Notes on the QED evolution to NLL accuracy

Abstract

In these notes, we discuss the details of the implementation of the QED collinear evolution up to next-to-leading logarithmic (NLL) accuracy in the variable-flavour-number scheme (VFNS) in the presence of both charged leptons and quarks.

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1 The structure of the DGLAP equation

Suppose one wants to study the coupled collinear evolution in pure QED of the photon distribution γ, the i-th down-type quark distribution with a certain colour (rgb) $d_i \in \{d^r, d^g, d^b, s^r, s^g, s^b, b^r, b^g, b^b\}$, the k-th down-type anti-quark distribution $\overline{d}_k \in \{\overline{d}^r, \overline{d}^g, \overline{d}^b, \overline{s}^r, \overline{s}^g, \overline{s}^b, \overline{b}^r, \overline{b}^g, \overline{b}^b\}$, the j-th up-type quark distribution $u_j \in \{u^r, u^g, u^b, c^r, c^g, c^b, t^r, t^g, t^b\}$, the h-th up-type anti-quark distribution $\overline{u}_k \in \{\overline{u}^r, \overline{u}^g, \overline{u}^b, \overline{c}^r, \overline{c}^g, \overline{c}^b, \overline{t}^r, \overline{t}^g, \overline{t}^b\}$, the α-th lepton distribution $\ell_\alpha \in \{e^-, \mu^-, \tau^-\}$, and the β-th anti-lepton distribution $\overline{\ell}_\beta \in \{e^+, \mu^+, \tau^+\}$. The most general form of the evolution equations reads:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} \ell_\alpha \\ u_j \\ d_i \\ \gamma \\ \overline{d}_k \\ \overline{u}_h \\ \overline{\ell}_\beta \end{pmatrix} = \sum_{e,l,m,n,\gamma,\delta} \begin{pmatrix} \mathcal{P}_{\ell_\alpha \ell_\gamma} & \mathcal{P}_{\ell_\alpha u_e} & \mathcal{P}_{\ell_\alpha d_l} & \mathcal{P}_{\ell_\alpha \gamma} & \mathcal{P}_{\ell_\alpha \overline{d}_m} & \mathcal{P}_{\ell_\alpha \overline{u}_n} & \mathcal{P}_{\ell_\alpha \overline{\ell}_\delta} \\ \mathcal{P}_{u_j \ell_\gamma} & \mathcal{P}_{u_j u_e} & \mathcal{P}_{u_j d_l} & \mathcal{P}_{u_j \gamma} & \mathcal{P}_{u_j \overline{d}_m} & \mathcal{P}_{u_j \overline{u}_n} & \mathcal{P}_{u_j \overline{\ell}_\delta} \\ \mathcal{P}_{d_i \ell_\gamma} & \mathcal{P}_{d_i u_e} & \mathcal{P}_{d_i d_l} & \mathcal{P}_{d_i \gamma} & \mathcal{P}_{d_i \overline{d}_m} & \mathcal{P}_{d_i \overline{u}_n} & \mathcal{P}_{d_i \overline{\ell}_\delta} \\ \mathcal{P}_{\gamma \ell_\gamma} & \mathcal{P}_{\gamma u_e} & \mathcal{P}_{\gamma d_l} & \mathcal{P}_{\gamma \gamma} & \mathcal{P}_{\gamma \overline{d}_m} & \mathcal{P}_{\gamma \overline{u}_n} & \mathcal{P}_{\gamma \overline{\ell}_\delta} \\ \mathcal{P}_{\overline{d}_k \ell_\gamma} & \mathcal{P}_{\overline{d}_k u_e} & \mathcal{P}_{\overline{d}_k d_l} & \mathcal{P}_{\overline{d}_k \gamma} & \mathcal{P}_{\overline{d}_k \overline{d}_m} & \mathcal{P}_{\overline{d}_k \overline{u}_n} & \mathcal{P}_{\overline{d}_k \overline{\ell}_\delta} \\ \mathcal{P}_{\overline{u}_h \ell_\gamma} & \mathcal{P}_{\overline{u}_h u_e} & \mathcal{P}_{\overline{u}_h d_l} & \mathcal{P}_{\overline{u}_h \gamma} & \mathcal{P}_{\overline{u}_h \overline{d}_m} & \mathcal{P}_{\overline{u}_h \overline{u}_n} & \mathcal{P}_{\overline{u}_h \overline{\ell}_\delta} \\ \mathcal{P}_{\overline{\ell}_\beta \ell_\gamma} & \mathcal{P}_{\overline{\ell}_\beta u_e} & \mathcal{P}_{\overline{\ell}_\beta d_l} & \mathcal{P}_{\overline{\ell}_\beta \gamma} & \mathcal{P}_{\overline{\ell}_\beta \overline{d}_m} & \mathcal{P}_{\overline{\ell}_\beta \overline{u}_n} & \mathcal{P}_{\overline{\ell}_\beta \overline{\ell}_\delta} \end{pmatrix} \begin{pmatrix} \ell_\gamma \\ u_e \\ d_l \\ \gamma \\ \overline{d}_m \\ \overline{u}_n \\ \overline{\ell}_\delta \end{pmatrix} , \tag{1.1}$$

