N° D'ORDRE : 7135

UNIVERSITÉ PARIS XI
UFR SCIENTIFIQUE D'ORSAY

THÈSE

présentée pour obtenir

le GRADE de DOCTEUR EN SCIENCES

DE L'UNIVERSITÉ PARIS XI – ORSAY

par

Vincent BEFFARA

Sujet : MOUVEMENT BROWNIEN PLAN, SLE, INVARIANCE CONFORME ET DIMENSIONS FRACTALES

Rapporteurs : M. Geoffrey R. GRIMMETT
M. Gregory F. LAWLER

Soutenue le 10 janvier 2003 devant la Commission d'examen composée de

M. Bertrand DUPLANTIER (Examinateur)
M. Richard KENYON (Examinateur)
M. Gregory F. LAWLER (Rapporteur)
M. Jean-François LE GALL (Examinateur)
M. Yves LE JAN (Président)
M. Wendelin WERNER (Directeur de Thèse)

Remerciements / Acknowledgments

Je tiens tout d'abord à remercier Wendelin Werner pour son soutien constant et ses conseils toujours judicieux. C'est peu de dire que cette thèse n'aurait pas pu se faire sans lui, et je ne sais quoi dire d'autre que ceci (une fois de plus, et toujours aussi sérieusement) : "Ah, patron, c'que vous êtes fort !"

I wish to thank Oded Schramm (who, despite what he might tell you, *is* the 'S' in SLE — but such things are not mine to decide after all) for many precious comments on my work. The week he invited me to spend in Redmond was very inspiring, and most of all he showed me the light when the sky was dim by convincing me that it was after all possible to compute SLE dimensions in the way I intended to do it.

I wish to thank Greg Lawler for everything he meant to this thesis. First of all, his ideas are omnipresent throughout the text, and many of the main arguments were his in one way or another. He never refused me his help and advice. Then he accepted to be one of the referees, and to participate in the jury. And, last but not least, he likes Simon and Garfunkel.

I wish to thank Geoffrey Grimmett for accepting to be one of the referees. I discovered "true" (as in "non-academic") probability in his book on percolation, and the magic is still going on. And I could discover in Jyväskylä that he has more stamina for teaching than I would even dare dream of having.

I wish to thank all the organizers and participants of the Mittag-Leffler program on probability and conformal mappings, for making those months in Djursholm both instructive and most enjoyable. Among them, I will fondly remember many enlightening discussions with our beloved "patriarch" Peter Jones ; I only regret that his insight on SLE did not lead to more than a few pages in this thesis.

(Bon, d'accord, j'arrête de remercier les gens en anglais ...) Je remercie Bertrand Duplantier, Richard Kenyon, Jean-François Le Gall et Yves Le Jan pour avoir accepté d'être dans mon jury. Chacun d'entre eux a contribué d'une façon ou d'une autre à l'élaboration de cette thèse — parfois sans le savoir, mais toujours de manière déterminante.

Je remercie tous ceux qui m'ont supporté et aidé à Orsay, et en particulier la bande de thésards passés et présents du bâtiment 430, de l'autre côté du parking. J'ai passé au bureau 110 (et trop souvent aussi dans la salle de café, mais ne le dites pas à Wendelin) des années géniales qui m'ont marqué tant du point de vue professionnel que personnel. Je remercie aussi les membres du DMA, qui m'accueillent depuis quelques mois.

Enfin, je souhaite dédier cette thèse à Stéphane Hoguet, qui m'a fait aimer les mathématiques pour les "bonnes" raisons ; tous ceux qui l'ont connu me comprendront.

Abstract

This thesis is dedicated to the study of various geometric properties of planar Brownian motion and the SLE process (also known as stochastic Loewner evolution, or sometimes as Schramm's process).

We prove that, on a typical planar Brownian path, there almost surely exist "pivoting" points, *i.e.* cut-points around which one half of the curve can rotate by a positive angle without ever intersecting the other half of the path; the set of all pivoting points of a given positive (small enough) angle is then of positive Hausdorff dimension. In fact, for every subset A of the complex plane, we describe an exceptional subset E_A of the path, defined in a geometric fashion. For each such A we define a *generalized intersection exponent* $\xi(A)$ and prove that $\dim_H(E_A) = 2 - \xi(A)$, so that E_A is non-empty as soon as $\xi(A) < 2$.

About SLE, the main result we obtain in this thesis is the computation of the Hausdorff dimension of its *trace* (*i.e.* of the curve generating it); that dimension is equal to $1 + \kappa/8$, where κ is the parameter of the SLE — and this holds for any positive parameter smaller than 8 and different from 4 (for $\kappa \geqslant 8$, the trace is a Peano curve hence has dimension 2). In passing we prove the almost sure existence of cut-points on every SLE with parameter smaller than 8.

We also study the problem of the generalization of the SLE process to non-simply connected domains; we show that the construction is doable for two particular values of the parameter ($\kappa = 6$ and $\kappa = 8/3$), using in each case specific properties of the corresponding SLE (respectively, the *restriction property* and *locality*), but the *universality* property of usual SLE is then lost.

Keywords : (Planar) Brownian motion, SLE, conformal invariance, critical exponents, Hausdorff dimension.

MSC2000 classification : 60D05, 60G17, 60G51, 60G57, 60G99, 28A80

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Chapitre 1

Introduction

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1.1 Position du problème

Une des principales questions qui se posent en mécanique statistique est celle de la détermination de limites d'échelle (*scaling limits*). D'une façon volontairement informelle, le problème est le suivant : on considère un modèle aléatoire discret défini sur un réseau, et on cherche à obtenir des informations sur le comportement à grande échelle du système — ou, ce qui est équivalent dans la plupart des cas, sur son comportement quand on fait tendre le pas du réseau vers zéro. Deux phénomènes peuvent alors apparaître :

- Ou bien le modèle devient déterministe, et on obtient des résultat comme des lois des grands nombres et des estimées de grandes déviations qui décrivent la convergence vers cet état déterministe ;
- Ou bien la limite reste aléatoire, ce qui signifie que le système donne naissance à un objet aléatoire continu que l'on cherche alors à identifier et à étudier de manière intrinsèque ; ses propriétés fournissent alors en retour des informations sur le système discret.

En pratique, le problème de l'existence même d'une limite ne semble pas avoir de solution générale, la convergence de chaque modèle particulier réclamant une preuve différente ; mais il est parfois possible, en admettant l'existence d'une limite, d'identifier cette dernière de manière exacte.

La distinction entre ces deux cas est particulièrement apparente dans l'étude des *transitions de phase* pour des systèmes de particules sur un réseau. Le système physique dépend alors d'un paramètre réel qui mesure la force d'interaction entre les particules. Intuitivement, si le paramètre est petit, le comportement à grande échelle du système est le même que celui de particules indépendantes, alors que s'il est grand, tous les sites vont avoir tendance à s'aligner. Dans les deux cas, on observe alors une limite d'échelle déterministe, mais pour des raisons différentes (on observe soit un comportement *moyen*, soit un comportement *commun*).

Pour certains modèles, il existe alors un valeur particulière du paramètre, dite *point critique*, correspondant à la transition entre les deux régimes, et pour lequel aucun de ces deux phénomènes ne se produit. La limite d'échelle du système au point critique produit alors un objet aléatoire continu.

Dans le cas ou le réseau considéré est de dimension 2, les physiciens ont développé des outils particuliers pour décrire le comportement de tels systèmes au point critique, comme par exemple les théories de champs conformes ([18]) et la gravitation quantique ([14]). Le résultat le plus surprenant est que l'objet limite ne dépend pas du choix du réseau mais seulement de la dimension du modèle. En particulier, il devient invariant pas rotation, et comme il est défini par une limite d'échelle il se transforme de manière simple par homothétie.

Cela a amené les physiciens à la notion d'*invariance conforme* : comme le comportement local d'une transformation conforme est essentiellement la composition d'une rotation et d'une homothétie, on obtient ainsi des informations sur l'image de l'objet continu par une transformation conforme du domaine où il est défini. En particulier, si cet objet est invariant par homothétie, ces considérations heuristiques donnent une bonne raison de croire qu'il est aussi invariant par transformation conforme.

Remarque : Le lien entre criticalité et invariance par changement d'échelle est particulièrement visible dans le cadre des *groupes de renormalisation*. Le cas le plus simple est l'opération de décimation dans $\mathbb{Z}^d$, qui consiste à définir le modèle sur $\mathbb{Z}^d$, pour un certain paramètre λ, et ensuite à ne conserver que les sites qui se trouvent sur un sous-réseau $(a\mathbb{Z})^d$, ou a un entier supérieur ou égal à 2. Cette opération correspond intuitivement à un changement d'échelle de facteur a pour la limite continue (toujours dans le cas où celle-ci existe) ; mais il se trouve que souvent le comportement du système discret sur le sous-réseau est proche de son comportement sur le réseau initial pour une autre valeur du paramètre, disons $\phi_a(\lambda)$. Il est alors naturel de considérer les paramètres qui sont les points fixes de ϕ_a, car ils seront les seuls à fournir une limite continue invariante par changement d'échelle — l'équation $\phi_a(\lambda) = \lambda$ sert alors de *définition* du point critique. (Cf. par exemple [13, ex. III.38, p. 527] pour un calcul explicite dans le cas du modèle d'Ising.)

En admettant l'existence de la limite d'échelle ainsi que l'invariance de celle-ci par transformation conforme, Schramm [43] prouve alors le résultat suivant : la limite est décrite par un processus aléatoire qu'il nomme *SLE* (pour *Stochastic Loewner Evolution*), et dont la loi ne dépend que d'un seul paramètre réel positif κ. Autrement dit, à tout système discret "raisonnable" (au sens où il admet une limite d'échelle qui est invariante par transformation conforme) il est possible d'associer une valeur de κ, qui joue le même rôle que la charge centrale dans le formalisme physique, de telle sorte que la limite d'échelle du système soit le processus SLE_κ.

La question de la description de la limite se ramène alors à trois problèmes *a priori* mieux posés : prouver qu'il y a effectivement convergence (en un sens à préciser), que le système est bien invariant par transformation conforme à la limite, et identifier la valeur du paramètre κ correspondant.

La majeure partie de cette thèse est consacrée à l'étude du processus SLE (en français, processus de Loewner stochastique ou processus de Schramm). Dans la suite de cette introduction, nous introduisons ce processus ainsi que certains objets discrets et continus dont les liens avec le SLE sont soit connus soit conjecturés ; puis nous présentons les résultats obtenus ainsi qu'une rapide description des outils mathématiques utilisés. Enfin nous donnons un plan général de la thèse et un résumé du contenu de chacun des chapitres.

1.2 Le mouvement brownien plan

1.2.1 Invariance conforme

Le cas le plus simple pour lequel on sait décrire une limite d'échelle continue est celui de la marche aléatoire simple. En effet, il est possible de prouver que, si l'on prend une marche aléatoire $(S_k)_{0\leqslant k\leqslant n}$ de longueur n dans le réseau carré $\mathbb{Z}^2$, issue de 0, et qu'on l'interpole par une fonction de $[0,1]$ dans $\mathbb{R}^2$ en posant

$$S^n(t) = \frac{S_{\lfloor nt\rfloor}}{\sqrt{n/2}},$$

alors la suite de fonctions $(S^n)_{n\geqslant 0}$ converge en loi, pour la topologie de Skorohod, vers un mouvement brownien plan $(B_t)_{t\in[0,1]}$ issu de 0 (*i.e.* le processus limite s'écrit $B_t = (B_t^1, B_t^2)$ où (B_t^1) et (B_t^2) sont des mouvements browniens réels standards issus de 0).

C'est ici qu'un miracle se produit : la loi du mouvement brownien plan est invariante par rotation (alors que celle de la marche aléatoire ne l'est pas, puisque le réseau lui-même ne l'est pas). De plus, l'effet d'un changement d'échelle de facteur $\lambda > 0$ sur le mouvement brownien est le même que celui d'un changement de temps linéaire de facteur λ^2 — et la loi de la courbe à paramétrisation près est donc invariante. On est donc dans le cadre exact où l'on peut espérer l'invariance du processus par transformation conforme (en identifiant $\mathbb{R}^2$ au plan complexe $\mathbb{C}$). C'est effectivement ce qui se produit :

Théorème 1.1 (Invariance conforme du mouvement brownien plan) :

(i). Soit $(B_t)_{t\geqslant 0}$ un mouvement brownien plan issu de 0, et soit Φ une fonction entière telle que $\Phi(0) = 0$. Alors, il existe un mouvement brownien plan $(W_t)_{t\geqslant 0}$ issu de 0 tel que
$$\forall t \geqslant 0 \quad \Phi(B_t) = W_{\int_0^t |\Phi'(B_s)|^2\, \mathrm{d}s} \; ;$$

(ii). Soient Ω et Ω' deux ouverts bornés simplement connexes de $\mathbb{C}$ contenant 0, Φ une application conforme de Ω sur Ω' fixant 0 et (B_t) un mouvement brownien plan issu de 0. Soit τ (resp. τ') le premier temps de sortie de Ω (resp. Ω') par

B. Alors,

$$\{\Phi(B_t), t \in [0, \tau]\} \stackrel{\text{(loi)}}{=} \{B_{t'}, t' \in [0, \tau']\}\,.$$

1.2.2 Points exceptionnels de la courbe brownienne

Il est alors naturel de s'intéresser à des propriétés géométriques de la courbe brownienne plane, en en particulier à celles qui sont préservées par transformation conforme. Nous reviendrons plus tard à la description de la frontière brownienne, pour donner ici la description de quelques sous-ensemble particuliers de la trajectoire. Soient donc à nouveau Ω un ouvert borné simplement connexe de $\mathbb{C}$ contenant 0, et $(B_t)_{t\geqslant 0}$ un mouvement brownien plan ; soit τ son premier temps de sortie de Ω. On notera $K = B_{[0,\tau]}$ la courbe décrite par B. K est un compact connexe de $\mathbb{C}$, et on sait qu'il est de dimension 2 et de mesure nulle.

Définition :

On dit que $B_t \in K$ est un *point de coupure* (resp. que $t \in [0, \tau]$ est un *temps de coupure*) de la trajectoire si $K \setminus \{B_t\}$ n'est pas connexe. (En particulier, 0 et τ ne sont *pas* des temps de coupure.)

Il est facile de voir que, pour tout temps $t \geqslant 0$ fixé, la probabilité que t soit un temps de coupure est égale à 0. Le théorème de Fubini nous dit alors que l'ensemble $\mathcal{T}$ des temps de coupure est presque sûrement de mesure nulle (on parle d'un *ensemble exceptionnel*), et on peut se demander s'il est vide ou non.

Théorème 1.2 (Burdzy [7] ; Lawler-Schramm-Werner [25, 31, 32]) :

L'ensemble $\mathcal{T}$ est presque sûrement non vide ; sa dimension de Hausdorff est presque sûrement égale à 3/8.

La preuve initiale du fait que $\mathcal{T}$ soit non vide, due à Burdzy, est très technique et ne donne pas d'information sur la dimension de $\mathcal{T}$. Nous décrivons ici celle de Lawler, Schramm et Werner, car la méthode générale est proche de celle que nous employons pour calculer la dimension du processus de Schramm. L'idée, qui est due à Lawler, est de calculer d'abord la dimension de Hausdorff de l'ensemble $\mathcal{T}$, puis de constater qu'elle est strictement positive, ce qui implique en particulier que $\mathcal{T}$ est non vide.

De manière générale, la détermination d'une borne supérieure pour la dimension d'un ensemble E (aléatoire ou non) est souvent plus facile que celle d'une minoration ; en effet, il suffit d'exhiber, pour tout $\varepsilon > 0$, un recouvrement de E par au plus $\varepsilon^{-\alpha}$ disques de rayon ε pour prouver que la dimension de Minkowski (et donc aussi celle de Hausdorff) de E est au plus égale à α. Dans le cas où E est un compact aléatoire contenu dans le carré $[0, 1]^2$, on peut procéder de la façon suivante.

Supposons que, pour tous $x \in [0, 1]^2$ et $\varepsilon > 0$, on ait

$$(H_1) \qquad P\left(E \cap \mathcal{B}(x, \varepsilon) \neq \varnothing\right) \asymp \varepsilon^s$$

(où le signe $\asymp$ signifie que le rapport des deux expressions est borné inférieurement et supérieurement par des constantes indépendantes de x et ε). Pour tout $\varepsilon > 0$ on peut fixer un recouvrement du carré par au plus $42\varepsilon^{-2}$ disques de rayon ε ; chacun de ces disques rencontre E avec une probabilité de l'ordre de ε^s, et par conséquent l'espérance du nombre de ces disques qui rencontrent E est de l'ordre de ε^{s-2}.

Soit alors $N_\varepsilon(E)$ le nombre minimal de disques de rayon ε nécessaires pour recouvrir E : si (H_1) est réalisée, on a donc, pour une certaine constante $C > 0$,

$$E(N_\varepsilon(E)) \leqslant C.\varepsilon^{s-2}.$$

(En fait on pourrait, sous les mêmes hypothèses, obtenir aussi une borne inférieure du même ordre — mais nous n'en aurons pas besoin ici.) Par conséquent, en appliquant l'inégalité de Bienaymé-Tchébychev, on obtient pour tout $\eta > 0$ l'estimation suivante :

$$P(N_\varepsilon(E) > \varepsilon^{s-2-\eta}) \leqslant C.\varepsilon^\eta.$$

En choisissant alors une suite de rayons (ε_k) qui décroisse assez rapidement (par exemple $\varepsilon_k = 2^{-k}$), on peut alors appliquer le théorème de Borel-Cantelli : presque sûrement, pour k assez grand, il est possible de recouvrir E par au plus $\varepsilon_k^{s-2-\eta}$ disques de rayon ε_k, et par conséquent, la dimension de E est presque sûrement inférieure ou égale à $2 - s + \eta$.

Comme cela est vrai dès que $\eta > 0$, on obtient une borne supérieure de la forme

$$(H_1) \quad \Rightarrow \quad P(\dim_H(E) \leqslant 2 - s) = 1$$

ainsi qu'une bonne indication du fait que la dimension de E devrait être égale à $2 - s$.

Pour obtenir une borne inférieure, la seule méthode praticable est la construction d'une *mesure de Frostman* portée par E — *i.e.*, d'une mesure positive μ, de masse finie non nulle, telle que pour tous $x \in [0,1]^2$ et $r > 0$, on ait

$$\mu(\mathcal{B}(x,r)) \leqslant C.r^\alpha$$

pour une certaine constante $C > 0$ et un certain exposant α. En effet il est facile de voir que s'il existe une telle mesure de support inclus dans E, alors la dimension de Hausdorff de E est au moins égale à α.

L'idée est alors la suivante : on a une famille de mesures "naturelles" μ_ε définies par

$$\mathrm{d}\mu_\varepsilon(x) = \varepsilon^{-s}\mathbf{1}_{\mathrm{d}(x,E)\leqslant\varepsilon}\,|\mathrm{d}x|$$

(où $|\mathrm{d}x|$ est la mesure de Lebesgue sur le carré), qui ont une masse d'ordre 1 par l'hypothèse (H_1), et qui satisfont, pour tous $x \in [0,1]^2$ et $r > 2\varepsilon$,

$$E\left(\mu_\varepsilon(\mathcal{B}(x,r)) | \mu_\varepsilon(\mathcal{B}(x,r)) > 0\right) \asymp \frac{E(\mu_\varepsilon(\mathcal{B}(x,r)))}{P(E \cap \mathcal{B}(x,r) \neq \varnothing)} \asymp \frac{r^2}{r^s} \asymp r^{2-s}.$$

Autrement dit, μ_ε se comporte en moyenne comme une mesure de Frostman d'exposant $2 - s$. Le but du jeu sera alors d'extraire de la famille (μ_ε) une sous-suite qui converge faiblement vers une mesure μ de masse totale positive, et de prouver que celle-ci est vraiment une mesure de Frostman portée par E.

Pour ce faire, on a besoin d'informations sur le comportement typique de μ_ε, connaissant son comportement moyen, autrement dit il nous faut une borne supérieure pour la variance de $\mu_\varepsilon(\mathcal{B}(x,r))$. Celle-ci sera fournie par une hypothèse sur les moments d'ordre deux de la loi de E ; plus précisément, si on a, pour tous x et y dans le carré et pour tout $\varepsilon > 0$,

$$(H_2) \qquad P\left(E \cap \mathcal{B}(x,\varepsilon) \neq \varnothing \text{ et } E \cap \mathcal{B}(y,\varepsilon) \neq \varnothing\right) \leqslant C.\varepsilon^s \wedge C.\frac{\varepsilon^{2s}}{|x-y|^s},$$

alors on peut prouver que la construction précédente peut être effectuée avec une probabilité positive :

$$(H_1, H_2) \quad \Rightarrow \quad P(\dim_H(E) = 2 - s) > 0.$$

On peut alors souvent obtenir un résultat presque sûr en appliquant une loi du zéro-un (souvent disponible puisque la définition de la dimension de Hausdorff est essentiellement locale).

Dans le cas des temps de coupure de la trajectoire brownienne, on doit en fait modifier légèrement la construction, en introduisant des temps de coupure approchés :

$$\mathcal{T}_\varepsilon \triangleq \{t \in [0, \tau] : B_{[0,t-\varepsilon]} \cap B_{[t+\varepsilon,\tau]} = \varnothing\}.$$

L'hypothèse (H_1) est alors remplacée par une estimation de la probabilité que t soit dans $\mathcal{T}_\varepsilon$, mais le principe général de la preuve est le même. En particulier, la condition (H_2) est une conséquence directe de (H_1) et de la propriété de Markov, et donc il "suffit" de prouver que (H_1) est satisfaite et de calculer la valeur de s.

Par un retournement du temps en t, puis en appliquant le scaling brownien, on obtient une définition équivalente de s, qui est la suivante. Soient B^1 et B^2 deux mouvements browniens plans indépendants, issus respectivement de 1 et -1 (ou de points uniforméments distribués sur le cercle-unité). On note T^1_R (resp. T^2_R) le premier temps d'atteinte du cercle $\mathcal{C}(0, R)$ par B^1 (resp. B^2). Alors,

$$P\left(B^1_{[0,T^1_R]} \cap B^2_{[0,T^2_R]} = \varnothing\right) \asymp R^{-2s}$$

(le doublement de l'exposant étant dû au fait que les temps d'arrêt utilisés ici sont définis à partir de propriétés spatiales de la trajectoire ; ils sont en effet de l'ordre de R^2). L'exposant $2s$ porte le nom d'*exposant d'intersection* brownien, et il est noté ailleurs $\xi(1, 1)$.

D'une manière générale, dans de nombreux modèles de mécanique statistique pris au point critique, la décroissance de certaines quantités (fonctions de corrélation, probabilités de certains événements) est également gouvernée par de tels exposants, dits *exposants critiques*. Au contraire, pour des systèmes en dehors du point critique, le comportement usuel de ces quantités présentera une décroissance exponentielle.

Il est possible, en utilisant un argument de sous-additivité, de prouver l'existence de s (la méthode est présentée dans le chapitre 2 de cette thèse dans un cas plus général) ; Lawler prouve alors dans [25] que l'on a effectivement, avec probabilité 1,

$$\dim_H(\mathcal{T}) = 1 - \frac{\xi(1, 1)}{2}.$$

Comme on peut montrer (cf. par exemple [46]) que s est strictement inférieur à 1 sans le calculer explicitement, cela prouve l'existence de points de coupure. Le calcul de la valeur exacte de l'exposant ($s = 5/8$ ici), et donc le calcul exact de la dimension de $\mathcal{T}$, repose sur les rapports entre le mouvement brownien plan et le processus SLE, que nous décrivons dans la prochaine section.

Par une méthode similaire, il est possible de relier la dimension de la frontière brownienne à la valeur d'un *exposant de déconnexion* η, défini de la façon suivante : soient toujours B^1 et B^2 deux mouvements browniens plans, on note $\Omega(t,t')$ l'unique composante connexe infinie du complémentairre de $B^1_{[0,t]} \cup B^2_{[0,t']}$. Alors, pour un certain $\eta \in (0,2)$, on a

$$P\left(0 \in \Omega(T^1_R, T^2_R)\right) \asymp R^{-\eta}$$

(*i.e.*, l'exposant η décrit la décroissance de la probabilité que les deux trajectoires ne séparent pas 0 de l'infini). On a alors un expression de la dimension de la frontière brownienne, elle aussi due à Lawler ([24]) :

$$\dim_H(\partial\Omega(1,0)) = 2 - \eta.$$

Le calcul de η repose alors également sur les liens entre mouvement brownien et SLE, qu'il est donc temps de décrire.

1.3 Le processus de Loewner stochastique

1.3.1 Le théorème de Loewner

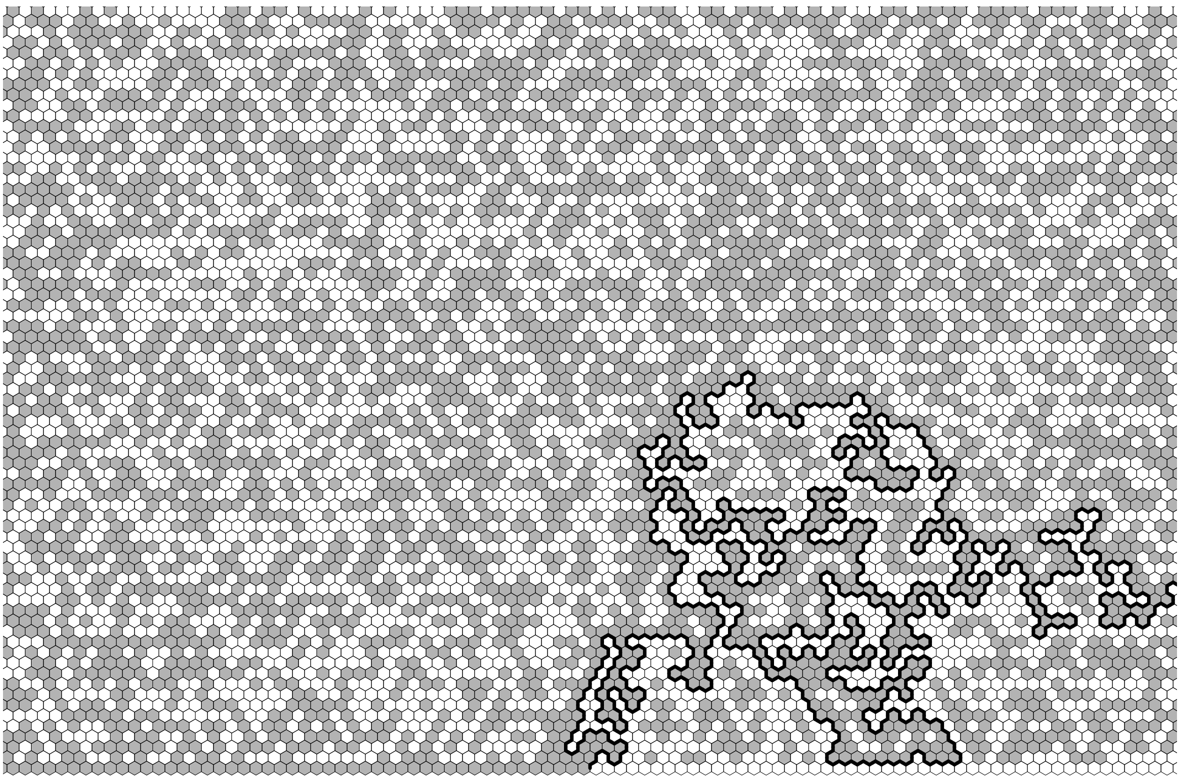

FIG. 1.1 – Exploration d'un modèle de percolation critique

L'intuition qui conduit à la construction du SLE provient de l'étude de *courbes d'exploration* définies à partir d'un système de mécanique statistique. Par exemple, considérons un modèle de percolation critique par sites sur une discrétisation du demi-plan supérieur par le réseau triangulaire (ce qui revient à considérer un réseau hexagonal et à en colorier chacune des faces, de manière indépendante, en blanc ou en noir avec probabilité 1/2 — cf. Fig. 1.1). Si l'on conditionne les hexagones situés le long de l'axe réel positif à être blancs et ceux situés le long de l'axe réel négatif à être noirs, ceci définit alors une frontière entre les clusters blancs touchant l'axe réel positif et les clusters noirs touchant l'axe réel négatifs (en gras sur la figure). Cette frontière est une courbe, dite courbe d'exploration de la frontière, et elle constitue un des objets pour lesquels on recherche une limite d'échelle.

On cherche alors un outil servant à décrire une courbe dans le demi-plan supérieur, issue de l'origine, et qui n'a pas de croisements (en un sens à préciser). Soit donc γ une fonction continue de $\mathbb{R}_+$ dans $\bar{\mathbb{H}} = \{z \in \mathbb{C} : \Im z \geqslant 0\}$. À chaque instant $t \geqslant 0$, le complémentaire de $\gamma([0,t])$ dans $\mathbb{H}$ est un ouvert qui a exactement une composante connexe infinie, H_t ; on note K_t le *remplissage* de $\gamma([0,t])$, qui est défini comme étant l'adhérence de $\mathbb{H} \setminus H_t$. La condition de non-croisement peut alors s'écrire :

$$\forall 0 < s < t, \qquad \gamma(t) \in \bar{H}_s$$

(ce qui signifie que la courbe après le temps s ne pénètre plus dans l'intérieur de K_s).

Pour tout $t \geqslant 0$, l'ouvert H_t est simplement connexe. Par conséquent, on peut appliquer le théorème de Riemann : il existe une unique application conforme g_t de H_t dans $\mathbb{H}$ ayant un développement asymptotique à l'infini de la forme $g_t(z) = z + o(1)$. Le terme suivant du développement asymptotique est alors

$$g_t(z) = z + \frac{2a(t)}{z} + \mathcal{O}(z^{-2}),$$

où a est une fonction continue croissante et positive ou nulle. Dans le cas où a est strictement croissante (ce qui se produit par exemple quand γ est une courbe simple), il est alors possible de faire un changement de temps de façon à avoir, pour tout $t \geqslant 0$, $a(t) = t$ — ce que nous supposerons dorénavant. Avec cette normalisation, la famille d'applications conformes $(g_t)_{t \geqslant 0}$ satisfait une équation différentielle dite *équation de Loewner dans le demi-plan* ; plus précisément on a le

Théorème 1.3 (Loewner) :

Il existe une fonction réelle continue $\beta : \mathbb{R}_+ \to \mathbb{R}$, avec $\beta(0) = 0$, telle que (g_t) soit le flot de l'équation différentielle ordinaire dans le demi-plan supérieur :

$$(L_\beta) \qquad y'(t) = \frac{2}{y(t) - \beta(t)}.$$

On dira que la courbe γ, ou la fonction β, *engendre le flot* (g_t).

Autrement dit, pour tout $z \in \mathbb{H}$ on a $g_0(z) = z$, et pour tous z, t tels que l'équation (L_β) avec condition initiale $y(0) = z$ ait une solution jusqu'au temps t, on a

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \beta(t)}.$$

Le théorème de Loewner permet donc de décrire un objet bidimensionnel (une courbe dans le plan complexe) par deux fonctions réelles, l'une décrivant une paramétrisation naturelle de la courbe et l'autre décrivant la croissance de cette courbe suivant cette paramétrisation. La plupart du temps on ne s'intéresse en fait à γ qu'à paramétrisation près, et on aboutit alors à une description de la courbe par *une* fonction réelle.

On trouvera par exemple dans [15] un énoncé plus général du théorème de Loewner (où l'on part d'une famille croissante de compacts (K_t) satisfaisant une condition technique "naturelle" exprimant que la croissance est locale, mais sans supposer l'existence de γ) ainsi que sa preuve.

Remarque : Il est possible de généraliser la construction précédente à une courbe à l'intérieur d'un domaine simplement connexe de $\mathbb{C}$, pour décrire une courbe joignant deux points du bord du domaine, en envoyant ce domaine de manière conforme sur le demi-plan supérieur (les deux points marqués correspondant alors à 0 et ∞). La courbe peut alors être décrite par une fonction continue à valeurs dans le bord du domaine.

Il existe également une version *radiale* de l'équation de Loewner, décrivant une courbe joignant un point du bord d'un domaine à un point de l'intérieur (la version précédente est dite *chordale*). Dans le cas du disque unité $\mathbb{U}$, quand le point de l'intérieur est l'origine du plan complexe, l'équation différentielle devient alors

$$(L'_\beta) \qquad y'(t) = y(t)\frac{y(t)+\beta(t)}{y(t)-\beta(t)},$$

avec $\beta : \mathbb{R}_+ \to \partial\mathbb{U}$, et toues les applications conformes g_t admettent 0 comme point fixe.

1.3.2 Définition du processus SLE

Admettons pour l'instant que la courbe d'exploration de la percolation critique décrite plus haut admette une limite continue qui soit une courbe dans le demi-plan supérieur. On peut alors paramétrer cette courbe de manière naturelle, et il existe alors une fonction réelle continue (aléatoire) β qui lui est associée par l'équation de Loewner. L'hypothèse d'invariance conforme (cf. par exemple [23]) peut alors s'exprimer de la façon suivante : la courbe γ sur l'intervalle de temps $[t, +\infty]$, conditionnellement à $\gamma([0,t])$, a la même loi que l'image de la courbe dans le demi-plan par l'application conforme de $\mathbb{H}$ dans H_t qui envoie 0 sur $\gamma(t)$.

Mais cette application n'est autre que $z \mapsto g_t^{-1}(z+\beta(t))$. Autrement dit, l'hypothèse d'invariance conforme entraîne naturellement la condition suivante :

$$g_{t+s} - \beta(t+s) \overset{\text{(loi)}}{=} [g_t - \beta(t)] \circ \left[\tilde{g}_s - \tilde{\beta}(s)\right], \tag{1.1}$$

où la famille $(\tilde{g}_s)$ est une copie indépendante de (g_t) et où $\tilde{\beta}$ est la fonction réelle associée à $(\tilde{g}_s)$. En considérant le développement asymptotique à l'infini de l'égalité précédente, on obtient

$$\beta(t+s) \overset{\text{(loi)}}{=} \beta(t) + \tilde{\beta}(s),$$

ce qui implique que β est stationnaire à accroissements indépendants. Comme de plus on sait que β est continue, et que β et $-\beta$ ont même loi (puisque la situation discrète fournit une courbe dont la loi est clairement symétrique), ceci est suffisant pour dire que β est un mouvement brownien réel, à un changement de temps linéaire près.

Ceci fournit une justification heuristique à la définition suivante :

Définition :

Soit $(B_t)_{t\geqslant 0}$ un mouvement brownien standard issu de 0, et soit κ un nombre réel positif fixé. On appelle *SLE chordal de paramètre κ dans $\mathbb{H}$*, ou SLE_κ dans $\mathbb{H}$, le flot associé à l'équation différentielle de Loewner (L_β) avec $\beta(t) = \sqrt{\kappa}B_t$.

La condition (1.1) joue alors le rôle d'une propriété de Markov pour le processus SLE.

Remarque : De manière équivalente, on pourra aussi appeler SLE la famille croissante de compacts (K_t) associée à (g_t).

Il est également possible de définir un *SLE radial* à partir de l'équation de Loewner radiale (L'_β), en prenant pour $(\beta(t))$ un mouvement brownien sur le cercle-unité — *i.e.* en posant $\beta(t) = \exp(i\sqrt{\kappa}W_t)$ où (W_t) est un mouvement brownien réel standard. Nous nous concentrerons ici essentiellement sur la version chordale ; les liens entre les deux versions sont profonds et encore mal compris (cf. [32]). En particulier, la plupart des propriétés géométriques du SLE chordal que nous décrivons par la suite sont vraies pour le SLE radial de même paramètre.

1.3.3 Quelques propriétés du SLE

Nous donnons ici sans démonstrations quelques propriété géométriques satisfaites par le SLE chordal. Les preuves se trouvent essentiellement dans [42] pour $\kappa \neq 8$ et dans [34] pour $\kappa = 8$.

Proposition 1.1 (Existence de la trace) :

Soit $\kappa \geqslant 0$, et soit (g_t) un SLE chordal de paramètre κ. Avec probabilité 1 il existe une courbe continue sans croisements $\gamma : \mathbb{R}_+ \to \bar{\mathbb{H}}$ qui engendre le flot (g_t), au sens du Théorème 1.3. Cette courbe est appelée *trace* du SLE.

Proposition 1.2 (Transitions de phase pour $\kappa = 4$ et $\kappa = 8$) :

Soit γ la trace d'un SLE_κ. Alors, presque sûrement :
- Si $0 \leqslant \kappa \leqslant 4$, la courbe γ est simple ;
- Si $4 < \kappa < 8$, la courbe γ a des points doubles mais elle est de mesure nulle ;
- Si $8 \leqslant \kappa$, la fonction γ est surjective de $\mathbb{R}_+$ sur $\bar{\mathbb{H}}$.

Ceci peut également se lire sur les compacts (K_t) associés à (g_t) : si $\kappa \leqslant 4$, alors $K_t = \gamma([0,t])$ est lui-même une courbe ; si $4 < \kappa < 8$, K_t est de mesure positive, et on a $\gamma([0,t]) \subsetneq K_t$, *i.e.* K_t est obtenu en prenant la réunion de $\gamma([0,t])$ et de toutes les composantes connexes bornées de son complémentaire (les "bulles" formées par la courbe) ; enfin si $8 \leqslant \kappa$ on a à nouveau $K_t = \gamma([0,t])$.

Le SLE a été introduit pour décrire les limites d'échelle de certains modèles de mécanique statistique en dimension 2 ; et de fait la convergence est connue dans un certain nombre de cas. (Toutes les convergences décrites ici sont en loi, dans un espace de courbes continues définies à paramétrisation près.)

Proposition 1.3 (Convergences vers le SLE) :

(i). La courbe d'exploration de la percolation critique par sites sur le réseau triangulaire (décrite plus haut) converge vers la trace d'un SLE dans $\mathbb{H}$ pour le paramètre $\kappa = 6$;

(ii). La marche à boucles effacées tuée à son premier temps de sortie du disque unité (cf. [34]) converge vers un SLE radial dans $\mathbb{U}$ pour le paramètre $\kappa = 2$;

(iii). La courbe de Peano uniforme, *i.e.* la courbe d'exploration d'un arbre couvrant uniforme (cf. [34] aussi) converge vers la trace d'un SLE pour le paramètre $\kappa = 8$.

De plus, on conjecture (cf. [35]) que la marche auto-évitante uniforme de longueur infinie dans le demi-plan (à supposer qu'elle existe) converge vers un SLE de paramètre $\kappa = 8/3$.

Enfin, il existe un lien profond entre SLE et mouvement brownien plan. L'expression exacte de ce lien nécessite l'introduction de plusieurs notations — mais "moralement" la courbe du $SLE_{8/3}$, la frontière du SLE_6 et la frontière extérieure du mouvement brownien plan ont la même géométrie locale. En particulier, trois modèles discrets *a priori* très différents ont presque la même limite d'échelle : la marche aléatoire simple, la marche auto-évitante uniforme et la percolation critique. Ceci n'est en fait pas surprenant du point de vue de la physique, puisque tous trois sont dans la même *classe d'universalité*, celle des modèles de charge centrale nulle.

Le lien entre $SLE_{8/3}$ et SLE_6 est une instance d'une relation plus générale : on conjecture que, pour tout $\kappa \in (4, 8]$, la géométrie locale de la frontière d'un SLE_κ est la même que celle de la courbe d'un $SLE_{16/\kappa}$. Cela est connu pour $\kappa = 6$ et pour $\kappa = 8$ — dans ce dernier cas, la preuve passe par la convergence des modèles discrets associés, et on peut voir la dualité comme une conséquence de l'algorithme de Wilson.

1.4 Résultats obtenus et plan général

1.4.1 Sur le mouvement brownien plan

Le chapitre 2 de cette thèse est largement indépendant des autres, il est consacré à l'étude de certains points exceptionnels sur la trajectoire d'un mouvement brownien plan, qui sont une généralisation de la notion de point de coupure. Pour $\alpha > 0$, on dit qu'un point B_t de la trajectoire brownienne $(B_s)_{s\in[0,1]}$ est un *point pivot d'angle* α si l'on a, pour tout $\theta \in [-\alpha/2, \alpha/2]$,

$$\Big[B_{[0,t)} - B_t\Big] \cap e^{i\theta} \Big[B_{(t,1]} - B_t\Big] = \varnothing.$$

Autrement dit, B_t est un point de coupure de la trajectoire, et l'image de l'une des deux moitiés par une rotation d'angle θ autour de B_t reste disjointe de l'autre moitié tant que $\theta \in [-\alpha/2, \alpha/2]$. (Cf. Fig. 1.2 pour une image dans le cas $\alpha = \pi/2$.)

On prouve alors le résultat suivant :

Théorème 1.4 :

Pour tout $\alpha > 0$ suffisamment petit, il existe presque sûrement sur la courbe brownienne plane des point pivots d'angle α, et ceux-ci forment un ensemble de dimension de Hausdorff strictement positive.

Si α_0 désigne le plus grand angle pour lequel de tels points existent, alors on a

$$\alpha_0 \geqslant \frac{(\log 2)^2}{2\pi}.$$

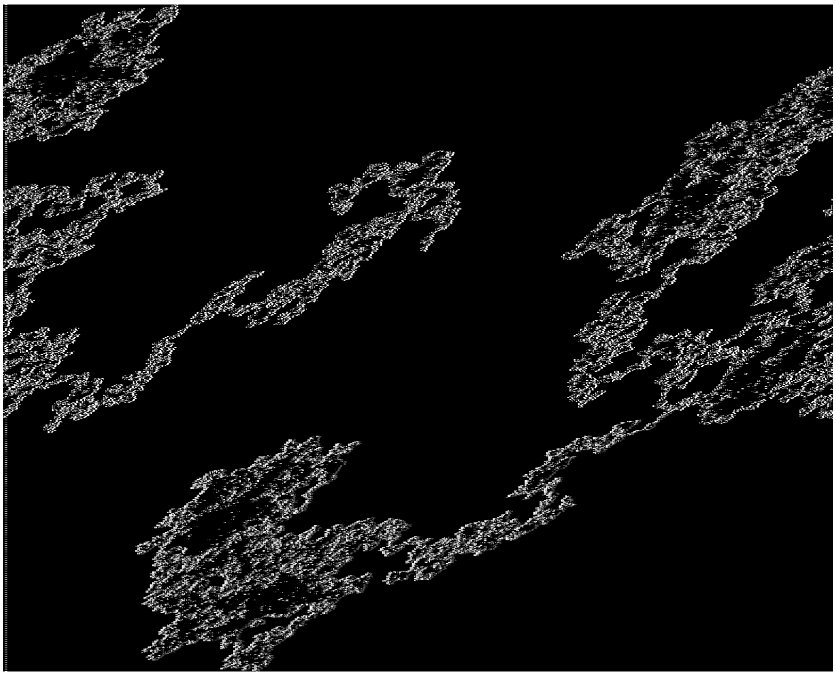

FIG. 1.2 – Un point pivot d'angle $\pi/2$
(En gris : l'image d'une moitié de la trajectoire par une rotation d'angle $+\pi/2$.)

Il semble que l'angle limite soit plutôt de l'ordre de $3\pi/4$, donc beaucoup plus grand que la borne obtenue de manière rigoureuse ici. La méthode générale est similaire à celle présentée plus haut dans le cas des points pivots, avec plusieurs complications techniques essentiellement dues au fait que le centre de la rotation qui intervient dans la définition est lui-même aléatoire.

On prouve en fait l'existence, pour tout α, d'un exposant d'intersection généralisé $\xi(\alpha)$, défini de la façon suivante. Soient (B^1_t) et (B^2_t) deux mouvements browniens plans, issus de -1 et $+1$ respectivement, et soit T^1_R (resp. T^2_R) le premier temps d'atteinte du cercle $\mathcal{C}(0,R)$ par B^1 (resp. B^2). Par un argument assez technique on prouve que la probabilité

$$p_R \triangleq P\left[B^1_{[0,T^1_R]} \cap \bigcup_{|\theta|\leqslant\alpha} e^{i\theta} B^2_{[0,T^2_R]} = \varnothing\right]$$

satisfait une relation de sous-multiplicativité "dans les deux sens", de la forme

$$c_- p_R p_{R'} \leqslant p_{RR'} \leqslant c_+ p_R p_{R'}$$

avec $0 < c_- < c_+ < \infty$. Il existe par conséquent un exposant $\xi(\alpha)$ décrivant la décroissance de p_R quand R tend vers $+\infty$, *i.e.* défini par

$$p_R \asymp R^{-\xi(\alpha)}.$$

On a alors à prouver que $\xi(\alpha)$ dépend de α de manière continue, et à utiliser ceci à deux reprises par la suite :

- Pour prouver que la dimension de l'ensemble des points pivots d'angle α est égale à $2-\xi(\alpha)$ (c'est la continuité de ξ qui permet de prendre en compte l'aspect aléatoire du centre de rotation) ;

– Pour prouver que $\xi(\alpha)$ est strictement inférieur à 2 pour α assez petit, en effet l'exposant $\xi(0)$ est l'exposant d'intersection pour deux mouvements browniens plans, noté ailleurs $\xi(1,1)$, et on sait (cf. [32]) qu'il est égal à $5/4$.

Puisqu'un point pivot est nécessairement sur la frontière extérieure de la trajectoire, et que l'on sait que la frontière brownienne est étroitement reliée à celle du SLE_6 et à la trajectoire du $SLE_{8/3}$, les résultats de ce chapitre s'appliquent également à ces deux objets ainsi qu'aux modèles discrets associés. En particulier, si on admet la convergence de la marche auto-évitante vers le $SLE_{8/3}$ et le fait que $\alpha_0 > \pi/2$, on a prouvé l'existence de "beaucoup" de points pivots d'angle $\pi/2$ (donc visibles au niveau discret) sur une marche auto-évitante typique.

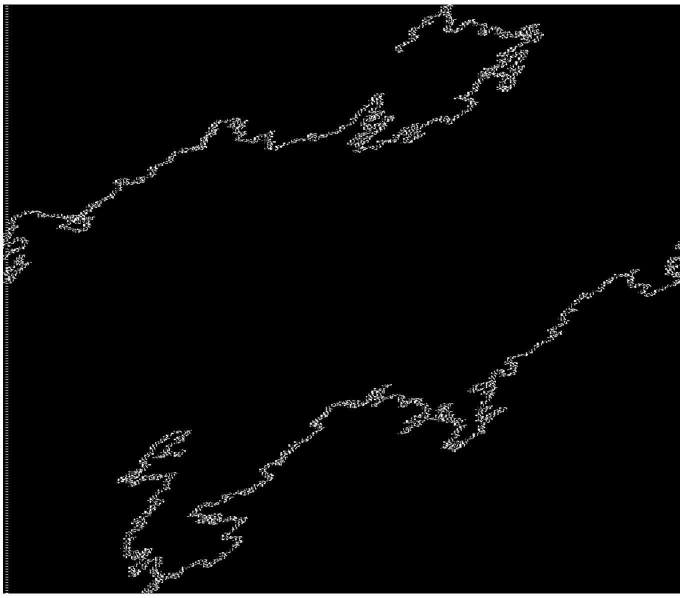

Fig. 1.3 – Une marche auto-évitante uniforme (obtenue par la méthode du pivot)

Cela donne alors des informations sur l'*algorithme du pivot*, qui est le seul algorithme efficace pour simuler une telle marche auto-évitante. Il s'agit d'un algorithme de Monte-Carlo, où l'on procède de la manière suivante. On part d'un chemin simple $(\omega_k)_{0\leqslant k\leqslant n}$ quelconque dans $\mathbb{Z}^2$, et à chaque étape on choisit un point ω_k uniformément sur ce chemin et un angle α uniformément dans $\{0, \pi/2, \pi, 3\pi/2\}$. Si, après rotation de $\{\omega_k, \ldots, \omega_n\}$ d'un angle α autour de ω_k, le chemin obtenu est encore simple, on le garde, et sinon on annule la rotation.

