Γ_1. For this example, we select the curves 0, ϕ^{-2}, $0, \phi^{-2}$, and $0, \phi^{-2}, 1$, giving rise to the linear system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} r_0 \\ g_0 \\ p_0 \\ b_0 \end{bmatrix} = \frac{1}{\alpha_1}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix}$$

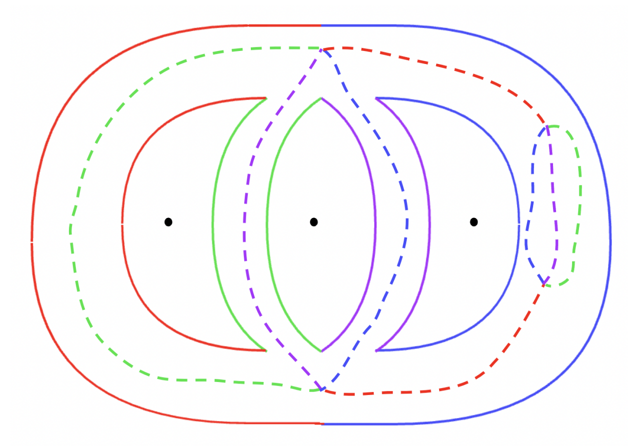
Considering the left and right components of this equality presents calculation of $\frac{1}{\alpha_1}$ as an eigenvalue problem with the metric (r_1, g_1, p_1, b_1). Then, solving for the PF eigenvalue of this system gives the inverse of our upper bound candidate, α_1. In solving this eigenvalue problem, one obtains

$$\alpha_1 \approx 0.739$$

Note that at the moment, this is only an upper bound candidate. We must check that all key inequalities are satisfied within the corresponding PF eigenvector, given by

$$v_1 = \begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \alpha_1^3 \\ \alpha_1^2 \\ \alpha_1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.404 \\ 0.546 \\ 0.739 \\ 1 \end{bmatrix}$$

Hence, $\sigma_1 : G_1 \to \Gamma_1$ endows Γ_1, which is homotopic to G_0, with lengths corresponding to this eigenvector. Seeing that the assumed inequalities $g_0 \leq p_0 \leq b_0$ is satisfied by this eigenvector, no contradictions arise in performing the length scaling operation $\Lambda_1 : \Gamma_1 \to G_0$. Finally, to prove this value as an upper bound for $\overline{E}^{\infty}$, we must show $g_k^1 : G_1 \to G_0$ to be a short map.



For a short map to be present is necessary that each edge in G_0 gets scaled by $\frac{1}{\alpha_1}$ or more in lifting to G_1.

In particular, it is clear from the above diagram that

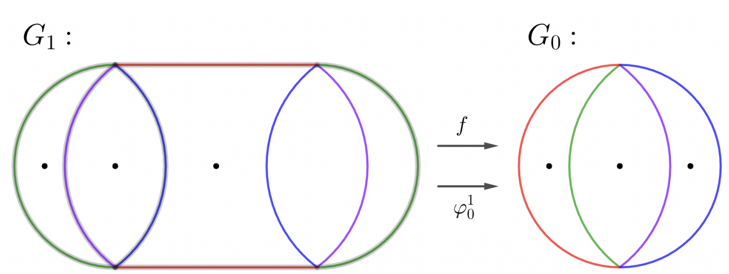
$$\frac{1}{\alpha_1} r_1 = g_1$$
$$\frac{1}{\alpha_1} g_1 = p_1$$
$$\frac{1}{\alpha_1} p_1 = b_1$$
$$\frac{1}{\alpha_1} b_1 = 2r_1 + \min\{g_1, p_1, b_1\}$$

Hence, the first three edges are mapped tautly while some of the edges mapping to blue have slack, and thus $g_k^1 : G_1 \to G_0$ is a short map. Therefore, it is clear that the positive solution to polynomial

$$2\alpha_1^4 + \alpha_1^3 - 1 = 0$$

or $\alpha \approx 0.739$ is a valid upper bound for $\overline{E}^\infty$.

Now we turn our attention to finding a lower bound of $\overline{E}^\infty$ using methods outlined in *(3)*. To do this, we rely on invariant graphs on $\mathcal{J}(f)$.