where the index l runs over the down-type quarks, the index m over the down-type anti-quarks, the index e runs over the up-type quarks, the index n over the up-type anti-quarks, the index γ over the leptons, and the index δ over the anti-leptons. In addition, the Mellin convolution integral between the splitting-function matrix and the vector of distributions in the r.h.s. is understood. Despite in general all entries of the splitting-function matrix are different from zero, in massless QED one can identify the following equalities for the splitting functions that that involve a photon:

$$\begin{aligned} \mathcal{P}_{\ell_\alpha \gamma} &= \mathcal{P}_{\overline{\ell}_\beta \gamma} \equiv \mathcal{P}_{\ell \gamma} , \\ \mathcal{P}_{u_j \gamma} &= \mathcal{P}_{\overline{u}_h \gamma} \equiv \mathcal{P}_{u \gamma} , \\ \mathcal{P}_{d_i \gamma} &= \mathcal{P}_{\overline{d}_k \gamma} \equiv \mathcal{P}_{d \gamma} , \end{aligned} \tag{1.2}$$

and:

$$\begin{aligned} \mathcal{P}_{\gamma \ell_\gamma} &= \mathcal{P}_{\gamma \overline{\ell}_\delta} \equiv \mathcal{P}_{\gamma \ell} , \\ \mathcal{P}_{\gamma u_e} &= \mathcal{P}_{\gamma \overline{u}_n} \equiv \mathcal{P}_{\gamma u} , \\ \mathcal{P}_{\gamma d_l} &= \mathcal{P}_{\gamma \overline{d}_m} \equiv \mathcal{P}_{\gamma d} . \end{aligned} \tag{1.3}$$

The splitting functions that only involve fermions (quarks and/or leptons) instead obey the following decompositions:

$$\begin{aligned}
&\mathcal{P}_{\ell_\alpha \ell_\gamma} = \mathcal{P}_{\bar{\ell}_\alpha \bar{\ell}_\gamma} \equiv \delta_{\alpha\gamma}\mathcal{P}^V_{\ell\ell} + \mathcal{P}^S_{\ell\ell}\,,\\
&\mathcal{P}_{\bar{\ell}_\beta \ell_\gamma} \equiv \mathcal{P}_{\ell_\beta \bar{\ell}_\gamma} = \delta_{\beta\gamma}\mathcal{P}^V_{\ell\bar{\ell}} + \mathcal{P}^S_{\ell\bar{\ell}}\,,\\
&\mathcal{P}_{u_j u_e} = \mathcal{P}_{\bar{u}_j \bar{u}_e} \equiv \delta_{je} P^V_{uu} + P^S_{uu}\,,\\
&\mathcal{P}_{\bar{u}_h u_e} = \mathcal{P}_{u_h \bar{u}_e} \equiv \delta_{he} P^V_{u\bar{u}} + P^S_{u\bar{u}}\,,\\
&\mathcal{P}_{d_i d_l} = \mathcal{P}_{\bar{d}_i \bar{d}_l} \equiv \delta_{il} P^V_{dd} + P^S_{dd}\,,\\
&\mathcal{P}_{\bar{d}_k d_l} = \mathcal{P}_{d_k \bar{d}_l} \equiv \delta_{kl} P^V_{d\bar{d}} + P^S_{d\bar{d}}\,,\\
&\mathcal{P}_{\ell_\alpha u_e} = \mathcal{P}_{\bar{\ell}_\alpha \bar{u}_e} \equiv \mathcal{P}^S_{\ell u}\,,\\
&\mathcal{P}_{\ell_\alpha d_l} = \mathcal{P}_{\bar{\ell}_\alpha \bar{d}_l} \equiv \mathcal{P}^S_{\ell d}\,,\\
&\mathcal{P}_{\ell_\alpha \bar{u}_e} = \mathcal{P}_{\bar{\ell}_\alpha u_e} \equiv \mathcal{P}^S_{\ell\bar{u}}\,,\\
&\mathcal{P}_{\ell_\alpha \bar{d}_l} = \mathcal{P}_{\bar{\ell}_\alpha d_l} \equiv \mathcal{P}^S_{\ell\bar{d}}\,,\\
&\mathcal{P}_{u_e \ell_\alpha} = \mathcal{P}_{\bar{u}_e \bar{\ell}_\alpha} \equiv \mathcal{P}^S_{u\ell}\,,\\
&\mathcal{P}_{d_l \ell_\alpha} = \mathcal{P}_{\bar{d}_l \bar{\ell}_\alpha} \equiv \mathcal{P}^S_{d\ell}\,,\\
&\mathcal{P}_{\bar{u}_e \ell_\alpha} = \mathcal{P}_{u_e \bar{\ell}_\alpha} \equiv \mathcal{P}^S_{\bar{u}\ell}\,,\\
&\mathcal{P}_{\bar{d}_l \ell_\alpha} = \mathcal{P}_{d_l \bar{\ell}_\alpha} \equiv \mathcal{P}^S_{\bar{d}\ell}\,,\\
&\mathcal{P}_{u_l d_k} = \mathcal{P}_{d_l u_k} = \mathcal{P}_{\bar{u}_l \bar{d}_k} = \mathcal{P}_{\bar{d}_l \bar{u}_k} \equiv \mathcal{P}^S_{ud}\,,\\
&\mathcal{P}_{u_l \bar{d}_k} = \mathcal{P}_{\bar{d}_l u_k} = \mathcal{P}_{\bar{u}_l d_k} = \mathcal{P}_{d_l \bar{u}_k} \equiv \mathcal{P}^S_{u\bar{d}}\,.
\end{aligned} \tag{1.4}$$