On obtient ainsi une chaîne de Markov dans l'espace Ω_n des chemins simples de longueur n dans $\mathbb{Z}^2$. Il est facile de voir que la mesure uniforme sur Ω_n est réversible pour cette chaîne de Markov ; il n'est *pas* facile de voir que la chaîne est irréductible (et en fait elle ne l'est pas si on exclut la rotation d'angle π — cf. [39]) mais c'est bien le cas. Par conséquent il y a convergence en loi vers la mesure uniforme. C'est ainsi que la figure 1.3 a été obtenue.

Le fait qu'il y ait "beaucoup" de pivots sur la courbe (de l'ordre d'une puissance de n) dit alors que la vitesse de convergence de la chaîne est assez rapide ; inversement, les estimations de la vitesse de l'algorithme présentées dans [39] suggèrent qu'on a effectivement

$\alpha_0 > \pi/2$ — ce qui est d'ailleurs cohérent avec les simulations présentées à la fin du chapitre 2.

1.4.2 Sur le processus de Schramm

Le principal résultat obtenu ici sur le processus SLE concerne la dimension de Hausdorff de la courbe γ. On prouve en effet le théorème suivant :

Théorème 1.5 :

Soit γ la trace d'un SLE_κ avec $\kappa \geqslant 0$, $\kappa \neq 4$. Presque sûrement, la dimension de Hausdorff de l'image $\gamma(\mathbb{R}_+)$ de la courbe est égale à

$$\dim_H \gamma(\mathbb{R}_+) = \left(1 + \frac{\kappa}{8}\right) \wedge 2.$$

On calcule également la dimension du bord du compact K_t dans certains cas :

Théorème 1.6 :

Soit (K_t) un SLE_κ avec $\kappa \in \{6, 8\}$. Alors pour tout $t > 0$, presque sûrement, la dimension de Hausdorff du bord de K_t est égale à

$$\dim_H \partial K_t = 1 + \frac{2}{\kappa}.$$

Dans le chapitre 3, nous étudions plus particulièrement le cas du processus de Schramm pour le paramètre $\kappa = 6$. Dans ce cas les preuves des conditions (H_1) et (H_2) (énoncées plus haut) dans le cas de la trace — et donc aussi la preuve du théorème 1.5 — sont rendues plus faciles par deux propriétés spécifiques au SLE_6, que nous décrivons ici rapidement.

• L'*équivalence entre SLE_6 radial et SLE_6 chordal* : tant que γ ne sépare pas un point marqué de l'intérieur du domaine d'un point marqué sur le bord du domaine, les deux versions correspondantes du SLE_6 ont même loi à changement de temps près. Ceci permet de réécrire l'événement présent dans la condition (H_1) (toucher une boule de centre x et de rayon ε) comme portant sur un SLE radial croissant en direction de x. La probabilité de l'événement peut alors s'interpréter comme probabilité de survie pour une diffusion dans un intervalle, ce qui se ramène à une détermination de la valeur propre principale du générateur associé.

• La *propriété de localité* du SLE_6 : elle exprime en substance que la croissance de (K_t) au temps t dans le cas $\kappa = 6$ ne dépend pas de la forme globale du domaine mais seulement de sa géométrie locale au voisinage de $\gamma(t)$. Cela permet de dire que les deux événements définissant (H_2) (toucher respectivement $\mathcal{B}(x, \varepsilon)$ et $\mathcal{B}(y, \varepsilon)$) sont "moralement indépendants", ce qui permet de voir la condition (H_2) comme conséquence de la condition (H_1).

Puis nous appliquons les mêmes propriétés du processus SLE_6 pour obtenir directement la dimension du bord de K_t dans ce cas. On a toujours $(H_1) \Rightarrow (H_2)$ par localité, mais le calcul de l'exposant s décrivant la probabilité de toucher un disque de rayon ε est ici plus problématique. On est en fait amené à étudier un problème annexe portant sur une diffusion réelle dans un intervalle, qui est assez naturel dans le cadre présenté ici mais ne semble pas avoir été traité indépendamment.

Le problème est le suivant. Soit (X_t) la diffusion sur $(-1, 1)$ définie par

$$(D) \qquad \mathrm{d}X_t = \sigma \mathrm{d}B_t + f(X_t)\mathrm{d}t,$$

avec $\sigma > 0$ et $f : (-1, 1) \to \mathbb{R}$ qui fasse de chaque extrémité de l'intervalle une frontière absorbante pour X, et satisfasse quelques conditions de régularité (une liste est donnée dans la section 3.1.2, mais elle est loin d'être optimale). Cette diffusion définit un flot (g_t), *i.e.* pour tout $t > 0$ l'application g_t est un difféomorphisme d'une partie I_t de I sur une partie J_t de I, de sorte que pour tout $x \in I$, $(g_t(x))$ soit une solution forte de (D) issue de x.

L'estimée usuelle porte sur la probabilité de survie en temps long ; on prouve de manière générale que $P(0 \in I_t)$ décroît exponentiellement vite,

$$P(0 \in I_t) \asymp e^{-\lambda t}$$

où $-\lambda$ est la valeur propre principale du générateur de la diffusion. On prouve également, en utilisant la formule de Feynman-Kac, que pour tout $b > 0$,

$$E\left((g_t'(0))^b\right) \asymp e^{-\lambda(b)t}$$

où cette fois $-\lambda(b)$ est la valeur propre principale de l'opérateur

$$\mathcal{L}_b : h \mapsto \frac{\sigma^2}{2} h'' + f(x)h' - bf'h$$

(de sorte que $\lambda(0) = \lambda$). On s'intéresse alors à l'image $J_t = g_t(I_t)$ du flot au temps t. Sa longueur l_t est égale à l'intégrale de g_t' le long de I_t, donc on peut utiliser l'inégalité de Jensen pour relier $E(l_t^b)$ à $E(g_t'(x)^b)$ (on obtient soit une majoration, soit une minoration suivant que b est inférieur ou supérieur à 1). En fait, on prouve ici le résultat plus fort suivant :

Proposition 1.4 :

Pour tout $b > 0$, on a l'estimation suivante quand t tend vers $+\infty$:

$$E\left(l_t^b\right) \asymp \exp\left(-\lambda(b).t\right).$$

La même diffusion que pour le cas de la trace γ, étudiée sous cet angle avec $b = 1/3$, fournit en fait l'exposant $\lambda(b) = 2/3$ qui permet d'obtenir la condition (H_1) dans le cas du bord d'un SLE_6. On obtient ainsi une preuve plus directe du fait, conjecturé par Mandelbrot et prouvé par Lawler, Schramm et Werner, que la dimension de la frontière brownienne (qui est égale à celle du bord d'un SLE_6) est presque sûrement égale à 4/3 — en particulier, on n'utilise pas ici les exposants d'intersection browniens.

Il existe un autre cas où la condition (H_1) implique la condition (H_2), celui de certains ensembles de temps exceptionnels. En effet, la propriété de Markov permet souvent de montrer que les deux événements définissant (H_2) sont vraiment indépendants, ce qui donne une estimation du bon ordre. Nous utilisons cette approche dans deux cas où les temps considérés ont une interprétation géométrique sur la courbe. Si γ est la trace d'un SLE_κ, on dit que t est un *temps de frontière* pour γ si $\gamma(t) \in \partial K_1$, et que t est un *temps de coupure* pour γ si $K_1 \setminus \{\gamma(t)\}$ n'est pas connexe.

Théorème 1.7 :

Soit γ la trace d'un SLE_κ ; soient $\mathcal{D}$ l'ensemble de ses temps de frontière $\mathcal{T}$ l'ensemble de ses temps de coupure. Alors, presque sûrement,

$$\dim_H(\mathcal{D}) = \frac{4+\kappa}{2\kappa} \wedge 1 \qquad \text{et} \qquad \dim_H(\mathcal{T}) = \left[\frac{8-\kappa}{4} \wedge 1\right] \vee 0.$$

En particulier, si $\kappa < 8$, $\mathcal{T}$ est non vide, ce qui prouve que K_1 a presque sûrement des points de coupure.

Une question naturelle est alors la suivante : étant donné un ensemble de temps $A \subset \mathbb{R}$ borélien, aléatoire ou non, progressivement mesurable ou non *a fortiori*, y a-t-il une relation simple entre $\dim_H A$ et $\dim_H \gamma(A)$? Dans le cas du mouvement brownien plan, on sait qu'une telle relation existe, et que la dimension de l'image est le double de celle de A (cf. [22]). Dans le cas du SLE, il semble qu'il n'existe pas en général de telle relation, car le comportement métrique de γ au temps t dépend fortement de la géométrie de K_t.

Il y a cependant un cas où ce problème ne se pose plus, celui où l'ensemble A est *markovien*, au sens suivant : pour tout t, l'ensemble $A \cap [t, \infty)$ est indépendant de $\sigma(\gamma(s), s \leqslant t)$ et a même loi que $\{a+t, a \in A\}$. Dans ce cas, et pour $\kappa = 6$, la méthode décrite dans le chapitre 3 pour le calcul de la dimension du bord donne une bonne motivation pour la conjecture suivante :

$$\dim_H(\gamma(A)) = \frac{7 + 8\dim_H(A) - \sqrt{49 - 48\dim_H(A)}}{8} \ ;$$

avec les notation précédentes, cela revient à un calcul explicite de $\lambda(b)$ où b serait l'exposant permettant de déterminer $\dim_H A$, *i.e.* :

$$\dim_H(\gamma(A)) = 2 - \lambda\left(1 - \dim_H(A)\right).$$

Dans le chapitre 4, nous achevons la preuve des théorèmes 1.5 et 1.6 dans le cas (presque) général $\kappa \neq 4$. Tout ce qui facilitait la preuve dans le cas précédent ($\kappa = 6$) devient faux, et en particulier (H_2) n'est plus une conséquence directe de (H_1). En fait, deux preuves séparées de (H_2) sont nécessaires, suivant que κ est dans $(0,4)$ ou dans $(4,8)$ (notons qu'il n'y a rien à démontrer dans le cas $\kappa \geqslant 8$ puisqu'alors γ est une courbe de Peano, donc de mesure pleine, et donc de dimension 2).

Dans le dernier cas ($\kappa = 4$) il n'est pas clair que (H_2) soit vraie. Ceci peut être interprété en termes de propriétés métriques de l'application conforme g_t : on sait en effet (cf. [42]) que celle-ci est höldérienne si et seulement si $\kappa \neq 4$, et il est possible de relier cela à des propriétés géométriques du bord de K_t (absence de "fjords" arbitrairement profonds) qui rappellent fortement les estimées servant à prouver (H_2). On trouvera en appendice une discussion plus formelle de ce lien, qui suggère l'existence d'une preuve plus simple du théorème 1.5 — au moins dans le cas $\kappa < 4$.

Au passage, le cas $\kappa = 8/3$ est particulièrement intéressant : on obtient en effet directement la dimension 4/3 du $SLE_{8/3}$, sans passer ni par le SLE_6 ni par les exposants browniens. On peut alors en déduire une troisième preuve, plus directe que les deux précédentes, du fait que la frontière brownienne est presque sûrement de dimension 4/3.

Dans le chapitre 5, nous étudions le problème de la généralisation du SLE au cas d'un domaine non simplement connexe. Cela pose problème puisque la définition du processus initial repose sur le théorème de Riemann — et donc sur l'existence d'un domaine de référence (le demi-plan supérieur, le disque-unité) qui pour tout t sera l'image de g_t. Cela permet alors d'exprimer la propriété de Markov du SLE de manière naturelle, et d'obtenir le résultat d'*universalité* au sens où la famille des processus obtenus est d'écrite par un seul paramètre réel κ, qui joue le même rôle que la charge centrale dans le formalisme des théories de champs conforems.

Nous montrons que quand le domaine Ω considéré est un ouvert multiplement connexe, il existe un analogue du processus de Schramm dans les cas $\kappa = 8/3$ et $\kappa = 6$. La construction utilise dans chacun des cas une propriété spécifique du processus correspondant dans un domaine simplement connexe (respectivement, la propriété de *restriction* et la propriété de *localité*).

Dans le premier cas, le processus est simplement un $SLE_{8/3}$ usuel dans le domaine obtenu en "remplissant les trous" de Ω, conditionné à rester dans Ω. La propriété de restriction montre alors que la courbe obtenue satisfait une propriété markovienne similaire à celle du SLE. Dans le second cas, on considère un SLE_6 dans le domaine rempli, jusqu'au premier instant τ (qui est fini presque sûrement) où K_t n'est plus contenu dans $\bar{\Omega}$ et on le prolonge par un SLE dans $\Omega \setminus K_\tau$. La propriété de localité permet de prouver qu'on a également une propriété markovienne dans ce cas.

Il est à noter toutefois que ces deux processus sont "artificiels" puisqu'ils nécessitent de considérer le domaine rempli (ils ne sont par définis de manière intrinsèque). En fait, dans le cas où Ω a la topologie d'un anneau, la famille des lois de courbes aléatoires, entre deux points de la même composante de $\partial\Omega$, satisfaisant la propriété de restriction, conserve un degré de liberté (alors qu'elle est réduite à $SLE_{8/3}$ dans le cas simplement connexe) : on n'a plus d'universalité dans ce cas.

Il est possible d'adapter la preuve de Smirnov ([44]) au cas d'un domaine non simplement connexe, et de prouver que la trace du SLE_6 généralisé est encore la limite d'échelle d'un modèle de percolation critique sur le domaine (avec les conditions au bord idoines le long des "trous" du domaine). Toutefois le problème de Dirichlet-Neumann qui apparaît dans la preuve n'est pas bien posé, puisque l'on peut fixer arbitrairement la valeur de la solution le long des trous — ce qui correspond encore une fois à un *défaut d'universalité* dans le cas des domaines non simplement connexes : la géométrie locale du modèle à la limite (ou, ce qui est équivalent par le théorème 1.5, sa charge centrale) ne détermine plus entièrement la loi de la limite d'échelle et on doit prendre en compte des paramètres globaux comme (la loi de) la classe d'homotopie de γ.

Nous décrivons dans ce même chapitre le comportement de SLE_κ quand le paramètre tend vers 0 ou vers $+\infty$. Dans le premier cas, la courbe $\gamma([0,1])$ converge (pour la topologie de Hausdorff) vers celle d'un SLE_0, qui est une courbe déterministe — un segment vertical dans le cas du demi-plan supérieur, une géodésique pour la géométrie hyperbolique dans le cas général — et si on la renormalise convenablement, on obtient à la limite une courbe d'équation $x = f(y)$, où f est la convolution de la fonction qui conduit le SLE avec un noyau déterministe que nous explicitons.

Le cas $\kappa \to \infty$ est plus intéressant. On doit alors renormaliser K_1 par un facteur $\sqrt{\kappa}$ dans la direction verticale, et par un facteur $1/\sqrt{\kappa}$ dans la direction horizontale. Le compact renormalisé converge alors, toujours dans la topologie de Hausdorff, vers l'hypographe du temps local du mouvement brownien réel qui conduit K. Plus précisément, si

(L_t^x) est une version bicontinue de ce temps local, le compact limite est

$$\tilde{K} = \{x + iy : x \in \mathbb{R},\ 0 < L_1^x,\ 0 \leqslant y \leqslant 2\pi L_1^x\}.$$

Ceci relie la trace du SLE quand κ tend vers $+\infty$ à la courbe de Peano du temps local, définie par

$$\tilde{\gamma}(t) = B_t + 2\pi i.L_t^{B_t}$$

(qui est bien une courbe continue surjective car L est bicontinu). Il est probable (mais pas encore démontré) que la trace γ du SLE converge en loi vers $\tilde{\gamma}$.

On présente également un objet amusant qui est une version discrète du SLE (ou plus exactement un SLE conduit par un processus discret, plus précisément par une interpolation constante par morceaux de la marche aléatoire simple dans $\mathbb{Z}$), et qui converge vers le SLE quand le pas de discrétisation tend vers 0. Ce "SLE discret" présente lui aussi une transition de phase (ou du moins un changement d'aspect) pour le paramètre $\kappa = 4$, qui semble similaire à la transition de phase du SLE usuel : on passe de "quelque chose qui ressemble à une courbe simple" à "quelque chose qui ne ressemble pas à une courbe simple".

Cet aspect de la géométrie de l'objet discret ne suffit pas à obtenir la transition de l'objet continu — ce qui est bien dommage, car l'idée d'obtenir des informations topologiques sur K_t à partir de propriétés algébriques issues du cadre discret était plutôt attirante. Toutefois, une telle reconstruction du SLE à partir de la composition aléatoire de déformations infinitésimales simples (ici de la forme $\sqrt{z^2 + 4\varepsilon}$) pourrait être plus facile à généraliser.

L'annexe A regroupe les preuves de quelques résultats techniques ainsi que deux lemmes sur les domaines höldériens qui pourraient constituer une part significative de la "vraie preuve" du résultat du chapitre 4 — mais qui ne sont pas utilisés dans cette thèse.

Enfin, l'annexe B contient la description d'un algorithme de simulation du SLE et les images obtenues pour différentes valeurs de κ, ainsi que le code source du programme utilisé et quelques images des processus discrets associés (marche à boucles effacées, marche auto-évitante uniforme, et différents clusters de percolation critique).

Chapter 2

On Conformally Invariant Subsets of the Planar Brownian Curve

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Introduction

Theoretical physicists have conjectured for more than twenty years that conformal invariance plays an important role in understanding the behaviour of critical two-dimensional models of statistical physics. They justify by a mathematically non-rigorous argument involving renormalization, that in the scaling limit these models behave in a conformally invariant way; they have then been able to classify them via a real-valued parameter corresponding to the central charge of the associated Virasoro algebra, and to predict the exact value of critical exponents that describe the behaviour of these systems. Different models (for instance, self-avoiding walks and percolation) with the same central charge have the same exponents.

Recently, Schramm ([43]) introduced a family of new mathematical objects that give insight into these conjectures. These are random, set-valued, increasing processes $(K_t)_{t\geqslant 0}$ which he named Stochastic Loewner Evolution processes. For each positive number κ, there exists one such process of parameter κ, denoted SLE_κ. He proved that for various models, if they have a conformally invariant scaling limit, then this limit can be interpreted in terms of one of the SLE_κ's — the parameter κ would then be related to the central charge of the model. One can then interpret the conjectures of the theoretical physicists in terms of properties of SLE.

In particular, Lawler, Schramm and Werner ([31, 32]) showed that for one specific value of the parameter κ (namely $\kappa = 6$) which conjecturally corresponded to the scaling limit of percolation cluster interfaces, the SLE_6 has the remarkable *restriction property* which relates its critical exponents to the so-called intersection exponents of planar Brownian motions. This led ([31, 32, 33, 36]) to the derivation of the exact value of the intersection exponents between planar Brownian paths. Furthermore, it was later shown ([48]) that, in fact, the outer boundary of a planar Brownian curve has exactly the same law as that of an SLE_6. In other words, the geometry of critical two-dimensional percolation clusters in their scaling limit should be exactly that of a planar Brownian outer frontier.

In a very recent paper, Smirnov ([44]) showed that critical site percolation in the triangular lattice is conformally invariant in the scaling limit, so that the geometry of critical two-dimensional percolation cluster boundaries, in their scaling limit, is identical to that of a planar Brownian outer frontier.

Before all these recent developments, geometric properties of planar Brownian paths had already been the subject of numerous studies (see e.g. [38] for references). In particular, the Hausdorff dimension of various geometrically defined subsets of the planar Brownian curve had been determined. For instance, Evans ([16]) showed that the Hausdorff dimension of the set of two-sided cone points of angle θ (*i.e.* points B_t such that both $B_{[0,t]}$ and $B_{[t,1]}$ are contained in the same cone of angle θ with endpoint at B_t) is $2-2\pi/\theta$. In a series of papers (see *e.g.* the review in [27]), Lawler proved that the dimension of various important subsets of the planar Brownian curve can be related to Brownian intersection exponents. In particular ([25]), he showed that the dimension of the set C of cut points (*i.e.* points B_t such that $B_{[0,1]} \setminus \{B_t\}$ is not connected) is $2-\xi$ where ξ is the Brownian intersection exponent defined by

$$p_R = P(B^1_{[0,T^1_R]} \cap B^2_{[0,T^2_R]} = \varnothing) = R^{-\xi+o(1)} \tag{2.1}$$

(for independent Brownian paths B^1 and B^2 starting respectively from 1 and -1, T^1_R and T^2_R standing for their respective hitting times of the circle $\mathcal{C}(0,R)$).

In order to derive such results, and in particular the more difficult lower bound $d \geqslant 2-\xi$, the strategy is first to refine the estimate (2.1) to $p_R \asymp R^{-\xi}$ (we shall use this notation to denote the existence of two positive constants c and c' such that $cR^{-\xi} \leqslant p_R \leqslant c'R^{-\xi}$), then to derive second-moment estimates and finally to use these estimates to construct a random measure of finite r-energy supported on C, for all $r < 2 - \xi$. The determination of the values of the critical exponents via SLE_6 ([31, 32]) then implies that the dimension of C is 3/4. Similarly ([24]), the Hausdorff dimension of the outer frontier of a Brownian path can be interpreted in terms of another critical exponent, and the determination of this exponent using SLE_6 then implies (see [30] for a review) that this dimension is 4/3 as conjectured by Mandelbrot.

In the present paper we define and study a family of generalizations of the Brownian intersection exponent ξ parameterized by subsets of the complex plane. For each $A \subset \mathbb{C}$, we define an exponent $\xi(A)$ as follows. Let B^1 and B^2 be two independent planar Brownian paths starting from uniformly distributed points on the unit circle. Then $\xi(A)$ is defined by

$$p_R(A) = P(B^1_{[0,T^1_R]} \cap A.B^2_{[0,T^2_R]} = \varnothing) = R^{-\xi(A)+o(1)} \tag{2.2}$$

(with the notation $E_1.E_2 = \{xy : x \in E_1, y \in E_2\}$). Note that the case $A = \{1\}$ corresponds to the usual intersection exponent. In Section 2.1 we first show that for a wide class of sets A

$$p_R(A) \asymp R^{-\xi(A)}. \tag{2.3}$$

In Section 2.2 we study regularity properties of the mapping $A \mapsto \xi(A)$. In particular we prove its uniform continuity (with respect to the Hausdorff metric) on certain families of sets. One important tool for this result is the fact that the constants implicit in (2.3) can in fact be taken uniform over these families of sets.

In Section 2.3 we associate to each set A a geometrically defined subset $\mathcal{E}_A$ of the planar Brownian curve:

$$\mathcal{E}_A = \{B_t : \exists\varepsilon > 0, (B_{[t-\varepsilon,t]} - B_t) \cap A.(B_{(t,t+\varepsilon]} - B_t) = \varnothing\}.$$

Using the strong approximation and continuity of the mapping $A \mapsto \xi(A)$, we then show that the Hausdorff dimension of this subset of the planar Brownian curve is almost surely $2-\xi(A)$ when $\xi(A) \leqslant 2$ (and is 0 when $\xi(A) > 2$). For example, when $A = \{e^{i\theta}, 0 \leqslant \theta \leqslant \alpha\}$ the corresponding subset C_α of the Brownian curve is the set of (local) pivoting points, *i.e.* points around which one half of the path can rotate by any angle smaller than α without intersecting the other half.

When $A \subset A'$, then $\mathcal{E}_{A'} \subset \mathcal{E}_A$. In particular, if A contains 1, then $\mathcal{E}_A$ is a subset of the set of (local) cut points and therefore the shape of the path in a neighbourhood of such a point is the same as that of the Brownian frontier in the neighbourhood of a cut-point. This shows in particular that (at least some of) the exponents $\xi(A)$ also describe the Hausdorff dimension of sets of exceptional points of the scaling limit of critical percolation clusters.

In Section 2.4 we derive some bounds on the exponents $\xi(A)$ for small sets A, by a technique similar to that used by Werner ([47]) to estimate disconnection exponents. In particular, for small α, we show that the exponent $\xi(C_\alpha)$ is strictly smaller than 2, which implies the existence of pivoting points of any angle less than $\alpha_0 > 0$ on the planar Brownian curve. We then briefly present results of simulations which suggest that the maximal angle α_0 is close to $3\pi/4$.

It is actually easy to define other "generalized" exponents in a similar fashion, by studying non-intersection properties between Brownian motions and some of their images under isometries and scalings, *i.e.* one can view A as a subset of the linear group $GL_2(\mathbb{R})$. One can also consider non-intersection properties between B and its image $f(B)$ by a conformal map. For instance it is easy to see using the function $z \mapsto z^2$ that the exponent describing the non-intersection between B and $-B$ is in fact twice the disconnection exponent. The methods of the present paper can then be adapted to such situations.

Similarly, one could extend the definitions to higher dimensions (the cases $d \geqslant 4$ can also be interesting if the set A is sufficiently large), but conformal invariance then cannot be used, so that some of the tools in the present paper do not apply.

Acknowledgments

I wish to thank Wendelin Werner for suggesting that I look for pivoting points on the planar Brownian curve and for never refusing help and advice.

Notations

Throughout this paper, we will use the following notations for the asymptotic behaviour of positive functions (and sequences, with the same meaning):

- $f \sim g$ if $\lim_{t\to\infty} \dfrac{f(t)}{g(t)} = 1$ — and f and g are said to be *equivalent*;
- $f \approx g$ if $\log f \sim \log g$, *i.e.* if $\lim_{t\to\infty} \dfrac{\log f(t)}{\log g(t)} = 1$ — f and g are then *logarithmically equivalent*;
- $f \asymp g$ if f/g is bounded above and below, *i.e.* if there exist two positive finite constants c and C such that for all t, $cg(t) \leqslant f(t) \leqslant Cg(t)$ — which we call *strong approximation* of f by g.

2.1 Generalized intersection exponents

2.1.1 Definition of the exponents

Proposition and Definition :

Let A be a non-empty subset of the complex plane and B^1, B^2 be two independent Brownian paths starting uniformly on the unit circle $\mathcal{C}(0,1)$; define the hitting time T^i_R of $\mathcal{C}(0,R)$ by B^i and let $\tau^i_n = T^i_{\exp(n)}$,

$$E_n = E_n(A) = \{B^1_{[0,\tau^1_n]} \cap AB^2_{[0,\tau^2_n]} = \varnothing\},$$

$$q_n(A) = P(E_n) \text{ and } p_R(A) = P(E_{\log R}).$$

Then, assuming the existence of positive constants c and C such that $p_R(A) \geqslant cR^{-C}$, there exists a real number $\xi(A)$ such that, when $R \to \infty$,

$$p_R(A) \approx R^{-\xi(A)}.$$

// This is a standard sub-multiplicativity argument. If B is a Brownian path starting on $\mathcal{C}(0,1)$ with *any* law μ, then the law of $B_{\tau_1(B)}$ on the circle $\mathcal{C}(0,e)$ has a density (relative to the Lebesgue measure) bounded and bounded away from zero by universal constants (*i.e.* independently of μ). Combining this remark with the Markov property at the hitting times of the circle of radius e^n shows that:

$$\forall m, n \geqslant 1 \qquad q_{m+n} \leqslant c q_n q_{m-1}.$$

Hence the family (cq_{n-1}) is sub-multiplicative, and using Proposition 2.7 we have $q_n \approx e^{-\xi n}$, with $\xi \in (0,\infty)$, as well as a lower bound $q_n \geqslant c^{-2}e^{-\xi(n+1)}$. //

Remarks: For some choices of A there is an easy geometric interpretation of the event $E_n(A)$: $\xi(\{1\})$ is the classical intersection exponent; if $A = (0,\infty)$, the $E_n(A)$ is the event that the paths stay in different wedges.

If A is such that no lower bound $p_R(A) \geqslant cR^{-C}$ holds, we let $\xi(A) = \infty$. However, in most of the results presented here, we will restrict ourselves to a class of sets A for which it is easy to derive such lower bounds:

Definition :

A non-empty subset A of the complex plane is said to be *nice* if it is contained in the intersection of an annulus $\{r < |z| < R\}$ (with $0 < r < R < \infty$) with a wedge of angle strictly less than 2π and vertex at 0.

Indeed, let A be such a set and let $\alpha < 2\pi$ be the angle of a wedge containing A: B^1 and AB^2 will not intersect provided each path remains in a well-chosen wedge of angle $(2\pi - \alpha)/2$, and then it is standard to derive the following bound:

$$p_R(A) \geqslant cR^{-4\pi/(2\pi-\alpha)}. \tag{2.4}$$

The fact that A be contained in an annulus will be needed in the following proof. The only usual case where this does not hold is when A is a wedge itself; but in this case a direct study is possible, based on the derivation of *cone exponents* in [16] and the exact value of ξ is then known (cf. next section for details).

We will often consider the case where A is a subset of the unit circle. For such sets, A is nice if and only if $\bar{A} \subsetneq \partial\mathbb{U}$ (it is in fact easy to prove that for $A \subset \partial\mathbb{U}$, $\xi(A) = \infty$ if and only if $\bar{A} = \partial\mathbb{U}$).

2.1.2 Strong approximation

This whole subsection will be dedicated to the refinement of $p_R \approx R^{-\xi}$ into $p_R \asymp R^{-\xi}$. This is *not* anecdotal, since this "strong" approximation will be needed on several occasions later.

Theorem 2.1 :

For every nice A, $p_R(A) \asymp R^{-\xi(A)}$, *i.e.* there exist positive constants $c(A) < C(A)$ such that

$$cR^{-\xi(A)} \leqslant p_R(A) \leqslant CR^{-\xi(A)}.$$

Moreover, the constants $c(A)$ and $C(A)$ can be taken uniformly on a collection $\mathcal{A}$ of subsets of the plane, provided the elements of $\mathcal{A}$ are contained in the same nice set.

// Note that since $A \in \mathcal{A}$ is nice, the exponents $\xi(A)$ exist and are uniformly bounded for $A \in \mathcal{A}$. The sub-additivity argument showed that $q_n \geqslant ce^{-\xi(A).(n+1)}$, which implies readily the lower bound in the theorem. It is more difficult to derive the upper bound. By Proposition 2.7, it will be sufficient to find a finite constant $c_-(A)$ (that can be bounded uniformly for $A \in \mathcal{A}$) such that

$$\forall n, n' \qquad q_{n+n'} \geqslant c_- q_n q_{n'}. \tag{2.5}$$

In order to make the proof more readable, it is carried out here for a fixed A; however it is easy to see that, at each step, the constants can be taken uniformly for all A contained in some fixed nice set A_0. Moreover, we shall first assume that A_0 is a subset of the unit circle: We briefly indicate at the end of the proof what are the few modifications needed to adapt it to the general case.

The basic method is adapted from Lawler's proof for non-intersection exponents in [26], with some technical simplifications made possible using the absence of the λ exponent. The main idea is to obtain a weak independence between the behaviour of the paths before and after they reach radius e^n. The first step is an estimate concerning the probability that the paths are "well separated" when they reach radius e^n (more precisely, that they remain in two separated wedges between radius e^{n-1} and radius e^n). Let $\mathcal{F}_n$ stands for the σ-field generated by both paths up to radius e^n (so that for instance E_n is in $\mathcal{F}_n$).

Lemma (Technical) :

Let $\eta > 0$ and $\alpha < 2\pi - \eta$ such that A is contained in a wedge of angle less than α. Define

$$W_\alpha = \left\{ re^{i\theta} : r > 0, |\theta| < \frac{\alpha}{2} \right\},$$

$\delta_n = e^{-n}[\mathrm{d}(B^1_{\tau^1_n}, AB^2_{[0,\tau^2_n]}) \wedge \mathrm{d}(AB^2_{\tau^2_n}, B^1_{[0,\tau^1_n]})]$ and the following events:

$$U^1_n = \left\{ B^1_{[0,\tau^1_n]} \cap \{|z| \geqslant e^{n-1}\} \subset -W_{2\pi-\alpha-\eta} \right\},$$
$$U^2_n = \left\{ AB^2_{[0,\tau^2_n]} \cap \{|z| \geqslant e^{n-1}\} \subset W_\alpha \right\},$$

and $U_n = U^1_n \cap U^2_n$. Then:

$$\exists c, \beta > 0 \; \forall \varepsilon > 0 \; \forall r \in \left[\frac{3}{2}, 3\right] \quad P(E_{n+r}, U_{n+r} | \mathcal{F}_n, E_n, \delta_n \geqslant \varepsilon) \geqslant c\varepsilon^\beta.$$

//// The proof is easy and uses only simple estimates on Brownian Motion in a wedge, we omit the details. ////

we now prove that paths conditioned not to intersect up to radius e^{n+2} have a good chance to be well separated at this radius, uniformly with respect to their behaviour up to radius e^n:

Lemma (End-separation) :

There exists $c > 0$ such that, for every $n > 0$:

$$P(U_{n+2} | E_{n+2}, \mathcal{F}_n) \geqslant c$$

(in other words, the essential lower bound of $P(U_{n+2} | E_{n+2}, \mathcal{F}_n)$, as an $\mathcal{F}_n$-measurable function, is not less than c).

//// The technical lemma states that start-separation occurs if the starting points are sufficiently far from each other; more precisely, applying it for $r = 2$, we obtain for all $\varepsilon > 0$:

$$P(U_{n+2} | E_{n+2}, \mathcal{F}_n, \delta_n \geqslant \varepsilon) \geqslant c\varepsilon^\beta. \tag{2.6}$$

Hence, what is to be proved is that two paths conditioned not to intersect have a positive probability to be far from each other after a relatively short time. To prove this fact, one has to use conditioning on the value of δ_n.

Fix $k > 0$, and assume that $2^{-(k+1)} \leqslant \delta_n < 2^{-k}$; let τ_k be the smallest r such that one of the following happens: either $\delta_{n+r} \geqslant 2^{-k}$, or E_{n+r} does not hold. It is easy to use scaling to prove that for some $\lambda > 0$,

$$P(\tau_k \geqslant 2^{-k}) \leqslant 2^{-\lambda},$$

meaning that with positive probability (independent of k and n) the paths separate or meet before reaching radius $e^{n+2^{-k}}$. Hence, by the strong Markov property applied k^2 times, this leads to

$$P(\tau_k \geqslant k^2 2^{-k} | 2^{-(k+1)} \leqslant \delta_n < 2^{-k}) \leqslant 2^{-\lambda k^2}. \tag{2.7}$$

The technical lemma states that $P(E_{n+2}|\delta_n \geqslant 2^{-(k+1)}) \geqslant c2^{-\beta k}$: combining both estimates then leads to

$$P(\tau_k \geqslant k^2 2^{-k} | E_{n+2}, \delta_n \geqslant 2^{-(k+1)}) \leqslant c2^{\beta k - \lambda k^2}. \tag{2.8}$$

Consider now a generic starting configuration at radius e^n, satisfying E_n and hence $\delta_n > 0$. Fix also $k_0 > 0$ and introduce the radii τ_k (for $k_0 \leqslant k < \infty$) defined by

$$\tau_k = \text{Inf}\{r : \delta_{n+r} \geqslant 2^{-k}\}$$

(so that $\tau_k = 0$ as long as $2^{-k} \leqslant \delta$). Equation (2.8) can be rewritten (using the fact that the technical lemma is valid for all $r \geqslant 3/2$) as

$$P(\tau_k - \tau_{k+1} \geqslant k^2 2^{-k} | E_{n+2}, \tau_{k+1} \leqslant \frac{1}{2}) \leqslant c2^{\beta k - \lambda k^2}. \tag{2.9}$$

Fix k_0 such that

$$\sum_{k=k_0}^{\infty} k^2 2^{-k} < \frac{1}{2},$$

and sum this estimate for $k_0 \leqslant k < \infty$: this leads to

$$P(\forall k \geqslant k_0, \tau_k - \tau_{k+1} \leqslant k^2 2^{-k} | E_{n+2}) \geqslant 1 - c \sum_{k=k_0}^{\infty} 2^{\beta k - \lambda k^2}.$$

In particular, if k_0 is taken large enough, this probability is greater than 1/2, and we obtain

$$P(\tau_{k_0} \leqslant \frac{1}{2} | E_{n+2}) \geqslant \frac{1}{2}.$$

It is then sufficient to combine this and Equation (2.6) to get

$$P(U_{n+2}|E_{n+2}) \geqslant c2^{-\beta k_0} > 0,$$

and is can be seen that the obtained constant does not depend on the configuration at radius e^n — provided E_n is satisfied. ///

The first consequence of the end-separation lemma is $P(E_n, U_n) \asymp q_n$; but it is easy to see, using estimates on Brownian motion in wedges again and the strong Markov property, that

$$P(E_{n+1}|E_n, U_n) \geqslant c > 0$$

(with c independent of n), and combining both estimates leads to $q_{n+1} \geqslant cq_n$, *i.e.* $q_{n+1} \asymp q_n$. Now if $\bar{q}_n$ stands for the upper bound for the non-intersection probabilities, namely

$$\bar{q}_n \triangleq \operatorname*{Sup}_{B_0^1, B_0^2 \in \mathbb{U}} P(E_n | B_0^1, B_0^2),$$

the previous remark concerning the law of $W_{\tau_1(W)}$ can be used to prove that $\bar{q}_n \leqslant c q_{n-1}$: hence,

$$\bar{q}_n \asymp q_n.$$

Now that we know that paths conditioned not to intersect have a good chance to exit a disk at a large distance from each other, what remains to be proved is that paths starting from distant points on $\mathcal{C}(0, e^n)$ remain well separated for a sufficiently long time and become (in a sense to be specified later) independent from their behaviour before radius e^n.

Lemma (Start-separation) :

Let α and η be as in the technical lemma, $\eta' = \eta/2$ and $\alpha' = (2\pi + \alpha)/2$; introduce

$$J_n^1 = \left\{ B^1_{[0,\tau_n^1]} \cap \mathcal{B}(0,2) \subset -W_{2\pi-\alpha'-\eta'} \setminus \mathcal{B}(0, 1-\eta') \right\},$$
$$J_n^2 = \left\{ AB^2_{[0,\tau_n^2]} \cap \mathcal{B}(0,2) \subset W_{\alpha'} \setminus \mathcal{B}(0, 1-\eta') \right\},$$

and $\tilde{E}_n = E_n \cap J_n^1 \cap J_n^2$. Define $\tilde{q}_n$ as

$$\tilde{q}_n(x,y) = P(\tilde{E}_n | B^1_0 = x, B^2_0 = y).$$

Then there exists $c > 0$ such that, for all $n \geqslant 2$ and uniformly on all pairs (x,y) satisfying U_0 (*i.e.*, both having modulus 1 and such that U_0 holds when $B^1_0 = x$ and $B^2_0 = y$):

$$\tilde{q}_n(x,y) \geqslant c q_n.$$

//// Introduce the following ("forbidden") sets:

$$K^1 = (\mathcal{B}(0,2) \setminus -W_{2\pi-\alpha'-\eta'}) \cup \mathcal{B}(0, 1-\eta');$$
$$K^2 = (\mathcal{B}(0,2) \setminus W_{\alpha'}) \cup \mathcal{B}(0, 1-\eta').$$

For all n we have $J_n^1 = \{B^1_{[0,\tau_n^1]} \cap K^1 = \varnothing\}$ and $J_n^2 = \{AB^2_{[0,\tau_n^2]} \cap K^2 = \varnothing\}$. For the rest of the proof we shall fix n, and condition the paths by their starting points; introduce the following stopping times (for positive values of k):

$$T^1_0 = \mathrm{Inf}\{t > 0 : B^1_{[0,t]} \cap \mathcal{C}(0,3) \neq \varnothing\},$$
$$S^1_k = \mathrm{Inf}\{t > T^1_{k-1} : B^1_{[T^1_{k-1},t]} \cap K^1 \neq \varnothing\},$$
$$T^1_k = \mathrm{Inf}\{t > S^1_k : B^1_{[S^1_k,t]} \cap \mathcal{C}(0,3) \neq \varnothing\},$$

and S^2_k, T^2_k similarly, replacing all occurrences of B^1 by AB^2 and K^1 by K^2. We shall also use the notation N^i for the *number of crossings* by B^1 (resp. AB^2) between K^i and $\mathcal{C}(0,3)$, defined as

$$N^i \triangleq \mathrm{Max}\{k : S^i_k < \tau^i_n\}.$$

With those notations, $J^i_n = J^i_1 \cap \{N^i = 0\}$ and a.s. $N^i < \infty$. Moreover, uniformly on the starting points considered here (satisfying the condition U_0), we have $P(J^i_1) \geqslant c > 0$ by the technical lemma, where c depends only on η.

First, we split the event E_n according to the value of, say, N^2: we write $P(E_n) = \sum_{k=0}^{\infty} P(E_n, N^2 = k)$. By the Beurling estimate, on $\{N^2 \geqslant k\}$, the probability that $B^1_{[0,\tau^1_n]}$ and $AB^2_{[S^2_k, T^2_k]}$ do not intersect is bounded by some universal constant $\lambda < 1$ (which can even be chosen independent of A), independently of B^1 and the two remaining parts of B^2. By the strong Markov property at time T^2_k, when $N^2 = k$ the probability that AB^2 after T^2_k does not intersect B^1 is bounded by $P(B^1 \cap AB^2_{[T^2_0, \tau^2_n]} = \varnothing, N^2 = 0)$ (*i.e.* the path after T^2_k when $N^2 = k$ is the same as the entire path when $N^2 = 0$). Introducing those two estimates in the sum leads to

$$P(E_n) \leqslant \sum_{k=0}^{\infty} \lambda^k P(E_n, N^2 = 0) = \frac{1}{1-\lambda} P(E_n, N^2 = 0).$$

Doing this decomposition again according to N^1 (with the same constant $\lambda < 1$) we then obtain

$$P(E_n) \leqslant \frac{1}{(1-\lambda)^2} P(E_n, N^1 = N^2 = 0),$$

i.e. $P(N^1 = N^2 = 0 | E_n) \geqslant (1-\lambda)^2 > 0$. This, and the previous remark that $P(J^i_n | N^i = 0)$ is bounded below by a constant provided that the starting points satisfy U_0, gives:

$$P(\tilde{E}_n | B^1_0 = x, B^2_0 = y) \geqslant c P(E_n | B^1_0 = x, B^2_0 = y). \tag{2.10}$$

Conditioning on B^2 shows that the map

$$f : x \mapsto P(E_n | B^1_0 = x, B^2_0 = 1) \tag{2.11}$$

is harmonic and does not vanish on the complement of $\overline{A}$. Moreover, its supremum on the unit circle is equal to $\bar{q}_n$ by definition: Applying the Harnack principle then proves that f is bounded below by cq_n on the set of x satisfying U_0, which completes the proof. ///

Another estimate can be obtained using the very same proof: Only keeping the conditions involving disks and relaxing those involving wedges, we obtain

$$P\left(B^1_{[0,\tau^1_n]} \cap B(0, 1-\eta) = \varnothing, AB^2_{[0,\tau^2_n]} \cap \mathcal{B}(0, 1-\eta) = \varnothing \,\middle|\, B^1_0, B^2_0, E_n \right) \geqslant c > 0, \tag{2.12}$$

where c does not depend on the initial positions B^1_0 and B^2_0, nor on n (it clearly depends on η, though, and a closer look at the proof shows that we can ensure $c > \eta^\beta$ as $\eta \to 0$, for some $\beta > 0$). This estimate will be needed in the derivation of Hausdorff dimensions, cf. Section 2.3.

We now have all the needed estimates to derive the lower bound in the sub-additivity condition, and hence the conclusion of the theorem. Take two paths with independent starting points uniformly distributed on the unit circle and killed at radius e^{m+n}, conditioned not to intersect between radii 1 and e^n. This happens with probability q_n. With large probability (*i.e.* with a positive probability, independent of m and n) the paths up to radius e^n end up "well separated" in the sense of the end-separation lemma. In particular, the points where they reach radius e^n, after suitable rescaling, satisfy the hypothesis of the start-separation lemma: Hence with probability greater that cq_m, the paths between radii e^n and e^{m+n} remain separated up to radius e^{n+1}, do not reach radius $(1-\eta)e^n$ anymore and do not intersect up to radius e^{m+n}. Under those conditions, it is easy to see that the paths do not meet at all. So $q_{m+n} \geqslant cq_m q_n$ for some positive c, and we get the conclusion.

Some adaptations are needed if A is included in an annulus, say $\{r < |z| < R\}$ with $r < 1 < R$. First, replace all occurrences of e by e_0, with e_0 chosen larger than $10R/r$, and in the start-separation lemma, replace $\mathcal{B}(0, 1-\eta)$ by $\mathcal{B}(0, r/2R)$ in the definition of the J_n. As long as r and R are fixed, this changes nothing to the proof, except that the constants we obtain will then depend on R/r — which itself is bounded provided A remains a subset of some fixed nice set.

A more serious problem arises if the complement of $\overline{A}$ is not connected (*i.e.*, if A has holes), since the natural domain of the function f (as defined by Equation (2.11)) is itself not connected. However, since A is nice, its complement has exactly one unbounded component, and it is easy to see that if x is not in this component then $f(x)$ vanishes for $n \geqslant 1$. Hence, nothing changes (as far as non-intersection properties are concerned) when A is replaced by the complement of the infinite component of its complement (*i.e.* when filling the holes in A). //

In fact, a stronger result can be derived: If the starting points B^1_0 and B^2_0 are fixed, then $P(E_n|B^1_0, B^2_0)$ is *equivalent* to $ce^{-n\xi(A)}$, where c is a function of B^1_0 and B^2_0 satisfying $c \leqslant c_0 d(B^1_0, AB^2_0)^\beta$. This estimate is related to a strong convergence result on the law of paths conditioned by $B^1 \cap AB^2 = \varnothing$. However, proving this result would be much more involved (cf. [37] for the proof in the case $A = \{1\}$).

2.2 Properties of the function $A \mapsto \xi(A)$

We first list a few simple properties of the function $A \mapsto \xi(A)$. For $p \in \mathbb{Z}$ and $A \subset \mathbb{C}$, introduce $A^p = \{z^p, z \in A\}$ and let $A^* = \{\bar{z}, z \in A\}$.

Proposition 2.1 :

Is these statements, all sets are assumed to be non-empty but do not need to be nice:

(i). ξ is *non-decreasing*: if $A \subset A'$ then $\xi(A) \leqslant \xi(A')$;

(ii). ξ is *homogeneous*: if $\lambda \in \mathbb{C}^*$ then $\xi(\lambda A) = \xi(A)$;

(iii). ξ is *symmetric*: $\xi(A^{-1}) = \xi(A^*) = \xi(A)$;

(iv). ξ has the following property: if $n \geqslant 1$ then

$$\xi\left(\bigcup e^{2ik\pi/n}A\right) = n\xi(A^n).$$

// (i): This is a trivial consequence of $p_R(A) \geqslant p_R(A')$.

