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Schramm-Loewner Evolution and other Conformally Invariant Objects

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Contents

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FOREWORD

These notes are not meant as a reference manual, but rather as an introduction combined with a kind of "user's guide" to the existing bibliography. I plan to keep them mostly self-contained in the sense that the reader will need no additional information to understand the majority of the statements; but they contain essentially no detailed proofs. In the case of very important results, I give indications about the main ideas of the demonstration, but of course that is hardly sufficient to a motivated student.

In each part, the most important section is therefore the extended bibliography at the end. I chose to gather all bibliographical references there, and to omit them from the main body of the text (in particular, the main results are only attributed to their respective authors, not to a particular publication). The point is to make reading through the text more natural; maybe it failed!

The notes were started while I was giving a graduate course in Lyon during the spring preceding the school. As a result, they cover a certain quantity of material in addition to what will was discussed in Buzios (mostly the parts about random-cluster models and convergence to SLE, which correspond more to Smirnov's course). These can constitute indications towards further reading, or can be ignored completely in a first reading.

For reference, here is a rough outline of the course schedule in Buzios; the contents of the exercise sessions matched these. However, to avoid too much overlap between these notes and the others from the school, and to make them more focused, some of the material is not included here (for instance, the exercise sheet about Brownian intersection exponents was left out).

- Course 1: Percolation and Cardy's formula.
- Course 2: Loop-erased random walks and uniform spanning trees.
- Course 3: Loewner chains in the radial case.
- Course 4: Chordal Loewner chains, and definition of SLE.
- Course 5: First properties of SLE.
- Course 6: The locality property and SLE_6.
- Course 7: The restriction property, $\mathrm{SLE}_{8/3}$ and restriction measures.
- Course 8: More exotic objects: CLE, loop soups, Gaussian fields...

Part I . A FEW DISCRETE MODELS

Introduction

The goal of these lectures is to provide a self-contained introduction to SLE and related objects, but some motivation is needed before introducing SLE as such; so it seems natural to start with a quick review of a few two-dimensional discrete models.

The focus of this part will be, for each model, to arrive at the question of scaling limits as quickly as possible, and to justify conformal invariance where it is known to hold in the limit. The proofs of actual convergence to SLE will of course have to be postponed (see Part III) — but providing the key arguments is our main objective here.

I.1. Lattice models

We start with what we want to call *lattice models* — even though that might not exactly be the usual sense of that word. Essentially, given a (two-dimensional) lattice embedded in the plane, a *configuration* is a map from the set of vertices and/or edges of the lattice into a finite alphabet, and a probability measure on the set of configurations is constructed by taking a thermodynamical limit from measures in finite boxes derived from a Hamiltonian.

We choose to limit ourselves to a few representative models, namely percolation, the Ising and Potts models, and the random-cluster model. The uniform spanning tree (UST) is an important case because it was one of the first two-dimensional models for which convergence to SLE was proved; we will briefly come back to it in the next section in association with the loop-erased random-walk.

Besides, we will mostly be interested in models taken at their critical point, and defined on specific lattices for which more is understood about their asymptotic behavior (*e.g.*, we limit our description of percolation to the case of site percolation on the triangular lattice) — even though of course a lot is known in a more general setting.

I.1.1. Percolation. The simplest lattice model to describe is *Bernoulli percolation*. Let $p \in (0,1)$ be a parameter; for each vertex of the triangular lattice $\mathcal{T}$, toss a coin and declare it to be *open* (resp. *closed*) with probability p (resp. $1-p$), independently of the others. Denote by P_p the corresponding probability measure on the set of configurations (it is simply a product measure). One can see a configuration as a random subgraph of the underlying lattice, obtained by keeping the open vertices and all the edges connecting two open vertices.

I.1.1.1. *Basic features of the model.* The question of interest is that of the connectivity structure of this subgraph. Let

$$\theta(p) := P_p\left[0 \leftrightarrow \infty\right]$$

be the probability that the origin belongs to an infinite connected component, or *cluster* (*i.e.*, that it is "connected to infinity"). It is easy to show that the function θ is non-decreasing, and using a simple counting argument (known as a *Peierls* *argument*), that for p small enough, $\theta(p)$ is equal to 0 and $\theta(1-p)$ is positive; in other words, defining

$$p_c := \inf\{p : \theta(p) > 0\} = \sup\{p : \theta(p) = 0\},$$

one has $0 < p_c < 1$. The value p_c is called the *critical point* of the model. Its value depends on the choice of the underlying lattice; in the case of the triangular lattice, by duality arguments it is equal to $1/2$.

The behavior of the system changes drastically across the critical point:

- If $p < p_c$, then almost surely all connected components are finite; moreover, they have finite expected volume, and the connection probabilities exhibit exponential decay: There exists $L(p) < \infty$ such that, for every x, $y \in \mathbb{Z}^2$,
$$P_p[x \leftrightarrow y] \leqslant C\, e^{-\|y-x\|/L(p)};$$
- If $p > p_c$, then almost surely there exists a *unique* infinite cluster and it has asymptotic density $\theta(p)$; but exponential decay still occurs for connectivity through finite clusters: There exists $L(p) < \infty$ such that for all x, $y \in \mathbb{Z}^2$,
$$P_p[x \leftrightarrow y; x \not\leftrightarrow \infty] \leqslant C\, e^{-\|y-x\|/L(p)};$$
- If $p = p_c$, there is no infinite cluster (*i.e.*, $\theta(p_c) = 0$) yet there is no finite characteristic length in the system; the two-point function has a power-law behavior, in the sense that for some $c > 0$ and for all x, $y \in \mathbb{Z}^2$,
$$c\,\|y-x\|^{-1/c} \leqslant P_p[x \leftrightarrow y] \leqslant c^{-1}\,\|y-x\|^{-c}.$$

FIGURE 1. Critical site-percolation on a rectangular region of the triangular lattice (the state of a vertex is represented by the color of the corresponding face of the dual lattice).

The last statement is an instance of what is known as *Russo-Seymour-Welsh theory*, or *RSW* for short: essentially, the largest cluster within a large box of a given size has a diameter of the same order as the size of the box, and it crosses it horizontally with a positive probability, uniformly in the actual size of the box.

To be more specific, if $\mathcal{R}$ is a rectangle aligned with the axes of the lattice, denote by $LR(\mathcal{R})$ the probability that, within (the intersection between $\mathbb{Z}^2$ and) $\mathcal{R}$, there is a path of open edges connecting its two vertical sides. Then, RSW states that for every $\lambda > 0$, there exists $\eta(\lambda) \in (0,1)$ such that, for every n large enough,

$$\eta(\lambda) \leqslant P_p\left[LR([0,\lambda n] \times [0,n])\right] \leqslant 1 - \eta(\lambda). \tag{1}$$

This can easily be used as a black box, but figuring out the proof is a good way to get intuition on the model, so we include it here in the form of an exercise.

Exercise I.1.1 (Proof of the RSW bounds).

(1) *Since we are working in the triangular lattice, it makes more sense to first prove* (1) *for parallelograms aligned with the lattice; it is easy to see why this is sufficient. In this whole exercise, we will thus use two lattice directions as coordinate axes, so that for instance what is denoted as* $[0,n]^2$ *is in fact a rhombus. What is the probability that there exists a horizontal crossing of* $[0,n]^2$*?*

(2) *Assume* $[0,n]^2$ *is crossed from left to right and set* Γ *to be the lowest horizontal crossing. Let* γ *be a deterministic path from left to right, prove that* $\{\Gamma = \gamma\}$ *is measurable with respect to the* σ*-algebra spanned by the sites* ***below*** γ *and the sites of* γ*. When conditioning on* $\{\Gamma = \gamma\}$*, what can be said about the law of sites above* γ*?*

(3) *Consider the shape in the following figure and assume that the left rectangle* $[0,n]^2$ *is crossed horizontally. Can you bound from below the probability that the lowest crossing* Γ *is connected to the bold part by a black path?* Hint: condition on $\{\Gamma = \gamma\}$ and consider the reflected path $\sigma(\gamma)$ with respect to the line $y = n + \frac{1}{2}$.

(4) *Deduce that the probability of crossing the rectangle* $[0,2n] \times [0,n]$ *horizontally is bounded away from 0 when* n *goes to infinity.*

(5) *a) Let* $\rho > 1$*. Deduce that the probability to cross the rectangle* $[0,\rho n] \times [0,n]$ *horizontally is (uniformly in* n*) bounded away from* 0*;*

b) Prove that the probability of a black circuit surrounding the origin in the annulus $[-2n,2n]^2 \setminus [-n,n]^2$ *remains bounded away from* 0 *when* n *goes to infinity;*

c) Show that almost surely there is no infinite cluster at $p = \frac{1}{2}$*;*

d) What can be said about $\mathbb{P}(0 \leftrightarrow \partial\Lambda_n)$*?*

e) (difficult) *Explain a strategy to prove that* $p_c = \frac{1}{2}$*.*

A natural question is then the following: Does the crossing probability above actually converge as $n \to \infty$? In fact, that question is still open in the general case, and in particular in the case of bond-percolation on the square lattice $\mathbb{Z}^2$, and it is not quite clear how many new ideas would be needed to prove convergence. But, conjecturally, the limit does exist and does not depend on the choice of the underlying lattice, provided that it has enough symmetry — this is part of what is known as *universality.*

I.1.1.2. *The Cardy-Smirnov formula.* We now turn to the main result which we want to present in this section. In the case of site-percolation on the triangular lattice (which is the one we are considering here), in fact the above probability does converge, and the limit is known explicitly. This was first conjectured by Cardy using mathematically non-rigorous arguments, and later proved by Smirnov.

Before stating the main theorem, we need some additional notation. Let Ω be a smooth, simply connected, bounded domain in the complex plane, and let a, b, c and d be four points on $\partial\Omega$, in this order if the boundary is oriented counterclockwise. Let $\delta > 0$, and consider the triangular lattice scaled by a factor of δ, which we will denote by $\mathcal{H}_\delta$; let $\mathcal{P}_\delta(\Omega, a, b, c, d)$ be the probability that, within percolation on $\Omega \cap \mathcal{H}_\delta$, there is an open path connecting the arc ab to the arc cd of the boundary.[1]

THEOREM I.1.1 (Cardy, Smirnov). *There exists a function f defined on the collection of all 5-tuples formed of a simply connected domain with four marked boundary points, satisfying the following:*

(1) *As $\delta \to 0$, $\mathcal{P}_\delta(\Omega, a, b, c, d)$ converges to $f(\Omega, a, b, c, d)$;*
(2) *f is* conformally invariant*, in the following sense: If Ω and Ω' are two simply connected domains and if Φ maps Ω conformally to Ω', then*
$$f(\Omega, a, b, c, d) = f(\Omega', \Phi(a), \Phi(b), \Phi(c), \Phi(d));$$
(3) *If $\mathcal{T}$ is an equilateral triangle, and if a, b and c are its vertices, then*
$$f(\mathcal{T}, a, b, c, d) = \frac{|cd|}{|ab|}$$
(which, together with the conformal invariance, characterizes f uniquely).

A complete proof of this theorem can be found in several places, so it does not make much sense to produce yet another one here; instead, we briefly describe the main steps of Smirnov's general strategy in some detail. The same overall approach (though obviously with a few modifications) will be applied to other models below; the main point each time will be to find the correct *observable*, *i.e.* a quantity derived from the discrete model and which is computable enough that its asymptotic behavior can be obtained (and is non-trivial).

Step 1: Definition of the observable. Let Ω be as above, and let z be a vertex of the dual lattice $\mathcal{H}_\delta^*$ (or equivalently, a face of the triangular lattice); denote by $E_\delta^a(z)$ the event that there is a simple path of open vertices joining two points on the boundary of Ω and separating a and z on one side and b and c on the other side, and by $H_\delta^a(z)$ the probability of $E_\delta^a(z)$. Define H_δ^b and H_δ^c accordingly. Notice that if we choose $z = d$, we get exactly the crossing probability:
$$\mathcal{P}_\delta(\Omega, a, b, c, d) = H_\delta^a(d).$$
In fact, we will compute the limit of H_δ^a as $\delta \to 0$ in the whole domain; the existence of f will follow directly.

Step 2: Tightness of the observable. Let z and z' be two points within the domain, and let $\mathcal{A}$ be an annulus contained in Ω and surrounding both z and z'. If $\mathcal{A}$ contains an open circuit, then either both of the events $E_\delta^a(z)$ and $E_\delta^a(z')$ occur, or neither of them does. The existence of such circuits in disjoint annuli are independent events, and if one fixes the modulus of the annuli, their probability is bounded below by RSW estimates (1). Besides, the number of such disjoint annuli

[1] Technically, this definition would require constructing a discrete approximation of the domain; we choose to skip over such considerations here, and refer the avid reader to the literature for more detail.

which can be fit around $\{z, z'\}$ is of order $-\log|z'-z|$. This implies a bound of the form

$$|H^a_\delta(z') - H^a_\delta(z)| \leqslant C\,|z'-z|^c$$

for some C, $c > 0$ depending only on the domain Ω, but not on δ. In other words, the functions H^a_δ are uniformly Hölder with the same exponent and the same norm; and this implies, by the Arzelà-Ascoli theorem, that they form a relatively compact family of continuous maps from Ω to $[0,1]$. In particular, one can always choose a sequence (δ_k) going to 0 along which $H^a_{\delta_k}$ (as well as $H^b_{\delta_k}$ and $H^c_{\delta_k}$) converges to some continuous function h^a (resp. h^b, h^c) defined on $\overline{\Omega}$. Proving convergence of (H^a_δ) then amounts to proving the uniqueness of such a sub-sequential limit, *i.e.*, all that remains to be done is to identify the function h^a.

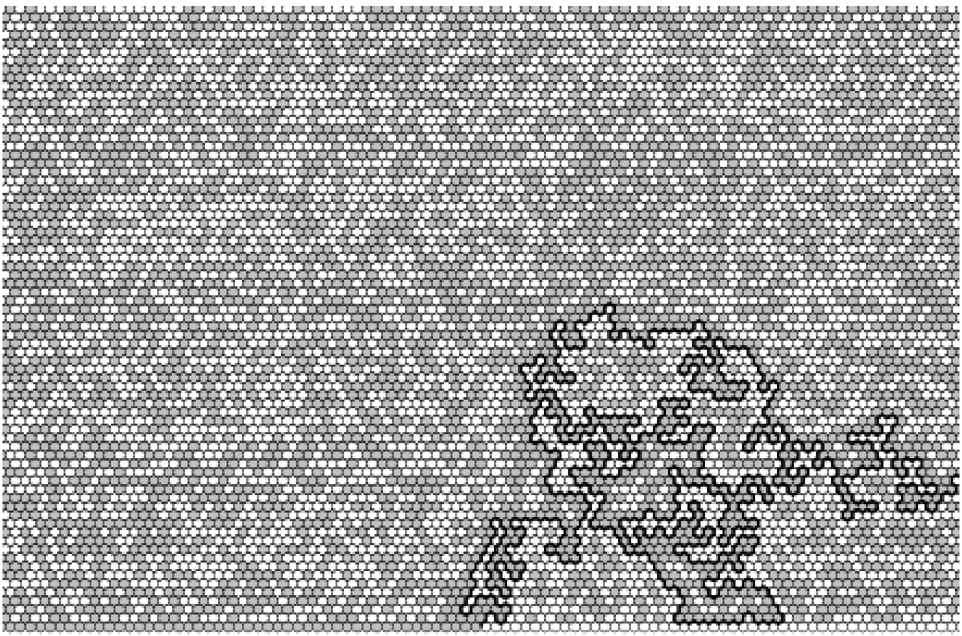

FIGURE 2. The exploration of a critical percolation interface in the upper-half plane.

Step 3: The exploration process. One key tool which we will need in what follows is an algorithmic way to measure features of percolation interfaces. In any simply connected planar domain different from the whole plane, split the boundary into two subsets, and assign "artificial" boundary conditions to be open on the first one and closed on the second one. From each contact point between the two, this creates an *interface* between open and closed vertices of the lattice, and one can follow it by looking at each site one after the other along the interface, turning left or right according to its color. This is easier drawn than formally described; see Figure 2. The outcome of the construction is a lattice path, known as the *exploration process* of the interface. We will use it in the next step, and again when we speak about the scaling limit of geometric objects (the percolation exploration process, as the lattice mesh goes to 0, converges to the trace of SLE_6).

Step 4: Local behavior of the observable. This is essentially the only place in the proof where one uses the fact that the underlying model is site-percolation on a triangulation. Let z be a vertex of $\mathcal{H}_\delta^*$, and let z_1, z_2 and z_3 denote its three neighbors, ordered counterclockwise; define $P_\delta^a(z,z_1)$ to be the probability that $E_\delta^a(z_1)$ occurs, but $E_\delta^a(z)$ does not. This is equivalent to the existence of three disjoint paths in $\mathcal{H}_\delta$, each joining one of the vertices of the triangle around z to one of the three boundary arcs delimited by a, b and c, and of appropriate states (two open, one closed — draw a picture!). The core of Smirnov's proof is then a wonderful relation between these quantities; namely:

$$P_\delta^a(z,z_1) = P_\delta^b(z,z_2) = P_\delta^c(z,z_3).$$

The argument is very simple, but not easy to write down formally; it goes as follows: Assuming the existence of three arms as above, it is possible to discover two of them by exploring the percolation configuration starting from c (say), and always staying on the interface between open and closed vertices. The exploration path reaches z if and only if two of the above arms exist; besides, it gives us no information about the state of the vertices which are not along it, because the underlying measure is a product measure. The key remark is then the *color-swapping argument*: changing the state of each of the vertices in the unexplored portion of Ω does not change the probability of the configuration (because we work at $p = p_c = 1/2$); but it does change the state of the third arm from open to closed. Swapping the colors of all the vertices in Ω (which still does not change probabilities) one then arrives at a configuration with three arms of the appropriate colors, but where the role of a (resp. z_1) is now taken by b (resp. z_2).