Each one of splitting functions on the r.h.s. of the $\equiv$ symbol above admits a perturbative expansion that truncated to next-to-leading order (as appropriate to achieve NLL accuracy at the level of the solution of the evolution equation), reads:

$$\mathcal{P} = \left(\frac{\alpha}{4\pi}\right)\mathcal{P}^{(0)} + \left(\frac{\alpha}{4\pi}\right)^2\mathcal{P}^{(1)}\,. \tag{1.5}$$

The coefficients $\mathcal{P}^{(0)}$ and $\mathcal{P}^{(1)}$ undergo further simplifications. Let us start with the leading order one $\mathcal{P}^{(0)}$. At this order all of the pure-singlet contributions vanish, that is $\mathcal{P}^{(0),S}_{xy} = 0$ for all pairs xy, and so do the valence fermion/anti-fermion splitting functions vanish, *i.e.* $\mathcal{P}^{(0),V}_{\ell\bar{\ell}} = \mathcal{P}^{(0),V}_{u\bar{u}} = \mathcal{P}^{(0),V}_{d\bar{d}} = 0$. In addition, the splitting functions that survive obey the following equalities:

$$\begin{aligned}
&\mathcal{P}^{(0)}_{\ell\gamma} = e^2_\ell P^{(0)}_{f\gamma}\,,\\
&\mathcal{P}^{(0)}_{u\gamma} = e^2_u P^{(0)}_{f\gamma}\,,\\
&\mathcal{P}^{(0)}_{d\gamma} = e^2_d P^{(0)}_{f\gamma}\,,\\[1ex]
&\mathcal{P}^{(0)}_{\gamma\ell} = e^2_\ell P^{(0)}_{\gamma f}\,,\\
&\mathcal{P}^{(0)}_{\gamma u} = e^2_u P^{(0)}_{\gamma f}\,,\\
&\mathcal{P}^{(0)}_{\gamma d} = e^2_d P^{(0)}_{\gamma f}\,,\\[1ex]
&\mathcal{P}^{V,(0)}_{\ell\ell} = e^2_\ell P^{V,(0)}_{ff}\,,\\
&\mathcal{P}^{V,(0)}_{uu} = e^2_u P^{V,(0)}_{ff}\,,\\
&\mathcal{P}^{V,(0)}_{dd} = e^2_d P^{V,(0)}_{ff}\,,
\end{aligned} \tag{1.6}$$

where $e^2_\ell = 1$, $e^2_u = 4/9$, and $e^2_d = 1/0$ are the electric charges of up- and down-type quarks, respectively.

