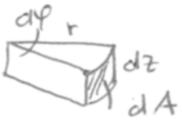
Aufgabe 1

(1) a) Flächennormale $\vec{n}_0$ auf Mantel

$$\vec{n} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \xrightarrow{\text{normieren}} \vec{n}_0 = \frac{1}{2}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \quad (\text{wobei } R=2)$$

b) $dA = r\,d\varphi\,dz$



c)
$$\phi_{Mantel} = \iint_{Mantel} \vec{v} \bullet \vec{n}_0 \cdot dA = \frac{1}{2}\iint (x^4 - y^2)\,dA$$

$$= \frac{1}{2}\int_{\varphi=0}^{2\pi}\int_{z=0}^{5} (r^4\cos^4\varphi - r^2\sin^2\varphi)\,r\,dr\,d\varphi = \underline{\cancel{80\pi}} = \underline{40\pi}$$

d)
$$\phi_{Deckel} = \iint \begin{pmatrix} x^3 \\ -y \\ z \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA = \iint z\,dA \overset{\text{Polarkoord. (und } z=5)}{=} 5\underbrace{\iint dA}_{R^2\pi} = \underline{20\pi}$$

$$\phi_{Boden} = \iint \begin{pmatrix} x^3 \\ -y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dA = \iint -z\,dA = 0 \quad \text{da } \underline{z=0}$$

$$\rightarrow \underline{\underline{\phi_{tot} = 60\pi}}$$

(2) Mit Satz von Gauss (geht, da Fläche geschlossen ist)

$$\phi = \iint_{A_{Zylind.}} \vec{v} \circ \vec{n}\,dA = \iiint_V \operatorname{div}\vec{v}\,dV$$

$$\operatorname{div}\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 3x^2 - 1 + 1 = 3x^2 \xrightarrow{\text{Zylinder}} 3 \cdot R^2 \cdot \cos^2\varphi$$

$$\phi = \int_{z=0}^{5}\int_{\varphi=0}^{2\pi}\int_{r=0}^{2} 3\cos^2\varphi\, r^3\,dr\,d\varphi\,dz = \underline{60\pi}$$

Aufgabe 2

i) Satz von Gauss: $\vec{v} = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix} \to \operatorname{div} \vec{v} = 3x^2 + 3y^2 + 3z^2$

$$\phi = \int_{V_{Kugel}} \operatorname{div} \vec{v} \, dV \underset{\text{Ku.Ko}}{\overset{!}{=}} 3 \cdot \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{r=1}^{R} r^2 (\sin^2\vartheta \cos^2\varphi + \sin^2\vartheta \sin^2\varphi + \cos^2\vartheta) \cdot r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi$$

$$= \underline{\underline{\frac{12}{5} \pi R^5}}$$

ii) Oberflächenintegral:

Normalenvektor: $\vec{n}_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\to \vec{n}_0 \cdot \vec{v} = x^4 + y^4 + z^4$$

Radius R ist fest (Ku.Ko)

$$\to \phi = \iint_{A_{Kugel}} (x^4 + y^4 + z^4) \, dA \overset{!}{=} R^4 \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} (\cancel{\cos}\,\sin^4\vartheta \cos^4\varphi + \cancel{\cos}\,\sin^4\vartheta \cdot \sin^4\varphi + \cos^4\vartheta) \cdot R \sin\vartheta \, d\vartheta \, d\varphi$$

$$= \underline{\underline{\frac{12}{5} \pi R^5}}$$

Aufgabe 3:

a) $\phi = \phi(x,y,z)$ $\quad \operatorname{div}(\operatorname{grad} \phi) = \operatorname{div} \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{pmatrix} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ ✓

b) 2d: $\phi(r) = \frac{1}{r} \to \phi(x,y) = (x^2 + y^2)^{-1/2}$

$$\frac{\partial \phi}{\partial x} = -\frac{x}{r^3}, \quad \frac{\partial \phi}{\partial y} = -\frac{y}{r^3}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{3x^2}{r^5} - \frac{1}{r^3}, \quad \frac{\partial^2 \phi}{\partial y^2} = \frac{3y^2}{r^5} - \frac{1}{r^3}$$

$$\to \Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{3\overbrace{(x^2 + y^2)}^{r^2}}{r^5} - \frac{2}{r^3} = \frac{3}{r^3} - \frac{2}{r^3} = \underline{\underline{\frac{1}{r}}}$$

3d: Analog folgt mit $\phi(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \to \underline{\underline{\Delta \phi = 0}}$

Aufgabe 4

Es gilt, dass es ein Vektorfeld $\vec{E}$ mit $\vec{F} = \text{rot}\,\vec{E}$ gibt.

Oberflächenintegral (geschl. Flächen):

$$\oiint_A \vec{F} \circ \vec{n}_0 \, dA = \underbrace{\oiint_A \text{rot}\,\vec{E} \circ \vec{n}_0 \, dA}_{\text{nach Definition}} \overset{\text{Gauss}}{=} \iiint_V \text{div}(\text{rot}\,\vec{E})\, dV$$

Was ist $\text{div}(\text{rot}\,\vec{E})$?

$$\vec{E} := \begin{pmatrix} E_x(x,y,z) \\ E_y(x,y,z) \\ E_z(x,y,z) \end{pmatrix} \qquad \text{rot}\,\vec{E} = \vec{\nabla} \times \vec{E} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times E$$

$$\rightarrow \text{rot}\,\vec{E} = \begin{pmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix}$$

$$\rightarrow \text{div}\,\text{rot}\,\vec{E} = \frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\text{div}(\text{rot}\,\vec{E}) = \frac{\partial^2 E_z}{\partial x \partial y} - \frac{\partial^2 E_y}{\partial x \partial z} + \frac{\partial^2 E_x}{\partial y \partial z} - \frac{\partial^2 E_z}{\partial x \partial y} + \frac{\partial^2 E_y}{\partial x \partial z} - \frac{\partial^2 E_x}{\partial y \partial z}$$

$=$

$= 0$, da sich alle Terme wegkürzen!

$$\rightarrow \iiint_V \underbrace{\text{div}(\text{rot}\,\vec{E})}_{= 0} dV = \underline{\underline{0}} \qquad \text{q.e.d}$$