Referencing G_0 and G_1, there are three obvious choices of invariant graphs on $\mathcal{J}(f)$ which are homotopic to G_0. Once again, we must choose which path out of g_0, p_0, and b_0 to take.

As g_0 has previously been determined as having the shortest length, we will choose to take g_0. Denote this invariant graph as being determined by $g_1^1 : G_0 \subset G_1 \to G_0$. This invariant graph is represented by Markov partition matrix

$$A_1^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

with PF eigenvalue determined by the polynomial

$$\lambda_1(g_1^1)^4 - \lambda_1(g_1^1) - 2 = 0$$

which, upon applying Lemmas 14 and 15, gives agreeing lower and upper bounds of $\overline{E}^\infty$. Thus,

$$\overline{E}^\infty \approx 0.739$$

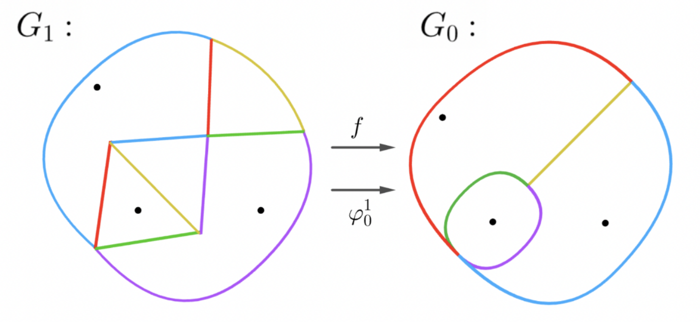
Example 2: Quadratic Map #7 (Green)
Consider the following PCF hyperbolic rational map along with its corresponding critical portrait:

$$f(z) = \frac{(z-1)^2}{(z-e^{2\pi i/3})^2}$$

$$\begin{tikzcd}
\frac{e^{\pi i/3}}{2} \arrow[dr] & & 2e^{\pi i/3} \arrow[dr] & & \\
& 1 \arrow[r, "2"] & 0 \arrow[r] & e^{2\pi i/3} \arrow[r, "2"] & \infty \arrow[lll, bend left]
\end{tikzcd}$$

Note $P(f) = \{0, 1, e^{2\pi i/3}, \infty\}$. Additionally, see that the critical portrait contains a cycle of type $C(4,4)$. From previous results, it is immediately clear that $2^{-1/2} \le \overline{E}^\infty < 1$.

We must choose a spine G_0. For quadratic census map #3.2, we select the following spine for G_0. We assign an arbitrary metric $(r_0, g_0, y_0, p_0, b_0)$ to the edges of this graph, not yet requiring any edge length ordering assumptions. We then pull back to G_1.



First, we will improve the upper bound of $\overline{E}^{\infty}$ using the methods outlined in *(2)*.

In constructing $\sigma_1 : G_1 \to \Gamma_1$, we consider the shortest separating curves around subsets of $P(f) \setminus \{\infty\}$ in G_1. In order to find the shortest curve around certain subsets of $P(f)$, we must assume *key inequalities* on the metric assigned to G_0.

Before finding shortest curves in G_1, we take inventory of all relevant edge length key inequalities in G_1, outlined for this choice of G_0 in the table below.

TABLE 2. Quadratic Census #6.1 Key Inequalities

$P(f)$ **Subset**	0	1	$e^{2\pi i/3}$	$0, 1$	$1, e^{2\pi i/3}$	$0, e^{2\pi i/3}$	$0, 1, e^{2\pi i/3}$
Key Inequalities	$y_0, p_0 + b_0$	$g_0, r_0 + y_0$	$r_0, g_0 + y_0$ $b_0, p_0 + y_0$	$b_0, p_0 + y_0$ $g_0, r_0 + y_0$	N/A	$b_0, p_0 + y_0$ $r_0, g_0 + y_0$	$y_0, r_0 + g_0$

Unfortunately, this map and spine give us many key inequalities to consider. Though not always true, it is good practice to assume one edge has less or equal length than a linear combination of multiple edges. We first will assume the key inequalities $y_0 \leq p_0 + b_0$, $g_0 \leq r_0 + y_0$, $r_0 \leq g_0 + y_0$, $p_0 \leq g_0 + y_0$, and $p_0 \leq b_0$ in G_0. We are now able to choose the precise shortest curve around any subset of $P(f) \setminus \{\infty\}$ in G_1. If the corresponding linear system which follows' PF eigenvector contradicts this choice of metric in any way, we will reject this initial assumption and restart the process of finding $\sigma_1 : G_1 \to \Gamma_1$ accordingly.