(ii): Applying the scaling property with factor $|\lambda|$ to B^2 proves that one can suppose $|\lambda| = 1$; in which case we have $p_R(A) = p_R(\lambda A)$ (because the starting points are uniformly distributed on the unit circle).

(iii): Simply exchange B^1 and B^2 for A^{-1}, and say that the complex conjugate of a Brownian path is still a Brownian path to get A^*.

(iv): This is a consequence of the analyticity of the mapping $z \mapsto z^n$ (hence the fact that $((W_t)^n)$ is a Brownian path if W is one) together with the remark that the existence of $s, t > 0$ and $z \in A^n$ with $(B^1_s)^n = z(B^2_t)^n$ is equivalent to the existence of z' in $\bigcup e^{2ik\pi/n}A$ with $B^1_t = z'B^2_t$ — note that the mapping also has an influence on R, hence the factor n. //

We now turn our attention toward regularity properties of the function $A \mapsto \xi(A)$ — the following result being a key step toward the derivation of dimensions in the next section. Introduce the Hausdorff distance between compact subsets of the plane (cf. Section 2.5 for details). It will be convenient here to define neighbourhoods by $V_r(A) = \{xe^z, x \in A, |z| < r\}$ instead of the usual $A + \mathcal{B}(0,r)$ — leading to the *logarithmic* Hausdorff distance. The (logarithmic) Hausdorff topology is the metric topology derived from this distance.

Proposition 2.2 :

ξ is continuous on the collection of nice sets, endowed with the logarithmic Hausdorff topology. For any nice set A_0, ξ is uniformly continuous in $\{A \ : \ A \subset A_0\}$.

// The proof relies on the uniformity of the strong approximation in Theorem 2.1: fix a nice set A_0 and assume all sets considered here are subsets of A_0. The constants c, c_- and c_+ appearing during the proof may only depend on A_0.

First, fix $R > 1$ and condition all events by $B^2_{[0,T^2_{R+1}]}$ — *i.e.* fix the second path. For all $A \subset A_0$, let

$$\mathrm{d}_R(A) = \mathrm{d}_H(B^1_{[0,T^1_R]}, AB^2_{[0,T^2_R]}) \ ;$$

for all $\varepsilon > 0$ introduce the stopping time

$$S_\varepsilon = \mathrm{Inf}\{t : \mathrm{d}_H(B^1_t, AB^2_{[0,T^2_R]}) < \varepsilon\}.$$

Note that $\{\mathrm{d}_R(A) < \varepsilon\} = \{S_\varepsilon < T^1_R\}$. On this event, the strong Markov property shows that $B^1_{S_\varepsilon+\cdot}$ is a Brownian path starting ε-close to AB^2. By Beurling's theorem, the probability that they do not meet before radius

$R+1$ is smaller than the corresponding probability for a path near a half line; hence,

$$P(B^1_{[S_\varepsilon, T^1_{R+1}]} \cap AB^2_{[0,T^2_{R+1}]} = \varnothing | \mathrm{d}_R(A) < \varepsilon) \leqslant \sqrt{\varepsilon},$$

so that, considering the whole path, $P(E_{R+1}|\mathrm{d}_R(A) < \varepsilon) \leqslant \sqrt{\varepsilon}$. Apply the Bayes formula:

$$P(\mathrm{d}_R(A) < \varepsilon | E_{R+1}) = \frac{P(\mathrm{d}_R(A) < \varepsilon)}{P(E_{R+1})} P(E_{R+1}|\mathrm{d}_R(A) < \varepsilon);$$

since we know that $P(E_{R+1}) \geqslant c_-(R+1)^{-\xi(A)}$ with $\xi(A) \leqslant \xi(A_0)$ we finally obtain

$$P(\mathrm{d}_R(A) < \varepsilon | E_{R+1}) \leqslant cR^{\xi(A_0)}\sqrt{\varepsilon}.$$

From now on, we shall assume that ε is sufficiently small to make the obtained bound smaller that 1. Taking the complement leads to

$$P(\mathrm{d}_R(A) \geqslant \varepsilon | E_{R+1}) \geqslant 1 - cR^{\xi(A_0)}\sqrt{\varepsilon}.$$

Now, remark that when $\mathrm{d}_R(A) \geqslant \varepsilon$ and $\mathrm{d}_H(A, A') < \varepsilon/R$, we have $B^1_{[0,T^1_R]} \cap A'B^2_{[0,T^2_R]} = \varnothing$: from this and the previous equation it follows that, as long as A and A' remain subsets of A_0,

$$\mathrm{d}_H(A, A') < \frac{\varepsilon}{R} \quad \Rightarrow \quad p_R(A') \geqslant \left(1 - cR^{\xi(A_0)}\sqrt{\varepsilon}\right) p_{R+1}(A).$$

We can apply the estimates on p_R we derived in Theorem 2.1 — *i.e.* $p_R(A) \asymp p_{R+1}(A) \asymp R^{-\xi(A)}$: still for $\mathrm{d}_H(A, A') < \varepsilon/R$ and A, A' inside A_0 we get

$$c_+ R^{-\xi(A')} \geqslant \left(1 - cR^{\xi(A_0)}\sqrt{\varepsilon}\right) c_- R^{-\xi(A)},$$

and taking the logarithm of each side of the inequality leads to

$$\log c_+ - \xi(A') \log R \geqslant \log c_- + \log\left(1 - cR^{\xi(A_0)}\sqrt{\varepsilon}\right) - \xi(A) \log R,$$

hence after suitable transformations:

$$\xi(A') \leqslant \xi(A) + \frac{c}{\log R} - \frac{\log\left(1 - cR^{\xi(A_0)}\sqrt{\varepsilon}\right)}{\log R}. \tag{2.13}$$

Fix $\eta > 0$, and choose R such that $c/\log R < \eta/2$. It is then possible to take ε sufficiently small so that $|\log(1 - cR^{\xi(A_0)}\sqrt{\varepsilon})| < (\eta \log R)/2$; for $\mathrm{d}_H(A, A') < \varepsilon/R$ we then have $\xi(A') \leqslant \xi(A) + \eta$, hence by symmetry $|\xi(A') - \xi(A)| \leqslant \eta$. This proves that ξ is uniformly continuous on $\mathcal{P}_c(A_0)$, for all A_0, hence continuous on the family of nice sets. //

Remark 1: Equation (2.13) allows the derivation of an explicit modulus of continuity for ξ inside A_0, of the form

$$|\xi(A') - \xi(A)| \leqslant \frac{C(A_0)}{|\log \mathrm{d}_H(A, A')|}$$

(take $R = d^{-1/2\xi(A_0)}$). But since $C(A_0)$ is not known, this does not provide numerical bounds for ξ.

Remark 2: Inside a nice set, the usual and logarithmic Hausdorff topologies are equivalent, so the introduction of "exponential neighbourhoods" in Proposition 2.2 can seem artificial; however, it leads to constants that do not vary when A is multiplied by some constant (as in Proposition 2.1, point (ii)), hence uniform continuity holds on the collection of nice sets contained in a fixed wedge and in some annulus $\{r < |z| < cr\}$ for fixed c — which is wrong for the usual Hausdorff topology, as a consequence of the homogeneity of ξ applied for small $|\lambda|$.

Note that uniform continuity cannot hold on the family of nice sets contained in a given annulus since ξ would then be bounded (by a compactness argument), which it is not: the exponent associated to a circle is infinite.

2.3 Hausdorff dimension of the corresponding subsets of the path

2.3.1 Conformally invariant subsets of the Brownian path

It is well-known that the Brownian path is invariant in law under conformal transformations; in this section, we study subsets of the Brownian curve that are also invariant under conformal maps. A first example is the set of so-called Brownian *cut-points*, *i.e.* points B_t such that $B_{[0,t)}$ and $B_{(t,1]}$ are disjoint; these points form a set of Hausdorff dimension $2 - \xi(\{1\}) = 3/4$. Related to those are *local cut-points*, *i.e.* points such that there exists $\varepsilon > 0$ satisfying $B_{[t-\varepsilon,t)} \cap B_{(t,t+\varepsilon]} = \varnothing$ — the dimension is the same as for global cut-points. Other examples are given by Lawler in [27]: in particular the set of *pioneer points* (such that B_t lies on the frontier of the infinite component of the complement of $B_{[0,t]}$), related to the disconnection exponent η_1; *frontier points* (points of the boundary of the infinite component of the complement of $B_{[0,1]}$), related to the disconnection exponent for two paths in the plane. Another exceptional subset of the path is the set of *cone points* (such that $B_{[0,t]}$ is contained in a cone of endpoint B_t), related to the *cone exponents* (studied in [38] for example).

We will use the exponents introduced in the previous sections to describe a family of exceptional sets, indexed by a subset A of the complex plane, having dimension $2 - \xi(A)$, and that are invariant under conformal transformations, as follows. Fix a Brownian path $B_{[0,1]}$, a subset A of the complex plane, and introduce the following times for all $t \in (0,1)$ and $r > 0$:

$$T_r(t) = \mathrm{Inf}\{s > t : |B_s - B_t| = r\}, \qquad S_r(t) = \mathrm{Sup}\{s < t : |B_s - B_t| = r\}.$$

Definition :

If $0 < \varepsilon < R$ and $t \in (0,1)$, let

$$Z_t^{[\varepsilon,R]}(B) = \left\{ \frac{B_s - B_t}{B_{s'} - B_t} : s \in [T_\varepsilon(t), T_R(t)], s' \in [S_R(t), S_\varepsilon(t)] \right\};$$

and introduce $\mathcal{E}_A^{[\varepsilon,R]} = \{B_t : Z_t^{[\varepsilon,R]} \cap A = \varnothing\}$. Then, letting ε go to 0:

$$Z_t^R = \bigcup_{\varepsilon>0}\!\uparrow Z_t^{[\varepsilon,R]}, \qquad Z_t = \bigcap_{R>0}\!\downarrow Z_t^R, \qquad \tilde{Z}_t = \bigcap_{R>0}\!\downarrow \overline{Z_t^R};$$

define $\mathcal{E}_A^R$, $\mathcal{E}_A$ and $\tilde{\mathcal{E}}_A$ accordingly.

We shall also use the notation $\mathcal{T}_A = \{t : B_t \in \mathcal{E}_A\}$, for the set of A-*exceptional times*, and $\tilde{\mathcal{T}}_A = \{t : B_t \in \tilde{\mathcal{E}}_A\}$, for the set of A-*strongly exceptional times*.

Note that, since 0 is polar for planar Brownian motion, Z is well-defined for almost any t. For $A = \{1\}$, $\mathcal{E}_A$ is the set of local cut-points; more generally, B_t is in $\mathcal{E}_A$ if, and only if, for some $\varepsilon > 0$, we have

$$(B_{(t,t+\varepsilon]} - B_t) \cap A.(B_{[t-\varepsilon,t)} - B_t) = \varnothing,$$

so the setup looks similar to the definition of the exponent $\xi(A)$. It is easy to see that for all fixed $t > 0$, a.s. $Z_t = \mathbb{C}^*$ and $\tilde{Z}_t = \mathbb{C}$, so that for $A \neq \varnothing$, $P(t \in \mathcal{T}_A) = 0$, leading to $E(\mu(\mathcal{T}_A)) = 0$ *i.e.* $\mu(\mathcal{T}_A) = 0$ almost surely — hence the term "exceptional points".

The set $\mathcal{E}_A$ of A-*exceptional points* is generally not conformally invariant. However, it is the case for strongly exceptional points:

Proposition 2.3 :

Let Φ be a conformal map on a neighbourhood Ω of 0, with $\Phi(0) = 0$, and let B^Ω be B stopped at its first hitting of $\partial\Omega$. By conformal invariance of planar Brownian motion, $\Phi(B^\Omega)$ is a Brownian path stopped at its first hitting of $\partial\Phi(\Omega)$. Moreover, we have

$$\tilde{\mathcal{E}}_A(\Phi(B^\Omega)) = \Phi(\tilde{\mathcal{E}}_A(B^\Omega)).$$

// We prove that $\tilde{Z}$ is invariant. It is sufficient to prove the following characterization:

$$z \in \tilde{Z}_t(B) \iff \exists (s_n) \downarrow 0,\ (s'_n) \downarrow 0 : \frac{B_{t+s_n} - B_t}{B_{t-s'_n} - B_t} \to z,$$

as conformal maps conserve the limits of such quotients. Such a sequence is easily constructed using the very definition of $\tilde{Z}$. //

Note that nothing in the preceding uses the fact that B be a Brownian path, except for the remark about $P(t \in \mathcal{T}_A)$. The remaining of the present section is dedicated to deriving the Hausdorff dimension of $\mathcal{E}_A$ and $\tilde{\mathcal{E}}_A$. It will be more convenient to work in the time set, so introduce

$$\mathcal{T}_A^{[\varepsilon,R]} = \{t \in [0,1] : (B_{[t-R,t-\varepsilon]} - B_t) \cap A.(B_{[t+\varepsilon,t+R]} - B_t) = \varnothing\}.$$

The scaling property of Brownian motion can then be used to show, as in [25, lemmas 3.14–3.16], that Theorem 2.1 implies the following, provided A is nice:

$$P(t \in \mathcal{T}_A^{[\varepsilon,R]}) \asymp \left(\frac{\varepsilon}{R}\right)^{\xi(A)/2}. \tag{2.14}$$

2.3.2 Second moments

Fix $R > 0$. The purpose of this subsection is to give an estimate of the probability that two times t and t' are A-exceptional times, *i.e.* are both in $\mathcal{T}_A^{[\varepsilon,R]}$. To get an upper bound on this probability, the idea will be to dissociate the microscopic and macroscopic scales, giving respectively the first and second factor in the following estimate:

$$P(t,t' \in \mathcal{T}_A^{[\varepsilon,R]}) \leqslant c\left[\frac{\varepsilon}{R}\right]^{\xi(A)}\left[1 \vee |t-t'|^{-\xi(A)/2}\right].$$

If $t < t'$ are two times, introduce the "mesoscopic" scale $d = |t' - t|$, and separate the following three cases:

- If $d > 2R$ (long-range interaction), the events $E_t \triangleq \{t \in \mathcal{T}_A^{[\varepsilon,R]}\}$ and $E_{t'}$ are independent, leading to the right second-order moment;
- If $R/2 \leqslant d \leqslant 2R$ (medium-range interaction), the trivial bound $P(E_x, E_y) \leqslant C(2\varepsilon/d)^{\xi(A)}$ (obtained by forgetting what happens after radius $d/2$) gives the needed contribution.
- If $d < R/2$ (short-range interaction), a little more work is required. Introduce the following times:

$$\tilde{T}_r(x) = \mathrm{Min}(x + r, \mathrm{Inf}\{x' > x : |B_x - B_{x'}| = r^{1/2}\}),$$
$$\tilde{S}_r(x) = \mathrm{Max}(x - r, \mathrm{Sup}\{x' < x : |B_x - B_{x'}| = r^{1/2}\}).$$

First, E_t and $E_{t'}$ imply two independent events:

$$E_1 : \left(B_{[t+\varepsilon,\tilde{T}_{d/2}(t)]} - B_t\right) \cap A.\left(B_{[\tilde{S}_{d/2}(t),t-\varepsilon]} - B_t\right) = \varnothing,$$
$$E_2 : \left(B_{[t'+\varepsilon,\tilde{T}_{d/2}(t')]} - B'_t\right) \cap A.\left(B_{[\tilde{S}_{d/2}(t'),t'-\varepsilon]} - B'_t\right) = \varnothing\ ;$$

as in [25], it can be proved that $P(E_1) \asymp P(t \in \mathcal{T}_A^{[\varepsilon,d/2]}) \asymp (\varepsilon/d)^{\xi(A)/2}$. Let

$$\delta = \mathrm{Max}\left((d/2)^{1/2}, |B_{\tilde{T}_{d/2}(t')}|\right).$$

δ is stochastically dominated by the sum of $(d/2)^{1/2}$ and a Gaussian variable $\mathcal{N}(0,d)$ (accounting for the behaviour of B between the times $\tilde{T}_{d/2}(t)$ and $\tilde{S}_{d/2}(t')$). Moreover, conditionally to the value of δ, the joint distribution of B at times $S_{2\delta}(t)$ and

$$T' \triangleq \mathrm{Inf}\{x > t' : |B_x - t| = 2\delta\}$$

is absolutely continuous with respect to the Lebesgue measure on $\mathcal{C}(0,2\delta)^2$, and its density is bounded above and below by absolute constants. Lastly, E_t and $E_{t'}$ imply that

$$\left(B_{[T',t+R]} - B_t\right) \cap A.\left(B_{[t-R,S_{2\delta}(t)]} - B_t\right) = \varnothing,$$

and (still conditionally on δ) the probability of this event is bounded above by $C.(2\delta)^{\xi(A)}$ by Theorem 2.1. But the previous remark on the law of δ shows that

$$E(\delta^{\xi(A)}) \leqslant C\ d^{\xi(A)/2},$$

hence finally the correct estimate:

$$P(E_t, E_{t'}) \leqslant C.\left(\frac{\varepsilon}{d}\right)^{\xi(A)} d^{\xi(A)/2} = C\,\frac{\varepsilon^{\xi(A)}}{|t-t'|^{\xi(A)/2}}.$$

So in the case of exceptional points defined locally, bounds on second moments are not difficult to derive (and this "scale separation" construction can be used in various setups). In contrast, if the whole path was to influence every single point, interactions would not be that easy to classify.

2.3.3 Hausdorff dimensions

The main result of this section is the following:

Theorem 2.2 :

Let $(B_t)_{t\in[0,1]}$ be a planar Brownian path. If A is any nice subset of the complex plane such that $\xi(A) \leq 2$, then almost surely

$$\dim_H(\mathcal{E}_A(B)) = \dim_H(\tilde{\mathcal{E}}_A(B)) = 2 - \xi(A).$$

In particular, both subsets are a.s. non-empty and dense in the path if $\xi(A) < 2$. If $\xi(A) > 2$, $\mathcal{E}_A(B) = \tilde{\mathcal{E}}_A(B) = \varnothing$ almost surely.

// The first step in the proof is the statement of a zero-one law:

Lemma 2.1 :

The dimension of the set of all A-exceptional points (resp. of A-strong exceptional points) has an almost sure value. More precisely, there exist δ_A and $\tilde{\delta}_A$ in $[0,2]$ such that

$$P(\dim_H(\mathcal{E}_A) = \delta_A) = P(\dim_H(\tilde{\mathcal{E}}_A) = \tilde{\delta}_A) = 1.$$

Moreover, the following holds with probability 1 (and the same for $\tilde{\mathcal{E}}_A$ also):

$$\forall s < t \quad \dim_H(\mathcal{E}_A(B_{[s,t]})) = \delta_A.$$

/// The proof is the same in both cases; we perform it here for δ_A.
Introduce the following random variables in $[0,2]$:

$$Z = \dim_H(\mathcal{E}_A), \qquad Z_- = \dim_H(\mathcal{E}_A(B_{[0,1/3]})), \qquad Z_+ = \dim_H(\mathcal{E}_A(B_{[2/3,1]})).$$

The scaling property, associated with the Markov property, shows that these three variables have the same law; basic properties of the Hausdorff dimension imply that $Z \geqslant Z_- \vee Z_+$; and locality proves that Z_- and Z_+ are independent.

$0 \leqslant Z_- \leqslant Z \leqslant 2$ with the same mean value: from here follows that $P(Z_- = Z) = 1$. By the same argument $P(Z_+ = Z) = 1$, hence $P(Z_- = Z_+) = 1$; Z_- and Z_+ being independent, this is only possible if they are deterministic: thus giving the existence of δ_A as their common almost sure value.

Now if $0 \leqslant s < t \leqslant 1$ the dimension of $\mathcal{E}_A(B_{[s,t]})$ is (almost surely) δ_A. This holds at the same time for all rational s, t; then it suffices to note that $\dim_H(\mathcal{E}_A(B_I))$ is increasing in I to extend the equality to all $s < t$. ///

From this lemma follows that as soon as $\mathcal{E}_A$ has positive dimension it is dense in the path.

For convenience we will prove the result in the time set, *i.e.* we shall compute the dimension of $\mathcal{T}_A$; it is known that planar Brownian motion doubles Hausdorff dimensions (*i.e.* with probability 1, for any Borel subset I of $[0,1]$, $\dim_H(B_I) = 2\dim_H(I)$ — cf. [22]), whence $\dim_H(\mathcal{E}_A) = 2\dim_H(\mathcal{T}_A)$. Moreover, to avoid problems near 0 and 1 we shall suppose that B is defined for $t \in \mathbb{R}$ — this will not change $\mathcal{T}_A$ since the definition is local.

First step: lower bound. Fix $R > 0$ and let A_n be the following set:

$$A_n = \{t : (B_{[t-R,t-2^{-n}]} - B_t) \cap A(B_{[t+2^{-n},t+R]} - B_t) = \varnothing\}.$$

For shorter notations, let $s = \xi(A)/2$; moreover, assume from now on that $s \in (0,1)$ (if $s \geqslant 1$ there is nothing to prove, and since $A \neq \varnothing$ we have $s > 0$ anyway). From the previous estimates for first- and second-moments, we obtain

$$E(\mathbf{1}_{A_n}(x)) \asymp 2^{-sn} \qquad E(\mathbf{1}_{A_n}(x)\,\mathbf{1}_{A_n}(y)) \leqslant c2^{-2sn}\left[1 \vee \frac{1}{|y-x|^s}\right].$$

Introduce the (random) measure μ_n having density $2^{sn}\mathbf{1}_{A_n}$ with respect to the Lebesgue measure. It is not hard to derive the following estimates:

$$E(\|\mu_n\|) = \int_{[0,1]} 2^{sn}E(\mathbf{1}_{A_n}(x))\,\mathrm{d}x \asymp 1, \tag{2.15}$$

$$\begin{aligned} E(\|\mu_n\|^2) &= \iint_{[0,1]^2} 2^{2sn}E(\mathbf{1}_{A_n}(x)\,\mathbf{1}_{A_n}(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant c2^{sn}\left[\int_0^1 \mathrm{d}x\int_x^{x+2^{-n}}\mathrm{d}y \;+\; \int_0^{1-2^{-n}}\mathrm{d}x\int_{x+2^{-n}}^1 \frac{2^{-sn}\mathrm{d}y}{(y-x)^s}\right] \\ &\leqslant c2^{(s-1)n} \;+\; c\int_0^{1-2^{-n}}\left(\frac{(1-x)^{1-s}}{1-s} - \frac{2^{(s-1)n}}{1-s}\right)\mathrm{d}x \\ &\leqslant c + c2^{(s-1)n} + c2^{(s-2)n} \leqslant c. \end{aligned} \tag{2.16}$$

Hence, $\|\mu_n\|$ has finite expectation and finite variance, independent of n: there exists $\varepsilon > 0$ satisfying $P(\|\mu_n\| > \varepsilon) > \varepsilon$ for all positive n. Consequently, it is possible, with positive probability, to extract a subsequence (μ_{n_k}) such that, for all k, $\|\mu_{n_k}\| \geqslant \varepsilon$. By a compactness argument, another extraction leads to a converging subsequence, the limit μ of which satisfies $\|\mu\| \geqslant \varepsilon$. μ is supported on the intersection of the A_n, this intersection is non-empty: hence $P(\bigcap A_n \neq \varnothing) > 0$.

Introduce then the notion of r-energy of a measure: if ν is some mass measure supported on a metric space X, let

$$\mathcal{E}_r(\nu) \triangleq \iint_{X^2} \frac{\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)}{\mathrm{d}(x,y)^r}.$$

It is known that if X supports a mass measure of finite r-energy, then its Hausdorff dimension is not less than r (cf. [21]). Let then $r \in (0, 1-s)$: a calculation analogous to the derivation of (2.16) leads to

$$E(\mathcal{E}_r(\mu_n)) \leqslant c + c2^{(r+s-1)n} + c2^{(r+s-2)n} \leqslant c. \tag{2.17}$$

Performing another subsequence extraction, it is possible to obtain μ supported on $\bigcap A_n$ and having finite r-energy: hence

$$\forall r < 1-s \quad P(\dim_H(\bigcap A_n) \geqslant r) > 0.$$

By definition $\mathcal{T}_A$ is the increasing union, for R going to 0, of $\bigcap_n A_n(R)$: hence for all $r < 1-s$ we have $P(\dim_H(\mathcal{T}_A) \geqslant r) > 0$. Combining this and the zero-one result (Lemma 2.1) then proves that almost surely $\dim_H(\mathcal{T}_A) \geqslant 1-s$.

Second step: upper bound. This step is usually the easier one, but in the present case a complication arises due to the fact that the "non-intersection" event we consider at B_t depends on the position of B_t — which is not the case for instance in the case of cut-points [27]. This explains why we need one more argument, namely the continuity of $\xi : A \mapsto \xi(A)$.

Fix a nice set A, $\varepsilon > 0$, $R > 0$ and a sequence $(\lambda_n)_{n \geqslant 0}$ of positive numbers, tending slowly to 0 (in the following sense: for all positive η, $2^{-\eta n} = o(\lambda_n)$ — for instance, take $\lambda_n = 1/n$). Now suppose some time t is in A_n. With positive probability, the following happens:

$$\begin{cases} B_{[t-\lambda_n 2^{-n}, t+\lambda_n 2^{-n}]} \subset \mathcal{B}(B_t, \lambda_n^{1/2} 2^{-n/2}) \\ |B_{t-2^{-n}} - B_t| \geqslant 2^{-n/2} \\ |B_{t+2^{-n}} - B_t| \geqslant 2^{-n/2} \\ (B_{[t-R, t-2^{-n}]} \cup B_{[t+2^{-n}, t+R]}) \cap \mathcal{B}(B_t, (1-\varepsilon)2^{-n/2}) = \varnothing \end{cases}$$

(the first three conditions are a consequence of scaling, and the fourth one is the start-separation lemma, more precisely the weakened version of it as stated in equation (2.12)). Introduce $A^{\eta_n} = \{az : a \in A, z \in \mathcal{B}(1, \eta_n)$: we have

$$\begin{aligned} P(B_{[t-R,t-2^{-n}]} - B_t) \cap A^{\eta_n}(B_{[t+2^{-n},t+R]} - B_t) = \varnothing \mid t \in A_n) \\ \asymp \frac{2^{-n\xi(A^{\eta_n})/2}}{2^{-n\xi(A)/2}} = 2^{-n[\xi(A^{\eta_n}) - \xi(A)]/2}. \end{aligned} \tag{2.18}$$

It is easy to see that under the previous conditions, if $t \in \mathcal{T}_{A^{\eta_n}}$, then every $t' \in [t - \lambda_n 2^{-n}, t + \lambda_n 2^{-n}]$ is in A_n, as soon as $\eta_n > 18\lambda_n/(1-\varepsilon)$. From now on we shall assume that this holds, and that $\eta_n \to 0$. Putting these estimates together, we obtain the following (where l is the Lebesgue measure on $\mathbb{R}$): for all interval I,

$$P(l(A_n \cap I) > \lambda_n 2^{-n} | A_n \cap I \neq \varnothing) \geqslant c.2^{-n[\xi(A^{\eta_n}) - \xi(A)]/2}. \tag{2.19}$$

The Markov inequality then states that

$$P(l(A_n \cap I) > \lambda_n 2^{-n}) \leqslant \frac{E(l(A_n \cap I))}{\lambda_n 2^{-n}},$$

and $E(l(A_n \cap I)) \asymp 2^{-n\xi(A)/2} l(I)$. From this and (2.19) follows that

$$P(A_n \cap I \neq \varnothing) \leqslant C \, \frac{2^{-n\xi(A)/2} l(I)}{\lambda_n 2^{-n}} \, \frac{1}{2^{-n[\xi(A^{\eta_n})-\xi(A)]/2}}. \tag{2.20}$$

By continuity of ξ, for large n we have $|\xi(A^{\eta_n})-\xi(A)| < \varepsilon$; by the hypothesis on λ_n, still for large n we have $\lambda_n \geqslant 2^{-\varepsilon n/2}$. Hence for large n:

$$P(A_n \cap I \neq \varnothing) \leqslant C \, 2^{\varepsilon n} \, 2^{-n\xi(A)/2} \, \frac{l(I)}{2^{-n}}. \tag{2.21}$$

Cover the interval $[0,1]$ with the $I_k^n = [k2^{-n}, (k+1)2^{-n}]$, and let X_n be the number of such intervals intersecting A_n. Then

$$E(X_n) = \sum_k P(I_k^n \cap \mathcal{T}_A \neq \varnothing) \leqslant 2^n \, C \, 2^{\varepsilon n} \, 2^{-n\xi(A)/2} \, \frac{l(I_0^n)}{2^{-n}} \leqslant C \, 2^{\varepsilon n} \, 2^{n[1-\xi(A)/2]}.$$

By another application of the Markov inequality,

$$P(X_n > 2^{n[1-\xi(A)/2+2\varepsilon]}) \leqslant C \, 2^{-\varepsilon n}.$$

Hence by the Borel-Cantelli theorem, for sufficiently large n, A_n is covered by at most $2^{n[1-\xi(A)/2+2\varepsilon]}$ intervals of length 2^{-n} — and this implies that $\dim_H(\bigcap A_n) \leqslant 1-\xi(A)/2+2\varepsilon$. Letting ε tend to 0 then leads to (a.s.) $\dim_H(\bigcap A_n) \leqslant 1-\xi(A)/2$. This is true for all $R>0$, hence remains true in the limit $R \to 0$: together with the first step of the proof this gives (a.s.) $\dim(\mathcal{T}_A) = 1-\xi(A)/2$ hence $\dim(\mathcal{E}_A) = 2-\xi(A)$.

Then, $\tilde{\mathcal{E}}_A$ is contained in $\mathcal{E}_A$ and besides it contains every $\mathcal{E}_{A^\eta}$ for positive η (with the previous notations): another use of the continuity of ξ then gives $\dim_H(\tilde{\mathcal{E}}_A) = \dim_H(\mathcal{E}_A) = 2-\xi(A)$. //

As a consequence, we get a second result:

Theorem 2.3 :

If A is any nice subset of the complex plane, then the set of *globally A-exceptional points*, *i.e.* points B_t satisfying

$$(B_{[0,t)} - B_t) \cap A.(B_{(t,1]} - B_t) = \varnothing,$$

has Hausdorff dimension $2-\xi(A)$ — and in particular it is a.s. non-empty for $\xi(A) < 2$, and a.s. empty for $\xi(A) > 2$.

// Again, extend B to $(B_t)_{t\in\mathbb{R}}$ defined on the entire real line. The set $\mathcal{T}_A^1$ of A-exceptional times up to the scale $R=1$ (as was introduced previously) in $[0,1]$ is exactly the set of globally exceptional points. Therefore, the previous proof can be applied directly. The upper bound is immediate: since every globally exceptional point is locally exceptional we have $\dim_H(\mathcal{T}_A^1) \leqslant \dim_H(\mathcal{T}_A) \leqslant 1-\xi(A)/2$ a.s.

The lower bound requires a little more work, indeed we do not have a zero-one law for the dimension of $\mathcal{T}_A^1$. It can be seen that in fact Equation (2.17) can be refined, the proof being exactly the same, into the following (with the same notations as previously):

$$\exists C>0 \quad \forall r \in (0,1-s) \quad \forall n>0 \qquad E(\mathcal{E}_r(\mu_n)) \leqslant \frac{C}{1-(r+s)},$$

where C may only depend on A. Hence, with the same constant and for all $\lambda > 1$:

$$P\left(\mathcal{E}_r(\mu_n) \leqslant \frac{\lambda C}{1-(r+s)}\right) \geqslant 1 - \frac{1}{\lambda}.$$

one can then perform the subsequence extraction (cf. proof of Theorem 2.2) in a way which ensures that, for all r,

$$P\left(\|\mu\| > 0 \text{ and } E_r(\mu) \leqslant \frac{\lambda C}{1-(r+s)}\right) \geqslant c, \tag{2.22}$$

with $c > 0$ and $\lambda > 1$ independent of r. Moreover, $\mathcal{E}_r(\mu)$ being a non-decreasing function of r (since the set $[0,1]$ is of diameter 1), we finally obtain, with positive probability, a mass measure μ supported on $\mathcal{T}_A$ satisfying

$$\forall r < 1-s \qquad \mathcal{E}_r(\mu) \leqslant \frac{\lambda C}{1-(r+s)} < \infty.$$

Hence, with positive probability, $\dim_H(\mathcal{T}_A) \geqslant 1 - s = 1 - \xi(A)/2$, and combining this to the previous paragraph leads to

$$P\left(\dim_H(\mathcal{T}_A) = 1 - \frac{\xi(A)}{2}\right) > 0.$$

It is then possible to conclude using the same method as in [25, pp. 8–9]. //

2.3.4 A remark about critical cases

In cases where $\xi(A) = 2$, the previous theorem is not sufficient to decide whether A-exceptional points exist. We shall see in the next paragraph that $\xi((-\infty,0)) = \xi((0,\infty)) = 2$. In fact these two cases are very different:

Proposition 2.4 :

Almost surely, $\mathcal{E}_A$ is empty for $A = (0,\infty)$ and non-empty (with Hausdorff dimension 0 though) for $A = (-\infty, 0)$.

// The second point is easier: if t is such that $\Re(B_t)$ is maximal in the path, then $B_{[0,1]}$ lies inside a half-plane whose border goes through B_t. Since a.s. B_t is the only point having this real part, this proves that $(B_s - B_t)/(B_{s'} - B_t)$ is never in $(-\infty, 0)$, which is precisely what we wanted.

The first point is more problematic. The method used to derive the value of ξ for a wedge with end-point at the origin (cf. next paragraph) allows to prove the following: Let α and β be in $(0, 2\pi)$, then the probability that, given independent paths B^1 and B^2 starting from the unit circle, there exist two wedges of angles α and β, and containing respectively B^1 and B^2 up to radius R, decreases as

$$p_R(\alpha,\beta) \approx R^{-(\pi/\alpha + \pi/\beta)}.$$

Hence, as soon as $\pi/\alpha + \pi/\beta$ is greater than 2, there is a.s. no point B_t on the path such that $B_{[0,t]}$ lies in a wedge of angle α and $B_{[t,1]}$ lies in a wedge of angle β (there is no "asymmetric two-sided cone point" of those angles on the path).

For all $\alpha \in (0, \pi)$, introduce $\alpha_1 = 2\pi - \alpha$ and α_2 as the biggest angle in $(0, 2\pi]$ satisfying $\pi/\alpha + \pi/\alpha_2 \geqslant 2$. Note that $\alpha_2 > \alpha_1$: denote then

$$\beta(\alpha) = \frac{\alpha_1 + \alpha_2}{2}.$$

Note that $\pi/\alpha + \pi/\beta(\alpha) > 2$ and $\beta(\alpha) + \alpha > 2\pi$ for all $\alpha \in (0, \pi)$. From this follows that, almost surely, for all $\alpha \in (0, \pi) \cap \mathbb{Q}$, there is no asymmetric cone point with angles α and $\beta(\alpha)$.

Let now $A = (0, \infty)$ and suppose there is a point B_t in $\mathcal{E}_A$. That is, there exist two half-lines starting from B_t whose reunion separates $B_{[0,t]}$ from $B_{[t,1]}$. Then we are in one of two cases:

- Either these half-lines form a straight line, *i.e.* there is a straight line cutting the path. This cannot happen, as recently proved by Bass and Burdzy [3] — and the proof is very difficult.

- Or there are disjoint wedges of angles $\alpha \in (0, \pi)$ and $2\pi - \alpha$, each containing one part of the path. Then, there exists $\alpha_0 \in \mathbb{Q}$ such that $\alpha_0 > \alpha$ and $\beta(\alpha_0) > 2\pi - \alpha$, and B_t is an asymmetric cone point with angles α_0 and $\beta(\alpha_0)$. We just saw that such a point cannot exist.

Hence $\mathcal{E}_A = \varnothing$. //

2.4 Bounds and conjectures on the exponent function

2.4.1 Known exact values of ξ

Proposition 2.5 :

(i). $\xi(\{1\}) = 5/4$, hence for all $z \neq 0$ and $n > 0$:

$$\xi\left(\{ze^{2ik\pi/n}, k = 1, \ldots, n\}\right) = 5n/4;$$

(ii). Letting W_α be a wedge of angle $0 \leqslant \alpha < 2\pi$:

$$\xi(W_\alpha) = \frac{4\pi}{2\pi - \alpha};$$

in particular $\xi((0, \infty)) = \xi((-\infty, 0)) = 2$;

// (i): The value of $\xi(\{1\}) = 5/4$ has recently been derived by Lawler, Schramm and Werner [32], and the proof is far beyond the scope of this paper. The result for all n is then a straightforward consequence of Proposition 2.1, point (iv).

(ii): Suppose $A = W_\alpha$ is centered around the positive axis, so that $A = \{re^{i\theta}, r > 0, |\theta| < \alpha/2\}$; introduce the symmetrical wedges $W'_\beta =$

$\{re^{i\theta}, r > 0, |\theta - \pi| < \beta/2\}$. If B^1 stays in $W_{\pi-\alpha/2}$ and B^2 remains in $W'_{\pi-\alpha/2}$, then $B^1 \cap AB^2 = \varnothing$: The probability of staying in a wedge of angle β until radius R being strongly approximated by $R^{-\pi/\beta}$ (the exponent is obtained through the gambler's ruin estimate combined with the analyticity of the exponential function; the strong approximation is true but in fact not needed here, cf. [16]), we get a lower bound:

$$p_R(W_\alpha) \geqslant c\left(R^{-\pi/(\pi-\alpha/2)}\right)^2,$$

hence $\xi(W_\alpha) \leqslant 4\pi/(2\pi - \alpha)$.

Now remark that the condition $B^1 \cap AB^2 = \varnothing$ means that the complement of the paths contains an "hourglass", *i.e.* the union of two disjoint wedges of angle $\alpha/2$. So introduce $\eta > 0$ and a (finite) family $(S_i)_{1\leqslant i\leqslant N}$ of hourglasses with angles $\alpha/2 - \eta$, such that any hourglass with angle $\alpha/2$ contains one of the S_i. If $q_R(i)$ is the probability that the paths are separated from each other by S_i, then $p_R(W_\alpha) \leqslant \sum q_R(i)$. Noticing that if β_i and β'_i are the angles of the wedges forming the complement of S_i, we obtain as previously $q_R(i) \asymp R^{-\pi/\beta_i - \pi/\beta'_i}$, and optimizing this under the constraint $\beta_i + \beta'_i = 2\pi - (\alpha - 2\eta)$ — where the greatest value is for $\beta = \beta'$ — we finally get the following estimate:

$$p_R(W_\alpha) \leqslant CN\, R^{-2\pi/(\pi+\eta-\alpha/2)}.$$

From this follows that $\xi(W_\alpha) \geqslant 4\pi/(2\pi + 2\eta - \alpha)$, and letting η go to 0 then gives the conclusion — at least for $\alpha > 0$. But in fact the same method still applies for $\alpha \geqslant 0$: simply inflate the complement of the hourglass instead of introducing angle $\alpha/2 - \eta$, the fact that the wedges to consider may overlap does not change anything to the proof. //

Remark: If we denote $A^\alpha = \{ze^{i\theta}, z \in A, |\theta| \leqslant \alpha/2\}$ (that is, A "thickened" by an angle α), then it can easily be proved that

$$\xi(A^\alpha) = \frac{h_A(\alpha)}{2\pi - \alpha}, \tag{2.23}$$

where h_A is continuous (until the angle $\alpha_0 \leqslant 2\pi$ when $\xi(A^\alpha)$ tends to infinity), non-decreasing, and satisfies $h_A(0) = 2\pi\xi(A)$; in the wedge case, h is constant.

2.4.2 An upper bound for the exponent

From continuity of ξ and the exact value $\xi(\{1\}) = 5/4 < 2$, one can deduce that there are "pivoting points" of any sufficiently small angle on the Brownian path (that is, points around which one half of the path can rotate of a small angle without intersecting the other half — the associated A being $\mathcal{C}_\alpha = \{e^{i\theta}, \theta \in [0, \alpha]\}$). The following proposition gives a (bad but) quantitative bound for such values of α — without usage of the exact value for $\alpha = 0$:

Proposition 2.6 :

For all positive α, we have the following upper bound:

$$\xi(\mathcal{C}_\alpha) \leqslant \frac{4\pi}{2\pi - \alpha}\left[1 - \frac{(\log 2)^2}{4\pi^2}\right].$$

// The proof is adapted from [46], where an upper bound for the classical disconnection exponent for one path, *i.e.* $\xi(1,0)$, was obtained. The method is the following: First, estimate the extremal length of a strip bounded by Lipschitz functions; then describe a sufficiently large subset of E_R, using such strips, and use the previous estimate to derive a bound for $P(E_R)$.

Lemma :

Let f be a continuous, M-Lipschitz function on $\mathbb{R}$, satisfying $f(x)+f(-x)=2f(0)$ for all x, and let $\beta>0$. Introduce the strip of width β and length $2r$ around f as

$$\mathcal{B}_f^\beta(r)=\left\{x+iy\ :\ |x|<r,\ |y-f(x)|<\frac{\beta}{2}\right\};$$

let W be a planar Brownian path starting at $if(0)$, and denote $A_f^\beta(r)$ the event that the point $x+iy$ where W first reaches $\partial\mathcal{B}_f^\beta(r)$ satisfies $|x|=r$ (*i.e.* W exits $\mathcal{B}$ by one of the vertical parts of its boundary). Then

$$P(A_f^\beta(r))\geqslant\frac{1}{\pi}\exp\left[-\frac{\pi r}{\beta}(1+M^2)\right].$$

/// This is an easy consequence of the following estimate, which can be found in [1] and is a consequence of Proposition 2.9: If L is the extremal distance between both vertical parts of $\partial\mathcal{B}$ in $\mathcal{B}$, then

$$L\leqslant\frac{2r}{\beta}(1+M^2);$$

using this together with the classical estimate for Brownian motion in a strip provides the right estimate. ///

For the rest of this proof, we shall consider paths in the logarithmic space, denoted by the letter W; the actual path B is obtained from W by applying the exponential map — conformal invariance of Brownian motion then proves that B is a Brownian path. Let f be a function such as in the lemma: it is clear that if W^1 remains in $\mathcal{B}_f^\pi(r)$ and W^2 stays in $\mathcal{B}_{f+\pi}^\pi(r)$, then B^1 and B^2 do not intersect up to the first time they reach radius e^r or e^{-r}. Together with the fact that $P(A_f^\pi(r))=P(A_{f+\pi}^\pi(r))$, this leads to $P(E_R(\{1\}))\geqslant(P(A_f^\pi(\log R))/2)^2$, hence using the lemma:

$$P(E_R(\{1\}))\geqslant cR^{-2(1+M^2)}. \tag{2.24}$$

Doing the same with strips of width $\beta=\pi-\alpha/2$ (for which it can be seen that B^1 and B^2 can rotate around 0 by an angle at least $\alpha/2$ in each direction) leads to

$$P(E_R(\mathcal{C}_\alpha))\geqslant c\exp\left[-\frac{4\pi}{2\pi-\alpha}(1+M^2)\log R\right], \tag{2.25}$$

hence, letting $f=0$, a first bound on the exponent:

$$\xi(\mathcal{C}_\alpha)\leqslant\frac{4\pi}{2\pi-\alpha}$$

(this is also a direct consequence of $\mathcal{C}_\alpha \subset W_\alpha$ and the exact value of $\xi(W_\alpha)$, which happens to be precisely the upper bound we just obtained). Note that the bound is never less than 2, hence we proved nothing useful yet.

We now want to consider families of strips. Keep $\beta = \pi - \alpha/2$ and fix $\gamma > 0$; let $U_N = \{\pm 1\}^N$ and for $u \in U_N$ let f_u be constructed as follows:

- $f_u(0) = 0$, and for $1 \leqslant n \leqslant N$, $f_u(n\gamma) = \dfrac{\beta}{2} \displaystyle\sum_{k=1}^{n} u_k$;
- f is affine on each $[n\gamma, (n+1)\gamma]$, satisfies $f_u(x) = f_u(N\gamma)$ for all $x > N\gamma$ and $f_u(-x) = -f_u(x)$ for all x.

Then for $u \neq u'$ the intersection of $\mathcal{B}^\beta_{f_u}$ and $\mathcal{B}^\beta_{f_{u'}}$ is not connected, hence $A^\beta_{f_u}$ and $A^\beta_{f_{u'}}$ are disjoint. This leads to

$$P(E_R(\mathcal{C}_\alpha)) \geqslant c \sum_{u \in U_N} \exp\left[-\frac{2\pi}{\beta}(1 + (\beta/2\gamma)^2) \log R\right]$$

for all N, where $R = e^{N\gamma}$. Then using $P(E_R(\mathcal{C}_\alpha)) \asymp R^{-\xi(\mathcal{C}_\alpha)}$, noticing that all the terms of the sum are equal (there are 2^N of them) and applying a logarithm:

$$\xi(\mathcal{C}_\alpha) N\gamma \leqslant \frac{2\pi}{\beta}(1 + (\beta/2\gamma)^2) N\gamma - N \log 2 - \log c. \tag{2.26}$$

Divide by $N\gamma$ and let N go to infinity to obtain

$$\xi(\mathcal{C}_\alpha) \leqslant \frac{\pi\beta}{2} \left(\frac{1}{\gamma}\right)^2 - \log 2 \left(\frac{1}{\gamma}\right) + \frac{2\pi}{\beta}. \tag{2.27}$$

This is true for all $\gamma > 0$; the optimal value is $\gamma = \pi\beta / \log 2$, leading to

$$\xi(\mathcal{C}_\alpha) \leqslant \frac{4\pi}{2\pi - \alpha} \left[1 - \frac{(\log 2)^2}{4\pi^2}\right],$$

which is precisely what we wanted. //

Remark: The same proof gives a bound on $\xi(A)$ if A is included in a small ball centered at 1, as a function of the radius. But since it does not make use of the value of $\xi(\{1\})$, no modulus of continuity for ξ can be obtained this way. Cf. however equation (2.23) for another bound, which does provide such a modulus but is not quantitative.

As a consequence of this bound, we obtain the following

Theorem 2.4 :

> For all $\alpha < \log^2 2/2\pi$, the following holds: With probability 1, the set of local pivoting points of angle α on a planar Brownian path is non-empty and has a positive Hausdorff dimension.

Remark: The bound given in the theorem ($\log^2 2/2\pi \simeq 0.076$) is certainly not the best one; simulations suggest that there are pivoting points of any angle less than $3\pi/4 \simeq 2.356$ — cf. next subsection for details and figure 2.1 for a picture of a pivot of angle $\pi/2$. In particular, the maximal angle is conjectured to be greater than $2\pi/3$, and this seems to indicate that a discrete analogue of (local) pivoting points will appear on the exploration process of a critical percolation cluster on the triangular lattice [43, 44].

