The liquid-gas interface in dimer models

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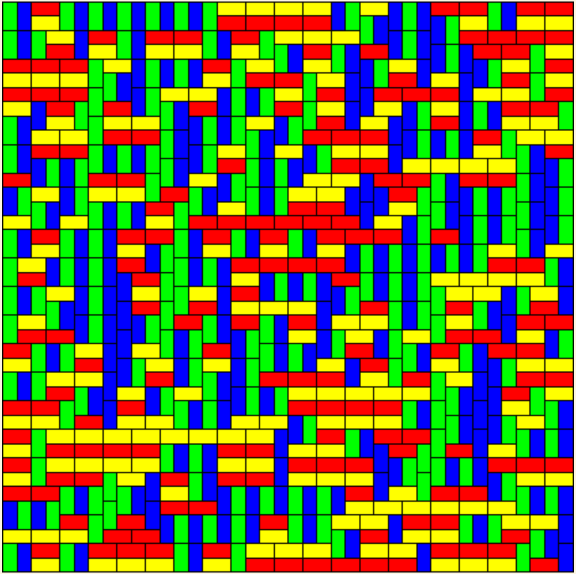
Rennes, 21/11/2016

Outline

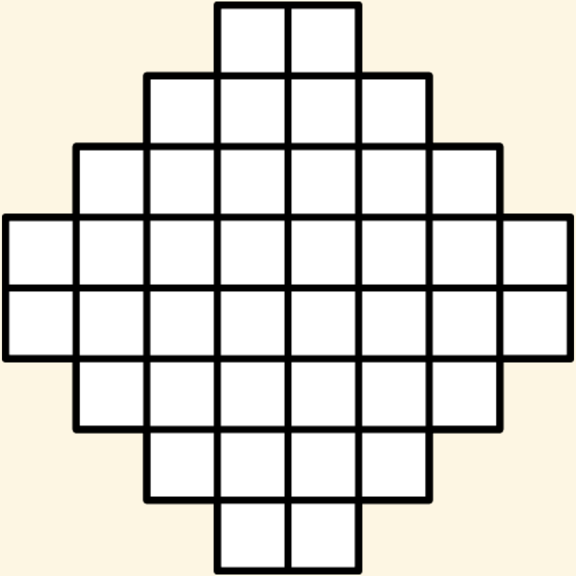
The model: uniform tiling of a planar region by dominos



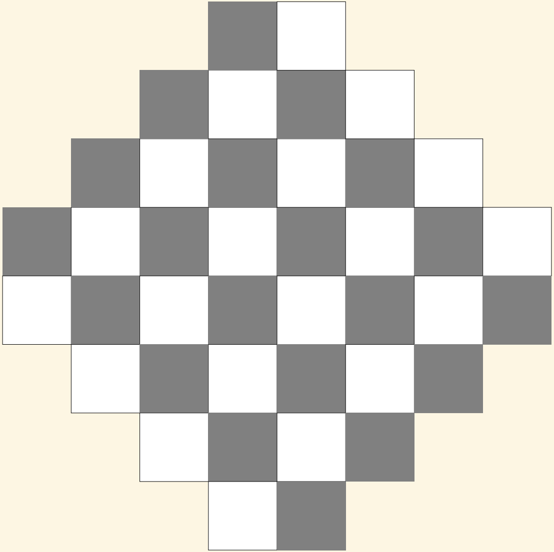
A few facts

- The number of such tilings, for even L, behaves like $\exp(cL^2)$ for some positive constant (free energy) c
- Correlations are explicit in terms of the Kasteleyn matrix, and have a power-law decay
- Can generate such a configuration by the flip dynamics which is ergodic
- This follows from the definition of the height function