In constructing $\sigma_1 : G_1 \to \Gamma_1$, we are searching for a metric $(r_1, g_1, y_1, p_1, b_1)$ on Γ_1 forcing curves in G_0 to realize efficient lifts. Symbolically, we force

$$\frac{1}{\alpha_1}\begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} r_0 \\ g_0 \\ y_0 \\ p_0 \\ b_0 \end{bmatrix}$$

In creating a well-determined linear system, we must choose five of the seven shortest curves in G_1 to equal to the corresponding curves in Γ_1. For this example, we select the curves 0, 1, $e^{2\pi i/3}$, $0, 1$, and $0, 1, e^{2\pi i/3}$, giving rise to the linear system

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ g_0 \\ y_0 \\ p_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ b_1 \end{bmatrix} = \frac{1}{\alpha_1} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ b_1 \end{bmatrix}$$

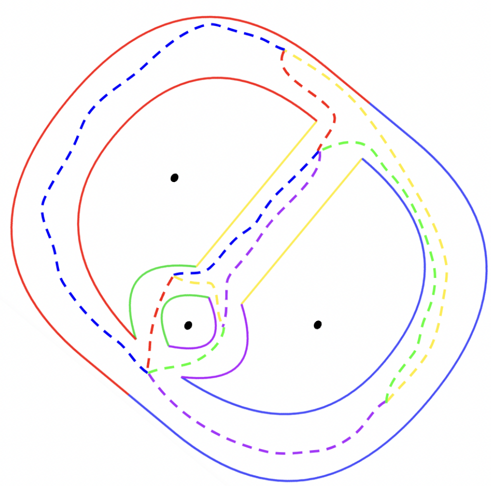
Considering the middle and right components of this equality presents calculation of $\frac{1}{\alpha_1}$ as an eigenvalue problem. In solving for the PF eigenvalue, we obtain

$$\alpha_1 = 2^{-1/2}$$

Note that at the moment, this is only an upper bound candidate. We must check that all key inequalities are satisfied within the corresponding PF eigenvector,

$$v_1 = \begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2\sqrt{2}-2 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $\sigma_1 : G_1 \to \Gamma_1$ endows Γ_1, which is homotopic to G_0, with lengths corresponding to this eigenvector. Upon inspection, it can be found that the assumed key inequalities satisfied by this eigenvector, and hence no contradictions arise in performing the length scaling operation $\Lambda_1 : \Gamma_1 \to G_0$. Finally, to prove this value as an upper bound for $\overline{E}^{\infty}$, we must show $g_k^1 : G_1 \to G_0$ to be a short map.



For a short map to be present is necessary that each edge in G_0 gets scaled by $\frac{1}{\alpha_1}$ or more in lifting to G_1.

In particular, each edge must be stretched as depicted above to satisfy this condition. Noting $y_1 = 2\sqrt{2}-2 = (\sqrt{2}-1)+(\sqrt{2}-1)$, we stretch the outer yellow edge in half as such. This gives, on the outer curves,

$$\frac{1}{\alpha_1} r_1 = \sqrt{2} = b_1 + \sqrt{2} - 1$$
$$\frac{1}{\alpha_1} b_1 = \sqrt{2} = p_1 + \sqrt{2} - 1$$

We must then stretch the outer red and green edges giving equivalent length as applied to the above equations. Then, the portion of red and green each contributing to $\frac{1}{\alpha_1} y_1$ is $2 - \sqrt{2}$.

