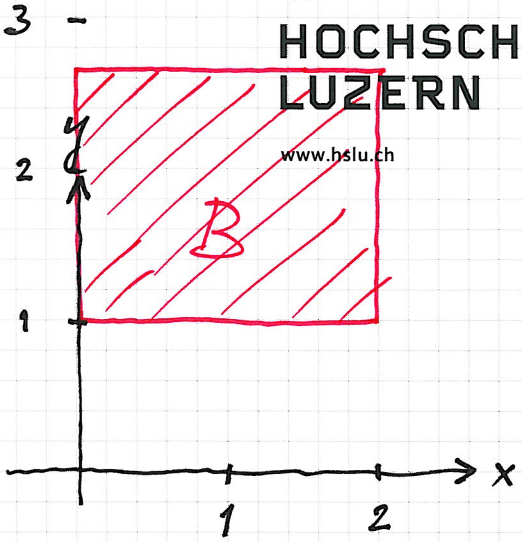
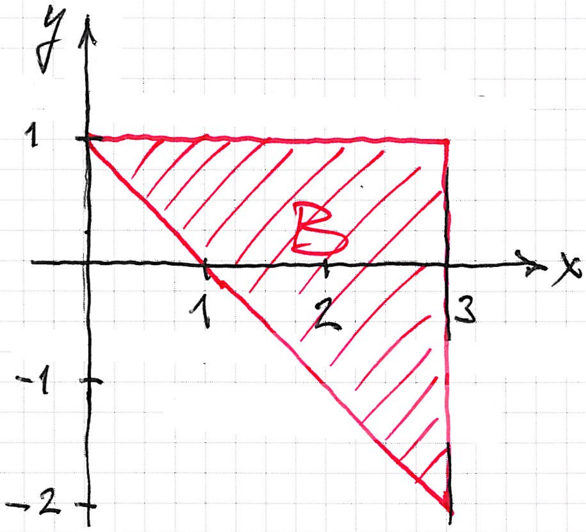


1 (a) $I = \int_0^2 \int_1^e \frac{x^2}{y}\,dy\,dx$

$$I = \int_0^2 x^2 \Big[\ln|y|\Big]_1^e dx$$
$$= \int_0^2 x^2 \big(\underbrace{\ln|e|}_{1} - \underbrace{\ln|1|}_{0}\big)\,dx$$
$$= \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} = \frac{8}{3}$$