Let us now move to the next-to-leading order coefficient $\mathcal{P}^{(1)}$. At this order, the pure-single splitting functions are different from zero but all the same up to an overall factor:

$$\mathcal{P}^{S,(1)}_{xy} = e^2_x e^2_y P^{S,(1)}_{ff} \tag{1.7}$$

where x and y are either a lepton, or a down-type quark, or an up-type quark. For the other splitting functions one finds:

$$
\begin{aligned}
\mathcal{P}^{(1)}_{\ell\gamma} &= e_\ell^4 P^{(1)}_{f\gamma}\,,\\
\mathcal{P}^{(1)}_{u\gamma} &= e_u^4 P^{(1)}_{f\gamma}\,,\\
\mathcal{P}^{(1)}_{d\gamma} &= e_d^4 P^{(1)}_{f\gamma}\,,\\
\\
\mathcal{P}^{(1)}_{\gamma\ell} &= e_\ell^4 P^{(1)}_{\gamma f} + e_\ell^2 P^{(1)}_{\gamma f,n}\,,\\
\mathcal{P}^{(1)}_{\gamma u} &= e_u^4 P^{(1)}_{\gamma f} + e_u^2 P^{(1)}_{\gamma f,n}\,,\\
\mathcal{P}^{(1)}_{\gamma d} &= e_d^4 P^{(1)}_{\gamma f} + e_d^2 P^{(1)}_{\gamma f,n}\,,\\
\\
\mathcal{P}^{V,(1)}_{\ell\ell} &= e_\ell^4 P^{V,(1)}_{ff} + e_\ell^2 P^{V,(1)}_{ff,n}\,,\\
\mathcal{P}^{V,(1)}_{uu} &= e_u^4 P^{V,(1)}_{ff} + e_u^2 P^{V,(1)}_{ff,n}\,,\\
\mathcal{P}^{V,(1)}_{dd} &= e_d^4 P^{V,(1)}_{ff} + e_d^2 P^{V,(1)}_{ff,n}\,,\\
\\
\mathcal{P}^{V,(1)}_{\ell\overline{\ell}} &= e_\ell^4 P^{V,(1)}_{f\overline{f}}\,,\\
\mathcal{P}^{V,(1)}_{u\overline{u}} &= e_u^4 P^{V,(1)}_{f\overline{f}}\,,\\
\mathcal{P}^{V,(1)}_{d\overline{d}} &= e_d^4 P^{V,(1)}_{f\overline{f}}\,.
\end{aligned}
\tag{1.8}
$$

We are now in a position to rewrite the evolution system in Eq. (1.1) exploiting the equalities identified above:

$$
\begin{aligned}
\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}\ell^+_\alpha\\ u^+_j\\ d^+_i\\ \gamma\end{pmatrix} = \Bigg[&\left(\frac{\alpha}{4\pi}\right)\begin{pmatrix}
e_\ell^2 P^{V,(0)}_{ff} & 0 & 0 & 2e_\ell^2 P^{(0)}_{f\gamma}\\
0 & e_u^2 P^{V,(0)}_{ff} & 0 & 2e_u^2 P^{(0)}_{f\gamma}\\
0 & 0 & e_d^2 P^{V,(0)}_{ff} & 2e_d^2 P^{(0)}_{f\gamma}\\
e_\ell^2 P^{(0)}_{\gamma f} & e_u^2 P^{(0)}_{\gamma f} & e_d^2 P^{(0)}_{\gamma f} & P^{(0)}_{\gamma\gamma}
\end{pmatrix}\begin{pmatrix}\ell^+_\alpha\\ u^+_j\\ d^+_i\\ \gamma\end{pmatrix}\\
&+\left(\frac{\alpha}{4\pi}\right)^2\begin{pmatrix}
e_\ell^4 P^{V,(1)}_{ff} + e_\ell^2 P^{V,(1)}_{ff,n} + e_\ell^4 P^{V,(1)}_{f\overline{f}} & 0 & 0 & 2e_\ell^4 P^{(1)}_{f\gamma}\\
0 & e_u^4 P^{V,(1)}_{ff} + e_u^2 P^{V,(1)}_{ff,n} + e_u^4 P^{V,(1)}_{f\overline{f}} & 0 & 2e_u^4 P^{(1)}_{f\gamma}\\
0 & 0 & e_d^4 P^{V,(1)}_{ff} + e_d^2 P^{V,(1)}_{ff,n} + e_d^4 P^{V,(1)}_{f\overline{f}} & 2e_d^4 P^{(1)}_{f\gamma}\\
e_\ell^4 P^{(1)}_{\gamma f} + e_\ell^2 P^{(1)}_{\gamma f,n} & e_u^4 P^{(1)}_{\gamma f} + e_u^2 P^{(1)}_{\gamma f,n} & e_d^4 P^{(1)}_{\gamma f} + e_d^2 P^{(1)}_{\gamma f,n} & P^{(1)}_{\gamma\gamma}
\end{pmatrix}\begin{pmatrix}\ell^+_\alpha\\ u^+_j\\ d^+_i\\ \gamma\end{pmatrix}\\
&+\left(\frac{\alpha}{4\pi}\right)^2 2P^{S,(1)}_{ff}\begin{pmatrix}
e_\ell^4 & e_\ell^2 e_u^2 & e_\ell^2 e_d^2 & 0\\
e_u^2 e_\ell^2 & e_u^4 & e_u^2 e_d^2 & 0\\
e_d^2 e_\ell^2 & e_d^2 e_u^2 & e_d^4 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix}\Sigma_\ell\\ \Sigma_u\\ \Sigma_d\\ \gamma\end{pmatrix}\Bigg]\,,
\end{aligned}
\tag{1.9}
$$