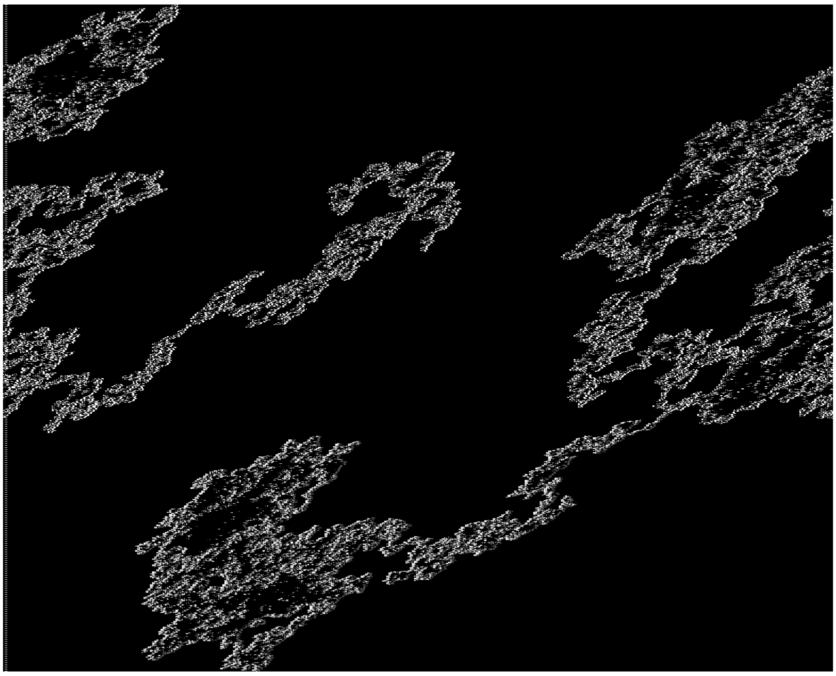

Figure 2.1: A pivoting point of angle $\pi/2$
(in gray is the image of one half of the path by a rotation of angle $+\pi/2$)

2.4.3 Conjectured and experimental values

Some exact values of $\xi(A)$ are known, cf. subsection 2.2. However, heuristic arguments seem to indicate that the formula giving the exponent for wedges is close to apply in other cases such as notably the "weak pivot" exponent, namely:

$$\xi(\{1, e^{i\theta}\}) \simeq \frac{5\pi/2}{2\pi - \theta}$$

for all $\theta \in [0, \pi]$ — corresponding to a continuous version of Proposition 2.1, point (iv). This is confirmed by simulations, at least for $\theta = \pi/2$ and $\theta = \mathrm{arctg}(3/4)$ (cf. table 2.1), based on the following

Conjecture

Let A be a bounded, non-empty subset of $\mathbb{Z}^2 \setminus \{0\}$; let B^1 and B^2 be independent Brownian paths starting respectively from 0 and 1, and S^1 and S^2 be independent standard random walks starting respectively from 0 and $(a, 0)$ with a sufficiently large (so as not to make the probability in the formula equal to 0). Then,

$$P(B^1_{[0,T]} \cap AB^2_{[0,T]} = \varnothing) \asymp P(S^1_{[0,T]} \cap AS^2_{[0,T]} = \varnothing) \asymp T^{-\xi(A)/2}.$$

// There is no known direct proof of the existence of a non-intersection exponent for random walks, the only way to obtain the desired behaviour is coupling with Brownian motion — cf. [28]. The present generalization can certainly be obtained in a similar way, note however that walks appear that are not standard simple random walks but take steps in $\{a, ia, -a, -ia\}$

for some $a \in \mathbb{C} \cap \mathbb{Z}^2$; exponents for such walks are the same as for SRW's (cf. [11]), but strong approximation in not yet proved. //

The most severe restriction is the assumption that $A \subset \mathbb{Z}^2 \setminus \{0\}$, in particular simulations cannot (yet) be performed if A is connected, except for very special cases such as wedges (where the exact exponent is known). However homogeneity can sometimes be used when $A \in \mathbb{Q}^2$ (as for $A = \{5, 4+3i\}$ which has the same exponent as $\{1, e^{i\theta}\}$ for $\theta = \mathrm{arctg}(3/4)$).

A	conjectured exponent	number of samples	computed exponent	relative error
$\{\pm 1\}$	~ 2.5	$2.6\,10^9$	2.501293	+0.05%
$\{1, i\}$	$\sim 5/3$	$3.0\,10^8$	1.662239	−0.27%
			1.668242∗	+0.09%
$\{5, 4+3i\}$	~ 1.392679	$1.2\,10^6$	1.382311	−0.74%
			1.394610∗	+0.14%
$\{5, 4+3i, 5i\}$	$\sim 5/3$	$1.6\,10^7$	1.662964	−0.22%
			1.665650∗	−0.06%

Table 2.1: Some simulated values of ξ
(100 000-step walks — exponents marked with a star are obtained after a non-rigorous correction)

2.5 Appendix

2.5.1 Sub-additivity

The following proposition is well known and included here only for completeness (note however that the bounds are not asymptotic and that the constants are exactly known, which is needed to derive continuity of ξ). A proof can be found *e.g.* in [12, Lemma 6.1.11].

Proposition 2.7 (Sub-additivity) :

Let $f : [1, \infty) \to (0, \infty)$ be some function such that:

- f is bounded and bounded away from 0 on any $[0, l]$, $l > 0$;
- There exist ε, A, c and C in $(0, \infty)$ such that for all $t \geqslant 1$, $ct^{-A} \leqslant f(t) \leqslant Ct^{-\varepsilon}$;
- There exist $0 \leqslant c_- \leqslant c_+ \leqslant \infty$, at least one of which finite and positive, such that
$$\forall t, t' \in [1, \infty) \quad c_- f(t) f(t') \leqslant f(tt') \leqslant c_+ f(t) f(t').$$

Then, there is a $\xi > 0$ such that $f(t) \approx t^{-\xi}$. Moreover, for all $t \geqslant 1$,
$$c_+^{-1} t^{-\xi} \leqslant f(t) \leqslant c_-^{-1} t^{-\xi}.$$

In particular, if both c_- and c_+ are in $(0, \infty)$ we get strong approximation: $f(t) \asymp t^{-\xi}$.

2.5.2 Extremal distance

Many of the known estimates for exponents (apart from cases where the exact value in known — such as the exponent of a cone here, and the intersection exponents in the half-plane in [31]) come from the corresponding estimates for Brownian paths in rectangles, using conformal invariance. The introduction of extremal distance generalizes the notion of aspect ratio of a rectangle and hence provides a natural parameter in this process.

Theorem and Definition :

Let Ω be an open, bounded, simply connected subset of $\mathbb{C}$, the frontier of which (oriented in the usual direct sense) is a Jordan curve $\gamma : [0,1] \to \partial\Omega$; fix four real numbers $0 < a < b < c < d < 1$. Then there exist a unique positive real number L and a unique conformal map $\Phi : \Omega \to (0, L) \times (0,1)$, with natural extension to $\bar{\Omega}$, such that $\Phi(\gamma(a)) = i$, $\Phi(\gamma(b)) = 0$, $\Phi(\gamma(c)) = L$ and $\Phi(\gamma(d)) = L + i$.

L is called *extremal distance* between $\partial_1 = \gamma([a,b])$ and $\partial_2 = \gamma([c,d])$ in Ω; it is denoted $\mathrm{d}_\Omega(\partial_1, \partial_2)$.

// For the proof of this result, and much more about conformal maps and related topics (including the proofs of Propositions 2.8 and 2.9), cf. [1]. //

Examples: The extremal distance between both sides of length a in an $a \times b$ rectangle is b/a. By the analyticity of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$, if $\Omega = \{\rho e^{i\theta} : r < \rho < R, 0 < \theta < \alpha\}$ with $0 < r < R < \infty$ and $0 < \alpha < 2\pi$, then the extremal distance in Ω between both circle arcs is $\alpha^{-1} \log(R/r)$. Finally, if L is the extremal distance in Ω between two connected parts ∂_1 and ∂_2 of $\partial\Omega$, then the extremal distance between the two components of $\partial\Omega \setminus (\partial_1 \cup \partial_2)$ is L^{-1}.

Proposition 2.8 :

Let $\rho : \Omega \to [0, \infty)$ be a continuous function, and denote $A_\rho(\Omega) = \iint_\Omega \rho^2$ and for any continuous arc γ in Ω, $L_\rho(\gamma) = \int_\gamma \rho(z)|\mathrm{d}z|$ (this defines the Riemannian metric associated with ρ). Then we have, thus giving a justification to the term *extremal length*, the following characterization of d_Ω:

$$\mathrm{d}_\Omega(\partial_1, \partial_2) = \operatorname*{Sup}_{\rho} \operatorname*{Inf}_{\gamma:\partial_1 \rightsquigarrow \partial_2} \frac{L_\rho(\gamma)^2}{A_\rho(\gamma)}$$

(where $\gamma : \partial_1 \rightsquigarrow \partial_2$ means that γ is a continuous path in Ω with first and second endpoints respectively in ∂_1 and ∂_2).

In many cases, it is sufficient to apply this with a finite family of ρ's to obtain a fairly good lower bound for d_Ω — usually even $\rho = 1$, *i.e.* taking the Euclidean metric, is sufficient. Another estimate for d_Ω is the following:

Proposition 2.9 :

Let L be a positive real number and f_1, $f_2 : [0, L] \to \mathbb{R}$ be two continuous functions such that for all t in $[0, L]$ we have $f_1(t) < f_2(t)$. Introduce $\Omega = \{x + iy : 0 < x < L, f_1(x) < y < f_2(x)\}$, and let ∂_1 and ∂_2 stand for the vertical components of $\partial\Omega$. Then:

$$\mathrm{d}_\Omega(\partial_1, \partial_2) \geqslant \int_0^L \frac{\mathrm{d}t}{f_2(t) - f_1(t)}.$$

Moreover, if f_1 has a continuous derivative and $f_2 = f_1 + a$, then

$$\mathrm{d}_\Omega(\partial_1, \partial_2) \leqslant \frac{L}{a}\left[1 + \|f_1'\|_\infty^2\right].$$

2.5.3 Some topological tools

In this section, all sets considered will be assumed non-empty.

Definition :

If A is a subset of the set $\mathbb{C}$ of complex numbers (or of any Banach space), note

$$V_r(A) = \{x \in \mathbb{C} : \mathrm{d}(x, A) < r\} = A + \mathcal{B}(0, r);$$

if A and B are two bounded subsets of $\mathbb{C}$, introduce the *Hausdorff distance* between A and B as

$$\mathrm{d}_H(A, B) = \mathrm{Inf}\{r : A \subset V_r(B), B \subset V_r(A)\}.$$

It is easy to see that d_H is nonnegative and satisfies the triangle inequality (namely $\mathrm{d}_H(A, B) \leqslant \mathrm{d}_H(A, C) + \mathrm{d}_H(C, B)$ for any A, B, C); moreover $\mathrm{d}_H(A, B) = 0$ if and only if $\bar{A} = \bar{B}$. Hence, d_H defines a metric topology on the set of compact subsets of $\mathbb{C}$, known as the *Hausdorff topology.*

We will need the following standard property about the Hausdorff topology on the subsets of some fixed set, describing the compact case:

Proposition 2.10 :

Let K be a compact subset of $\mathbb{C}$. Then the set $\mathcal{P}_c(K)$ of all (non-empty) closed subsets of K, equipped with the topology induced by the Hausdorff distance, is compact.

Remark: It is still true (and the proof is basically the same) that for any complete space E the set $\mathcal{P}_c(E)$ is complete. Moreover, if E is locally compact, so is $\mathcal{P}_c(E)$. However, it is generally not bounded, hence not compact.

Chapter 3

Hausdorff dimensions for SLE_6

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Introduction

It has been conjectured by theoretical physicists that various lattice models in statistical physics (such as percolation, Potts model, Ising, uniform spanning trees), taken at their critical point, have a continuous conformally invariant scaling limit when the mesh of the lattice tends to 0. Recently, Oded Schramm [43] introduced a family of random processes called Stochastic Loewner Evolutions (or SLE) which are the only possible conformally invariant scaling limits of random cluster interfaces (which are very closely related to all above-mentioned models).

An SLE process is defined using the usual Loewner equation, where the driving function is a time-changed Brownian motion: More specifically, in the present paper we will be mainly concerned with SLE in the upper-half plane (sometimes called chordal SLE), defined by the following PDE:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z \tag{3.1}$$

where (B_t) is a standard Brownian motion on the real line and κ is a positive parameter. It can be shown that this equation defines a family (g_t) of conformal mappings from simply connected domains (H_t) contained in the upper-half plane, onto $\mathbb{H}$. We shall denote by K_t the complement of H_t in $\mathbb{H}$: then for all $t > 0$, K_t is a compact subset of $\mathbb{H}$ and the family (K_t) is increasing. For each value $\kappa > 0$, this defines a random process denoted by SLE_κ (see e.g. [42] for more details on SLE).

In three cases, it has now been proved that SLE_κ is the scaling limit of a discrete model. Smirnov [44] proved that SLE_6 (which is one of the processes we will focus on in the present paper) is the scaling limit of critical site percolation interfaces on the triangular grid, and Lawler-Schramm-Werner [34] have proved that SLE_2 and SLE_8 are the respective scaling limits of planar loop-erased random walks and uniform Peano curves. In fact, we will use Smirnov's result as a key argument in the present paper.

It is natural to study the geometry of SLE_κ, and in particular, its dependence on κ. It is now known (see Rohde and Schramm [42] for $\kappa \neq 8$ and Lawler-Schramm-Werner [34] for $\kappa = 8$) that there almost surely exists a continuous curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ (called the *trace* of the SLE) that generates K_t, in the following sense: H_t is the infinite component of $\mathbb{H} \setminus \gamma([0, t])$. Furthermore (see [42]), γ is a simple curve when $\kappa \leq 4$, and it is a space-filling curve when $\kappa \geq 8$.

It is possible, for each $x \in \mathbb{H}$, to evaluate the asymptotics when $\varepsilon \to 0$ of the probability that γ intersects the disk of radius ε around x. When $\kappa < 8$, this probability decays like ε^α for some $\alpha = \alpha(\kappa) > 0$. This (loosely speaking) shows that the expected number of balls of radius ε needed to cover $\gamma[0, 1]$ (say) is of the order of $\varepsilon^{-2+\alpha}$, and implies that the Hausdorff dimension of γ is not larger than $2 - \alpha$. Rohde and Schramm [42] used this strategy to show that almost surely the Hausdorff dimension of the SLE_κ trace is not larger than $1 + \kappa/8$ when $\kappa \leq 8$.

This exponent α and various other exponents describing exceptional subsets of γ are closely related to critical exponents that describe the behaviour near the critical point of some functionals of the related statistical physics model. Actually, in the physics literature, the derivation of the exponent is often announced in terms of (almost sure) fractal dimension, thereby omitting to prove the lower bound on the dimension. Indeed, it may a priori be the case that the value $\varepsilon^{-2+\alpha}$ is due to exceptional realizations of SLE_κ with exceptionally many visited balls of radius ε, while "typical" realizations of SLE_κ meet much less disks. One usual way to exclude such a possibility and to prove that $2 - \alpha$ corresponds to the almost sure dimension of a random fractal is to estimate second-moments, *i.e.* given *two* balls of radius ε, to estimate the probability that the SLE trace intersects both of them.

It is conjectured that for all $\kappa \in [0, 8]$, the Hausdorff dimension of the trace of SLE_κ is indeed almost surely $1 + \kappa/8$. Up to the present paper, this is known to hold for $\kappa = 8/3$

for reasons that will be described below. We prove that it is the case for $\kappa = 6$:

Theorem 3.1 :

Almost surely, the dimension of the SLE_6 trace is $7/4$.

Note that the discrete analog of this Theorem in terms of percolation is an open problem, while it is known that the expected number of steps of a discrete exploration process is $N^{7/4}$ (cf. [45] for further reference).

Another natural object is the *boundary* of an SLE, namely $\partial K_t \cap \mathbb{H}$. For $\kappa \leqslant 4$, since γ is a simple curve, the boundary of the SLE is the SLE itself; for $\kappa > 4$, it is a strict subset of the trace, and its dimension is conjectured to have dimension $1 + 2/\kappa$. Again, the first moment estimate is known to hold for all κ, but the only value of $\kappa > 4$ for which the dimension is known rigorously is $\kappa = 6$:

Theorem 3.2 (Lawler-Schramm-Werner [30]) :

Almost surely, the dimension of the SLE_6 boundary is $4/3$.

It is known that SLE_6 is closely related to planar Brownian motion, so that this theorem is equivalent to the same statement for the exterior boundary of a Brownian path. It was first conjectured by Mandelbrot that the fractal dimension of the boundary should be $4/3$; the first mathematical proof is due to Lawler, Schramm and Werner (cf. [30] for a review) and goes as follows.

First, note that to each point of the Brownian path, two independent Brownian motions can be associated (the past and the future), and that this point is on the boundary of the complete path iff the union of these two processes does not disconnect it from infinity. This remark provides a relation between the dimension of the boundary and the non-disconnection exponent for two paths. It is then necessary to compute the value of this exponent, and this requires a long and very technical proof. In particular, it uses the fact that the Brownian intersection exponents are analytic [33] and sharp estimates for the probabilities of non-disconnection events (these estimates, up to the value of the exponents, were obtained earlier by Lawler in a series of clever and technical papers).

It is conjectured (see [42] for a discussion) that the boundary of SLE_κ, $\kappa > 4$ is very similar to the trace of $SLE_{16/\kappa}$, and a precise statement of this *duality* is known for $\kappa = 6$ [29]: this and Theorem 3.2 provide the dimension of $SLE_{8/3}$, namely: With probability 1, the dimension of the $SLE_{8/3}$ trace is $4/3$.

In the present paper, we will reprove, without using the relation to planar Brownian motion, that the dimension of the outer frontier of SLE_6 is almost surely $4/3$. Combining this with the previously mentioned universality arguments, this implies also that the dimension of the $SLE_{8/3}$ trace and that of the outer frontier of planar Brownian motion are almost surely $4/3$ and gives a shorter proof of these results. We should also mention here that $SLE_{8/3}$ is the natural candidate for the scaling limit of self-avoiding walks [35] and therefore also an interesting object.

Theorem 3.1 can be related to the dimension of *pioneer points* on a Brownian path (*i.e.* points B_t that are on the boundary at time t): It is known [32] that the set of pioneer points has dimension $7/4$, the same as the SLE_6 trace, and this is not surprising since they play similar roles. However, it can be proved that they are different (note for instance that Brownian motion can enter its past hull and the SLE trace cannot).

The method described here cannot be extended directly to other values of κ. Indeed, two properties that are specific to SLE_6 are used, namely the chordal/radial equivalence (in the computation of the hitting probabilities) and the locality property (in the derivation of second moments). It should be possible to obtain second moments using only the Markov property (at the cost of a more technical proof); however, the derivation of the hitting probabilities will need a different approach.

It is also possible to compute the dimension of exceptional time-sets. This is in fact easier than for subsets of the upper-half plane, since the distortion of space due to the past does not influence the probability estimates — and this makes it possible to compute dimensions for every $\kappa \geqslant 0$. In the last section we compute the dimension of the set of boundary times and that of the set of cut-times (*i.e.* times t such that $\gamma(t)$ is, respectively, a boundary point or a cut-point of K). In particular, we prove the following

Theorem 3.3 :

Let (K_t) be an SLE_κ for $\kappa < 8$. Then, almost surely, K_1 has cut-points.

Acknowledgments

I wish to thank Greg Lawler, Oded Schramm and Wendelin Werner for very useful discussions.

3.1 Ingredients

We provide in this section several estimates and tools which will be needed in the subsequent proofs, but are also (maybe) of more general interest.

3.1.1 Hausdorff dimension of random sets

We will use the following result to derive the value of Hausdorff dimensions from the values of exponents. It is stated here in dimension $d \geq 1$, but we will use it only for $d = 1$ (for time sets) or $d = 2$ (for subsets of the complex plane).

Suppose that λ denotes the Lebesgue measure in $[0,1]^d$. Let $(C_\varepsilon)_{\varepsilon>0}$ be a family of random Borelian subsets of the cube $[0,1]^d$. Assume that for $\varepsilon < \varepsilon'$ we have $C_\varepsilon \subseteq C_{\varepsilon'}$, and let $C = \bigcap C_\varepsilon$. Define the following conditions (where $f \asymp g$ means that there exist positive numbers c_- and c_+ such that $c_- g \leqslant f \leqslant c_+ g$, and where the constants do not depend on ε, x nor y):

1. For all $x \in [0,1]^d$,
$$P(x \in C_\varepsilon) \asymp \varepsilon^s \ ;$$

2. There exists $c > 0$ such that for all $x \in [0,1]^d$ and ε,
$$P(\lambda(C_\varepsilon \cap \mathcal{B}(x,\varepsilon)) > c\varepsilon^d | x \in C_\varepsilon) \geqslant c > 0 \ ;$$

3. There exists $c > 0$ such that for all x, $y \in [0,1]^d$ and ε,
$$P(\{x,y\} \subset C_\varepsilon) \leqslant c\varepsilon^{2s}|x-y|^{-s}.$$

Proposition 3.1 :

(i). If conditions 1. and 2. hold, then a.s. $\dim_H(C) \leqslant d - s$;

(ii). If conditions 1. and 3. hold, then with positive probability $\dim_H(C) \geqslant d - s$.

// A detailed proof of this Proposition can be found in [5] (Theorem 2). The outline goes as follows. First, if conditions 1. and 2. hold, they provide an upper bound on the expected number of balls of radius ε needed to cover C_ε, hence C. By Borel-Cantelli, this gives an upper bound on the Minkowski dimension of C, which is valid with probability 1.

To derive a lower bound one introduces the random measures μ_ε having density $\varepsilon^{-s}\mathbf{1}_{C_\varepsilon}$ with respect to the Lebesgue measure in $[0,1]^d$. If 1. and 3. hold, with positive probability it is possible to extract a sub-sequence μ_{ε_k} converging to some measure μ supported on C, and to prove that with positive probability μ is a Frostman measure with dimension $d-s$, which implies that the Hausdorff dimension of the support of μ is at least $d-s$. //

Each time we will derive almost sure Hausdorff dimension, we will in fact check these three conditions and use a zero-one law to conclude.

Remark: A similar proposition can be found in [27], stated in a discrete setup in which condition 2. does not appear. Indeed, in most cases, this condition is a direct consequence of condition 1. and the definition of C_ε (for instance, if C_ε is a union of balls of radius ε).

3.1.2 An estimate for diffusions

We will need estimates for stochastic flows in an interval, that we now state and prove. For background on this topic cf. for instance [2].

Let (X_t) be the diffusion process on the interval $I = [-1,1]$ defined by the following stochastic differential equation:

$$\mathrm{d}X_t = \sigma \mathrm{d}B_t + f(X_t)\mathrm{d}t, \tag{3.2}$$

where $\sigma > 0$ and f is a given smooth function satisfying $f' \leqslant -a < 0$ and:

$$-f(-1+x) \sim f(1-x) \sim -C_0.x^{-1}, \tag{3.3}$$

$$f'(-1+x) \sim f'(1-x) \sim -C_1.x^{-2}, \tag{3.4}$$

$$-f''(-1+x) \sim f''(1-x) \sim -C_2.x^{-3} \tag{3.5}$$

as $x \to 0+$ for some positive constants C_1, C_2, C_3.

Let (g_t) be the stochastic flow associated to this stochastic differential equation, *i.e.* $(g_t)_{t\geqslant 0}$ is the family of random functions from I to itself such that $g_t(x)$ is the value at time t of the solution of (3.2) starting from x at $t=0$. Note that X is absorbed on the points 1 and -1. This implies that, with probability 1, for all $t > 0$, there is an interval $I_t \subset I$ such that

$$g_t(I) = \{-1,1\} \cup I_t.$$

We want to estimate the length l_t of I_t. Consider the following family of partial differential equations, indexed by $b \geqslant 0$:

$$(E_b) \quad \dot{h}(t,x) = \frac{\sigma^2}{2} h''(t,x) + f(x)h'(t,x) - bf'(h)h(t,x).$$

Assume that for each $b \geq 0$, (E_b) has a positive solution $h_b(t,x)$ satisfying

$$h_b(t,x) \asymp [(1+x)(1-x)]^{q(b)} e^{-\lambda(b)t} \tag{3.6}$$

It is then possible, using the Feynman-Kac formula (following exactly [32]), to prove that if $b > 0$,

$$E((g_t'(x))^b) \asymp e^{-\lambda(b)t}[(1+x)(1-x)]^{q(b)} \tag{3.7}$$

(where as usual we let $g_t'(x) = 0$ if the path starting from x is absorbed by the boundary before time t). For all x, let

$$\tau_x = \mathrm{Inf}\{t : g_t(x) \in \{-1,1\}\} = \mathrm{Inf}\{t : g_t'(x) = 0\}.$$

Lemma 3.1 :

In the previous setup,

$$\forall b > 0 \quad E(l_t^b \mathbf{1}_{\tau_0 > t}) \asymp e^{-\lambda(b)t}.$$

Note that this type of result does not seem to be standard in the literature on diffusions. The natural way to obtain estimates on the length of I_t is to use Jensen's inequality, and depending on the value of b it can give a lower bound (if $b < 1$) or an upper bound (if $b > 1$) of the right form. Another way to obtain a lower bound is given in [32], and consists in computing the length of the image of a small interval around 0, thus giving a lower bound in terms of $g'(0)$ which is valid for all $b > 0$. Hence, all that needs to be done to complete the proof is to derive the upper bound in the case $b < 1$.

// The idea is to write the length l_t as the integral of g_t' over I and to obtain an upper bound on $g_t'(x)$. Two cases contribute to the estimate:

- If $g_s(x)$ stays away from the boundary for $s \leq t$, then $g_t''(x)$ is bounded by above and it will be possible to compare $g_t'(x)$ to $g_t'(0)$ and use (3.7);
- If $g_s(x)$ comes close to the boundary for $s \leq t$, then $g_t'(x)$ becomes very small and does not contribute to l_t anymore.

The definition of g implies that for all $x \in I$,

$$g_t'(x) = \exp\left[\int_0^t f'(g_s(x))\,\mathrm{d}s\right], \tag{3.8}$$

and differentiating this with respect to x leads to

$$\frac{g''(x)}{g'(x)} = \int_0^t g_s'(x) f''(g_s(x))\,\mathrm{d}s. \tag{3.9}$$

Moreover, since f' is bounded by $-a < 0$, Equation (3.8) also proves that almost surely, for all $t > 0$ and for all $x \in I$,

$$g_t'(x) \leqslant e^{-at} \tag{3.10}$$

and in particular $l_t \leqslant 2e^{-at}$.

Let $\alpha > 0$ and $J_s = [-1+\alpha e^{-as/4}, 1-\alpha e^{-as/4}]$: If for all $s > 0$, $g_s(x) \in J_s$, then condition (3.5) leads to $|f''(g_s(x))| \leqslant C_2 \alpha^{-3} e^{3as/4}$, hence

$$\left| \frac{g_t''(x)}{g_t'(x)} \right| \leqslant \int_0^t C_2 e^{-as} e^{3as/4} \, \mathrm{d}s \leqslant 4C_2 a^{-1} \alpha^{-3}.$$

Assume that for all $s \in [0, t]$, $g_s(0) \in J_s$ (so that the previous estimate applies for $x = 0$). For all $x \in (-1, 0)$ such that $\tau_x > t$, write

$$\int_0^t f'(g_s(x))\mathrm{d}s = \int_0^t f'(g_s(x))\mathbf{1}_{g_s(x)\in J_s}\mathrm{d}s + \int_0^t f'(g_s(x))\mathbf{1}_{g_s(x)\notin J_s}\mathrm{d}s.$$

In the first integral, integrating f'' over $[g_s(x), g_s(0)]$ (which is a subset of J_s) and using (3.5) shows that

$$|f'(g_s(x)) - f'(g_s(0))| \leqslant Ce^{-as}(\alpha e^{-as/4})^{-3} = C\alpha^{-3}e^{-as/4}. \tag{3.11}$$

In the second one, since g_s is monotone, $g_s(x)$ can only be in $[-1, -1 + \alpha e^{-as/4}]$, on which f' is negative and increasing. Hence, $f'(g_s(x)) \leqslant f'(-1+\alpha e^{-as/4})$, and integrating f'' between $-1 + \alpha e^{-as/4}$ and $g_s(0)$ as previously leads to

$$|f'(-1 + \alpha e^{-as/4}) - f'(g_s(0))| \leqslant C\alpha^{-3}e^{-as/4}.$$

In both cases we finally obtain

$$f'(g_s(x)) \leqslant f'(g_s(0)) + Ce^{-as}(\alpha e^{-as/4})^{-3}$$

and integrating over $s \in [0, t]$ then proves that

$$g_t'(x) \leqslant \exp\left[C\alpha^{-3} + \int_0^t f'(g_s(0))\mathrm{d}s\right] \leqslant Kg_t'(0).$$

A similar computation shows that this also holds for $x \in (0, 1)$. Integrating this inequality leads to $l_t \leqslant 2Kg_t'(0)$, hence to the desired conclusion — on the event $\{\forall s \in [0, t], g_s(0) \in J_s\}$.

The very same argument can be applied on the interval $[t_1, t_2]$, starting from $g_{t_1}(0)$ instead of 0 (but the estimate remains valid). It shows that conditionally to the fact that $g_s(0)$ stays in J_s for all s in this interval, we have

$$l_{t_2} \leqslant Kl_{t_1}(g_{t_2} \circ g_{t_1}^{-1})'(g_{t_1}(0)) = Kl_{t_1}\frac{g_{t_2}'(0)}{g_{t_1}'(0)}. \tag{3.12}$$

Besides, the Markov property at time t_1 shows that the quotient $g'_{t_2}(0)/g'_{t_1}(0)$ is independent of $\mathcal{F}_{t_1}$ given the value of $g_{t_1}(0)$. Moreover, choosing α large enough can make K arbitrarily close to 1.

We now have to consider the "bad" case where $g_s(0)$ exits J_s (and this will happen in particular for small values of s, for which J_s can even be empty if α is large enough). For this, we shall count the number of times it does it and use the previous estimate (3.12) between those times. More precisely, let $t_n = \log n$. Scaling shows that if $g_s(0)$ is outside J_s for some $s_0 \in [t_n, t_{n+1}]$, then with positive probability (independent of n and α) it stays outside of $[-1+\alpha^{1/2}e^{-as_0/8}, 1-\alpha^{1/2}e^{-as_0/8}]$ longer than $t_{n+1}-t_n$. If this is the case, using the condition (3.4) together with (3.10), we obtain $l_{t_{n+2}} \leqslant c.l_{t_n}$ with positive probability, for some constant $c<1$ (still independent of α). Hence, since we know *a priori* that l^b is decreasing (*e.g.* by (3.10)), its expected value also decreases by a factor $c_0 < 1$.

Now let $n_1 < \ldots < n_{N_t}$ be the integers n such that $g_s(0)$ exits J_s in $[t_n, t_{n+1}]$, and satisfying $t_n \leqslant t$. At each t_n the expectation of l^b decreases by a factor c_0; and between two such times the previous method can be applied, providing a factor $K.e^{-\nu(t_{n_{k+1}}-t_{n_k+2})}$. Putting all the slices together, we obtain

$$E(l_t^b|N_t, n_1, \ldots, n_{N_t}) \leqslant (c_0K)^{N_t}e^{-\nu(t-L_t)}$$

where L_t is the total length of the "bad intervals", *i.e.*

$$L_t = \sum_{k=1}^{N_t}(t_{n_k+2}-t_{n_k}).$$

But the sequence $(t_{n+1}-t_n)$ is decreasing: Hence $L_t \leqslant t_{2N_t}$, whatever the exact values of the n_k's. Hence for all N we obtain

$$E(l_t^b|N_t = N) \leqslant N^{\beta}(c_0K)^N e^{-\nu(b)t}.$$

Since $c_0 < 1$ it is now possible to choose α so as to ensure that $c_0K < 1$, in which case we can sum the previous estimate over all possible values of N to finally obtain the correct upper bound on $E(l_t^b)$. //

Lemma 3.2 :

In the same setup, the probability that a given point x survives up to time $t>0$ is

$$P(\tau_x > t) = P(g'_t(x) > 0) \asymp e^{-\lambda(0)t}.$$

// We know that $E(h_0(0, g_t(x))) = h_0(t,x) \asymp e^{-\lambda(0)t}$. On the other hand, since h_0 is bounded, we have $E(h_0(0,g_t(x))) \leqslant \|h_0(0,.)\|_\infty P(\tau_x > t)$: hence

$$P(\tau_x > t) \geqslant \frac{ce^{-\lambda(0)t}}{\|h_0(0,.)\|_\infty}.$$

Conversely, consider the distribution of $g_1(x)$. It is easy to see that, except for Dirac masses at -1 and 1, it has a bounded density p_1 with respect to the Lebesgue measure. Since h is positive, we know that $-\lambda(0)$ is the largest eigenvalue of the generator of the diffusion, and that it is simple; hence, $\|p_t\|_2 \leqslant \|p_1\|_2 \exp(-(t-1)\lambda(0))$. It is then a direct application of the Cauchy-Schwarz inequality to see that $\|p_t\|_1 \leqslant C.e^{-\lambda(0)t}$, and since we have $\|p_t\|_1 = P(\tau_x > t)$ this completes the proof of the Lemma. //

3.2 Dimension of the trace of SLE_6

3.2.1 Construction of the trace

Let K be a chordal SLE in the upper-half plane and C be the intersection of its trace with the square $[-1,1] \times [1,3]$. In order to apply Proposition 3.1, introduce

$$C_\varepsilon = \{z \in [-1,1] \times [1,3] : \mathrm{d}(z, C) \leqslant \varepsilon\}.$$

Since C is a compact set, we have $C = \bigcap C_\varepsilon$. Moreover, we make the following remark: Let z be some point in $[-1,1] \times [1,3]$, $\varepsilon > 0$, and assume that z is at distance greater than ε from the boundary of the square. Let $T_{\mathcal{B}(z,\varepsilon)}$ be the hitting time defined as usual as

$$T_{\mathcal{B}(z,\varepsilon)} \stackrel{\triangle}{=} \mathrm{Inf}\{t : K_t \cap \mathcal{B}(z,\varepsilon) \neq \varnothing\}.$$

Then, we have the following equivalence:

$$z \in C_\varepsilon \quad \Longleftrightarrow \quad \mathcal{B}(z,\varepsilon) \not\subset K_{T_{\mathcal{B}(z,\varepsilon)}}. \tag{3.13}$$

We call the second part of the equivalence *non-disconnection*: Indeed, the condition is equivalent to the fact that $K_{T_{\mathcal{B}(z,\varepsilon)}}$ does not disconnect z from ∞. Note the similarity with the definition of Brownian pioneer points [27].

3.2.2 The (non-)disconnection exponent

The proofs in this section rely on the equivalence between chordal and radial SLE for $\kappa = 6$ that have been proved in [32]. More precisely, there are two versions of SLE in the unit disk. The first one (chordal SLE in the disk) is obtained by mapping chordal SLE in the upper-half plane to the disk by a conformal map — so that it grows toward a point on the unit circle. The second version is called *radial SLE*, and it corresponds to the case where K grows toward 0 instead of a point on the boundary. It is defined by the following PDE (if $(\tilde{g}_t)$ is the corresponding family of conformal maps):

$$\partial_t \tilde{g}_t(z) = \tilde{g}_t(z) \frac{\tilde{g}_t(z) + \beta_t}{\tilde{g}_t(z) - \beta_t}$$

where $\beta_t = e^{i\sqrt{\kappa}t}$ is a time-scaled Brownian motion on the unit circle.

Chordal/radial equivalence is stated as follows. Let (K_t) be a *chordal SLE_6* in the unit disk, starting at 1 and aiming at -1, and $(\tilde{K}_t)$ be a *radial SLE_6* in the unit disk, starting from 1 and aiming at 0. Let T (resp. $\tilde{T}$) be the first time when K (resp. $\tilde{K}$) separates -1 from 0. Then, K_{T^-} and $\tilde{K}_{\tilde{T}^-}$ have the same law, and so do $(K_{t \wedge T})_{t>0}$ and

$(\tilde{K}_{t\wedge\tilde{T}})_{t>0}$ up to a (random) time change. For complete references about this, cf. [32]. Note that this is specific to the case $\kappa = 6$.

Proposition 3.2 :

Let (K_t) be a chordal SLE_6 in the unit disk, starting from 1 and growing toward -1, and T_r the first time when K_t hits the ball with radius r centered at 0. Then K_{T_r} disconnects this disk from the unit circle if and only if $\mathcal{B}(0,r) \subset K_{T_r}$, and as r tends to 0,

$$p(r) \stackrel{\triangle}{=} P(\mathcal{B}(0,r) \not\subset K_{T_r}) \asymp r^{1/4}.$$

// This estimate is similar to theorem 3.1 in [32], of which it is the natural counterpart in the case $b = 0$. Let K' be a *radial* SLE_6 in the unit disk, aiming at 0, and T_r' be the first time when it reaches $\mathcal{B}(0,r)$. The chordal/radial equivalence shows that $p(r)$ is equal to the probability that $K'_{T_r'}$ does not disconnect $\mathcal{B}(0,r)$ from -1 — *i.e.* the probability that $-1 \notin K'_{T_r'}$.

Let $W_t = e^{i\sqrt{6}B_t}$ be the (time-scaled) Brownian motion on $\partial\mathbb{U}$ driving (K_t') (where (B_t) is a standard Brownian motion on $\mathbb{R}$), and Y_t be the continuous determination of the argument of $g_t(-1)/W_t$ starting at π. Y_t is well defined as long as K' does not reach -1. Loewner's differential equation and Itô's formula show that

$$\mathrm{d}Y_t = \sqrt{6}\,\mathrm{d}B_t + \mathrm{cotg}(Y_t/2)\,\mathrm{d}t,$$

i.e. (Y_t) is a diffusion process with diffusion $\sqrt{6}$ and drift $\mathrm{cotg}(\cdot/2)$, absorbed by $\{0, 2\pi\}$ when -1 is absorbed by K_t'. Straightforward calculations prove that $f_t(x) = e^{-t/4}(\sin y/2)^{1/3}$ satisfies $\partial_t f_t = Lf_t = -\frac{1}{4}f_t$; we can now apply Lemma 3.2 and obtain

$$P(-1 \notin K_t') \asymp e^{-t/4}. \tag{3.14}$$

But Köbe's distortion theorem [41] states that, if $r(t) = \mathrm{d}(0, K_t)$, then

$$\frac{e^{-t}}{4} \leqslant r(t) \leqslant e^{-t},$$

which, combined with estimate (3.14), proves the Proposition (details of the last step are the same as in [32]). //

Corollary 3.1 :

Fix $\eta > 0$, and let $\mathcal{B} = \mathcal{B}(z,r)$ be some disk contained in $\mathbb{U}$, where $|z| < 1 - 2\eta$ and $r < \eta$; let (K_t) be a chordal SLE_6 in the unit disk, starting from 1 and aiming at -1. If $T_\mathcal{B}$ denotes the first time when K_t reaches $\mathcal{B}$, then the probability $p(\mathcal{B})$ that $K_{T_\mathcal{B}}$ does not disconnect $\mathcal{B}$ from -1 satisfies

$$p(\mathcal{B}) \asymp r^{1/4},$$

where the implicit constants depend only on η.

// There exists exactly one Möbius transform $\Phi : \mathbb{U} \to \mathbb{U}$ mapping 1 to itself and $\mathcal{B}$ to a disk centered at 0. The radius of $\Phi(\mathcal{B})$ is then

$$\rho(z,r) = \frac{(1 + r^2 - |z|^2) - \sqrt{(1 + r^2 - |z|^2)^2 - 4r^2}}{2r} \asymp r.$$

$\Phi(K)$ is then a chordal SLE in the disk starting from 1 and aiming at $\Phi(-1)$. Moreover, $|\Phi(-1)-1|$ is bounded away from 0 by a constant. The proof of Proposition 3.2 can then be adapted (only changing the position of the end point) to show that

$$p(\mathcal{B}) \asymp p(\rho(z,r)) \asymp p(r) \asymp r^{-1/4},$$

with constants depending only on η. //

It is then easy, by mapping the disk to the upper half-plane and using (3.13), to turn this corollary into the first condition of Proposition 3.1, *i.e.*:

$$\forall z \in [-1,1]\times[1,3] \quad P(z \in C_\varepsilon) \asymp \varepsilon^{1/4}. \tag{3.15}$$

It then follows from the definition of C_ε that condition 2. holds: If $z \in C_\varepsilon$, let $z' \in C$ such that $|z-z'| = \varepsilon$ (which exists by a compactness argument), then the disk with diameter $[zz']$ is contained in $\mathcal{B}(z,\varepsilon)\cap C_\varepsilon$ and it has area $\pi\varepsilon^2/4$.

3.2.3 Percolation and second moments

We now turn our attention to the derivation of second moments for the hitting probability of disks by the SLE_6 trace, namely condition 3. in Proposition 3.1. Again we will make strong use of the fact that we are in the case $\kappa = 6$, and in fact the decay of correlations we obtain is a consequence of the locality property of SLE_6. It has been proved [44, 45] that the exploration process of critical percolation on the triangular lattice converges to the SLE_6 trace; in particular, consider critical percolation on a discretization of the upper-half plane with mesh $\delta > 0$ and the usual boundary conditions (*i.e.* wired on $[0,+\infty)$ and free on $(-\infty,0)$): Then the probability that the discrete exploration process γ_δ hits the ball $\mathcal{B}(i,\varepsilon)$ satisfies:

$$P\left(\gamma_\delta \cap \mathcal{B}(i,\varepsilon) \neq \varnothing\right) \underset{\delta\to 0}{\longrightarrow} P(i \in C_\varepsilon) \asymp \varepsilon^{1/4}. \tag{3.16}$$

But the fact that the discrete exploration process touches this disk is equivalent to the existence of both a closed path connecting the disk to $[0,+\infty)$ and an open path connecting the disk to $(-\infty,0)$. Applying the results in [45], this leads to the following

Corollary 3.2 :

Let A_ε be the annulus centered at 0, with radii ε and 1. For all $\delta > 0$, consider critical site-percolation on the intersection of A_ε with the triangular lattice of mesh δ. Let $p(\varepsilon,\delta)$ be the probability that $\mathcal{C}(0,\varepsilon)$ is connected to $\mathcal{C}(0,1)$ both by a path of open vertices and a path of closed vertices in A_ε. Then, as δ tends to 0, $p(\varepsilon,\delta)$ converges to some $p(\varepsilon)$ satisfying

$$p(\varepsilon) \asymp \varepsilon^{1/4}.$$

Note that this says nothing about the speed of convergence, and hence does not provide useful estimates for the probability of the discrete event itself — but it is sufficient for our purpose here.

Now fix z, $z' \in [-1,1]\times[1,3]$ and $\varepsilon < |z-z'|/2$. Again, the probability that the SLE_6 trace touches both $\mathcal{B}(z,\varepsilon)$ and $\mathcal{B}(z',\varepsilon)$ can be written as the limit, as δ goes to 0, of the corresponding probability for critical site-percolation on the triangular lattice with mesh δ. But this implies the following:

- There exist a path of open vertices and a path of closed vertices, both connecting $\mathcal{C}(z,\varepsilon)$ to $\mathcal{C}(z,|z-z'|/2)$ inside $\mathcal{B}(z,|z-z'|/2)$;

- There exist a path of open vertices and a path of closed vertices, both connecting $\mathcal{C}(z',\varepsilon)$ to $\mathcal{C}(z',|z-z'|/2)$ inside $\mathcal{B}(z',|z-z'|/2)$;

- There exist a path of open vertices and a path of closed vertices, both connecting $\mathcal{C}((z+z')/2,|z-z'|)$ to the real axis outside $\mathcal{B}((z+z')/2,|z-z'|)$.

Those three events are independent, since they describe the behaviour of pairwise disjoint sets of vertices; besides, the probability of each of them can be estimated using Corollary 3.2 and converges, as $\delta \to 0$ and up to universal multiplicative constants, respectively to $(\varepsilon/d)^{1/4}$, $(\varepsilon/d)^{1/4}$ and $d^{1/4}$, with $d=|z-z'|$. Hence, letting δ go to 0, we obtain the following estimate:

$$P(\{z,z'\} \subset C-\varepsilon) \leqslant C.\left(\frac{\varepsilon}{|z-z'|}\right)^{1/2} |z-z'|^{1/4} = C\,\frac{\varepsilon^{1/2}}{|z-z'|^{1/4}}, \tag{3.17}$$

which is exactly condition 3. in Proposition 3.1 with $s=1/4$, as we wanted.

3.2.4 Conclusion

It is now possible to apply Proposition 3.1 with $s=1/4$: We obtain

$$P(\dim_H(C) \leqslant 7/4) = 1, \qquad P(\dim_H(C) = 7/4) > 0.$$

Now let $\mathcal{H}_\infty$ be the complete trace of K. Since $C \subset \mathcal{H}_\infty$, we obtain the same results for $\mathcal{H}_\infty$. Theorem 3.1 is then a consequence of the following

Lemma 3.3 (0–1 law for the trace) :

For all $d \in [0,2]$, we have

$$P(\dim_H(\mathcal{H}_\infty) = d) \in \{0,1\}.$$

// For all $n \in \mathbb{Z}$, let $D_n = \dim_H(\mathcal{H}_{2^n})$. For all n, we then have $D_{n+1} \geqslant D_n$ (because $(\mathcal{H}_t)$ is increasing) and besides D_n and D_{n+1} have the same law (by the scaling property). Hence, almost surely, for all m, n, we have $D_n = D_m$. Taking this to the limit gives $P(\dim_H(\mathcal{H}_\infty) = D_n) = 1$, hence the random variable $\dim_H(\mathcal{H}_\infty)$ is $\mathcal{F}_{2^n}$-measurable for all n. Hence it is $\mathcal{F}_{0+}$-measurable, and we know by Blumenthal's zero-one law that this σ-field is trivial. //

3.3 Dimension of the boundary of SLE_6

In this section we adapt the previous proof to compute the Hausdorff dimension of the boundary of K at some fixed time.

3.3.1 The escape probability

Proposition 3.3 :

Let (K'_t) be an SLE_6 in the half-plane, and T'_R be the first time it reaches radius R. Then, as R goes to infinity,

$$P(i \notin K'_{T'_R}) \asymp R^{-1/3}.$$

Note that the corresponding result for $P(1 \notin K'_{T'_R})$ has been derived in [32].