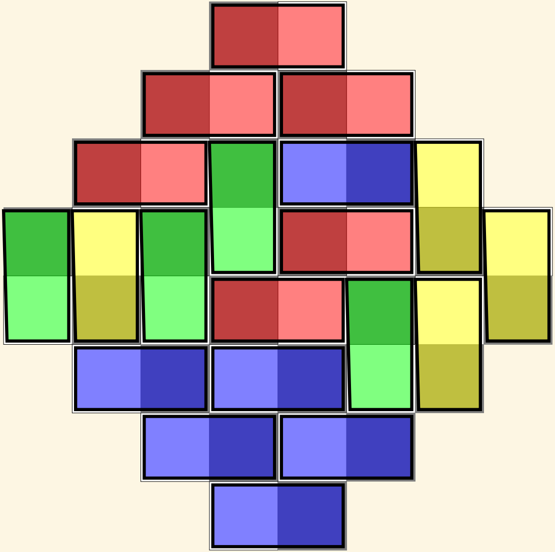
The "Aztec diamond"



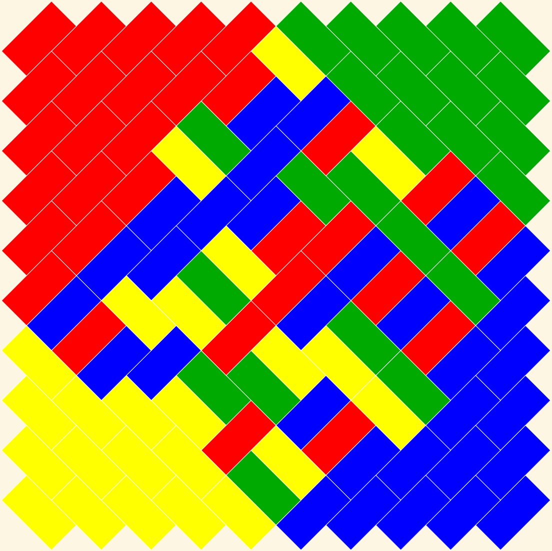
The “Aztec diamond” is bipartite



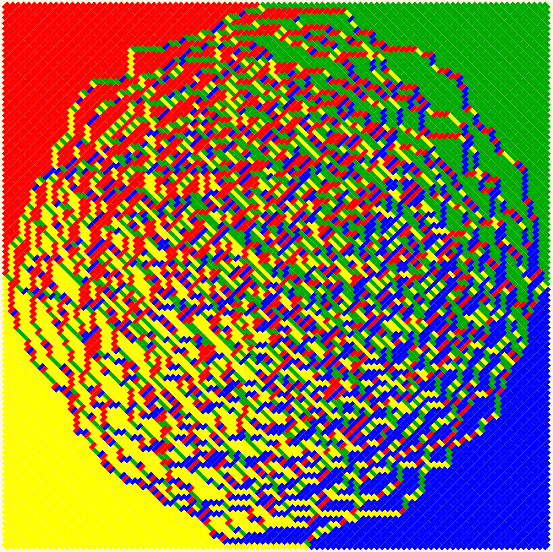
The "Aztec diamond" is bipartite and tilable



