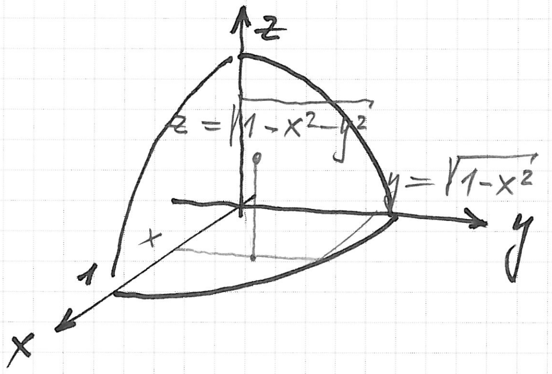
1

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2}\, dz$$



Kugel mit Zentrum in $(0,0,0)$ und Radius $R=1$.

In Kugelkoord.

$0 \leq \theta \leq \pi/2$

$0 \leq \varphi \leq \pi/2$

$0 \leq r \leq 1$

$$\underline{I} = \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi \int_0^1 r^2 \sin\varphi \, \frac{1}{1+r^2}\, dr$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sin\varphi \, d\varphi \underbrace{\int_0^1 \frac{r^2\, dr}{1+r^2}}_{\left[r - \arctan(r)\right]_0^1 = 1-\frac{\pi}{4}}$$

$$= \frac{\pi}{2}\left(1-\frac{\pi}{4}\right) \underbrace{\left[-\cos\varphi\right]_0^{\pi/2}}_{1-0} = \underline{\frac{\pi}{2}\left(1-\frac{\pi}{4}\right)}$$

Das entspricht der Masse der 8-el Kugel

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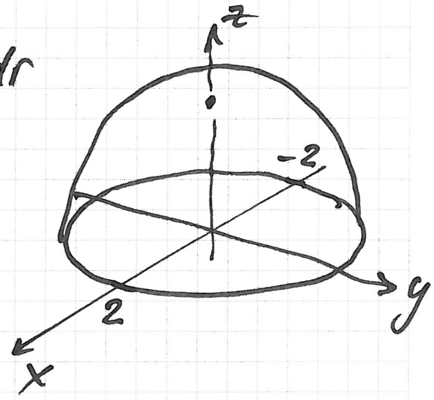
(2) Hier geht's um eine Halbkugel $(z \geq 0)$ mit Radius 2 und Zentrum in $(0,0,0)$.

In Kugelkoord. hat man für das Integral

$$\underline{I} = \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \int_0^2 r^2 \sin\varphi \, (r\cos\varphi)^2 \, r \, dr$$



$$= 2\pi \int_0^{\pi/2} d\varphi \sin\varphi \cos^2\varphi \underbrace{\int_0^2 r^5 dr}_{2^6/6}$$

$$= \frac{2^7\pi}{6} \int_0^{\pi/2} \cos^2\varphi \, d(-\cos\varphi)$$

$$= \frac{2^7\pi}{6} \int_0^1 u^2 du$$

$$= \frac{2^7\pi}{6} \cdot \frac{1}{3} = \frac{2^6\pi}{9} = \underline{\frac{64\pi}{9}}$$

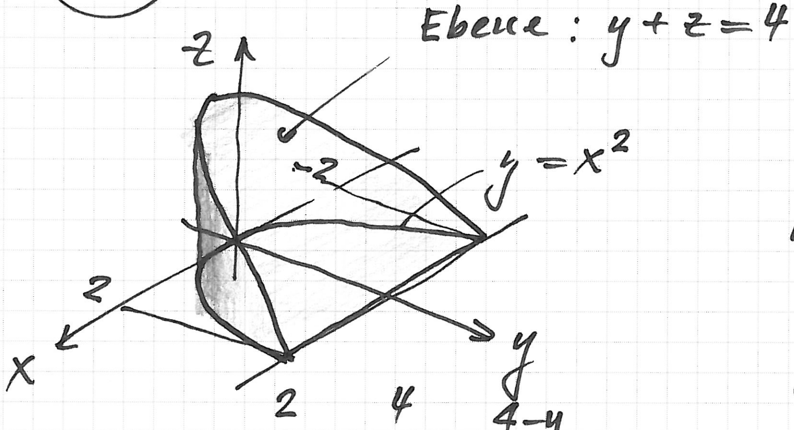
Mittlere "Dichte" $(z^2 r)$ ist $\sim 1^2 \cdot 1 = 1$.

Multipliziert mit dem Kugelvol./2 hat man

$$I \sim \frac{4\pi 2^2}{3\cdot 2} \cdot 1 = \frac{8\pi}{3} = \frac{24\pi}{9}$$

was ein bisschen weniger als 50% des Resultat ist: die Grössenordnung stimmt.