Finally, we must apply a standard $\Delta - Y$ transform as outlined in [DDT22] to the central (y_1, b_1, p_1)-triangle. Doing this, one obtains the rest of the equations:

$$\frac{1}{\alpha_1} g_1 = \sqrt{2} = r_1 + \sqrt{2} - 1$$
$$\frac{1}{\alpha_1} p_1 = \sqrt{2} = g_1 + \sqrt{2} - 1$$
$$\frac{1}{\alpha_1} y_1 = 4 - 2\sqrt{2} < (2 - \sqrt{2}) + (4 - 2\sqrt{2})$$

where in the last inequality $2 - \sqrt{2}$ comes from the outer red and green edges and $4 - 2\sqrt{2}$ comes from the $\Delta - Y$ transform. Not that this last inequality denotes the edges mapping to yellow having slack.

Hence, we have a short map. Therefore,

$$\overline{E}^{\infty} \leq 2^{-1/2}$$

As this upper bound is equal to the immediate lower bound, it follows that, without need for the lower bound methods from *3*, that

$$\overline{E}^{\infty} = \frac{1}{\sqrt{2}}$$

Example 3: Quadratic Map #3.2 (Yellow)

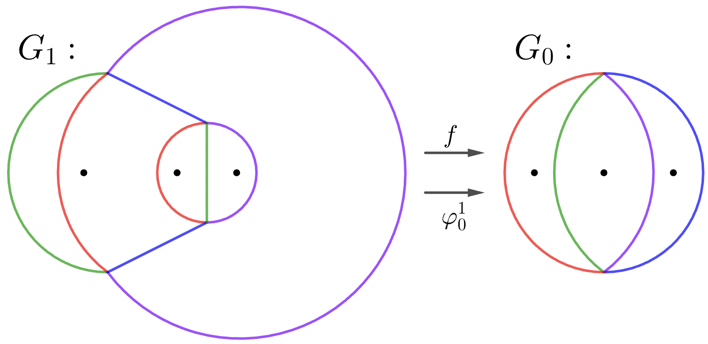
Consider the following PCF hyperbolic rational map along with its corresponding critical portrait:

$$f(z) \approx \frac{13.532z - 13.532}{(z + 2.383)^2}$$

$$\begin{tikzcd}
 & & & -10.445 \arrow[dr] & & \\
4.382 \arrow[r, "2"] & 1 \arrow[r] & 0 \arrow[rr] & & -2.382 \arrow[r, "2"] & \infty \arrow[llll, bend left]
\end{tikzcd}$$

Note $P(f) = \{-2.382, 0, 1, \infty\}$ and that the precise values of constants and coefficients can be found in [BBL+00]. Additionally, see that the critical portrait contains a cycle of type $C(2,3)$. From previous results, it is immediately clear that $2^{-1/3} \leq \overline{E}^{\infty} < 1$.

We must choose a spine G_0. For quadratic census map #3.2, we select the "eye spine" for G_0. We assign an arbitrary metric (r_0, g_0, p_0, b_0) to the edges of this graph, not yet requiring any edge length ordering assumptions. We then pull back to G_1.



First, we will improve the upper bound of $\overline{E}^{\infty}$ using the methods outlined in *(2)*.

In constructing $\sigma_1 : G_1 \to \Gamma_1$, we consider the shortest separating curves around subsets of $P(f) \setminus \{\infty\}$ in G_1. In order to find these curves, we must assume some key inequalities on the metric assigned to G_0.

In particular, finding the shortest curve around -2.383 in G_1 requires finding $\min\{r_0, g_0\}$. Before finding shortest curves in G_1, we take inventory of all relevant edge length key inequalities in G_1, outlined for this choice of G_0 in the table below.

Table 3. Quadratic Census #6.1 Key Inequalities

$P(f)$ **Subset**	-2.383	0	1	$-2.383, 0$	$0, 1$	$-2.383, 1$	$-2.383, 0, 1$
Key Inequalities	r_0, g_0	None	None	r_0, g_0	None	N/A	r_0, g_0

Fortunately, quadratic map #3.2 with the eye spine provides us with only one key inequality to consider. We first will assume $g_0 \leq r_0$ in G_0. We are now able to choose the precise shortest curve around any subset of $P(f) \setminus \{\infty\}$ in G_1. If the corresponding linear system which follows' PF eigenvector contradicts this choice of metric, we will reject this initial assumption and restart the process of finding $\sigma_1 : G_1 \to \Gamma_1$ accordingly.

