1 Measuring the running of α_s from HERA DIS inclusive data

In this set of notes we collect a number of ideas to measure the running of the strong coupling α_s from the HERA I+II combined DIS inclusive data. This will be carried out by a simultaneous fit of PDFs to this data. Throughout these notes cross sections are assumed to be renormalised and factorised in the $\overline{\mathrm{MS}}$ scheme. We will focus on two main strategies that should lead to consistent results as they are both based on QCD factorisation.

1.1 Strategy one

We start by observing that in fit of PDFs the strong coupling α_s enters both the computation of the partonic cross sections, which are perturbative quantities and thus are typically expressed as truncated series in powers of α_s, and the DGLAP evolution where terms of the form $\alpha_s^n \ln^n(Q^2/Q_0^2)$ are resummed to all orders. Focusing on DIS, the prediction for a certain cross section in terms of initial scale PDFs is given by:

$$\sigma(Q) = \hat{\sigma}(Q) \otimes \Gamma(Q, Q_0) \otimes f_0 \,. \tag{1}$$

In Eq. (1) the partonic cross section $\hat{\sigma}$ allows for the perturbative expansion:

$$\hat{\sigma}(Q) = \hat{\sigma}_0 + \hat{\alpha}_s \hat{\sigma}_1 + \hat{\alpha}_s^2 \hat{\sigma}_2 + \dots \tag{2}$$

where we have defined:

$$\hat{\alpha}_s \equiv \alpha_s(Q) \quad \Longrightarrow \quad \hat{\sigma}(Q) \equiv \hat{\sigma}(\hat{\alpha}_s) \,. \tag{3}$$

It should be noticed that the single perturbative terms $\hat{\sigma}_i$ do depend explicitly on Q through the ratios Q/μ_R, Q/μ_F, and Q/m_H but this is not relevant in the following discussion because we are only looking at the dependence on α_s. The evolution kernel Γ in Eq. (1) is instead determined by solving the DGLAP equation:

$$Q^2 \frac{d}{dQ^2} \Gamma(Q, Q_0) = P(\alpha_s(Q)) \otimes \Gamma(Q, Q_0) \,. \tag{4}$$

with P being the DGLAP splitting functions which are perturbatively computable and are currently known to $\mathcal{O}(\alpha_s^3)$, *i.e.* NNLO accuracy. Using the "obvious" boundary condition:

$$\Gamma(Q_0, Q_0) = 1 \,, \tag{5}$$

one can schematically write the solution to Eq. (4) as follows:

$$\Gamma(Q, Q_0) = \exp\left[\int_{\ln Q_0^2}^{\ln Q^2} P(\alpha_s(Q'))\, d\ln Q'^2\right] \,. \tag{6}$$

It is clear from Eq. (6) that the evolution operator Γ depends on all the values that α_s takes between Q_0 and Q. In other words, we need to know how α_s evolves in this range. But this only depends on one single input that is the value of α_s at some reference scale. Such value can be chosen to be $\hat{\alpha}_s$ that thus means:

$$\Gamma \equiv \Gamma(\hat{\alpha}_s)\,, \tag{7}$$

where we have omitted the dependence on Q and Q_0 that we consider fixed here. Finally, we notice that f_0 represents PDFs at the initial scale. Since this quantity by definition does not undergo any evolution, it only depends on the (fixed) initial scale Q_0 and not on the value of α_s in Q_0. As a consequence, this quantity can also be considered constant.

In the end of the day we can write Eq. (1) as:

$$\sigma(Q) = \hat{\sigma}(\hat{\alpha}_s) \otimes \Gamma(\hat{\alpha}_s) \otimes f_0 \equiv \sigma(\hat{\alpha}_s)\,, \tag{8}$$

that is to say that the prediction for $\sigma(Q)$ only depends on $\hat{\alpha}_s$.

Now, suppose to have a set of measurements corresponding to a number of energies Q_i:

$$\{\sigma_i = \sigma(Q_i) = \sigma(\hat{\alpha}_{s,i})\}\,. \tag{9}$$

In principle, assuming that everything else is fixed (*i.e.* perturbative order, PDFs, heavy quark masses, etc.) one can use each of these measurements to determine $\hat{\alpha}_{s,i}$. This is equivalent to determine the value of α_s for the different values of the scale Q_i which in turn means determining the running of the strong coupling. There is a caveat though. In particular, Eq. (6) assumes that the running of the strong couplings follows the RGE evolution. In fact, what we really aim at here is not a direct measurement of the running of α_s but rather whether the scaling violations of the measurements we are considering, which are driven by the strong coupling, are consistent with the QCD expectations.

As is well know, there is a close interplay between the value of α_s and PDFs. Consequently, extracting the values of $\hat{\alpha}_{s,i}$ keeping PDFs fixed represents an approximation that might not be appropriate. To overcome this problem, one can parametrize the initial scale PDFs f_0 by means of one of the usual functional forms:

$$f_0 \equiv f_0(\{a_j\}) \tag{10}$$

and fit the free parameters a_j to data as commonly done in the standard PDF determinations. As clear from Eq. (8), f_0 is common to all predictions for σ_i and thus one can no longer determine the value of $\hat{\alpha}_{s,i}$ from σ_i only independently of the other measurements because now there is a cross-talk between predictions due to f_0. Therefore, one needs to perform a "global fit" to the set of measurements $\{\sigma_i\}$ and the result of such a fit would be the determination of the two sets of free parameters:

$$\{\hat{\alpha}_{s,i}\} \quad \text{and} \quad \{a_j\}\,. \tag{11}$$

This will (should) allow us to perform a simultaneous determination of PDFs and the running of α_s avoiding the bias that a determination of the running of the strong coupling obtained with fixed PDFs might lead to.

It is interesting to notice that the approach described above represents a generalisation of the standard procedure. As a matter of fact, the reference values $\hat{\alpha}_{s,i}$ do not have to be necessarily equal $\alpha_s(Q_i)$, but they can all be take to be equal to the coupling to any arbitrary scale. The value of α_s at any scale Q can then be obtained by evolution. In particular, one can take all $\hat{\alpha}_{s,i}$ to be equal to one particular value. This is what is typically done to determine the value of $\alpha_s(M_Z)$, that is by simply setting:

$$\hat{\alpha}_{s,i} = \alpha_s(M_Z) \tag{12}$$

for all measurements included in the fit. This clearly reduces the set of free parameters to be determined the a global fit from Eq. (11) to:

$$\alpha_s(M_Z) \quad \text{and} \quad \{a_j\}\,. \tag{13}$$

While on the one hand this provides a simplification, this comes at the price of assuming that the RGE running of the strong coupling is used in a *maximal* way also to completely determine the value of the partonic cross section in Eq. (2) at the relevant scale of the single measurements.