Step 5: Holomorphicity in the scaling limit. Now, we need to exhibit a holomorphic function built out of h^a, h^b and h^c; following the symmetry of order 3 in the setup, it is natural to define

$$H_\delta(z) := H_\delta^a(z) + \tau H_\delta^b(z) + \tau^2 H_\delta^c(z)$$

and $h := h^a + \tau h^b + \tau^2 h^c$ accordingly, where $\tau = e^{2\pi i/3}$. To prove that h is holomorphic, it is enough to show that, along every smooth curve γ contained in Ω, one has

$$\oint_\gamma h(z)\,\mathrm{d}z = 0$$

(by Morera's theorem); and to show that, it is enough to pick a sequence of suitable discretizations of γ and estimate the integral using H_δ, and to show that the discrete estimate vanishes as δ goes to 0. It is always possible to approach γ by a discrete path $\gamma_\delta = (z_0^\delta, z_1^\delta, \dots, z_{L_\delta}^\delta = z_0^\delta)$ on $\mathcal{H}_\delta^*$ in such a way that $L_\delta = \mathcal{O}\big(\delta^{-1}\big)$, and one then has

$$\oint_\gamma h(z)\mathrm{d}z = \sum_{j=0}^{L_\delta - 1} \frac{H_\delta(z_j^\delta) + H_\delta(z_{j+1}^\delta)}{2} (z_{j+1}^\delta - z_j^\delta) + \mathcal{O}(\delta^c)$$

with $c > 0$ by the previous tightness estimate. One can then apply a discrete analog of Green's formula to make discrete derivatives of H_δ appear, and write these in terms of P_δ^a, P_δ^b and P_δ^c: after elementary calculus, one gets

$$\oint_\gamma h(z)\mathrm{d}z = \frac{i\delta\sqrt{3}}{2} \sum_{z\sim z'} \left[P_\delta^a(z,z') + \tau P_\delta^b(z,z') + \tau^2 P_\delta^c(z,z')\right](z'-z) + \mathcal{O}(\delta^c),$$

where the sum extends to all pairs of nearest neighbors in the interior of γ. Applying Smirnov's identity to write everything in terms of P_δ^a only then leads to

$$\oint_\gamma h(z)\mathrm{d}z = \frac{i\sqrt{3}}{2}\sum_{z\sim z'}\left[P_\delta^a(z,z')\sum_{j=0}^{2}\tau^j(z_j - z)\right] + \mathcal{O}(\delta^c)$$

(where the z_j are the neighbors of z, numbered counterclockwise in such a way that $z_0 = z'$). It is then easy to see that the inner sum is identically equal to 0 (because it is always proportional to $1+\tau^2+\tau^4$).

Step 6: Boundary conditions and identification. The same computation as above can be performed starting with $S_\delta := H_\delta^a + H_\delta^b + H_\delta^c$, and the conclusion is the same: The (sub-sequential) limit $s := h^a + h^b + h^c$ is holomorphic as well. But because it is real-valued, this leads to the conclusion that it is constant, equal to 1 by looking at the point $z = a$. This means that the triple (h^a, h^b, h^c) can be seen as the barycentric coordinates of $h(z)$ relative to the points 1, τ and τ^2, respectively, meaning that h maps Ω to the interior of the corresponding equilateral triangle $\mathcal{T}$. Since it sends boundary to boundary in a one-to-one way (the variations of h^a on the boundary are easy to determine), it has to be conformal, and so it has to be the unique conformal map from Ω to $\mathcal{T}$ mapping a (resp. b, c) to 1 (resp. τ, τ^2). Because the sub-sequential limit is thus identified uniquely, one obtains convergence of (H_δ) itself to h, and it is not difficult to conclude the proof.

This concludes the few features of percolation which we will need in the following parts; we will come back to it (and say a little bit more about the exploration process) in Part III. For now, the relevant piece of information to remember is that, at criticality, the scaling limit of percolation (in any reasonable sense) is non-trivial and exhibits conformal invariance.

I.1.2. The random-cluster model. Percolation is very easy to describe, because the states of the vertices are independent of each other; but it is not very physically realistic. We now focus our attention on the *random-cluster model* (sometimes also referred to as *FK-percolation* or simply the FK model, for the names of its inventors, Fortuin and Kasteleyn). It is a dependent variant of *bond* percolation. We choose to keep this section shorter than it could be; the interested reader will find all the details in the notes for Smirnov's course at the same summer school [**38**].

I.1.2.1. *Definitions and first properties.* Let $G = (V, E)$ be a finite graph, and let $q \in [1, +\infty)$ and $p \in (0,1)$ be two parameters. The random-cluster measure on G is defined on the set of subgraphs of G, seen as subsets of E, by

$$P_{p,q,G}[\{\omega\}] := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}}{Z_{p,q,G}},$$

where $o(\omega)$ is the number of open edges in ω, $c(\omega)$ the number of closed edges, and $k(\omega)$ the number of connected components of the subgraph (counting isolated vertices). The *partition function* $Z_{p,q,G}$ is chosen so as to make the measure a probability measure. Notice that the case $q = 1$ is exactly that of a product measure, in other words it is Bernoulli bond-percolation on G.

The definition in the case of an infinite graph needs a little more care — of course defining the measure as above makes little sense since the terms $o(\omega)$, $c(\omega)$ and $k(\omega)$ would typically be infinite (as well as the partition function). The first step is to define *boundary conditions*, which in this case amounts to introducing additional edges whose state is fixed (either open or closed); if ξ denotes such a choice, then $P^{\xi}_{p,q,G}$ denotes the corresponding measure. Notice that the only effect ξ has is in the counting of connected components within G.

Now consider the square lattice, and a sequence of increasing boxes $\Lambda_n := [-n,n]^2$. We will consider two types of boundary conditions for the random-cluster model on Λ_n: *free* (*i.e.*, ξ is empty) and *wired* (*i.e.*, all the vertices on the boundary of Λ_n are assumed to be connected). We denote these boundary conditions by f and w, respectively. A third boundary condition is known as the *Dobrushin* boundary condition, and consists in wiring the vertices of one boundary arc of the box (with prescribed endpoints) together while leaving the rest of the boundary free.

If $q \geqslant 1$, the model exhibits positive correlations (in the form of the FKG inequality). This implies that, if $n < N$, the restriction of the wired (resp. free) measure on Λ_N to Λ_n is stochastically smaller (resp. larger) than the corresponding measure defined on Λ_n directly. As n goes to infinity, this allows for the definition of infinite-volume measures as monotonic limits of both sequences, which we will denote by $P^w_{p,q}$ and $P^f_{p,q}$.

For fixed q and either free or wired boundary conditions, these two measure families are stochastically ordered in p; this implies the existence of a critical point $p_c(q)$ (the same in both cases, as it turns out) such that, as in the case of Bernoulli percolation, there is a.s. no infinite cluster (resp. a unique infinite cluster) if $p < p_c$ (resp. $p > p_c$).

It has long been conjectured, and was recently proved [**5**], that for every $q \geqslant 1$,

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{2}$$

This comes from the following duality construction. Let for now G be a graph embedded into the 2-sphere, and let G^* be its dual graph. To any configuration ω of the random-cluster model on G, one can associate a configuration ω^* on G^* by declaring a dual bond to be open if and only if the corresponding primal bond is closed (see figure 3). As it turns out, if ω is distributed as $P_{p,q,G}$, then ω^* is distributed as P_{p^*,q^*,G^*}, *i.e.* it is a random-cluster model configuration, with

$$q^* = q \quad \text{and} \quad \frac{p^*}{1-p^*} = q\frac{1-p}{p}.$$

It is easy to see that there is a unique value p_{sd} of p satisfying $p_{sd} = (p_{sd})^*$, and it is then natural to expect that $p_c = p_{sd}$, leading to the value above.

Exercise I.1.2. *Prove the duality statement.*

Answer: Use Euler's formula to relate the number of open and closed bonds in the primal and the dual configurations with their numbers of faces and clusters. It helps to rewrite the weight of a configuration as

$$\left(\frac{p}{1-p}\right)^{o(\omega)} q^{k(\omega)}$$

and to notice that $o(\omega) + o(\omega^*)$ does not depend on the configuration.

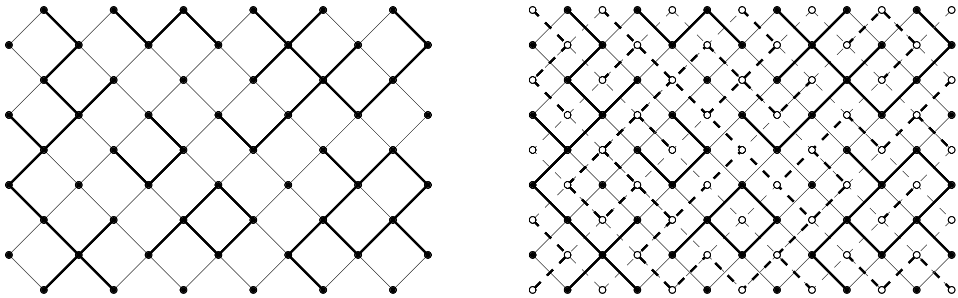

FIGURE 3. Duality between random-cluster configurations.

I.1.2.2. *The para-fermionic observable.* As before, we want to introduce an observable defined from the discrete model and giving us enough information to determine asymptotic properties in the scaling limit. From now on, let Ω be a smooth, simply connected domain in the complex plane, with two marked points a and b on its boundary. Let G_δ be the graph obtained as the intersection of Ω with $\delta\mathbb{Z}^2$ for some mesh $\delta > 0$; we will consider the random-cluster model with parameters $q \geqslant 1$ and $p = p_{sd}(q)$ on G_δ, with Dobrushin boundary conditions, wired on the (positively oriented) arc ab — and we will denote the corresponding measure simply by P (or later by P_δ when we insist on the scaling behavior as $\delta \to 0$).

As in the case of percolation, we briefly describe the main steps in Smirnov's proof of conformal invariance. A big difference is that the statement of convergence needs more notation, so we will have to postpone it a little bit.

Step 1: The loop representation. In addition to the graph G_δ and its dual G_δ^*, we need a third one known as the *medial graph* and denoted by $G_\delta^\diamond$; it is defined as follows. The vertices of the medial graph are in bijection with the bonds of either G_δ or G_δ^* (which are in bijection), and one can think of them as being at the intersection of each primal bond with its dual; there is an edge between two vertices of $G_\delta^\diamond$ if and only if the two corresponding primal edges share an endpoint and the two corresponding dual edges do as well.

One can encode a random-cluster configuration on G_δ using the medial graph, by following the boundary of each of its clusters (or equivalently, each of the clusters of the dual configuration). This leads to a covering of all the bonds of $G_\delta^\diamond$ by a family of edge-disjoint paths, one joining a to b (which we will call the *interface* and denote by γ), and the others being loops. If $l(\omega)$ denotes the number of loops obtained this way, it is possible to rewrite the probability of a configuration ω as

$$P[\{\omega\}] = \frac{x^{o(\omega)}(\sqrt{q})^{l(\omega)}}{Z_{x,q}} \quad \text{with} \quad x := \frac{p}{(1-p)\sqrt{q}}.$$

Since we work at the self-dual point, in fact we have $x = 1$ and the weight of a configuration is written as a function of only the number of loops in its loop representation.

EXERCISE I.1.3. *Prove the equivalence of the random-cluster representation and the loop representation.*

Answer: It works exactly the same way as the previous exercise, use Euler's formula in the natural way and it will work.

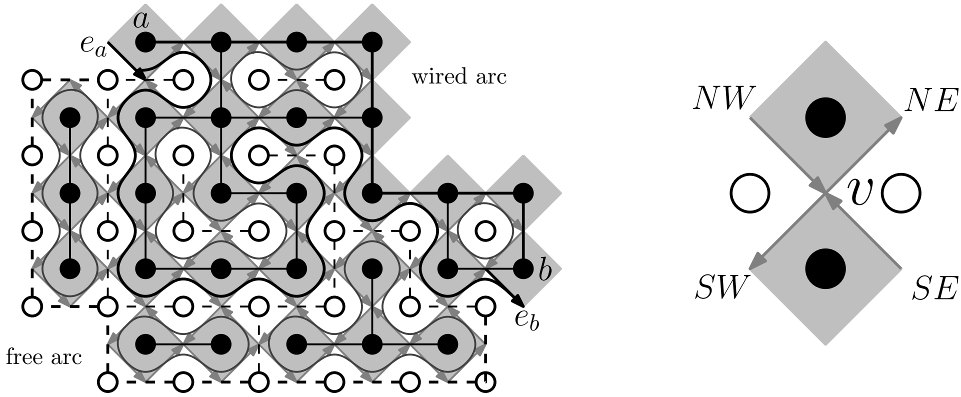

FIGURE 4. The loop representation of a random-cluster configuration, and the possible orientations of the interface γ around a vertex.

Step 2: Definition of the para-fermionic observable. Let e be an edge of the medial graph. We would like to be able to compute the probability that the interface passes through e; unfortunately, this seems to be out of reach of current methods, so we need an alternative. Assume that γ does go through e. Then, it is possible to follow it from a to the midpoint of e, and to follow the variation of the angle of the tangent vector along the way: it increases (resp. decreases) by $\pi/2$ whenever γ turns left (resp. right). The *winding* of the curve at e is the value one gets when reaching e; it is in $(\pi/2)\mathbb{Z}$ and we denote it by $W(e)$. If γ does not pass through e, define $W(e)$ to be an arbitrary value, as it will not be relevant.

The para-fermionic observable is then defined as

$$F_\delta(e) := E\left[e^{-i\sigma W(e)}\mathbb{1}_{e\in\gamma}\right] \quad \text{where } \sigma \text{ satisfies} \quad \sin\frac{\sigma\pi}{2} = \frac{\sqrt{q}}{2}.$$

Notice the difference here between the cases $q \leqslant 4$ (when σ is real) and $q > 4$ (when it is pure imaginary); we will come back to this distinction shortly. The parameter σ is known as the *spin* of the model. Morally, the main convergence result is that, at the self-dual point, $\delta^{-\sigma}F_\delta$ converges (to an explicit limit) as $\delta \to 0$; but giving a precise sense to that statement requires a little more preparation.

Step 3: Local behavior of the observable. Let ω be a configuration, and let e again be an edge of the medial lattice. We will denote by $F_\delta(e,\omega)$ the contribution of ω to the observable, so that $F_\delta(e) = \sum F_\delta(e,\omega)$. Besides, let ℓ be a bond of the primal lattice which is incident to e. There is a natural involution on the set of configurations given by changing the state of the bond ℓ without changing anything else — denote this involution by s_ℓ. It is easy to see how $F_\delta(e,\omega)$ and $F_\delta(e,s_\ell(\omega))$ differ: If both are non-zero, then the winding term is the same and their ratio is therefore either $x\sqrt{q}$ or $x/\sqrt{q}$ according to whether opening ℓ creates or destroys a loop.

Notice that the medial lattice can be oriented in a natural way, by declaring that its faces corresponding to dual vertices are oriented positively. The definitions above ensure that γ always follows the orientation of the bonds it uses. We will use the notation $e \to \ell$ (resp. $\ell \to e$) to mean that the bond e is oriented towards (resp. away from) its intersection with ℓ. The above observations imply that, for

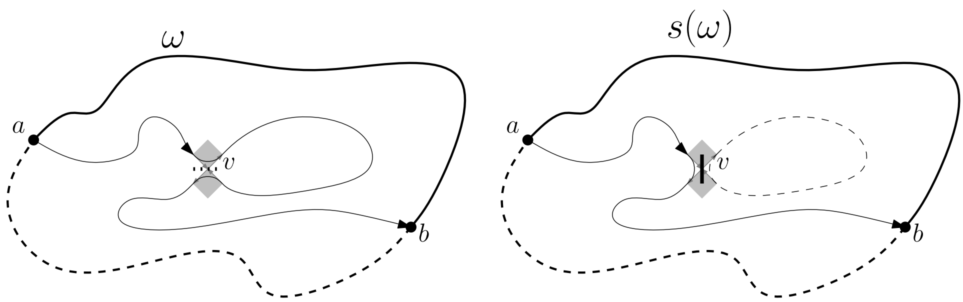

FIGURE 5. One pair of configurations contributing to (3).

every ℓ in the primal lattice and every $x > 0$,

$$\sum_{e \to \ell} \big[F_\delta(e,\omega) + F_\delta(e, s_\ell(\omega))\big] = \frac{e^{\sigma\pi i/2} + x}{1 + x e^{\sigma\pi i/2}} \sum_{e \to \ell} \big[F_\delta(e,\omega) + F_\delta(e, s_\ell(\omega))\big]. \tag{3}$$

The prefactor on the right-hand side is a complex number of modulus 1 if $q \leqslant 4$; it is real and positive if $q > 4$; and in both cases, it is equal to 1 if and only if $x = 1$ — in other words, exactly at the self-dual point. Summing the previous relation over ω, we get

$$\sum_{e \to \ell} F_\delta(e) = \frac{e^{\sigma\pi i/2} + x}{1 + x e^{\sigma\pi i/2}} \sum_{e \to \ell} F_\delta(e)$$

and in particular, at the self-dual point, this relation boils down to the flow condition:

$$\sum_{e \to \ell} F_\delta(e) = \sum_{e \to \ell} F_\delta(e).$$

This is the basis from which all the rest of the proof is built up: we now need to interpret this relation as the vanishing of a divergence, and in turn as the (discrete) holomorphicity of a well-chosen function.

I.1.2.3. *Interlude.* Here we have to stop for a moment. All the preceding reasoning is perfectly general, and the intuition behind it is rather clear. However, a worrisome remark is that we get one linear relation per bond of the primal lattice, but one unknown (the value of F_δ) per bond of the *medial* lattice, of which there are twice as many. That means that these relations cannot possibly characterize F_δ uniquely, and indicates that something more is needed; and in fact, the proof of conformal invariance is indeed not known in all generality.