$$
\begin{aligned}
\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}d^-_i\\ u^-_j\\ \ell^-_\alpha\end{pmatrix} = \Bigg[&\left(\frac{\alpha}{4\pi}\right)\begin{pmatrix}
e_d^2 P^{V,(0)}_{ff} & 0 & 0\\
0 & e_u^2 P^{V,(0)}_{ff} & 0\\
0 & 0 & e_\ell^2 P^{V,(0)}_{ff}
\end{pmatrix}\begin{pmatrix}d^-_i\\ u^-_j\\ \ell^-_\alpha\end{pmatrix}\\
&+\left(\frac{\alpha}{4\pi}\right)^2\begin{pmatrix}
e_d^4 P^{V,(1)}_{ff} + e_d^2 P^{V,(1)}_{ff,n} - e_d^4 P^{V,(1)}_{f\overline{f}} & 0 & 0\\
0 & e_u^4 P^{V,(1)}_{ff} + e_u^2 P^{V,(1)}_{ff,n} - e_u^4 P^{V,(1)}_{f\overline{f}} & 0\\
0 & 0 & e_\ell^4 P^{V,(1)}_{ff} + e_\ell^2 P^{V,(1)}_{ff,n} - e_\ell^4 P^{V,(1)}_{f\overline{f}}
\end{pmatrix}\begin{pmatrix}d^-_i\\ u^-_j\\ \ell^-_\alpha\end{pmatrix}\Bigg]\,,
\end{aligned}
\tag{1.10}
$$

where we have used the following definitions:

$$
\begin{aligned}
\ell^\pm_\alpha &= \ell_\alpha \pm \overline{\ell}_\alpha\\
u^\pm_j &= u_j \pm \overline{u}_j\\
d^\pm_i &= d_i \pm \overline{d}_i
\end{aligned}
\tag{1.11}
$$

and also introduced:

$$
\Sigma_\ell = \sum_{\alpha=e,\mu,\tau} \ell^+_\alpha \qquad \Sigma_u = \sum_{k=i}^{N_c n_u} u^+_k \qquad \Sigma_d = \sum_{k=i}^{N_c n_d} d^+_k\,. \tag{1.12}
$$

For completeness we also introduce:

$$
V_\ell = \sum_{\alpha=e,\mu,\tau} \ell^-_\alpha \qquad V_u = \sum_{k=i}^{N_c n_u} u^-_k \qquad V_d = \sum_{k=i}^{N_c n_d} d^-_k\,. \tag{1.13}
$$

Notice that quarks are summed also over their colour degree of freedom.

In order to diagonalise as much as possible the evolution matrix in the presence of QED corrections avoiding unnecessary couplings between parton distributions, we propose the following evolution basis:

$$\begin{array}{ll} 1)\ \gamma & \\ 2)\ \Sigma_\ell & 11)\ V_\ell \\ 3)\ \Sigma_u & 12)\ V_u \\ 4)\ \Sigma_d & 13)\ V_d \\ 5)\ T_1^\ell = \ell_e^+ - \ell_\mu^+ & 14)\ T_2^\ell = \ell_e^+ + \ell_\mu^+ - 2\ell_\tau^+ \\ 6)\ V_1^\ell = \ell_e^- - \ell_\mu^- & 15)\ V_2^\ell = \ell_e^- + \ell_\mu^- - 2\ell_\tau^- \\ 7)\ T_1^u = u^+ - c^+ & 16)\ V_1^u = u^- - c^- \\ 8)\ T_2^u = u^+ + c^+ - 2t^+ & 17)\ V_2^u = u^- + c^- - 2t^- \\ 9)\ T_1^d = d^+ - s^+ & 18)\ V_1^d = d^- - s^- \\ 10)\ T_2^d = d^+ + s^+ - 2b^+ & 19)\ V_2^d = d^- + s^- - 2b^- \end{array} \tag{1.14}$$

