Structure functions

Contents

In this part the technology developed to numerically compute Mellin convolutions exploiting interpolation functions will be applied to the computation of totally inclusive structure functions. On top of the numerical aspects of the computation, more "theoretical" aspects will be touched such as the computation of scale variations and of the target-mass corrections.

1 Zero-mass structure functions

A generic structure function in the zero-mass (ZM) scheme, in which the finitness of the quark mass is neglected, computed at the value x of the Bjorken variable and at the (absolute) value Q of the vector-boson virtuality is given by the following Mellin convolution:

$$F(x,Q) = \sum_{i=g,q} x \int_x^1 \frac{dy}{y} C_i\left(y, \alpha_s(Q)\right) q_i\left(\frac{x}{y}, Q\right) = \sum_{i=g,q} \int_x^1 dy\, C_i\left(y, \alpha_s(Q)\right) \frac{x}{y} q_i\left(\frac{x}{y}, Q\right). \tag{1.1}$$

The functions C_i, often dubbed coefficient functions, are perturbatively computable and thus, to N-th order, admit the expansion:

$$C_i\left(y, \alpha_s(Q)\right) = \sum_{n=0}^{N} \left(\frac{\alpha_s(Q)}{4\pi}\right)^n C_i^{(n)}(x)\,. \tag{1.2}$$

The integral in Eq. (1.1) can be computed using the numerical techniques developed in the part of this documentation devoted to the structure of the Mellin-convolution integrals. Specifically, introducing a grid $\{x_\alpha\}$ having a set of interpolating functions of degree k, $\{w_\alpha^{(k)}(y)\}$, associated, the structure function on the grid point is computed as:

$$F(x_\beta,Q) = \sum_{i=g,q} \sum_{\alpha=0}^{N_x} \sum_{n=0}^{N} \left(\frac{\alpha_s(Q)}{4\pi}\right)^n \underbrace{\int_{\max(x_\beta, x_\beta/x_{\alpha+1})}^{\min(1, x_\beta/x_{\alpha-k})} dy\, C_i^{(n)}(y)\, w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right)}_{\Gamma_{i,\beta\alpha}^{(n)}} \overline{q}_\alpha\,. \tag{1.3}$$

with $\overline{q}_\alpha = x_\alpha q(x_\alpha, Q)$. The value of the structure function for any generic value of x can then be obtained through interpolation. The integrals $\Gamma_{i,\beta\alpha}^{(n)}$ are efficiently computed according to the procedure discussed in the part on the integration. Notice, that this integrals are totally scale independent in that the Q dependence of the structure functions is totally driven by α_s and the PDFs q. This allows for a pre-computation of these integrals that can be used to compute structure functions at any scale Q.

Before moving to considering the massive structure functions, it is useful to point out that in the ZM scheme there is no need to distinguish between charged- and neutral-current coefficient functions. In other words, when considering the exchange of a neutral vector boson Z/γ^* or of a charged vector boson $W^\pm$, the difference in

functional form of the coefficient functions C_i only amounts to an overall factor associated to the electroweak charge.

2 Massive structure functions

When computing structure functions retaining the mass of the quarks, the convolution integrals become more complicated. In addition, contrary to the massless case, the single precomputed intergrals $\Gamma^{(n)}_{i,\beta\alpha}$, introduced in Eq. (1.3), will depend on the scale Q. This prevents a scale independent pre-computation of the coefficient functions on the x-space grid. A possible solution to this problem relies on interpolation. Specifically, the integrals $\Gamma^{(n)}_{i,\beta\alpha}$ are tabulated over a grid in the variable:

$$\xi = \frac{Q^2}{m_H^2}\,, \tag{2.1}$$

where m_H is the mass of the heavy quark under consideration, and subsequently interpolated to obtain the structure function for any value of Q.

2.1 Neutral Current Coefficient Functions

As far as the neutral-current coefficient functions are concerned, explicit expressions beyond $\mathcal{O}(\alpha_s)$ are complicated and thus not suitable for fast numerical computations. Fortunately, the authors of Ref. [1] have tabulated these functions and published the corresponding interpolating routines. It is these routines that APFEL++ uses for the $\mathcal{O}(\alpha_s^2)$ corrections to the massive neutral current coefficient functions. The only exception is the pure-singlet $\mathcal{O}(\alpha_s^2)$ coefficient functions (sometimes called gluon-radiation terms) in which case the analytical expressions given in Appendix A of Ref. [2] are used. At $\mathcal{O}(\alpha_s)$ the analytic expressions are instead used.[1]

In the implementation of a so-called General-Mass (GM) scheme, it is usually required to know the massless limit of the massive coefficient functions analytically. To be more precise, this limit prescribes to set to zero all mass-suppressed contributions and retain only the mass-independent and the logarithmically enhanced contributions. In this case, exact expressions up to $\mathcal{O}(\alpha_s^2)$ have been computed in Ref. [2] and reported in Appendix D of this paper. These expressions are implemented in APFEL++.

As a final remark, the massive coefficient functions for the neutral current structure functions are presently known only for F_2 and F_L. For the parity-violating structure function F_3 the massless coefficient functions are instead used. This is usually acceptable because the neutral-current F_3 in the typical kinematics covered by modern experiments is sizeable at large Q, *i.e.* where mass effects are negligible.

2.2 Charged Current Coefficient Functions

We can now consider the charged-current sector in which massive coefficient functions are know up to $\mathcal{O}(\alpha_s^2)$. However, the $\mathcal{O}(\alpha_s^2)$ corrections, recently computed in Ref. [4], are not publicly available. Therefore a computation of charged-current structure functions at NNLO in APFEL++ is currently impossible.

The structure functions associated to the heavy quark H in the approximation of diagonal CKM matrix[2] are given in terms of the following convolutions:

$$F_1^H(x,Q,m_H) = \frac{1}{2}\int_\chi^1 \frac{dy}{y}\left[C_{1,q}(y,Q)s\left(\frac{\chi}{y},Q\right) + C_{1,g}(y,Q)g\left(\frac{\chi}{y},Q\right)\right] \tag{2.2}$$

$$F_2^H(x,Q,m_H) = \chi\int_\chi^1 \frac{dy}{y}\left[C_{2,q}(y,Q)s\left(\frac{\chi}{y},Q\right) + C_{2,g}(y,Q)g\left(\frac{\chi}{y},Q\right)\right] \tag{2.3}$$

$$F_3^H(x,Q,m_H) = \int_\chi^1 \frac{dy}{y}\left[C_{3,q}(y,Q)s\left(\frac{\chi}{y},Q\right) + C_{3,g}(y,Q)g\left(\frac{\chi}{y},Q\right)\right] \tag{2.4}$$

with:

$$\chi = x\left(1+\frac{m_H^2}{Q^2}\right) = \frac{x}{\lambda}\,, \tag{2.5}$$

[1] Actually, if neglecting intrinsic heavy-quark components, at $\mathcal{O}(\alpha_s)$ there is one single coefficient function for each structure function, available for example in available in Ref. [3], that is convoluted with the gluon distribution.

[2] This approximation will be released later.

where:

$$\lambda = \frac{Q^2}{Q^2 + m_H^2} = \frac{\xi}{1+\xi}\,, \tag{2.6}$$

with ξ given in Eq. (2.1). Defining:

$$F_L^H(x,Q,m_H) = F_2^H(x,Q,m_H) - 2xF_1^H(x,Q,m_H)\,, \tag{2.7}$$

one has that:

$$F_L^H(x,Q,m_H) = \chi\int_\chi^1 \frac{dy}{y}\left[C_{L,q}(y,Q)s\left(\frac{\chi}{y},Q\right) + C_{L,g}(y,Q)g\left(\frac{\chi}{y},Q\right)\right]\,, \tag{2.8}$$

with:

$$C_{L,q(g)}(y,Q) = C_{2,q(g)}(y,Q) - \lambda C_{1,q(g)}(y,Q) \tag{2.9}$$

As usual, the coefficient functions admit a perturbative expansion that at $\mathrm{N}^N\mathrm{LO}$ reads:

$$C_{k,q(g)}(y,Q) = \sum_{n=0}^{N}\left(\frac{\alpha_s(Q)}{4\pi}\right)^n C_{k,q(g)}^{(n)}(y,\xi)\,,\quad k = 1,2,3,L\,. \tag{2.10}$$

In the following we will truncate the expansion at $N = 1$, *i.e.* at NLO.