The first "easy" case is that of $q > 4$. Here the observable is real-valued, and therefore lends itself to more analytic techniques, mostly inequalities. This is quite fruitful if one aims for the value of the critical point, and indeed in that case one can show that $p_c = p_{sd}$ using only elementary calculus. The use of inequalities is not optimal though, as it is not precise enough to derive any information at the critical point — much less to prove conformal invariance.

The second "easy" case is that of $q = 2$, for which $\sigma = 1/2$. Recall that γ always traverses edges in their positive direction; this means that $W(e)$ is known in advance up to a multiple of 2π. In turn, this means that the argument of $e^{i\sigma W(e)}$ is known up to an integer multiple of π; in other words, for every (oriented) edge e, the observable $F_\delta(e)$ takes its value on a line in the complex plane depending only on the direction of e (essentially, viewing e as a complex number, $F_\delta(e)/\sqrt{e}$ is a real number). The previous obstruction can then be bypassed: while we get a (complex) linear relation per primal bond, we only need to determine a *real* unknown per medial bond, which is the same quantity of information. This is morally why a complete derivation of conformal invariance is known only in that particular case, to which we restrict ourselves from now on.

I.1.2.4. *The rest of the proof when $q = 2$ and $x = 1$.*

Step 4: The integrated observable. We would like to prove that $\delta^{-1/2} F_\delta$ converges and be done with it. However, this is not very realistic, in particular because the argument of F_δ oscillates wildly (with a period of one lattice mesh). A solution to that is to consider an integrated version of it, or rather of its square. There is (up to an additive constant) a unique function H_δ, defined on the vertices of both the primal and dual lattices, satisfying the following condition: Whenever W (resp. B) is a vertex of the primal (resp. dual) lattice, and they are adjacent and separated by the bond e_{WB} of the primal lattice, then

$$H_\delta(B) - H_\delta(W) = |F_\delta(e_{WB})|^2 .$$

In some sense, one can think of H_δ as a discrete integral of $|F_\delta|^2$; proving its existence is a matter of checking that the sum of the prescribed increments around a vertex of the medial lattice vanishes; and this in turn is a direct consequence (via the Pythagorean theorem) of the flow condition, noticing that the values of $F_\delta(e)$ for $e \to \ell$ (resp. $\ell \to e$) are always orthogonal. Besides, it is easy to check that H_δ is constant along both boundary arcs of the domain, and that its discontinuity at a (and hence also at b) is exactly equal to 1 — because the interface *has* to pass through a with no winding. From now on, we will thus assume that H_δ is equal to 0 (resp. 1) on the arc ab (resp. ba).

Now, H_δ has two natural restrictions, H_δ^w to the primal vertices and H_δ^b to the dual ones. These two restrictions have nice properties: H_δ^w is superharmonic (its discrete Laplacian is non-positive) while H_δ^b is subharmonic (its Laplacian is non-negative). Besides, assuming that F_δ is small, they differ by very little, so that any sub-sequential scaling limit of H_δ has to be harmonic — in fact, has to be the unique harmonic function h with boundary values 0 on ab and 1 on ba. That is already a non-empty statement, but extracting useful information from H_δ is not easy ...

EXERCISE I.1.4. *Prove that H_δ^w is indeed superharmonic.*

Answer: This actually takes some doing (the last, very tedious 4 pages of Smirnov's article [**37**]), but it is completely elementary. Simply expand the discrete Laplacian in terms of F_δ, and use the flow relation repeatedly to eliminate terms (it allows one to express each value of F_δ in terms of its values at 3 neighboring edges, but projecting on lines brings this down to 2).

Step 5: Tying up the loose ends. The above arguments are rather convincing, but a lot is missing which would not fit comfortably in these notes. Following the scheme of the percolation argument, the main ingredient is relative compactness in the shape of uniform continuity; here, it follows from classical results about the Ising model (essentially, from the fact that the phase transition is of second order, or equivalently that the magnetization of the critical 2D Ising model vanishes) and it does imply the convergence of H_δ, as $\delta \to 0$, to h as defined above.

Then, one needs to come back down from H_δ to F_δ, which involves taking a derivative (the same way H_δ was obtained by integration). Notice in passing that

$|F_\delta|^2$ is an increment of H_δ, so it is expected to be of the same order as the lattice spacing, and thus F_δ itself should be — and, indeed, is — of order $\sqrt{\delta}$, and related to the square root of the gradient of H_δ. The objection we raised before about the argument of F_δ still stands; we need to make the following adjustments.

Each vertex of the medial graph has four *corners* (one per adjacent face); by "rounding up" γ at each of its turns, it is possible to naturally extend F_δ to all such corners, with a winding at a corner defined to be the midpoint between that on the incident edge and that of the exiting one. Then, define $F_\delta^\diamond$ on each medial vertex to be the sum of F_δ over all adjacent corners. $F_\delta^\diamond$ is now a *bona fide* complex-valued function, and in fact F_δ can be recovered from it by appropriate projections.

We are now poised to state the main convergence result. Let Φ be a conformal map from Ω to the horizontal strip $\mathbb{R}\times(0,1)$, mapping a to $-\infty$ and b to $+\infty$. Such a map is unique up to a horizontal translation, which will not matter here since we will look at derivatives anyway; notice that h is simply the imaginary part of Φ.

THEOREM I.1.2 (Smirnov). *As $\delta \to 0$, and uniformly on compact subsets of Ω,*

$$\delta^{-1/2} F_\delta^\diamond \to \sqrt{2\Phi'}.$$

In particular, the scaling limit of $F_\delta^\diamond$ is conformally invariant.

I.1.3. The Ising model. The Ising model might be the best known and most studied model of statistical mechanics; it is amenable to the same kind of study as the random-cluster model, through the use of a similar observable. The two-point function of the Ising model, which encodes the spin-spin correlations, is closely related to the connection probability for the $q = 2$ random-cluster model, through the Edwards-Sokal coupling: starting from a random-cluster configuration with parameter p, color each cluster black or white independently of the others with probability $1/2$; this leads to a dependent coloring of the vertices of the lattice, which is distributed according to the Ising model with inverse temperature $\beta = -\log(1-p)$.

The observable is a little different, because the spin interfaces are not completely well-defined as simple loops (think for instance about the case of a checkerboard configuration); this is the main reason why the construction is more specific. The interested reader will find a detailed description in the notes for the mini-course of Stanislav Smirnov in the same school [**38**].

I.2. Path models

Maybe the simplest model for which conformal invariance is well understood is that of the *simple random walk* on a periodic lattice, say $\mathbb{Z}^2$. Indeed, as the mesh of the lattice goes to 0, the random walk path converges in distribution to that of a Brownian motion, and this in turn is conformally invariant.

More precisely, let Ω be a (bounded, smooth, simply connected) domain of the complex plane, and let $z \in \Omega$; let (B_t) be a standard planar Brownian motion started from z, τ be its hitting time of the boundary of Ω. Besides, let Φ be a conformal map from Ω onto a simply connected domain Ω', and let (W_s) be a Brownian motion started from $\Phi(z)$ and σ its hitting time of $\partial\Omega'$.

It is not true that (W_s) and $(\Phi(B_t))$ have the same distribution in general, because their time parameterizations will be different, but in terms of the path considered as a subset of the plane, they do; the following statement is another instance of conformal invariance:

Theorem I.2.1 (P. Lévy). *The random compact sets* $\{\Phi(B_t) : t \in [0,\tau]\}$ *and* $\{W_s : s \in [0,\sigma]\}$ *have the same distribution.*

A consequence of this and the study of SLE processes will be (among others) a very detailed description of the *Brownian frontier*, *i.e.* of the boundary of the connected component of infinity in the complement of $B_{[0,\tau]}$. However, the frontier is not visited by the Brownian path in chronological order, and that makes the direct use of planar Brownian motion problematic; it seems that things would be simpler if the random curve had no double point, and correspondingly if the underlying discrete path were self-avoiding.

The most natural way to generate a self-avoiding path in a discretized simply connected domain, from an inside point z to the boundary, would be to notice that there are finitely many such paths and to define a probability measure on the set of paths (morally, uniform given the length of the path). This leads to the definition of the *self-avoiding walk*, but unfortunately not much is known about its scaling limit, so we turn our attention to a different object which is a bit more difficult to define but much easier to study.

I.2.1. Loop-erased random walk. Let again Ω be a simply connected domain in the plane, and let $\delta > 0$; let Ω_δ be an appropriate discretization of Ω by $\delta\mathbb{Z}^2$ (say, the largest component of their intersection), and let z_δ be a vertex in Ω_δ. In addition, let (X_n) be a discrete-time random walk on $\delta\mathbb{Z}^2$, starting from z_δ, and let

$$\tau := \inf\{n : X_n \notin \Omega_\delta\}$$

its exit time from Ω_δ. The loop-erasure $LE(X)$ is defined, as the name indicates, by removing the loops from (X_n) as they are created. Formally, define the (n_i) inductively by letting $n_0 = 0$ and, as long as $n_i < \tau$,

$$n_{i+1} := \max\{n \leqslant \tau : X_n = X_{n_i}\} + 1.$$

Then, $LE(X)_i := X_{n_i}$.

Clearly, the loop-erasure of a discrete path is a self-avoiding path, as the same vertex cannot appear twice in $LE(X)$; when as above X is a simple random walk, $LE(X)$ is known as the *loop-erased random walk* (from z_δ to $\partial\Omega_\delta$ in Ω_δ). If b is a boundary point of Ω_δ, one can condition X to leave Ω_δ at b and the loop-erasure of that conditioned random walk is called the loop-erased random walk from a to b in Ω_δ.

The profound link between the loop-erased random walk and the simple random walk itself will be instrumental in the study of its asymptotic properties as δ goes to 0. For instance, the distribution of the exit point of a simple random walk or an unconditioned loop-erased walk is the same (it is the discrete harmonic measure from z_δ).

The counterpart of RSW for loop-erased walks (in the sense that it is one of the basic building blocks in proofs of convergence) will be a statement that $LE(X)$ does not "almost close a loop" — so that in particular, if it does have a scaling limit, the limit will be supported on simple curves. We defer the exact statement to a later section, but essentially what happens is the following: for $LE(X)$ to form a fjord, without closing it, X itself needs to approach its past path and then proceed to the boundary of the domain without actually closing the loop (as this would

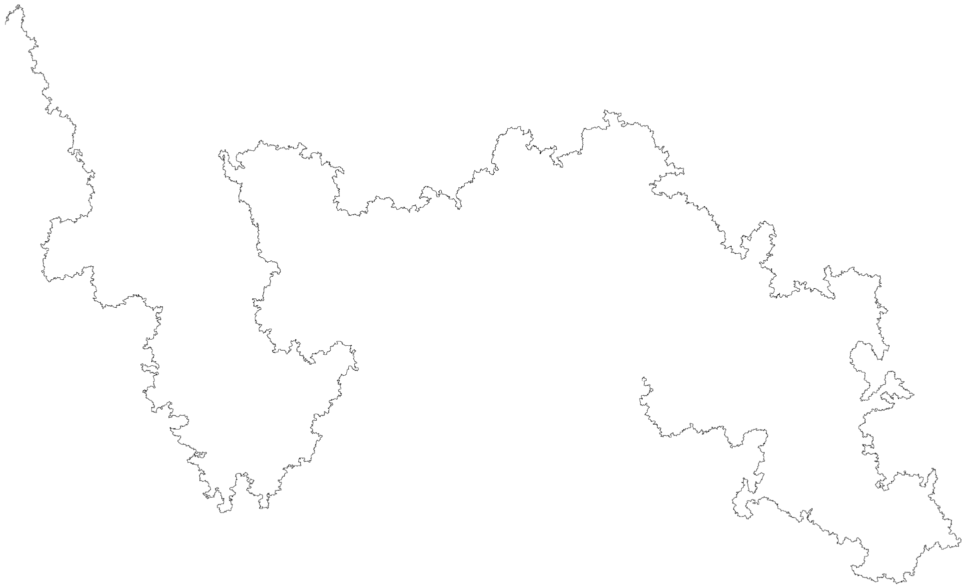

FIGURE 6. A loop-erased random walk.

vanish in $LE(X)$); the escape itself is very unlikely to happen as a consequence of Beurling's estimate.

We finish this definition of the loop-erased random walk with a statement of the *domain Markov property*. Let again Ω be a simply connected domain in the plane, discretized as $\Omega_\delta = \Omega \cap \delta\mathbb{Z}^2$, and let $a \in \delta\mathbb{Z}^2$ be an interior point of Ω and $b \in \delta\mathbb{Z}^2$ be a boundary point (*i.e.*, a point outside Ω with at least one neighbor inside Ω). Consider a loop-erased random walk path γ from a to b in Ω, and label it backwards as $(\gamma_i)_{0 \leqslant i \leqslant \ell(\gamma)}$ where $\ell(\gamma)$ is the number of steps in γ, and where $\gamma_0 = b$ and $\gamma_{\ell(\gamma)} = a$.

This defines a sequence of discrete domains $\Omega_i := (\Omega \cap \mathbb{Z}^2) \setminus \{\gamma_j : j < i\}$, and by definition, γ is a loop-erased random walk from a to γ_0 in Ω_0. The statement of the domain Markov property is then the following: for every $k > 0$, the conditional distribution of $(\gamma_i)_{i \geqslant k}$ given $(\gamma_i)_{i \leqslant k}$ is the same as that of a loop-erased random walk from a to γ_k in Ω_k. In other words, the decreasing sequence of domains (Ω_k) can be seen as a Markov chain.

One way of proving the Markov property is, for a given path γ, to write the probability that a loop-erased random walk from a to b follows γ in terms of a product of Green functions and transition probabilities. The outcome of the proof is an alternative description of the backwards loop-erased random walk as a *Laplacian walk*, which is a growth process defined in terms of harmonic functions.

Fix $k \geqslant 0$, and assume that the first k steps of γ do not contain a. Let f_k be the unique function which is harmonic on Ω_k, equal to 0 outside Ω_k, except at γ_k where it is equal to 1. There is a finite, non-empty family $\{z_1, \ldots, z_\ell\}$ of neighbors of γ_k at which f_k is positive, which are exactly the possible locations for γ_{k+1}, and one has

$$P\left[\gamma_{k+1} = z_i | \gamma_0, \dots, \gamma_k\right] = \frac{f_k(z_i)}{f_k(z_1) + \cdots + f_k(z_\ell)}.$$

In words, the growth distribution of γ is proportional to the value of f_k at the neighbors of γ_k. Yet another restatement of the same fact is that backwards loop-erased random walk is the same as "DLA conditioned not to branch." [Do not take this statement as a hope that a given branch in the DLA tree would look like a loop-erased walk; this is not true at all, unfortunately.]

I.2.2. Uniform spanning trees and Wilson's algorithm. Let $G = (V, E)$ be a finite graph; let v_∂ be a vertex of G (the "boundary" of the graph). A *spanning tree* of G is a connected subgraph of G containing all its vertices and no loop (a subgraph with all the vertices and no loop is called a *spanning forest*, and a tree is a connected forest). The set of spanning trees of G is finite; a *uniform spanning tree* is a random tree with the uniform distribution on that set.

Given a vertex $v \neq v_\partial$, we now have two ways of constructing a random self-avoiding path from v to v_∂:

- The loop-erased random walk in G from v to v_∂ (defined exactly as in the case of the square lattice above);
- The (unique) branch of a uniform spanning tree joining v to v_∂.

As it turns out, these two random paths have the same distribution. In particular, because in the second definition the roles of v and v_∂ are symmetric, we get an extremely non-obvious feature of loop-erased random walks: the time-reversal of the loop-erased walk from v to v_∂ is exactly the loop-erased walk from v_∂ to v. This is instrumental in the proof of convergence of the loop-erased walk to SLE_2 in the scaling limit.

As an aside, loop-erased walks provide a very efficient method for sampling a uniform spanning tree, which is due to David Wilson. Essentially: pick a point v_1, and run a loop-erased walk γ_1 from it to v_∂; then, pick a vertex v_2 which is not on γ_1 (if there is such a vertex) and run a loop-erased walk γ_2 from v_2 to γ_1; proceed until all the vertices of V are exhausted, each time building a loop-erased walk from a vertex to the union of all the previous walks. When the construction stops, one is facing a random spanning tree of G; and as it happens, the distribution of this tree is that of a uniform spanning tree.

I.2.3. The self-avoiding walk. Another, perhaps more natural probability measure supported on self-avoiding paths in a lattice is simply the uniform measure on paths of a given length. More specifically, let Ω_n be the set of n-step nearest-neighbor, self-avoiding path in $\mathbb{Z}^2$, starting at the origin, and let P_n be the uniform measure on Ω_n: we are interested in the behavior of a path sampled according to P_n, asymptotically as $n \to \infty$. The measure P_n is known as the *self-avoiding walk* of length n.

Obviously the first question coming to mind is that of the cardinality of Ω_n. By a simple sub-multiplicativity argument, there exists a constant $\mu \in [2, 3]$ known as the *connectivity constant* of the lattice such that

$$\frac{1}{n} \log |\Omega_n| \to \log \mu$$

FIGURE 7. A self-avoiding walk.

(which we will denote in short by $|\Omega_n| \approx \mu^n$). In fact, the behavior of $|\Omega_n|$ is conjecturally given by

$$|\Omega_n| \sim C\mu_n n^{\gamma-1}$$

for some exponent γ which, in two dimensions, is expected to be equal to

$$\gamma = \frac{43}{32}.$$

The value of μ depends on the chosen lattice, and is not expected to take a particularly relevant value in most cases; it is only known in the case of the hexagonal lattice, for which it is equal to $(2+\sqrt{2})^{1/2}$ — we refer the reader to the notes for the course of Gordon Slade in this same volume for a proof of this fact. The value of γ however is expected to be *universal* and depend only on the dimension.