we the introduce the following splitting functions:

$$\begin{aligned} P_p^\pm &= \left(\frac{\alpha}{4\pi}\right) e_p^2 P_{ff}^{V,(0)} + \left(\frac{\alpha}{4\pi}\right)^2 \left[e_p^4 P_{ff}^{V,(1)} + e_p^2 P_{ff,n}^{V,(1)} \pm e_p^4 P_{f\bar{f}}^{V,(1)}\right] \\ P_{p\gamma} &= \left(\frac{\alpha}{4\pi}\right) 2N_c^{(p)} n_p e_p^2 P_{f\gamma}^{V,(0)} + \left(\frac{\alpha}{4\pi}\right)^2 2N_c^{(p)} n_p e_p^4 P_{f\gamma}^{V,(1)} \\ P_{\gamma p} &= \left(\frac{\alpha}{4\pi}\right) e_p^2 P_{\gamma f}^{V,(0)} + \left(\frac{\alpha}{4\pi}\right)^2 \left[e_p^4 P_{\gamma f}^{V,(1)} + e_p^2 P_{\gamma f,n}^{V,(1)}\right] \\ P_{pp'}^{\mathrm{PS}} &= \left(\frac{\alpha}{4\pi}\right)^2 2N_c^{(p)} n_p e_p^2 e_{p'}^2 P_{ff}^{S,(1)} \\ P_{pp} &= P_p^+ + P_{pp}^{\mathrm{PS}} \\ P_{\gamma\gamma} &= \left(\frac{\alpha}{4\pi}\right) P_{\gamma\gamma}^{V,(0)} + \left(\frac{\alpha}{4\pi}\right)^2 P_{\gamma\gamma}^{V,(1)} \end{aligned} \tag{1.15}$$

with $p, p' = \ell, u, d$, and $N_c^{(\ell)} = 1$ and $N_c^{(u)} = N_c^{(d)} = N_c = 3$. With these definitions at hand we can finally write the full set of systems of evolution equation using the basis given in Eq. (1.14):

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} \gamma \\ \Sigma_\ell \\ \Sigma_u \\ \Sigma_d \end{pmatrix} = \begin{pmatrix} P_{\gamma\gamma} & P_{\gamma\ell} & P_{\gamma u} & P_{\gamma d} \\ P_{\ell\gamma} & P_{\ell\ell} & P_{\ell u}^{\mathrm{PS}} & P_{\ell d}^{\mathrm{PS}} \\ P_{u\gamma} & P_{u\ell}^{\mathrm{PS}} & P_{uu} & P_{ud}^{\mathrm{PS}} \\ P_{d\gamma} & P_{d\ell}^{\mathrm{PS}} & P_{du}^{\mathrm{PS}} & P_{dd} \end{pmatrix} \begin{pmatrix} \gamma \\ \Sigma_\ell \\ \Sigma_u \\ \Sigma_d \end{pmatrix} \tag{1.16}$$

$$\mu^2 \frac{\partial V_p}{\partial \mu^2} = P_p^- V_p \tag{1.17}$$

$$\mu^2 \frac{\partial T_{1,2}^p}{\partial \mu^2} = P_p^+ T_{1,2}^p \tag{1.18}$$

$$\mu^2 \frac{\partial V_{1,2}^p}{\partial \mu^2} = P_p^- T_{1,2}^p \tag{1.19}$$

2 Threshold crossing

In order to implement the evolution in the variable-flavour-number scheme (VFNS), it is necessary to introduce thresholds. Thresholds need to be ordered in value. To do so, we use the PDG mass values:

$$\begin{aligned} m_e &= 0.5109989461 \cdot 10^{-3}\ \mathrm{GeV} \\ m_u &= 2.16 \cdot 10^{-3}\ \mathrm{GeV} \\ m_d &= 4.67 \cdot 10^{-3}\ \mathrm{GeV} \\ m_s &= 0.093\ \mathrm{GeV} \\ m_\mu &= 0.1056583745\ \mathrm{GeV} \\ m_c &= 1.27\ \mathrm{GeV} \\ m_\tau &= 1.77686\ \mathrm{GeV} \\ m_b &= 4.18\ \mathrm{GeV} \\ m_t &= 172.76\ \mathrm{GeV} \end{aligned} \tag{2.1}$$

