Expansivity on Julia Sets

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Presentation Overview

Julia Sets

Consider $f(z) = \frac{P(z)}{Q(z)}$ in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Definition. Denote the Fatou set of f, $\mathcal{F}(f)$, by the set of points $z \in \hat{\mathbb{C}}$ having some neighborhood $D(z, \epsilon)$ where the sequence of iterates f^n has a subsequence converging uniformly.

Definition. Denote the Julia set of f, $\mathcal{J}(f)$, as $\hat{\mathbb{C}} \setminus \mathcal{F}(f)$. It is characterized by "chaotic" behavior upon iterating f on any neighborhood.

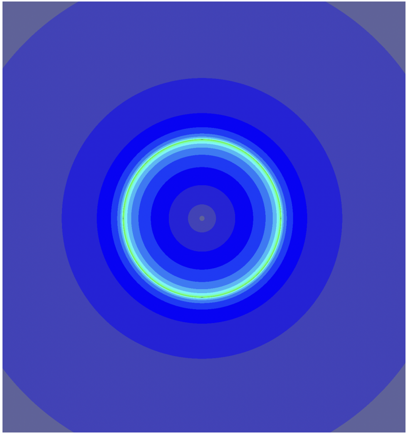


Figure: Julia set of $f(z) = z^2$

Example: $f(z) = z^2$

Consider three cases of iterating f: $|z| < 1$, $|z| = 1$, and $|z| > 1$

Iterating on any z with $|z| = 1$.

The subset of $D(z, \epsilon)$ inside the unit circle converges to 0 and the subset outside the unit circle goes to ∞.

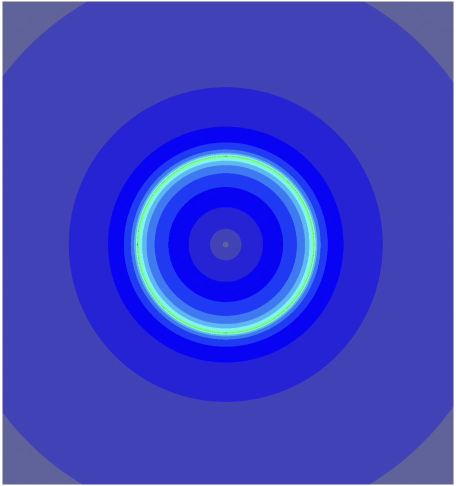


Figure: $\mathcal{J}(f)$ is the unit circle in $\hat{\mathbb{C}}$

Example: $f(z) = z^2 - 1$

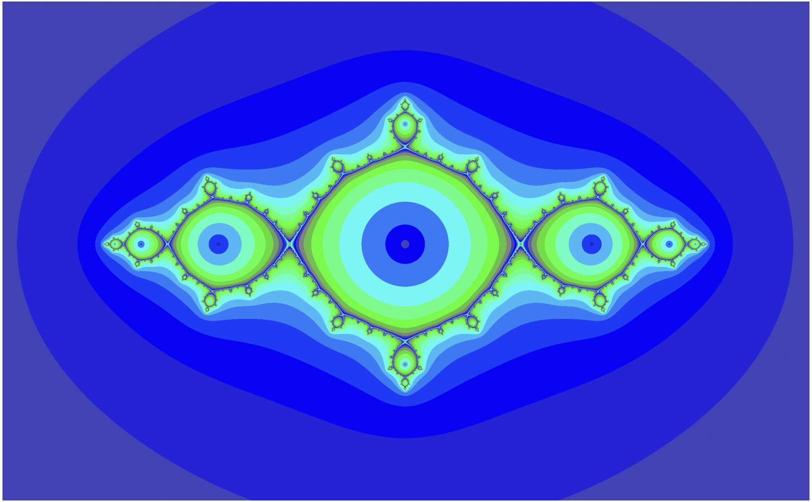


Figure: Julia set of $f(z) = z^2 - 1$

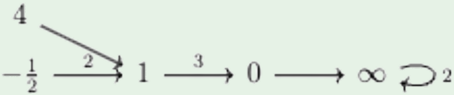
Studying $\mathcal{J}(f)$ for post-critically finite maps

Definition. The *post-critical set* $P(f)$ is $\bigcup_{n>0} f^n(C(f))$ with $C(f)$ being critical points of f.