Now, let $\omega = (\omega_0, \dots, \omega_n)$ be a self-avoiding path distributed according to P_n. We are still interested in scaling limits as $n \to \infty$, and for that the second relevant piece of information would be the appropriate scaling to apply to ω. One way to determine it is to look at the law of $\|\omega_n\|$; conjecturally, there exists a *scaling exponent* $\nu \in (0,1)$ such that

$$E\left[\|\omega_n\|^2\right] \approx n^{2\nu}.$$

Again, the value of ν is expected to be universal and depend only on the dimension. It is known in high dimension that $\nu = 1/2$, meaning that the self-avoiding walk is diffusive in that case and behaves like the simple random walk; this is the main focus of the course of Gordon Slade in the same summer school, so we simply again refer the reader to the corresponding notes. In the two-dimensional case, it is believed that

$$\nu = \frac{3}{4};$$

we will come back to this after we talk about convergence to SLE.

In the case of the self-avoiding walk (as we will see later also happens for percolation), there is a natural way to bypass the question of the relevant scaling and still be able to define a natural limit as the lattice mesh vanishes. Let U be a bounded, simply connected domain in the complex plane, with smooth (enough) boundary, and let a and b be two points on ∂U. For every $\delta > 0$, let $U_\delta = \delta\mathbb{Z}^2 \cap U$ and let a_δ and b_δ be approximations of a and b in the same connected component of U_δ; let Ω_δ^U be the set of self-avoiding paths from a_δ to b_δ in U_δ.

Since the elements of Ω_δ^U have various lengths, it is not that natural to consider the uniform measure on it. Instead, let $x > 0$ and define a measure $\mu_{x,\delta}^U$ on Ω_δ^U by letting

$$\mu_{x,\delta}^U(\{\omega\}) = \frac{x^{\ell(\omega)}}{Z_{x,\delta}^U}$$

where $\ell(\omega)$ denotes the length of ω and $Z_{x,\delta}^U$ is a normalizing constant (the *partition function* in physical parlance). Then, as $\delta \to 0$, one expects the asymptotic behavior of a walk ω distributed according to $\mu_{x,\delta}^U$ to strongly depend on the value of x. More precisely, letting $x_c := 1/\mu$ where μ is the connective constant of the lattice[2]:

- If $x < x_c$, then ω converges in distribution to a deterministic measure supported on the shortest path joining a to b in $\overline{U}$; its fluctuations around the limiting path are Gaussian and of order $\delta^{1/2}$, and the scaling limit of ω after the corresponding rescaling in the transverse direction is a Brownian bridge.
- If $x > x_c$, then the scaling limit of ω is a random space-filling curve in U, which is conjectured to be be the same as the scaling limit of the exploration path of the uniform spanning tree in U (*i.e.*, SLE_8 for those reading ahead).
- If $x = x_c$, then the scaling limit is believed to be a non-trivial random curve from a to b in $\overline{U}$, and to be conformally invariant. It is known that **if** this is the case, then the scaling limit is $\mathrm{SLE}_{8/3}$, and the previously mentioned conjectures about the values of ν and γ hold.

One key remark about the measures defined above is the following. Let $U' \subset U$ be another simply connected domain of the plane, such that a and b are on $\partial U'$ as

[2]It is unfortunate that every second object in this section seems to be called μ, but each of these notations seems to be classical ... hopefully this is not too confusing for the reader.

well. One can define two probability measures on $\Omega_\delta^{U'}$ in a natural way: the first one is $\mu_{x,\delta}^{U'}$, and the other one is the *restriction* of $\mu_{x,\delta}^{U}$ to $\Omega_\delta^{U'} \subset \Omega_\delta^{U}$, renormalized to be a probability measure. It is very easy to check that those two measures are in fact exactly identical; we will say that the self-avoiding walk has the (discrete) *restriction property*, and we morally expect the scaling limit to exhibit something similar.

We will see in the last part that there is exactly one conformally invariant measure supported on simple curves which has the restriction property, namely $\mathrm{SLE}_{8/3}$; meaning that, **if** the self-avoiding walk converges to a conformally invariant random simple curve, it has to be $\mathrm{SLE}_{8/3}$. In other words, as in the case of percolation, we can *predict* the value of the parameter κ for the scaling limit. Note however that an actual proof of convergence might not use this fact at all (as again is the case for percolation).

I.3. Bibliographical notes

Section I.1.1. A very complete review of percolation theory is Grimmett's book [**13**]; it contains everything mentioned in these notes except for Cardy's formula. Its bibliography section is far more complete than I could hope to gather here, so I will just list a few key papers. An alternative, which is a bit hard to find but well worth reading, is the book of Kesten [**18**]. For more recent progress and conformal invariance (and more exercises), one can *e.g.* consult the lecture notes for Werner's lectures [**41**].

Besides the anecdotal quotation from [**43**], the first proper introduction of percolation as a mathematical model is the article of Broadbent and Hammersley [**10**]. Exponential decay (up to the critical point) was derived in a very general setting by Menshikov [**27**]. The first derivation of the value of a critical parameter was obtained (for bond-percolation on the square lattice) by Kesten [**17**]; RSW estimates were obtained independently by Russo [**31**] and by Seymour and Welsh [**33**].

Cardy's formula was first conjectured by — well, Cardy [**12**], and then proved on the triangular lattice by Smirnov [**35**]. A slightly simplified exposition of the proof (which is the one we followed here) can be found in [**3**], and a very (very!) detailed one in the book of Bollobás and Riordan [**9**].

Section I.1.2. Here again, the reader is advised to refer to the book of Grimmett [**14**] (and references therein) for a general introduction to random-cluster models, including most of the results which are mentioned in this section. The proof of conformal invariance for the $q = 2$ critical random-cluster model was first obtained by Smirnov [**37**]; the approach we follow here is very close to the original, but some notation is borrowed from [**6**] (and technically, the notation $F_\delta^\diamond$ is only used here). All the details can be found in the notes for Smirnov's course in Buzios [**38**].

The equality $p_c = p_{sd}$ is related to the so-called Kramers-Wannier duality [**19**]; while still open in the general case, it is known to hold in the case $q = 1$ (where it is exactly Kesten's result on the percolation critical point in [**17**]); in the case $q = 2$ (where it is related to the derivation of the critical temperature of the two-dimensional Ising model by Onsager [**28**] — see also [**6**]); and in the general case as proved in [**5**].

Section I.2.1. For the contents of this section, and an introduction (possibly the best introduction), one can have a look at Schramm's original paper on LERWs and USTs [**32**]. Wilson's article [**42**] complements it nicely; and for more quantitative results, parts of the paper by Lawler, Schramm and Werner [**25**] can be read without any prior knowledge of SLE.

Section I.2.3. As was apparent in the text, we strongly recommend that the interested reader have a look at the notes for the lectures of Gordon Slade, in this same volume; it contains all we could possibly mention here and more.

Part II . SCHRAMM-LOEWNER EVOLUTION

Introduction

The previous part introduced a few discrete models, and for each of them we saw that, in the scaling limit as the lattice mesh goes to zero, a particular *observable* converges to a conformally invariant limit. It is natural to hope that convergence will actually occur in a much stronger sense, and in particular that the *interfaces* of the discrete model will have a continuous counterpart described as random curves in a planar domain.

Schramm's insight was to realize that, under mild (and reasonable) assumptions in addition to conformal invariance, the limit has to be distributed as one of a one-parameter family of measures on curves, which he named Stochastic Loewner Evolutions. They are now universally known as *Schramm-Loewner Evolutions.* The aim of this part is to define these random curves and give a few of their fundamental properties.

II.1. Definition of SLE

II.1.1. Loewner evolution in the half-plane. Let $\mathbb{H}$ denote the open upper half-plane, seen as a subset of the complex plane, and (for now) let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be a continuous, simple curve. In keeping with probabilistic tradition, we will denote the position of γ at time t by γ_t instead of $\gamma(t)$; besides, we will assume that γ satisfies the following conditions:

- $\gamma_0 = 0$;
- For every $t > 0$, $\gamma_t \in \mathbb{H}$ (or in other words, $\gamma_t \notin \mathbb{R}$);
- $|\gamma_t| \to \infty$ as $t \to \infty$.

The results we will state in this section are actually valid in much more generality, but the intuition is not fundamentally different in the general case.

Let $H_t := \mathbb{H} \setminus \gamma_{[0,t]}$ be the complement of the path up to time t. Our assumptions ensure that H_t is a simply connected domain, and therefore Riemann's mapping theorem can be applied to show that there exists a conformal map

$$g_t : H_t \to \mathbb{H}$$

(we refer the reader to Appendix IV for a refresher on complex analysis, if needed). The map g_t is uniquely determined if one imposes the *hydrodynamic normalization*, which amounts to fixing the following asymptotic behavior at infinity:

$$g_t(z) = z + \frac{a(t)}{z} + \mathcal{O}\left(\frac{1}{z^2}\right).$$

With this notation, it is not hard to prove that a is a strictly increasing, continuous function; it need not go to infinity with t, but we will add this as an assumption on the curve. The function a can therefore be used to define a "natural" time parametrization of the curve: up to reparametrization, it is always possible to ensure that $a(t) = 2t$ for all $t > 0$. From now on, we shall assume that γ is indeed parametrized that way.

EXERCISE II.1.1. *Prove the statements made so far in the section, and in particular prove that a is indeed continuous and strictly increasing. Give an example of a curve going to infinity, but for which a is bounded.*

Answer: It is enough to show that $a(t)$ is strictly positive for every $t > 0$ — look at what happens under composition. One can then use Schwarz' Lemma to conclude. a will remain bounded if γ remains close enough to the real line.

The normalizations of g and t are chosen in such a way that the behavior of $g_t(z)$ as a function of t is then easy to describe:

THEOREM II.1.1 (Loewner). *There exists a continuous function $\beta : [0, \infty) \to \mathbb{R}$ such that, for every $t \geqslant 0$ and every $z \in H_t$,*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \beta_t}.$$

This differential equation is known as *Loewner's equation (in the half-plane).* The gain is substantial: We were able to encode the whole geometry of γ, up to reparametrization, in terms of a single real-valued function. Indeed, it is not difficult to show that the construction up to now is essentially reversible: Given β, one can solve Loewner's equation to recover (g_t), and hence (H_t) and (γ_t) as well[3].

II.1.2. Chordal SLE. Consider, say, critical site-percolation on the triangular lattice in the upper half-plane, with boundary conditions open to the right of the origin and close to the left — this corresponds to the case in Section I.1.1 with one of the boundary points at infinity. This creates an interface starting from the origin, which is the path of the exploration process and satisfies the previous hypotheses on the curve γ.

Now, assume that, as the lattice mesh goes to 0, the exploration curve converges in distribution to a (still random) curve in the upper half-plane. This scaling limit can then be encoded into a real-valued process β using Loewner's equation; of course, β will be random as well. The question is now whether we can use the results of the previous part to identify β.

Let $R > 0$, and stop the exploration process at the first time τ_R when it reaches the circle of radius R centered at 0. Conditionally on its path so far, the next steps are exactly the exploration process of percolation in a new domain H_{τ_R}, namely the unbounded connected component of the complement of the current path: this is known as the *domain Markov property.*[4]

Cardy's formula being conformally invariant, it is natural to expect that the scaling limit of the exploration process would be as well, or in other words, that

[3]Well, some care is needed here: It is *not* true that one can plug any function β into Loewner's equation and obtain a Jordan curve γ out of it. It is true if β is Hölder with exponent 1/2 and small enough norm, but a sharp condition is not known. Obviously everything works out fine if β comes from the above construction in the first place!

[4]The counterpart in statistical physics would be the DLR conditions for Gibbsian fields.

the path after time τ_R would be distributed as the conformal image of the path in $\mathbb{H}$ by a map sending $\mathbb{H}$ to the appropriate domain.

We almost have such a map at our disposal, from Loewner's equation: The map g_{τ_R} sends H_{τ_R} to $\mathbb{H}$, so its inverse map looks like what we are looking for. The only difference is that $g_{\tau_R}(\gamma_{\tau_R})$ is equal to β_{τ_R} instead of 0. Taking this into account, we get the following property (assuming of course the existence of the scaling limit): **The image of $(\gamma_t)_{t\geqslant\tau_R}$ by $g_{\tau_R}-\beta_{\tau_R}$ has the same distribution as $(\gamma_t)_{t\geqslant 0}$.**

Besides, all information coming from the path up to time τ_R is forgotten in this map: Only the shape of H_{τ_R} is relevant (because such is the case at the discrete level), and the dependence on that shape vanishes by conformal invariance. **The image of $(\gamma_t)_{t\geqslant\tau_R}$ by $g_{\tau_R}-\beta_{\tau_R}$ is independent of $(\gamma_t)_{0\leqslant t\leqslant\tau_R}$.**

It remains to investigate what these two properties translate to in terms of the process (β_t). First, notice that the coefficient in $1/z$ in the asymptotic expansion at infinity which we are using is additive under composition. Letting $s, t > 0$, the previous reasoning, applied at time t, leads to the following:

$$g_{t+s} = \beta_t + \tilde{g}_s \circ (g_t - \beta_t),$$

where equality holds in distribution and where $\tilde{g}$ is an independent copy of g_s; the addition of β_t takes care of the normalization at infinity.

Differentiating in s and using Loewner's equation, this leads to

$$\beta_{t+s} = \beta_t + \tilde{\beta}_s,$$

where again equality holds in distribution and $\tilde{\beta}$ is an independent copy of β. **The process $(\beta_t)_{t\geqslant 0}$ has independent and stationary increments.**

Besides, the distribution of γ is certainly invariant under vertical reflection (because this holds at the discrete level), so β and $-\beta$ have the same distribution. So, we arrive at the following characterization: **Under the hypotheses of conformal invariance and domain Markov property, there exist a constant $\kappa \geqslant 0$ and a standard Brownian motion (B_t) such that**

$$(\beta_t)_{t\geqslant 0} = (\sqrt{\kappa} B_t)_{t\geqslant 0}.$$

II.1.3. Radial SLE. We just say a few words here about the case of radial Loewner chains, since not much needs to be changed from the chordal setup. Here, we are given a continuous, Jordan curve γ in the unit disk $\mathbb{D}$, satisfying $\gamma_0 = 1$, $\gamma_t \neq 0$ for all $t > 0$ and $\gamma_t \to 0$ as $t \to 0$. In other words, the reference domain is not the upper half-plane with two marked boundary points, but the unit disk with one marked boundary point and one marked interior point.

Let D_t be the complement of $\gamma_{[0,t]}$ in the unit disk; notice that 0 is in the interior of D_t, so there exists a conformal map g_t from D_t onto $\mathbb{D}$ fixing 0; this map is unique if one requires in addition that $g_t'(0) \in \mathbb{R}_+$, which we will do from now on.

The natural parametrization of the curve still needs to be additive under composition of conformal maps; here, the only choice (up to a multiplicative constant) is the logarithm of $g_t'(0)$: up to reparametrization, we can ensure that for every $t > 0$, $g_t'(0) = e^t$. With this choice, we have the following:

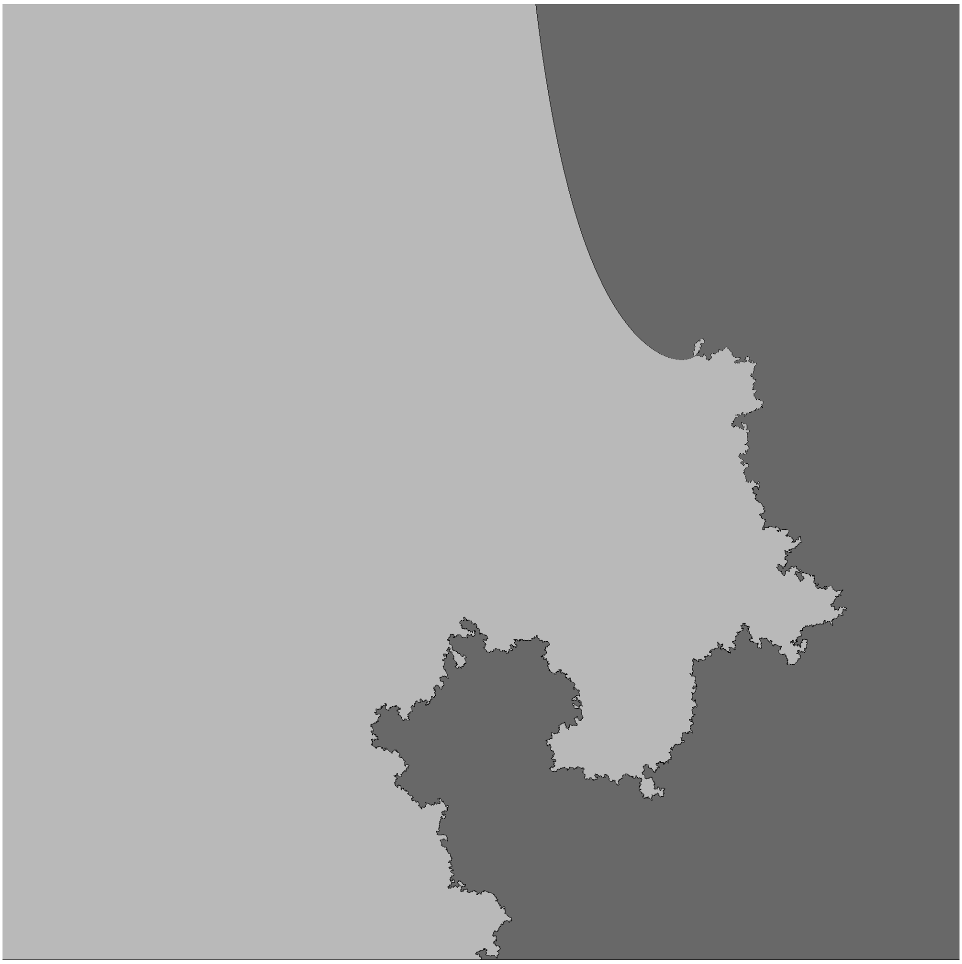

FIGURE 8. A chordal SLE process with parameter $\kappa = 2$. (The driving Brownian motion is stopped at time 1, which explains the smooth "tail" of the curve.)