Assuming this order, the number of active leptons n_ℓ, dow-type quark n_d and up-type quarks n_u changes as follows as the scale increases. In addition, while Eq. (1.16) and (1.17) are valid as they are at all scales, the $T^p_{1,2}$ and $V^p_{1,2}$ evolves as in Eqs. (1.18) and (1.19) only for $n_p = 3$. Assuming no intrinsic contributions, they reduce to the corresponding singlet Σ_p or total valence V_p distributions:

- $Q < m_e$:
  - $n_\ell = 0$, $n_d = 0$, $n_u = 0$
  - $T_1^\ell = T_2^\ell = \Sigma_\ell$
  - $T_1^d = T_2^d = \Sigma_d$
  - $T_1^u = T_2^u = \Sigma_u$
- $m_e \leq Q < m_u$:
  - $n_\ell = 1$, $n_d = 0$, $n_u = 0$
  - $T_1^\ell = T_2^\ell = \Sigma_\ell$
  - $T_1^d = T_2^d = \Sigma_d$
  - $T_1^u = T_2^u = \Sigma_u$
- $m_u \leq Q < m_d$:
  - $n_\ell = 1$, $n_d = 0$, $n_u = 1$
  - $T_1^\ell = T_2^\ell = \Sigma_\ell$
  - $T_1^d = T_2^d = \Sigma_d$
  - $T_1^u = T_2^u = \Sigma_u$
- $m_d \leq Q < m_s$:
  - $n_\ell = 1$, $n_d = 1$, $n_u = 1$
  - $T_1^\ell = T_2^\ell = \Sigma_\ell$
  - $T_1^d = T_2^d = \Sigma_d$
  - $T_1^u = T_2^u = \Sigma_u$
- $m_s \leq Q < m_\mu$:
  - $n_\ell = 1$, $n_d = 2$, $n_u = 1$
  - $T_1^\ell = T_2^\ell = \Sigma_\ell$
  - $T_2^d = \Sigma_d$
  - $T_1^u = T_2^u = \Sigma_u$
- $m_\mu \leq Q < m_c$:
  - $n_\ell = 2$, $n_d = 2$, $n_u = 1$
  - $T_2^\ell = \Sigma_\ell$
  - $T_2^d = \Sigma_d$
  - $T_1^u = T_2^u = \Sigma_u$
- $m_c \leq Q < m_\tau$:
  - $n_\ell = 2$, $n_d = 2$, $n_u = 2$
  - $T_2^\ell = \Sigma_\ell$
  - $T_2^d = \Sigma_d$
  - $T_2^u = \Sigma_u$
- $m_\tau \leq Q < m_b$:

  - $n_\ell = 3$, $n_d = 2$, $n_u = 2$
  - $T_2^d = \Sigma_d$
  - $T_2^u = \Sigma_u$

- $m_b \leq Q < m_t$:
  - $n_\ell = 3$, $n_d = 3$, $n_u = 2$
  - $T_2^u = \Sigma_u$

- $Q \geq m_t$:
  - $n_\ell = 3$, $n_d = 3$, $n_u = 3$

3 The Δ scheme

In order to implemet the Δ scheme, we need to apply a tranformation to the matrix of evolution kernels. The transformed system of evolution equations reads:

$$\mu^2 \frac{d}{d\mu^2}\mathbf{f} = \beta(a(\mu))\mathbb{J}(1 + a(\mu)\mathbb{J})^{-1} + (1 + a(\mu)\mathbb{J})\mathcal{P}(a(\mu))(1 + a(\mu)\mathbb{J})^{-1}\mathbf{f}\,, \tag{3.1}$$

where $\mathbf{f}$ is the vector of PDFs, $\mathcal{P}$ the matrix of splitting functions, and $\mathbb{J}$ the matrix that parametrises the transformation from the $\overline{\text{MS}}$ to the Δ scheme. Since the evolution is effectivelly written as differential in a, one finds that:

$$\frac{d}{da}\mathbf{f} = \mathbb{J}(1 + a(\mu)\mathbb{J})^{-1} + (1 + a(\mu)\mathbb{J})\frac{\mathcal{P}(a(\mu))}{\beta(a(\mu))}(1 + a(\mu)\mathbb{J})^{-1}\mathbf{f}\,. \tag{3.2}$$

Now we need to specify the form of the matrix $\mathbb{J}$. In the evolution basis in Eq. (1.1), the explicit form of $\mathbb{J}$ is:

$$\mathbb{J} = \begin{pmatrix} \delta_{\ell_\alpha e^-}\delta_{\ell_\gamma e^-}J_{\ell\ell} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_{\ell_\gamma e^-}J_{\gamma\ell} & 0 & 0 & 0 & 0 & 0 & \delta_{\ell_\delta e^+}J_{\gamma\ell} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_{\ell_\beta e^+}\delta_{\ell_\delta e^+}J_{\ell\ell} \end{pmatrix}. \tag{3.3}$$

