Applied Monte Carlo Numerical Integration Method Demonstration.

Practical application on a toroid section.

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***Abstract* — this paper explains the systematic process of obtaining the integral of an n-dimension function based on the stochastic approach of the Monte Carlo method to approximate the value applied to a section of a toroid without executing the symbolic analysis of the function, which is valuable especially if the analysis has an increased space complexity.**

Keywords — Monte Carlo; Integral; Stochastic; N-dimension.

I. Introduction

According to Bellman (1957), the curse of dimensionality is a common problem faced by the most diverse study areas known to humankind. Its effects can be both space and time complexity far greater than the current processing capacity of the best supercomputers available nowadays. This limitation of processing power demands alternatives to the way we face this problems, generally, it is necessary to approach from a different perspective i.e. reducing dimensions or variables, partitioning the main problem in smaller dependent sub problems or in some cases accepting sub optimal solutions.

The integral of a single dimension function represents the area between the function and the respective axis. The more dimensions we add to the analysis, results in an increase in complexity, and so becomes harder, or even impossible, to execute the symbolic analysis of said integrals, as said by Press, et al (1996).

In this paper, we are going to explore the applied method of the Monte Carlo integration based on taking randomly distributed samples of the function in a determined interval and averaging the result. As defined by Grindstead and Snell (2006), statistically, the more samples we take, more accurate the average becomes, this being the definition of the law of large numbers.

II. Monte carlo method

The Monte Carlo Methods are strongly stochastic and probabilistic methods based mostly on exploration and randomly generating states to evaluate the resultants and by the law of large numbers, which states that for a sufficient large number of samples, the experimental probability distribution will converge to the real probability. In an example given by Grindstead and Snell(2006), the coin tossing case, for a small number of tries the distribution of both heads and tails can be very different, but if we assume an n-number of tries big enough, the average with converge to ½ on both distributions (GRINDSTEAD; SNELL, 2006).

A. Pseudo-random Numbers

As said by Lotstedt (200-?) an important part of the method is the use of random numbers to assign values to variables so the analysis can be consistent to the definition of the method. As true random number generators are still a unsolvable computational problem we must recur to the use of pseudo-random number generators that meet the requirement that if the sequence is repeated after some time, this time must be long enough that it doesn`t affect the results.

B. Mersenne Twister – Pseudo-Random Number Generator (PRNG)

The Mersenne Twister PRNG was proposed by Matsumoto and Nishimura (1997) and states that "for a particular choice of parameters, the algorithm provides a super astronomical period of $2^{19937} - 1$ and 623-dimensional equidistribution up to 32 bits accuracy, while consuming a working area of only 624 words". That grants that the algorithm is as fast as the current PRNGs available while seeking to address most flaws found in other PRNGs (MATSUMOTO; NISHIMURA, 1997).

III. Monte carlo Integration

In this section, we will go through the process of estimating the integral of a generic one-dimension function to exemplify the steps and explain the theory before we face the toroid section case. The following steps were described by Mathews and Fink (2005):

A. Generating random samples

The first step is to use the PRNG to generate a number n of randomly distributed points X1, X2, X3… Xn in the interval [a,b].

B. Averaging the samples

The following step is to calculate the function value for each one of the n-samples generated in the previous step, sum every function result and divide by the number of samples. The formula is:

$$\langle f \rangle = \frac{1}{n} * \sum_{i=0}^{n-1} f\,(Xi) \qquad (1)$$

Being $\langle f \rangle$ the average of the samples, n the number of samples and f (Xi) the result of the function for the value Xi.

C. Estimating the integral

We estimate the integral value by multiplying the average of the samples by the interval of integration as show below:

$$I = \int_a^b f(x)dx \approx V * \langle f \rangle \qquad (2)$$

In this equation, I represents the integral value and V the total volume or interval of integration. If the interval is one-dimensional, V is only the difference between a and b, but when there are more dimensions we need to multiply the intervals to get the total volume of integration.

D. Estimating the Error of Approximation

The error E is estimated by the following formula:

$$E = V * \sqrt{\frac{\langle f^2 \rangle - \langle f \rangle^2}{n}} \qquad (3)$$

With $\langle f^2 \rangle$ being:

$$\langle f^2 \rangle = \frac{1}{n} * \sum_{i=0}^{n-1} f^2 (Xi) \qquad (4)$$

IV. Integrating the Toroidal Section

Figure (1) represents the toroid section that we intend to calculate the volume.

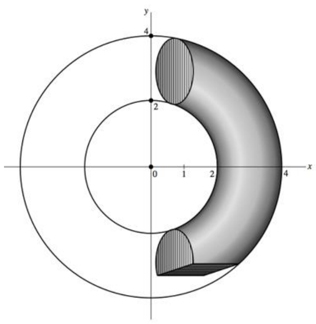

Fig. 1. Toroidal Section.

Figure (1) represents the toroid section that we intend to calculate the volume.

When we use more than one dimension, we need to define the interval where the randomly generated points are inside the toroid, and when they are not. With this rate, we can then multiply by the total volume of the intervals of integration to estimate the toroid volume, which is valid for more dimensions but is not practical to illustrate. The following formula defines if a point is inside or outside the toroid:

$$\langle f \rangle = \frac{1}{n} * \sum_{i=0}^{n-1} f\,(Xi, Yi, Zi) \qquad (4)$$

With $f\,(Xi, Yi, Zi)$ defined by:

$$f(Xi, Yi, Zi) = \begin{cases} 1 \; if \; z^2 + (\sqrt{x^2 + y^2} - 3)^2 \leq 1 \\ else \; 0 \end{cases} \qquad (5)$$

The interval of integration in this toroid is: $1 \leq x \leq 4; -3 \leq y \leq 4 \; and -1 \leq z \leq 1$

Resulting in a volume V of:

$$V = (x2 - x1) * (y2 - y1) * (z2 - z1) \qquad (6)$$

$$V = (4 - 1) * (4 - (-3)) * (1 - (-1) = 42 \qquad (7)$$

Running the algorithm in an external code compiler we have the following results for the rate of points inside the toroid, $\langle f \rangle$, of different n, the estimated volumes I for each n, and finally the estimated error.

TABLE I. *Results of the algorithm*

Values of $\langle f \rangle$ for different number of samples n.			
n	$\langle f \rangle$	***I***	***Error***
10	0.7	29.4	10.37
100	0.52	21.84	3.612
1000	0.546	22.93	1.113
10000	0.526	22.09	0.357
100000	0.5268	22.125	0.112

a. Results

V. Conclusion

In conclusion, this procedure is able to integrate complex functions that would be much harder if we would take the symbolic analysis approach. The error estimate is enough to evaluate the quantity of samples and improve the results as needed. As most problems do not need the exact result to be practical as a good estimate is more than enough to solve most problems in engineering and other fields of study.

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