(b) $I = \int_0^3 \int_1^{1-x} (2xy - x^2 - y^2)\,dy\,dx$

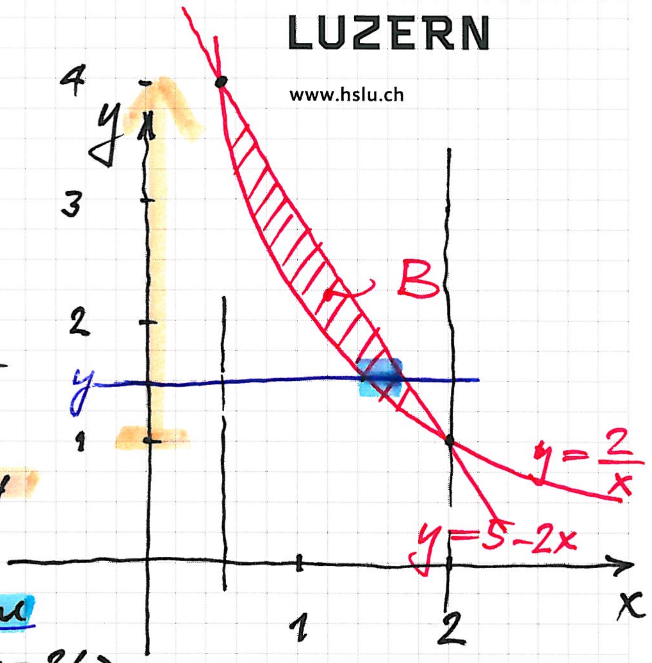
Damit



$$I = \int_0^3 \left[xy^2 - x^2y - \frac{y^3}{3}\right]_1^{1-x} dx$$
$$= \int_0^3 \Big(\frac{6x}{6}(1-x)^2 - \frac{6x^2}{6}(1-x) - \frac{2(1-x)^3}{2\cdot 3} - \frac{6x}{6} + \frac{6x^2}{6} + \frac{2\cdot 1}{3\cdot 2}\Big)\,dx$$
$$= \frac{1}{6}\int_0^3 \big(6x(1-x)^2 - 6x^2(1-x) - 2(1-x)^3 - 6x + 6x^2 + 2\big)\,dx$$
$$= \frac{1}{6}\int_0^3 \big(6x - 12x^2 + 6x^3 + 6x^3 \cancel{-2} + 6x - 6x^2 + 2x^3 \cancel{-6x} \cancel{+2}\big)\,dx$$
$$= \frac{1}{6}\int_0^3 (6x - 18x^2 + 14x^3)\,dx$$
$$= \frac{1}{6}\left[\cancel{6}^{\,3}\frac{x^2}{\cancel{2}} - \cancel{18}^{\,6}\frac{x^3}{\cancel{3}} + \cancel{14}^{\,7}\frac{x^4}{\cancel{4}_{\,2}}\right]_0^3 = \frac{1}{6}\left(3^3 - 2\cdot 3^4 + \frac{7}{2}3^4\right)$$
$$= \frac{1}{6}\left(3^3 + \frac{3}{2}\cdot 3^4\right) = \frac{1}{12}\left(2\cdot 3^3 + 3^2\cdot 3^3\right) = \frac{3^3}{12}\cdot 11 = \frac{3^2\cdot 11}{4} = \frac{99}{4}$$

(2)

$$I = \int_{1/2}^{2} \int_{2/x}^{5-2x} \frac{x}{\sqrt{y}}\, dy\, dx$$



Vertauschen der Integrationsgrenzen:

- wir lassen y von 1 bis 4 laufen.
- für ein festes y geht dann x von $2/y$ (folgt aus $y = 2/x$) bis zu $(5-y)/2$ (folgt sofort aus $y = 5-2x$ durch Auflösen nach x)

Somit lautet das gesuchte Integral

$$I = \int_1^4 \int_{2/y}^{(5-y)/2} \frac{x}{\sqrt{y}}\, dx\, dy$$

$$= \int_1^4 \frac{1}{\sqrt{y}} \left[\frac{x^2}{2}\right]_{2/y}^{\frac{5-y}{2}} dy = \int_1^4 \frac{1}{\sqrt{y}} \left(\frac{\overbrace{(5-y)^2}^{25-10y+y^2}}{8} - \frac{2}{y^2}\right) dy$$

$$= \int_1^4 \left(\frac{25}{8} y^{-1/2} - \frac{10}{8} y^{1/2} + \frac{1}{8} y^{3/2} - 2 y^{-5/2}\right) dy$$

$$= \left[\frac{25}{\cancel{8}_4} \frac{y^{1/2}}{\cancel{1/2}} - \frac{10}{\cancel{8}_{12}} \frac{y^{3/2}}{\cancel{3/2}} + \frac{1}{\cancel{8}_{20}} \frac{y^{5/2}}{\cancel{5/2}} - 2\, \frac{y^{-3/2}}{-3/2}\right]_1^4$$

$$= \frac{25}{4}(2-1) - \frac{\cancel{10}^{5}}{\cancel{12}_{6}}(8-1) + \frac{1}{20}(32-1) + \frac{4}{3}\left(\frac{1}{8} - 1\right)$$

$$= \frac{25}{4} - \frac{35}{6} + \frac{31}{20} - \frac{\cancel{4}}{3}\frac{7}{\cancel{8}_2} = \frac{125+31}{20} - \frac{42}{6} = \frac{156}{20} - 7$$

$$= \frac{156-140}{20} = \frac{16}{20} = \frac{4}{5}$$

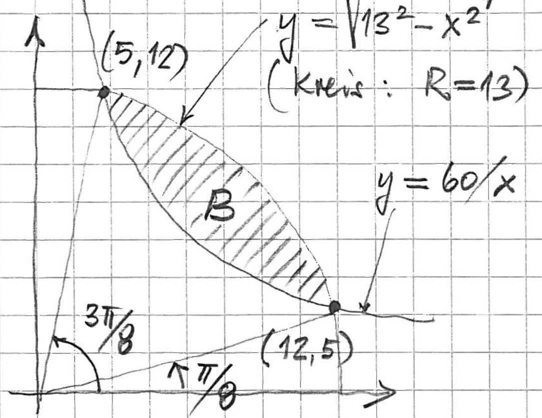
(3) Man findet sofort, dass sich die beiden Kurven

$$y = \frac{60}{x}$$

und

$$y = \sqrt{13^2 - x^2}$$

in $(5,12)$ und $(12,5)$ schneiden.