3



Wir verwenden $\rho = 1$. Für den Schwerpunkt gilt $x_S = 0$ (aus Symmetriegründen)

$$\underline{m} = 2\int_0^2 dx \underbrace{\int_{x^2}^4 dy \underbrace{\int_0^{4-y} dz}_{4-y}}_{(4y - y^2/2)}$$

$$= 2\int_0^2 dx \left(16 - 8 - 4x^2 + \frac{x^4}{2}\right)$$

$$= 2\left[8x - \frac{4}{3}x^3 + \frac{x^5}{10}\right]_0^2$$

$$= 2\left(16 - \frac{32}{3} + \frac{32}{10}\right)$$

$$= \frac{2}{30}\left(\underbrace{\overbrace{16\cdot 30}^{480} - 320}_{160} + 96\right)$$

$$= \frac{2\cdot 256}{30} = \underline{\frac{2^9}{30} = \frac{2^8}{15}}$$

$$\underline{z_S} = \frac{1}{m} 2 \int_0^2 dx \int_{x^2}^4 dy \underbrace{\int_0^{4-y} z\,dz}_{\left[z^2/2\right]_0^{4-y}}$$

$$= \frac{2 \cdot 15}{2^8} \int_0^2 dx \int_{x^2}^4 dy \, \frac{(4-y)^2}{2}$$

$$= \frac{15}{2^8} \int_0^2 dx \left[16y - 4y^2 + \frac{y^3}{3}\right]_{x^2}^4$$

$$= \frac{15}{2^8} \int_0^2 \left(\frac{2^6}{3} - 16x^2 + 4x^4 - \frac{x^6}{3}\right) dx$$

$$= \frac{15}{2^8} \left[\frac{2^6 x}{3} - \frac{16}{3}x^3 + \frac{4}{5}x^5 - \frac{x^7}{21}\right]_0^2$$

$$= \frac{15}{2^8} \left(\cancel{\frac{2^7}{3}} - \cancel{\frac{2^7}{3}} + \frac{2^7}{5} - \frac{2^7}{21}\right)$$

$$= \frac{\cancel{15}^{\,1}}{\cancel{2^8}\,2} \cancel{2^7} \, \frac{21-5}{\cancel{105}\,7} = \frac{16}{2 \cdot 7} = \frac{2^3}{7} = \underline{\frac{8}{7}}$$

$$\underline{y_S} = \frac{2 \cdot 15}{2^8} \int_0^2 dx \int_{x^2}^4 y\,dy \underbrace{\int_0^{4-y} dz}_{4-y}$$

$$\underbrace{\left[2y^2 - y^3/3\right]_{x^2}^4} = \frac{2}{3} 4^2 - 2x^4 + \frac{x^6}{3}$$

$$= \frac{15}{2^7} \left[\frac{2^5}{3} x - \frac{2}{5} x^5 + \frac{x^7}{21}\right]_0^2 = \frac{15}{2^7} \left[\frac{2^6}{3} - \frac{2^6}{5} + \frac{2 \cdot 2^6}{21}\right]$$

$$= \frac{15}{2^7\,105} \left(35 \cdot 2^6 - 21 \cdot 2^6 + 10 \cdot 2^6\right) = \frac{1}{2 \cdot 7} (24) = \underline{\frac{12}{7}}$$

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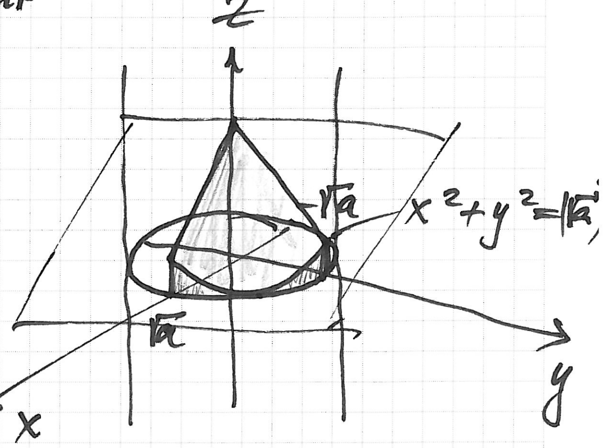
(4) $x^2+y^2 \leq a$: Zylinder mit Achse parallel z-Achse durch $(0,0,0)$ und Radius $\sqrt{a}$