In constructing $\sigma_1 : G_1 \to \Gamma_1$, we are searching for a metric (r_1, g_1, p_1, b_1) on Γ_1 forcing curves in G_0 to realize efficient lifts. Symbolically, we force

$$\frac{1}{\alpha_1}\begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} r_0 \\ g_0 \\ p_0 \\ b_0 \end{bmatrix}$$

In creating a well-determined linear system, we must choose four of the seven shortest curves in G_1 to equal to the corresponding curves in Γ_1. For this example, we select the curves (-2.383), (0), $(-2.383, 0)$, and $(-2.383, 0, 1)$, giving rise to the linear system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ g_0 \\ p_0 \\ b_0 \end{bmatrix} = \frac{1}{\alpha_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix}$$

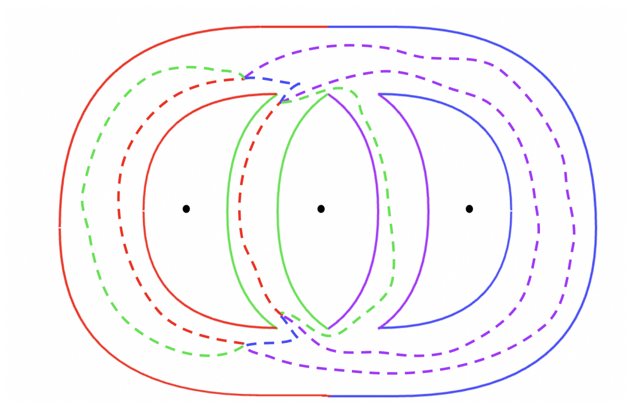
Considering the left and right components of this equality presents calculation of $\frac{1}{\alpha_1}$ as an eigenvalue problem. Then, solving for the PF eigenvalue of this system gives

$$\alpha_1 = 2^{-1/3}$$

Note that at the moment, this is only an upper bound candidate. We must check that all key inequalities are satisfied within the corresponding PF eigenvector,

$$v_1 = \begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} \\ 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} \\ 1 + \frac{1}{\alpha_1} \\ 1 \end{bmatrix}$$

Hence, $\sigma_1 : G_1 \to \Gamma_1$ endows Γ_1, which is homotopic to G_0, with lengths corresponding to this eigenvector. Seeing that the assumed inequality $g_0 \leq r_0$ is satisfied by this eigenvector, no contradictions arise in performing the length scaling operation $\Lambda_1 : \Gamma_1 \to G_0$. Finally, to prove this value as an upper bound for $\overline{E}^\infty$, we must show $g_k^1 : G_1 \to G_0$ to be a short map.



For a short map to be present is necessary that each edge in G_0 gets scaled by $\frac{1}{\alpha_1}$ or more in lifting to G_1.

In particular, it is clear from the above diagram that

$$\begin{aligned} \frac{1}{\alpha_1} r_1 = g_1 + 1 &= r_1 + 1 \\ \frac{1}{\alpha_1} g_1 &= r_1 + 1 \\ \frac{1}{\alpha_1} p_1 &= g_1 - 1 \\ \frac{1}{\alpha_1} b_1 &= p_1 - 1 \end{aligned}$$

Thus $g_k^1 : G_1 \to G_0$ is a short map. Therefore, it is clear that

$$\overline{E}^\infty \leq 2^{-1/3}$$

As this upper bound is equal to the immediate lower bound, it follows that, without need for the lower bound methods from *3*, that

$$\overline{E}^\infty = 2^{-1/3}$$

Example 4: Cubic Map #51 (Red)