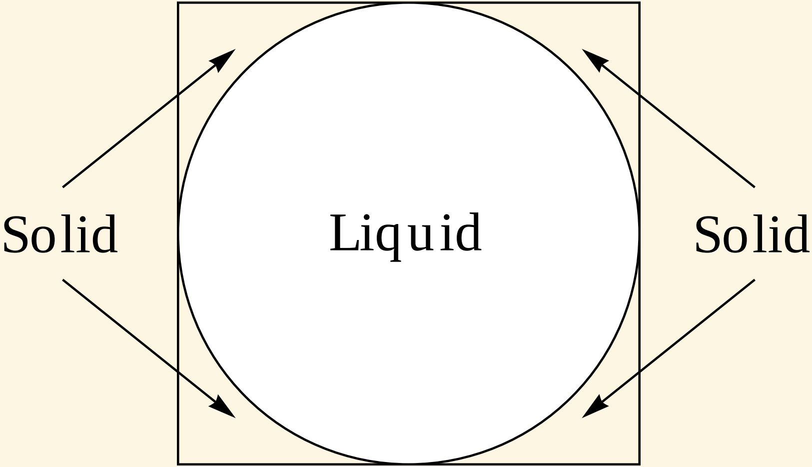
Uniform tiling of a large Aztec diamond



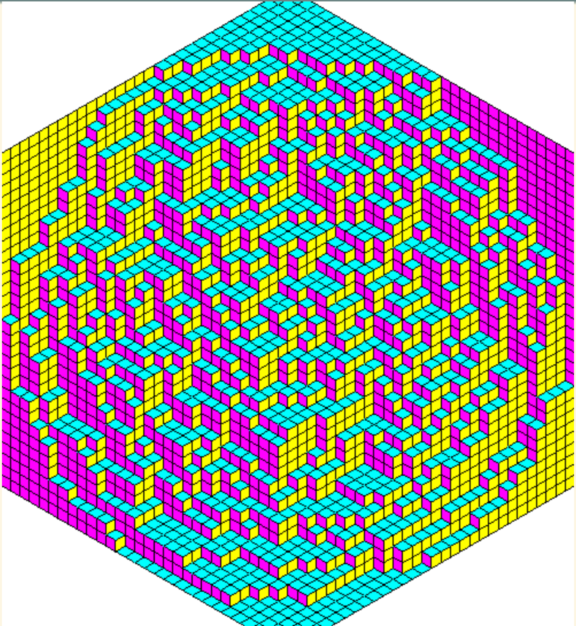
Uniform tiling of a larger Aztec diamond



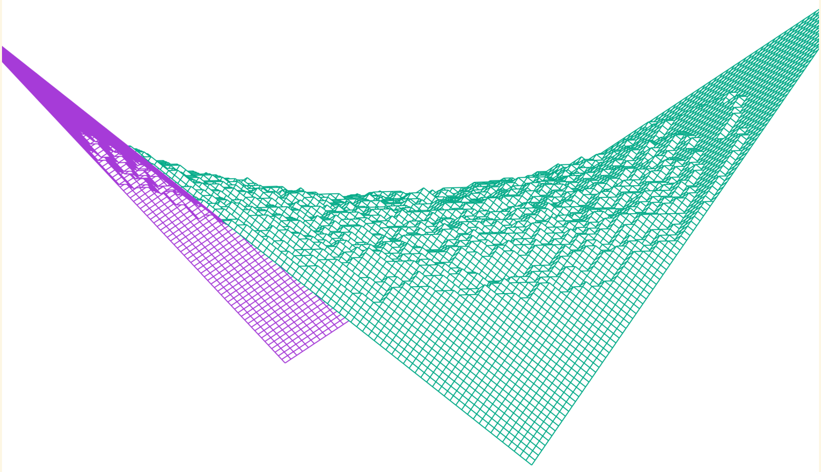
Limit shape as $L \to \infty$



Rhombus tiling of a hexagon: height function



Asymptotic height function



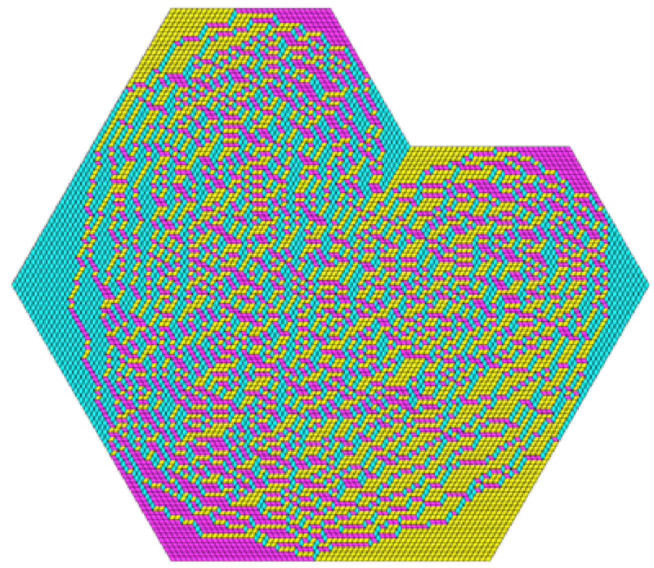
Steps of the proof

- Compute the free energy at a given slope
- Solve a variational problem to get the typical height