At LO the coefficient functions read:

$$\begin{aligned}
C_{1,q}^{(0)}(x,\xi) &= \delta(1-x)\,,\\
C_{2,q}^{(0)}(x,\xi) &= \delta(1-x)\,,\\
C_{3,q}^{(0)}(x,\xi) &= \delta(1-x)\,,\\
C_{L,q}^{(0)}(x,\xi) &= (1-\lambda)\delta(1-x)\,,\\
C_{k,g}^{(0)}(y,\xi) &= 0\,,\quad k = 1,2,3,L\,.
\end{aligned} \tag{2.11}$$

The $\mathcal{O}(\alpha_s)$ (NLO) charged-current massive coefficient have been computed and reported in Appendix A of Ref. [5]. However, their implementation in APFEL++ requires some manipulations. We start by defining:

$$K_A = \frac{1}{\lambda}(1-\lambda)\ln(1-\lambda)\quad\text{and}\quad K_F = \frac{Q}{\mu_F}\,. \tag{2.12}$$

The explicit expressions of the NLO quark coefficient functions then read:

$$\begin{aligned}
C_{1,q}^{(1)} &= 2C_F\Bigg\{\left(-4-\frac{1}{2\lambda}-2\zeta_2-\frac{1+3\lambda}{2\lambda}K_A+\frac{3}{2}\ln\frac{K_F^2}{\lambda}\right)\delta(1-z)\\
&- \frac{(1+z^2)\ln z}{1-z}+\left(-\ln\frac{K_F^2}{\lambda}-2\ln(1-z)+\ln(1-\lambda z)\right)(1+z)+(3-z)+\frac{1}{\lambda^2}+\frac{z-1}{\lambda}\\
&+ 2\left[\frac{2\ln(1-z)-\ln(1-\lambda z)}{1-z}\right]_+ + 2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\left[\frac{1}{1-z}\right]_+\\
&+ \frac{\lambda-1}{\lambda^2}\left[\frac{1}{1-\lambda z}\right]_+ + \frac{1}{2}\left[\frac{1-z}{(1-\lambda z)^2}\right]_+\Bigg\}\,,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
C_{2,q}^{(1)} &= 2C_F\Bigg\{\left(-4-\frac{1}{2\lambda}-2\zeta_2-\frac{1+\lambda}{2\lambda}K_A+\frac{3}{2}\ln\frac{K_F^2}{\lambda}\right)\delta(1-z)\\
&- \frac{(1+z^2)\ln z}{1-z}+\left(2-\ln\frac{K_F^2}{\lambda}-2\ln(1-z)+\ln(1-\lambda z)\right)(1+z)+\frac{1}{\lambda}\\
&+ 2\left[\frac{2\ln(1-z)-\ln(1-\lambda z)}{1-z}\right]_+ + 2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\left[\frac{1}{1-z}\right]_+\\
&+ \frac{2\lambda^2-\lambda-1}{\lambda}\left[\frac{1}{1-\lambda z}\right]_+ + \frac{1}{2}\left[\frac{1-z}{(1-\lambda z)^2}\right]_+\Bigg\}\,,
\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
C_{3,q}^{(1)} &= 2C_F\Bigg\{\Bigg(-4-\frac{1}{2\lambda}-2\zeta_2-\frac{1+3\lambda}{2\lambda}K_A+\frac{3}{2}\ln\frac{K_F^2}{\lambda}\Bigg)\delta(1-z)\\
&- \frac{(1+z^2)\ln z}{1-z}+\left(1-\ln\frac{K_F^2}{\lambda}-2\ln(1-z)+\ln(1-\lambda z)\right)(1+z)+\frac{1}{\lambda}\\
&+ 2\left[\frac{2\ln(1-z)-\ln(1-\lambda z)}{1-z}\right]_+ +2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\left[\frac{1}{1-z}\right]_+\\
&+ \frac{\lambda-1}{\lambda}\left[\frac{1}{1-\lambda z}\right]_+ +\frac{1}{2}\left[\frac{1-z}{(1-\lambda z)^2}\right]_+\Bigg\},\\
C_{L,q}^{(1)} &= 2C_F(1-\lambda)\Bigg\{\Bigg(-4-\frac{1}{2\lambda}-2\zeta_2-\frac{1+\lambda}{2\lambda}K_A+\frac{3}{2}\ln\frac{K_F^2}{\lambda}\Bigg)\delta(1-z)\\
&- \frac{(1+z^2)\ln z}{1-z}+\left(-\ln\frac{K_F^2}{\lambda}-2\ln(1-z)+\ln(1-\lambda z)\right)(1+z)+3\\
&+ 2\left[\frac{2\ln(1-z)-\ln(1-\lambda z)}{1-z}\right]_+ +2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\left[\frac{1}{1-z}\right]_+\\
&- 2\left[\frac{1}{1-\lambda z}\right]_+ +\frac{1}{2}\left[\frac{1-z}{(1-\lambda z)^2}\right]_+\Bigg\}+2C_F\left[\lambda K_A\delta(1-z)+(1+\lambda)z\right].
\end{aligned}
\tag{2.15, 2.16}
$$

In order to proceed, it is useful to work out the effect of the +-prescription in the presence of an incomplete integration:

$$
\int_x^1 dz\,[f(z)]_+\,g(z)=\int_x^1 dz\,f(z)\,[g(z)-g(1)]-g(1)\underbrace{\int_0^x dz\,f(z)}_{-R_f(x)}=\int_x^1 dz\,\{[f(z)]_x+R_f(x)\delta(1-z)\}\,g(z)\,. \tag{2.17}
$$

where the x-prescription in the r.h.s. of the equation above should be understood in as a usual +-prescription regardless of the integration bounds.

Despite most of the times the residual function $R_f(x)$ can be evaluated analytically, sometimes it needs to be evaluated numerically. The +-prescribed functions present in the expressions above give rise to the following residual functions that can be computed analytically:

$$
-\int_0^x\frac{dz}{1-z}=\ln(1-x)\,, \tag{2.18}
$$

$$
-\int_0^x dz\frac{\ln(1-z)}{1-z}=\frac{1}{2}\ln^2(1-x)\,, \tag{2.19}
$$

$$
-\int_0^x\frac{dz}{1-\lambda z}=\frac{1}{\lambda}\ln(1-\lambda x)\,, \tag{2.20}
$$

$$
-\int_0^x dz\frac{1-z}{(1-\lambda z)^2}=\frac{1}{\lambda^2}\ln(1-\lambda x)+\frac{1-\lambda}{\lambda}\frac{x}{1-\lambda x}\,, \tag{2.21}
$$

$$
-\int_0^x dz\frac{\ln(1-\lambda z)}{1-z}=R(x)=-\mathrm{Li}_2\left(\frac{1}{1-\lambda}\right)+\mathrm{Li}_2\left(\frac{1-x\lambda}{1-\lambda}\right)+\ln\left(\frac{\lambda(1-x)}{1-\lambda}\right)\ln(1-\lambda x)\,. \tag{2.22}
$$

The advantage of the x-prescription is that, when convoluting the coefficient functions above with PDFs at the point x, one can treat the +-prescribed functions using the standard definition at the price of adding to the local terms the following functions:

$$
\begin{aligned}
C_{1,q}^{(1)} \;\to\; & C_{1,q}^{(1)}+2C_F\Bigg[2\ln^2(1-x)-2R(x)+2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\ln(1-x)\\
&+\frac{\lambda-1}{\lambda^3}\ln(1-\lambda x)+\frac{1}{2\lambda^2}\ln(1-\lambda x)+\frac{1-\lambda}{2\lambda}\frac{x}{1-\lambda x}\Bigg]\delta(1-z)\,,
\end{aligned}
\tag{2.23}
$$

$$C_{2,q}^{(1)} \to C_{2,q}^{(1)} + 2C_F\Bigg[2\ln^2(1-x) - 2R(x) + 2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\ln(1-x) + \frac{2\lambda^2-\lambda-1}{\lambda^2}\ln(1-\lambda x) + \frac{1}{2\lambda^2}\ln(1-\lambda x) + \frac{1-\lambda}{2\lambda}\frac{x}{1-\lambda x}\Bigg]\delta(1-z) \quad , \tag{2.24}$$

$$C_{3,q}^{(1)} \to C_{3,q}^{(1)} + 2C_F\Bigg[2\ln^2(1-x) - 2R(x) + 2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\ln(1-x) + \frac{\lambda-1}{\lambda^2}\ln(1-\lambda x) + \frac{1}{2\lambda^2}\ln(1-\lambda x) + \frac{1-\lambda}{2\lambda}\frac{x}{1-\lambda x}\Bigg]\delta(1-z)\,, \tag{2.25}$$

$$C_{L,q}^{(1)} \to C_{L,q}^{(1)} + 2C_F(1-\lambda)\Bigg[2\ln^2(1-x) - 2R(x) + 2\left(-1+\ln\frac{K_F^2}{\lambda}\right)\ln(1-x) - \frac{2}{\lambda}\ln(1-\lambda x) + \frac{1}{2\lambda^2}\ln(1-\lambda x) + \frac{1-\lambda}{2\lambda}\frac{x}{1-\lambda x}\Bigg]\delta(1-z)\,. \tag{2.26}$$