THEOREM II.1.2 (Loewner). *There exists a continuous function $\theta : [0,\infty) \to \mathbb{R}$ such that, for every $t \geqslant 0$ and every $z \in D_t$,*

$$\partial_t g_t(z) = \frac{e^{i\theta_t} + g_t(z)}{e^{i\theta_t} - g_t(z)} g_t(z).$$

This is known as Loewner's equation in the disk.

Everything we just saw in the chordal case extends to the radial case. In particular, if the curve is related to a conformally invariant model (say, if it is the scaling limit of the loop-erased random walk), then under the same hypothesis of domain Markov property, one gets that there must exist $\kappa > 0$ such that

$$(\theta_t)_{t\geqslant 0} = (\sqrt{\kappa} B_t)_{t\geqslant 0}$$

(where again (B_t) is a standard real-valued Brownian motion). Solving Loewner's equation in the disk with such a driving function defines *radial* SLE_κ.

Remark. The local behavior of this equation around the singularity at $z = e^{i\theta_t}$ involves a numerator of norm 2; it is the same 2 as in Loewner's equation in the upper half-plane, in the sense that the local behavior of the solution for the same value of κ will then be the same on both sides.

II.1.4. SLE in other domains. We end the definition of the various kinds of SLE processes by a remark on the general case of simply connected domains. If Ω is such a domain and if a and b are two boundary points, then there is a conformal map Φ from Ω to $\mathbb{H}$ sending a to 0 and b to ∞; *chordal* SLE$_\kappa$ *from a to b in Ω* is simply the pullback of chordal SLE$_\kappa$ in $\mathbb{H}$ through Φ. One potential obstruction is that Φ is not uniquely defined; however, this is harmless because all such conformal maps are scalings of each other, and chordal SLE is *scale-invariant*.

The radial case is treated in a similar, and actually easier way: given a boundary point a and an interior point c in Ω, there is a *unique* conformal map Ψ from Ω to the unit disk mapping a to 1 and c to 0. *Radial* SLE$_\kappa$ *from a to c in Ω* is simply defined as the pullback of radial SLE$_\kappa$ in the disk through Ψ.

II.2. First properties of SLE

II.2.1. Geometry. The first, very non-trivial question arising about SLE is whether it actually fits the above derivation, which more specifically means whether the curve γ exists. In the case of regular enough driving functions, namely Hölder with exponent $1/2$ and small enough norm, Marshall and Rohde [**26**] proved that it does, but it is possible to construct counterexamples.

It turns out to indeed be the case, up to one notable change. Recall that one can always solve Loewner's equation to obtain $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ where K_t is the (relatively compact) set of points in the upper half-plane from which the solution blows up before time t. Then:

Theorem II.2.1 (Rohde-Schramm, Lawler-Schramm-Werner). *For every $\kappa > 0$,* SLE$_\kappa$ *is generated by a curve, in the following sense: there exists a (random) continuous curve γ in the closure of the upper half-plane $\overline{\mathbb{H}}$, called the* SLE trace, *such that, for every $t > 0$, $\mathbb{H} \setminus K_t$ and $\mathbb{H} \setminus \gamma_{[0,t]}$ have the same unbounded connected component.*

What this really means is that K_t can be obtained starting up from $\gamma_{[0,t]}$ and then filling up every bounded "bubble" it forms, if there is any; technically, the proof in the article by Rohde and Schramm [**30**] covers all cases but one, namely $\kappa = 8$, for which the existence of the trace is only known as a consequence of the convergence of the UST contour to SLE$_8$.

Given the existence of the trace, it is natural to ask whether K itself is a curve or not. Whether this happens depends on the value of κ:

Theorem II.2.2 (Rohde-Schramm). *The topology of the* SLE *trace undergoes two transitions:*

- *If $\kappa \leqslant 4$, then γ is almost surely a simple curve, and besides $\gamma_t \in \mathbb{H}$ for every $t > 0$;*
- *If $4 < \kappa < 8$, then γ does have double points, and $\gamma_{[0,t]} \subsetneq K_t$;*
- *If $8 \leqslant \kappa$, then γ is almost surely a space-filling curve,* i.e. $\gamma_{[0,\infty)} = \overline{\mathbb{H}}$.

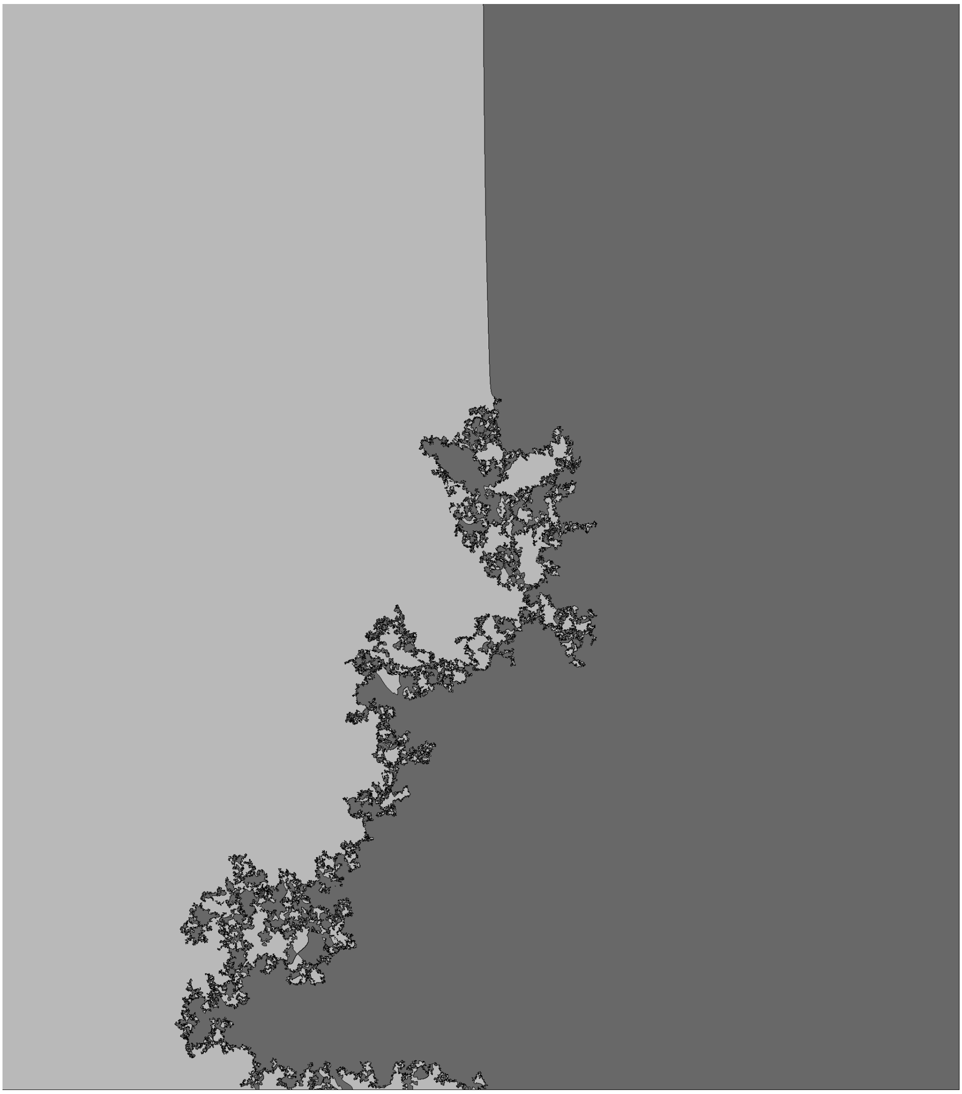

FIGURE 9. A chordal SLE with parameter $\kappa = 6$.

PROOF. The proof of this theorem involves the first use of SLE in computations. Let us start with the transition across $\kappa = 4$. Let $x > 0$, and trace the evolution of x under the (chordal) SLE flow by defining

$$Y_t^x := g_t(x) - \beta_t.$$

From Loewner's equation, one gets

$$\mathrm{d}Y_t^x = \partial_t g_t(x)\mathrm{d}t - \mathrm{d}\beta_t = \frac{2}{g_t(x) - \beta_t}\mathrm{d}t - \mathrm{d}\beta_t = \frac{2\mathrm{d}t}{Y_t^x} - \sqrt{\kappa}\mathrm{d}B_t.$$

Up to a linear time change, this is exactly a *Bessel process of dimension* $1+4/\kappa$. In particular, it will hit the origin (meaning that x is swallowed by the curve in finite time) if and only it the dimension of the process is less than 2, if and only if $\kappa > 4$.

The transition across $\kappa = 8$ is a bit more problematic, and involves estimates for the probability of hitting a ball inside the domain, proving that this probability is equal to 1 if and only if $\kappa \geqslant 8$; we leave that as an exercise, which can be skipped on first reading. □

Exercise II.2.1 (Conformal radius, space-filling SLE and Hausdorff dimension). *We consider chordal SLE(κ) in the upper half-plane for $\kappa > 0$. As usual, g_t is the conformal map from H_t to $\mathbb{H}$ with hydrodynamic renormalization. Define*

$$\tilde{g}_t(z) := \frac{g_t(z) - g_t(z_0)}{g_t(z) - \overline{g_t(z_0)}} \quad \textit{for all} \quad z \in \mathbb{H}.$$

(1) *Let $z_0 \in \mathbb{H}$. Prove that there exist constants $0 < c_1, c_2 < \infty$ such that*
$$c_1 d\big[z_0, \gamma[0,t)\big] \leqslant |\tilde{g}_t'(z_0)|^{-1} \leqslant c_2 d\big[z_0, \gamma[0,t)\big]$$
for any $t < \tau(z_0)$.

(2) *a) Assume that $t < \tau(z_0)$. Explain how one could derive the following equality – we do not ask for the (straightforward yet messy) computation.*
$$\partial_t \tilde{g}_t(z) = a_t \times \frac{\tilde{\beta}_t \tilde{g}_t(z)(\tilde{g}_t(z) - 1)}{(1 - \tilde{\beta}_t)(\tilde{g}_t(z) - \tilde{\beta}_t)},$$
where
$$\tilde{\beta}_t = \frac{\beta_t - g_t(z_0)}{\beta_t - \overline{g_t(z_0)}} \qquad \textit{and} \qquad a_t = \frac{2(\tilde{\beta}_t - 1)^4}{(g_t(z_0) - \overline{g_t(z_0)})^2 \tilde{\beta}_t^2}.$$
Check that $\tilde{\beta}_t \in \partial\mathbb{D}$ and $a_t > 0$ for every $t < \tau(z_0)$.

b) We introduce the time-change $s := \int_0^t a_u du$. Show that $h_s = \tilde{g}_{t(s)}$ satisfies a 'radial Loewner-like' equation with 'driving process' α_s where $\exp(i\alpha_s) = \tilde{\beta}_{t(s)}$.

c) By differentiating (with respect to z) the previous equation at z_0, show that
$$\partial_s h_s'(z_0) = \frac{2h_s'(z_0)}{1 - \tilde{\beta}_s}.$$
Deduce that $|h_s'(z_0)| = |h_0'(z_0)|e^{-s}$ when $s < s(\tau(z_0))$.

(3) *Explain how one could prove (we do not ask for the computation)*
$$\alpha_0^x = x \quad \textit{and} \quad d\alpha_t^x = \sqrt{\kappa} dB_t + \frac{\kappa - 4}{2} \cot(\alpha_t^x/2) dt.$$

(4) *To which event for the diffusion α does $\{t(s) = \tau(z_0)\}$ correspond? Show that $d(z_0, \gamma[0,\infty)) \asymp e^{-S}$ where S is the survival time of the diffusion α.*

(5) *Prove that the* SLE(κ) *is dense whenever $\kappa \geqslant 8$. We assume that it is generated by a transient continuous curve. Prove that it is space-filling.*

(6) *What can be said about the Hausdorff dimension of* SLE(κ) *when $\kappa < 8$?*

Much more can be said about the topological and metric properties of γ and the related boundary behavior of the maps g_t; they are the topic of G. Lawler's mini-course in this summer school, so we don't dwell on it much further, simply ending on the following result:

Theorem II.2.3 (Beffara). *For every $\kappa \in [0, 8]$, the Hausdorff dimension of the* SLE *trace is almost surely equal to $1 + \kappa/8$.*

II.2.2. Probability. We now turn to uses of SLE in computing the probabilities of various events which are of particular interest in the framework of scaling limits of discrete models; we focus on two kinds of estimates, for crossings and for arm events. Before doing that, though, we first mention two special properties which were already mentioned in a discrete setting, for percolation and the self-avoiding walk respectively.

II.2.2.1. *Locality and the restriction property.* Let A be a non-empty relatively compact subset of $\mathbb{H}$, at a positive distance from the origin, and with simply connected complement in $\mathbb{H}$ — such a set is called a *hull* in the SLE literature, though the reason for the choice of this term is not clear. We are in the presence of two simply connected domains, the upper half-plane $\mathbb{H}$ and a subdomain $\mathbb{H} \setminus A$; we wish to compare SLE in these two domains.

Let (K_t) be an SLE_κ in $\mathbb{H}$, and let γ be its trace; let $(\tilde{K}_t)$ be an SLE_κ in $\mathbb{H} \setminus A$, and let $\tilde{\gamma}$ be its trace. Let τ (resp. $\tilde{\tau}$) be the first hitting time of A by (K_t) (resp. by $(\tilde{K}_t)$) — for now, assume that $\kappa > 4$ so that both of these stopping times are a.s. finite.

Theorem II.2.4 (Locality property of SLE_6). *In the case $\kappa = 6$, with the previous notation, the two random sets*

$$K_{\tau-} := \bigcup_{t<\tau} K_t \quad \textit{and} \quad \tilde{K}_{\tilde{\tau}-} := \bigcup_{t<\tilde{\tau}} \tilde{K}_t$$

have the same distribution.

In other words, as long as γ does not touch the boundary of the domain, it does not "know" whether it is growing within $\mathbb{H}$ or $\mathbb{H} \setminus A$, hence the name of this property. Notice that the exploration process of percolation (see section I.1.1 of the previous part) satisfies the same property at the discrete level; this is one way to predict that its scaling limit has to be SLE_6. One interesting corollary of locality is the following:

Corollary II.2.1. *Let Ω be a simply connected domain in the plane, and let a, b and c be three points on $\partial\Omega$. Then, until their first hitting time of the boundary arc bc, an SLE_6 in Ω from a to b and an SLE_6 in Ω from a to c have (up to time-change) the same distribution.*

So, not only does SLE_6 not know in which domain it is growing, it does not know where it is going to either. This allows for language shortcuts such as "SLE_6 in Ω from a to the arc bc" which will be useful very soon.

With the same notation as above, assume now that $\kappa < 4$, so that in particular $\tilde{K}$ a.s. never hits A and K avoids it with positive probability. This provides us with two probability distributions on simple curves in the complement of A in the upper half-plane: $\tilde{\gamma}$ on one hand, and on the other hand, γ conditioned not to hit A, which we will (temporarily) denote by $\hat{\gamma}$.

Theorem II.2.5 (Restriction property of $\mathrm{SLE}_{8/3}$). *In the case $\kappa = 8/3$, with the previous notation, the two random sets*

$$\tilde{K}_\infty := \{\tilde{\gamma}_t : t > 0\} \quad \textit{and} \quad \hat{K}_\infty := \{\hat{\gamma}_t : t > 0\}$$

have the same distribution; the curves $\tilde{\gamma}$ and $\hat{\gamma}$ themselves have the same distribution as well, up to appropriate time-change.

We mentioned already that the self-avoiding walk measure has the same property at the discrete level; that is one reason to predict that it converges to SLE(8/3) in the scaling limit. However, this presumes the existence of a scaling limit and its conformal invariance, which in that case remain mostly mysterious.

II.2.2.2. *Interlude: restriction measures.* As an aside to the main text, we now give a short description of another family of measures on random sets which have strong links to SLE. Let K be a closed, connected subset of $\mathbb{H} \cup \{0\}$, containing 0, and having its complement in $\mathbb{H}$ consisting of exactly two (open) connected components, both unbounded, one having $\mathbb{R}_+$ on its boundary and the other, $\mathbb{R}_-$. For the duration of this interlude, let us call such a set a *nice* set.

Let A be a compact subset of $\overline{\mathbb{H}}$; we will say that A is a *hull* if the distance $d(0, A)$ is positive and if $\mathbb{H} \setminus A$ is simply connected. If A is a hull, we will denote by Ψ_A the unique conformal map from $\mathbb{H} \setminus A$ to $\mathbb{H}$ sending 0 to 0, ∞ to ∞ and such that $\Psi(z)/z$ tends to 1 at infinity. (Notice that this is not exactly the same normalization as that of g_t in the case of SLE; they differ by a real constant term.) If A is a hull, K a nice set and if $K \cap A = \varnothing$, then $\Psi_A(K)$ is again a nice set.

Now, let P be a probability measure supported on nice sets. We say that P is a *restriction measure* if the following happens: if K is distributed according to K, then so is λK for every $\lambda > 0$, and moreover, for every hull A, conditionally on the event $K \cap A = \varnothing$, the nice set $\Psi_A(K)$ is also distributed according to P.

We already saw that the whole trace of $\mathrm{SLE}_{8/3}$ is an example of such a restriction measure; in fact, there is a very simple structure theorem:

THEOREM II.2.6. *Let P be a restriction measure: there exists a real $\alpha \geqslant 5/8$ such that, if K is distributed according to P, then for every hull A,*

$$P[K \cap A = \varnothing] = \Psi_A'(0)^\alpha.$$

Moreover, for every real $\alpha \geqslant 5/8$, there is a unique restriction measure P_α satisfying the previous relation for every hull A. The only restriction measure supported on simple curves is $P_{5/8}$, which corresponds to $\mathrm{SLE}_{8/3}$.