Examples

Consider the following rational map with three critical points and its critical portrait

$$f(z) = \frac{4(z-1)^3}{27z}$$



Note this map is "post-critically finite" (PCF). f is hyperbolic as every cycle contains a critical point.

Elastic Graphs

We can approximate the Julia set of a PCF rational map by isolating the Fatou basins of each element of P_f. We can then take pre-images which has limit homeomorphic to the Julia set of f [Nek14].

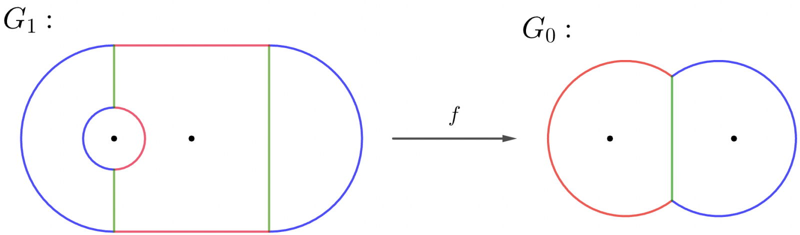


Figure: The first pullback of the function $f(z) = \frac{4(z-1)^3}{27z}$.

Example of G_4

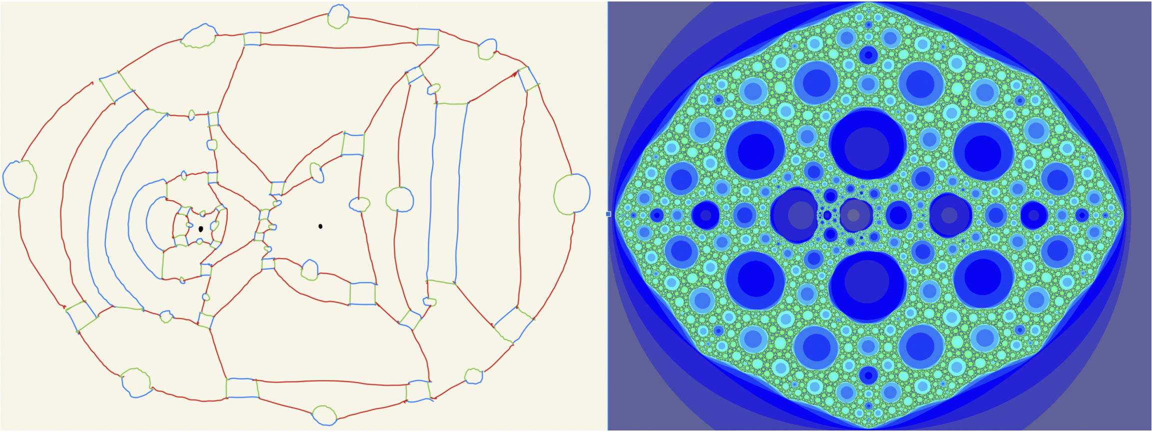


Figure: Julia set of $f(z) = z^2 - 1$

The inclusion map

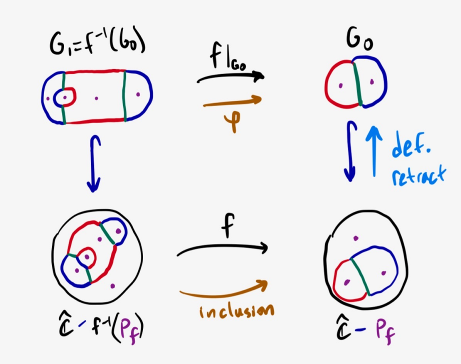


Figure: We contruct a map φ_0^n between graphs G_0 and G_1 to use in defining energies. This was shown in [Thu16, Thu19, Thu20].