Now let us consider the gluon coefficient functions. They read:

$$C_{1,g}^{(1)} = 2T_R\Bigg\{[z^2+(1-z)^2]\left[\ln\left(\frac{1-z}{z}\right) - \frac{1}{2}\ln(1-\lambda) + \frac{1}{2}\ln\frac{K_F^2}{\lambda}\right] + 4z(1-z) - 1 + (1-\lambda)\left[-4z(1-z) + \frac{z}{1-\lambda z} + 2z(1-2\lambda z)\ln\frac{1-\lambda z}{(1-\lambda)z}\right]\Bigg\}\,, \tag{2.27}$$

$$C_{2,g}^{(1)} = 2T_R\Bigg\{[z^2+(1-z)^2]\left[\ln\left(\frac{1-z}{z}\right) - \frac{1}{2}\ln(1-\lambda) + \frac{1}{2}\ln\frac{K_F^2}{\lambda}\right] + 8z(1-z) - 1 + (1-\lambda)\left[-6(1+2\lambda)z(1-z) + \frac{1}{1-\lambda z} + 6\lambda z(1-2\lambda z)\ln\frac{1-\lambda z}{(1-\lambda)z}\right]\Bigg\}\,, \tag{2.28}$$

$$C_{3,g}^{(1)} = 2T_R\Bigg\{[z^2+(1-z)^2]\left[2\ln\left(\frac{1-z}{1-\lambda z}\right) + \frac{1}{2}\ln(1-\lambda) + \frac{1}{2}\ln\frac{K_F^2}{\lambda}\right] + (1-\lambda)\left[2z(1-z) - 2z[1-(1+\lambda)z]\ln\frac{1-\lambda z}{(1-\lambda)z}\right]\Bigg\}\,, \tag{2.29}$$

$$C_{L,g}^{(1)} = 2T_R\Bigg\{(1-\lambda)[z^2+(1-z)^2]\left[\ln\left(\frac{1-z}{z}\right) - \frac{1}{2}\ln(1-\lambda) + \frac{1}{2}\ln\frac{K_F^2}{\lambda}\right] + 4(2-\lambda)z(1-z) + (1-\lambda)\left[-2(3+4\lambda)z(1-z) + 4\lambda z(1-2\lambda z)\ln\frac{1-\lambda z}{(1-\lambda)z}\right]\Bigg\}\,. \tag{2.30}$$

Since these functions do not contain any +-prescribed functions, they can be implemented as they are.

We now consider the massless limit of the above massive coefficient functions. In the limit $m_H \to 0$, one has:

$$\lambda \to 1 \quad \text{and} \quad K_A \to 0 \quad . \tag{2.31}$$

For the quark coefficient functions this gives:

$$\begin{aligned} C_{1,q}^{(1)} \underset{m_H\to 0}{\longrightarrow} C_{1,q}^{0,(1)} &= 2C_F\Bigg\{ -\left(\frac{9}{2}+2\zeta_2-\frac{3}{2}\ln K_F^2\right)\delta(1-z) \\ &- \frac{(1+z^2)\ln z}{1-z} - \left(\ln(1-z)+\ln K_F^2\right)(1+z)+3 \\ &+ 2\left[\frac{\ln(1-z)}{1-z}\right]_x - \left(\frac{3}{2}-2\ln K_F^2\right)\left[\frac{1}{1-z}\right]_x\Bigg\}, \end{aligned} \tag{2.32}$$

$$\begin{aligned} C_{2,q}^{(1)} \underset{m_H\to 0}{\longrightarrow} C_{2,q}^{0,(1)} &= 2C_F\Bigg\{ -\left(\frac{9}{2}+2\zeta_2-\frac{3}{2}\ln K_F^2\right)\delta(1-z) \\ &- \frac{(1+z^2)\ln z}{1-z} - \left(\ln(1-z)+\ln K_F^2\right)(1+z)+2z+3 \\ &+ 2\left[\frac{\ln(1-z)}{1-z}\right]_x - \left(\frac{3}{2}-2\ln K_F^2\right)\left[\frac{1}{1-z}\right]_x\Bigg\}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} C_{3,q}^{(1)} \underset{m_H\to 0}{\longrightarrow} C_{3,q}^{0,(1)} &= 2C_F\Bigg\{ -\left(\frac{9}{2}+2\zeta_2-\frac{3}{2}\ln K_F^2\right)\delta(1-z) \\ &- \frac{(1+z^2)\ln z}{1-z} - \left(\ln(1-z)+\ln K_F^2\right)(1+z)+z+2 \\ &+ 2\left[\frac{\ln(1-z)}{1-z}\right]_x - \left(\frac{3}{2}-2\ln K_F^2\right)\left[\frac{1}{1-z}\right]_x\Bigg\}, \end{aligned} \tag{2.34}$$

$$C_{L,q}^{(1)} \underset{m_H\to 0}{\longrightarrow} C_{L,q}^{0,(1)} = 4C_F z\,. \tag{2.35}$$

Considering that:

$$R(x) \underset{m_H\to 0}{\longrightarrow} \frac{1}{2}\ln(1-x)^2\,, \tag{2.36}$$

the local terms to be added to the quark coefficient functions are:

$$C_{1,q}^{0,(1)} \to C_{1,q}^{0,(1)} + 2C_F\left[\ln^2(1-x) - \left(\frac{3}{2}-2\ln K_F^2\right)\ln(1-x)\right]\delta(1-z)\,, \tag{2.37}$$

$$C_{2,q}^{0,(1)} \to C_{2,q}^{0,(1)} + 2C_F\left[\ln^2(1-x) - \left(\frac{3}{2}-2\ln K_F^2\right)\ln(1-x)\right]\delta(1-z)\,, \tag{2.38}$$

$$C_{3,q}^{0,(1)} \to C_{3,q}^{0,(1)} + 2C_F\left[\ln^2(1-x) - \left(\frac{3}{2}-2\ln K_F^2\right)\ln(1-x)\right]\delta(1-z)\,, \tag{2.39}$$

while no local term needs to be added to $C_{L,q}^{0,(1)}$.

Now we turn to consider the zero-mass limit of the gluon coefficient functions for which we need to know that:

$$\ln(1-\lambda) \underset{m_H\to 0}{\longrightarrow} -\ln\left(\frac{Q^2}{m_H^2}\right)\,. \tag{2.40}$$

This term is clearly divergent in the zero-mass limit and embeds a collinear divergence typical of any massless calculations. However, we retain all such terms so that:

$$C_{1,g}^{(1)} \underset{m_H\to 0}{\longrightarrow} C_{1,g}^{0,(1)} = 2T_R\left\{[z^2+(1-z)^2]\left[\ln\left(\frac{1-z}{z}\right)+\frac{1}{2}\ln\left(\frac{Q^2}{m_H^2}\right)+\frac{1}{2}\ln K_F^2\right]+4z(1-z)-1\right\}\,, \tag{2.41}$$

$$C_{2,g}^{(1)} \underset{m_H\to 0}{\longrightarrow} C_{2,g}^{0,(1)} = 2T_R\left\{[z^2+(1-z)^2]\left[\ln\left(\frac{1-z}{z}\right)+\frac{1}{2}\ln\left(\frac{Q^2}{m_H^2}\right)+\frac{1}{2}\ln K_F^2\right]+8z(1-z)-1\right\}\,, \tag{2.42}$$

$$C_{3,g}^{(1)} \underset{m_H\to 0}{\longrightarrow} C_{3,g}^{0,(1)} = 2T_R[z^2+(1-z)^2]\left[-\frac{1}{2}\ln\left(\frac{Q^2}{m_H^2}\right)+\frac{1}{2}\ln K_F^2\right]\,, \tag{2.43}$$

$$C_{L,g}^{(1)} \underset{m_H \to 0}{\longrightarrow} C_{L,g}^{0,(1)} = 2T_R \left[4z(1-z)\right] . \tag{2.44}$$

We also note that in the limit $m_H \to 0$, the convolution integrals in Eqs. (2.2), (2.3), (2.4) and (2.8) will extend from x to 1 rather than from χ to 1.

The massive structure functions will eventually need to be combined to the massless and the massive-zero ones to compute the GM structure functions. Since the latter are computed through Mellin convolutions whose integral extends from x to 1, it is convenient to rewrite Eqs. (2.2), (2.3), (2.4) and (2.8) in such a way that the lower integration bound is also x rather than χ. To this end, let us consider the integral:

$$I = \int_\chi^1 \frac{dy}{y} C(y) f\left(\frac{\chi}{y}\right) , \tag{2.45}$$

where $\chi = x/\lambda$. By changing of integration variable $z = \lambda y$, integral above becomes:

$$I = \int_x^\lambda \frac{dz}{z} C\left(\frac{z}{\lambda}\right) f\left(\frac{x}{y}\right) = \int_x^1 \frac{dz}{z} \widetilde{C}(z,\lambda) f\left(\frac{x}{y}\right) , \tag{2.46}$$

where:

$$\widetilde{C}(z,\lambda) = \theta(\lambda - z) C\left(\frac{z}{\lambda}\right) . \tag{2.47}$$

In this way we have achieved the goal of expressing the "reduced" convolution in Eqs. (2.2), (2.3), (2.4) and (2.8) as a "standard" convolution between x and 1. As already mentioned above, this does not need to be done in the massive-zero case as the convolution already extends between x and 1.