There is another one of these restriction measures which is easy to describe and has to do with planar Brownian motion; essentially it is "Brownian motion in the half-plane, conditioned not to touch the boundary", or a variation of a *planar Brownian excursion.*

The easiest description is as follows: Let $\varepsilon > 0$, $R > 0$ and let B be Brownian motion started at $i\varepsilon$, conditioned to reach imaginary part R before hitting the real axis; by the gambler's ruin estimate, the probability of the conditioning event is ε/R. Let $K_{\varepsilon,R}$ be the path of that Brownian motion, up to the hitting time of imaginary part R. It is not difficult to prove that as ε goes to 0 and R to infinity, $K_{\varepsilon,R}$ converges in distribution to a random locally compact set K which intersects the real axis exactly at the origin. For short, while in this interlude we will refer to K as a *Brownian excursion.*

Mapping the picture through Ψ_A as ε goes to 0 and R to infinity sends $i\varepsilon$ to a point close to $i\varepsilon\Psi'(A)$ while it leaves the horizontal line $iR + \mathbb{R}$ close to invariant (because Ψ_A is normalized to be close to a horizontal translation near infinity). From this remark, conformal invariance of the Brownian path, and the gambler's ruin estimate, one directly obtains the fact that K is in fact distributed as the restriction measure P_1.

One final remark is in order. Let α_1 and α_2 both be real numbers at least equal to $5/8$; let K_1 and K_2 be independent and distributed as P_{α_1} and P_{α_2}, respectively. Typically, the union $K_1 \cup K_2$ is not a nice set, but there is a well-defined minimal nice set, say $K_1 \oplus K_2$, containing both of them (one can think of it as being obtained from their union by "filling its holes", so we will call it the *filled union* of K_1 and K_2). By independence, it is obvious that $K_1 \oplus K_2$ is distributed as $P_{\alpha_1+\alpha_2}$.

This has a very interesting consequence: since 8 times $5/8$ is equal to 5, the filled union of 8 independent realizations of $\mathrm{SLE}_{8/3}$ has the same distribution as the filled union of 5 independent Brownian excursions. In particular, the boundaries of those two sets have the same distribution. On the other hand, the boundary of the first one looks locally like the boundary of one of them, *i.e.* like an $\mathrm{SLE}_{8/3}$, while the boundary of the second one looks like that of a single Brownian excursion, which is th same as the Brownian frontier. This is one possible way to prove Mandelbrot's conjecture that the Brownian frontier has dimension $4/3$.

II.2.2.3. *Crossing probabilities and Cardy's formula.* The initial motivation behind the definition of SLE was the conformal invariance of some scaling limits; here we looked particularly at percolation through Cardy's formula. We still have to identify the value of κ though, and crossing probabilities are a natural way to do it: one can compute them in terms of κ and match the result with Smirnov's theorem.[5]

THEOREM II.2.7. *Let $\kappa > 4$, $a < 0 < c$, and let $E_{a,c}$ be the event that the* SLE_κ *trace visits $[c, +\infty)$ before $(-\infty, a]$. Then,*

$$P\left[E_{a,c}\right] = F\left(\frac{-a}{c-a}\right) \quad \textit{where} \quad F(x) = \int_0^x \frac{\mathrm{d}u}{u^{4/\kappa}(1-u)^{4/\kappa}}.$$

This is equal to the Cardy-Smirnov result in the case $\kappa = 6$.

PROOF. The proof follows essentially the same lines as in the exercise on Bessel processes, but it is rather instructive, so we still give a very rough outline here for the benefit of the serious reader willing to do the computation. Let $A_t := g_t(a)$, $C_t = g_t(c)$ and

$$Z_t := \frac{\beta_t - A_t}{C_t - A_t}.$$

From Itô's formula, it is straightforward to obtain

$$\mathrm{d}Z_t = \frac{\sqrt{\kappa}\mathrm{d}B_t}{C_t - A_t} + \frac{2\mathrm{d}t}{(C_t - A_t)^2}\left(\frac{1}{Z_t} - \frac{1}{1-Z_t}\right),$$

which after the time-change $\mathrm{d}s = \mathrm{d}t/(C_t - A_t)^2$, $\tilde{Z}_s = Z_t$ leads to

$$\mathrm{d}\tilde{Z}_s = \sqrt{\kappa}\mathrm{d}\tilde{B}_s + 2\left(\frac{1}{\tilde{Z}_s} - \frac{1}{1-\tilde{Z}_s}\right).$$

Finding F now amounts to writing that the drift term of $F(Z_{\tilde{s}})$ should vanish, thus leading to the following differential equation:

$$\frac{\kappa}{4}F''(x) + \left(\frac{1}{x} - \frac{1}{1-x}\right)F'(x) = 0.$$

Proceeding from this is left as an exercise. □

[5]Historically, the first tool used to predict which value of the parameter κ corresponds to which model was to derive estimates for the winding numbers of the SLE curves as a function of κ; this has the advantage of being more general, but if one is only interested in the case of percolation, crossing probabilities give a shorter route.

II.2.2.4. *Arm events and critical exponents.* We now turn to radial SLE in the disk. For the remaining of the section, fix $\kappa > 4$ (as the estimates we are going to consider would be trivial in the case $\kappa \leqslant 4$). We will compute the *one-arm* and *two-arm exponents* of radial SLE_κ; the names should become clear as soon as one sees the SLE trace as an exploration process ...

For $\varepsilon > 0$, let τ_ε be the first time the radial SLE_κ trace visits the circle of radius ε around 0. Besides, let T be the first time t when K_t contains the whole unit circle. In addition, let U be the first time t when $\gamma_{[0,t]}$ contains both a clockwise and a counterclockwise loop separating 0 from the unit circle; note that almost surely $0 < T < U < \infty$.

THEOREM II.2.8. *As $\varepsilon \to 0$, the* one-arm probability *scales like*

$$P\left[\tau_\varepsilon < U\right] = \varepsilon^{\lambda_\kappa^{(1)} + o(1)} \quad \textit{with} \quad \lambda_\kappa^{(1)} = \frac{\kappa^2 - 16}{32\kappa};$$

the two-arm probability *(or* non-disconnection probability*) behaves like*

$$P\left[\tau_\varepsilon < T\right] = \varepsilon^{\lambda_k^{(2)} + o(1)} \quad \textit{with} \quad \lambda_k^{(2)} = \frac{\kappa - 4}{8}.$$

The case $\kappa = 6$ is of particular interest for us because of its ties to critical percolation: from these SLE estimates, one gets that the one-arm exponent of 2D percolation is $5/48$ and that the two-arm exponent is equal to $1/4$.

The proof is presented in detail in the form of an exercise (in the case $\kappa = 6$, but this is nothing special here); the overall idea is the same as that of the proof of Cardy's formula, *i.e.* to derive a PDE from Loewner's equation, to identify boundary conditions, and to exhibit a positive eigenfunction.

EXERCISE II.2.2 (Disconnection exponent for SLE_6). *Let $x \in (0, 2\pi)$ and let $\mathcal{H}(x,t)$ be the event that one radial* SLE(6) *starting from 1 does not disconnect e^{ix} from 0 before time t. The goal of this exercise is to show that there exists $c > 0$ universal such that*

$$e^{-\frac{t}{4}}\left(\sin\frac{x}{2}\right)^{\frac{1}{3}} \leqslant \mathbb{P}\left[\mathcal{H}(x,t)\right] \leqslant c e^{-\frac{t}{4}}\left(\sin\frac{x}{2}\right)^{\frac{1}{3}}.$$

(1) *Let ζ_t be the driving process of the* SLE *(it is $\sqrt{6}$ times a standard Brownian motion). Why can one define a real valued process Y_t^x such that $g_t(e^{ix}) = \zeta_t \exp(iY_t^x)$ and $Y_0^x = x$ for every $t < \tau(e^{ix})$ ($\tau(z)$ is the disconnecting time)? Show that*

$$dY_t^x = \sqrt{6} dB_t + cotg(Y_t^x/2)dt.$$

Hint. Recall that the argument is the imaginary part of the logarithm. Moreover, what is $\mathrm{Im}\left[\partial_t \log g_t(e^{ix})\right]$?

(2) *Let $\tau_x := \inf\{t \geqslant 0 : Y_t^x \in \{0, 2\pi\}\}$, prove that $\mathbb{P}[\mathcal{H}(x,t)] = \mathbb{P}[\tau^x > t]$.*

(3) *Assume that $f(x,t) := \mathbb{P}\left[\mathcal{H}(x,t)\right]$ is smooth on $(0,2\pi)\times[0,\infty)$ (the general theory of diffusion processes guarantees that), show that*

$$3f'' + \cot(x/2)f' = \partial_t f$$

and that $\lim_{x\to 0^+} f(x,t) = \lim_{x\to 2\pi^-} f(x,t) = 0$ and $f(x,0) = 1$.

(4) *Define* $F(x,t) := \mathbb{E}\big[\mathbb{1}_{\mathcal{H}(x,t)}(\sin\frac{Y_t^x}{2})^{\frac{1}{3}}\big]$ *and show that*

$$F(x,t) = e^{-\frac{t}{4}}\big(\sin\frac{x}{2}\big)^{\frac{1}{3}}.$$

Hint. We can assume that the solutions of the PDE are determined by boundary conditions.

(5) *Conclude the proof.*

EXERCISE II.2.3 (Disconnection exponent II). *Let* $x \in (0,2\pi)$ *and let* $\mathcal{J}(t)$ *be the event that one radial* SLE(6) *starting from 1 does not close any counterclockwise loop before time* t*. Let* ∂_t^1 *be the part of* $\partial K_t \setminus \partial\mathbb{U}$ *lying on the left of the endpoint* γ_t*. We set* Y_t *to be the arc-length of* $g_t(\partial_t^1)$*.*

(1) *Find an* SDE *which is satisfied by* Y *and express* $\mathbb{P}\big[\mathcal{J}(t)\big]$ *in terms of the survival time of* Y*.*
(2) *Find a* PDE *which is satisfied by the function* $h(x,t) := \int_0^1 f(x,t+s)ds$ *where* $f(x,t) := \mathbb{P}(2\pi \notin Y[0,t]|Y_0 = x)$*. What are the boundary conditions?* Hint. One can use a relation with discrete models in order to prove that $h(x,t) - h(0,t) = o(x)$; this relation can be assumed to hold.
(3) *Explain how one could prove that there exist* $0 < c_1, c_2 < \infty$ *such that*

$$c_1 e^{-\frac{5t}{48}} \leqslant \mathbb{P}\big[\mathcal{J}(t)\big] \leqslant c_2 e^{-\frac{5t}{48}}.$$

(4) *Show that the probability of a radial* SLE(6) *starting from 1 does not close any counterclockwise loop before touching the circle of radius* ε *is of order* $\varepsilon^{5/48}$.

II.3. Bibliographical notes

It is still difficult to find a self-contained reference on SLE processes. Lawler's book [**20**] is a good start, and contains both the basics of stochastic calculus and complex analysis. Werner's Saint-Flour lecture notes [**40**] assume more preliminary knowledge.

Of course, it is always a good idea to have a look at the articles themselves. The very first paper where SLE was introduced by Schramm, together with the reasoning at the beginning of the part, is about loop-erased walks [**32**]; a reference for the complex-analytic statements on Loewner chains is the book of Pommerenke [**29**].

One can then consult the whole series of articles by Lawler, Schramm and Werner [**21, 22, 23, 24**], as well as the (very technical) article of Rohde and Schramm [**30**] for the existence of the trace. The Hausdorff dimension of the trace is derived in [**2, 4**].

Part III . CONVERGENCE TO SLE

In this part, we gather two things: first, information about convergence of discrete objects to SLE in the scaling limit, *i.e.*, a description of the method of proof and its application in a few cases; and second, uses of convergence to get estimates about the discrete models themselves, mainly in the form of the values of some critical exponents.

III.1. Convergence: the general argument

We already have a few convergence results for *observables* of discrete models; let us focus for now on percolation and Cardy's formula. The question is whether the convergence of crossing probabilities is enough to obtain the convergence of the exploration process to the trace of an SLE (in that case, SLE_6 because of the locality property, or by matching the crossing probabilities). As it turns out, this is not an easy question; it seems that this information in itself is not quite enough to conclude.[6]

Moreover, in other cases such as the random-cluster model, the information contained in the observable is not directly of a geometric nature, and it is not clear at first how to extract geometry from it.

III.1.1. First attempt, percolation. The most natural approach (which is briefly described by Smirnov in [**35, 36**] and can indeed be applied in full rigor in the case of percolation) is the following. Let $\varepsilon > 0$ and let $\delta \in (0, \varepsilon)$; look at critical site-percolation on the triangular lattice of mesh size δ, in the upper-half plane. Fix boundary conditions to be open to the right of the origin and closed to the left, and let γ^δ be the corresponding exploration curve; let τ be its first exit time of the disk of radius ε around 0.

The distribution of $\gamma^\delta(\tau)$, asymptotically as $\delta \to 0$, is precisely given in terms of Cardy's formula in a half-disk. Indeed, for fixed δ, the probability that γ^δ exits the disk to the left of a point $z \in \mathbb{H} \cap \mathcal{C}(0, \varepsilon)$ is exactly the probability that there is an open crossing of the half-disk between the boundary intervals $(0, \varepsilon)$ and $(z, -\varepsilon)$, which we know converges to an explicit limit as $\delta \to 0$ by Theorem I.1.1.

In fact, this distribution is also the same, through the locality property and the computation of SLE crossing probabilities, as that of the exit point of the half-disk by an SLE_6 process. This means that morally, at the scale ε and as $\delta \to 0$, the beginning of an SLE_6 and that of a percolation exploration process look "very similar."

There is a more specific statement about SLE_6 which we will not in fact need later but which is very nice anyway. Let Ω be a simply connected domain in the plane, and let a, b and c be three points (in that order) on $\partial\Omega$. From the locality property, we know that we can define an SLE_6 from a to the boundary arc bc, until it touches that boundary arc; let K be its shape just before it touches. [Technically, recall that this SLE is defined as SLE_6 from a to either b or c, and that these two processes agree until γ disconnects b from c, which happens exactly when the trace hits bc.]

The set K is a relatively compact subset of Ω; its complement in Ω has two connected components, one containing b on its boundary and the other, c. Now, let A be a hull in Ω (*i.e.*, in this case, a relatively compact subset of Ω which does not disconnect a from bc). It is easy to compute the probability that K does not intersect A: indeed, again by the locality property, it is exactly equal to the probability that SLE_6 from a to $bc \setminus \overline{A}$ in $\Omega \setminus A$ touches bc before it touches ∂A.

[6] It might be enough if one can derive relative compactness from it directly; see for instance [**15**], [**16**] and possibly [**34**] (after reading the rest of this section) for current progress in this direction.

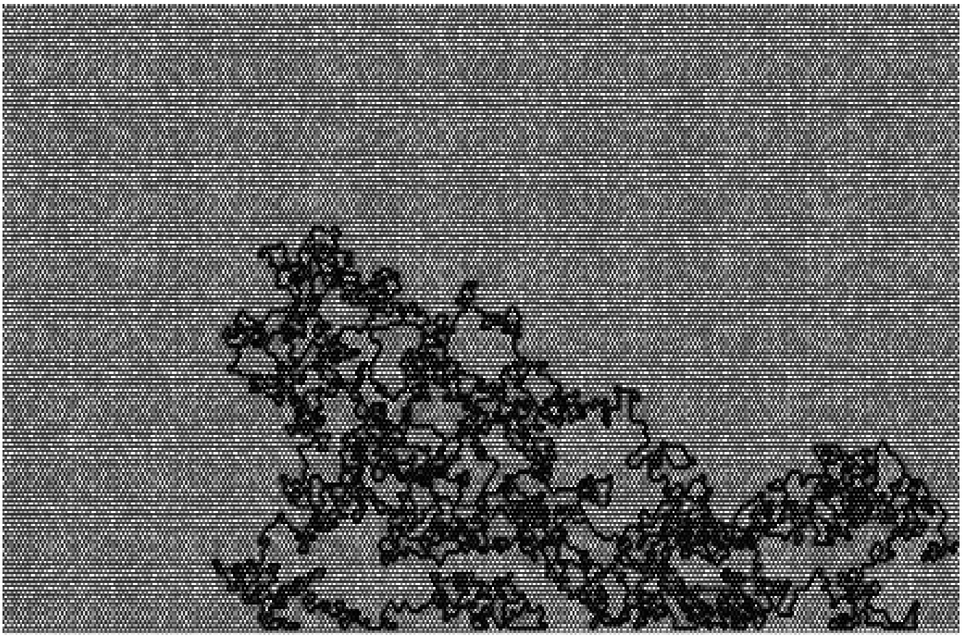

FIGURE 10. The same exploration curve of critical percolation as before, with a smaller lattice mesh.

This last probability can be expressed in terms of crossing probabilities within $\Omega \setminus A$; on the other hand, by a Donsker-class type argument, the data of the non-intersection probabilities $P[K \cap A = \varnothing]$ for all hulls A characterizes the distribution of the set K itself. In other words: morally, the shape of SLE_6 "as seen from the outside" is characterized by its crossing probabilities.

Now, Theorem I.1.1 gives us convergence of percolation crossing probabilities to those of SLE_6, and this implies the convergence of the exploration process itself to the SLE process — still *seen from the outside*. In particular, the outer shape of a large percolation cluster can be described by the boundary of an SLE_6. All this however says very little about the convergence of the exploration process *seen as a curve.*

Coming back to our proof attempt: at the time when they both exit the ball of radius ε around the origin, the exploration process of percolation at mesh δ and the trace of SLE_6 look very similar. In addition to that, they both satisfy the domain Markov property; if z_1 denotes the hitting point of the circle of radius ε by either of them, and K_1 the (filled) shape at the hitting time τ_1, then the distribution of the process after time τ_1 is the same as the initial process started from z_1 in the domain $\mathbb{H} \setminus K_1$.

One can then look at the hitting time τ_2 of the circle of radius ε around z_1: the two processes will "live" in very similar domains, so their outer shape K_2 will still be very similar at time τ_2. Inductively, one can then couple a percolation exploration with an SLE_6 process through a chain of disks of radius ε.