// We shall suppose that K' starts at 3 instead of 0 — it is easy to see that this only changes the estimates up to a fixed constant. For each $t \geqslant 0$ such that $0 \notin K_t$, the intersection of $\mathcal{C}(0,2)$ with H_t is an at most countable union of open arcs, one of which covers the angles from some $\alpha_t \in (0,\pi)$ up to π (the "leftmost arc"). Introduce the following stopping times (where $S_0 = T_0 = 0$), for all $n > 0$:

$$S_n = \mathrm{Inf}\{\ t > T_{n-1} : \alpha_t > \alpha_{T_{n-1}} \},$$
$$T_n = \mathrm{Inf}\{\ t > S_n : K'_t \cap \mathcal{C}(0,3) \neq K'_{S_n} \cap \mathcal{C}(0,3)\ \}.$$

(Loosely speaking, S_n is the first time after T_{n-1} when the process "touches" the circle of radius 2 and T_n the first time after S_n when it returns to the circle of radius 3.) Moreover, let $T = \mathrm{Inf}\{t : K'_t \cap \mathcal{C}(0,R) \neq \varnothing\}$. Then, almost surely, the integer $N = \mathrm{Sup}\{n : T_n < T\}$ is finite and we have

$$0 = T_0 < S_1 < T_1 < \ldots < S_N < T_N < T < \infty$$

(*i.e.* K' crosses the annulus between radii 2 and 3 only finitely many times before reaching radius R). In the Brownian case, N would be exponential with parameter $\log(3/2)/\log(R/2)$. Let E_R and E'_R be defined as

$$E_R \triangleq \{i \notin K_T\} \qquad E'_R \triangleq \{[0,i] \cap K_T = \varnothing\}.$$

We have to estimate $P(E_R)$; from Theorem 3.1 in [31], $P(E'_R) \asymp R^{-1/3}$, and we have $P(E_R) \geqslant P(E'_R)$.

We shall decompose E_R according to the value of N: we can write $P(E_R) = \sum_{n=0}^{\infty} P(E_R, N = n)$. For fixed n, make the following remark: if there is not disconnection before T, then there is not disconnection for t inside any $[S_k, T_k]$, for all $k \geqslant n$. Applying the strong Markov property at time S_k and the locality property of SLE_6, we see that the conditional probability of non-disconnexion between $[S_k, T_k]$ is a decreasing function of the extremal length between $\mathcal{C}(0,R)$ and the boundary component spanning from 0 to $2e^{i\alpha_{S_k}}$, in the domain $H_{S_k} \cap \mathcal{B}(0,R)$.

But Beurling's inequality shows that, as soon as $R \geqslant 4$ (say), this extremal length is bounded below by an absolute constant (namely $\log(2)/\pi$ but this is unimportant), for which the disconnexion probability is strictly positive (because $\kappa > 4$). Thus, for all k, the probability that there is no disconnection between times S_k and T_k is bounded by some $\lambda < 1$, independent of $K_{T_{k-1}}$ and R; moreover, for the last part of the path, the probability that no disconnection occurs after time T_N is bounded by $P(E_R, N = 0)$ (by the strong Markov property at time T_N and the Beurling inequality). Hence,

$$P(E_R) \leqslant \sum_{k=0}^{\infty} \lambda^k P(E_R, N = 0) \leqslant \frac{1}{1-\lambda} P(E_R, N = 0) \leqslant \frac{1}{1-\lambda} P(E_R).$$

Written more synthetically, this becomes

$$P(E_R) \asymp P(E_R, N = 0),$$

i.e. knowing that K' does not disconnect i there is a positive probability that it does not even touch the disk of radius 2. The very same proof applied this time to E' leads to $P(E'_R) \asymp P(E'_R, N = 0)$.

Now, it is easy to see that $\{E_R, N = 0\} = \{E'_R, N = 0\}$, meaning that if K'_T does not intersect $\mathcal{B}(0, 2)$, the conditions $i \notin K_T$ and $[0, i] \cap K_T = \varnothing$ are equivalent. Hence $P(E_R, N = 0) = P(E'_R, N = 0)$, from which we can conclude that

$$P(E'_R) \asymp P(E_R) \asymp R^{-1/3}.$$

//

3.3.2 Exponent for $b = 1/3$

Proposition 3.4 :

Let (K_t) be a chordal SLE_6 in the unit disk, starting from 1 and growing toward -1, and T_r the first time when K_t hits the ball with radius r centered at 0. Let L_{T_r} be π times the extremal distance in $\mathbb{U} \setminus K_{T_r}$ between $\mathcal{C}(0, r)$ and $\partial\mathbb{U}$. Then, as r tends to 0,

$$E(e^{-L_{T_r}/3}) \asymp r^{2/3}.$$

// As previously, let K' be a radial SLE_6 in $\mathbb{U}$, starting from 1 and aimed at 0. Then, since all the involved events satisfy non-disconnection between $\mathcal{C}(0, r)$ and -1 ($L_{T_r} = \infty$ iff there is disconnection), we have:

$$q(r) \triangleq E\left(e^{-L_{T_r}/3}\right) = E\left(e^{-L_{T_r}/3}\, \mathbf{1}_{L_{T_r} < \infty}\right) = E\left(e^{-L'_{T'_r}/3}\, \mathbf{1}_{-1 \notin K'_{T'_r}}\right). \tag{3.18}$$

We shall estimate the third term, again following the steps of the proof of Theorem 3.1 in [32]. From now on, fix $b = 1/3$ and $\nu = \nu(\kappa, b) = 2/3$: since $b < 1$, we need a separate proof here. Let l_t be the Euclidean length of the arc $g_t(\partial\mathbb{U} \setminus K'_t)$. The only place in [32] where $b \geqslant 1$ was needed was in the derivation of

$$E(l_t^b) \asymp \exp\left(-\nu t\right). \tag{3.19}$$

But this is exactly what Lemma 3.1 shows, after suitable rescaling. //

3.3.3 Construction of the boundary

Again we describe the studied set as the decreasing intersection of a family B_ε of subsets of the plane. Here, let

$$B_\varepsilon = \{z \notin K_1 : \mathrm{d}(z, K_1) < \varepsilon\}.$$

In order for z to be in B_ε the following must happen: First there is some point in $\mathcal{H}$ at a distance less than ε from z; letting $T(z,\varepsilon) = \mathrm{Inf}\{t : \mathrm{d}(z, K_t) < \varepsilon\}$, and introducing the extremal distance $L(z,\varepsilon)$ between $\mathcal{B}(z,\varepsilon)$ and $\partial\mathbb{U}$ in $\mathbb{U}\setminus K_{T(z,\varepsilon)}$, this condition is equivalent to

$$L(z,\varepsilon) < \infty.$$

Then, the *SLE* after $T(z,\varepsilon)$ and up to time 1 must not disconnect z from "infinity" (*i.e.* from $\partial\mathbb{U}$), and conditionally to $K_{T(z,\varepsilon)}$ this happens with probability of order

$$e^{-L(z,\varepsilon)/3}.$$

Proposition 3.4 then states that $P(z \in B_\varepsilon) \asymp \varepsilon^{2/3}$. Second moments can be obtained in the same fashion as for the trace; in this case, the relevant estimate (describing in which conditions a disk intersects the boundary of the discrete exploration process) is the following: First, two crossings of different colors must ensure that the exploration process touches the disk; then a third path, disjoint of the first two, will prevent it from closing a loop around it. Hence the following consequence of Proposition 3.4:

Corollary 3.3 :

Let A_ε be the annulus centered at 0, with radii ε and 1. For all $\delta > 0$, consider critical site-percolation on the intersection of A_ε with the triangular lattice of mesh δ. Let $\tilde{p}(\varepsilon,\delta)$ be the probability that $\mathcal{C}(0,\varepsilon)$ is connected to $\mathcal{C}(0,1)$ both by a path of open vertices and two disjoint paths of closed vertices in A_ε. Then, as δ tends to 0, $\tilde{p}(\varepsilon,\delta)$ converges to some $\tilde{p}(\varepsilon)$ satisfying

$$\tilde{p}(\varepsilon) \asymp \varepsilon^{2/3}.$$

The rest of the construction is the same, and we obtain sufficient estimates to apply Proposition 3.1, this time with $s = 2/3$. We obtain

$$P(\dim_H(\partial K_1) \leqslant \frac{4}{3}) = 1, \qquad P(\dim_H(\partial K_1) = \frac{4}{3}) > 0,$$

and once more we need a zero-one law:

Lemma 3.4 (0–1 law for the boundary) :

For all $d \in [0,2]$, we have

$$P(\dim_H(\partial K_1) = d) \in \{0,1\}.$$

// Let $D_t = \dim_H(\partial K_t)$. As previously in the case of the trace, scaling shows that the law of D_t does not depend on $t > 0$. However here (∂K_t) is not increasing anymore, so we need another argument. Let t, $t' > 0$ and consider the boundary of $K_{t+t'}$. It has two parts, namely the "new" part $\partial_1 = \partial K_{t+t'} \setminus K_t$, and the "old" part $\partial_2 = \partial K_{t+t'} \cap K_t \subset \partial K_t$. It is clear that

$$D_{t+t'} = \dim_H(\partial_1) \vee \dim_H(\partial_2),$$

hence in particular $\dim_H(\partial_1) \leqslant D_{t+t'}$. Besides, conformal mapping shows that $\dim_H(\partial_1)$ has the same law as $D_{t'}$, hence the same law as $D_{t+t'}$. Hence, with probability 1, $D_{t+t'} = \dim_H(\partial_2)$.

Moreover, conformal mapping also shows that $\dim_H(\partial_2)$ is independent of $\mathcal{F}_t$. This proves that for all $t, t' > 0$ the dimension of $\partial K_{t+t'}$ is independent of $\mathcal{F}_t$. It is then a direct application of Blumenthal's zero-one law that D_t has an almost sure value. //

This concludes the proof of Theorem 3.1.

3.3.4 Dimension of $SLE_{8/3}$

It should be theoretically possible to apply the previous construction to other values of κ, but some of the main tools that we used (namely, the radial/chordal equivalence and the restriction property) do hold only for $\kappa = 6$ so that additional arguments would be required.

For the special value $\kappa = 8/3$, the result on the frontier of SLE_6 makes it possible to show that the dimension of $SLE_{8/3}$ is almost surely 4/3. More precisely, Lawler-Schramm-Werner [29] have shown that the outer boundary of the union of 8 $SLE_{8/3}$'s has the same law as that of the union of 5 Brownian excursions. The zero-one laws previously proved for both the trace and the boundary of SLE extends to this object: Its boundary has a.s. the same dimension as the boundary of SLE_6 and also a.s. the same dimension as $SLE_{8/3}$. Hence these dimensions are equal, and the result follows.

3.4 Time-sets for SLE_κ

We now turn our attention to the dimension of sets of exceptional times. Note that time corresponds to the Loewner parameterization of the trace, which is in a way not the most canonical: It is not clear for instance whether it behaves nicely under time-reversal. More precisely, how smoothly does the Riemann map from $\mathbb{H} \setminus \gamma([t, \infty))$ to $\mathbb{H}$ evolve as t increases?

A natural question that also arises is the following. Let A be some (random) subset of $[0, \infty]$, and $\gamma(A)$ be its image by the trace of a chordal SLE in the upper-half plane. Is it possible, knowing the Hausdorff dimension of A, to obtain that of $\gamma(A)$? Such a relation holds for Brownian motion [22], namely the dimension of the image is a.s. equal to twice the dimension of A. It is expected that such a relation cannot hold for SLE without additional requirements on A, however a few cases can be treated entirely (in the sense that both the time and space dimensions can be computed in independent ways), at least for $\kappa = 6$: the trace itself, cut-points, and the boundary.

3.4.1 Boundary times

In the previous section, we derived the dimension of the boundary of SLE_6. The dimension of the corresponding time-set can also be computed (and it should be noted that the following is true even for $\kappa \neq 6$):

Theorem 3.4 :

Let K be an SLE in the upper-half plane, with $\kappa > 4$, and let D be the set of *boundary times* in $[0,1]$ — *i.e.* the set of times t such that $\gamma(t) \in \partial K_1$. Then, with probability 1,

$$\dim_H(D) = \frac{4+\kappa}{2\kappa}.$$

/// It is clearly sufficient to compute the dimension of left-boundary times, namely times t such that $g_1(\gamma(t)) \in (-\infty, \beta_t)$ where β is the process driving K. Introduce the sets of approximate left-boundary times between ε and a, defined by

$$D_{\varepsilon,a} = \{t : \mathrm{Inf}(\mathbb{R} \cap g_t(K_{t+\varepsilon})) = \mathrm{Inf}(\mathbb{R} \cap g_t(K_{t+a}))\}$$

(*i.e.*, γ may touch the real line on the right side of K but not on the left side). Let D^a be the intersection of the $D_{\varepsilon,a}$ when $\varepsilon \to 0$. Scaling and the Markov property show that $P(t \in D_{\varepsilon,a})$ depends only on ε/a. Hence, to obtain condition 1. in Proposition 3.1, with $s = (\kappa-4)/2\kappa$, it is sufficient to obtain the following estimate:

Lemma :

Let (K_t) be a chordal SLE_κ ($\kappa > 4$) in the upper-half plane: then as t goes to infinity,

$$p_t \triangleq P(\mathrm{Inf}(\mathbb{R} \cap K_t) = \mathrm{Inf}(\mathbb{R} \cap K_1)) \asymp t^{(4-\kappa)/2\kappa}.$$

//// First, apply the Markov property of SLE at time 1 and map the picture to the upper-half plane by $\Phi = g_1 - \beta_1$. Let $Y_{_0} \leqslant 0$ be the image of $\mathrm{Inf}(\mathbb{R} \cap K_1)$ by Φ. The process $(\tilde{K}_u) = (\Phi(K_{1+u}))$ is an SLE_κ, and the probability we are interested in is then given by

$$p_t = P(Y_{_0} \notin \tilde{K}_{t-1}).$$

Let $(\tilde{\beta}_u)$ and $\tilde{g}_u : \mathbb{H} \setminus \tilde{K}_u \to \mathbb{H}$ be respectively the process driving $\tilde{K}$ and the associated conformal maps; let $Y_u = \tilde{g}_u(Y_{_0}) - \tilde{\beta}_u$. It is easy to see, using Itô's formula and the definition of chordal SLE, that Y satisfies the following SDE (where B is a standard Brownian motion):

$$\mathrm{d}Y_u = \sqrt{\kappa}\, \mathrm{d}B_u + \frac{2}{Y_t}\, \mathrm{d}u; \tag{3.20}$$

i.e., up to a linear time change, Y is a Bessel process of dimension $b = 1+4/\kappa < 2$ starting from $Y_{_0}$. Hence, it is known that the probability that it does not hit 0 up to time u behaves like $(u/Y_{_0}^2)^{-\nu}$ where $\nu = (\kappa-4)/2\kappa > 0$ is the index of the process. Hence,

$$p_t \asymp t^{-\nu} E(Y_{_0}^{2\nu}) \asymp t^{-\nu},$$

as we wanted. ////

This provides the right estimate:

$$P(t \in D_{\varepsilon,a}) \asymp \left[\frac{\varepsilon}{a}\right]^s$$

where the implicit constants depend only on κ. Notice that if $t+\varepsilon$ is in $D_{\varepsilon,a}$, then $t \in D_{2\varepsilon,a}$ (because $K_t \subset K_{t+\varepsilon}$) and even $[t, t+\varepsilon] \subset D_{2\varepsilon,a}$. This and the previous estimate provide

$$P\left([t,t+\varepsilon] \subset D_{2\varepsilon,a} | t \in D_{2\varepsilon,a}\right) \geqslant \frac{P(t+\varepsilon \in D_{\varepsilon,a})}{P(t \in D_{2\varepsilon,a})} \geqslant c > 0,$$

which is condition 2. It remains to obtain second moments, and these are given by the Markov property, as follows.

Let $x < y$ be two times in $[0,1]$. If x and y are in $D_{\varepsilon,a}$ with $a > y-x$, then in particular $x \in D_{\varepsilon,y-x}$ and $y \in D_{\varepsilon,a}$. By the Markov property of SLE, applied at time y, those two events are independent. Hence we obtain

$$\begin{aligned} P(x,y \in D_{\varepsilon,a}) &\leqslant P(x \in D_{\varepsilon,y-x})P(y \in D_{\varepsilon,a}) \\ &\leqslant C\left[\frac{\varepsilon}{y-x}\right]^s \left[\frac{\varepsilon}{a}\right]^s \leqslant C\,\frac{\varepsilon^{2s}}{(y-x)^s}, \end{aligned}$$

still with $s = (\kappa-4)/2\kappa$. This is exactly condition 3. If $a \leqslant y-x$ then the events $x \in D_{\varepsilon,a}$ and $y \in D_{\varepsilon,a}$ are themselves independent and the same method applies. Hence, everything is ready to apply Proposition 3.1: For all $a > 0$, with positive probability,

$$\dim_H(D^a) = 1 - \frac{\kappa-4}{4} = 2 - \frac{\kappa}{4}.$$

Noticing then that $D^1 \subset D \subset D^2$ hence provides

$$P\left(\dim_H(D) = 2 - \frac{\kappa}{4}\right) > 0.$$

It is then easy to apply the same proof as that of Lemma 3.4 and obtain a zero-one law for $\dim_H(D)$, thus completing the proof. //

Remark: In particular, the dimension of boundary-times is never less than 1/2, even when $\kappa \to \infty$. Note that in this case, the dimension of the Bessel process appearing in the proof tends to 1, so the exponent 1/2 is the same as in the usual gambler's ruin estimate.

This is not surprising since, when κ tends to ∞, the trace of an SLE_κ converges, after suitable rescaling, to

$$\gamma_\infty : t \mapsto (B_t, L_t^{B_t}),$$

where B is standard Brownian motion and (L_t^x) denotes its local time at point x (cf. [6]). In the limit, the boundary times correspond to last-passage times, which have dimension 1/2 by a reflection argument.

3.4.2 Cut-times and the existence of cut-points

We saw in the previous sections how the dimension of the trace of SLE was related to non-disconnection exponents: Here, we follow the analogy with Brownian motion to describe cut-points on the SLE trace. Let K be a chordal SLE_κ and C be the set of cut-points of K_2 in K_1 (*i.e.*, the set of points $z \in K_1$ such that $K_2 \setminus \{z\}$ is not connected). Such a point is on the boundary of K_1, hence if γ is the trace of K every cut-point is on $\gamma([0,1])$. We say that t is a cut-time if $\gamma(t)$ is a cut-point, and note $\tilde{C}$ the set of cut-times.

Theorem 3.5 :

(i). If $0 \leqslant \kappa \leqslant 4$, then $\tilde{C} = [0,1]$ and $C = K_1$;

(ii). If $4 < \kappa < 8$, then with positive probability $\tilde{C}$ has Hausdorff dimension $(8-\kappa)/4$, in particular it is non-empty, hence $C \neq \varnothing$;

(iii). If $\kappa > 8$ then a.s. $\tilde{C} = \varnothing$ and K_1 has no cut-point.

// (i) is a direct consequence of the fact that γ be a simple path [42], and (iii) is proved exactly like (ii) with the usual convention that a set of negative dimension is empty. Hence, we may assume that $4 < \kappa < 8$. Again we are going to apply Proposition 3.1, and the proof will be very similar to that of Theorem 3.4.

Introduce the set of approximate cut-times between ε and a defined as

$$C_{\varepsilon,a} = \{t \in [0,1] : \gamma([t+\varepsilon, t+a]) \cap (K_t \cup \mathbb{R}) = \varnothing\}\,.$$

Define C^a as the (indeed non-increasing) intersection of the $C_{\varepsilon,a}$. By the Markov property at time t, follows that $P(t \in C_{\varepsilon,a})$ does not depend on t. Moreover, scaling shows that it is a function of ε/a. Hence, to obtain condition 1. in Proposition 3.1 with $s = (\kappa-4)/4$, it suffices to prove the following:

Lemma :

Let K be an SLE_κ in the upper-half plane, starting at $x \in (0,1)$, with $\kappa > 4$. Then, when $t \to \infty$,

$$P(\{0,1\} \cap K_t = \varnothing) \asymp t^{(4-\kappa)/4}.$$

/// The proof of this Lemma is very similar to that of Theorem 3.1 in [36]. Two things have to be done: First, extend this theorem to the (easier) case where $w_1 = w_2 = 0$; second, to translate it back to an estimate for SLE at a fixed time. Introduce the following processes: $X_t = g_t(1) - \beta_t$, $Y_t = g_t(0) - \beta_t$, where (β_t) is the time-scaled Brownian motion driving K. As was seen previously, X and Y satisfy the following SDE's:

$$\mathrm{d}X_t = \frac{2}{X_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \mathrm{d}Y_t = \frac{2}{Y_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t,$$

where B is a standard real Brownian motion. Let $L_t = X_t - Y_t$ be the length of the image interval, and let $R_t = X_t/L_t$. Tedious application of

Itô's formula leads to

$$\mathrm{d}L_t = \frac{2\mathrm{d}t}{L_t R_t(1-R_t)}, \qquad \mathrm{d}R_t = \frac{2(1-2R_t)}{L_t^2 R_t(1-R_t)}\mathrm{d}t + \frac{\sqrt{\kappa}}{L_t}\mathrm{d}B_t.$$

Introduce the following random time-change:

$$\mathrm{d}t(s) = \frac{L_{t(s)}^2 R_{t(s)}\left(1-R_{t(s)}\right)}{2}\mathrm{d}s,$$

then the previous system reduces to $\mathrm{d}L_{t(s)} = L_{t(s)}\mathrm{d}s$, *i.e.* almost surely $L_{t(s)} = e^s$, and, letting $Z_s = R_{t(s)}$,

$$\mathrm{d}Z_s = (1-2Z_s)\mathrm{d}s + \sqrt{\frac{\kappa Z_s(1-Z_s)}{2}}\mathrm{d}B_s \tag{3.21}$$

as in [36]. Now introduce the following stopping times:

$$S = \mathrm{Inf}\left\{s : Z_s \in \{0,1\}\right\}, \qquad T = t(S) = \mathrm{Inf}\left\{t : R_t \in \{0,1\}\right\}.$$

The counterpart of Theorem 3.1 in [36] for the case $w_1 = w_2 = 0$ is obtained as Lemma 3.2 in the present paper, it gives the following estimate:

$$P(S > s) \asymp \exp(-\lambda(0,0)s) = \exp\left(-\frac{\kappa-4}{2}s\right). \tag{3.22}$$

It remains to transfer this estimate to deterministic values of t. Recall that we have $2\mathrm{d}t(s) = e^{2s}Z_s(1-Z_s)\mathrm{d}s$. This already proves that $\mathrm{d}t(s) \leqslant e^{2s}/8\mathrm{d}s$ *i.e.* $t(s) \leqslant e^{2s}/16$ or $s \geqslant \log(16t(s))/2$. Hence,

$$P(T > t) \leqslant P\left(S > \frac{\log(16t)}{2}\right) \asymp t^{-(\kappa-4)/4}.$$

To obtain the lower bound, note that the proof of Theorem 3.1 in [36] also gives the distribution of R_s knowing that $S > s$ — which is the eigenfunction associated to the eigenvalue $\lambda(0,0)$ for the generator of R, namely

$$c.[x(1-x)]^{(\kappa-4)/\kappa}.$$

In particular, conditionally to the fact that $S > s$, there is a positive probability that $Z_s \in [1/4, 3/4]$. Comparison with Brownian motion then shows that

$$P\left(\forall s \in [s_0, s_0+1], Z_s \in \left[\frac{1}{8}, \frac{7}{8}\right] \middle| Z_{s_0} \in \left[\frac{1}{4}, \frac{3}{4}\right]\right) \geqslant c > 0$$

and combining this with (3.22) provides, for all $s_0 > 42$:

$$P\left(\forall s \in [s_0-1, s_0], Z_s \in \left[\frac{1}{8}, \frac{7}{8}\right] \middle| S > s_0\right) \geqslant c > 0.$$

Now on this event, we obtain

$$t(s_0) \geqslant \int_{s_0-1}^{s_0} \frac{e^{2s}}{128} \mathrm{d}s \geqslant c_0.e^{2s_0},$$

from which the lower bound follows:

$$P(T > t) \geqslant c.P\left(S > \frac{\log(t/c_0)}{2}\right) \geqslant c.t^{-(\kappa-4)/4}.$$

///

The end of the proof is exactly the same as that of the previous theorem, so we do not repeat it here. //

Remarks: For $\kappa = 8$ (where the obtained dimension is 0), to our knowledge the question of existence is open. Oded Schramm conjectures that there is no cut-point on SLE_8. Note that in this case the existence of the trace requires a separate proof [34]; the trace is then the scaling limit of the UST Peano curve, hence it is itself a Peano curve, but this is not sufficient to prove that there is no cut-point on K.

If $\kappa = 6$, we get that the dimension of cut-times is $1/2$. It is known in this case (using Brownian exponents) that the dimension of cut-points is $2 - 5/4 = 3/4$ (cf. [32]). For the other values of $\kappa \in (4, 8)$, the dimension of C is not known.

Chapter 4

Hausdorff dimensions in the general case

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Introduction

In this chapter we derive the dimension of the SLE trace in the general case, *i.e.* we prove the following theorem, which was conjectured in [42]:

Theorem 4.1 :

Let γ be the trace of an SLE_κ, where $\kappa \in (0, 8)$, $\kappa \neq 4$. Then, almost surely, the path $\gamma([0, +\infty])$ has Hausdorff dimension $1 + \kappa/8$.

Not surprisingly, we are going to follow the steps of the previous proof (which was specific to the case $\kappa = 6$), and in particular our main tool will still be Proposition 3.1. Note that the zero-one laws stated in the previous chapter still hold here, so it will be sufficient to obtain Hausdorff dimensions with positive probability.

We will keep the notation $\mathcal{H}$ for the complete trace of the process and

$$C_\varepsilon = \{z \in [-1, 1] \times [1, 3] : \mathrm{d}(z, \mathcal{H}) \leqslant \varepsilon\}.$$

Then again, $\mathcal{H} \cap [-1, 1] \times [1, 3]$ is the non-increasing intersection of the C_ε as ε goes to 0, and the setup is exactly the same as in the previous chapter; the event $\{z \in C_\varepsilon\}$ is can still be written in terms of non-disconnection, and Condition 2. in Proposition 3.1 is still an automatic consequence of the definition of C_ε.

However, we cannot use the chordal/radial equivalence anymore, nor apply the relation between SLE_6 and percolation to obtain the correct decay of correlations, hence we need a different approach to obtain conditions 1. and 3. The general principle leading to the first moment estimate is the same, namely we will look at the growing compact K_t from the point of view of a fixed point in the upper-half plane; but the proof of Condition 3. is a pain in the neck.

4.1 The first moment estimate

Fix $\kappa > 0$ and $z_0 \in \mathbb{H}$; let γ be the trace of a chordal SLE_κ in $\mathbb{H}$, and let $\mathcal{H} = \gamma([0,\infty))$ be the image of γ. We want to compute the probability that $\mathcal{H}$ touches the disk $\mathcal{B}(z_0, \varepsilon)$ for $\varepsilon > 0$.

Proposition 4.1 :

Let $\alpha(z_0) \in (0, \pi)$ be the argument of z_0. Then, if $\kappa \in (0, 8)$, we have the following estimate:

$$P(\mathcal{B}(z_0, \varepsilon) \cap \mathcal{H} \neq \varnothing) \asymp \left(\frac{\varepsilon}{\Im(z_0)} \right)^{1-\kappa/8} (\sin \alpha(z_0))^{8/\kappa - 1}.$$

If $\kappa \geqslant 8$, then this probability is equal to 1 for all $\varepsilon > 0$.

Remark: We know that $\mathcal{H}$ is a closed subset of $\bar{\mathbb{H}}$ (indeed, this is a consequence of the transience of γ — cf. [42]). For $\kappa \geqslant 8$, this proves that for all $z \in \bar{\mathbb{H}}$, $P(z \in \mathcal{H}) = 1$, hence $\mathcal{H}$ almost surely has full measure. And since it is closed, this implies that with probability 1, γ is space-filling, as was already proved by Rohde and Schramm ([42]) for $\kappa > 8$ and by Lawler, Schramm and Werner ([34]) for $\kappa = 8$ (for which a separate proof is needed for the existence of γ).

// The idea of the following proof is originally due to Oded Schramm. Let δ_t be the Euclidean distance between z_0 and K_t. (δ_t) is then a non-increasing process, and its limit when t goes to $+\infty$ is the distance between z_0 and $\mathcal{H}$. Besides, we can apply the Köbe 1/4 theorem to the map g_t: this leads to the estimate

$$\delta_t \asymp \frac{\Im(g_t(z_0))}{|g_t'(z_0)|} \tag{4.1}$$

(where the implicit constants are universal — namely, 1/4 and 4).

It will be more convenient to fix the image of z_0 under the random conformal map. Hence, introduce the following map:

$$\tilde{g}_t : z \mapsto \frac{g_t(z) - g_t(z_0)}{g_t(z) - \overline{g_t(z_0)}}.$$

It is easy to see that $\tilde{g}_t$ maps $\mathbb{H} \setminus K_t$ conformally onto the unit disk $\mathbb{U}$, and maps infinity to 1 and z_0 to 0. In other words, the map

$$w \mapsto \tilde{g}_t \left(\frac{w\overline{g_t(z_0)} - g_t(z_0)}{w - 1} \right)$$

maps the complement of some compact $\tilde{K}_t$ in $\mathbb{U}$ onto $\mathbb{U}$, fixing 0 and 1 (in all this proof, z will stand for an element of $\mathbb{H}$ and w for an element of $\mathbb{U}$). Moreover, in this setup Equation 4.1 becomes simpler (because the distance between 0 and the unit circle is fixed): Namely,

$$\delta_t \asymp \frac{1}{|\tilde{g}_t'(z_0)|}. \tag{4.2}$$

Differentiating $\tilde{g}_t(z)$ with respect to t (which is a little messy and error-prone, but straightforward) leads to the following differential equation:

$$\partial_t \tilde{g}_t(z) = \frac{2(\tilde{\beta}_t - 1)^3}{\left(g_t(z_0) - \overline{g_t(z_0)}\right)^2 \tilde{\beta}_t^2} \cdot \frac{\tilde{\beta}_t \tilde{g}_t(z)(\tilde{g}_t(z) - 1)}{\tilde{g}_t(z) - \tilde{\beta}_t}, \tag{4.3}$$

where $(\tilde{\beta}_t)$ is the process on the unit circle defined by

$$\tilde{\beta}_t = \frac{\beta_t - g_t(z_0)}{\beta_t - \overline{g_t(z_0)}}.$$

Now the structure of the expression for $\partial_t \tilde{g}_t(z)$ (Equation (4.3)) is quite nice: The first factor does not depend on z and the second one only depends on z_0 through $\tilde{\beta}$. Hence, let us define a (random) time change by taking the real part of the first factor; namely let

$$\mathrm{d}s = \frac{(\tilde{\beta}_t - 1)^4}{\left|g_t(z_0) - \overline{g_t(z_0)}\right|^2 \tilde{\beta}_t^2} \,\mathrm{d}t,$$

and introduce $h_s = \tilde{g}_{t(s)}$.

Then Equation (4.3) becomes similar to a radial Loewner equation, *i.e.* it can be written as

$$\partial_s h_s(z) = \tilde{X}(\tilde{\beta}_{t(s)}, h_s(z)), \tag{4.4}$$

where $\tilde{X}$ is the vector field in $\mathbb{U}$ defined as

$$\tilde{X}(\zeta, w) = \frac{2\zeta w(w-1)}{(1-\zeta)(w-\zeta)}. \tag{4.5}$$

The only missing part is now the description of the driving process $\tilde{\beta}$. Applying Itô's formula (now *this* is an ugly computation) and then the previous time-change, we see that $\tilde{\beta}_{t(s)}$ can be written as $\exp(i\alpha_s)$ where (α_s) is a diffusion process on the interval $(0, 2\pi)$ satisfying the equation

$$\mathrm{d}\alpha_s = \sqrt{\kappa}\,\mathrm{d}B_s + \frac{\kappa - 4}{2} \operatorname{cotg} \frac{\alpha_s}{2} \,\mathrm{d}s \tag{4.6}$$

with the initial condition $\alpha_0 = 2\alpha(z_0)$.

The above construction is licit as long as z_0 remains inside the domain of g_t. While this holds, differentiating (4.4) with respect to z at $z = z_0$ yields

$$\partial_s h'_s(z_0) = \frac{2h'_s(z_0)}{1 - \tilde{\beta}_s},$$

so that dividing by $h'_s(z_0) \neq 0$ and taking the real parts of both sides we get

$$\partial_s \log |h'_s(z_0)| = 1,$$

i.e. almost surely, for all $s > 0$, $|h'_s(z_0)| = |h'_0(z_0)|e^s$. Combining this with (4.2) shows that

$$\delta_{t(s)} \asymp \delta_0 e^{-s} \asymp \Im(z_0)\; e^{-s}.$$

Finally, let us look at what happens at the stopping time

$$\tau_{z_0} = \mathrm{Inf}\{t : z_0 \in K_t\}.$$

We are in one out of two situations: Either z_0 is on the trace: in this case δ_t goes to 0, meaning that s goes to ∞, and the diffusion (α_s) does not touch $\{0, 2\pi\}$. Or, z_0 is not on the trace: then δ_t tends to $\mathrm{d}(z_0, \mathcal{H}) > 0$, and the diffusion (α_s) reaches the boundary of the interval $(0, 2\pi)$ at time

$$s_0 \triangleq \log \delta_0 - \log \mathrm{d}(z_0, \mathcal{H}) + \mathcal{O}(1).$$

Let S be the surviving time of (α_s): the previous construction then shows that

$$\mathrm{d}(z_0, \mathcal{H}) \asymp \delta_0\; e^{-S},$$

and estimating the probability that z_0 is ε-close to the trace becomes equivalent to estimating the probability that (α_s) survives up to time $\log(\delta_0/\varepsilon)$.

Assume for a moment that $\kappa > 4$. The behaviour of $\mathrm{cotg}\, \alpha/2$ when α is close to 0 shows that (α_s) can be compared to the diffusion $(\bar{\alpha}_s)$ generated by

$$\mathrm{d}\bar{\alpha}_s = \sqrt{\kappa}\, \mathrm{d}B_s + (\kappa - 4)\, \frac{\mathrm{d}s}{\bar{\alpha}_s},$$

which (up to a linear time-change) is a Bessel process of dimension

$$d = \frac{3\kappa - 8}{8}.$$

More precisely: $(\bar{\alpha}_s)$ survives almost surely, if and only if (α_s) survives almost surely. But it is known that a Bessel process of dimension d survives almost surely if $d \geqslant 2$, and dies almost surely if $d < 2$. Hence, we already obtain the phase transition at $\kappa = 8$:

- If $\kappa \geqslant 8$, then $d \geqslant 2$, and (α_s) survives almost surely. Hence, almost surely $\mathrm{d}(z_0, \mathcal{H}) = 0$, and for all $\varepsilon > 0$ the trace will almost surely touch $\mathcal{B}(z_0, \varepsilon)$;

- If $\kappa < 8$, then $d < 2$ and (α_s) dies almost surely in finite time. Hence, almost surely $\mathrm{d}(z_0, \mathcal{H}) > 0$.

So, there is nothing left to prove for $\kappa \geqslant 8$. From now on, we shall then suppose that $\kappa \in (0,8)$. If $\kappa \leqslant 4$ then the drift of (α_s) is toward the boundary, hence comparing it to standard Brownian motion shows that it dies almost surely in finite time as for $\kappa \in (4,8)$. We want to apply Lemma 3.2 to (α_s) and for that we need to know the principal eigenvalue of the generator L_κ of the diffusion. It can be seen that the function

$$(\sin(x/2))^{8/\kappa-1}$$

is a positive eigenfunction of L_κ, with eigenvalue $1-\kappa/8$: hence we already obtain that, if α_0 is far from the boundary, $P(S > s) \asymp \exp(-(1-\kappa/8)s)$ *i.e.*

$$P(\mathrm{d}(z_0, \mathcal{H}) \leqslant \varepsilon) \asymp e^{(1-\kappa/8)\log(\varepsilon/\delta_0)} \asymp \left(\frac{\varepsilon}{\delta_0}\right)^{1-\kappa/8}, \tag{4.7}$$

which is the correct estimate. It remains to take the value of α_0 into account.

Introduce the following process:

$$X_s \triangleq \sin\left(\frac{\alpha_s}{2}\right)^{8/\kappa-1} e^{(1-\kappa/8)s}$$

(and $X_s = 0$ if $s \geqslant S$). Applying the Itô formula shows that (X_s) is a local martingale (in fact this is the same statement as saying that $\sin(x/2)^{8/\kappa-1}$ is an eigenfunction of the generator), and it is bounded on any bounded time interval. Hence, taking the expected value of X at times 0 and s shows that

$$\sin\left(\frac{\alpha_0}{2}\right)^{8/\kappa-1} = e^{(1-\kappa/8)s}\ P(S \geqslant s)\ E\left[\sin\left(\frac{\alpha_s}{2}\right)^{8/\kappa-1} \middle| S \geqslant s\right]. \tag{4.8}$$

The same proof as that of Lemma 3.2 shows that, for all $s \geqslant 1$,

$$P(\alpha_s \in [\pi/2, 3\pi/2] | S \geqslant s) > 0$$

with constants depending only on κ; combining this with (4.8) then provides

$$P(S \geqslant s) \asymp e^{-(1-\kappa/8)s} \sin\left(\frac{\alpha_0}{2}\right)^{8/\kappa-1},$$

again with constants depending only on κ. Applying the same computation as for Equation (4.7) ends the proof. //

Corollary 4.1 :

Let $D \subsetneq \mathbb{C}$ be a simply connected domain, a and b be two points on the boundary of D, and γ be the path of a chordal SLE_κ in D from a to b, with $\kappa \in (0,8)$. Then, for all $z \in D$ and $\varepsilon < \mathrm{d}(z, \partial D)/2$, we have

$$P(\gamma \cap \mathcal{B}(z,\varepsilon) \neq \varnothing) \asymp \left(\frac{\varepsilon}{\mathrm{d}(z,\partial D)}\right)^{1-\kappa/8} (\omega_z(ab) \wedge \omega_z(ba))^{8/\kappa-1},$$

where ω_z is the harmonic measure on ∂D seen from z and ab is the positively oriented arc from a to b along ∂D.

// This is easily seen by considering a conformal map Φ mapping D to the upper-half plane, a to 0 and b to ∞: Since the harmonic measure from z in D is mapped to the harmonic measure from $\Phi(z)$ in $\mathbb{H}$, it is sufficient to prove that for all $z \in \mathbb{H}$,

$$\omega_z(\mathbb{R}_+) \wedge \omega_z(\mathbb{R}_-) \asymp \sin(\arg z) ;$$

and $\omega_z(\mathbb{R}_+)$ can be explicitly computed, because ω_z is a Cauchy distribution on the real line:

$$\omega_{x+iy}(\mathbb{R}_+) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}u/y}{1+(u-x)^2/y^2} = \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg}(x/y).$$

When x tends to $-\infty$, this behaves like $-y/\pi x$ which is equivalent to $\sin(\arg(x+iy))/\pi$. //

This “intrinsic” formulation of the hitting probability will make the derivation of the second moment estimate more readable.

4.2 The second moment estimate

We still have to derive condition 3. in Proposition 3.1. For $\kappa = 6$ it was obtained using the locality property, but this does not hold for other values of κ, so we can rely only on the Markov property. The general idea is as follows. Fix two points z and z' in the upper half plane, and $\varepsilon < |z'-z|/2$. We want to estimate the probability that the trace γ visits both $\mathcal{B}(z,\varepsilon)$ and $\mathcal{B}(z',\varepsilon)$. Assume that it touches, say, the first one (and this happens with probability of order $\varepsilon^{1-\kappa/8}$), and that it does so before touching the other.

Apply the Markov property at the first hitting time $T_\varepsilon(z)$ of $\mathcal{B}(z,\varepsilon)$: If everything is going fine and we are lucky, the distance between z' and $K_{T_\varepsilon(z)}$ will still be of order $|z'-z|$. Hence, applying the first moment estimate to this situation shows that the conditional probability that γ hits $\mathcal{B}(z',\varepsilon)$ is not greater than $C.(\varepsilon/|z'-z|)^{1-\kappa/8}$ (it might actually be much smaller, if the real part of $g_{T_\varepsilon(z)}(z')$ is large, but this is not a problem since we only need an upper bound), and this gives the right estimate for the second moments:

$$C\,\frac{\varepsilon^{2-\kappa/4}}{|z'-z|^{1-\kappa/8}}.$$

The whole point is then to prove that this is the main contribution to the second moment probability.

Small print: You probably don't want to read the rest of this section. It is full of ugly notations and rather dull. You can jump directly to Section 4.3.

4.2.1 The setup

Let z and z' be two points in the upper-half plane, and let $\delta = \mathrm{d}(z,z')/2$. Let

$$\mathcal{E} = \mathcal{C}\left(\frac{z+z'}{2}, 2\delta\right) \cup \{w \in \mathbb{H} : \mathrm{d}(w,z) = \mathrm{d}(w,z') \leqslant \delta\sqrt{5}\}$$

be a "separator set" between z and z' (cf. Figure 4.1).

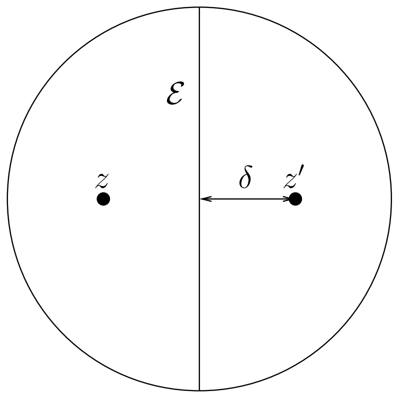

Figure 4.1: Second moments: the setup

Introduce a small constant $a \in (0,1)$ (to be determined later), and let $r_n = \delta.a^n$. We will condition the path with respect to the order in which it visits the circles $\mathcal{C}_n = \mathcal{C}(z, r_n)$ and $\mathcal{C}'_n = \mathcal{C}(z', r_n)$, so introduce the following families of stopping times:

$$\begin{aligned} T_n &= \mathrm{Inf}\{t : \gamma(t) \in C_n\}; \\ \tilde{T}_n &= \mathrm{Inf}\{t > T_n : \gamma(t) \in \mathcal{E}\}; \\ T'_n &= \mathrm{Inf}\{t : \gamma(t) \in C'_n\}; \\ \tilde{T}'_n &= \mathrm{Inf}\{t > T'_n : \gamma(t) \in \mathcal{E}\}. \end{aligned}$$

Moreover, define inductively $N_0 = N'_0 = 0$ and

$$N_{i+1} = \mathrm{Min}\{k : \tilde{T}_k > \tilde{T}_{N_i}\},$$

$$N'_{i+1} = \mathrm{Min}\{k : \tilde{T}'_k > \tilde{T}'_{N'_i}\}$$

(so that the $(N_{i+1} - N_i)$ are the successive numbers of circles around z that γ crosses for the first time between returns on $\mathcal{E}$). Lastly, let K be the number of times γ returns on $\mathcal{E}$ before touching $\mathcal{B}(z,\varepsilon)$ (that is, $\mathrm{Max}\{k : r_{N_k} > \varepsilon\}$) and K' accordingly around z'.

Splitting the event that γ touches both $\mathcal{B}(z,\varepsilon)$ and $\mathcal{B}(z',\varepsilon)$ according to the values of K, and then according to the values of the $(N_{i+1} - N_i)$ for $0 \leqslant i < K$ and $(N_{j+1} - N_j)$ for $0 \leqslant j < K'$, we have

$$P(\gamma \cap \mathcal{B}(z,\varepsilon) \neq \varnothing,\ \gamma \cap \mathcal{B}(z',\varepsilon) \neq \varnothing) = \\ \sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{n_1,\ldots,n_k>0}\ \sum_{n'_1,\ldots,n'_{k'}>0} P\begin{pmatrix} \gamma \cap \mathcal{B}(z,\varepsilon) \neq \varnothing,\ \gamma \cap \mathcal{B}(z',\varepsilon) \neq \varnothing \\ K = k, N_1 - N_0 = n_1, \ldots, N_k - N_{k-1} = n_k \\ K' = k', N'_1 - N'_0 = n'_1, \ldots, N'_{k'} - N'_{k'-1} = n'_{k'} \end{pmatrix}. \quad (4.9)$$

Note that we say nothing about the order in which the sequences T_{N_i} and $T'_{N'_i}$ are intertwined, all we know is that they are both increasing. So each term in the summation can in turn be written as a sum of "elementary probabilities" over all possible such orderings. There are

$$\binom{k+k'}{k} \leqslant 2^{k+k'}$$

of them; so if we obtain an estimate for each term of the sum restricted to a given ordering, which does not depend on this ordering, we only get an additional factor $2^{k+k'}$ in front of each term in (4.9).

We write each of the elementary probabilities as a chain of telescopic conditional probabilities, conditioning first on the fact that γ hits $\mathcal{E}$ and then using the strong Markov property at the times $\tilde{T}_{N_i}$ and $\tilde{T}'_{N'_i}$, in increasing order: This leads to a product of conditional probabilities that $N_i - N_{i-1} = n_i$ for all $1 \leqslant i \leqslant k$, and corresponding terms around z' (we do not write this product explicitly because it would require even more tedious notations).

Each of these factors is smaller that the probability that $N_i - N_{i-1} \geqslant n_i$, which in turn is smaller than the conditional probability to ever touch the circle $\mathcal{C}(z, r_{N_{i-1}+n_i})$ — and this is exactly the kind of probability which we estimated in the previous section. We already know that the first term will be

$$P\left(\frac{z+z'}{2} \in C_\delta\right) \asymp \delta^{1-\kappa/8},$$

so we will estimate the other factors separately, using Corollary 4.1, and see what happens.

We will have to distinguish between two cases, depending on whether $\kappa < 4$ or $\kappa > 4$, because the reason why the sum in (4.9) converges is different in both situations. If $\kappa = 4$, the method does not seem to work directly — cf. the end of the section for some discussion about this problem.

4.2.2 First case: $\kappa < 4$

In this whole subsection, we assume that $\kappa \in (0,4)$ — *i.e.* that γ is a simple curve. Conditionally on the whole process up to time $\tilde{T}_{N_{i-1}}$, it is clear that $\mathrm{d}(z, K_{\tilde{T}_{N_{i-1}}})$ is at least $a.r_{r_{N_{i-1}}}$. Introduce the following notation for the harmonic measures appearing in the statement of Corollary 4.1:

$$\omega_i = \omega_z(g^{-1}_{\tilde{T}_{N_{i-1}}}((\beta_{T_{N_{i-1}}}, \infty))) \wedge \omega_z(g^{-1}_{\tilde{T}_{N_{i-1}}}((-\infty, \beta_{T_{N_{i-1}}})))$$

(*i.e.*, at time $\tilde{T}_{N_{i-1}}$, the smaller of the harmonic measures of the two "sides" of the curve, seen from z), and ω'_j similarly around z'. The first moment estimate then provides a factor not greater than

$$C\left(\frac{r_{N_i}}{a.r_{N_{i-1}}}\right)^s \omega_i^\eta = C a^{s(n_i-1)} \omega_i^\eta \tag{4.10}$$

(where we let $s = 1 - \kappa/8$ and $\eta = 8/\kappa - 1$). The last such term will correspond to the probability of touching $\mathcal{B}(z, \varepsilon)$ in the end; it will not be greater than

$$C\left(\frac{\varepsilon}{ar_{N_k}}\right)^s \omega_{k+1}^\eta = C(\varepsilon/\delta)^s a^{-s(n_1+\cdots+n_k+1)} \omega_{k+1}^\eta. \tag{4.11}$$

So taking everything into account we see that the a^{sn_i} will cancel out, and we get the estimate:

$$P(z,z'\in C_\varepsilon)\leqslant C\frac{\varepsilon^{2s}}{\delta^s}\sum E\left[2^{k+k'}\left(\frac{C}{a^s}\right)^{k+k'+2}\prod_{i=1}^{k+1}(\omega_i)^\eta\prod_{j=1}^{k'+1}(\omega'_j)^\eta\right] \tag{4.12}$$

(the summation here ranges on the same set as in the huge sum (4.9), and the expectation is understood as a conditional expectation knowing the values of K, K', the n_i and n'_j). Since C is a generic constant, we can multiply it by 2 to take care of the factor $2^{k+k'}$, which we shall not write down anymore.