3 Target Mass Corrections

Kinematic corrections due to the finite mass of the hadron M_p which recoils against the vector boson might be relevant in the small-Q region. The leading contributions to these corrections have been computed long time ago in Ref. [6] and, denoting the target-mass corrected structure functions with the symbol $\widetilde{\ }$, they take the form:

$$\begin{aligned}
\widetilde{F}_2(x,Q) &= \frac{x^2}{\xi^2 \tau^{3/2}} F_2(\xi,Q) + \frac{6\rho x^3}{\tau^2} I_2(\xi,Q) , \\
\widetilde{F}_L(x,Q) &= F_L(\xi,Q) + \frac{x^2(1-\tau)}{\xi^2 \tau^{3/2}} F_2(\xi,Q) + \frac{\rho x^3 (6-2\tau)}{\tau^2} I_2(\xi,Q) , \\
x\widetilde{F}_3(x,Q) &= \frac{x^2}{\xi^2 \tau} \xi F_3(\xi,Q) + \frac{2\rho x^3}{\tau^{3/2}} I_3(\xi,Q) ,
\end{aligned} \tag{3.1}$$

where:

$$\rho = \frac{M_p^2}{Q^2} , \qquad \tau = 1 + 4\rho x^2 , \qquad \xi = \frac{2x}{1+\sqrt{\tau}} , \tag{3.2}$$

and:

$$I_2(\xi,Q) = \int_\xi^1 dy \frac{F_2(y,Q)}{y^2} , \qquad I_3(\xi,Q) = \int_\xi^1 dy \frac{y F_3(y,Q)}{y^2} . \tag{3.3}$$

These integrals can be efficiently computed relying on the procedure based on the interpolating functions discussed in the part of the documentation devoted to the interpolation. From the equations above, it is clear that in the limit $M_p \to 0$, that implies $\rho \to 0$, $\tau \to 1$, and $\xi \to x$, all structure functions reduce to the usual formulas.

4 Renormalisation and factorisation scale variations

In the previous sections, when discussing the implementation of the structure functions in APFEL++, we implicitly assumed that the renormalisation scale μ_R and the factorisation scale μ_F were identified with the scale Q. The purpose of this section is to relax this assumption. To do so, it is necessary to consider the expansion of the DGLAP and of the renormalisation-group (RG) equation for α_s up to $\mathcal{O}(\alpha_s^2)$ that is the maximum order considered in the structure functions discussed above. These equations read:

$$\frac{\partial f_i}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{4\pi} \left[P_{ij}^{(0)}(x) + \frac{\alpha_s(\mu_F)}{4\pi} P_{ij}^{(1)}(x) + \dots \right] \otimes f_j(x,\mu_F) , \tag{4.1}$$

where a summation over repeated indices is understood, and:

$$\frac{\partial}{\partial \ln \mu_R^2}\left(\frac{\alpha_s}{4\pi}\right) = -\left(\frac{\alpha_s(\mu_R)}{4\pi}\right)^2 \left[\beta_0 + \frac{\alpha_s(\mu_R)}{4\pi}\beta_1 + \dots\right] . \tag{4.2}$$

Defining:

$$\xi_R \equiv \frac{\mu_R}{Q}\,, \quad \xi_F \equiv \frac{\mu_F}{Q} \quad \text{and} \quad a_s = \frac{\alpha_s}{4\pi} \tag{4.3}$$

and:

$$t_R \equiv \ln \xi_R^2 \quad \text{and} \quad t_F \equiv \ln \xi_F^2\,, \tag{4.4}$$

they can be written more compactly as:

$$\frac{\partial f_i}{\partial t_F} = a_s(t_F)\left[P_{ij}^{(0)} + a_s(t_F)P_{ij}^{(1)} + \dots\right] \otimes f_j(t_F)\,, \tag{4.5}$$

and:

$$\frac{\partial a_s}{\partial t_R} = -a_s^2(t_R)\left[\beta_0 + a_s(t_R)\beta_1 + \dots\right] . \tag{4.6}$$

Now, the Taylor expansion of $f_i(t)$ around $t = t_F$, up to second order is:

$$f_i(t) = f_i(t_F) + \left.\frac{\partial f_i}{\partial t}\right|_{t=t_F}(t - t_F) + \frac{1}{2}\left.\frac{\partial^2 f_i}{\partial t^2}\right|_{t=t_F}(t - t_F)^2 + \dots \tag{4.7}$$

Using Eqs. (4.5) and (4.6), we have that:

$$\begin{aligned}
\left.\frac{\partial f_i}{\partial t}\right|_{t=t_F} &= \left[a_s(t_F)P_{ij}^{(0)} + a_s^2(t_F)P_{ij}^{(1)}\right] \otimes f_j(t_F) + \mathcal{O}(a_s^3) \\
\left.\frac{\partial^2 f_i}{\partial t^2}\right|_{t=t_F} &= a_s^2(t_F)\left[P_{il}^{(0)} \otimes P_{lj}^{(0)} - \beta_0 P_{ij}^{(0)}\right] \otimes f_j(t_F) + \mathcal{O}(a_s^3)
\end{aligned} \tag{4.8}$$

Choosing $t = 0$ in Eq. (4.7), that is equivalent to setting $\mu_F = Q$, gives:

$$\begin{aligned}
f_i(0) &= \left\{1 - a_s(t_F)t_F P_{ij}^{(0)} + a_s^2(t_F)\left[-t_F P_{ij}^{(1)} + t_F^2\frac{1}{2}\left(P_{il}^{(0)} \otimes P_{lj}^{(0)} - \beta_0 P_{ij}^{(0)}\right)\right]\right. \\
&+ a_s^3(t_F)\left[\frac{1}{2}t_F^2\left(P_{il}^{(0)} \otimes P_{lj}^{(1)} + P_{il}^{(1)} \otimes P_{lj}^{(0)}\right) - \frac{1}{2}t_F^2\beta_1 P_{ij}^{(0)} + \frac{1}{2}t_F^3\beta_0 P_{il}^{(0)} \otimes P_{lj}^{(0)}\right. \\
&- \left.\left.\frac{1}{3}t_F^3\beta_0^2 P_{ij}^{(0)} - \frac{1}{6}t_F^3 P_{il}^{(0)} \otimes P_{lk}^{(0)} \otimes P_{kj}^{(0)} - t_F^2\beta_0 P_{ij}^{(1)} - t_F P_{ij}^{(2)}\right]\right\} \otimes f_j(t_F) + \mathcal{O}(a_s^4)\,.
\end{aligned} \tag{4.9}$$

In addition, using Eq. (4.6), one easily finds:

$$a_s(t_F) = a_s(t_R)\left[1 + a_s(t_R)\beta_0(t_R - t_F) + a_s^2(t_R)\left(\beta_1(t_R - t_F) + \beta_0^2(t_R - t_F)^2\right) + \mathcal{O}(a_s^3)\right] , \tag{4.10}$$

which can be plugged into Eq. (4.9) to give:

$$\begin{aligned}
f_i(0) &= \left\{1 - a_s(t_R)t_F P_{ij}^{(0)} + a_s^2(t_R)\left[-t_F P_{ij}^{(1)} + t_F^2\frac{1}{2}\left(P_{il}^{(0)} \otimes P_{lj}^{(0)} + \beta_0 P_{ij}^{(0)}\right) - t_F t_R\beta_0 P_{ij}^{(0)}\right]\right. \\
&+ a_s^3(t_R)\left[\frac{1}{2}t_F^2\left(P_{il}^{(0)}P_{lj}^{(1)} + P_{il}^{(1)}P_{lj}^{(0)}\right) + \frac{1}{2}t_F^2\beta_1 P_{ij}^{(0)} - t_F t_R\beta_1 P_{ij}^{(0)} - \frac{1}{2}t_F^3\beta_0 P_{il}^{(0)} \otimes P_{lj}^{(0)}\right. \\
&- \frac{1}{3}t_F^3\beta_0^2 P_{ij}^{(0)} - \frac{1}{6}t_F^3 P_{il}^{(0)} \otimes P_{lk}^{(0)} \otimes P_{kj}^{(0)} + t_F^2 t_R\beta_0 P_{ik}^{(0)} \otimes P_{kj}^{(0)} + t_F^2 t_R\beta_0^2 P_{ij}^{(0)} - t_F t_R^2\beta_0^2 P_{ij}^{(0)} \\
&+ \left.\left.t_F^2\beta_0 P_{ij}^{(1)} - 2t_F t_R\beta_0 P_{ij}^{(1)} - t_F P_{ij}^{(2)}\right]\right\} \otimes f_j(t_F) + \mathcal{O}(a_s^4)\,.
\end{aligned} \tag{4.11}$$

This equality can be conveniently written as:

$$f_i(0) = \sum_{k=0}^{3} a_s^k(t_R) f_i^{[k]}(t_R, t_F) + \mathcal{O}(a_s^4)\,, \tag{4.12}$$

where the coefficients $f_i^{[k]}(t_R, t_F)$ can be read off from Eq. (4.11).