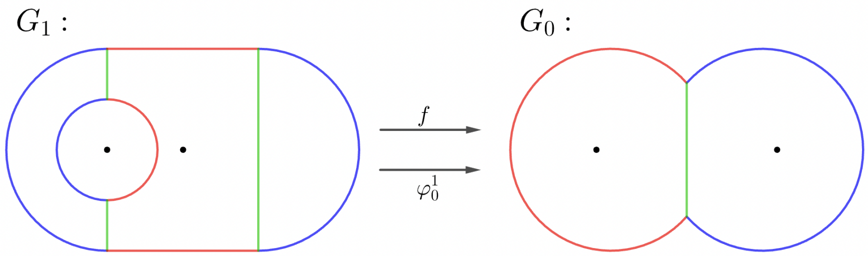
Consider the following PCF hyperbolic rational map along with its corresponding critical portrait:

$$f(z) = \frac{4(z-1)^3}{27z}$$

$$\begin{tikzcd} 4 \arrow[dr] & & & \\ -\frac{1}{2} \arrow[r, "2"] & 1 \arrow[r, "3"] & 0 \arrow[r] & \infty \arrow[loop right, "2"] \end{tikzcd}$$

Note $P(f) = \{0, 1, \infty\}$ and more precise information about this rational map can be found in [BBL$^+$00]. Additionally, see that the critical portrait contains a cycle of type $C(2,2)$. From previous results, it is immediately clear that $\frac{1}{\sqrt{2}} \leq \overline{E}^\infty < 1$.

One problem arising when $|P(f)| = 4$ is that there are infinitely many initial spines G_0 to choose from. Hence, it is difficult to find any "good spines." However, when $|P(f)| = 3$, there are only seven initial spines G_0 to choose from. Hence, choosing an ideal initial spine when $|P(f)| = 3$ is much easier. For $f_{3,51}$, we choose the "theta spines."

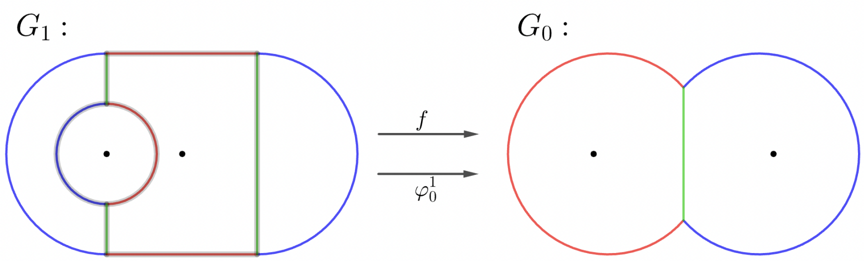


To construct $(\sigma_1, \Gamma_1, \Lambda_1)$, the only key inequality one must be establish is $\min\{g_0, b_0\}$. Choosing $g_0 \leq b_0$, and the only three separating curves possible in G_1, we obtain the following eigenvalue problem:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ g_0 \\ b_0 \end{bmatrix} = \frac{1}{\alpha_1} \begin{bmatrix} r_0 \\ g_0 \\ b_0 \end{bmatrix}$$

Which yields $\alpha_1 = \phi$. It is not hard to check that σ_0 is a taut map (since green is smaller than blue, the left most blue edge won't be stretched more than green, and the green edges of the square do not suffer any stretch). This yields the metric $(1, \phi - 1, \phi)$.

The problem in this example arrives when we try to compute improvements on the lower bound $d^{-\frac{1}{p}} = \frac{1}{\sqrt{2}}$. The best invariant graph one can chose for this example is shown bellow.



The Markov matrix (which is fairly simple in this particular example so the details are left to the reader) yields $\frac{1}{\lambda_1(g_1^1)} = 0.5281...$, with the characteristic equation being $\lambda_1^3 - 2\lambda_1 - 3 = 0$.

Upon exhaustively pulling back and finding invariant graphs in G_4, we have obtained a lower bound of 0.553.... Our best upper bound, 0.563..., is obtained via finding a σ-triplet for (G_0, G_3).

5. Results and Discussion

We have collected our results for quadratic $|P(f)| = 4$ maps in Table 5. For some of the maps, we have not been able to find matching lower and upper bounds, but have found other ad. hoc. arguments where able to provide us with a value for $\overline{E}^{\infty}$.

The main issue with the problematic rational maps seems to be that, in these cases, the upper and lower bounds we obtain via our described methods do not agree on G_n for finite n. This seems to be the case for ex. #51, since were not able to find matching bounds for the four pullbacks we tried (the pullbacks quickly become unmanageable, so pulling back to G_5 and further did not seem to be productive).