The Lipschitz energy

Definition. The *Lipschitz energy* $E^\infty[\varphi_0^n]$ of a map $\varphi_0^n : G_n \to G_0$ is:

$$E^\infty[\varphi_0^n] := \inf_{\varphi_0^n \in [\psi_0^n]} \left(\sup_{x \in G_n} |(\varphi_0^n)'(x)| \right)$$

Theorem 1.

Bestvina-White [Bes11, Prop. 2.1] For any $\varphi_0^n : G_n \to G_0$, we have

$$E^\infty[\varphi_0^n] = \sup_{C \subset G_n} \frac{\ell_0(\varphi_0^n(C))}{\ell_n(C)}$$

Definition. The *asymptotic Lipschitz energy* $\overline{E}^\infty(f)$ of a PCF rational map f is:

$$\overline{E}^\infty(f) := \lim_{n \to \infty} (E^\infty[\varphi_0^n])^{1/n}$$

Note that, $\overline{E}^\infty(f)$ does not depend on the choice of G_0 and φ.

Expansivity of f on its Julia Set

Definition. Fix a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Consider the dynamical system given by $f : \mathcal{J} \to \mathcal{J}$. For a given metric d on $\mathcal{J}$, expansion factor $\lambda > 1$, and $\epsilon > 0$, we say f is (λ, ϵ)-*expansive* on its Julia set with respect to d if, for every $z_1, z_2 \in \mathcal{J}$ with $|z_1 - z_2| < \epsilon$, we have

$$|f(z_1) - f(z_2)| \geq \lambda |z_1 - z_2|.$$

Definition. We say that f is λ-*expansive* on its Julia set if it is (λ, ϵ)-expansive for some $\epsilon > 0$.

Definition. Define the *expansivity of f* on its Julia set as

$$\lambda(f) := \sup\{\lambda \mid \text{there is a metric } d \text{ on } \mathcal{J} \text{ so } f \text{ is } \lambda\text{-expansive wrt } d.\}$$

Results

- In studying census map #51 [BBL+00], resistant to Summer 2022 techniques, which were non-linear, we developed novel, linear methods to bound and compute $\overline{E}^{\infty}$ through upper and lower bounds.
- Proved the relation $\overline{E}^{\infty} \geq \frac{1}{\lambda(f)}$
- Improved #51 bound from $[0.5, 0.618]$ to $[0.553, 0.563]$.
- Calculated $\overline{E}^{\infty}$ for another class of PCF hyperbolic maps.

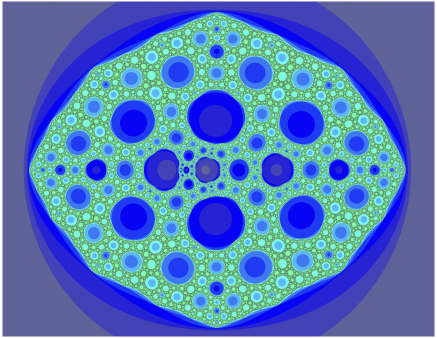


Figure: Julia Set for #51, open problem from last summer

Upper Bounds

For obtaining upper bounds, we simply need to construct a graph Γ_n in a way that the following diagram commutes, building off the work in [DDT22] to optimize the metric in G_0.

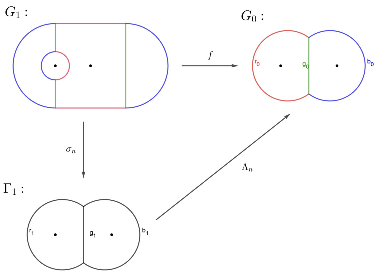


Figure: Here we require that $|\sigma_n'(x)| \leq 1$ for all $x \in G_n$, and Λ_n to be a map with constant derivative $\alpha_n < 1$, i.e. a scaling of graphs isomorphic to the identity.