It is also useful to consider $t_R = 0$ in Eq. (4.10), that is equivalent to set $\mu_R = Q$, which gives:

$$a_s(0) = a_s(t_R)\left[1 + a_s(t_R)\beta_0 t_R + a_s^2(t_R)\left(\beta_1 t_R + \beta_0^2 t_R^2\right)\right] + \mathcal{O}(a_s^4)\,. \tag{4.13}$$

We are now ready to use these equations to derive the scale variation terms to be included in ZM structure functions. Truncating the perturbative series to $\mathcal{O}(\alpha_s^2)$, they are written in terms of PDFs and coefficient functions as:

$$F(t_R, t_F)/x = \left[\sum_{k=0}^{3} a_s^k(t_R)\widetilde{\mathcal{C}}_i^{(k)}(t_R, t_F)\right] \otimes f_i(t_F) + \mathcal{O}(a_s^4)\,, \tag{4.14}$$

where the symbol $\otimes$ represents the convolution. Since structure functions are physically observable, they must be renormalisation and factorisation scale invariant, that is:

$$F(t_R, t_F) = F(0, 0)\,, \tag{4.15}$$

order by order in perturbation theory. Since:

$$F(0,0)/x = \left[\sum_{k=0}^{3} a_s^k(0)\widetilde{C}_i^{(k)}\right] \otimes f_i(0) + \mathcal{O}(a_s^4)\,, \tag{4.16}$$

where $\widetilde{C}_i^{(k)}$ are the usual perturbative contributions to the ZM coefficient functions, one can plug Eqs. (4.11) and (4.13) into Eq. (4.16) and impose the identity in Eq. (4.15). By doing so, one finds the explicit expression of the "generalised" coefficient functions $\widetilde{\mathcal{C}}_i^{(k)}(t_R, t_F)$ that include also the scale variation terms. It is convenient to first compute renormalisation scale variations while leaving $t_F = 0$:

$$\begin{aligned}
\widetilde{\mathcal{C}}_j^{(0)}(t_R, 0) &= \widetilde{C}_j^{(0)} \\
\widetilde{\mathcal{C}}_j^{(1)}(t_R, 0) &= \widetilde{C}_j^{(1)} \\
\widetilde{\mathcal{C}}_j^{(2)}(t_R, 0) &= \widetilde{C}_j^{(2)} + t_R\beta_0\widetilde{C}_j^{(1)}\,, \\
\widetilde{\mathcal{C}}_j^{(3)}(t_R, 0) &= \widetilde{C}_j^{(3)} + 2t_R\beta_0\widetilde{C}_j^{(2)} + t_R\left(\beta_1 + \beta_0^2 t_R\right)\widetilde{C}_j^{(1)}\,,
\end{aligned} \tag{4.17}$$

so that:

$$F(t_R, 0)/x = \left[\sum_{k=0}^{3} a_s^k(t_R)\widetilde{\mathcal{C}}_i^{(k)}(t_R, 0)\right] \otimes f_i(0) + \mathcal{O}(a_s^4)\,. \tag{4.18}$$

We can now use Eq. (4.12) to express the PDF $f(0)$ in terms of $f(t_F)$ finally obtaining:

$$F(t_R, 0)/x = \sum_{k=0}^{3} a_s^k(t_R) \sum_{j=0}^{k} \widetilde{\mathcal{C}}_i^{(k-j)}(t_R, 0) \otimes f_i^{[j]}(t_R, t_F) + \mathcal{O}(a_s^4)\,. \tag{4.19}$$

In the absence of factorisation scale variations, only the term with $j = 0$ contributes to the inner sum.

In order to provide an operative formulation of scale variations, it is necessary to specify the basis in which PDFs are expressed. The preferred choice in APFEL++ is the so-called QCD evolution basis:

$$\{g, \Sigma, V, T_3, V_3, T_8, V_8, T_{15}, V_{15}, T_{24}, V_{24}, T_{35}, V_{35}\}\,. \tag{4.20}$$

The distributions in the QCD evolution basis can be written in terms of distributions in the more familiar "physical" basis, *i.e.* $\{\bar{t},\bar{b},\bar{c},\bar{s},\bar{u},\bar{d},d,u,s,c,b,t\}$, as follows:

$$\begin{array}{rcl}
\Sigma &=& \sum_q q^+\,,\\
V &=& \sum_q q^-\,,\\
T_3 &=& u^+ - d^+\,,\\
V_3 &=& u^- - d^-\,,\\
T_8 &=& u^+ + d^+ - 2s^+\,,\\
V_8 &=& u^- + d^- - 2s^-\,,\\
T_{15} &=& u^+ + d^+ + s^+ - 3c^+\,,\\
V_{15} &=& u^- + d^- + s^- - 3c^-\,,\\
T_{24} &=& u^+ + d^+ + s^+ + c^+ - 4b^+\,,\\
V_{24} &=& u^- + d^- + s^- + c^- - 4b^-\,,\\
T_{35} &=& u^+ + d^+ + s^+ + c^+ + b^+ - 5t^+\,,\\
V_{35} &=& u^- + d^- + s^- + c^- + b^- - 5t^-\,.
\end{array} \tag{4.21}$$

where the notation $q^\pm \equiv q \pm \bar{q}$ is used. As the name suggests, the QCD evolution basis is particularly useful when evolving PDFs because in this basis the DGLAP evolution equations take a maximally diagonalised form. Adopting the QCD evolution basis implies that the indices i, j, and l in Eq. (**??**) run between 0 and 12 over this basis with 0 corresponding to the gluon component.

In the following, we will consider a ZM neutral current structure function for $Q \ll M_Z$ in such a way that only the photon contributes. We will extend the treatment to the general neutral-current case and to the charge-current one below. Omitting for simplicity the convolution sign and an overall factor x, the starting point is the definition of the structure function in terms of the PDFs in the physical basis that reads:

$$F = \langle e_q^2\rangle \left\{ C_g g + \sum_{i=u}^{t} \underbrace{\theta(Q^2-m_i^2)\left[C_{\rm PS} + \frac{e_i^2}{\langle e_q^2\rangle} C_+\right]}_{\hat{C}_i} q_i^+ \right\}\,, \tag{4.22}$$

where C_+ and $C_{\rm PS}$ correspond to the non-singlet and pure-singlet coefficient functions, that are usually the quantities computed in perturbation theory, and where:

$$\langle e_q^2\rangle = \sum_{i=u}^{t} e_i^2 \theta(Q^2-m_i^2)\,. \tag{4.23}$$

with m_i is the mass of the i-th quark flavour. Now, in order to express the structure function in Eq. (4.22) in the evolution basis, we need to find the tranformation T such that:

$$q_i^+ = \sum_{j=1}^{6} T_{ij} f_j\,, \tag{4.24}$$

where f_j belongs to the evolution basis, that is: $f_1 = \Sigma$, $f_2 = T_3$, $f_3 = T_8$ and so on. One can show that the trasformation matrix T takes the form:

$$\begin{array}{l}
T_{ij} = \theta_{ji}\dfrac{1-\delta_{ij}j}{j(j-1)} \quad j \geq 2\,,\\[2ex]
T_{i1} = \dfrac{1}{6}\,,
\end{array} \tag{4.25}$$

with $\theta_{ji} = 1$ for $j \geq i$ and zero otherwise. In addition, one can show that:

$$\sum_{j=1}^{6} T_{ij} = 0\,, \quad \text{and} \quad \sum_{i=1}^{6} T_{ij} = \delta_{1j}\,. \tag{4.26}$$

Now, we can plug Eq. (4.24) into Eq. (4.22) and, using Eq. (4.25), we get:

$$F = \langle e_q^2\rangle \left\{ C_g g + \frac{1}{6}\left(C_+ + n_f C_{\rm PS}\right)\Sigma + \sum_{j=2}^{6} \frac{1}{j(j-1)}\left[\sum_{i=1}^{j}\hat{C}_i - j\hat{C}_j\right] f_j \right\}\,, \tag{4.27}$$

where we have transmuted the sum over u, d and so on into a sum between 1 and 6 and we have defined the number of active flavours n_f as:

$$n_f = \sum_{i=1}^{6} \theta(Q^2 - m_i^2)\,. \tag{4.28}$$

We can now express the term in square brackets in terms of C_+ and $C_{\rm PS}$. In particular:

$$\sum_{i=1}^{j} \hat{C}_i - j\hat{C}_j = \sum_{i=1}^{j} \theta(Q^2 - m_i^2) \left(C_{\rm PS} + \frac{e_i^2}{\langle e_q^2 \rangle} C_+ \right) - j\theta(Q^2 - m_j^2) \left(C_{\rm PS} + \frac{e_j^2}{\langle e_q^2 \rangle} C_+ \right) . \tag{4.29}$$

We can distinguish two cases. The first is $Q^2 < m_j^2$ and under this assumption we have:

$$\sum_{i=1}^{j} \hat{C}_i - j\hat{C}_j = C_+ + n_f C_{\rm PS}\,. \tag{4.30}$$

If instead $Q^2 \geq m_j^2$, then:

$$\sum_{i=1}^{j} \hat{C}_i - j\hat{C}_j = K_j C_+\,, \tag{4.31}$$

with:

$$K_j = \frac{1}{\langle e_q^2 \rangle} \left(\sum_{i=1}^{j} e_i^2 - j e_j^2 \right) = \frac{1}{\langle e_q^2 \rangle} \left(\sum_{i=1}^{j-1} e_i^2 - (j-1) e_j^2 \right) . \tag{4.32}$$

Both cases can be gathered in one single formula as:

$$\sum_{i=1}^{j} \hat{C}_i - j\hat{C}_j = \theta(m_j^2 - Q^2 - \epsilon) \left[C_+ + n_f C_{\rm PS} \right] + \theta(Q^2 - m_j^2) \left[K_j C_+ \right] . \tag{4.33}$$

where ϵ is an infinitesimal positive number that ensures that the case $Q^2 = m_j^2$ is included in the second term of the r.h.s. of Eq. (4.33). Eq. (4.33) can be rewritten as:

$$\sum_{i=1}^{j} \hat{C}_i - j\hat{C}_j = \theta_{n_f j} \left[K_j C_+ \right] + \theta_{j,n_f-1} \left[C_+ + n_f C_{\rm PS} \right] . \tag{4.34}$$