Note that, from the definition of Lipschitz energy, our $\alpha_n = E^\infty[\varphi_0^n, f^{\circ n}]$ assumes the following form:

$$\alpha_n = \xi_n(\overline{E}^\infty)^n$$

Where $(\xi_n)^{\frac{1}{n}} \to 1$ as $n \to \infty$. Since $(\overline{E}^\infty)^n$ depends only on f, ξ_n carries all the information of the inclusion map φ_0^n.It also is responsible for the submultiplicativity of $E^\infty[\varphi_0^n, f^{\circ n}]$. To see the latter notice that $\alpha_{n+m} \leq \alpha_n\alpha_m$, thus $\xi_{n+m}(\overline{E}^\infty)^{n+m} \leq \xi_n\xi_m(\overline{E}^\infty)^{n+m}$, which entails $\xi_{m+n} \leq \xi_m\xi_n$. Moreover, $\xi_n \geq 1$ for all n.

It is clear that, for our successful attempts, $\xi_1 = 1$. In general, one would need some k for which any $n \geq k$, $\xi_n = 1$ so that our techniques would yield an exact value for $\overline{E}^\infty$; we call PCF maps with such property *finitely realizable*. There are currently no ways of determining if a given PCF f map is finitely realizable a priori, nor there are any established methods of computing ξ_n exactly.

It may well be that #51 is finitely realizable for some $n > 4$, however, for every single example which yielded a successful computation of $\overline{E}^\infty$ we were able to emit some G_0 with $\xi_1 = 1$. The next steps would be developing methods to understand when can we have a finitely realizable PCF map, and how to go about to finding $\overline{E}^\infty$ for the non finitely realizable cases. We end with the following conjecture:

Conjecture A PCF rational map f is finitely realizable if and only if the exists some G_0 such that $\xi_1 = 1$.

6. Acknowledgements

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We thank Dylan Thurston for advising us during this summer. We also thank Caroline Davis and Kevin Pilgrim for useful conversations.

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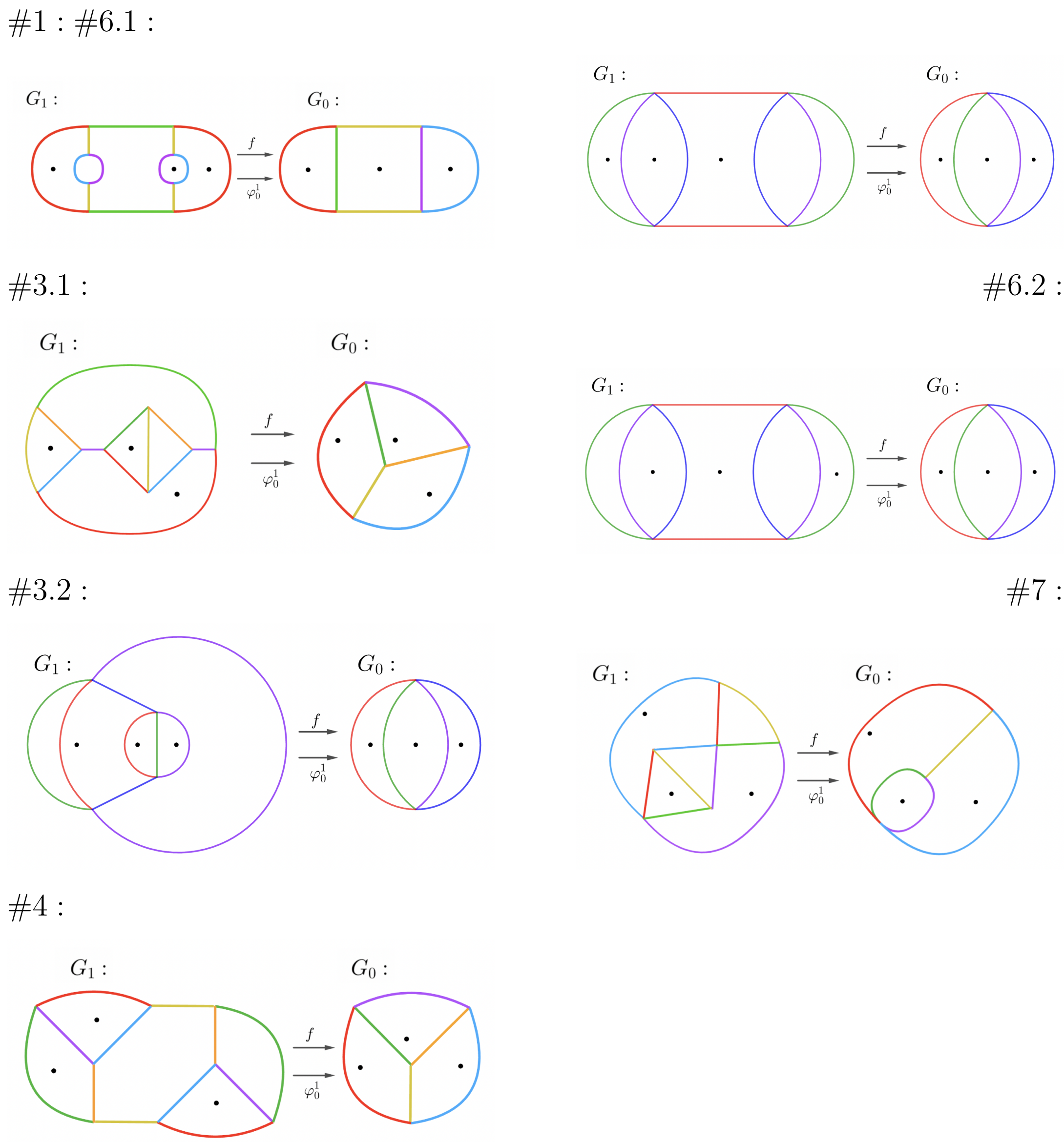