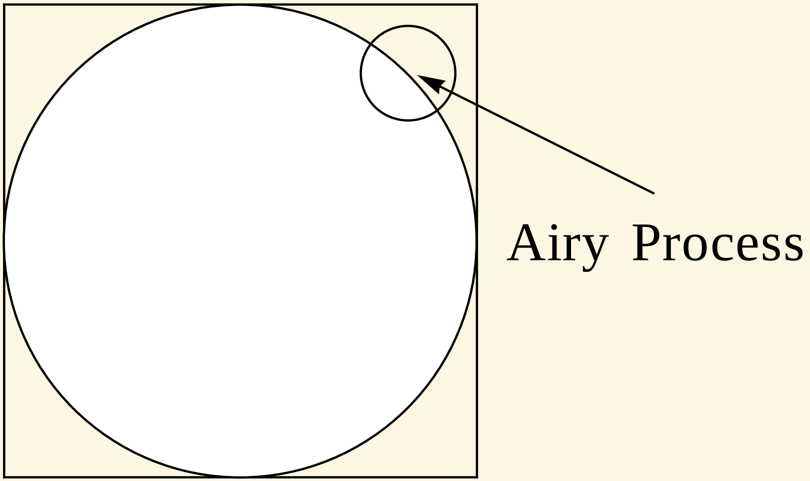
Reference: Kenyon–Okounkov–Sheffield, *Dimers and Amoeba*

The proof is very general: on any periodic lattice, with periodic weights, one gets algebraic curves separating phases. For uniform tilings on the Aztec diamond, it is a circle.

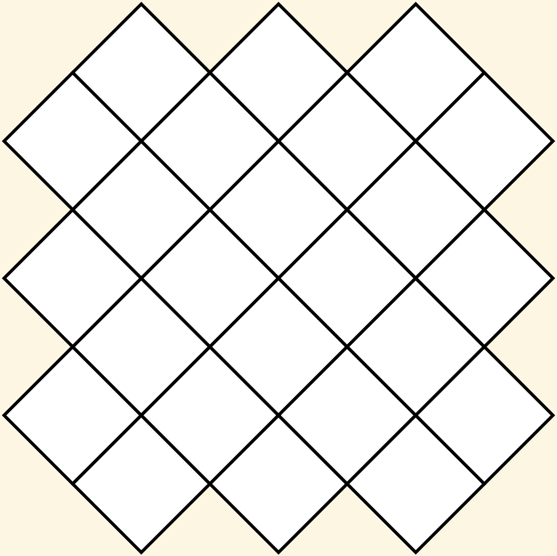
Rhombus tiling of a polygonal region



Fluctuations around the asymptotic shape



Dominos into Dimers



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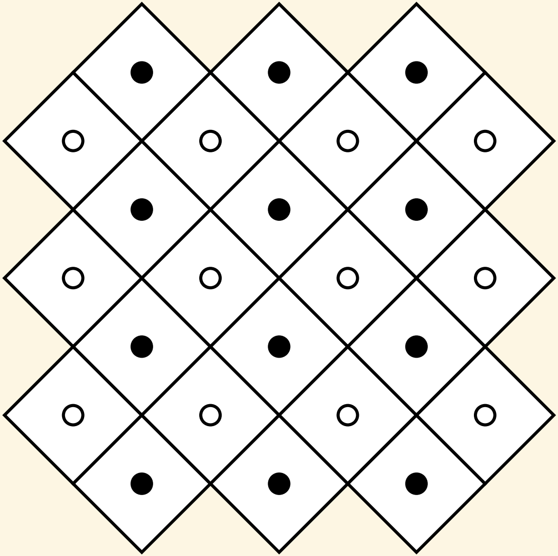