Dann lautet das gesuchte Integral:

$$\tan\varphi_1 = \frac{5}{12} \Rightarrow \varphi_1 = \frac{\pi}{8}$$

$$\tan\varphi_2 = \frac{12}{5} \Rightarrow \varphi_2 = \frac{3\pi}{8}$$

$$(1) \quad I = \int_{\pi/8}^{3\pi/8} \int_{\sqrt{\frac{120}{\sin(2\varphi)}}}^{13} r^3 \, dr \, d\varphi$$

Hier haben wir verwendet, dass $r^2 = x^2 + y^2 = x^2 + \frac{60^2}{x^2}$, also mit $x = r\cos\varphi$ (Polarkoord.)

$$r^2 r^2 \cos^2\varphi = r^4\cos^4\varphi + 60^2$$

$$r^4 \cos^2\varphi\,(1 - \cos^2\varphi) = 60^2$$

$$r^4 \cos^2\varphi \sin^2\varphi = 60^2$$

$$r^4 \sin^2(2\varphi) = (2\cdot 60)^2$$

Also $r = \sqrt{120/\sin(2\varphi)}$ (beachte $0 \le \varphi \le \pi/2$).

Nun folgt aus (1) nacheinander

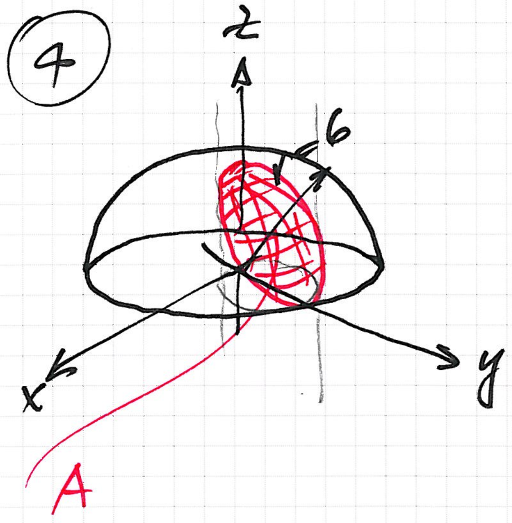
$$I = \int_{\pi/8}^{3\pi/8} \frac{r^4}{4}\Bigg|_{\sqrt{\frac{120}{\sin(2\varphi)}}}^{13} d\varphi = \frac{1}{4}\left(13^4 \frac{\pi}{4} - 120^2 \int_{\pi/8}^{3\pi/8} \frac{d\varphi}{\sin^2(2\varphi)}\right)$$

$$= \left(\frac{13}{2}\right)^4 \pi - 60^2 \left(-\frac{1}{2}\frac{\cos(2\varphi)}{\sin(2\varphi)}\right)\Bigg|_{\pi/8}^{3\pi/8}$$

$$= \left(\frac{13}{2}\right)^4 \pi + \frac{60^2}{2}\left[\frac{-1/\sqrt{2}}{1/\sqrt{2}} - \frac{1/\sqrt{2}}{1/\sqrt{2}}\right]$$

$$\underline{I = \left(\frac{13}{2}\right)^4 \pi - 60^2} = 2007.94\ldots$$

4



$$x^2+(y-3)^2=9$$

Zylinder mit Radius $R=3$ und Achse parallel zur z-Achse durch $(0,3,0)$.