$x, y, z \geq 0$: 1. Oktant



Masse m bei $\rho = 1$:

$$\underline{m} = \int_0^{\pi/2} d\theta \int_0^{\sqrt{a}} dr\, r \underbrace{\int_0^{\sqrt{2}a - r(\sin\theta+\cos\theta)} dz}_{\sqrt{2}a - r(\sin\theta+\cos\theta)}$$

$$= \int_0^{\pi/2} d\theta \underbrace{\left[\sqrt{2}\,a\frac{r^2}{2} - (\sin\theta+\cos\theta)\frac{r^3}{3}\right]_0^{\sqrt{a}}}_{\frac{a^2}{\sqrt{2}} - (\sin\theta+\cos\theta)\frac{a\sqrt{a}}{3}}$$

$$= \frac{a^2}{\sqrt{2}}\frac{\pi}{2} - \frac{a\sqrt{a}}{3}\underbrace{\left[-\cos\theta + \sin\theta\right]_0^{\pi/2}}_{1+1}$$

$$= \underline{\frac{a^2}{\sqrt{2}}\frac{\pi}{2} - \frac{a\sqrt{a}}{3}\cdot 2}$$

(war eigentlich nicht gefragt ... aber macht Spass)

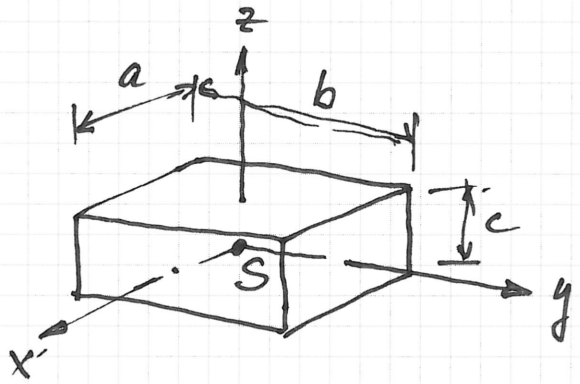
$$\underline{I_z} = \iiint_\Omega r^2\, dm = \int_0^{\pi/2} d\theta \int_0^{\sqrt{a}} dr\, r \underbrace{\int_0^{\sqrt{2}a - r(\sin\theta+\cos\theta)} r^2 dz}_{\sqrt{2}a - r(\sin\theta+\cos\theta)}$$

$$\overbrace{\left[\sqrt{2}\,a\frac{r^4}{4} - \frac{r^5}{5}(\cos\theta+\sin\theta)\right]_0^{\sqrt{a}}}$$

$$= \frac{\pi}{2}\frac{\sqrt{2}\,a\,a^2}{4} - \frac{a^{5/2}}{5}\underbrace{\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta}_{\left[+\sin\theta - \cos\theta\right]_0^{\pi/2} = 1+1=2}$$

$$= \frac{\pi a^3}{4\sqrt{2}} - \frac{a^2\sqrt{a}\,2}{5}$$

5



$\rho = \rho_c$: konstant

$$I_{ik} = \iiint_{\Omega} (r^2 \delta_{ik} - r_i r_k) \rho_c \, dV$$

$$= \rho_c \int_{-a/2}^{a/2} dr_1 \int_{-b/2}^{b/2} dr_2 \int_{-c/2}^{c/2} dr_3 \underbrace{\left[(r_1^2 + r_2^2 + r_3^2) \delta_{ik} - r_i r_k \right]}_{\begin{pmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ & r_1^2 + r_3^2 & -r_2 r_3 \\ & & r_1^2 + r_2^2 \end{pmatrix}}$$

$$I_{11} = \rho_c a \int_{-b/2}^{b/2} dr_2 \underbrace{\int_{-c/2}^{c/2} (r_2^2 + r_3^2) \, dr_3}$$

$$\left[r_2^2 r_3 + \frac{r_3^3}{3} \right]_{-c/2}^{c/2} = r_2^2 c + 2 \frac{(c/2)^3}{3} = r_2^2 c + \frac{c^3}{12}$$

$$= \rho_c a \int_{-b/2}^{b/2} dr_2 \left(r_2^2 c + \frac{c^3}{12} \right) = ac \left[2 \cdot \frac{(b/2)^3}{3} + b \frac{c^2}{12} \right]$$

$$= \rho_c acb \left[\frac{b^2}{12} + \frac{c^2}{12} \right] = m \left(\frac{b^2}{12} + \frac{c^2}{12} \right)$$

Aus Symmetrieüberlegungen folgt:

$$I_{22} = m\left(\frac{a^2}{12} + \frac{c^2}{12}\right)$$

$$I_{33} = m\left(\frac{a^2}{12} + \frac{b^2}{12}\right)$$

Wir berechnen nun

$$I_{12} = \rho_c \int_{-a/2}^{a/2} dr_1 \int_{-b/2}^{b/2} dr_2 \int_{-c/2}^{c/2} dr_3 \,(-r_1 r_2)$$

$$= \rho_c \int_{-a/2}^{a/2} (-r_1)\, dr_1 \underbrace{\int_{-b/2}^{b/2} r_2\, dr_2}_{\left[r_2^2/2\right]_{-b/2}^{b/2} = 0} \underbrace{\int_{-c/2}^{c/2} dr_3}_{c}$$

$$= 0$$

Analog findet man $I_{13} = I_{23} = 0$.

Also:

$$I = \frac{m}{12}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$