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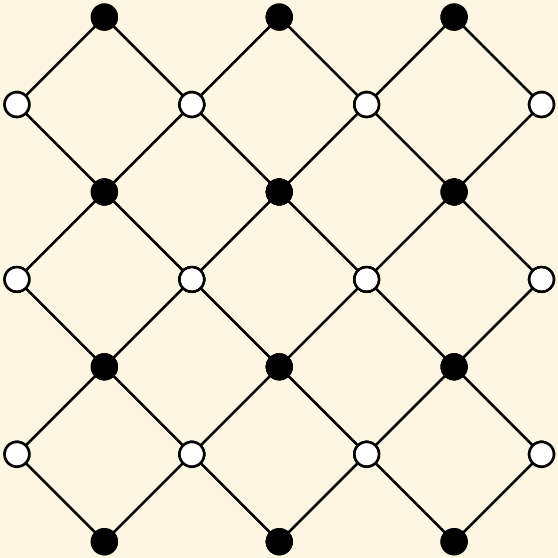

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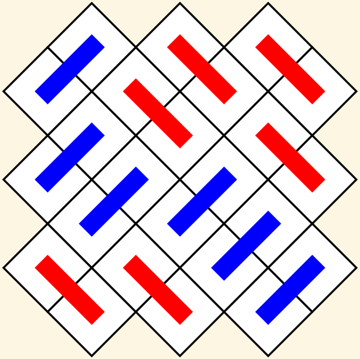
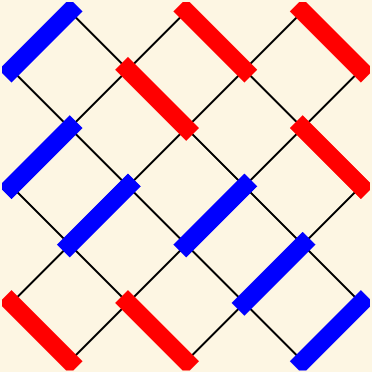
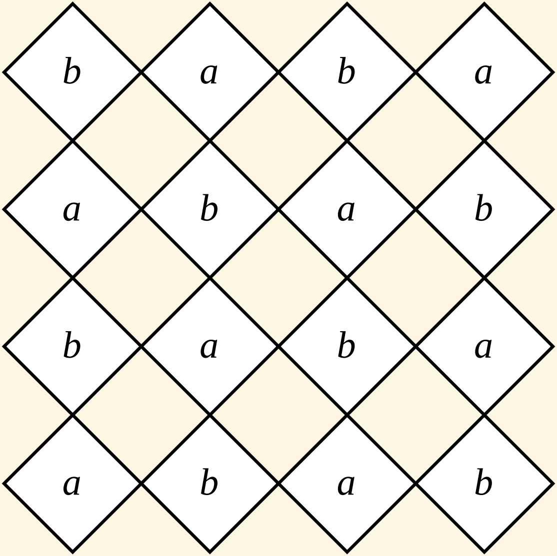
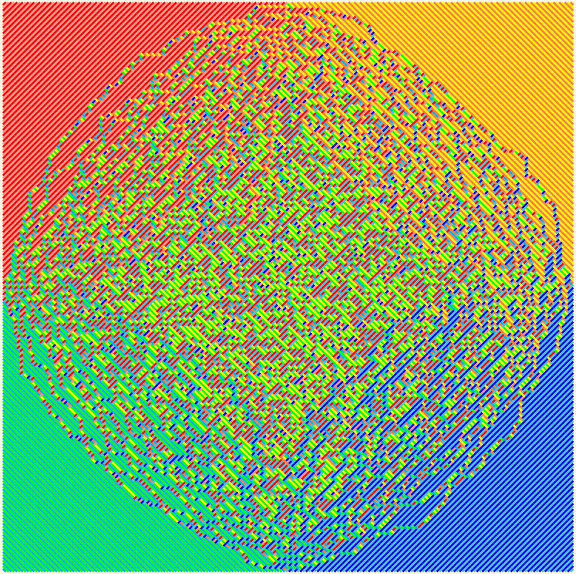




Two-periodic weights



Two-periodic weights ($a = 1/2$)