Letting $\varepsilon \to 0$, on the SLE side this gives the whole information about the trace of the process, so it should be possible to leverage the construction into a proof of pathwise convergence. Unfortunately, the main piece missing from the puzzle here is an estimate of the speed of convergence in Theorem I.1.1, meaning that at each step of the process there is an error term which we cannot estimate (and the errors accumulate as the construction proceeds); moreover, even if one manages to produce a fully formal proof, it would rely too strongly on the locality property to be of any more general use. We need another idea.

III.1.2. Proving convergence using an observable. Let us consider the exploration process of percolation a bit more. One way to represent it graphically is to see percolation itself as a random coloring of the faces of the hexagonal lattice, in which case the exploration curve can be seen as a collection of edges of the hexagonal lattice separating hexagons of different colors. In other words, it can be seen as a piecewise linear curve γ^δ in the upper half-plane.

Being a curve in the half-plane, it is amenable to the previous general construction of Loewner chains, which gives it a natural continuous time-parametrization, a family of conformal maps (g_t^δ), and encodes it into a real-valued driving process, say $(\beta_t^\delta)_{t\geqslant 0}$ where again δ is the mesh of the lattice. Since γ^δ is piecewise linear, β^δ is piecewise smooth.

Morally, Loewner chains should depend continuously on their driving functions. This means that a natural notion of convergence of γ^δ to an SLE_6 is the convergence of β^δ to a Brownian motion of appropriate variance as $\delta \to 0$. This is a good plan of attack, for two reasons:

- Proving convergence in distribution of a sequence of real-valued processes is a classical problem, and there are several well-known techniques to choose from;
- Convergence at the level of the driving processes does not seem at first sufficient to obtain pathwise convergence, but what is missing is an *a priori* estimate of interface regularity, similar to the Aizenman-Burchard precompactness criterion [**1**]; and in fact, in most cases such regularity *can* indeed be extracted from convergence (though we won't say more about this here — see [**34**] for details).

So now we need a tool to prove convergence of the "discrete driving process" to the appropriate Brownian motion as $\delta \to 0$. Here is a general framework; the actual implementation will depend on the model. For all $n > 0$, let Γ_n^δ be the discrete exploration up to its n-th step and let $H_n^\delta := \mathbb{H} \setminus \Gamma_n^\delta$; let τ_n^δ be the corresponding time-parameter, so that $g^\delta_{\tau_n^\delta}$ maps H_n^δ conformally back to $\mathbb{H}$.

Let z be a point in the upper half-plane, and let A_z^δ be the event that γ^δ passes to the left of z — to fix ideas; the observable of choice can vary from model to model, but the general argument will be the same in all cases. Let X_n^δ be the conditional probability of A_z^δ, given Γ_n. The key remark is that the sequence (X_n^δ) is a martingale; it converges almost surely, and its limit is either 0 or 1 according to whether γ^δ passes to the left or to the right of z.

The main assumption we will make is that we know how to compute the limit

$$\varphi(z) := \lim_{\delta\to 0} P[A_z^\delta] = \lim_{\delta \to 0} X_0^\delta,$$

that the function φ is smooth, and that we have conformal invariance (in the same sense as in Theorem I.1.1). Notice that, for percolation, this is not exactly the kind of probability that Theorem I.1.1 gives, but it is close enough to present the gist of the argument.

Because of the domain Markov property of the exploration process, X_n^δ is the probability that the exploration curve *defined in the domain* H_n passes to the left of z; by conformal invariance, this is (close to) the probability that the exploration in the initial domain passes to the left of $g^\delta_{\tau_n^\delta}(z) - \beta^\delta_{\tau_n^\delta}$. In other terms, morally

$$X_n^\delta \simeq \varphi\left(g^\delta_{\tau_n^\delta}(z) - \beta^\delta_{\tau_n^\delta}\right).$$

Now let ε be small, and let N be the first time at which either τ_n^δ is larger than ε^2, or $|\beta^\delta_{\tau_n^\delta}|$ is larger than ε; let $\sigma := \tau_N^\delta$. If δ is taken small enough, σ cannot be much larger than ε^2, and $|\beta^\delta_\sigma|$ cannot be much larger than ε (only the N-th step of the exploration process needs to be accounted for). Since σ is small, Loewner's equation gives

$$g^\delta_\sigma(z) \simeq z - \frac{2\sigma}{z}$$

so that, by the previous paragraph,

$$X_N^\delta \simeq \varphi\left(z - \beta^\delta_\sigma - \frac{2\sigma}{z}\right)$$

(the first "small" term β^δ_σ being of order ε, and the second one $2\sigma/z$ of order ε^2). Because X is a martingale, this boils down to

$$E\left[\varphi\left(z - \beta^\delta_\sigma - \frac{2\sigma}{z}\right)\right] \simeq \varphi(z).$$

The point here is that, since we know φ explicitly, we can power-expand it around z inside the expectation and then match the two sides of the relation. This will provide a relation between the powers of β^z_σ, those of σ, and explicit functions of z coming from the appropriate derivatives of φ. This is not enough to identify the driving process, but one can always write the same relation for various values of z, and this typically leads to a pair of equations of the form

$$\left\{\begin{array}{rcl} E[\beta^\delta_\sigma] & \simeq & 0 \\ E[(\beta^\delta_\sigma)^2 - \kappa\sigma] & \simeq & 0 \end{array}\right.$$

(where κ is a constant coming out of the computation, which will be the parameter of the SLE in the scaling limit, and where of course the symbol '$\simeq$' means that equality holds up to error terms, which have to be controlled along all the previous steps).

The last step of the proof involves what is known as *Skorokhod embedding*. The basic statement is the following: given a square-integrable random variable Z such that $E[Z] = 0$, there exists a standard Brownian motion (B_t) and a stopping time T such that B_T has the same distribution as Z, and satisfying the equality $E[T] = E[Z^2]$. Applying this to β^δ_σ above, we find that we can write it (still up to error terms) as $B_{\kappa T_1}$ where $E[T_1] = E[\sigma]$.

It remains to iterate the process. Once the discrete interface is explored up to capacity σ, what remains is a random discrete domain H_N^δ in which the exploration can be extended, thus extending β^δ from time $\sigma = \sigma_1$ to some $\sigma_2 > \sigma_1$; by the very same argument as above, we get

$$\left\{\begin{array}{rcl} E[\beta^\delta_{\sigma_2 - \sigma_1}] & \simeq & 0 \\ E[(\beta_{\sigma_2} - \beta^\delta_{\sigma_1})^2 - \kappa(\sigma_2 - \sigma_1)] & \simeq & 0 \end{array}\right.$$

and iteratively, we can construct an increasing sequence (σ_k) of stopping times and the corresponding sequence (T_k) through repeated use of Skorokhod embedding.

The increments $(T_{k+1} - T_k)$ are essentially independent and identically distributed, because of the domain Markov property of the underlying discrete model; thus a law of large numbers applies, stating that T_k is approximately equal to its expectation. Combining this with the previous remark, we arrive at the fact that

$$\beta^\delta_{\sigma_k} \simeq B_{\kappa\sigma_k}$$

where again B is a standard Brownian motion.

In other words, provided that δ is small, the discrete driving process is very close to being a Brownian motion with variance parameter κ; and as $\delta \to 0$, the error terms will vanish and we get

$$(\beta^\delta_t) \underset{\delta\to 0}{\longrightarrow} (B_{\kappa t}).$$

One final remark is in order. What we just did in the case of an observable depending on a point z within the domain does not in fact depend directly on the existence of z; only the martingale (X^δ_n) is relevant. Of course, if (X^δ_n) is not defined in terms of z, the function φ will have to be replaced accordingly, and we will see an example of this below in the case of the UST contour. Nevertheless, the core of the argument is the same in all cases.

Notice how conformal invariance of the scaling limit, and statements of convergence in other domains, come "for free" with the rest of the argument as soon as the scaling limit for φ is itself conformally invariant: since SLE in another domain is defined via conformal mappings anyway, the discrete driving process itself drives an SLE in the upper half-plane, and all that is needed in addition of the above argument is a composition by the conformal map from the domain to $\mathbb{H}$.

III.2. The proof of convergence in a few cases

What remains to be done now is to apply the above strategy to a few actual models. This amounts to two things to do for each model: find an appropriate observable (which is what the reader will find below); and refine driving process convergence into pathwise convergence (which, being of a much more technical nature, will be kept out of these notes and can be found in the literature).

III.2.1. The UST Peano curve. The simplest case to state is that of the uniform spanning tree in the upper half-plane, on a square lattice of mesh δ, with wired boundary conditions to the right of the origin and free boundary conditions to the left. The dual of that tree is a uniform spanning tree with reversed boundary conditions (wired on the left and free on the right); and the curve γ^δ winding between the two is known as the *UST Peano curve*, or UST contour curve. It is a simple exercise to check that the curve γ^δ satisfies the domain Markov property.

Let $z \in \mathbb{H} \cap \delta\mathbb{Z}^2$ and let $x \in \mathbb{R}_+$; let A^δ_z be the event that the branch of the UST containing z lands on the real axis somewhere on the interval $[0, 1]$, and let $\varphi_\delta(z)$ be the probability of this event. The key remark is the following: from Wilson's algorithm, $\varphi_\delta(z)$ is exactly equal to the (discrete) harmonic measure of the interval $[0, 1]$ in $\mathbb{H}$ seen from z, with reflecting boundary conditions on the negative real axis; or equivalently, to the harmonic measure of the interval $[0, 1]$ in the slit plane $(\mathbb{C} \cap \delta\mathbb{Z}^2) \setminus \mathbb{R}_+$, seen from z.

This converges to the continuous counterpart of that harmonic measure, which is easily computable: if $z = re^{i\theta}$,

$$\begin{aligned}\varphi_\delta(z) \to \varphi(z) &:= \omega^z_{\mathbb{C}\setminus\mathbb{R}_+}([0,1]) = \omega^{z^{1/2}}_{\mathbb{H}}([-1,1]) \\ &= \frac{1}{\pi}\cot^{-1}\left(\frac{|z|-1}{2\operatorname{Im}\sqrt{z}}\right) = \frac{1}{\pi}\cot^{-1}\left(\frac{r-1}{2\sqrt{r}\sin(\theta/2)}\right)\end{aligned}$$

where the determination of $\cot^{-1}$ is taken in the interval $(0,\pi)$. Because harmonic measure (or equivalently, planar Brownian motion up to time-reparametrization) is conformally invariant, so is φ, in the sense that if we defined the UST in another domain, with 3 boundary intervals as above, the scaling limit of the hitting probability would be conjugated to φ by the appropriate conformal map.

Exercise III.2.1. *Show the convergence of φ_δ to φ and prove the formulas giving the value of $\varphi(z)$ as a function of z.*

Answer: The convergence can be obtained by coupling a discrete with a planar Brownian motion and writing the continuous harmonic measure in terms of hitting probabilities. To prove the formula, it is enough to show that the function as given takes values between 0 and 1, and satisfies appropriate boundary conditions: equal to 1 on $[0,1]$, 0 on $[1,+\infty)$ with vanishing normal derivatives along $(-\infty,0)$.

Now, all that remains to do to show convergence of the driving process to a Brownian motion is to apply the strategy described above; one gets $\kappa = 8$ from the computation, so the UST contour curve converges (in the driving-process topology) to SLE_8 in the scaling limit.

III.2.2. The loop-erased random walk. The case of the loop-erased random walk is a little bit more involved. First of all, convergence will be to *radial* SLE rather than chordal as in the case of the UST contour curve — though this is a minor point, as the scheme of the proof is exactly the same in both cases. Hence, we will work primarily with the loop-erased walk from the origin to 1 in the unit disk $\mathbb{U}$; let Γ be the path of a simple random walk from 0 to 1 in $\mathbb{U}\cap\delta\mathbb{Z}^2$ (*i.e.*, conditioned to exit the domain at 1); let γ be the loop-erasure of its time-reversal. As in the first part, this defines a decreasing sequence of domains $\Omega_n := (\mathbb{U}\cap\delta\mathbb{Z}^2)\setminus\{\gamma_i : i<n\}$; besides, γ turns out to have exactly the same distribution as the time-reversal of the loop-erasure of Γ (though of course they differ in general).

Fix $v \in \mathbb{U}\cap\delta\mathbb{Z}^2$, neither too close to the boundary nor to the origin (say for instance $d(0,v)\in[1/3,2/3]$), and let Z be the number of visits of v by Γ before it exits $\mathbb{U}$; this will replace the event defining the observable in the previous section: $X^\delta_n := E[Z|\gamma_0,\dots,\gamma_n]$. As above, the key is to write the fact that (X^δ_n) is a martingale, or in other words that

$$E[X^\delta_n] \quad = \quad E[X^\delta_0] \quad (= E[Z]),$$

and to estimate both sides to the appropriate precision. Notice that $E[Z]$ is nothing but a Green function.

Conditionally on $(\gamma_j)_{j\leqslant\ell}$, let n_j be the first hitting time of γ_j by Γ and let Γ^j be the portion of Γ between times n_j and n_{j-1}; let Z_j be the number of visits of v by Γ^j. The key argument is then that the distribution of Γ up to time n_j is the same as that of a simple random walk conditioned to exit Ω_j through γ_j, so that one can write Z as the sum of the Z_j plus the number of visits to v before time n_ℓ, and this last term has the same distribution as that of the version of Z as defined in Ω_ℓ.

Now, the structure of Γ^1 is simple: it is a finite sequence of random walk excursions in Ω based at γ_1, followed by a jump from γ_1 to γ_0. This enables the computation of the expectation of Z_1 in terms of (discrete) Green functions and hitting probabilities within the domain Ω. Asymptotically as the lattice mesh goes to 0, the behavior of those is well understood (if technical to obtain to the right precision); all the limiting quantities as $\delta \to 0$ are conformally invariant and explicit.

Since we want to focus here on the heuristics of the proof, we refer the more serious readers to the initial article of Lawler, Schramm and Werner [**25**] for the full details of the proof.

III.2.3. Percolation. In the case of percolation, we again have convergence of a discrete observable (the crossing probability of a conformal rectangle) to an explicit scaling limit, so the general framework of the proof is still the same.

Here, the natural way to set up the computation is the following: let $d < 0 < b$, and let $A^\delta_{b,d}$ be the event that the critical percolation exploration curve in the upper-half plane, started at the origin, touches the half-line $[b, +\infty)$ before the half-line $(-\infty, d]$ (so, instead of depending on the location of one point in Ω, it depends on the location of two points on its boundary). The limit of $P[A^\delta_{b,d}]$ as $\delta \to 0$ can be computed from Theorem I.1.1, and is conformally invariant.

III.2.4. The Ising model. The observable in this case is given by the para-fermionic observable introduced in the previous part, and the general scheme of the proof is once again the same; one key difference is that the observable as defined initially goes to 0 with the lattice mesh (as $\delta^{1/2}$), so it has to be normalized accordingly. The reader can find all the details in the notes of Smirnov's course on that very topic at the same school.

III.3. One application of convergence: critical exponents

We saw that the disconnection exponent for radial SLE_6 is equal to 1/4. This has a natural counterpart (and generalization) in terms of critical site-percolation on the triangular lattice:

Theorem III.3.1 (Arm exponents for critical percolation). *Consider critical site-percolation on the triangular lattice in the plane; recall that Λ_n is the intersection of the lattice with the ball of radius n. As $n \to \infty$,*

$$P\left[0 \leftrightarrow \partial\Lambda_n\right] \approx n^{-5/48}.$$

Besides, let $k > 1$, fix a sequence of k colors $\sigma = (\sigma_i)_{1\leqslant i\leqslant k}$ and let $A_{k,\sigma}(n)$ be the event that there exist, between the circle of radius k and that of radius n, k disjoint paths (γ_i) (in that order), such that all the vertices along γ_i are of color σ_i. As $n \to \infty$,

$$P[A_{k,\sigma}(n)] \approx n^{-\alpha_k} \quad \textit{where} \quad \alpha_k := \frac{k^2-1}{12}.$$

The exponents α_k are called *polychromatic k-arm exponents*; notice that the non-disconnection exponent of SLE_6 corresponds to α_2. The value of α_k does not depend on the precise sequence of colors σ, as long as both colors are present. The main reason for this already appeared in the proof of the Cardy-Smirnov formula (see section I.1.1.2 in Part I): conditioned on the existence of 2 arms of different colors, it is possible to discover those two arms using the exploration process — this actually provides an algorithm to find the leftmost white arm and the rightmost black arm, say. The exploration process depends only on the states of the vertices between those two arms, so one can swap the state of all non-explored vertices, and this gives a correspondence between the events $A_{k,\sigma}(n)$ and $A_{k,\tilde{\sigma}}(n)$ where $k-2$ colors differ between σ and $\tilde{\sigma}$. Variations of the argument allow to relate any two sequences of colors, as long as the exploration process is available, *i.e.* as soon as both colors are present.

In fact, it can be shown that the situation is indeed similar but with different exponents in the monochromatic case (see [**8**]):

THEOREM III.3.2 (Monochromatic arm exponents for percolation). *Consider critical site-percolation on the triangular lattice in the plane; let $k > 1$, and let $B_k(n)$ be the event that there exist, between the circle of radius k and that of radius n, k disjoint open paths. As $n \to \infty$,*

$$P[B_k(n)] \approx n^{-\tilde{\alpha}_k} \quad \textit{for some} \quad \tilde{\alpha}_k \in (\alpha_k, \alpha_{k+1}).$$

The values of the exponents $\tilde{\alpha}_k$ are not known for $k > 1$. The case of $\tilde{\alpha}_2$ is of particular interest; that exponent is known as the *backbone exponent* in the physics literature, and its value is very close to 17/48 (see [**7**]). Whether that is its actual value remains open.

As we did before, we present the proof of Theorem III.3.1 in the form of two exercises (one for the one-arm exponent, and one for the others).

EXERCISE III.3.1 (One-arm exponent for percolation). *Consider critical site percolation on the triangular lattice; we want to prove that*

$$\mathbb{P}(0 \leftrightarrow \partial\Lambda_n) = n^{-\frac{5}{48}+o(1)}$$

where $\partial\Lambda_n$ is the boundary of the box of size n. We assume that the result of exercise 3 is known.