This works since $|(\varphi_0^n)'(x)| = |\Lambda'(\sigma_n)\sigma_n'(x)| \leq \alpha_n$ for all $x \in G_n$. We've developed an iterative process to construct such a graph.

Upper bound example

- To construct such a graph, we look at certain subsets of $P_f \setminus \{\infty\}$ and compute what is the shortest curve around the given subset. This generally comes with assumptions regarding the metric on G_0.
- In this example, choosing the appropriate curves yields:

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ g_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} r_1 \\ g_1 \\ b_1 \end{bmatrix} = \vec{c}_1$$

- We obtain $\alpha_1 = \frac{1}{\phi}$ as an upper bound, and the metric $\vec{c}_1 = [1, \phi, 2\phi]^T$.

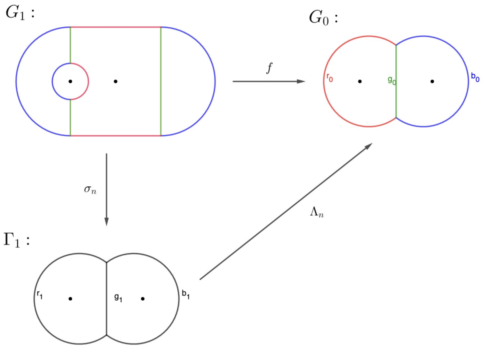


Figure: Construction of a graph Γ_1 for example #51.

Lower Bounds

Lemma 1.

If $f : J \to J$ has an f-invariant graph Γ with induced map $j : \Gamma \to \Gamma$ with Markov partition matrix A, then $\lambda(f) \leq \lambda_{\uparrow}(j)$, where $\lambda_{\uparrow}(j)$ is the Perron-Frobenius eigenvalue of A.

Lemma 2.

Let a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be PCF hyperbolic. Then,

$$\overline{E}^{\infty}[\pi, \phi] \geq \frac{1}{\lambda(f)}.$$

Moreover, if the conditions for Lemma 1 are met, then

$$\overline{E}^{\infty}[\pi, \phi] \geq \frac{1}{\lambda_{\uparrow}(j)}.$$

Invariant Graphs

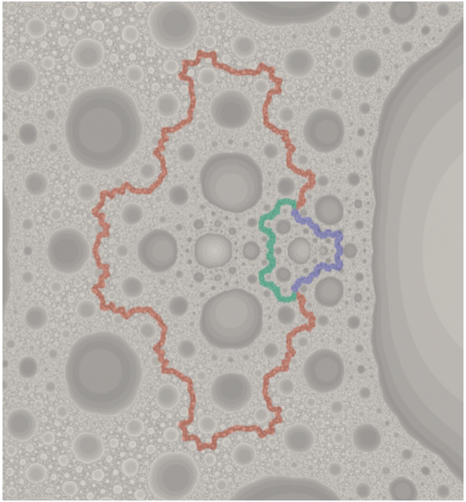
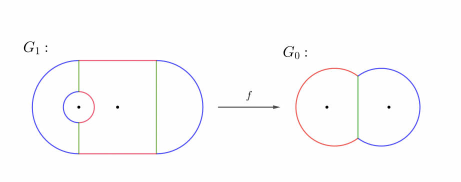


Figure: Invariant graph is embedded in $\mathcal{J}$–this is a sub-dynamical system of $f : \mathcal{J} \to \mathcal{J}$. Figure from [DDT22]

Lower Bound Example

#51 Spine



Examples

Use the map $j : G_0 \to G_0 \subset G_1$ given by the matrix below. Recalling that these invariant graphs are literally a subset of $\mathcal{J}(f)$.

$$\lambda_{\uparrow}(j) \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} r \\ g \\ b \end{bmatrix}$$

Seeing $\lambda_{\uparrow}(j)$ as an "expansion factor" of G_0 gives $0.528 \leq \overline{E}^{\infty}$. Best l.b. was found in G_4.

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Thank you!

Thanks for listening!

Questions?