FIGURE 1. Pullbacks for all quadratic $|P(f)| = 4$ maps.

	function	$\overline{\alpha}$ algebraically	$\overline{\alpha}$ numerically	lengths	category
1	$\frac{4}{3z(4-3z)}$	$2^{-1/2}$	0.707	NOTE: G_2, NOT G_1 $\begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$	Y
3.1	$\frac{a_0(z-1)}{(z-P_0)^2}$	$2^{-1/3}$	0.794	$\begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ o_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 3\alpha_1^2 \\ 2+\alpha_1-\alpha_1^2 \\ 3\alpha_1 \\ \alpha_1^2+\frac{1}{\alpha_1^2}-2 \\ \frac{2}{\alpha_1}+1-\alpha_1 \\ 3 \end{bmatrix}$	Y
3.2	$\frac{a_0(z-1)}{(z-P_0)^2}$	$2^{-1/3}$	0.794	$\begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{\alpha_1}+\frac{1}{\alpha_1^2} \\ 1+\frac{1}{\alpha_1}+\frac{1}{\alpha_1^2} \\ 1+\frac{1}{\alpha_1} \\ 1 \end{bmatrix}$	Y
4	$\frac{-1.2+1.6i}{(10z-2-4i)(z-1)}$	$2^{-1/3}$	0.794	$\begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ o_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 6\alpha_1^2-7\alpha_1+15 \\ 30\alpha_1^2+6\alpha_1-7 \\ 21\alpha_1-4-18\alpha_1^2 \\ 3+15\alpha_1-7\alpha_1^2 \\ 3(8-\alpha_1-5\alpha_1^2) \\ 41\alpha_1^2 \end{bmatrix}$	Y
6.1	$\frac{\phi^{-2}}{(z-1)^2}$	$2\alpha_1^4+\alpha_1^3-1=0$	0.739	$\begin{bmatrix} r_1 \\ g_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \alpha_1^3 \\ \alpha_1^2 \\ \alpha_1 \\ 1 \end{bmatrix}$	B
6.2	$\frac{\phi^{2}}{(z-1)^2}$			NOTE: G_2, NOT G_1	
7	$\frac{(z-1)^2}{(z-e^{2\pi i/3})^2}$	$2^{-1/2}$	0.707	$\begin{bmatrix} r_1 \\ g_1 \\ y_1 \\ p_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{2}{\alpha_1}-2 \\ 1 \\ 1 \end{bmatrix}$	G

TABLE 4. Results for $|P(f)| = 4$ maps