(1) *Let $\pi(r,R)$ be the probability that there exists a path between $\partial\Lambda_r$ and $\partial\Lambda_R$. Show that there exists a constant $c > 0$ uniform in r and R such that*

$$c\pi(r_1,r_2)\pi(r_2,r_3) \leqslant \pi(r_1,r_3) \leqslant \pi(r_1,r_2)\pi(r_2,r_3).$$

Hint. For the left side of the inequality, use RSW.

(2) *Consider percolation on a finite subgraph of the triangular lattice with circular shape. Explain how to define a natural exploration process at a discrete level. Towards which process should it converge (we do not ask for a proof!!)?*

(3) *What is the event associated to $\{\partial\Lambda_r \leftrightarrow \partial\Lambda_R\}$ for the exploration process? Show that there exist $0 < c_1, c_2 < \infty$ such that for every R,*

$$c_1 R^{-\frac{5}{48}} \leqslant \pi(n, nR) \leqslant c_2 R^{-\frac{5}{48}}$$

when n is large enough.

(4) *Conclude the proof.*

EXERCISE III.3.2 (Universal exponents). *Let σ be a finite sequence of colors (B for black, W for white). We associate to $n > 0$ and $\sigma = \{\sigma_1, \ldots, \sigma_k\}$ the event $A_\sigma(n)$ that there exist paths $\gamma_1, \ldots, \gamma_k$ such that:*

- *γ_i has color σ_i*
- *γ_i connects the origin to the boundary of $[-n,n]^2$. When $k>2$, we require only that the paths connect $[-k,k]^2$ to the boundary of $[-n,n]^2$.*
- *$\gamma_1,\dots,\gamma_k$ can be found in counterclockwise order.*

We define the same event in the upper half-plane (which we denote by $A^{\mathbb{H}}_\sigma(n)$). In this case, the paths must be found in counterclockwise order, starting from the right.

(1) *a) Prove that $\mathbb{P}(A^{\mathbb{H}}_{BW}(n)) \geqslant \frac{c}{n}$ for some universal constant c.* Hint. Use the RSW theorem to construct a point in $\{0\}\times[-n/2,n/2]$ which is connected to the boundary of the box by two arms of distinct colors.

b) Assume $A^{\mathbb{H}}_{BW}(n)$ holds. We require that the site on the left of 0 is white and that it is the start of the white path, and the site on the right is black and is the start of the black path. Show that one can explore the interface between the black and the white paths without exploring any other site;

c) Let $B(n)$ be the event that there exist a white path connected to the left side of $[-n,n]\times[0,n]$ and a black path connected to the right side. Show that there exists a universal $c_1>0$ such that

$$\mathbb{P}(A^{\mathbb{H}}_{BW}(n)) \leqslant c_1 \mathbb{P}(B(n));$$

d) Deduce that there exists $c_2>0$ such that

$$\mathbb{P}(A^{\mathbb{H}}_{BW}(n)) \leqslant \frac{c_2}{n}.$$

What was proved?

(2) *Prove that the exponent for $BWBWW$ in the plane is 2.*

(3) (difficult) *Prove that the exponent for $BWBW$ in the plane is smaller than 2.*

III.4. Bibliographical notes

Section III.1. The argument outlined in the beginning of the section is that initially described by Smirnov [**35, 36**], and more details, including the necessary technicalities involved in the proof of convergence of the exploration process to the trace of SLE_6, are in the article of Camia and Newman [**11**]. The general method is fully described in a paper by Lawler, Schramm and Werner [**25**], applied in the cases of the LERW and the UST Peano curve.

Section III.2. In addition to the previously mentioned articles, we simply refer the reader to the notes for Smirnov's Buzios course [**38**] again.

Section III.3. The polychromatic percolation exponents were obtained from SLE by Werner and Smirnov [**39**]; the existence of the monochromatic ones was proved in [**8**].

Part IV . MATHEMATICAL TOOLBOX

We gather in this section a few exercises from the sessions in Buzios, which provide some mathematical background for the main body of these notes. Somehow every result proved here is classical in some mathematical communities, but which community depends on the result — SLE itself being at the interface between probability and complex analysis.

IV.1. Probabilistic tools

IV.1.1. Stochastic calculus and Itô's formula. In this section, B is a standard one-dimensional Brownian motion. Let $(\mathcal{F}_t)$ be the filtration associated to the Brownian motion, *i.e.* $\mathcal{F}_t = \sigma(B_s, s < t)$. The process (M_t) is a *martingale* (with respect to $\mathcal{F}_t$) if for each $s < t$, $\mathbb{E}[|M_t|] < \infty$ and $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$.

EXERCISE IV.1.1 (integration with respect to Brownian motion).

(1) *We call H a* simple process *if it is of the form*

$$H_s = \sum_{j=1}^{n} C_j \mathbb{1}_{[t_{j-1},t_j)}(s)$$

where (t_j) is increasing and C_j is $\mathcal{F}_{t_{j-1}}$-measurable;

a) For a (random) process $H = C\mathbb{1}_{[s,t)}$, where C is $\mathcal{F}_s$-measurable, find a natural candidate for the integral of H against Brownian motion B, in other words, what could $\int_0^\infty H_s dB_s$ be? How could the notion of integral be extended to any simple process?

b) We assume that the integral has been constructed as above. For any simple process H, check that

$$\mathbb{E}\left[\left(\int_0^\infty H_s dB_s\right)^2\right] = \int_0^\infty \mathbb{E}[H_s^2]ds.$$

(2) *Let $\mathcal{L}^2$ the set of square integrable adapted processes (*i.e., *processes (H_s) satisfying $\int_0^\infty \mathbb{E}[H_s^2]ds < \infty$). Explain how to extend the definition of integral to $\mathcal{L}^2$.*

(3) *For a bounded adapted process H, we define $\int_0^t H_s dB_s$ as $\int_0^\infty H_s \mathbb{1}_{[0,t)} dB_s$. Show that $M_t = \int_0^t H_s dB_s$ is an $\mathcal{F}_t$-martingale.* Hint: Check it in the case of simple processes first. *** Show that it is a continuous process.*

Remark: *Note that for any bounded adapted process a, $\int_0^t a_s ds$ is straightforward to define. It is also possible to check that $H_t = \int_0^t a_s dB_s + \int_0^t \sigma_s ds$ is a martingale if and only if $\sigma = 0$.*

(4) *a) Let H_s be a bounded continuous adapted process and $t > 0$. Considering subdivisions $0 = t_1^n < \cdots < t_n^n = t$ with $\max(t_{i+1}^n - t_i^n) \to 0$, show that*

$$\sum_{i=1}^{n-1} H_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n}) \xrightarrow{\mathcal{L}^2} \int_0^t H_s dB_s.$$

b) Let H_s be a bounded continuous adapted process and $t > 0$. Considering subdivisions $0 = t_1^n < \cdots < t_n^n = t$ with $\max(t_{i+1}^n - t_i^n) \to 0$, show that

$$\sum_{i=1}^{n-1} H_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n})^2 \xrightarrow{\mathcal{L}^2} \int_0^t H_s ds.$$

Hint: Recall that $B_t^2 - t$ is a martingale.

c) Prove Itô's formula: For any a, σ bounded adapted processes and $t > 0$, we set $Y_t = \int_0^t a_s dB_s + \int_0^t \sigma_s ds$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function twice continuously derivable; then

$$\varphi(Y_t) = \varphi(Y_0) + \int_0^t \varphi'(Y_s) a_s dB_s + \int_0^t \left[\varphi'(Y_s)\sigma_s + \frac{1}{2}\varphi''(Y_s) a_s^2\right] ds.$$

Remark: *In order to write the equality*

$$H_t = x + \int_0^t a_s dB_s + \int_0^t \sigma_s ds$$

in a concise way, we often write

$$H_0 = x \quad \text{and} \quad dH_t = a_t dB_t + \sigma_t dt.$$

IV.1.2. An application: Bessel processes.

EXERCISE IV.1.2. *Let $d > 0$. We* ***assume*** *without proof that there exists a unique process, denoted X_t^x, which solves the following stochastic differential equation:*

$$dX_t^x = dB_t + \frac{d-1}{2X_t^x} dt, \quad X_0^x = x$$

*up to time $T_x \coloneqq \inf\{t : X_t^x = 0\}$. This process is called a d-*dimensional Bessel process*. For integer values of d, this process is the norm of a d-dimensional vector with independent Brownian entries. Let $0 < a < x < b < \infty$, τ the first exit time of the set $[a, b]$, and $\varphi(x) = \mathbb{P}(X_\tau^x = a)$.*

(1) *Show that $\varphi(X_{t\wedge\tau}^x)$ is a martingale with respect to $\mathcal{F}_{t\wedge\tau}$.*
(2) *a) Assume φ is twice continuously differentiable. Using Itô's formula, deduce that*

$$\frac{1}{2}\varphi''(x) + \frac{d-1}{2x}\varphi'(x) = 0, \quad a < x < b,$$

and compute φ when $d \geqslant 2$,
b) When $d > 0$, compute $\mathbb{P}(X_\tau^x = a)$. What can you deduce?
(3) (difficult) *Using Itô's formula, show that $\psi(x, t) = \mathbb{P}_x(\tau > t)$ is the solution of a partial differential equation. Deduce an estimate for $\mathbb{P}_x(\tau > t)$ when t goes to infinity.*

IV.2. Complex analytic tools

IV.2.1. Conformal maps.

EXERCISE IV.2.1 (Around the Riemann mapping theorem). *Recall the statement of the RMT: Let D and D' be two simply connected domains included in $\mathbb{C}$ and different from $\mathbb{C}$, there exists a conformal map (*i.e.* a bijection differentiable in the complex variable) between D and D'.*

(1) *Find a conformal map between the following domains:*
  - *from $\mathbb{R}\times]0, \pi[$ to $\mathbb{H} = \{z, \mathrm{Im}(z) > 0\}$;*
  - *from the disk $\mathbb{D} = \{z, |z| < 1\}$ to $\mathbb{H}$;*
  - *from $\mathbb{H} \setminus [0, ir]$ to $\mathbb{H}$;*
  - *from $\mathbb{D}$ to $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$;*
  - *from $S_\epsilon = \big(\mathbb{R} \times (0, 2)\big) \setminus \big((i - \infty, i - \varepsilon] \cup [i + \varepsilon, i + \infty)\big)$ to $\mathbb{H}$*
  - *from $\mathbb{H}$ to an equilateral triangle.*
(2) *a) Show that there is no conformal map from $D(0, 1)$ to $\mathbb{C}$.* It confirms that the assumption $D \neq \mathbb{C}$ is necessary.
b) Let D be a simply connected domain and f be a conformal map; why is $f(D)$ simply connected?

(3) *What are the conformal maps from $D(0,1)$ into $D(0,1)$?* Hint. One can guess what they are and make sure none is omitted using Schwarz's Lemma. *Deduce that there are three (real) degrees of freedom in the choice of a conformal map between two domains in the following sense:*
- *one can fix the image of one point on the boundary and the image of one point inside the domain;*
- *one can fix the image of one point inside the domain and the direction of the derivative;*
- *one can fix the image of three points on the boundary (keeping the order).*

Exercise IV.2.2 (Estimates for conformal maps).

(1) ***Schwarz' Lemma:*** *Let f be a continuous map from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ such that $f(0) = 0$ and f is holomorphic inside $\mathbb{D}$. Show that $|f(z)| \leqslant |z|$.* Hint. Think about the maximum principle. *Study the case where $|f'(0)| = 1$.*

(2) ***Koebe's*** $1/4$***-theorem:*** *Let*

$$\mathcal{S} := \{f : \mathbb{D} \to \mathbb{C}, \textit{analytic, one-to-one with } f(0) = 0 \textit{ and } f'(0) = 1\}$$

a) (Area theorem) Let $f \in \mathcal{S}$ and $K = \mathbb{C} \setminus \{1/z, z \in f(\mathbb{D})\}$, prove that

$$\mathit{area}(K) = \pi \left[1 - \sum_{n=1}^{\infty} n|b_n|^2 \right]$$

where $1/f(1/z) = z + b_0 + \sum_{n\geqslant 1} \frac{b_n}{z^n}$. Note that it implies $|b_1| \leqslant 1$.

b) Prove that if $f = z + a_2 z + \cdots$ is in $\mathcal{S}$, then $|a_2| \leqslant 2$. Hint: construct a function $h \in \mathcal{S}$ such that $h(z) = z + \frac{a_2}{2} z^3 + \cdots$ and conclude.

c) Deduce Koebe's $1/4$-theorem: if $f \in \mathcal{S}$, then $B(0, \frac{1}{4}) \subset f(\mathbb{D})$.

d) Suppose $f : D \to D'$ is a conformal transformation with $f(z) = z'$. Then

$$\frac{1}{4} \frac{d(z', \partial D')}{d(z, \partial D)} \leqslant |f'(z)| \leqslant 4 \frac{d(z', \partial D')}{d(z, \partial D)}.$$

IV.2.2. Interaction with Brownian motion.

Exercise IV.2.3 (Conformal invariance of Brownian motion). *Consider B a Brownian motion in the plane, and for a domain U, $\tau_U := \inf\{t \geqslant 0 : B_t \notin U\}$ the exit time of U. A conformal map is a bijective biholomorphic map. In this exercise, we prove the following theorem:*

Theorem IV.2.1. *Let $z \in U$, and let $f : U \to V$ be conformal. The law of $\{f(B_t), t \leqslant \tau_U\}$ is the same as the law of the trace of a Brownian motion in V from $f(z)$ to the boundary.*

Let $\tilde{B}$ be an independent Brownian motion in the plane; introduce the time changes

$$\zeta_s := \int_0^s |f'(B_u)|^2 du \quad \textit{and} \quad \sigma_t := \inf\{s \geqslant 0 : \zeta_s \geqslant t\}.$$

Define

$$W_t = f(B(\sigma_t \wedge \tau_U)) + \tilde{B}(t) - \tilde{B}(\zeta[\sigma_t \wedge \tau_U]), \quad t \geqslant 0.$$

It is sufficient to prove that W is a Brownian motion.

(1) *What does the previous construction boil down to?*

(2) *Show that W is continuous. Let $\mathcal{G}_s$ be the σ-algebra generated by the set $\{W_u, u \leqslant s\}$. Show that W is a $(\mathcal{G}_s)$-Brownian motion if and only if*

$$\mathbb{E}\left[\ e^{\langle \lambda, W_t\rangle} \mid W_s = f(z)\ \right] = \exp\left(\ \frac{1}{2}|\lambda|^2(t-s) + \langle \lambda, f(z)\rangle\ \right)$$

for every $z \in U$. The quantity $\langle u, v\rangle$ corresponds to the usual scalar product between two complex numbers. In order to simplify, we prove the statement only for $s = 0$. Consider a conformal map f from U to $f(U)$. Assume first that f and f' are uniformly bounded.

(3) *a) Show that for every $z \in U$,*

$$\mathbb{E}\left[\ e^{\langle \lambda, W_t\rangle} \mid W(0) = f(z)\ \right]$$
$$= \mathbb{E}_z\left[\exp\left(\ \frac{1}{2}|\lambda|^2(t - \zeta(\sigma_t \wedge \tau_U)) + \langle \lambda, f(B(\sigma_t \wedge \tau_U))\rangle\ \right)\right]$$

where $\mathbb{P}_z$ is the law of a Brownian motion starting at z.

b) Prove that for every $z \in U$,

$$\Delta e^{\langle \lambda, f(z)\rangle} = |\lambda|^2 |f'(z)|^2 e^{\langle \lambda, f(z)\rangle}.$$

c) Using the two-dimensional Itô formula (see below), show that

$$M_s = \exp\left(\frac{1}{2}|\lambda|^2[t - \zeta(s \wedge \tau_U)] + \langle \lambda, f(B(s \wedge \tau_U))\rangle\right)$$

is a bounded martingale and conclude.

(4) *Where did we use the assumption that f and f' are uniformly bounded? How could one get rid of the assumption on f and f'?*

(5) *What is the probability that a Brownian motion starting at ϵ exits the domain $\mathbb{D} \setminus [-1, 0]$ through $\partial\mathbb{D}$?*

(6) *a) Explain how one could define the Brownian motion in a simply connected domain D between two boundary points a and b. We denote by $\mathbb{P}^{BM}_{(D,a,b)}$ this measure;*

b) Sketch a proof of the following conformal invariance property: let (D, a, b) be a simply connected domain and f a conformal map; then

$$f \circ \mathbb{P}^{BM}_{(D,a,b)} = \mathbb{P}^{BM}_{(f(D),f(a),f(b))};$$

c) Make explicit a construction when $D = \mathbb{H}$, $a = 0$ and $b = \infty$;

d) Let K be a compact set such that $H = \mathbb{H} \setminus K$ is a simply connected domain containing 0 (the set H is called a hull*). Prove that*

$$\mathbb{P}^{BM}_{(\mathbb{H},0,\infty)}(B \textit{ stays inside } H) = \varphi'_H(0)$$

where φ_H is the map from H to $\mathbb{H}$ that maps 0 to 0 and with $\varphi_H(z) \sim z$ when z goes to infinity.

THEOREM IV.2.2 (Itô's formula in dimension 2). *Let $B = B^{(1)} + iB^{(2)}$ be a two-dimensional Brownian motion. Let Σ be an increasing continuous adapted process and*

$$X_t = \int_0^t a_s^{(1)} dB_s^{(1)} + i \int_0^t a_s^{(2)} dB_s^{(2)}.$$

Let $f : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{R}$ be a smooth function; then

$$f(X_t, \Sigma_t) = f(X_0, \Sigma_0) + \sum_{i=1}^{2} \int_0^t \partial_i f(X_s, \Sigma_s) a_s^{(i)} dB_s^{(i)} \\ + \int_0^t \partial_t f(X_s, \Sigma_s) d\Sigma_s + \frac{1}{2} \sum_{i=1}^{2} \int_0^t \partial_{ii}^2 f(X_s, \Sigma_s) \big(a_s^{(i)}\big)^2 ds.$$

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