In addition, neglecting intrinsic heavy-quark contributions, one can easily see that:

$$f_j = \theta_{n_f j} f_j + \theta_{j,n_f+1} \Sigma\,, \tag{4.35}$$

and thus:

$$\sum_{j=2}^{6} \frac{1}{j(j-1)} \left[\sum_{i=1}^{j} \hat{C}_i - j\hat{C}_j \right] f_j = C_+ \left[\sum_{j=2}^{n_f} \frac{K_j}{j(j-1)} f_j \right] + \left[\sum_{j=n_f+1}^{6} \frac{1}{j(j-1)} \right] \left[C_+ + n_f C_{\rm PS} \right] \Sigma\,. \tag{4.36}$$

But:

$$\sum_{j=n_f+1}^{6} \frac{1}{j(j-1)} = \frac{1}{n_f} - \frac{1}{6}\,, \tag{4.37}$$

and moreover:

$$\frac{K_j}{j(j-1)} = \frac{1}{\langle e_q^2 \rangle} \frac{1}{j(j-1)} \left(\sum_{i=1}^{j} e_i^2 - j e_j^2 \right) = \frac{1}{\langle e_q^2 \rangle} \underbrace{\frac{1}{j(j-1)} \sum_{i=1}^{6} e_i^2 \left[\theta_{ji} - j\delta_{ij} \right]}_{d_j} . \tag{4.38}$$

Finally, putting all pieces together, gives:

$$\begin{aligned} F &= \langle e_q^2 \rangle \left[C_g g + \left(C_{\rm PS} + \frac{1}{n_f} C_+ \right) \Sigma \right] + C_+ \sum_{j=2}^{n_f} d_j f_j \\ &= \langle e_q^2 \rangle \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] + C_+ \sum_{j=2}^{6} d_j f_j\,. \end{aligned} \tag{4.39}$$

The second version of the equation above is particularly useful for the implementation because the flavours structure does not have any explicit dependence on n_f and does not rely on the absence of intrinsic heavy-quark contributions (Eq. (4.35)). It is useful to separate the contributions deriving from the coupling of the vector boson to the different quark flavours. To do so, one just needs to select the contributions proportional to, say, the k-th charge e_k^2. This is easily done with the following replacement:

$$e_i^2 \to \delta_{ik} e_i^2 \,. \tag{4.40}$$

This gives:

$$\langle e_q^2 \rangle \to \theta(Q^2 - m_k^2) e_k^2 \,, \tag{4.41}$$

and:

$$d_j \to \frac{e_k^2 \left[\theta_{jk} - j\delta_{kj}\right]}{j(j-1)} = \theta(Q^2 - m_k^2) e_k^2 \frac{\left[\theta_{jk} - j\delta_{kj}\right]}{j(j-1)} \,, \tag{4.42}$$

so that the component of the structure function F associated to the k-th quark flavour is:

$$\begin{aligned} F^{(k)} &= \theta(Q^2 - m_k^2) e_k^2 \left\{ \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] + C_+ \sum_{j=2}^{6} \frac{\left[\theta_{jk} - j\delta_{kj}\right]}{j(j-1)} f_j \right\} \\ &= \theta(Q^2 - m_k^2) e_k^2 \left\{ \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] - \frac{1}{k} C_+ f_k + C_+ \sum_{j=k+1}^{6} \frac{1}{j(j-1)} f_j \right\} \end{aligned} \tag{4.43}$$

and it is such that:

$$F = \sum_{k=1}^{6} F^{(k)} \,. \tag{4.44}$$

Phenomenologically relevant combinations are the light component, that includes the contribution of down, up, and strange, and the three heavy quark components (even though the top component is typically not relevant). The light component is defined as:

$$F^l = \sum_{k=1}^{3} F^{(k)} = \langle e_l^2 \rangle \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] + C_+ \sum_{j=2}^{6} d_j^{(l)} f_j \,. \tag{4.45}$$

where:

$$\langle e_l^2 \rangle = \sum_{i=1}^{3} e_i^2 \,, \tag{4.46}$$

and:

$$d_j^{(l)} = \frac{1}{j(j-1)} \sum_{i=1}^{3} e_i^2 \left[\theta_{ji} - j\delta_{ij}\right] = \begin{cases} \frac{1}{2}(e_u^2 - e_d^2) \,, & j = 2 \\ \frac{1}{6}(e_u^2 + e_d^2 - 2e_s^2) \,, & j = 3 \\ \frac{\langle e_l^2 \rangle}{j(j-1)} \,, & j \geq 4 \end{cases} \,, \tag{4.47}$$

no need of the θ-functions as the scale Q will always be above the strange threshold. Therefore, the explicit form of F^l is:

$$F^l = \langle e_l^2 \rangle \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] + \frac{1}{2}(e_u^2 - e_d^2) C_+ T_3 + \frac{1}{6}(e_u^2 + e_d^2 - 2e_s^2) C_+ T_8 + \langle e_l^2 \rangle C_+ \sum_{j=4}^{6} \frac{1}{j(j-1)} f_j \,. \tag{4.48}$$

The heavy-quark components are instead defined as:

$$\begin{aligned} F^c &= \theta(Q^2 - m_c^2) e_c^2 \left\{ \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] - \frac{1}{4} C_+ T_{15} + C_+ \sum_{j=5}^{6} \frac{1}{j(j-1)} f_j \right\} \,, \\ F^b &= \theta(Q^2 - m_b^2) e_b^2 \left\{ \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] - \frac{1}{5} C_+ T_{24} + C_+ \sum_{j=6}^{6} \frac{1}{j(j-1)} f_j \right\} \,, \\ F^t &= \theta(Q^2 - m_t^2) e_t^2 \left\{ \left[C_g g + \frac{1}{6} \left(C_+ + 6 C_{\rm PS} \right) \Sigma \right] - \frac{1}{6} C_+ T_{35} \right\} \,. \end{aligned} \tag{4.49}$$

To conclude the treatment of all the structure functions, it should be mentioned that Eq. (4.39) is valid only for F_2 and F_L. However, F_3, that appears when weak contributions are included, can be easily derived following the same steps with the only differences being that:

- the distributions $\{\Sigma, T_3, T_8, T_{15}, T_{24}, T_{35}\}$ must be replaced with $\{V, V_3, V_8, V_{15}, V_{24}, V_{35}\}$,
- C_+ must be replaced with C_-,
- the pure-singlet coefficient function $C_{\rm PS}$ is to be replaced with the total-valence coefficient function C_V,
- the gluon coefficient function is identically zero,
- the squared electric charges must be replaced with the appropriate electroweak charges c_i.

Following this recipe, one finds:

$$F_3 = \langle c_q^2 \rangle \frac{1}{6} \left(C_- + 6 C_V \right) V + C_- \sum_{j=2}^{6} d_j g_j \,, \tag{4.50}$$

where g_j runs over $\{V, V_3, V_8, V_{15}, V_{24}, V_{35}\}$.

Despite Eq. (4.43) has been derived in the ZM scheme, it can be generalised to the massive scheme. The complication is that, due to the explicit dependence of the coefficient functions on the k-th quark mass,[3] the k-th structure function will read:

$$F^{{\rm M},(k)} = e_k^2 \left\{ \left[C_g^{(k)} g + \frac{1}{6} \left(C_+^{(k)} + 6 C_{\rm PS}^{(k)} \right) \Sigma \right] + C_+^{(k)} \sum_{j=2}^{6} \frac{\left[\theta_{jk} - j \delta_{kj} \right]}{j(j-1)} f_j \right\} \,, \tag{4.51}$$

where an explicit dependence of the index k has been introduced in the coefficient functions. This dependence invalidates the equality in Eq. (4.44) where F is given in Eq. (4.39). Therefore, in order to compute inclusive structure functions in the massive case a different combination is to be taken. Specifically one defines:

$$F^{\rm M} = \sum_{k=1}^{6} F^{{\rm M},(k)} = \left[\langle C_g \rangle g + \frac{1}{6} \langle C_q \rangle \Sigma \right] + \sum_{j=2}^{6} \frac{\langle C_+ \rangle_j}{j(j-1)} f_j \,, \tag{4.52}$$

with the definitions:

$$\begin{aligned} \langle C_g \rangle &= \sum_{k=1}^{6} e_k^2 C_g^{(k)} \,, \\ \langle C_q \rangle &= \sum_{k=1}^{6} e_k^2 \left(C_+^{(k)} + 6 C_{\rm PS}^{(k)} \right) \,, \\ \langle C_+ \rangle_j &= \sum_{k=1}^{6} e_k^2 \left[\theta_{jk} - j \delta_{kj} \right] C_+^{(k)} = \sum_{k=1}^{j-1} e_k^2 C_+^{(k)} + e_j^2 (1-j) C_+^{(j)} \,. \end{aligned} \tag{4.53}$$

Eq. (4.39) is the result that allows us to implement the scale variation formulae given in Eq. (**??**) in APFEL++. An important aspect of Eq. (4.39) is that it is written in terms of the fundamental coefficient functions C_g, C_+ and $C_{\rm PS}$ and PDFs appear in the evolution basis in which the splitting-function matrix diagonalises. In particular, up to $\mathcal{O}(\alpha_s^2)$, one has that:

$$\begin{array}{ll} P_{ij}^{(k)} \to P_{ij}^{(k)} & i,j = g, q(\Sigma) \\ P_{ij}^{(k)} \to \delta_{ij} P_+^{(k)} & i,j = T_3, T_8, V_{15}, T_{24}, T_{35} \\ P_{ij}^{(k)} \to \delta_{ij} P_-^{(k)} & i,j = V, V_3, V_8, V_{15}, V_{24}, V_{35} \end{array} \tag{4.54}$$

[3] In general, the k-th structure function will not only depend on the k-th mass but also on the other masses. However, in the neutral-current case and neglecting intrinsic heavy-quark contributions, the dependence on masses other than the k-th one in the coefficient functions only enters at $\mathcal{O}(\alpha_s^3)$ and therefore will be neglected here.