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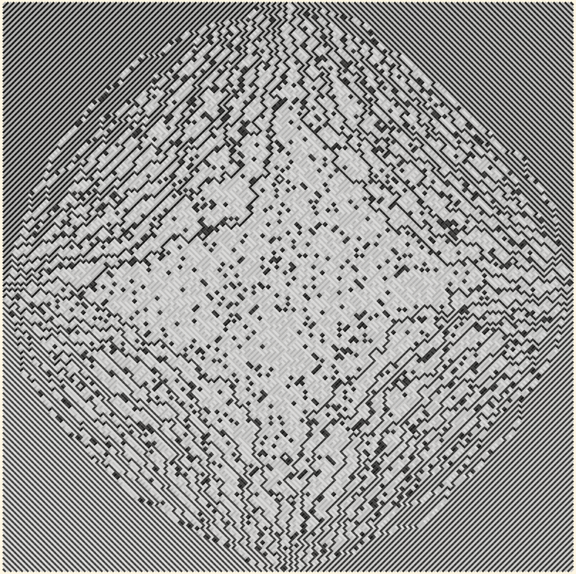
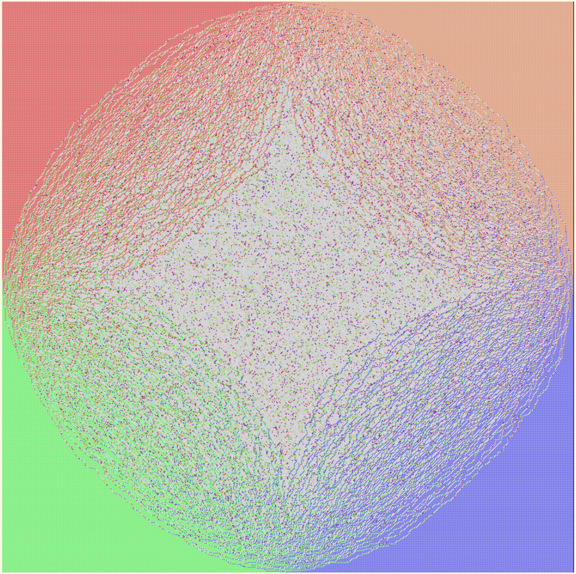
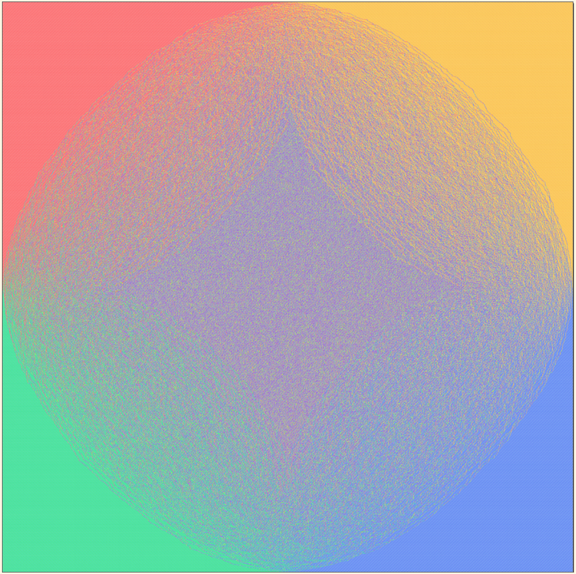


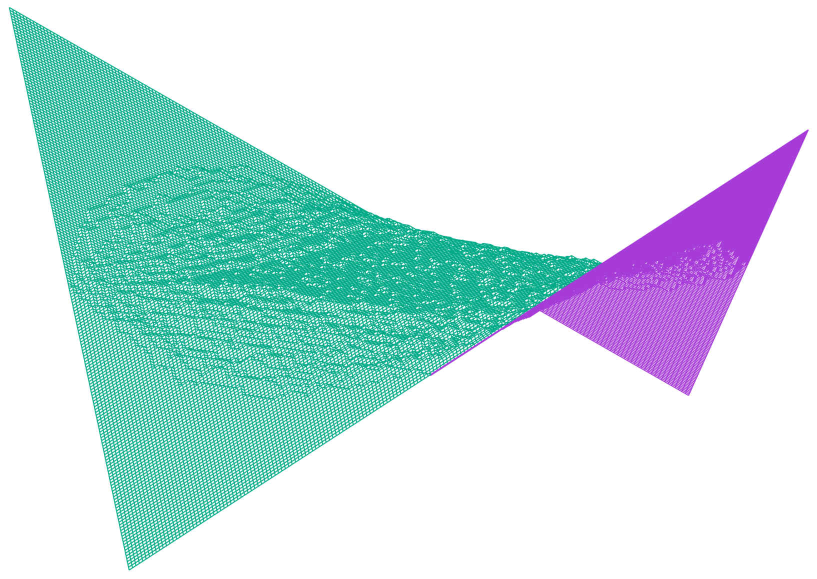
Larger Aztec diamond (still $a = 1/2$)