Next, we shall estimate the (ω_i). At time $\tilde T_{N_i}$, the intersection of γ with the disc $\mathcal{B}(z,a.r_{N_{i-1}})$ is at distance at most r_{N_i} of z and hence, by the Beurling estimate, it has harmonic measure (seen from z) at least $1-c.a^{(n_i-1)/2}$; and this lower bound actually holds for the harmonic measure of one of the sides of the curve. Hence an upper bound on ω_{i+1}, which is not greater than $c.a^{(n_i-1)/2}$. So, picking any $\eta'<\eta/2$, we get a factor in (4.12) which is not greater than $C_0.a^{\eta' n_i}$, as soon as

$$n_i\geqslant\bar n=\frac{\eta}{\eta-2\eta'}.$$

The only "bad case" in the previous computation is therefore the case when $n_i\leqslant\bar n$ *and* the harmonic measure from z charges the two sides of $\gamma[0,\tilde T_{N_i}]$ with about the same mass. Assume i satisfies this. We may assume that $\bar n=1$ (it is fixed anyway) as well as continue γ until the first time $\tau\geqslant\tilde T_{N_i}$ when it touches the intersection of $\mathcal{E}$ with the boundary of the connected component of z' in $\mathbb{H}\setminus(\mathcal{E}\cup\gamma[0,\tilde T_{N_i}])$ — quite often this corresponds to doing nothing at all; and it has to happen if γ is to eventually hit $\mathcal{B}(z',\varepsilon)$. Assume that we still are in a bad situation, namely that each of the sides of γ has harmonic measure greater than $C_0.a^{\eta' n_i}$ (with the same constant C_0 as in the previous paragraph). The only possibility for this to happen is when $\gamma[0,T_{N_i}]\cup\mathcal{B}(z,r_{N_i}a^{-\bar n})$ separates z' from infinity.

Assuming that we are in this case, we can refine our estimate of the harmonic measures of the two sides of $\gamma[0,T_{N_i}]$ seen from z': Indeed, at least one of them is smaller than that of $\mathcal{B}(z,r_{N_i}a^{-\bar n})$ in the domain Ω enclosed by $\gamma[0,T_{N_i}]\cup\mathcal{B}(z,r_{N_i}a^{-\bar n})$. Consider a planar Brownian motion W started at z', and estimate the probability that it exits Ω on $\mathcal{B}(z,r_{N_i}a^{-\bar n})$. First it has to reach $\mathcal{E}$: by the Beurling estimate, this happens with probability not greater than $c(\mathrm{d}(z',\partial\Omega)/\delta)^{1/2}\leqslant c.a^{(N'_j-1)/2}$, where j is the number of crossings toward z' so far. Then the Brownian motion has to touch $\mathcal{B}(z,r_{N_i}a^{-\bar n})$ before the other parts of $\partial\Omega$, and this too, conditionally on the point at which W touches $\mathcal{E}$, has a probability that can be bounded by Beurling's estimate: this conditional probability is not greater than $c.(r_{N_i}a^{-\bar n}/\delta)^{1/2}$. Hence we can replace the estimate of ω'_j in this case by the following:

$$\omega'_j\leqslant c.a^{(N_i+N'_j-\bar n-1)/2}.$$

But we know that $N_i\geqslant n_i+i-1$ and $N'_j\geqslant n'_j$: So, as soon as $i\geqslant\bar n+2$, this term is smaller than $C.a^{(n_i+n'_j)/2}$, and up to an additional factor in (4.9) which depends only on a and κ, we may assume that it is the case for all i: So, in the case $n_i\leqslant\bar n$, we also obtain the factor $C.a^{\eta' n_i}$ in (4.9).

Collecting all the terms we obtained, the upper bound on the probability that γ hits both $\mathcal{B}(z,\varepsilon)$ and $\mathcal{B}(z',\varepsilon)$ becomes

$$P(z,z'\in C_\varepsilon)\leqslant C\frac{\varepsilon^{2s}}{\delta^s}\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{n_1,\ldots,n_k>0}\sum_{n'_1,\ldots,n'_{k'}>0}\left(\frac{C}{a^s}\right)^{k+k'+2}a^{\eta'\sum n_i}a^{\eta'\sum n'_i}$$
$$\leqslant Ca^{-2s}\frac{\varepsilon^{2s}}{\delta^s}\left(\sum_{k=0}^{\infty}\left[Ca^{-s}\sum_{n=1}^{\infty}a^{\eta' n}\right]^k\right)^2=Ca^{-2s}\frac{\varepsilon^{2s}}{\delta^s}\left(\sum_{k=0}^{\infty}\left[\frac{C.a^{\eta'-s}}{1-a^{\eta'}}\right]^k\right)^2.$$

Remember that we are in the case $\kappa<4$: this implies that

$$\frac{\eta}{2}-s=\frac{1}{2}\left(\frac{8}{\kappa}-1\right)-\left(1-\frac{\kappa}{8}\right)=\frac{(8-\kappa)(4-\kappa)}{8\kappa}>0.$$

Hence, by choosing η' close enough to $\eta/2$, we may assume that $\eta'-s>0$; then picking a small enough, we may assume that $C.a^{\eta'-s}<1-a^{\eta'}$, and in this case the sum in the last term converges, and this leads to

$$P(z,z'\in C_\varepsilon)\leqslant C\frac{\varepsilon^{2s}}{\delta^s},$$

which is exactly Condition 3. in Proposition 3.1.

4.2.3 Second case: $\kappa>4$

Assume now that $\kappa\in(4,8)$. The previous proof does not work anymore, but we know that γ will not be a simple curve. It is actually easy to see that the following holds: Let $0<r_1<r_2$, $t\geqslant 0$, and let $A_{r_1}(t)$ be the connected component of $\mathcal{B}(\gamma_t,r_1)\setminus K_t$ which contains γ_t on its boundary. Then the conditional probability, knowing γ up to time t, that γ separates $A_{r_1}(t)$ from infinity before it reaches the circle $\mathcal{C}(\gamma_t,r_2)$ is bounded below by $1-c.(r_1/r_2)^\alpha$ for some $c,\alpha>0$ depending only on κ.

Consider the same decomposition of the event $\{z,z'\in C_\varepsilon\}$ as in the previous case. We may add another condition, which is that for each $i>0$, $z\notin K_{\tilde{T}_{N_i}}$ (*i.e.*, γ does not separate z from ∞ while heading out towards $\mathcal{E}$). Indeed, if this is not the case, then there is no way for γ to ever reach $\mathcal{B}(z,\varepsilon)$. And the conditional probability, knowing γ up to T_{N_i}, that $z\notin K_{\tilde{T}_{N_i}}$, is not greater than $c.r^\alpha_{N_i}$. So the previous decomposition leads this time to

$$P(z,z'\in C_\varepsilon)\leqslant C\frac{\varepsilon^{2s}}{\delta^s}\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{n_1,\ldots,n_k>0}\sum_{n'_1,\ldots,n'_{k'}>0}\left(\frac{C}{a^s}\right)^{k+k'+2}a^{\alpha\sum N_i}a^{\alpha\sum N'_i}$$
$$=C\frac{\varepsilon^{2s}}{\delta^s}\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{n_1,\ldots,n_k>0}\sum_{n'_1,\ldots,n'_{k'}>0}\left(\frac{C}{a^s}\right)^{k+k'+2}a^{\alpha\sum(k-i+1)n_i}a^{\alpha\sum(k'-i+1)n'_i}$$
$$=C\frac{\varepsilon^{2s}}{\delta^s}\left(\sum_{k=0}^{\infty}C^ka^{-sk}\prod_{l=1}^{k}\sum_{n=1}^{\infty}a^{\alpha ln}\right)^2=C\frac{\varepsilon^{2s}}{\delta^s}\left(\sum_{k=0}^{\infty}\prod_{l=1}^{k}\frac{Ca^{l\alpha-s}}{1-a^{l\alpha}}\right)^2.$$

Since $a < 1$, the product $\prod(1 - a^{l\alpha})$ is convergent, hence the previous expression can be written as

$$\begin{aligned} P(z, z' \in C_\varepsilon) &\leqslant C\frac{\varepsilon^{2s}}{\delta^s}\left(\sum_{k=0}^{\infty}\prod_{l=1}^{k} Ca^{l\alpha - s}\right)^2 \\ &\leqslant C\frac{\varepsilon^{2s}}{\delta^s}\left(\sum_{k=0}^{\infty} C^k a^{-sk} a^{\alpha k(k+1)/2}\right)^2 \\ &\leqslant C\frac{\varepsilon^{2s}}{\delta^s}\left(\sum_{k=0}^{\infty} a^{\alpha k^2/2 - ck}\right)^2. \end{aligned}$$

This last sum is convergent, so in this case too we obtain the right estimate for $P(z, z' \in C_\varepsilon)$. This concludes the proof of the theorem for all $\kappa \neq 4$.

4.2.4 Comments about the case $\kappa = 4$

The proof for $\kappa > 4$ does not work when $\kappa = 4$ because SLE_4 is a simple curve and does not close any loops.

Our proof in the case $\kappa < 4$ has to do with the fact the domain H_t is Hölder for all $t > 0$ ([42]). Indeed, this implies (cf. for instance [10]) that, intuitively, γ cannot go back and forth too many times between two given points, and cannot create too many deep fjords — which is exactly what we proved here. However, it is not clear how to use this fact directly to obtain second moments, because it would require a quantitative version of this intuition, which is not known (yet). Still, it is an indication about the reason why this proof does not extend to the case $\kappa = 4$, for which H_t is *not* a Hölder domain anymore (cf. [42]). Equivalently, the "boundary exponent" $8/\kappa - 1$ is not large enough compared to the "bulk exponent" $1 - \kappa/8$. In fact, it is not even clear whether condition 3. holds for $\kappa = 4$ (note that the value of the constant in the upper bound on the second moment depends on κ and seems to explode when κ tends to 4). There might be a logarithmic correction term.

The right way to prove that $\dim_H(\gamma) = 3/2$ here might be to use the case $\kappa < 4$ and let κ increase 4, but it is not sure whether this can be done in a simple way. Anyway, it is *a posteriori* not so surprising that we need something more, since the existence of γ itself requires a separate proof in the case $\kappa = 4$ (see [42]).

4.3 The occupation density measure

As a side remark, let us consider the proof of the lower bound for the dimension (cf. Section A.1). It is based on the construction of a Frostman measure μ supported on the path, constructed as a subsequential limit of the family (μ_ε) defined by their densities with respect to the Lebesgue measure on the upper-half plane:

$$\mathrm{d}\mu_\varepsilon(z) = \varepsilon^{-s}\mathbf{1}_{z\in C_\varepsilon}|\mathrm{d}z|.$$

Then, μ is a random measure with correlations between $\mu(A)$ and $\mu(B)$, for disjoint compact sets A and B, decaying as a power of their inverse distance. So, at least formally, it behaves in this respect like a conformal field: the one-point function (corresponding to the density of μ) is not well-defined, because μ is singular to the Lebesgue measure, but the two-point correlation

$$\lim_{\delta\to 0} \delta^{-4} \operatorname{Cov}\left(\mu(\mathcal{B}(z,\delta)), \mu(\mathcal{B}(z',\delta))\right)$$

behaves like $\mathrm{d}(z,z')^{-1+\kappa/8}$.

A little more can be said about this measure, or about its expectation. The proof of the estimate for $P(\gamma \cap \mathcal{B}(z,\varepsilon) \neq \varnothing)$ can be refined in the following way: When we apply the stopping theorem (4.8), saying that the diffusion conditioned to survive has a limiting distribution shows that

$$E\left[\sin\left(\frac{\alpha_s}{2}\right)^{8/\kappa-1} \middle| S \geqslant s\right]$$

has a limit λ when $s \to \infty$, and that this limit depends only on κ. So what we get out of the construction in Section 4.1 is

$$P\left(\exists t > 0 : |g_t'(z)| \geqslant \frac{\Im(z)}{\varepsilon}\right) \underset{\varepsilon\to 0}{\sim} \lambda(\kappa) \left(\frac{\varepsilon}{\Im z}\right)^{1-\kappa/8} (\sin(\arg(z)))^{8/\kappa-1}.$$

This lead us to an estimate on $P(\mathrm{d}(z,\gamma) < \varepsilon)$ by the Köbe 1/4 Theorem; but it is also natural to measure the distance to γ by the modulus of g'. We can now define

$$\phi_1(z) = \lim_{\varepsilon\to 0} \varepsilon^{\kappa/8-1} P\left(\exists t > 0 : |g_t'(z)| \geqslant \frac{\Im(z)}{\varepsilon}\right) :$$

the previous estimate boils down to

$$\phi_1(z) = \lambda(\kappa) \Im(z)^{\kappa/8-1} \sin(\arg z)^{8/\kappa-1},$$

and by the construction of μ, we obtain that for every Borel subset A of the upper-half plane,

$$E(\mu(A)) \asymp \int_A \phi_1(z)\, |\mathrm{d}z|$$

with universal constants.

It is then possible to do this construction for several points; note first that the second moment estimate can actually be written as

$$P(\{z,z'\} \subset C_\varepsilon) \asymp \frac{\varepsilon^{2(1-\kappa/8)}}{|z-z'|^{1-\kappa/8} \Im((z+z')/2)^{1-\kappa/8}}$$

as long as both $\Im(z)$ and $\Im(z')$ are bounded below by $|z-z'|/M$ for some fixed $M > 0$. Indeed, the upper bound is exactly what we derived in the previous section, and the lower bound is provided by the term $k = k' = 0$ in (4.9). Hence, any subsequential limit $\psi(z,z')$, as ε vanishes, of

$$\varepsilon^{2(\kappa/8-1)} P\left(\{z,z'\} \subset C_\varepsilon\right)$$

satisfies $\psi(z,z') \asymp \phi_2(z,z')$ for some fixed function ϕ_2, with constants depending only on κ. The second moment estimate then shows that

$$\phi_2(z,z') \underset{z'\to z}{\asymp} \frac{\phi_1(z)}{|z-z'|^{1-\kappa/8}},$$

i.e. ϕ_2 behaves like a correlation function when z and z' are close to each other.

The general case of n points, $n \geqslant 2$, can be treated in the same fashion. First, the derivation of second moments admits a generalization to n points, as follows. Let $(z_i)_{1 \leqslant i \leqslant n}$ be n distinct points in $\mathbb{H}$, such that their imaginary parts are large enough (bigger than, say, $18n$ times the maximal distance between two of them). We use them to construct a Voronoi tessellation of the plane; denote by C_i the face containing z_i, and by δ_i the (Euclidean) distance between z_i and ∂C_i. Let $\mathcal{C}(z_0, \delta_0)$ be the smallest circle containing all the discs $\mathcal{B}(z_i, \delta_i)$. Lastly, let $\mathcal{E}$ be the "separator set" between the z_i's, defined as

$$\mathcal{E} = \mathcal{C}(z_0, \delta_0) \cup \left[\left(\bigcup_{i=1}^{n} \partial C_i \right) \cap \mathcal{B}(z_0, \delta_0) \right].$$

It is the same as defined previously in the case $n = 2$.

The previous proof can then be adapted to show that

$$P(\{z_1, \dots, z_n\} \subset C_\varepsilon) \asymp \left(\frac{\delta_0 \varepsilon^n}{\prod \delta_i} \right)^{1-\kappa/8}$$

(using radii $\delta_i a^k$ for the circles around z_i). In the case $n = 2$, we have $\delta_1 = \delta_2 = \delta_0/2$, so this estimate is exactly the same as previously. So, it makes sense to take a (subsequential) limit, as ε tends to 0, of

$$\varepsilon^{n(\kappa/8-1)} P(\{z_1, \dots, z_n\} \subset C_\varepsilon),$$

and all possible subsequential limits are comparable to a fixed symmetric function ϕ_n.

The behaviour of $\phi_n(z_1, \dots, z_n)$ when z_n approaches the boundary is then given by the boundary term in Proposition 4.1, *i.e.* ϕ behaves like $(\Im z_n)^{8/\kappa-1}$ there. Lastly, it is easy to see that, when z_n tends to z_1, $\phi_n(z_1, \dots, z_n)$ has a singularity which is comparable to $|z_n - z_1|^{\kappa/8-1}$; in other words, we have a recursive relation between all the ϕ_n's, given by

$$\phi_n(z_1, \dots, z_n) \underset{z_n \to z_1}{\asymp} \frac{\phi_{n-1}(z_1, \dots, z_{n-1})}{|z_n - z_1|^{1-\kappa/8}}, \tag{4.13}$$

$$\phi_n(z_1, \dots, z_n) \underset{\Im z_n \to 0}{\asymp} \phi_{n-1}(z_1, \dots, z_{n-1}).(\Im z_n)^{8/\kappa-1}. \tag{4.14}$$

These relations are very similar to some of those satisfied by the correlation functions in conformal field theory. In fact it is possible to push the relation further, in two ways. First, we can look at the evolution of the system in time. This corresponds to mapping the whole picture by the map $g_t - \beta_t$, and this map acts on the discs of small radius around the z_i's like a multiplication of factor $|g_t'(z_i)|$ (as long as K_t remains far away from the z_i's, which we may assume if t is small enough). Hence, the process

$$Y_t^n \triangleq \left(\prod |g_t'(z_i)|^{1-\kappa/8} \right) \phi_n(g_t(z_1) - \beta_t, \dots, g_t(z_n) - \beta_t)$$

(defined as long as all the z_i's remain outside K_t) is a local martingale. We can apply Itô's formula to compute $\mathrm{d}Y_t^n$, and write that the drift term has to be 0 at time 0 to obtain a PDE satisfied by ϕ_n.

Note though that the formula involves the modulus of g_t', meaning that the equation we would obtain cannot be expressed in terms of complex derivatives of g_t only, and that we have to introduce derivatives with respect to the coordinates. This is also the case for the second-order term in Itô's formula: Since β is a real process, we would obtain terms involving second derivatives of ϕ_n with respect to the x-coordinates of the arguments. To sum it up, it would be an ugly formula without the correct formalism — which is why we do not put it here. The formula is much nicer when considering points on the boundary of the domain — cf. [17].

The last thing we can do is study what happens if we add one point z_{n+1} to the picture. This will add one multiplicative factor, corresponding (at least intuitively) to the conditional probability to hit z_{n+1} knowing that we touch the first n points already. In the case $\kappa = 8/3$ and for points on the boundary of the domain, this can be computed using the restriction property, and it leads to Ward's equations (cf. [17]). In the "bulk" (*i.e.* for points inside the domain), or for other values of κ, it is not clear yet how to do it.

4.4 The boundary

A natural question is the determination of the dimension of the boundary of K_t for some fixed t, in the case $\kappa > 4$. The conjectured value is

$$\dim_H(\partial K_t) = 1 + \frac{2}{\kappa},$$

and this can now be proved for a few values of κ for which the boundary of K can be related to the path of an $SLE_{\kappa'}$ with $\kappa' = 16/\kappa$. In fact, this relation is only known in the cases where convergence of a discrete model to SLE is known, namely:

- $\kappa = 6$, where actually both ∂K_t and the path of the $SLE_{\kappa'}$ are closely related to the Brownian frontier. Hence we obtain a third derivation of the dimension of the Brownian frontier, this time through $SLE_{8/3}$.
- $\kappa = 8$: Here, SLE_8 is known to be the scaling limit of the uniform Peano curve and SLE_2 that of the loop-erased random walk (cf. [34]). Since these two discrete objects are closely related through Wilson's algorithm, this shows that the local structure of the SLE_2 curve and the SLE_8 boundary are the same, and in particular they have the same dimension.

So we obtain one additional result here:

Corollary 4.2 :

Let (K_t) be a chordal SLE_8 in the upper-half plane: Then, for all $t > 0$, the boundary of K_t almost surely has Hausdorff dimension $5/4$.

It would be nice to have a direct derivation of the general result, without using the "duality" between SLE_κ and $SLE_{16/\kappa}$. All that is needed is probably a precise estimate of the probability that a given ball intersects the boundary of K_1: The previous proof in the case $\kappa < 4$ can be applied directly if we know that, for all $z \in \mathbb{H}$, we have

$$P\left(\mathcal{B}(z,\varepsilon) \cap \partial K_1 \neq \varnothing\right) \asymp \left(\frac{\varepsilon}{\Im z}\right)^{1-2/\kappa} \sin(\arg z)^\eta$$

with $\eta/2 > 1-2/\kappa$. The argument we used in the case $\kappa > 4$ cannot work though, because the fact that γ closes loops is exactly what will provide the difference between $\dim\gamma$ and $\dim\partial K$.

Chapter 5

Variations around SLE

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5.1 SLE in a non-simply connected domain

One of the first questions that arise when trying to extend the definition of SLE, especially when it is seen as the (conjectured) scaling limit of a discrete model, is whether there is a natural definition of it in a more general domain of the complex plane. Indeed, it is easy to define *e.g.* a percolation model or a loop-erased random walk in a discrete approximation of, say, a multiply connected open subset of $\mathbb{C}$, and the scaling limits of these models, if any, will share many properties with the corresponding usual SLE's.

For instance, the locality property of SLE_6 (corresponding to that of the percolation exploration process) basically states that, locally, the process does not "see" the shape of the domain; in particular, the local geometry of the curve should not be affected by the connectivity of the domain, and we would expect locality to hold for the corresponding process in a multiply connected domain.

A fair amount of the arguments used to prove convergence in the simply connected case actually do not use the fact that the domain is simply connected. For instance, the "reversed Markov property" of loop-erased random walks and Wilson's algorithm (used in [34]) are valid in any connected graph, in particular they apply in the case of the discretization of a multiply connected subset of the complex plane. It is therefore natural to expect some sort of Markov property, similar to SLE's, in the scaling limit. Besides, the close relation between loop-erased walks and Brownian motion (through the simple random walk) gives an intuition that the scaling limit should also exhibit some kind of conformal invariance.

5.1.1 SLE in the annulus

The construction of SLE relies heavily on Riemann's Theorem, *i.e.* on the existence of a reference domain: There is a natural statement of the Markov property if we can map $\mathbb{H} \setminus K_t$ conformally onto $\mathbb{H}$, by saying that

$$g_{t+s} - \beta_{t+s} \overset{\text{(law)}}{=} (\tilde{g}_s - \tilde{\beta}_s) \circ (g_t - \beta_t),$$

where $(\tilde{g}_s)$ is an independent copy of (g_t) with driving process $(\tilde{\beta}_s)$. In other words, the natural setup is that of a semi-group of conformal maps, in which SLE can be written as an infinitely divisible process.

Suppose now that we want to construct a simple random curve (γ_t) in an annulus. More precisely, for all $r \in (0,1)$, let

$$A_r = \{z \in \mathbb{C} : r < |z| < 1\},$$

then for all such r and all $a, b \in \partial\mathbb{U}$, we want a law on curves from a to b in $\bar{A}_r$. Assume that we have such a curve, and pick $t > 0$ such that γ_t is defined and different from b. Consider the domain $A_r(t) = A_r \setminus \gamma([0,t])$. Is is easy to see that its modulus is strictly smaller than that of A_r (*i.e.* $-\log r$). Writing this modulus as $-\log(r(t))$, there exists exactly one conformal map g_t from $A_r(t)$ to $A_{r(t)}$, fixing b; and, when the curve reaches b (which can happen in finite and infinite time), $r(t)$ converges to a limiting value r' such that $-\log r'$ is the modulus of the complement of the whole curve in $\mathcal{A}_r$.

Up to reparameterization of γ, we may suppose that for all $t \leqslant \log(r'/r)$, $r(t) = re^t$: Then, the situation is comparable to the simply connected case, *i.e.* γ is characterized by the function $t \mapsto \beta_t = g_t(\gamma_t) \in \partial\mathbb{U}$ and $(g_t(z))$ satisfies a differential equation similar to Loewner's evolution, with a vector field on each $A_{r(t)}$ depending only on b and β_t and related to Villat's kernel ([49]). So it is tempting construct a natural law on (β_t) and then run the differential equation to obtain (g_t), and hopefully (γ_t), as it is done in the usual simply connected setup.

If a Markov property is to be looked for, we have to construct the law of (β_t) simultaneously for all values of a, b and r. Let $\mathcal{L}(a,b,r)$ be this law: It is supported on

$$\mathcal{C}_{a,r} = \{\gamma \in \mathcal{C}([0, \log r'/r), \partial\mathbb{U}) : r' \in (r,1), \beta_0 = a\}.$$

With these notations, the Markov property can be seen as a compatibility relation between these laws: If β it a random function distributed according to $\mathcal{L}(a,b,r)$, defined on the time range $[0, t(\beta))$, and if $t \in (0, -\log r)$, then, conditionally to the fact that $t(\beta)$ at least t and to β up to time t, we have

$$(\beta_{t+s})_{s\in[0,)} \sim \mathcal{L}(\beta_t, b, re^t) \tag{5.1}$$

In other words, (re^t, β_t) should be a continuous Markov process.

This is the point where a miracle occurred in the simply connected case: r did not appear because we had a common domain, and a few heuristic considerations were sufficient to obtain enough information on β (namely, in the upper-half plane, that it had to be continuous with independent increments and symmetric) to conclude that it had to be a Brownian motion with a linear time-change. Hence, we obtained "universality", in the sense that there is only a one-parameter family of laws, described by the parameter κ.

Here, we might write $\beta_t = be^{i\alpha_t}$ where α_t is a continuous diffusion on $(0, 2\pi)$ generated by

$$\mathrm{d}\alpha_t = \sigma(re^t, \alpha_t)\,\mathrm{d}B_t + v(re^t, \alpha_t)\,\mathrm{d}t,$$

and tailor σ and v so that, locally around a, (g_t) looks like a (time-changed) chordal SLE_κ from a to b in the whole disk. However, there seems to be no simple reason why κ should not depend on r, or why the drift of α (if any) should be the same.

Actually there is one case where we do have a fixed reference domain, namely the disk punctured at 0 (corresponding to the case $r = 0$ in the previous setup). In this case, we need conformal maps from the punctured disk minus a compact set touching the boundary, onto the punctured disk. But it is easy to see that such a map can be continued at the puncture, and that this implies that the continued map will be conformal on a neighborhood of the origin. Hence, what we are interested in is conformal maps fixing the origin and one marked point on the boundary (the target of the process, *i.e.* b with the previous notations). And this is exactly sufficient to ensure existence and uniqueness of the maps involved in the construction.

With this normalisation, the vector field defining the Loewner equation toward b, growing at W_t, is the one we used in the proof of Proposition 4.1, when we looked at the growing compact set from a fixed point in the domain; namely, the family of conformal maps (g_t) satisfies the equation

$$\partial_t g_t(z) = \frac{2W_t g_t(z)(g_t(z) - b)}{(b - W_t)(g_t(z) - W_t\zeta)}$$

(this is exactly Equation (4.4)). The time parameterization given by $r(t) = re^t$ becomes $|g_t'(0)| = e^t$ in this case, *i.e.* the time parameterization is similar to that of a radial SLE.

In other words, for $\kappa \leqslant 4$, it is possible to define a law on random curves from a to b in the punctured disk by simply taking the curve of an SLE from a to b in the unit disk, and saying that it almost surely does not go through 0. Then the driving process for the growing curve is the diffusion on the circle which we described in the previous chapter, generated by the following SDE:

$$\mathrm{d}\alpha_t = \sqrt{\kappa}\,\mathrm{d}B_t + \frac{\kappa - 4}{2}\operatorname{cotg}\frac{\alpha_t}{2}\,\mathrm{d}t.$$

The Markov property in the punctured disk is a direct consequence of the usual Markov property for chordal SLE.

The stochastic differential equation giving α and the ODE satisfied by g_t still make sense in the case $\kappa \in (4, 8)$ (for which the diffusion a.s. touches b in finite time); but the construction stops in finite (SLE-)time, when the SLE curve closes a bubble around 0 — which it does with probability 1. Then we obtain a curve in the punctured disk that separates the puncture from b. It is perfectly possible to continue it by appending to it a standard SLE_κ from the point where the loop was closed to b, in which case the complete curve is that of a standard SLE_κ from a to b in $\mathbb{U}$; but the nature of the process changes when the bubble is closed, the statement of the Markov property is different, and so is the time parameterization.

This will certainly happen in the more general setup of a domain with finitely many punctures (or holes — or even in the case of a Riemann surface of positive genus): If we manage to define a counterpart to SLE_κ with $4 < \kappa < 8$, it will close loops around the punctures and the "relevant domain" at time t (the connected component of the complement of $\gamma([0,t])$ that has b on its boundary) will have a non-increasing genus $h(t)$ which will go down each time such a bubble is closed. But in any case, as long as $h(t) = h(0)$, the "natural" curve is exactly the trace of a usual SLE in the filled-in domain.

Going back to the case of the annulus with $r > 0$, there are a few cases where a natural measure on curves can be described, using known properties of usual chordal SLE for specific values of the parameter κ.

5.1.2 Using the restriction property : $SLE_{8/3}$ in the annulus

The first working approach is to view SLE for $\kappa = 8/3$ as a restriction measure ([29]), and study whether it makes sense to generalize the definition to an annulus. So, define the random curve γ from a to b in A_r as a chordal SLE fro a to b in the unit disk, conditioned not to touch the disk of radius r centered at 0. (Note that this happens with positive probability.) We want to prove that it satisfies the compatibility relation (5.1).

Let $\delta = \gamma([0,\tau])$, where $\tau > 0$ is some finite stopping time; Let r' be such that the modulus of $A_r \setminus \gamma([0,\tau])$ is equal to $-\log r'$. Let $\tilde\gamma$ be the image of $\gamma \setminus \delta$ by the conformal map Φ from $A_r \setminus \delta$ to $A_{r'}$ fixing b. And let $\bar\gamma$ be a chordal $SLE_{8/3}$ from $\Phi(\gamma_\tau)$ to b in $\mathbb{U}$, conditioned not to touch $\mathcal{B}(0,r')$. What we have to show is that, conditionally on δ, $\tilde\gamma$ and $\bar\gamma$ have the same law.

This is a consequence of the restriction property, as follows. Let B be a crossing of A_r (*i.e.* a locally compact subset of A_r such that $A_r \setminus B$ is simply connected), containing neither a nor b in its closure. Then we know that $SLE_{8/3}$ conditioned not to hit $\bar B$ is an $SLE_{8/3}$ in $\mathbb{U} \setminus \bar B$, and the same happens if we replace $\bar B$ by $B \cup \bar{\mathcal{B}}(0,r)$. In particular, γ conditioned not to touch B is a plain old chordal $SLE_{8/3}$ in $A_r \setminus B$.

Now apply the Markov property to the unconditioned SLE at time τ. It states that conditionally to δ, $\gamma \setminus \delta$ is a chordal $SLE_{8/3}$ in $\mathbb{U} \setminus \delta$. Combining this and the restriction property shows that, conditionally to $\delta \cap B = \varnothing$, $\gamma \setminus \delta$ conditioned not to hit B is a chordal SLE in $A_r \setminus B$.

Notice then that Φ induces a conformal map from $A_r \setminus B$ to $A_{r'} \setminus \Phi(B)$, and apply the conformal invariance of SLE: Conditionally to $\delta \cap B = \varnothing$, $\tilde\gamma$ conditioned not to touch $\Phi(B)$ is an $SLE_{8/3}$ in $A_{r'} \setminus \Phi(B)$. But, conditionally to δ, Φ maps crossings of A_r not intersecting δ to crossings of $A_{r'}$, so that $\tilde\gamma$ satisfies the following condition: For every crossing $\tilde B$ of $A_{r'}$, $\tilde\gamma$ conditioned not to touch $\tilde B$ is a chordal $SLE_{8/3}$ from $\Phi(a)$ to b in $A_{r'} \setminus \tilde B$. But the previous proof shows that this is exactly the law of $\bar\gamma$ conditioned not to touch $\tilde B$: What we finally obtain is that, for every crossing $\tilde B$ of $A_{r'}$, $\tilde\gamma$ conditioned not to touch $\tilde B$ and $\bar\gamma$ conditioned not to touch $\tilde B$ have the same law.

From now on, for every crossing $\tilde{B}$, let $\tilde{p}(B)$ be the probability that $\tilde{\gamma}$ does not hit $\tilde{B}$ and similarly $\bar{p}$ for $\bar{\gamma}$. If B_1 and B_2 are two crossings, we denote by $B_1 \cup B_2$ the smallest crossing containing both of them, or $A_{r'}$ if such a crossing does not exist. We just proved that

$$P(\tilde{\gamma} \cap B_2 = \varnothing | \tilde{\gamma} \cap B_1 = \varnothing) = P(\bar{\gamma} \cap B_2 = \varnothing | \bar{\gamma} \cap B_1 = \varnothing),$$

and this can be written as

$$\frac{\tilde{p}(B_1 \cup B_2)}{\tilde{p}(B_1)} = \frac{\bar{p}(B_1 \cup B_2)}{\bar{p}(B_1)}.$$

So, for all B_1 and B_2 such that $B_1 \cup B_2$ is a crossing, conditioning first on B_2 instead of B_1 and dividing, we obtain

$$\tilde{p}(B_1)/\tilde{p}(B_2) = \bar{p}(B_1)/\bar{p}(B_2), \tag{5.2}$$

so on any collection of crossings that is stable by union, $\tilde{p}$ and $\bar{p}$ are proportional.

Let $\mathcal{R}$ be the collection of all crossings on the right (*i.e.* the ones which touch $\partial\mathbb{U}$ on the arc that goes from $\Phi(\gamma_t)$ to b in trigonometric orientation). The previous derivation shows that $\tilde{p}$ and $\bar{p}$ are proportional on $\mathcal{R}$. But the event that $\tilde{\gamma}$ goes to the left of the hole $\mathcal{B}(0, r')$ is the union of the events that $\tilde{\gamma}$ does not meet B, over all $B \in \mathcal{R}$: Using an inclusion-exclusion decomposition, this shows that the proportionality ratio between $\tilde{p}$ and $\bar{p}$ on $\mathcal{R}$ is the ratio between the probability that $\tilde{\gamma}$ passes to the left, and that that $\bar{\gamma}$ passes to the left. So, we now know that $\tilde{\gamma}$ and $\bar{\gamma}$ have the same law when conditioned to pass on the left of $\mathcal{B}(0, r')$. All that remains to do is prove that they have the same probability to pass to the left of the hole.

Restate the problem as follows. Let K and K' be two simply connected compact subsets of the upper-half plane $\mathbb{H}$, such that $\mathbb{H} \setminus K$ and $\mathbb{H} \setminus K'$ have the same modulus; and let Ψ be the unique conformal map from $\mathbb{H} \setminus K$ to $\mathbb{H} \setminus K'$ tending to infinity at infinity; let $\lambda > 0$ be such that $\Psi(z) \sim \lambda z$ at infinity. We will use the usual convention $\Psi'(\infty) = 1/\lambda$. Up to a translation of K' in the horizontal direction, we may assume that Ψ sends 0 to itself. Let γ be the curve of a chordal $SLE_{8/3}$ in $\mathbb{H}$: The only thing we have left to prove is that we have

$$P(\gamma \text{ left of } K | \gamma \cap K = \varnothing) = P(\gamma \text{ left of } K' | \gamma \cap K' = \varnothing). \tag{5.3}$$

Again, let B be a crossing from K to $\mathbb{R}_+$ in $\mathbb{H} \setminus K$; and let Φ_1 be the conformal map from $\mathbb{H} \setminus (K \cup B)$ to $\mathbb{H}$ fixing 0 and satisfying $\Phi_1(z) \sim z$ at infinity. Then, we know that

$$P(\gamma \cap (K \cup B) = \varnothing) = |\Phi_1'(0)|^{5/8}.$$

If Φ_2 is the conformal map from $\mathbb{H} \setminus (K' \cup \Psi(B))$ with the same normalization:

$$P(\gamma \cap (K' \cup \Phi(B)) = \varnothing) = |\Phi_2'(0)|^{5/8}.$$

Last, the uniqueness of all the maps involved here shows that

$$\Phi_1 = \lambda^{-1} \Phi_2 \circ \Psi :$$

hence, taking derivatives at 0, we obtain

$$\Phi_1'(0) = \frac{\Psi'(0)}{\lambda} \Phi_2'(0).$$

Remark that the factor $\Psi'(0)/\lambda$ does not depend on the choice of B, so the same considerations as previously, using the fact that γ passes to the left of K if and only if there is such a crossing and writing an inclusion-exclusion decomposition of this event, prove the following

Proposition 5.1 :

Let K and K' be compact subsets of the upper-half plane, such that there is a conformal map Ψ from $\mathbb{H} \setminus K$ to $\mathbb{H} \setminus K'$ fixing 0 and ∞; and let γ be the curve of a chordal $SLE_{8/3}$ in $\mathbb{H}$. Then, we have

$$\frac{P(\gamma \text{ left of } K)}{P(\gamma \text{ left of } K')} = (\Psi'(0)\Psi'(\infty))^{5/8}.$$

Remark: If Ψ could be extended to the whole upper-half plane (this is the case for instance if both K and K' are disks, in which case Ψ is a Möbius transform), then we would have $\Psi(z) = \lambda z$ so the product $\Psi'(0)\Psi'(\infty)$ would be equal to 1, concluding the proof — but the result was clear in this case, due to the scale invariance of chordal SLE.

Conditioning γ not to touch K (resp. K'), we can rewrite the conclusion of Proposition 5.1 as

$$\frac{P(\gamma \text{ left of } K | \gamma \cap K = \varnothing)}{P(\gamma \text{ left of } K' | \gamma \cap K' = \varnothing)} = (\Psi'(0)\Psi'(\infty))^{5/8} \frac{P(\gamma \cap K' = \varnothing)}{P(\gamma \cap K = \varnothing)}. \tag{5.4}$$

Notice that we could do exactly the same construction to compute the probability that γ goes to the *right* of K (resp. K'), and that the right-hand term in the last equation would be the same; hence,

$$\frac{P(\gamma \text{ left of } K | \gamma \cap K = \varnothing)}{P(\gamma \text{ left of } K' | \gamma \cap K' = \varnothing)} = \frac{P(\gamma \text{ right of } K | \gamma \cap K = \varnothing)}{P(\gamma \text{ right of } K' | \gamma \cap K' = \varnothing)}. \tag{5.5}$$

And since in any case γ conditioned not to hit K passes either to the left, or to the right of K, we have

$$P(\gamma \text{ left of } K | \gamma \cap K = \varnothing) + P(\gamma \text{ right of } K | \gamma \cap K = \varnothing) = 1$$

and the same around K': The only case where (5.5) can hold is then when both ratios are equal to 1, and in particular we obtain (5.3), concluding the proof. As a side-result, we also obtain the following corollary:

Proposition 5.2 :

With the same hypotheses and notations as in Proposition 5.1,

$$\frac{P(\gamma \cap K = \varnothing)}{P(\gamma \cap K' = \varnothing)} = (\Psi'(0)\Psi'(\infty))^{5/8}.$$

This could be used to study the behaviour of the n-point correlation function described in the previous chapter when a point is added (which in the case of points on the boundary corresponds to a conditioning, by the restriction property of $SLE_{8/3}$, as is shown in [17]); *i.e.*, it could be the first step of the derivation of Ward's equations in the bulk for $SLE_{8/3}$.

Some remarks about the construction

Almost none of the tools we just used is available if $\kappa \neq 8/3$, in particular we do not have the restriction property and we cannot compute the probability not to touch a hull as a power the derivative of a conformal map. On the other hand, if we try to apply the method to the general case of conformal restriction measures as defined in [29], we can perform a substantial part of the construction; in particular, everything we stated in the upper-half plane is still valid, hence the measures on subsets of conformal annuli defined by conditioning restriction measures in the half plane are invariant under conformal maps of the annuli, exactly by the same proof — only the value of the exponent $5/8$ needs to be replaced by the correct one. In particular, Propositions 5.1 and 5.2 still holds. It is clear also that the obtained measures satisfy the restriction property, but since removing a hull from the domain here changes its modulus, the restriction property does not make as much sense as in the simply connected case.

However, it is not clear what the statement of the Markov property should be in this case, since in the general case the restriction measures are not supported on simple paths.

The same proof can be applied to the general case of a finitely connected domain: If $\Omega \subsetneq \mathbb{C}$ is an open simply connected set, a and b are two points on its boundary and K is a finite union of disjoint simply connected compact subsets of Ω, we can define a law

$$\mathcal{L}(a \to b, \Omega \setminus K)$$

supported on simple curves from a to b in Ω by conditioning an $SLE_{8/3}$ from a to b in Ω not to touch K. Then the family of measures we obtain satisfies the same conditions as SLE, *i.e.*:

- Conformal invariance: If γ is distributed as $\mathcal{L}(a \to b, \Omega \setminus K)$ and Φ is a conformal map from $\Omega \setminus K$ to $\Omega' \setminus K'$, then $\Phi(\gamma)$ is distributed as $\mathcal{L}(\Phi(a) \to \Phi(b), \Omega' \setminus K')$;
- Restriction property: If A is a hull (*i.e.* a compact set such that $\Omega \setminus A$ is simply connected) containing neither a nor b, then $\mathcal{L}(a \to b, \Omega \setminus K)$ conditioned not to hit A is the same as $\mathcal{L}(a \to b, \Omega \setminus (A \cup K))$;
- Markov property: If δ is an connected subset of γ containing a and not b, and a' is its other end (the intersection of $\bar{\delta}$ and $\overline{\gamma \setminus \delta}$), then conditionally on δ, $\gamma \setminus \delta$ is distributed as $\mathcal{L}(a' \to b, \Omega \setminus (K \cup \delta))$.

Conformal invariance and the restriction property still hold in the general case of restriction measures, as well as Proposition 5.2. Note however that the conditions in which it applies are very restrictive, because of the number of conformal invariants involved in the multiply connected case.

It is possible to define an artificial "twisted" measure on subsets of $\Omega \setminus K$, as follows (here we suppose that $\Omega \setminus K$ is a topological annulus): First, take a restriction measure, and condition it not to touch K; call μ this measure. Then define $\tilde{\mu}$ by its density with respect to μ, where the density is constant (but not 1 *a priori*) on the collection of sets passing to the left (resp. to the right) of K. Equivalently, this corresponds to fixing the probability of going to the left of the hole instead of taking that of $SLE_{8/3}$. For instance, we might consider $SLE_{8/3}$ conditioned to pass to the left of K.

If the weighing is invariant under conformal map, *i.e.* if the densities only depend on the conformal type of the domain, then clearly the family of laws we obtain is still conformally invariant. Moreover, in the case of curves (*i.e.* if we start with $SLE_{8/3}$), we can this time perform the first part of the previous proof: we still obtain the fact that $\tilde{\gamma}$ and $\bar{\gamma}$ become the same when conditioned to pass to the left of the hole; and actually, it is then possible to choose the weights depending on the domain in such a way that they have the same probability to pass to the left, in which case the curve we obtain has the Markovian property of SLE. There is even nothing left to prove in the case of $SLE_{8/3}$ conditioned to pass to the left of K.

This twisted process is probably not interesting in itself, but it stresses the problem of what a generalization of SLE should be: even if we want the random object to have the same local geometry as SLE (*i.e.* if we solve the "changing κ" question by a geometric argument), there is still one global degree of freedom preventing "universality" in this setup.

5.1.3 Using the locality property : SLE_6 in the annulus

The other case where SLE exhibits a particularly nice behaviour is when $\kappa = 6$, where the SLE curve is the scaling limit of the exploration process of critical site-percolation on the triangular lattice (and also probably of any "reasonable" critical percolation model). This allows us to construct a random curve in a natural way, as follows. Let $\mathbb{U}$ stand for the unit disk, let $r \in (0,1)$ be fixed and let $A = \mathcal{B}(0,r)$ be the hole of the annulus $\mathcal{A}_r$, and fix a, b two points on the unit circle.

We could take γ to be the curve of an SLE_6 from a to b in $\mathbb{U}$ and condition it not to touch A, as we did in the previous section. But this is problematic, for two reasons:

- This conditioning is very global, and this was fine when we wanted to use the restriction property of $SLE_{8/3}$, which is global too. But the locality of SLE_6, as the name indicates, describes the *local* behaviour of the SLE_6 curve. And actually, it tends to say that as long as γ does not touch A, there should be no modification to γ whatsoever — this is definitely not the case if the conditioning is not Markovian;

- Seeing the curve as the (conjectured) limit of an exploration process shows that it should touch A with positive probability: Indeed, by Russo-Seymour-Welsh we know in advance that with positive probability there are crossings between ∂U and A in the annulus.

In the discrete setup, we want to consider the hole as wired (*i.e.* we discretize the whole unit disk with a triangular lattice and we identify all the sites lying inside A). Coloring the discretization of the direct arc from a to b in black and the indirect arc in white, we can explore the interface between white on the left and black on the right, in the usual way, as soon as the hole is given a color.

In the continuous case we do something similar, *i.e.* we chose the color of the hole, either black or white. This can be done either deterministically or randomly, but in any case we do the construction conditionally to the chosen color. The morale is that we construct an SLE_6 that bounces off A as if ∂A were part of the boundary of a simply connected, in a direction that is determined by the color we picked.

So, let (K_t) be an SLE_6 in the unit disk, and $\tilde{\gamma}$ be its trace. If $\tilde{\gamma}$ does not touch A, we do nothing and define $\gamma = \tilde{\gamma}$. If it does, let τ be the first time when it happens and let $\Omega = \mathcal{A}_r \setminus K_\tau$. Now Ω is simply connected, so we just construct another SLE_6 from γ_τ to b in Ω, and let γ be the concatenation of $\gamma[0,\tau]$ with the trace of the new SLE.

The only problem here is to chose on which side the new SLE starts. Indeed, $\Omega \cap \mathcal{B}(\gamma_\tau, r)$ has two connected components, say Ω_1 and Ω_2; γ_τ corresponds to two prime ends in Ω. This is where the color of the hole is used. Note that the boundary of K_τ can be divided into two parts, corresponding to the two components of $\partial\mathbb{U} \setminus \{W_\tau, b\}$ in the image; color in black the one corresponding to the direct arc from W_t to b and in white the one corresponding to the indirect arc. Now for $i \in \{1,2\}$, define $\partial_i = \partial\Omega \cap \bar{\Omega}_i$. One of the ∂_i's is all of the same color, and the other, say ∂_{i_0}, has subsets of both colors. Then, the starting point of the new SLE will be the prime end at γ_τ which is in ∂_{i_0}.