Also, defining:

$$C_q = C_{\rm PS} + \frac{1}{n_f} C_+ \,, \tag{4.55}$$

one can connect Eq. (4.16) and Eq. (4.39) by observing that:

$$\begin{array}{ll} \widetilde{C}_j^{(k)} \to \langle e_q^2 \rangle C_j^{(k)} & j = g, q(\Sigma) \\ \widetilde{C}_j^{(k)} \to d_j C_+^{(k)} & j = T_3, T_8, T_{15}, T_{24}, T_{35} \\ \widetilde{C}_j^{(k)} \to d_j C_-^{(k)} & j = V_3, V_8, V_{15}, V_{24}, V_{35} \end{array} \tag{4.56}$$

where we have also considered the "minus" distributions that appear in the F_3 structure function. Of course, the same relations must hold also for Eq. (4.14):

$$\begin{array}{ll} \widetilde{\mathcal{C}}_j^{(k)} \to \langle e_q^2 \rangle \mathcal{C}_j^{(k)} & j = g, q(\Sigma) \\ \widetilde{\mathcal{C}}_j^{(k)} \to d_j \mathcal{C}_+^{(k)} & j = T_3, T_8, T_{15}, T_{24}, T_{35} \\ \widetilde{\mathcal{C}}_j^{(k)} \to d_j \mathcal{C}_-^{(k)} & j = V_3, V_8, V_{15}, V_{24}, V_{35} \end{array} \tag{4.57}$$

with

$$\mathcal{C}_q = \mathcal{C}_{\rm PS} + \frac{1}{n_f} \mathcal{C}_+ \,. \tag{4.58}$$

In addition, in the following we will make use of the following identity:

$$P_-^{(0)} = P_+^{(0)} = P_{qq}^{(0)} \,. \tag{4.59}$$

Now, considering that $C_{\rm PS}$ starts at $\mathcal{O}(\alpha_s^2)$, we can write:

$$\begin{array}{l} C_-^{(0)}(x) = C_+^{(0)}(x) = \Delta_{\rm SF} \delta(1-x) \\ C_j^{(0)}(x) = (\Delta_{\rm SF}/n_f)\, \delta_{qj} \delta(1-x) \quad \text{for } j = q, g \end{array} \tag{4.60}$$

where $\Delta_{\rm SF} = 1$ for F_2 and F_3 and $\Delta_{\rm SF} = 0$ for F_L. From Eq. (**??**) it follows that:

$$\begin{aligned} \mathcal{C}_\pm^{(0)}(t_R, t_F) &= \Delta_{\rm SF} \delta(1-x) \\ \mathcal{C}_\pm^{(1)}(t_R, t_F) &= C_\pm^{(1)} - \Delta_{\rm SF} t_F P_{qq}^{(0)} \\ \mathcal{C}_\pm^{(2)}(t_R, t_F) &= C_\pm^{(2)} + t_R \beta_0 C_\pm^{(1)} - t_F \left(\Delta_{\rm SF} P_\pm^{(1)} + C_\pm^{(1)} \otimes P_{qq}^{(0)} \right) \\ &+ \Delta_{\rm SF} \frac{t_F^2}{2} \left(P_{qq}^{(0)} \otimes P_{qq}^{(0)} + \beta_0 P_{qq}^{(0)} \right) - \Delta_{\rm SF} t_F t_R \beta_0 P_{qq}^{(0)} \,, \end{aligned} \tag{4.61}$$

that can be rearranged as:

$$\begin{aligned} \mathcal{C}_\pm^{(0)}(t_R, t_F) &= \Delta_{\rm SF} \delta(1-x) \\ \mathcal{C}_\pm^{(1)}(t_R, t_F) &= C_\pm^{(1)} - \Delta_{\rm SF} t_F P_{qq}^{(0)} \\ \mathcal{C}_\pm^{(2)}(t_R, t_F) &= C_\pm^{(2)} + t_R \beta_0 C_\pm^{(1)} - t_F C_\pm^{(1)} \otimes P_{qq}^{(0)} \\ &+ \Delta_{\rm SF} \frac{t_F^2}{2} \left(P_{qq}^{(0)} \otimes P_{qq}^{(0)} - \beta_0 P_{qq}^{(0)} \right) - \Delta_{\rm SF} t_F \left[P_\pm^{(1)} - (t_F - t_R) \beta_0 P_{qq}^{(0)} \right] \,. \end{aligned} \tag{4.62}$$

The term in square brackets in the r.h.s. of the third line corresponds to what we define $\widetilde{\mathcal{P}}_\pm^{(1)}(t_R, t_F)$, that is the NLO contribution to the non-singlet splitting functions in the presence of scale variations ($\mu_R \neq \mu_F$).

Let us now consider the singlet sector that becomes:

$$\begin{aligned}
\mathcal{C}_j^{(0)}(t_R,t_F) &= \frac{\Delta_{\rm SF}}{n_f}\delta_{qj}\delta(1-x)\\
\mathcal{C}_j^{(1)}(t_R,t_F) &= C_j^{(1)} - \frac{\Delta_{\rm SF}}{n_f}t_F P_{qj}^{(0)}\\
\mathcal{C}_j^{(2)}(t_R,t_F) &= C_j^{(2)} + t_R\beta_0 C_j^{(1)} - t_F C_i^{(1)}\otimes P_{ij}^{(0)}\\
&+ \frac{\Delta_{\rm SF}}{n_f}\frac{t_F^2}{2}\left(P_{qi}^{(0)}\otimes P_{ij}^{(0)} - \beta_0 P_{qj}^{(0)}\right) - \frac{\Delta_{\rm SF}}{n_f}t_F\widetilde{P}_{qj}^{(1)}\,,
\end{aligned} \tag{4.63}$$

for $j = g, q$. Taking into account Eq. (4.62) and considering also that $C_{\rm PS}^{(0)} = C_{\rm PS}^{(1)} = 0$, it is easy to see that:

$$\begin{aligned}
\mathcal{C}_{\rm PS}^{(0)}(t_R,t_F) &= 0\\
\mathcal{C}_{\rm PS}^{(1)}(t_R,t_F) &= 0\\
\mathcal{C}_{\rm PS}^{(2)}(t_R,t_F) &= C_{\rm PS}^{(2)} - t_F C_g^{(1)}\otimes P_{gq}^{(0)} + \frac{\Delta_{\rm SF}}{n_f}\frac{t_F^2}{2}P_{qg}^{(0)}\otimes P_{gq}^{(0)} - \frac{\Delta_{\rm SF}}{n_f}t_F\left[\widetilde{P}_{qq}^{(1)} - \widetilde{P}_{+}^{(1)}\right]\,.
\end{aligned} \tag{4.64}$$

Now we consider the massive case. In the neutral-current sector the leading-order coefficient functions $C_i^{(0)}$ are identically zero which substantially simplifies the structure of the coefficient functions in the presence of scale variations:

$$\begin{aligned}
\mathcal{C}_j^{(0)}(t_R,t_F) &= 0\\
\mathcal{C}_j^{(1)}(t_R,t_F) &= C_j^{(1)}\\
\mathcal{C}_j^{(2)}(t_R,t_F) &= C_j^{(2)} + t_R\beta_0 C_j^{(1)} - t_F C_i^{(1)}\otimes P_{ij}^{(0)}\,.
\end{aligned} \tag{4.65}$$

In addition, the factorisation scale variation terms are already present in the implementation of the massive coefficient functions in APFEL++. As a consequence, only the renormalisation variation terms need to be implemented.

As far as the massive charged-current sector is concerned, no $\mathcal{O}(\alpha_s^2)$ are presently available and thus only the first two lines of Eq. (**??**) are actually required. Also in this case the factorisation scale variation terms are already present in the implementation of the coefficient functions and again this avoids the pre-computation of additional terms.

5 Single-inclusive e^+e^- annihilation structure functions

The implementation of the Single-Inclusive e^+e^- Annihilation (SIA) structure functions in APFEL++ is not very complicated. The reason is that SIA is structurally identical to the case of deep-inelastic scattering (DIS) that was (implicitly) discussed above. As a matter fact, one can regard SIA as the time-like counterpart of DIS and the differences are only at the level of coefficient functions and splitting functions. Presently, the coefficient functions for SIA are known up to $\mathcal{O}(\alpha_s^2)$ (NNLO) in the zero-mass scheme and they have been computed in Ref. [7] and the x-space expressions reported in Appendix C of that paper.