Even larger Aztec diamond (still $a = 1/2$)



Asymptotic height function



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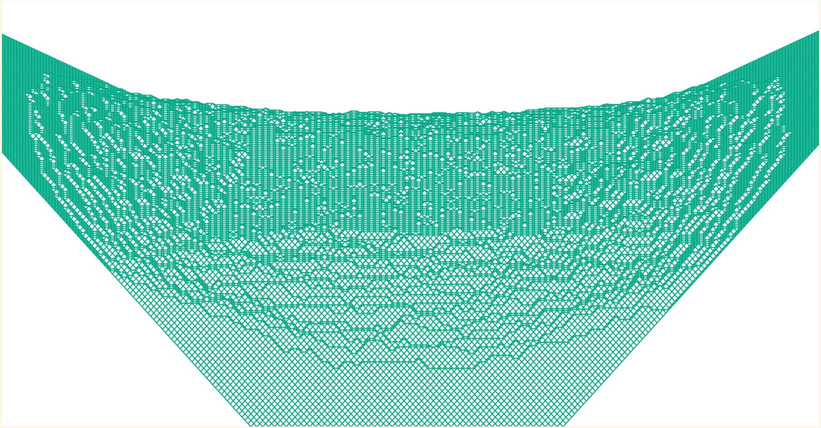

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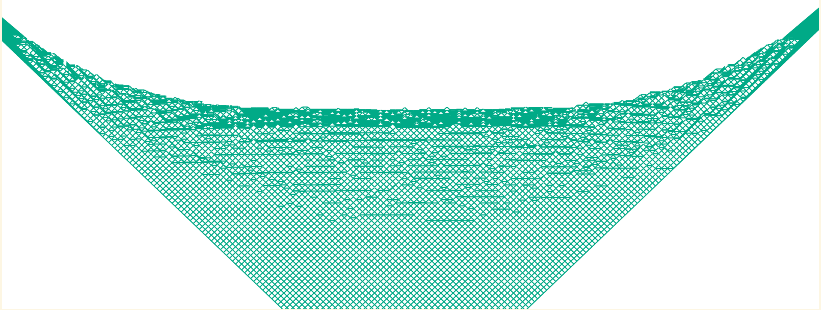
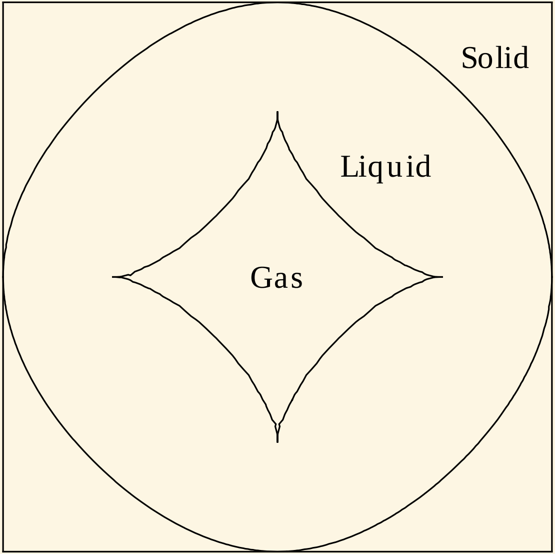


"Phase diagram"



Some results about the model

There are 3 phases, with distinctive local behavior:

- Solid which is ordered, with maximal height slope;
- Liquid with polynomial decay of correlations and nonzero slope;
- Gas with exponential decay and flat height function.

The interface is an algebraic curve of degree 8.

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Main question: what happens at the Liquid/Gas interface?