Finally, in the Δ scheme, the system of evolution equations takes the form:

$$\frac{\partial}{\partial a}\begin{pmatrix}\gamma \\ \Sigma_\ell \\ \Sigma_u \\ \Sigma_d\end{pmatrix} = \left[\begin{pmatrix} 0 & \frac{J_{\gamma\ell}}{1+aJ_{\ell\ell}} & 0 & 0 \\ 0 & \frac{J_{\ell\ell}}{1+aJ_{\ell\ell}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{\beta}\begin{pmatrix} 1 & aJ_{\gamma\ell} & 0 & 0 \\ 0 & 1+aJ_{\ell\ell} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} P_{\gamma\gamma} & P_{\gamma\ell} & P_{\gamma u} & P_{\gamma d} \\ P_{\ell\gamma} & P_{\ell\ell} & P^{\text{PS}}_{\ell u} & P^{\text{PS}}_{\ell d} \\ P_{u\gamma} & P^{\text{PS}}_{u\ell} & P_{uu} & P^{\text{PS}}_{ud} \\ P_{d\gamma} & P^{\text{PS}}_{d\ell} & P^{\text{PS}}_{du} & P_{dd} \end{pmatrix}\begin{pmatrix} 1 & -\frac{aJ_{\gamma\ell}}{1+aJ_{\ell\ell}} & 0 & 0 \\ 0 & \frac{1}{1+aJ_{\ell\ell}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix}\gamma \\ \Sigma_\ell \\ \Sigma_u \\ \Sigma_d\end{pmatrix} \tag{3.4}$$

$$\frac{\partial V_p}{\partial a} = \left[\frac{J_{\ell\ell}}{1+aJ_{\ell\ell}} + \frac{P_p^-}{\beta}\right]V_p \tag{3.5}$$

$$\frac{\partial T^p_{1,2}}{\partial a} = \left[\frac{J_{\ell\ell}}{1+aJ_{\ell\ell}} + \frac{P_p^+}{\beta}\right]T^p_{1,2} \tag{3.6}$$

$$\frac{\partial V^p_{1,2}}{\partial a} = \left[\frac{J_{\ell\ell}}{1+aJ_{\ell\ell}} + \frac{P_p^-}{\beta}\right]T^p_{1,2} \tag{3.7}$$

4 Evolution using the α_{M_Z} scheme

The evolution equations when using the α_{M_Z} scheme takes the following form:

$$\frac{d\mathbf{f}}{d\ln\mu^2} = \left[\left(\frac{\alpha_{M_Z}}{4\pi}\right)\mathbf{P}^{(0)} + \left(\frac{\alpha_{M_Z}}{4\pi}\right)^2\widetilde{\mathbf{P}}^{(1)}(\mu)\right]\mathbf{f}\,, \tag{4.1}$$

where α_{M_Z} is a constant, $\mathbf{P}^{(0)}$ is the usual (μ-independent) one-loop splitting function, while $\widetilde{\mathbf{P}}^{(1)}(\mu)$ does depend on μ and its expression reads:

$$\widetilde{\mathbf{P}}^{(1)}(\mu) = \mathbf{P}^{(1)} + \frac{20}{9} C^{(2)} \mathbf{P}^{(0)} + \beta_0 \mathbf{P}^{(0)} \ln \frac{\mu^2}{M_Z^2} \,, \tag{4.2}$$

where $\mathbf{P}^{(1)}$ is now the usual two-loop splitting function and:

$$C^{(2)} = N_C n_u e_u^2 + N_C n_d e_d^2 + n_\ell e_\ell^2 \,. \tag{4.3}$$

Given the explicit dependence on μ, Eq. (4.1) admits a simple closed-form solution that reads:

$$\mathbf{f}(\mu) = \mathbf{f}(\mu_0) \exp\left\{ \left[\left(\frac{\alpha_{M_Z}}{4\pi}\right) \mathbf{P}^{(0)} + \left(\frac{\alpha_{M_Z}}{4\pi}\right)^2 \left(\mathbf{P}^{(1)} + \frac{20}{9} C^{(2)} \mathbf{P}^{(0)} + \beta_0 \mathbf{P}^{(0)} \ln \frac{\mu_0 \mu}{M_Z^2} \right) \right] \ln \frac{\mu^2}{\mu_0^2} \right\} \,. \tag{4.4}$$