The way in which the SIA expressions are reported is slightly different w.r.t. DIS.It is then useful to reduce the SIA expressions to the same form of DIS in such a way to use the DIS-based structure of APFEL++ also for SIA. In particular, the SIA cross section in Ref. [7] is expressed in terms of the three structure functions: F_T, F_L and F_A. However, comparing the SIA cross section with the DIS one it is easy to realise that defining:

$$\begin{aligned}
&F_2(x,Q) = F_T(x,Q) + F_L(x,Q)\,,\\
&F_L(x,Q) = F_L(x,Q)\,,\\
&F_A(x,Q) = xF_3(x,Q)\,,
\end{aligned} \tag{5.1}$$

the SIA cross section reduces to the same structure of DIS. Upon this identification, the usual factorised form for the structure functions applies:[4]

$$F_k(x,Q) = \sum_{j=q,g} x \int_x^1 \frac{dy}{y} c_{k,j}(\alpha_s(Q),x)\mathcal{D}_j\left(\frac{x}{y},Q\right), \quad \text{with} \quad k = 2,L,3\,, \tag{5.2}$$

where $\mathcal{D}_j$ is the fragmentation function of the flavour j and the coefficient functions $c_{k,j}$ admit the perturbative expansion:

$$c_{k,j}(\alpha_s(Q),x) = \sum_{n=0}^{N} \left(\frac{\alpha_s(Q)}{4\pi}\right)^n c_{k,j}^{(n)}(x)\,. \tag{5.3}$$

The leading-order coefficient functions are trivially:

$$\begin{aligned} &c_{k,g}^{(0)}(x) = 0\,, \quad k = 2,L,3\,,\\ &c_{L,q}^{(0)}(x) = 0\,,\\ &c_{2,q}^{(0)}(x) = c_{3,q}^{(0)}(x) = \delta(1-x)\,. \end{aligned} \tag{5.4}$$

Now we consider the NLO coefficient functions. Their explicit expressions are give in Eqs. (C.13)-(C.17) of Ref. [7] but, in order to write them in a form suitable for the implementation in APFEL++, we need to isolate regular, singular, and local terms and finally combine them according to Eq. (5.1).

$$\begin{aligned} &c_{L,q}^{(1)}(x) = 2C_F\,,\\ &c_{L,g}^{(1)}(x) = 2C_F\frac{4(1-x)}{x}\,,\\ &c_{2,q}^{(1)}(x) = c_{T,q}^{(1)}(x) + c_{L,q}^{(1)}(x) &=& 2C_F\left[2\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - (1+x)\ln(1-x)\right.\\ &&&\left. +2\frac{1+x^2}{1-x}\ln x + \frac{5}{2} - \frac{3}{2}x + \left(4\zeta_2 - \frac{9}{2}\right)\delta(1-x)\right],\\ &c_{2,g}^{(1)}(x) = c_{T,g}^{(1)}(x) + c_{L,g}^{(1)}(x) &=& 4C_F\frac{1+(1-x)^2}{x}\ln[x^2(1-x)]\,,\\ &c_{3,q}^{(1)}(x) &=& 2C_F\left[2\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - (1+x)\ln(1-x)\right.\\ &&&\left. +2\frac{1+x^2}{1-x}\ln x + \frac{1}{2} - \frac{1}{2}x + \left(4\zeta_2 - \frac{9}{2}\right)\delta(1-x)\right],\\ &c_{3,g}^{(1)}(x) = 0\,. \end{aligned} \tag{5.5}$$

The NNLO coefficient functions, despite implemented in APFEL++, are not reported here because too lengthy.

6 Longitudinally polarised structure functions

Let us now consider the differential cross sections for unpolarised and polarised DIS (see *e.g.* Eq. (19.16) of Sec. 19 in Ref. [8]):

$$\begin{aligned} \frac{d^2\sigma^i}{dxdy} &= \frac{2\pi\alpha^2}{xyQ^2}\eta^i\left[+Y_+F_2^i \mp Y_- xF_3^i - y^2F_L^i\right]\\ \frac{d^2\Delta\sigma^i}{dxdy} &= \frac{2\pi\alpha^2}{xyQ^2}\eta^i\left[-Y_+g_4^i \mp Y_- 2xg_1^i + y^2g_L^i\right], \end{aligned} \tag{6.1}$$

[4] Notice that, to uniform the notation, we understood the factor x in front of F_3.

where $i = \mathrm{NC}, \mathrm{CC}$, $Y_\pm = 1 \pm (1-y)^2$, $\eta^{\mathrm{NC}} = 1$, $\eta^{\mathrm{CC}} = (1 \pm \lambda)^2 \eta_W$ (with $\lambda = \pm 1$ is the helicity of the incoming lepton and $\eta_W = \frac{1}{2}\left(\frac{G_F M_W}{4\pi\alpha}\frac{Q^2}{Q^2+M_W^2}\right)^2$), and

$$\begin{aligned} F_L^i &= F_2^i - 2xF_1^i \\ F_L^i &= g_4^i - 2xg_5^i \,. \end{aligned} \tag{6.2}$$

Because the same tensor structure occurs in the spin-dependent and spin-independent parts of the DIS hadronic tensor (in the limit $M^2/Q^2 \to 0$), the polarised cross section can be obtained from the unpolarised cross section with the following replacement

$$F_2^i \to -2g_4^i \qquad F_3^i \to +4g_1^i \qquad F_L^i \to -2g_L^i \,. \tag{6.3}$$

Note that the extra factor two is due to the fact that the total cross section is an average over initial-state polarisations.

The *polarised* structure functions g_4, g_1 and g_L are expressed as a convolution of coefficient functions, $\Delta c_{k,j}$, and polarised PDFs, Δf_j, (summed over all flavors j)

$$g_k(x,Q) = \sum_{j=q,g} x \int_x^1 \frac{dy}{y} \Delta c_{k,j}(\alpha_s(Q), x) \Delta f_j\left(\frac{x}{y}, Q\right) , \qquad \text{with} \quad k = 4, 1, L \,. \tag{6.4}$$

The coefficient functions $\Delta c_{k,j}$ allow for the usual perturbative expansion

$$\Delta c_{k,j}(\alpha_s(Q), x) = \sum_{n=0}^{N} \left(\frac{\alpha_s(Q)}{4\pi}\right)^n \Delta c_{k,j}^{(n)}(x) \,, \tag{6.5}$$

where the coefficients $\Delta c_{k,j}^{(n)}(x)$ are known up to NLO, *i.e.* $n = 1$ (see *e.g.* [9] and references therein). At LO they are

$$\begin{aligned} \Delta c_{4,q}^{(0)}(x) = \Delta c_{1,q}^{(0)}(x) &= \delta(1-x) \\ \Delta c_{L,q}^{(0)}(x) &= 0 \,, \\ \Delta c_{k,g}^{(0)}(x) &= 0 \qquad \text{with } k = 4, 1, L \,. \end{aligned} \tag{6.6}$$

At NLO they read

$$\begin{aligned} \Delta c_{4,q}^{(1)}(x) &= 2C_F \left\{ 2\left[\frac{\ln(1-x)}{1-x}\right]_+ - \frac{3}{2}\left[\frac{1}{1-x}\right]_+ - (1+x)\ln(1-x) \right. \\ &\quad \left. - \frac{1+x^2}{1-x}\ln x + 3 + 2x - \left(\frac{9}{2} + 2\zeta_2\right)\delta(1-x) \right\} , \\ \Delta c_{4,g}^{(1)}(x) &= 0 \,, \\ \Delta c_{1,q}^{(1)}(x) &= 2C_F \left\{ 2\left[\frac{\ln(1-x)}{1-x}\right]_+ - \frac{3}{2}\left[\frac{1}{1-x}\right]_+ - (1+x)\ln(1-x) \right. \\ &\quad \left. - \frac{1+x^2}{1-x}\ln x + 2 + x - \left(\frac{9}{2} + 2\zeta_2\right)\delta(1-x) \right\} , \\ \Delta c_{1,g}^{(1)}(x) &= 4T_R \left\{ (2x-1)\ln\frac{1-x}{x} - 4x + 3 \right\} , \\ \Delta c_{L,q}^{(1)}(x) &= 2C_F\, 2x \,, \\ \Delta c_{L,g}^{(1)}(x) &= 0 \,. \end{aligned} \tag{6.7}$$

In the NC case the couplings can be written as:

$$\begin{aligned} B_q(Q^2) &= -e_q A_q (V_e \pm \lambda A_e) P_Z + V_q A_q (V_e^2 + A_e^2 \pm 2\lambda V_e A_e) P_Z^2 \,, \\ D_q(Q^2) &= \pm\frac{1}{2}\lambda e_q^2 - e_q V_q (A_e \pm \lambda V_e) P_Z + \frac{1}{2}(V_q^2 + A_q^2)\left[2V_e A_e \pm \lambda (V_e^2 + A_e^2)\right] P_Z^2 \,. \end{aligned} \tag{6.8}$$

where λ corresponds to the polarisation of the incoming lepton. It should be stressed that B_q multiplies g_4 and g_L while D_q multiplies g_1.

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