Claim: the scaling limit is the same there as at the Solid/Liquid boundary

Peierls estimate in (and existence of) the gas phase

One can use a percolation-style argument to show the following bound on the size of loops in the model:

Theorem

For small enough a, there exists a constant $C(a) < \infty$ such that, with probability going to 1 as $L \to \infty$, every connected component of dimers of type a not touching the domain boundary has diameter less than $C(a) \log L$.

Proof: path-counting argument.

This implies exponential decay of correlations in the gas.

What exactly has a scaling limit?

The main problem in the statement of the scaling limit theorem at the Liquid/Gas interface, is to define the discrete object that has a limit. In fact there are 3 natural candidates:

- The dark part in the pictures, *i.e.* dominos of type a;
- The level lines of the height function;
- The tree lines in Temperley's bijection.

Because of the bulk disorder in the gas phase, these 3 objects are not exactly the same.

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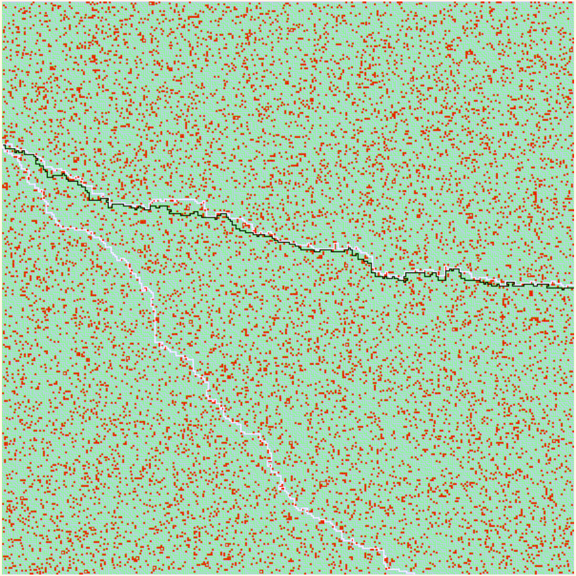
Theorem

After suitable smoothening, the level lines of the height function at the liquid-gas boundary converge in the scaling limit to the Airy process.

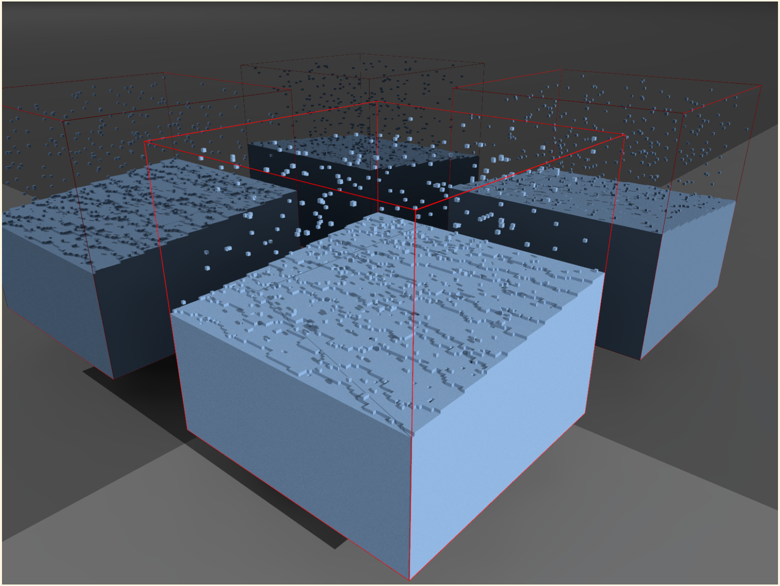
Local description of the interface



Steps of the construction

- Temperley's bijection between domino/dimer configurations and pairs of spanning trees (or forests)
- From the two-periodic model, this leads to a weighted measure on spanning trees with appropriate conditioning
- This tree measure can be realized using (a biased version of) Wilson's algorithm
- Conditioning on being on an interface line translates into conditioning in the Wilson construction in a simple way
- End-result: each interface is a loop-erased biased random walk
- From there: renewal structure and Airy process convergence

Open question: link with the 3d Ising model?



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