$$f(x,y)=z=\sqrt{36-x^2-y^2}$$

$$f_x=\frac{-2x}{2\sqrt{36-x^2-y^2}}$$

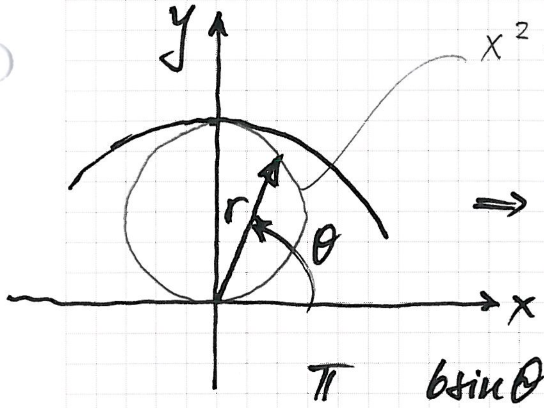
$$f_y=\frac{-2y}{2\sqrt{36-x^2-y^2}}$$

$$\sqrt{f_x^2+f_y^2+1}=\frac{6}{\sqrt{36-x^2-y^2}}$$

In Polarkoord.:

$$\sqrt{\cdots}=\frac{6}{\sqrt{36-r^2}}$$

Grundriss:



$$x^2+y^2=6y$$

$$r^2=6r\sin\theta$$

$$\Rightarrow\quad r=6\sin\theta$$

$$\underline{A}=\int_0^{\pi}d\theta\int_0^{6\sin\theta}\frac{6r\,dr}{\sqrt{36-r^2}}$$

$$=2\int_0^{\pi/2}d\theta\int_{36}^{36\cos^2\theta}\frac{-3\,dz}{\sqrt{z}}=6\int_0^{\pi/2}d\theta\left[2\sqrt{z}\right]_{36\cos^2\theta}^{36}$$

$z=36-r^2$
$dz=-2r\,dr$
$r=0\Rightarrow z=36$
$r=6\sin\theta\Rightarrow z=36(1-\sin^2\theta)\overset{!}{=}36\cos^2\theta$

$$=12\int_0^{\pi/2}d\theta\,(6-6\cos\theta)$$

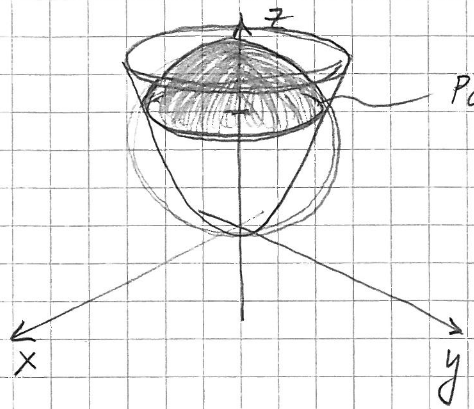
$$=72\left(\pi/2-\sin\theta\Big|_0^{\pi/2}\right)$$

$$=\underline{72\left(\frac{\pi}{2}-1\right)}$$

5 Die Kugel

$$x^2+y^2+(z-2)^2 = 4 = 2^2$$

hat das Zentrum $(0,0,2)$ und den Radius 2.



Paraboloid : $z = x^2+y^2$

Wir berechnen die Punkte welche sowohl auf der Kugel wie auch auf dem Paraboloid liegen! Dort gilt

$$\underbrace{x^2+y^2}_{z}+z^2 = 4z$$

$$z^2-3z = z(z-3) = 0 \Rightarrow \underline{z=0 \vee z=3}$$

Dort gilt $\quad x^2+y^2 = (\sqrt{3})^2$

In Polarkoord.

$$A = \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r\,dr \sqrt{(f_r)^2+1}$$

$$z = 2+\sqrt{4-r^2} = f(r,\theta) = f(r)$$

$$f_r = \frac{-r}{\sqrt{4-r^2}}$$

$$(f_r)^2 = \frac{r^2}{4-r^2}$$

$$\sqrt{(f_r)^2+1} = \frac{2}{\sqrt{4-r^2}}$$

$$A = 2\pi \cdot \cancel{2} \int_0^{\sqrt{3}} \frac{1}{\cancel{2}} \frac{d(r^2)}{\sqrt{4-r^2}}$$

$$= 2\pi \int_0^3 \frac{du}{\sqrt{4-u}} = 2\pi \left[-2\sqrt{4-u} \right]_0^3$$

$$= 4\pi (2-1) = 4\pi$$

$$\underline{A = 4\pi}$$