The same construction can be done in any conformal annulus with two marked points on the same component of its boundary. We call the obtained curve SLE_6 from a to b in the annulus. It is then easy to check that, conditionally to the color of the hole, the obtained curve it conformally invariant and has the same locality property as SLE_6 in a simply connected domain, namely: If B is a compact set which contains neither a nor b, and such that $A_r \setminus B$ is either a conformal annulus or a simply connected domain, then up to their first hitting time of B, the trace of an SLE_6 from a to b in $\mathcal{A}_r$ and that of an SLE_6 from a to b in $\mathcal{A}_r \setminus B$ have the same distribution.

Still conditionally to the color of the hole, we can now use the convergence of critical site-percolation on the triangular lattice to SLE_6 in a simply connected domain, twice, and the locality property of both SLE_6 and the percolation exploration process to obtain the following

Theorem 5.1 :

Let $r \in (0,1)$ and $\delta > 0$. Let γ^δ be the discrete exploration curve of the percolation interface from a to b in a discretization of $\mathcal{A}_r$ by a triangular lattice of mesh δ, as described previously, with a wired hole. Then, as $\delta \to 0$ and conditionally to the color of the hole, the law of γ^δ converges to that of the trace of an SLE_6 from a to b in $\mathcal{A}_r$.

But the same problem as previously arises here, in that we can still chose the color of the hole in any number of ways. In the critical percolation picture it will be natural to pick it white or black with the same probability 1/2; but in general we will obtain a one parameter family of laws on curves that all exhibit the same local geometry and the locality property. So, universality does not hold here either.

Remark: The same construction can of course be performed if the domain has finitely many holes. The only thing that changes is that each hole has to be colored.

In the case $\kappa \neq 6$, we might want to do the same construction; but since we do not have the locality property, the obtained law on curves will not be conformally invariant.

5.1.4 Percolation in the annulus

There is something else we can do related to critical percolation in an annulus, namely we can try to obtain crossing probabilities for rectangles with holes in them. In the simply connected case, this was done by Smirnov ([44]), and we will study what his proof can say about crossings of an annulus.

So, consider a simply connected bounded domain Ω and let A be a compact subset of Ω such that $\Omega \setminus A$ is a conformal annulus, and split the boundary of Ω into three intervals ∂_1, ∂_2 and ∂_3 (and define $\partial_4 = \partial_1$, $\partial_5 = \partial_2$ for easier notations).

Consider critical site-percolation on a discretization of $\Omega \setminus A$ by a triangular lattice, with wired boundary conditions along ∂A, and as in Smirnov's paper, if z is the center of a face of the lattice, define $H_i^\delta(z)$ to be the probability that in this discretization there is a closed simple path joining ∂_{i+1} to ∂_{i+2} and separating ∂_i from z.

We can apply the same arguments as in the triangle: By the Russo-Seymour-Welsh technology (cf. [19]), all the H_i^δ can be interpolated into uniformly Hölder functions on $\Omega \setminus A$, so they form a relatively compact family and it is sufficient to prove that there is exactly one possible subsequential limit to obtain convergence. And besides, any subsequential scaling limit (h_1, h_2, h_3) of the triple $(H_1^\delta, H_2^\delta, H_3^\delta)$ as δ goes to 0 is a "harmonic conjugate triple" in $\Omega \setminus A$, by exactly the same proof. We also obtain the same boundary conditions along $\partial\Omega$, namely h_i is identically 0 along ∂_i, and on ∂_{i+1} and ∂_{i+2} it has Neumann boundary conditions with angle $2\pi/3$ away from ∂_i.

The new fact here is the behaviour of the H_i^δ on the boundary of A. Using the Russo-Seymour-Welsh technology, we obtain uniform continuity on the boundary as $\delta \to 0$; and besides, if z and z' are the centers of two adjacent faces on ∂A, we clearly have $H_i^\delta(z) = H_i^\delta(z')$ for all i and δ. Hence, if (h_1, h_2, h_3) is any subsequential scaling limit of the triple $(H_1^\delta, H_2^\delta, H_3^\delta)$, then each of the h_i's is constant along ∂A (this would be "tangential Neumann conditions"). Note that the constant here corresponds the probability that there is a closed path from ∂_{i+1} to ∂_{i+2} separating A from ∂_i, but none of the arguments we used up to now seems to give a way to compute it explicitly.

In short, and stated in the equilateral triangle T with vertices $a = -i$, $b = (1+i\sqrt{3})/2$ and $c = (-1+i\sqrt{3})/2$, with $\partial_1 = [bc]$, $\partial_2 = [ca]$ and $\partial_3 = [ab]$, h has to be a solution to the following problem:

$$\begin{cases} h_1(a) = 1; & \\ h_1(z) = 0 & \text{for all } z \in [bc]; \\ \partial h_1/\partial x = 0 & \text{on } [ab] \cup [ac]; \\ h_1(z) = h_1(z') & \text{for all } z,\, z' \in \partial A. \end{cases} \tag{5.6}$$

But this problem is not well-posed, and in fact it is easy to use the maximum principle to show that for every $u \in \mathbb{R}$ it has exactly one solution taking the value u on ∂A. So, the method is not sufficient to compute crossing probabilities — however, it will suffice if we know how to compute the value of the constant by another method, typically using SLE_6 in the disk.

Actually, it seems that this degree of freedom on the value of h_1 along ∂A plays the same role as the ones which appeared in the construction of $SLE_{8/3}$ and SLE_6 in the annulus, and it is related to the absence of universality in the non-simply connected case — or, which is equivalent, to the ability to perturb a measure on random curves globally according to the side of A on which it passes.

In the general case of a domain of genus $k \in \mathbb{N}$, if a and b are two points on the exterior boundary of the domain, there are exactly 2^k homotopy classes of simple curves from a

to b, so the set of probability measures on such curves with the correct Markov property should be a simplex of dimension $2^k - 1$, *i.e.* we will obtain $2^k - 1$ degrees of freedom instead of 1. There will always be particular cases such as $SLE_{8/3}$ in the whole domain conditioned not to touch the holes, but they all require a way to fill the holes in order to retrieve universality from the simply connected case.

5.2 SLE as κ tends to 0 or ∞

We investigate in this section the behaviour of SLE when its parameter tends to 0 or ∞, at time 1; and since SLE is defined pathwise with respect to the driving function, we will do the same here and consider the Loewner chain driven by $\sqrt{\kappa}B_t$, letting κ tend to 0 or ∞ for a fixed B.

It is easy to see that in the first case, K_1 converges a.s. to the vertical segment $[0, 2i]$ in the Hausdorff metric, whereas in the second case it will spread along the real axis. Hence, we will have to renormalize it differently in both directions if we want to describe a nontrivial limit — and in particular this limit will not be conformally invariant at all. It will be more convenient to run the equation backward, *i.e.* to write it as

$$\partial_t g_t(z) = \frac{-2}{g_t(z) - \sqrt{\kappa}B_t}$$

where B is a standard Brownian motion. Indeed, this ensures that $g_t(z)$ is defined and differentiable in all variables on the domain $z \in \mathbb{H}$, $t \geqslant 0$. Note that, due to the time-reversibility of Brownian motion, the conditional distribution of g_t knowing β_t here is the same as that of the reciprocal map in the usual setup, up to a translation by β_t — so any information we can obtain on the image of g_t will actually hold for standard SLE.

5.2.1 Small values of κ

Let us consider first the case $\kappa \to 0$ — which is both easier and less interesting. Let $\varepsilon = \sqrt{\kappa}$, and to make things nicer, map the upper-half plane to the slitted plane by the map $z \mapsto z^2$. Then the conjugate $\tilde{g}$ of g by this application satisfies the following equation (SLE in the slit plane $\mathbb{C} \setminus \mathbb{R}$):

$$\partial_t \tilde{g}_t^\varepsilon(z) = \frac{-4}{1 - \frac{\varepsilon B_t}{\sqrt{\tilde{g}_t^\varepsilon(z)}}}. \tag{5.7}$$

Note that in this case, the solution for a constant driving function (*i.e.* when $\varepsilon = 0$) is given by $g_t^0(z) = z - 4t$; for small values of ε we will obtain a perturbation of this solution.

On the domain $z \notin [0, +\infty)$, $t > 0$ the solution is differentiable in all variables; differentiating (5.7) with respect to ε, at the point $\varepsilon = 0$, gives

$$\partial_\varepsilon \partial_t \tilde{g}_t^\varepsilon(z)|_{\varepsilon=0} = \frac{4B_t}{\sqrt{z - 4t}}.$$

Integrating then with respect to t provides the following expansion:

$$\tilde{g}_t^\varepsilon(z) = z - 4t + \varepsilon \int_0^t \frac{4B_s}{\sqrt{z - 4s}} \,\mathrm{d}s + o(\varepsilon), \tag{5.8}$$

where the term $o(\varepsilon)$ can be seen to be uniform in z. Now renormalize by simply multiplying the imaginary part by ε^{-1}, not touching the real part. All perturbative terms then disappear in the limit, except for the imaginary part of the above integral. Focusing on points $z \in [0,\infty)$ (which are mapped to $\tilde{K}_t \cup [0,\infty)$ by $\tilde{g}_t$), this imaginary part is non-zero only for $s > z/4$. Hence, in the limit, the renormalized version of $\tilde{K}_t$ has the following parameterization:

$$x = z - 4t, \quad y = \int_{z/4}^{t} \frac{4B_s}{\sqrt{4s - z}} \, \mathrm{d}s, \qquad z \in [0, 4t]. \tag{5.9}$$

For nicer notations, define the following functions:

$$b_t(s) = 4B_{s/4} \mathbf{1}_{s \leqslant t}, \qquad \varphi(s) = \frac{\mathbf{1}_{x<0}}{\sqrt{|x|}}.$$

With these notations, we proved that the renormalized curve converges, in the Hausdorff metric, to the curve of equation

$$y = (b_t * \varphi)(x + 4t), \quad x \in [-4t, 0]$$

where $*$ is the usual convolution operator. In other words, the curve of an SLE_κ for small values of κ, when renormalized correctly, converges to the graph of the convolution of its driving Brownian motion by a fixed kernel.

In particular, this renormalized SLE is the graph of a continuous function. An interesting question arises here: Under which conditions on the driving function β does the Loewner evolution lead to a graph? The same approach as above shows that, if we let $\beta_t = \varepsilon f(t)$ for some fixed continuous function f, the rescaled trace converges to the graph of the convolution of f and the same kernel φ. But it is also possible to look at the real part of $\tilde{g}_t^\varepsilon(z)$: If f is Hölder with exponent greater than 1/2, then for small ε this real part becomes monotonous in x along horizontal lines $\{x + iy\}$ for fixed y. In particular, K_t itself is a graph for sufficiently small values of ε.

In the case of SLE, the opposite happens, because B is not smooth enough. It is actually possible to compute the winding exponent of the curve as a function of κ, and to prove that for all values of κ this exponent is positive. Since it is equal to 0 for a graph, this proves that for every $\kappa > 0$, with probability 1, K_t is *not* a graph.

5.2.2 Large values of κ

When κ tends to ∞, SLE_κ tends to spread along the real axis, so that (at fixed time) its width is of order $\sqrt{\kappa}$ and its height of order $1/\sqrt{\kappa}$. Let K_t^κ be K_t renormalized so as to cancel this spreading; namely,

$$\Phi_\kappa(z) = \frac{\Re(z)}{\sqrt{\kappa}} + i\Im(z)\sqrt{\kappa},$$

$$K_t^\kappa = \Phi_\kappa(K_t).$$

We renormalize g_t in the same fashion, *i.e.*:

$$g_t^\kappa = \Phi_\kappa \circ g_t \circ \Phi_\kappa^{-1}.$$

Note that the renormalized g_t^κ is not conformal anymore; separating the real and imaginary part and writing $g_t^\kappa(x+iy) = X_t^\kappa + iY_t^\kappa$, then (X_t^κ, Y_t^κ) is the the unique solution of the following system of ODE's:

$$\partial_t X_t = \frac{-2(X_t - B_t)}{\kappa(X_t - B_t)^2 + Y_t^2/\kappa}, \qquad \partial_t Y_t = \frac{2Y_t}{\kappa(X_t - B_t)^2 + Y_t^2/\kappa}$$

with initial conditions $X_0^\kappa = x$ and $Y_0^\kappa = y$. The problem here is that we cannot describe a solution "at $\kappa = \infty$" and apply the same perturbative method as in the case $\kappa \to 0$, so we need to do everything by hand.

Proposition 5.3 :

Let (L_t^x) be a bicontinuous version of the local time of B at point x and time t. Then, as κ tends to infinity, we have almost surely

$$X_t = X_0 + o(1), \qquad Y_t = Y_0 + 2\pi L_t^{X_0} + o(1),$$

where the terms $o(1)$ are a.s. uniformly small on all the sets

$$A_{t_0,y_0} = \{(x,y,t) : t \leqslant t_0, y \geqslant y_0 > 0\}.$$

// First, replace the differential system by its integral counterpart, *i.e.* introduce the following operator:

$$[\mathcal{L}_\kappa(X,Y)]_t = \left(X_0 - \int_0^t \frac{2(X_u - B_u)}{\kappa(X_u - B_u)^2 + Y_u^2/\kappa} \mathrm{d}u, Y_0 + \int_0^t \frac{2Y_u}{\kappa(X_u - B_u)^2 + Y_u^2/\kappa} \mathrm{d}u \right).$$

Then, conditionally to B, (X^κ, Y^κ) is the unique fixed point of $\mathcal{L}_\kappa$. So we need to study the behaviour of $\mathcal{L}_\kappa$ as the parameter κ goes to infinity and from that to obtain information on its fixed point.

The morale is then the following: it is very easy to see that if (f_k) is a sequence of continuous functions on $[0,1]$, each having a unique fixed point x_k, and if the sequence (f_k) converges uniformly to a constant x, then (x_k) converges to x also. We lack several of the hypotheses to apply such a result directly here, but will prove that indeed the operators $\mathcal{L}_\kappa$ do converge to a constant, and that this is sufficient to conclude.

Fix $x + iy \in \mathbb{H}$ and a pair (X,Y) of continuous functions with $X_0 = x$, $Y_0 = y$ and such that Y is positive. The first remark is that the second coordinate of $\mathcal{L}_\kappa(X,Y)$ is always increasing, and since we are interested in fixed points we shall assume from now on that Y itself is non-decreasing. In particular, for all $t \geqslant 0$ we can assume $Y_t \geqslant y$.

We first study the first component of $\mathcal{L}_\kappa(X,Y)$, which we will denote by $\tilde{X}$. From the definition of $\mathcal{L}$, we know that $\tilde{X}$ is differentiable with respect to t and we first obtain (using the reknowned $2ab \leqslant a^2 + b^2$ inequality):

$$|\partial_t \tilde{X}_t| \leqslant \frac{1}{Y_t} \frac{2\left(|X_t - B_t|\sqrt{\kappa}\right)\left(Y_t/\sqrt{\kappa}\right)}{\kappa(X_u - B_u)^2 + Y_u^2/\kappa} \leqslant \frac{1}{y}. \tag{5.10}$$

Hence, $\tilde{X}$ is Lipschitz with a constant that depends only on y. So from now on, we may assume that X too is $(1/y)$-Lipschitz. We will need the following result:

Lemma 5.1 :

Let $(B_t)_{t\in\mathbb{R}}$ be a standard real-valued Brownian motion, and let C be a fixed positive real number. Then, with probability 1, there exists $K > 0$ such that, for every C-Lipschitz function $f : [0,1] \to \mathbb{R}$ and every $\varepsilon > 0$, we have

$$\lambda \{t \in [0,1] : |f(t) - B_t| \leqslant \varepsilon\} \leqslant K\varepsilon$$

(where λ is the Lebesgue measure on $[0,1]$).

/// This is a direct consequence of a result by Bass and Burdzy on Brownian local times along Hölder curves. More precisely, they prove in [4] that L_t^f defined for every continuous function f as the limit, when ε vanishes, of

$$L_t^f(\varepsilon) = \frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{|B_s - f(s)|<\varepsilon}\,\mathrm{d}s$$

is almost surely bounded on the class $\mathcal{S}_\alpha$ of all Hölder functions of some fixed exponent $\alpha > 1/2$ from $[0,1]$ to $[-1,1]$, and that it is jointly continuous as a function of $(t,f) \in [0,1] \times \mathcal{S}_\alpha$. Hence, by a compactness argument, the collection of all $L_t^f(\varepsilon)$ over $(t,f,\varepsilon) \in [0,1] \times \mathcal{S}_\alpha \times (0,1)$ is also bounded, thus proving the Lemma. ///

Let $\alpha \in (0,1)$, and $D_\kappa(X) = \{t \in [0,1] : |X_t - B_t| < \kappa^{-\alpha}\}$. Split the integral defining $\tilde{X}_t - X_0$ into two parts:

$$|\tilde{X}_t - X_0| \leqslant \int_0^t \frac{2\,|X_u - B_u|}{\kappa(X_u - B_u)^2 + Y_u^2/\kappa}\mathrm{d}u = \int_{[0,t]\cap D_\kappa} + \int_{[0,t]\setminus D_\kappa} = (1) + (2).$$

The first term can be estimated using the previous computation: it is indeed not greater than the integral of $1/y$ over $D_\kappa(X)$, and by Lemma 5.1 we obtain

$$(1) \leqslant \frac{1}{y}\lambda(D_\kappa(X)) \leqslant \frac{K}{y}\kappa^{-\alpha} \underset{\kappa\to\infty}{\longrightarrow} 0.$$

The second integral is also bounded above from the very definition of $D_\kappa(X)$:

$$(2) \leqslant \int_{[0,t]\setminus D_\kappa} \frac{2\,\mathrm{d}u}{\kappa(X_u - B_u)} \leqslant 2t\kappa^{\alpha-1} \underset{\kappa\to\infty}{\longrightarrow} 0.$$

So, with probability 1, $\tilde{X} - X_0$ converges to 0 uniformly in t and X as κ goes to infinity. Hence we get the first part of the announced result in a refined version: as $\kappa \to \infty$, almost surely,

$$X_t^\kappa = X_0^\kappa + \mathcal{O}(\kappa^{-1/2}) \tag{5.11}$$

by taking $\alpha = 1/2$ in the previous estimates. Moreover the implied constants are uniform in $t \in [0,1]$ and $x \in \mathbb{R}$ (but they strongly depend on y).

We still have to obtain the behaviour of the second component $\tilde{Y}$ of $\mathcal{L}_\kappa(X,Y)$. Look first at what happens when X and Y are constant. By the occupation times formula, we have almost surely

$$\tilde{Y}_t - Y_0 = \int_0^t \frac{2Y_0}{\kappa(X_0 - B_u)^2 + Y_0^2/\kappa}\,\mathrm{d}u = \int_{-\infty}^{\infty} \frac{2Y_0}{\kappa(X_0 - x)^2 + Y_0^2/\kappa} L_t^x \mathrm{d}x.$$

By the change of variable $x = X_0 + zY_0/\kappa$, we obtain

$$\tilde{Y}_t - Y_0 = \int_{-\infty}^{\infty} \frac{2L_t^{X_0 + zY_0/\kappa}}{z^2+1}\,\mathrm{d}z,$$

and by dominated convergence, as κ tends to ∞, this leads to

$$\tilde{Y}_t - Y_0 \underset{\kappa\to\infty}{\longrightarrow} 2L_t^{X_0} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{z^2+1} = 2\pi L_t^{X_0}.$$

Since we know that $\tilde{Y}$ is increasing, Dini's theorem can be applied to show that the convergences is almost surely uniform in t.

In fact, the same proof applies to the case when Y is not constant, approximating it by a piecewise constant function and noticing that the above limit did not depend on the value of Y_0. If X is not constant, using the definitions in [4] and the fact that X is supposed to be Lipschitz anyway, the same computation actually shows that

$$\tilde{Y}_t - Y_0 \underset{\kappa\to\infty}{\longrightarrow} 2\pi L_t^X$$

(or, and this is equivalent since we are assuming that X is Lipschitz, still by the results in [4], we can write this limit as the local time at 0 of the process $(B_u - X_u)$). It is then a consequence of the continuity of L_t^X in X that, if X now depends on κ in such a way that $\|X - x\|_\infty$ tends to 0 when κ tends to ∞, we obtain the same limit for $\tilde{Y}_t - Y_0$ as in the case when X was constant.

So, combining the convergence of $\tilde{X}$ to a constant and that of $\tilde{Y}$ to a known function when X is constant, we obtain the first step of the proof: Let $X_t^\infty = x$ and $Y_t^\infty = y + 2\pi L_t^x$, then for each pair (X,Y) of continuous functions with Y positive and increasing, with probability 1,

$$[\mathcal{L}_\kappa \circ \mathcal{L}_\kappa](X,Y) \to (X^\infty, Y^\infty)$$

as $\kappa \to \infty$, uniformly in $t \in [0,1]$.

It is easy to use the same method as when proving that $\tilde{X}$ is Lipschitz, to obtain the following estimate: For all continuous functions X, Y, $\bar{X}$ and $\bar{Y}$ with the usual restrictions and initial conditions, for all $t \in [0,1]$, and for every norm $\|\cdot\|$ on $\mathbb{R}^2$, we have

$$\|[\mathcal{L}_\kappa(\bar{X},\bar{Y})]_t - [\mathcal{L}_\kappa(X,Y)]_t\| \leqslant C_\kappa \int_0^t \|(\bar{X}_s - X_s, \bar{Y}_s - Y_s)\|\,\mathrm{d}u,$$

where the constant C_κ depends only on κ and y. This shows that the operator $\mathcal{L}_\kappa$ is locally Lipschitz with respect to the supremum norm.

The inequality can then be applied recursively (as when proving the Cauchy-Lipschitz theorem): If $\mathcal{L}_\kappa^n$ denotes $\mathcal{L}_\kappa$ composed n times, we obtain, for all $t > 0$ and $n \in \mathbb{N}$,

$$\|[\mathcal{L}_\kappa^n(\bar{X}, \bar{Y})]_t - [\mathcal{L}_\kappa^n(X, Y)]_t\| \leqslant \frac{C_\kappa^n t^n}{n!} \|(\bar{X}, \bar{Y}) - (X, Y)\|_\infty,$$

thus proving that in fact $\mathcal{L}_\kappa^n$ is Lipschitz with constant $C_\kappa^n/n!$ on the space $\mathcal{E}_{1/y}$ of pairs (X, Y) of continuous real functions on $[0, 1]$ with $Y \geqslant Y_0$. Hence, it is possible to chose $n_\kappa \geqslant 2$ for all κ in such a way that $\mathcal{G}_\kappa \triangleq \mathcal{L}_\kappa^{n_\kappa}$ be $2^{-\kappa}$-Lipschitz. The same proof as previously then shows that for all (X, Y), with probability 1,

$$\mathcal{G}_\kappa(X, Y) \to (\tilde{X}, \tilde{Y})$$

uniformly in $t \in [0, 1]$.

Recall that (X^κ, Y^κ) is the unique fixed point of $\mathcal{L}_\kappa$. Then, (X^κ, Y^κ) is also the unique fixed point of $\mathcal{G}_\kappa$ (which is contracting because $\kappa > 0$). So we obtain:

$$\begin{aligned}
\|(X^\kappa, Y^\kappa) - (\tilde{X}, \tilde{Y})\|_\infty &= \|\mathcal{G}_\kappa(X^\kappa, Y^\kappa) - (\tilde{X}, \tilde{Y})\|_\infty \\
&\leqslant \|\mathcal{G}_\kappa(X^\kappa, Y^\kappa) - \mathcal{G}_\kappa(x, y)\|_\infty + \|\mathcal{G}_\kappa(x, y) - (\tilde{X}, \tilde{Y})\|_\infty \\
&\leqslant 2^{-\kappa}\|(X^\kappa, Y^\kappa) - (x, y)\|_\infty + \|\mathcal{G}_\kappa(x, y) - (\tilde{X}, \tilde{Y})\|_\infty \\
&\leqslant 2^{-\kappa}\|(X^\kappa, Y^\kappa) - (\tilde{X}, \tilde{Y})\|_\infty + 2^{-\kappa}\|(\tilde{X}, \tilde{Y}) - (x, y)\|_\infty + \|\mathcal{G}_\kappa(x, y) - (\tilde{X}, \tilde{Y})\|_\infty.
\end{aligned}$$

As soon as $2^{-\kappa} \leqslant 1/2$, *i.e.* $\kappa \geqslant 1$, this leads to

$$\|(X^\kappa, Y^\kappa) - (\tilde{X}, \tilde{Y})\|_\infty \leqslant 2^{1-\kappa}\|(\tilde{X}, \tilde{Y}) - (x, y)\|_\infty + 2\|\mathcal{G}_\kappa(x, y) - (\tilde{X}, \tilde{Y})\|_\infty,$$

and we know that the right-hand term of this inequality tends almost surely to 0 as κ tends to infinity: So we obtain the announced result, that with probability 1, (X^κ, Y^κ) converges to $(\tilde{X}, \tilde{Y})$, uniformly in $t \in [0, 1]$.

Uniformity in (X, Y) then follows from the fact that all the estimates we used were indeed uniform, and that all the constants depended only on y. //

Now, let $\tilde{K}$ be the *local time shape* of B, defined as

$$\tilde{K} \triangleq \{(x + iy) : L_1^x > 0, 0 \leqslant y \leqslant L_1^x\}$$

and let $z = x + iy$ be a given point in $\mathbb{H} \setminus \tilde{K}$. Let $y_0 = y - 2\pi L_1^x$. From the previous Proposition, we know that there a.s. exists $\kappa_0 > 0$ such that, for each $\kappa > \kappa_0$ and each w satisfying $\Im w \geqslant y_0/2$, we have $|g_1^\kappa(w) - w + 2\pi i L_1^{\Re y}| < y_0/4$. In particular, this implies that the image of the line of equation $\Im w = y_0/2$ under g_1^κ passes below z, hence z is not in K_1^κ. This proves that the lim sup of the K_1^κ is contained in $\tilde{K}$.

To prove that (K_1^κ) actually converges to $\tilde{K}$ in the Hausdorff topology, we still have to prove that it fills up $\tilde{K}$. Here is a brief description of how to do it. Let $z \in \mathbb{H}$ be such that there exists a sequence (κ_k) tending to infinity satisfying, for all k, $z \notin K_1^{\kappa_k}$. This means that we can look at the backward differential equation (*i.e.*, the usual SLE up to a horizontal shift by β_1) starting at z for the parameters κ_k, and that this differential equation has a solution up to time 1.

So, let $(\varphi_{z,k})$ be the backward solution, defined by

$$\varphi_{z,k}(t) = \left[g_{1-t}^{\kappa_k} \circ (g_1^{\kappa_k})^{-1}\right](z)$$

(so that $\varphi_{z,k}(0) = z$ and $g_1^{\kappa_k}(\varphi_{z,k}(1)) = z$). The methods used in the proof of Proposition 5.3 can be adapted (and this is where the details are still a little sketchy) to prove that, as k goes to infinity, we have almost surely, for every $t \in [0,1]$,

$$\varphi_{z,k}(t) \to z - 2\pi i \left(L_1^{\Re z} - L_{1-t}^{\Re z}\right)$$

or, in other words, that the backward flow in the limit involves the local time of the time reversal of the driving process. Looking at the imaginary part and letting $t = 1$ then shows that $\Im z \geqslant 2\pi L_1^{\Re z}$, *i.e.* that z is not in the interior of $\tilde{K}$. So, the lim inf of the K_1^κ contains the interior of $\tilde{K}$, thus completing the proof.

As a side remark, one can look at the time parameterization of the usual SLE. Recall that, if K is a hull in $\mathbb{H}$, (Y_t) a planar Brownian motion and T the first hitting time of $\mathbb{R} \cup K$ by Y, we can define

$$a(K) \triangleq \lim_{y \to +\infty} \frac{y}{2}\, E^{iy}\left(\Im Y_T\right)$$

(where iy is the starting point of Y), and that SLE is then parameterized by $a(K_t) = t$. Now, if f is a nonnegative continuous function with compact support, define the *hypograph* of f as

$$K_f = \{x + iy \in \mathbb{H} : f(x) > 0, 0 \leqslant y \leqslant f(x)\}.$$

Heuristically, if $K = K_f$ and the supremum of f is very small, then the distribution of the real part of Y_T is close to the harmonic measure on $\mathbb{R}$ seen from iy, *i.e.* it is close to a Cauchy distribution with density

$$\rho_y(x) = \frac{y/\pi}{x^2 + y^2}.$$

This shows that the capacity of K_f can be estimated by

$$a(K_f) \simeq \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, \frac{y^2}{x^2 + y^2}\, \mathrm{d}x,$$

and by dominated convergence this last integral converges, as y goes to ∞, to the integral of f against the Lebesgue measure. So, still heuristically, if K_f is a very flat hypograph along the real axis, we have

$$a(K_f) \simeq \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = \frac{\mathrm{Area}(K_f)}{2\pi}.$$

In the case of SLE for a large value of κ, we have $a(K_1) = 1$ by definition, and the convergence to the hypograph of the local time implies that the area of K_1 converges to the integral of $2\pi L_1^x$ — which is equal to 2π because the local time is the density of the occupation measure.

5.3 Discretized SLE

We describe in this section a discrete version of the SLE process. The idea is to approximate the driving process β_t by a random walk, interpolating it by constants between the jumps (so that in particular the driving function is not continuous anymore). From here on, $(S_n)_{n\geqslant 0}$ will be a standard RW on $\mathbb{Z}$, starting from 0, with steps in $\{\pm 1\}$, each with probability 1/2.

Definition :

Let $\varepsilon > 0$, and define β^ε as follows:

$$\beta^\varepsilon_t = \sqrt{\varepsilon} S_{\lfloor t/\varepsilon \rfloor}.$$

Let $\kappa > 0$, and K^ε be the Loewner chain with driving function $\sqrt{\kappa}\beta^\varepsilon$. We call K^ε a *discretized SLE process* with parameter κ and scale ε.

It is easy to use Gronwall's Lemma to prove the following approximation result. Let (β^n) be a sequence of càdlàg functions converging uniformly to β, and (g^n_t) (resp. (g_t)) be the Loewner chain with driving function β^n (resp. β). Then for each $z \in \mathbb{H} \setminus K_t$, $g^n_t(z)$ is well defined for n large enough and

$$g^n_t(z) \to g_t(z).$$

Moreover, the convergence is uniform on every compact subset of $\mathbb{H} \setminus K_t$. Hence the sequence of domains $(\mathbb{H} \setminus K^n_t)$ converges to $\mathbb{H} \setminus K_t$ in the sense of Caratheodory.

Using Skorohod embedding, we can now fix a decreasing sequence (ε_n) tending to 0 and couple an SLE_κ and a sequence (K^{ε_n}) of discretized SLE_k's with scales ε_n, in such a way that their driving functions converge to that of the SLE when n tends to infinity, a.s. uniformly on any bounded time interval. Then, at each time $t > 0$, the sequence of discretized SLE's converges to the usual SLE (still in the sense of Caratheodory).

It is therefore natural to look at the geometry of discretized SLE and in particular how it depends on the value of κ. Note first that, by the scaling property of Loewner chains in the half-plane, the law of $\varepsilon^{-1/2} K^\varepsilon_{t\varepsilon}$ does not depend on ε — so we shall fix $\varepsilon = 1$ in what follows, and look at K^1_t for integer values of t; so let $\hat{K}_n = K^1_n$ and let $\hat{g}_n$ be the corresponding conformal map.

At time 1, we always have $\hat{g}_1(z) = \sqrt{z^2+4}$, *i.e.* $\hat{K}_1$ is the vertical segment $[0, 2i]$. Then $\hat{K}$ continues to grow from $\hat{g}_1^{-1}(\sqrt{\kappa} S_1)$. Note that $\hat{g}_1$ maps $\hat{K}_1$ to the horizontal segment $[-2, 2]$: depending on the value of κ, $\sqrt{\kappa} S_1$ will either be in or outside of $[-2, 2]$, and the shape of K_2 will be different in both cases. More precisely:

- If $\kappa < 4$: $\hat{g}_1^{-1}(\sqrt{\kappa} S_1)$ is on $\hat{K}_1$, hence $\hat{K}_2$ looks like a tree with two branches. Inductively, K_{n+1} will be obtained from K_n by adding a branch on the last branch of K_n, and K_n will look like a broken line with spines. In particular, $\hat{K}_n \cap \mathbb{R}$ is always reduced to the origin.

- If $\kappa = 4$: The second branch of $\hat{K}_2$ will start from 0, and inductively, so will every subsequent branch of $\hat{K}$. Hence, $\hat{K}_n$ will be a union of disjoint curves in the upper-half plane, all starting from 0.

- If $\kappa > 4$: The second branch of $\hat{K}_2$ will start from $\pm\sqrt{\kappa-4}$, *i.e.* $\hat{K}_2$ consists of two disjoint curves, starting from two different points on the real axis. In this case, $\hat{K}_n$ will be a forest of (many) disjoint trees in the upper half plane. In particular $\hat{K}_n \cap \mathbb{R}$ is never a single point for $n > 1$.

In particular, we observe a change of geometry when κ gets bigger than 4, similar to the transition between a simple curve and a curve with double points for the trace of a standard *SLE*; it is interesting to notice that they both happen for the same value of κ.

However, simply looking at the geometry of the discretized *SLE cannot* provide a proof of the existence of a transition at $\kappa = 4$, for the following reason. Let (X_n) be a Markov chain in $\{\pm 1\}$, with transition matrix

$$\begin{bmatrix} 1/2+\alpha & 1/2-\alpha \\ 1/2-\alpha & 1/2+\alpha \end{bmatrix}$$

and with initial distribution $P(X_1 = 1) = P(X_1 = -1) = 1/2$, where α is a parameter in $(-1/2, 1/2)$, and as previously let $S_n = X_1 + \cdots + X_n$. We can define a Loewner chain with driving function

$$\beta_t^{(\varepsilon,\alpha)} = \sqrt{4\varepsilon} S_{\lfloor t/\varepsilon \rfloor},$$

and the geometry of the associated compact will be the same as that of discretized SLE_4 (*i.e.* it will be a union of continuous curves in $\mathbb{H}$ starting from 0).

But when ε vanishes, $\beta^{\varepsilon,\alpha}$ converges in distribution to a time-changed Brownian motion $(B_{\kappa(\alpha)t})_{t>0}$ with

$$\kappa(\alpha) = 4\,\frac{1+2\alpha}{1-2\alpha}\ ;$$

so that choosing the parameter α accordingly, we can obtain any standard SLE_κ as a limit of discrete processes which all have the same structure.

It might still be possible to obtain precise results about standard *SLE* starting from this discrete model, especially in the iid case; but it would probably be a hidden application of Itô's formula, *i.e.* a transcription of the usual proof to the discrete setup.

Remark: Another way of seeing this construction is to write the conformal map $g_n(z) - \beta_n$ as the composition of n conformal maps, each of which is one of the following two elementary maps:

$$g_\pm^\kappa(z) = \sqrt{z^2+4} \pm \sqrt{\kappa}\ ;$$

if the maps are chosen independently, the composition converges to an *SLE* in the scaling limit. However, for the same reason as previously, studying the semi-group generated by these two maps is not sufficient to obtain informations on *SLE* itself. In particular, it is probably not possible to derive locality at $\kappa = 6$ by just studying the interactions between the $g_\pm^6$. Which is a pity, because it was a very natural thing to try ...

Appendix A

All that did not fit in the main text.

A.1 Hausdorff dimension of random sets

We give here a self-contained proof of Proposition 3.1. It is easy to extract this proof from that of Theorem 2.2, or from the proof of the dyadic analog to be found in [27], but the statement of condition 2. used here makes the upper bound easier. This relation between exponents and dimensions was first obtained by Lawler.

Suppose that λ denotes the Lebesgue measure in $[0,1]^d$. Let $(C_\varepsilon)_{\varepsilon>0}$ be a family of random Borelian subsets of the cube $[0,1]^d$. Assume that for $\varepsilon < \varepsilon'$ we have $C_\varepsilon \subseteq C_{\varepsilon'}$, and let $C = \bigcap C_\varepsilon$. Define the following conditions (where $f \asymp g$ means that there exist positive numbers c_- and c_+ such that $c_- g \leqslant f \leqslant c_+ g$, and where the constants do not depend on ε, x nor y):

1. For all $x \in [0,1]^d$, $P(x \in C_\varepsilon) \asymp \varepsilon^s$;

2. There exists $c > 0$ such that for all $x \in [0,1]^d$ and ε,

$$P(\lambda(C_\varepsilon \cap \mathcal{B}(x,\varepsilon)) > c\varepsilon^d | x \in C_\varepsilon) \geqslant c > 0 ;$$

3. There exists $c > 0$ such that for all x, $y \in [0,1]^d$ and ε,

$$P(\{x,y\} \subset C_\varepsilon) \leqslant c\varepsilon^{2s}|x-y|^{-s}.$$

Proposition 3.1

(i). If conditions 1. and 2. hold, then a.s. $\dim_H(C) \leqslant d - s$;

(ii). If conditions 1. and 3. hold, then with positive probability $\dim_H(C) \geqslant d - s$.

// As usual, the proof of the upper bound is done by giving an explicit covering of C by small balls, and the lower bound is obtained by constructing a measure supported on C.

(i). Fix $\varepsilon > 0$, and a covering (B_i) of the cube $[0,1]^d$ by $2^d\varepsilon^{-d}$ balls of radius ε. Combining conditions 1. and 2. shows that for all i, the probability that C_ε touches B_i is not greater than $C.\varepsilon^s$. Hence, if N_ε is the minimum number of balls of radius ε needed to cover C_ε, we obtain

$$E(N_\varepsilon) \leqslant C.\varepsilon^{-2}\varepsilon^s.$$

Applying the Markov inequality proves that, for all $\eta > 0$,

$$P(N_\varepsilon \geqslant \varepsilon^{s-2-\eta}) \leqslant C\varepsilon^\eta.$$

Now, let $\varepsilon = 2^{-n}$ for $n \in \mathbb{N}$. Since the sequence $(2^{-n\eta})$ is summable, we may apply Borel-Cantelli: Almost surely, there exists n_0 such that, for all $n \geqslant n_0$, we have $N_{2^{-n}} \leqslant 2^{(2-s+\eta)n}$.

Since we are assuming that the family (C_ε) is decreasing, any covering of C_ε is also a covering of C. Hence, the previous estimate can be expressed as follows: Almost surely, for all n large enough, it is possible to cover C with at most $2^{(2-s+\eta)n}$ balls of radius 2^{-n}. Hence the box dimension of C is a.s. not greater than $2 - s + \eta$. Letting η go to zero, we finally obtain that, with probability 1,

$$\dim_H C \leqslant \dim_{\text{box}} C \leqslant 2 - s.$$

(ii). This is exactly the same proof as that of the lower bound in Theorem 2.2, so we only state the main steps of the proof. Let $(\mu_\varepsilon)_{\varepsilon>0}$ be measures defined by their density with respect to the Lebesgue measure :

$$\mathrm{d}\mu_\varepsilon(x) = \varepsilon^{-s}\mathbf{1}_{x\in C_\varepsilon}\,\mathrm{d}^d x.$$

Condition 1. leads to $E(\|\mu_\varepsilon\|) \asymp 1$, and it is straightforward to apply conditions 1. and 3. to derive

$$\mathrm{Var}(\|\mu_\varepsilon\|) \leqslant E(\|\mu_\varepsilon\|^2) = \mathcal{O}(1)$$

as ε goes to 0. Hence, for α small enough, we have $P(\|\mu_\varepsilon\| > \alpha) > \alpha$ and with probability at least α we can extract a subsequence (μ_{ε_k}) converging weakly to a measure of mass at least α supported on C. Hence C is not empty (which in itself was not clear).

Now, for each $r > 0$, define the r-energy of μ_ε as

$$\mathcal{E}_r(\mu_\varepsilon) \triangleq \iint \frac{\mathrm{d}\mu_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(y)}{|y-x|^r}.$$

Again, condition 3. can be used to show that, for every $r < d - s$, the expectation of $E_r(\mu_\varepsilon)$ is bounded when ε goes to 0, hence it is smaller than C/α with probability at least $1 - \alpha/2$ if C is taken large enough. So with probability at least $\alpha/2$ we can extract a subsequence (μ_{ε_k}) of measures all having mass at least α and r-energy at most C/α. Up to an additional extraction we can assume that this subsequence converges to a measure μ supported on C and having the same characteristics.

But it is known that a set supporting a positive measure of finite r-energy has Hausdorff dimension at least r (because such a measure is automatically a Frostman measure of dimension r — cf. for instance [40]): So with probability at least $\alpha/2$, we have $\dim_H(C) \geqslant r$, and this holds for each $r < d-s$; hence, still with probability at least $\alpha/2 > 0$, we have $\dim_H(C) \geqslant d-s$, as we wanted. //

A.2 SLE and Hölder domains

We present in this short section two lemmas about Hölder domains, together with a construction (essentially due to Peter Jones) of a natural measure supported on the boundary of such a domain. It might be possible to exploit this construction to obtain a Frostman measure of the correct dimension, and hence to derive the dimention of an SLE boundary in this way — thus providing a better (as in "more natural for an analyst") proof of Theorem 4.1, at least in the case $\kappa < 4$. The tools presented here are not new, but neither are they widely known among probabilists.

A.2.1 Whitney decompositions of Hölder domains

Let Ω denote a simply connected, open and bounded Hölder domain with exponent $\alpha > 0$, containing the open disk $\mathcal{B}(0,1)$ (meaning that the conformal maps from the open unit disk onto Ω are all Hölder with exponent α). A *Whitney decomposition* of Ω is a family of dyadic squares

$$Q_j = \left[\frac{k_j}{2^{n_j}}, \frac{k_j+1}{2^{n_j}}\right] \times \left[\frac{l_j}{2^{n_j}}, \frac{l_j+1}{2^{n_j}}\right]$$

whose interiors are pairwise disjoint, whose union is dense in Ω, and such that the ratios $l(Q_j)/\mathrm{d}(Q_j, \partial\Omega)$ are bounded above and below. Note that this implies that the family is locally finite in Ω, and hence that the union of the Q_j is actually *equal* to Ω. We call 2^{-n_j} the *size* of Q_j and denote it by $l(Q_j)$; moreover we will introduce the *center* z_j of Q_j, defined by

$$z_j \triangleq \frac{2k_j+1}{2^{n_j+1}} + i\frac{2l_j+1}{2^{n_j+1}},$$

and for each $C > 1$ we can now define the enlarging of Q_j by a factor C, as

$$CQ_j \triangleq \{z_j + C(z - z_j),\ z \in Q_j\}.$$

The first Lemma states that we can cover the boundary of Ω by enlarging all the Whitney cubes of a given approximate size:

Lemma A.1 (Jones *et al.* [20]) :

Let $\{Q_j\}$ be a Whitney decomposition of Ω. Then, for all $\varepsilon > 0$ there exist $C > 0$ and $n_0 > 0$ such that, for all $n > n_0$,

$$\partial\Omega \subset \bigcup_{2^{-(1+\varepsilon)n} \leqslant l(Q_j) \leqslant 2^{-n}} CQ_j.$$

// Let Ω be a simply connected domain, and for all $z \in \Omega$, let $\delta(z) = \mathrm{d}(z, \partial\Omega)$ be the (Euclidean) distance between z and the boundary of Ω.

Assume without loss of generality that $0 \in \Omega$ and $\delta(0) \geqslant 1$. Let $G(z)$ be the Green function in Ω with pole at 0. It is a general result [9, Theorem 7] that, for all $z \in \Omega \setminus \mathcal{B}(0,1/2)$,

$$G(z) \leqslant C_0 \exp\left[-\frac{1}{2}\int_{\gamma_z} \frac{|\mathrm{d}\zeta|}{\delta(\zeta)}\right] \tag{A.1}$$

where γ_z is the geodesic from 0 to z in Ω — even if Ω is not Hölder.

We prove the Lemma by contradiction; assume that there is $\varepsilon > 0$ such that, for all $B > 0$ and all $n_0 > 0$, there exist $n \geqslant n_0$ and $x \in \partial\Omega$ satisfying

$$\forall z \in \gamma_x, \qquad 2^{-(1+\varepsilon)n} \leqslant |z-x| \leqslant 2^{-n} \implies \delta(z) \leqslant \frac{1}{B}|z-x| \tag{A.2}$$

(so that x is not in the Whitney cube at z enlarged by a factor B). Let $\lambda = 1+\varepsilon$ and $z' \in \gamma_x$ such that $|z'-x| = 2^{-\lambda n}$. By (A.1), we have

$$\begin{aligned} G(z') &\leqslant C_0 \exp\left[-\frac{1}{2}\int_{2^{-(1+\varepsilon)n}}^{2^{-n}} \frac{B\,|\mathrm{d}\zeta|}{|\zeta - x|}\right] \\ &\leqslant C_0 \exp\left[-\frac{B}{2}\int_{2^{-(1+\varepsilon)n}}^{2^{-n}} \frac{\mathrm{d}s}{s}\right] \\ &= C_0 \exp\left[-\frac{B}{2}\, n\varepsilon \log 2\right] = C_0 2^{-nB\varepsilon/2}. \end{aligned} \tag{A.3}$$

But it is easy to see that, since Ω is a Hölder domain,

$$\forall z \in \gamma_x, \qquad G(z) \geqslant C|z-x|^{1/\alpha} \tag{A.4}$$

(because we know that the Green function in the unit disk decays linearly near the boundary, and that it is mapped to G by any conformal map from $\mathbb{U}$ onto Ω fixing the origin). Applying this at point z' and using (A.3) leads to

$$C.2^{-\lambda n/\alpha} \leqslant C_0.2^{-nB\varepsilon/2}.$$

Since this happens for arbitrarily large values of n, it implies that

$$B \leqslant \frac{2(1+\varepsilon)}{\alpha\varepsilon}. \tag{A.5}$$

Hence the assumption cannot hold for all $B > 0$, and this proves the Lemma for $C = 4/\alpha\varepsilon$. //

Remark: The minimal value of C such that the lemma holds is difficult to determine in the general case, because the bounds we used in the proof, especially Equation (A.1), are far from being optimal in the Hölder case. The constant we obtain is of order $1/\varepsilon\alpha$, and this might lead to trouble when we apply the construction to SLE — because the value of α and then that of the Hölder norm (related to C in Equation (A.4)) are unknown. In particular, it is not clear how to state the lemma for a random domain; the nicest version would be the existence of $C > 0$ such that the union of the CQ_j over the same collection of cubes covers $\partial\Omega$ with probability 1, but this is hoping for too much ...

For every subset A of Ω, introduce the *shadow* of A as

$$\mathrm{Shadow}(A) = \{z \in \Omega : \gamma_z \cap A \neq \varnothing\}$$

where γ_z is the hyperbolic geodesic from 0 to z in Ω. Then the second Lemma says that the number of Whitney cubes of given size whose shadow touches a given ball centered on $\partial\Omega$ it bounded above:

Lemma A.2 (Jones *et al.*) :

Let $\{Q_j\}$ be a Whitney decomposition of Ω. Then, for all $\varepsilon > 0$, there exists $C > 0$ such that the following happens: For all $x \in \partial\Omega$ and $r > 0$, the family $\mathcal{F}_{x,r,\varepsilon}$ of all the cubes in $\{Q_j\}$ with sizes in $[r, r^{1-\varepsilon}]$, whose shadow touches $\Omega \cap \mathcal{B}(x,r)$, has at most $r^{-2\varepsilon}$ elements.

Moreover, $\mathcal{B}(x,r)$ in entirely contained in the union of the shadows the elements of $\mathcal{F}_{x,r,\varepsilon}$.

// Let x be a point in $\partial\Omega$. For any $z \in \Omega \cap \mathcal{B}(x,r)$, the construction in the proof of the previous Lemma provides a Whitney cube $Q_{j(z)}$ with center in $\mathcal{B}(z, r^{1-\varepsilon})$ and size in $[r, r^{1-\varepsilon}]$, whose shadow contains z. Hence the union of the $Q_{j(z)}$ contains $\Omega \cap \mathcal{B}(x,r)$. Each of these cubes has area at least r^2, and they are all contained in $\mathcal{B}(x, r^{1-\varepsilon})$ by the triangle inequality. Since they are pairwise disjoint, this implies that there are at most $r^{-2\varepsilon}$ of them, as we wanted. //

A.2.2 Construction of the Frostman measure

Now let f be a conformal map from the upper-half plane $\mathbb{H}$ onto Ω, and let $\{Q_j\}$ be the standard dyadic Whitney decomposition of $\mathbb{H}$. As usual, z_j will denote the center of Q_j. If $n > 0$ and Q_j, Q_{j_0} are two Whitney cubes, write $j \prec_n j_0$ if Q_j if below Q_{j_0} and $l(Q_j) = 2^{-n} l(Q_{j_0})$ (*i.e.* Q_j is in the n-th generation below Q_{j_0}). Introduce the following notations: For each cube Q_j, let fQ_j be its image under f (that is, $fQ_j = f(Q_j)$, but we keep the former to agree with the usual notations); and if Q_{j_0} is a Whitney square and if $d > 0$,

$$D(j_0, n, d) \triangleq 2^{-nd} \left|f'(z_{j_0})\right|^{-d} \sum_{j \prec_n j_0} \left|f'(z_j)\right|^d \asymp l(fQ_{j_0})^{-d} \sum_{j \prec_n j_0} l(fQ_j)^d.$$

Assume first that for some j_0 and $d > 0$, $D(j_0, n, d)$ tends to 0 when n tends to infinity. We can apply the definition of a Whitney cube and Köbe's 1/4 Theorem to show that the diameter of fQ_j is of order $2^{-n}|f'(z_j)|$, so that Lemma A.1 provides us with an explicit covering of the shadow of fQ_{j_0} on $\partial\Omega$ by sets (A_k) of uniformly small diameter, and satisfying

$$\sum \mathrm{diam}(A_k)^{d+\varepsilon} \leqslant C.D(j_0, n, d).$$

Letting n go to infinity, and then ε to 0, this shows that the Hausdorff dimension of the shadow of fQ_{j_0} on $\partial\Omega$ is at most equal to d. If this holds for every j_0, we finally obtain $\dim_H(\partial\Omega) \leqslant d$.

We are going to argue that, under some assumptions on f, it is possible to prove the opposite implication — namely, if $D(j_0, n, d) \not\to 0$, then $\dim_H(\partial\Omega) \geqslant d$. Note that this is most certainly false in the general case.

If n is given, the following holds:

$$\begin{aligned} D(j_0, 2n, d) &= 2^{-2nd} \left|f'(z_{j_0})\right|^{-d} \sum_{k \prec_n j_0} \sum_{j \prec_n k} \left|f'(z_j)\right|^d \\ &= 2^{-nd} \left|f'(z_{j_0})\right|^{-d} \sum_{k \prec_n j_0} \left|f'(z_k)\right|^d \left[2^{-nd} \left|f'(z_k)\right|^{-d} \sum_{j \prec_n k} \left|f'(z_j)\right|^d\right] \\ &= 2^{-nd} \left|f'(z_{j_0})\right|^{-d} \sum_{k \prec_n j_0} \left|f'(z_k)\right|^d D(k, n, d). \end{aligned}$$

Assuming that $D(j, n, d)$ does not depend on j (which is *quite* natural if Ω is a fractal) this would imply that $D(j, 2n, d) = D(j, n, d)^2$. In the case of SLE, the natural version of the hypothesis would state that the $D(j, n, d)$ have the same law and are not strongly correlated (in a sense to be specified eventually, and which will likely be similar to condition 2. in Proposition 3.1 with milder requirements). For now, assume that the following holds:

$$(C) \qquad \exists d, \quad \exists n_0, \quad \exists j_0, \quad \forall n \geqslant n_0, \quad \forall j \prec j_0, \qquad D(j, n, d) \geqslant 1.$$

We then construct a Frostman measure on $\partial\Omega$, as follows. Fix d, n and j_0 according to condition (C), and let μ_0 be the Lebesgue measure on fQ_{j_0} normalized to have mass 1. We construct a sequence of measures (μ_k) inductively, as follows. Assume μ_k is constructed. Then μ_{k+1} is the unique measure supported on

$$\mathrm{Supp}(\mu_{k+1}) = \bigcup_{j \prec_{(k+1)n} j_0} fQ_j,$$

proportional to the Lebesgue measure on each of the fQ_j's and such that, if $j \prec_n l \prec_{kn} j_0$,

$$\mu_{k+1}(fQ_j) = \frac{l(fQ_j)^d}{\sum_{j' \prec_n l} l(fQ_{j'})^d} \, \mu_k(fQ_l).$$

In particular, for all k, μ_k has total mass 1, and besides any subsequential limit of (μ_k) is supported on $\partial\Omega$.

Proposition A.1 :

Under condition (C), the following hold:

(i). For all $\varepsilon > 0$ there exist $k_0 > 0$ and $C > 0$ such that, for all $k \geqslant k_0$ and all $j \prec_{kn} j_0$,

$$\mu_k(fQ_j) \leqslant C \, l(fQ_j)^{d-\varepsilon};$$

(ii). The Hausdorff dimension of $\partial\Omega$ is not less than d.

// (i). We prove this by induction on k. Note that the denominator in the definition of μ_{k+1} is equal to

$$\begin{aligned}\sum_{j'\prec_n l} l(fQ_{j'})^d &\asymp \sum_{j'\prec_n l} |f'(z_j)|^d\, l(Q_j)^d \\ &\asymp l(Q_l)^d f'(z_l)^d.\left[2^{-nd} f'(z_l)^{-d} \sum_{j'\prec_n l} f'(z_j)^d\right] \\ &\asymp l(fQ_l)^d D(l,n,d)\end{aligned}$$

so that, dividing by $l(fQ_j)^d$ in the definition of μ_{k+1},

$$\frac{\mu_{k+1}(fQ_j)}{l(fQ_j)^d} \leqslant \frac{1}{D(l,n,d)}.\frac{\mu_k(fQ_l)}{l(fQ_l)^d}. \tag{A.6}$$

Now we are assuming that $D(l,n,d) \geqslant 1$ for all l, and the estimate follows.

(ii). It is then easy to apply Lemma A.2 to prove that any subsequential limit of the sequence (μ_k) is then a Frostman measure of dimension $d-(3+d)\varepsilon$ supported on $\partial\Omega$. But we know in advance, by a compactness argument, that such a subsequential limit always exists — thus proving that there exists such a Frostman measure supported on $\partial\Omega$.

Hence, for all $\varepsilon > 0$, we have

$$\dim_H \partial\Omega \geqslant d-(3+d)\varepsilon,$$

hence $\dim_H \partial\Omega \geqslant d$, as we wanted. //

Remark: With the particular statement of condition (C) we kept, the first item in this proposition also holds for $\varepsilon = 0$, and indeed the proof does not even mention ε. We present it this way to show that this weaker version still gives a correct lower bound on Hausdorff dimensions, so that we could replace condition (C) by a much weaker estimate. Looking at Equation (A.6), the factor on which we need an upper bound will in fact be of the form

$$\frac{1}{D(l,n,d)}\left(\frac{l(fQ_j)}{l(fQ_l)}\right)^{\varepsilon}$$

with $j \prec_n l$, so that $l(fQ_j)$ will tend to be much smaller than $l(fQ_l)$, allowing $D(l,n,d)$ itself to be smaller than 1.

A.2.3 Application to SLE

(Beware that this last subsection does not contain any real math and is only a loose attempt at giving the embryo of the skeletton of the indication of a proof.)

In the case of a random domain, and hence of a random map, the $D(l,n,d)$ are random variables; and the proof presented above will work assuming that almost all of these variables are greater than 1, or even if for “most” chains

$$j_0 \succ_n j_1 \succ_n j_2 \succ_n \cdots,$$

the product of the $D(j_i, n, d)$ tends to $+\infty$. This is typically the case for a random snowflake — but then again in the case of snowflakes it is probably easier to attack the problem using harmonic measure and symbolic dynamics, cf. for instance [8, 9, 10].

In the specific case of SLE, a good indication that the method might work is given in [42, Theorem 8.3], namely it is stated that if f is the reciprocal of g_1 in the case of an SLE_κ with $\kappa < 4$, then the expected value of

$$S(a) \triangleq \sum_j l(Q_j)^a \asymp \sum_j 2^{-an_j} f'(z_j)^a$$

is finite if $a > \delta(\kappa) = 1+\kappa/8$ and infinite if $a \geqslant \delta(\kappa)$. The idea is then to express this sum (restricted to cubes sitting below a fixed one Q_{j_0} — call it S_{j_0}) in terms of the $D(j_0, n, a)$: namely we have

$$S_{j_0}(a) \asymp |f'(z_{j_0})|^a \sum_{n=1}^{\infty} D(j_0, n, a).$$

Now fix n_0 and write this sum as

$$S_{j_0}(a) \asymp |f'(z_{j_0})|^a \sum_{k=1}^{n_0} \sum_{n \equiv k[n_0]} D(j_0, n, a).$$

The above considerations then show that each of the n_0 sums appearing should behave like a geometric sum of ratio $D(j_0, n_0, a)$, meaning that, for large values of n_0, the expectation of $D(j_0, n_0, a)$ should be smaller than 1 if $a > \delta(\kappa)$ and bigger than 1 if $a < \delta(\kappa)$.

Assume that we are in the second case. For all $\varepsilon > 0$, we can write

$$D(j_0, n_0, a-\varepsilon) \asymp l(fQ_{j_0})^{-(a-\varepsilon)} \sum_{j \prec_{n_0} j_0} l(fQ_j)^{a-\varepsilon} \geqslant cD(j_0, n_0, a) \left[\max_{j \prec_{n_0} j_0} l(fQ_j)\right]^{-\varepsilon}.$$

The maximum in the last term tends to 0 as n_0 goes to infinity, and this proves that as soon as $E(D(j_0, n_0, a))$ is bounded below, $E(D(j_0, n_0, a-\varepsilon))$ goes to infinity with n_0. With some luck, this will be sufficient to apply the construction of the Frostman measure presented in the previous subsection, for $d = a - \varepsilon$ (it does not imply condition (C), though).

So if everything worked out, we would obtain $\dim_H K_t \geqslant a - \varepsilon$ for each $a < 1 + \kappa/8$ and $\varepsilon > 0$ — hence $\dim_H \partial K_t \geqslant 1 + \kappa/8$, which was the difficult part of Theorem 4.1. The good point here is that the method seems to be more robust, because we do not need to work precisely at $d = \delta(\kappa)$ and we may add as many shifts by $-\varepsilon$ as we wish (to make all the terms big enough) and still obtain a lower bound of the form $\delta(\kappa)$ minus many times ε, which is still very fine. In comparison, the method we used to prove Theorem 4.1 strongly relies on the fact that the exponent s is exactly the same in the conditions 1. and 3. of Proposition 3.1.

Appendix B

Simulations and Pictures

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We gather in this chapter the results obtained by an SLE simulation program, which we describe briefly. The aim is to provide a picture of the object, and to see whether its aspect is what we would expect from comparison with discrete models. So we present, close to each other:

- An SLE_2 with a long loop-erased random walk;
- An $SLE_{8/3}$ with a long uniform self-avoiding walk (obtained by the pivot algorithm as described in [39]);
- An SLE_6 with various percolation-type pictures (namely, and in order: critical site-percolation on the trianular lattice; critical site- and bond-percolation on the square lattice; and gradient percolation).

We end it by an SLE for a value of κ bigger than 10 (indeed having a smoother boundary and exhibiting no cut point).

The method of simulation is the most stupid one, using a classical Euler scheme and discretizing the driving process into a simple random walk with steps of $\pm\sqrt{\kappa\varepsilon}$ over time-intervalls of length ε. The process is stopped at time 1 and constant afterwards (hence the "tail" on the pictures); this is a trick used split H_t into two components (the left- and right-hand sides of the tail), whose boundaries are then explored to draw the picture. We do have convergence to SLE in the Caratheodory topology, by Gronwall's lemma, but it is not very fast so there are artifacts due to the discretizations (they are especially visible on the $SLE_{8/3}$ picture).

Note that Marshall also produced pictures of SLE using his "zipper" algorithm (which follows the Loewner chain as a composition of infinitesimal deformations of the domain); his images are nicer for $\kappa < 4$, but crappier for $\kappa > 4$ — because the zipper method always produces a slit domain, *i.e.* a simple curve.

B.1 Loop-Erased Random Walk and SLE_2

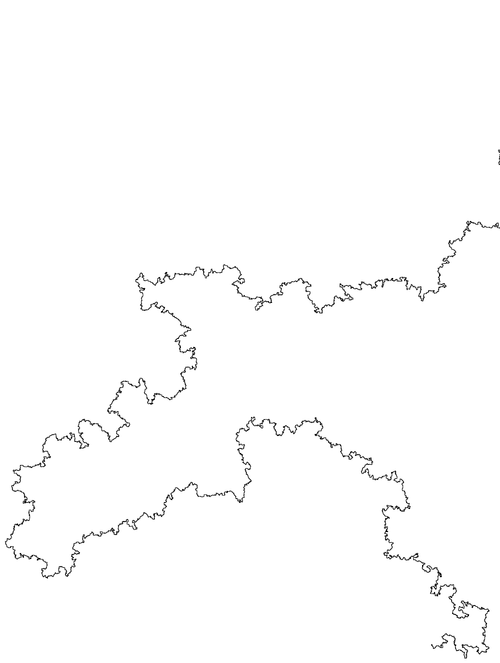

Figure B.1: A Loop-Erased Random Walk (LERW)

Figure B.2: The path of an SLE_2

B.2 Self-Avoiding Walk and $SLE_{8/3}$

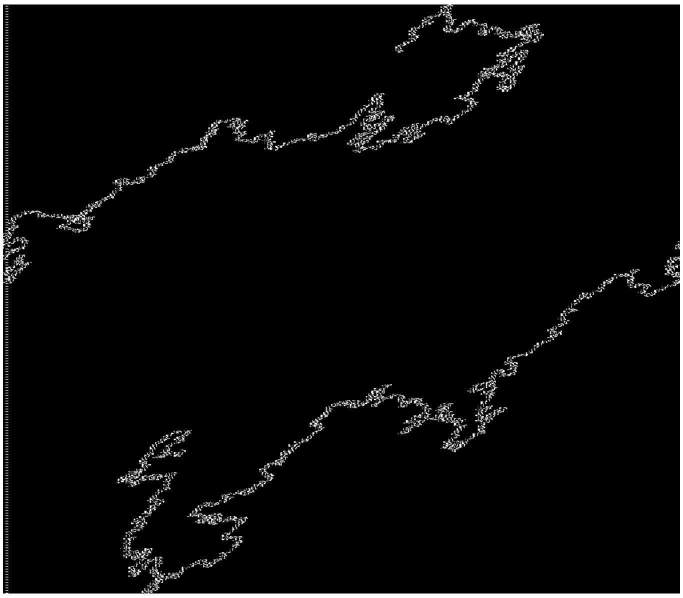

Figure B.3: A Self-Avoiding Walk (SAW)

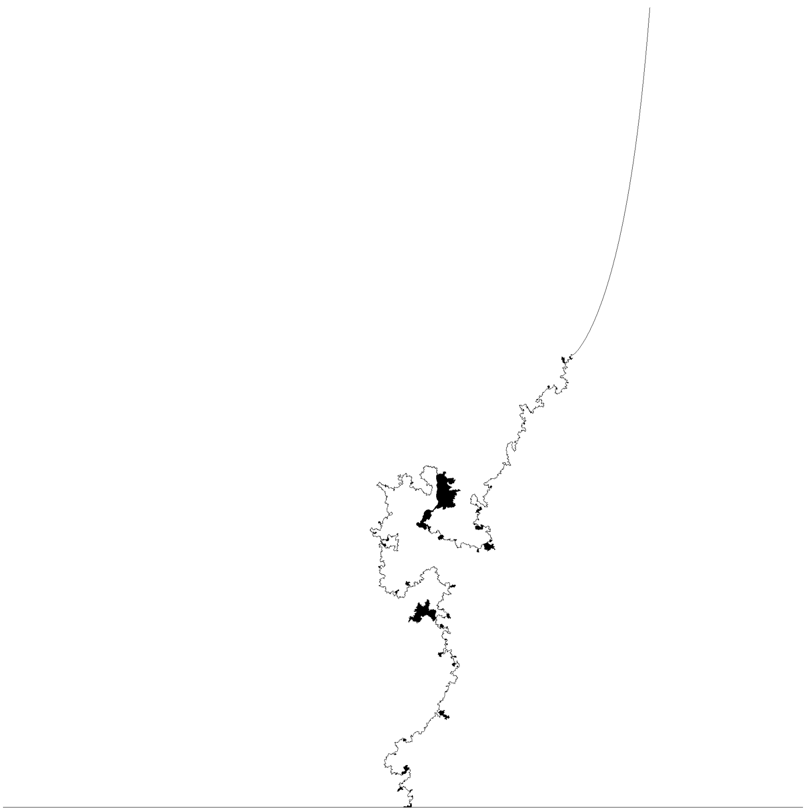

Figure B.4: The path of an $SLE_{8/3}$

B.3 Critical Percolation and SLE_6

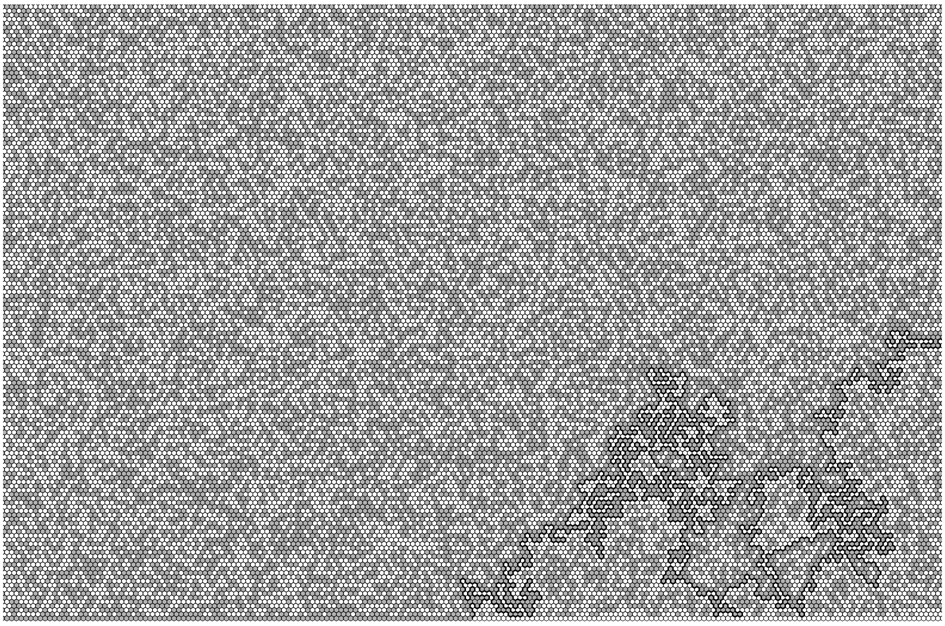

Figure B.5: A critical percolation exploration process (picture by Oded Schramm)

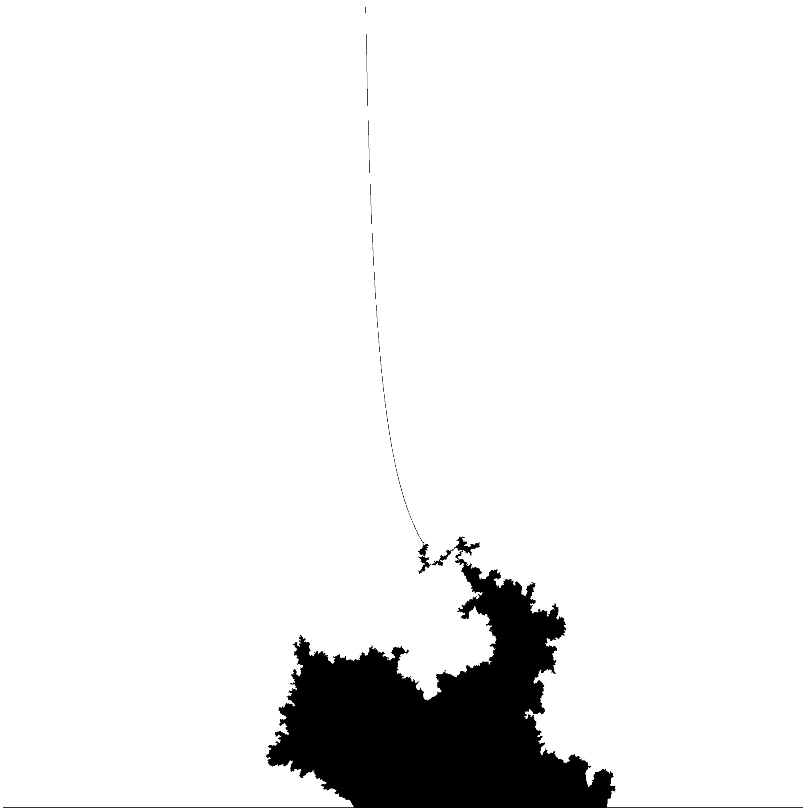

Figure B.6: The path of an SLE_6

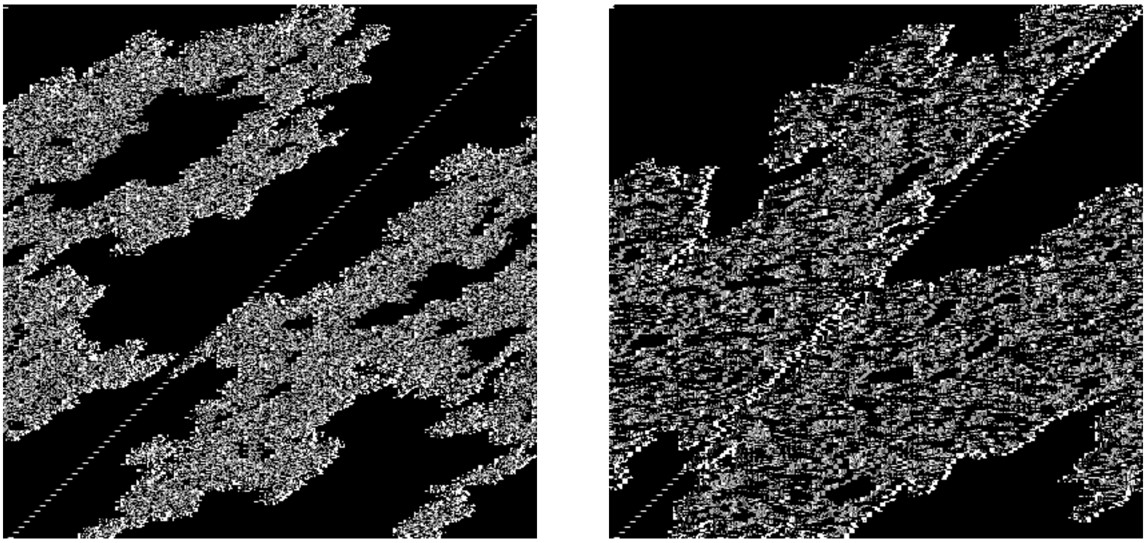

Figure B.7: Big critical percolation clusters on the square lattice
(left: site-percolation, right: bond-percolation)

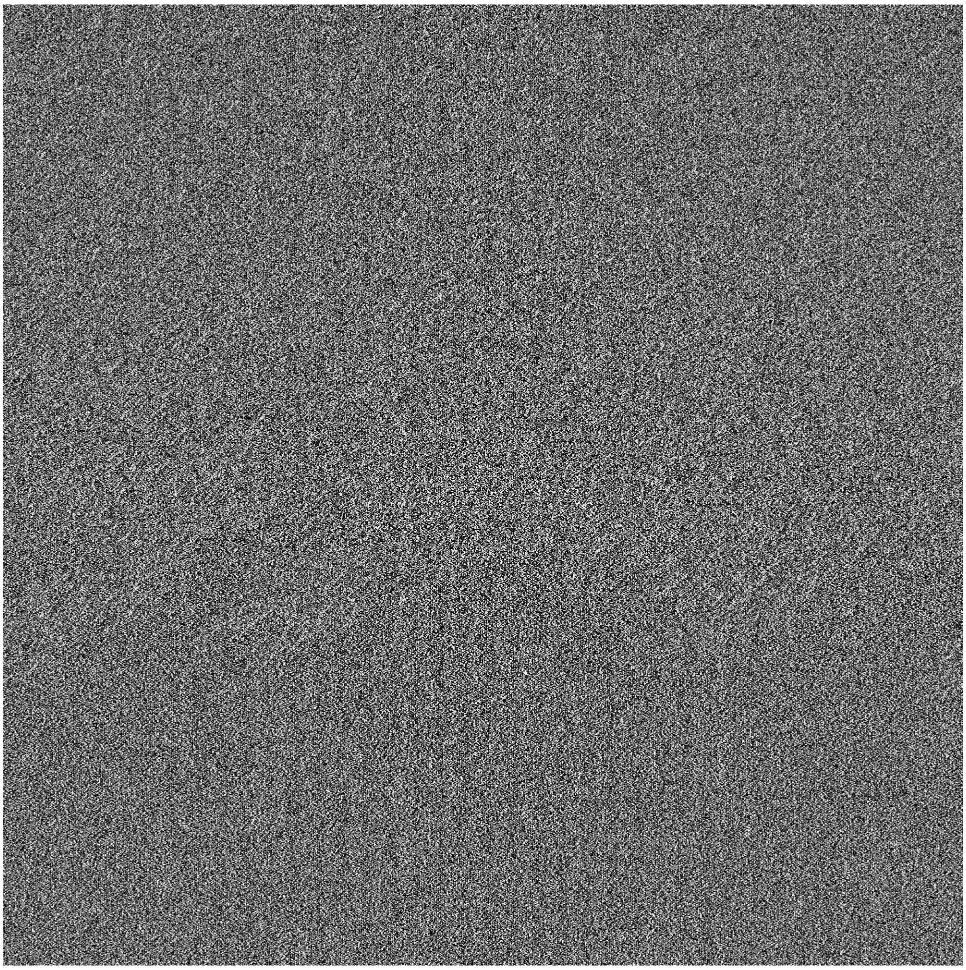

Figure B.8: “Gradient-percolation” on the square lattice

B.4 SLE for bigger values of κ

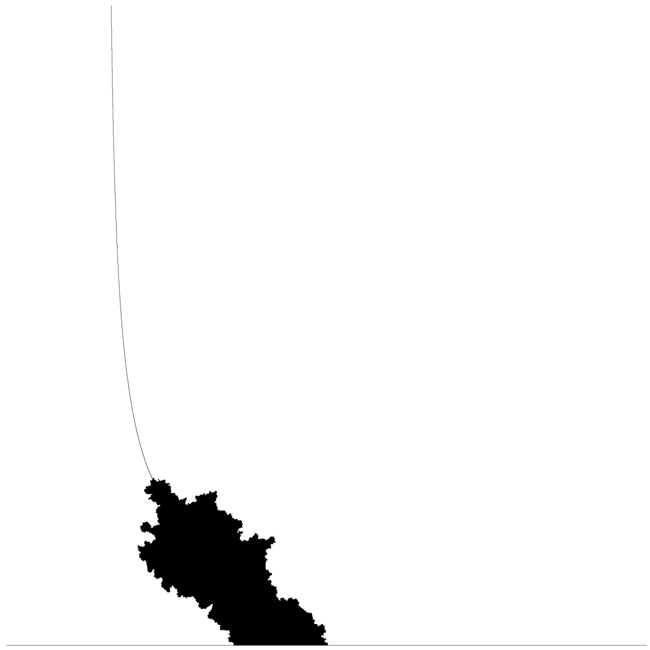

Figure B.9: The path of an SLE_{10}

B.5 The code

All the programs used to produced the pictures in this thesis (except for the percolation exploration curve on the honeycomb lattice) are available on the web at the address

`http://vbeffara.free.fr/boulot/`

but we include here the code used to generate the SLE images.

File `Schramm.c`: the simulation itself

```
#include <config.h>
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <printout.h>

#define COO(a,b) (2*nn*(a)+nn+(b))
#define EC fprintf(stderr,".\n")

#define DONTKNOW  0
#define INSIDE    1
#define LEFTSIDE  2
#define RIGHTSIDE 3

int n,nn,jmax,cnt;
image *img;
double *c;
double kappa;

int time (int *tloc);

char real_t(int i, int j)
{
  double a,b,d;
  int k;

  a=i; b=j; d=0;

  fprintf (stderr, "  (%d;%d)        \r", i, j);

  for (k=0;(k<n)&&(b>0);k++) {
    d=kappa / ((a-c[k])*(a-c[k]) + b*b);
    a += (a-c[k])*d;
    b -= b*d;
  }
  if (k<n) return INSIDE;
  else if (a<c[n-1]) return LEFTSIDE;
  else return RIGHTSIDE;
}

inline char clever_t(int i, int j)
{
  char tmp;

  if (i>=nn)   i=nn-1;
  if (i<-nn)   i=-nn;
  if (j<0)     j=0;
  if (j>=2*nn) j=2*nn-1;
```

```
  tmp=img->t[2*nn*j+nn+i];

  if (tmp==0)
    PutPoint (img,i+nn,j,tmp=real_t(i,j));

  return tmp;
}

void bord(void)
{
  const int dx[4]={1,0,-1,0};
  const int dy[4]={0,1,0,-1};
  const int blackmagic[27]={
    1,1,1, 1,1,1, 1,1,1,            // INSIDE
    2,2,0, 2,2,0, 0,0,0,            // LEFTSIDE
    3,0,3, 0,0,0, 3,0,3,            // RIGHTSIDE
  };
  const char tokeep[4] = {1,1,0,0};

  int x,y,d,k;
  char tmp,col;
  char self,right,down;

  // Follow the left side ...

  x=-nn; y=1; d=0;
  while ((clever_t(x,y)!=LEFTSIDE)&&(y<2*nn)) y++;
  while (y<2*nn) {
    x+=dx[d]; y+=dy[d]; d=(d+3)&3;
    while (clever_t(x+dx[d],y+dy[d])!=LEFTSIDE) d=(d+1)&3;
    cnt++;
  }

  // Then follow the right side ...

  x=nn-1; y=1; d=2;
  while ((clever_t(x,y)!=RIGHTSIDE)&&(y<2*nn)) y++;
  while (y<2*nn) {
    x+=dx[d]; y+=dy[d]; d=(d+1)&3;
    while (clever_t(x+dx[d],y+dy[d])!=RIGHTSIDE) d=(d+3)&3;
    cnt++;
  }

  // And now  for the dark side:  Catch the thin parts ie LEFT->RIGHT
  // and RIGHT->LEFT in addition to the fat parts (INSIDE).

  // First some stupid filling of "UNKNOWN" points :

  for (x=0;x<2*nn;x++)
    img->t[x]=INSIDE;
  for (y=1;y<2*nn;y++) {
    tmp=LEFTSIDE;
    for (x=-nn;x<nn;x++) {
      k=2*nn*y+nn+x;
      col=img->t[k];
      if (col==DONTKNOW) img->t[k]=tmp;
      else tmp=col;
    }
  }

  // OK, we have  a nice picture. Now for the  real black magic, Edge
  // Detection (US$0.02 version). 0=added points.  (ie: among itself
  // and its right- and down-neighbours lie at least one LEFTSIDE and
  // one RIGHTSIDE)

  for (y=2*nn-1;y>0;y--) {
    for (x=-nn;x<nn-1;x++) {
      self = img->t[2*nn*y+nn+x];
      right = img->t[2*nn*y+nn+x+1];
      down = img->t[2*nn*y-nn+x];
      img->t[2*nn*y+nn+x] = blackmagic[(9*self)+(3*right)+down-13];
    }
  }

  // Last sweep

  for (x=0;x<4*nn*nn;x++)
    img->t[x] = tokeep[(int)img->t[x]];
}

int main(int argc, char ** argv)
{
  int i;
  char s[80];
  double d, cd;

  /* arguments -> kappa et n */

  if (argc < 3) {
    fprintf(stderr, "Syntaxe : %s <kappa> <sqrt(n)> [seed]\n",
    argv[0]);
    exit(1);
  }
  sscanf (argv[1],"%lf",&kappa);
  sscanf (argv[2],"%d",&nn);
  if (argc>=4) {                 // Si on donne une initialisation :
    sscanf (argv[3],"%d",&n);
    srand48(n);
  } else {                       // Sinon, aleatoire :
    n = time(0);
    fprintf(stderr,"Random seed                   = %d\n",n);
    srand48(n);
  }

  sprintf(s,"Schramm's SLE Process (kappa=%f)",kappa);

  n=nn*nn; kappa=2/kappa;

  img = new_image (2*nn,2*nn,2,s);
  if (!img) exit(1);
  for (i=0;i<2*nn;i++) img->t[i]=1;

#ifdef HAVE_SDL
  OnScreen (img);
#endif

  /* Brownien qui conduit le SLE - kappa n'apparait pas ici*/

  c = (double *) malloc (n*sizeof(double));
  c[0]=0; cd=0;
  for (i=1;i<n;i++) {
    d = 2*sqrt(3)*drand48() - sqrt(3); // E=0, Var=1
    c[i] = c[i-1] + d;
    cd += (d*d);
  }

  fprintf (stderr,"End value        (normalized) = %f\n",c[n-1]/nn);
  fprintf (stderr,"Square variation (normalized) = %f\n",cd/n);

  /*
   * simulation du SLE : inutile sans strategie siouxe (clever_t est
   * malin), sinon au choix longer le bord ou dichotomie. Longer le
   * bord est beaucoup plus efficace : sur jade, pour 7 et 100 pts :
   *   sans strategie : 31.29s
   *   dichotomie     : 4.75s
   *   suivre le bord : 0.98s
   */

  fprintf (stderr, "Doing the hard work ...\n");

  bord();

  fprintf (stderr, "Estimated boundary dimension  = %f (%f)\n",
   log(cnt)/log(2*nn),
   (kappa>0.5?1+1/(4*kappa):1+kappa));

  fprintf (stderr, "Exporting EPS file.\n");

  /* affichage du resultat */

  img->dp = 1;
  printout_eps (img,0,0,2*nn,2*nn);

  fprintf (stderr, "Good bye, have a nice day.\n");

  free (img->t); free (img->title); free (img);

  return 0;
}
```

File `libprintout.c`: the eps output

```
//
// libprintout.c - v1.1 - © 2001 VB - GPL
//

#include <config.h>
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <errno.h>
#include <printout.h>

#ifdef HAVE_SDL
#include <SDL.h>
```

```
void DrawPixel (SDL_Surface *screen, int x, int y,
Uint8 R, Uint8 G, Uint8 B)
{
  /*
   * Taken directly from the SDL documentation ...
   */

  Uint32 color = SDL_MapRGB(screen->format, R, G, B);

  if ( SDL_MUSTLOCK(screen) ) {
    if ( SDL_LockSurface(screen) < 0 ) {
      return;
}
  }

  switch (screen->format->BytesPerPixel) {
  case 1: { /* Assuming 8-bpp */
    Uint8 *bufp;

    bufp = (Uint8 *)screen->pixels + y*screen->pitch + x;
    *bufp = color;
  }
    break;

  case 2: { /* Probably 15-bpp or 16-bpp */
    Uint16 *bufp;

    bufp = (Uint16 *)screen->pixels + y*screen->pitch/2 + x;
    *bufp = color;
  }
    break;

  case 3: { /* Slow 24-bpp mode, usually not used */
    Uint8 *bufp;

    bufp = (Uint8 *)screen->pixels + y*screen->pitch + x * 3;
    if(SDL_BYTEORDER == SDL_LIL_ENDIAN) {
      bufp[0] = color;
      bufp[1] = color >> 8;
      bufp[2] = color >> 16;
    } else {
      bufp[2] = color;
      bufp[1] = color >> 8;
      bufp[0] = color >> 16;
    }
  }
    break;

  case 4: { /* Probably 32-bpp */
    Uint32 *bufp;

    bufp = (Uint32 *)screen->pixels + y*screen->pitch/4 + x;
    *bufp = color;
  }
    break;
  }

  if ( SDL_MUSTLOCK(screen) ) {
    SDL_UnlockSurface(screen);
  }
}
#endif

int PutPoint (image *img, int x, int y, int c) {
  if (x<0) return -1;
  if (y<0) return -1;
  if (x>=img->wd) return -1;
  if (y>=img->ht) return -1;

  img -> t[x+y*img->wd] = c&255;

#ifdef HAVE_SDL
  if (img->screen) {
    DrawPixel (img->screen,x,y,img->palette[c&255][0],
       img->palette[c&255][1],img->palette[c&255][2]);
    if (c&PRINTOUT_FULL_UPDATE)
      SDL_UpdateRect(img->screen,0,0,img->wd,img->ht);
    else if (!(c&PRINTOUT_NO_UPDATE))
      SDL_UpdateRect(img->screen,x,y,1,1);
  }
#endif

  return c;
}

image *new_image (int wd, int ht, int dp, char *title)
{
  image *img;

  if ((dp!=1)&&(dp!=2)&&(dp!=4)) {
    fprintf (stderr, "printout library error : invalid depth");
    fprintf (stderr, " (only 1, 2 and 4 bpp allowed).\n");
    return NULL;
  }
  img = (image *) malloc(sizeof(image));

  img->title = (char*) calloc(80,sizeof(char));
  strncpy(img->title,title,80);
  img->wd=wd;
  img->ht=ht;
  img->dp=dp;
  img->t=(char*) calloc (wd*ht,sizeof(char));

  if (!(img->t)) {
    fprintf (stderr,"printout library error : image too large.\n");
    free (img->title);
    free (img);
    return NULL;
  }

#ifdef HAVE_SDL
  { int i,pstep;
  pstep = 255 / ((1<<dp)-1);
  for (i=0;i<(1<<dp);i++) {
    img->palette[i][0] = i*pstep;
    img->palette[i][1] = i*pstep;
    img->palette[i][2] = i*pstep;
  }}
#endif

  img->cropped=0;

  return img;
}

int OnScreen (image *img)
{
#ifdef HAVE_SDL
  int i,j;

  fprintf(stderr,"printout library : Mapping SDL window ...\n");

  SDL_Init(SDL_INIT_VIDEO);
  atexit(SDL_Quit);

  img->screen=SDL_SetVideoMode(img->wd,img->ht,0,SDL_SWSURFACE);
  if (!img->screen) {
    fprintf (stderr,"printout library error :");
    fprintf (stderr,"Couldn't map it ! Continuing without.\n");
    return 0;
  }

  SDL_WM_SetCaption (img->title,"Simulation");

  for (i=0;i<img->wd;i++)
    for (j=0;j<img->ht;j++)
      DrawPixel (img->screen,i,j,
 img->palette[(int)img->t[i+j*img->wd]][0],
 img->palette[(int)img->t[i+j*img->wd]][1],
 img->palette[(int)img->t[i+j*img->wd]][2]);

  SDL_UpdateRect (img->screen,0,0,img->wd,img->ht);
  return 1;
#else
  fprintf (stderr,"printout library : I can't do that, Dave.\n");
  return 0;
#endif
}

inline char trans (int i)
{
  static char *trans = "0123456789ABCDEF";

  if (i<0) i=0;
  if (i>15) i=15;
  return trans[i];
}

int range_check (int wd, int ht, int x, int y, int dx, int dy)
{
  if ((wd<0)  || (ht<0)    ||
      (x<0)   || (x>=wd)   || (y<0)  || (y>=ht) ||
      (dx<=0) || (x+dx>wd) || (dy<0) || (y+dy>ht)) {
    fprintf (stderr,"printout library error : invalid range.\n");
    return 1;
  }
  return 0;
}

int printout_eps (image *img, int x, int y, int dx, int dy)
{
  int acc,dec,i,j,bits;
  int xmin,xmax,ymin,ymax;

  if (range_check(img->wd,img->ht,x,y,dx,dy)) return 1;

  if ((img->dp!=1)&&(img->dp!=2)&&(img->dp!=4)) {
    fprintf (stderr,"printout library error : invalid depth");
    fprintf (stderr," (only 1, 2 and 4 bpp allowed).\n");
```

```
    return 1;
  }

  if (img->cropped != 0) {
    xmin=x+dx-1; ymin=y+dy-1; xmax=x; ymax=y;
    for (i=x;i<x+dx;i++)
      for (j=y;j<y+dy;j++)
if ((img->t[i+img->wd*j]&15) != 0) {
  if (i<xmin) xmin=i;
  if (i>xmax) xmax=i;
  if (j<ymin) ymin=j;
  if (j>ymax) ymax=j;
}
    x=xmin; dx=xmax-xmin+1; y=ymin; dy=ymax-ymin+1;
  }

  bits = dx * img->dp;

  // EPS header

  printf ("%%!PS-Adobe-2.0 EPSF-2.0\n");
  //  printf ("%%%%Title: %s\n",img->title);
  //  printf ("%%%%Creator: libprintout - v%s - © 2001 VB - GPL\n",
  //   VERSION);
  printf ("%%%%Creator:   Mail: Vincent.Beffara@math.u-psud.fr\n");
  printf ("%%%%Creator:   Web:  <http://vbeffara.free.fr/>\n");
  printf ("%%%%BoundingBox: 0 0 %d %d\n\n", dx, dy);

  // Commands

  printf ("save 20 dict begin /xpixels %d def /ypixels %d def\n",
  dx, dy);
  printf ("/pix %d string def xpixels ypixels scale\n", (bits+7)/8 );
  printf ("xpixels ypixels %d [xpixels 0 0 ypixels 0 0]\n",img->dp);
  printf ("{currentfile pix readhexstring pop} image\n");

  // Image

  for (j=y;j<y+dy;j++) {
    acc=15; dec=16>>img->dp;
    for (i=x; i<x+dx; i++) {
      acc -= dec*(img->t[i+img->wd*j]&15);
      if (dec==1) {
printf ("%c",trans(acc));
acc=15; dec=16>>img->dp;
      } else dec >>= img->dp;
      if (!((i-x+1)%(512>>img->dp))) printf ("\n");
    }
    if ((dx<<img->dp)%8) printf ("%c",trans(acc));
    if (((dx<<img->dp)%16)&&(((dx<<img->dp)%16)<=8)) printf ("F");
    if (dx%(512>>img->dp))
      printf ("\n");
  }

  // End of file

  printf ("end restore\n");
  return 0;
}

int printout_path (char *p, int l, char *title)
{
  const char *dirs = "ENWS";
  const int dx[4] = {1,0,-1,0};
  const int dy[4] = {0,1,0,-1};

  int i, imin,imax,  jmin,jmax, x,y;

  /* Step 1 = cropping */

  imin=0; imax=0; jmin=0; jmax=0; x=0; y=0;
  for (i=0;i<l;i++) {
    x+=dx[(int)p[(int)i]];
    y+=dy[(int)p[(int)i]];
    if (x<imin) imin=x;
    if (x>imax) imax=x;
    if (y<jmin) jmin=y;
    if (y>jmax) jmax=y;
  }

  /* Step 2 = printing */

  // Header

  printf ("%%!PS-Adobe-2.0 EPSF-2.0\n");
  printf ("%%%%Title: %s\n",title);
  printf ("%%%%Creator: libprintout - v%s - © 2001 VB - GPL\n",
  VERSION);
  printf ("%%%%Creator:   Mail: Vincent.Beffara@math.u-psud.fr\n");
  printf ("%%%%Creator:   Web:  <http://vbeffara.free.fr/>\n");
  printf ("%%%%BoundingBox: 0 0 %d %d\n\n",
  3*(imax-imin)+6, 3*(jmax-jmin)+6);

  // "Code" ;-)

  printf ("save 20 dict begin\n");
  printf ("/E {3 0 rlineto} bind def /W {-3 0 rlineto} bind def\n");
  printf ("/N {0 3 rlineto} bind def /S {0 -3 rlineto} bind def\n");
  printf ("newpath %d %d moveto\n", 3-3*imin, 3-3*jmin);

  for (i=0;i<l;) {
    printf ("%c", dirs[(int)p[(int)i]]);
    if (!(++i%40)) printf ("\n");
    else printf (" ");
  }
  if (i%40) printf ("\n");

  printf ("stroke end restore\n");
  return 0;
}
```

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Mouvement brownien plan, SLE, invariance conforme et dimensions fractales

Vincent BEFFARA

Résumé : Cette thèse est consacrée à l'étude de quelques propriétés géométriques du mouvement brownien plan et du processus SLE (ou processus de Loewner stochastique, ou encore processus de Schramm).

On prouve qu'il existe presque sûrement sur la trajectoire brownienne plane des points "pivots", *i.e.* des points de coupure autour desquels l'une des moitiés de la trajectoire peut pivoter d'un angle strictement positif sans jamais rencontrer l'autre moitié ; l'ensemble des point pivots d'angle donné (suffisamment petit) est alors de dimension de Hausdorff strictement positive. On décrit en fait, pour toute partie A du plan complexe, un ensemble E_A de points exceptionnels de la trajectoire définis de manière géométrique. À tout tel A est associé un *exposant de non-intersection généralisé* $\xi(A)$, et on prouve que $\dim_H(E_A) = 2 - \xi(A)$, si bien que E_A est non vide dès que $\xi(A) < 2$.

Concernant le SLE, le principal résultat obtenu dans cette thèse est le calcul de la dimension de Hausdorff de sa *trace* (*i.e.* de la courbe qui l'engendre) ; cette dimension est égale à $1 + \kappa/8$, où κ est le paramètre du SLE — et ceci pour tout paramètre positif différent de 4 et inférieur à 8 (pour $\kappa \geqslant 8$, la trace est une courbe de Peano donc de dimension 2). On prouve au passage l'existence presque sûre de points de coupure sur tout SLE de paramètre strictement inférieur à 8.

On s'intéresse également au problème de la généralisation du processus SLE à des ouverts non simplement connexes ; on montre que cela est faisable pour deux valeurs particulières du paramètre ($\kappa = 6$ et $\kappa = 8/3$), en utilisant à chaque fois des propriétés spécifiques du SLE considéré (respectivement, la *propriété de restriction* et la *localité*), mais que l'on perd la propriété d'*universalité* du SLE usuel.

Mots-clés : Mouvement brownien (plan), SLE, invariance conforme, exposants critiques, dimension de Hausdorff.

Classification MSC2000 : 60D05, 60G17, 60G51, 60G57, 60G99, 28A80