Robust design of experiments for Bayesian estimation of seismic fragility curves based on sensitivity indices

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THEORY

I/ Information theory in the Bayesian framework

Within the Bayesian framework, the prior distribution is updated by the observed statistics to provide a posterior distribution thanks to Bayes' theorem.

Dataset $\mathbf{y} = (y_i)_{i=1}^k \in \mathcal{Y}^k \longrightarrow$ likelihood $\ell_k(\mathbf{y}|\theta)$

Parameter $\theta \in \Theta \longrightarrow$ prior $\pi(\theta)$

$\longrightarrow$ posterior $p(\theta|\mathbf{y}) \propto \ell_k(\mathbf{y}|\theta)\pi(\theta)$

In practical studies, the influence of the prior distribution on the posterior distribution has to be assessed. We propose to use sensitivity analysis tools for achieving this task.

Links between reference priors and sensitivity analysis

What? A revisit of the reference prior theory (Berger et al. 2009).
Why? To interpret the information issued from the prior, and to determine an objective one.
How? With a refinement of the mutual information definition with sensitivity indices.

Definition: Mutual information as a sensitivity index

$$I_D(\pi|k) = \mathbb{E}\left[D(\mathbb{P}_{\mathbf{y}}||\mathbb{P}_{\mathbf{y}|\theta})\right]$$

Reference prior theory principle: maximize $I_D(\pi|k)$ w.r.t. π, i.e., maximize the impact of θ onto the distribution of the data.

Our main result

We take $D = D_f$ an f-divergence with $f \underset{x\to 0^+}{=} \gamma x^\delta + o(x^\delta)$, $\delta < 1$.

Theorem: Sensitivity index reference prior (Van Biesbroeck 2023b)

Consider $\mathcal{P}$: the class of priors admitting a continuous and positive p.d.f. on Θ.

For any prior in $\mathcal{P}$ with density π: $\lim_{k\to\infty} k^{d\delta/2} I_{D_f}(\pi|k) = \gamma C_\delta \int_\Theta \pi(\theta)^{1+\delta} J(\theta)^{-\delta} d\theta.$

Moreover, if $\gamma(\delta+1) > 0$, then: $\lim_{k\to\infty} k^{d\delta/2}(I_{D_f}(J|k) - I_{D_f}(\pi|k)) \geq 0,$

where $J(\theta)$ is the Jeffreys prior p.d.f. The equality stands iff $\pi = J$.

▶ This theorem states that the Jeffreys prior is the optimal reference prior under our sensitivity index based mutual information.

Proper reference priors under generalized moment constraints

What? A suggestion of a framework to derive priors simultaneously objective and proper.
Why? Usual reference priors often lead to improper priors, jeopardizing their implementation.
How? Relying on the sensitivity–index–reference–prior theory, and considering the decay rates of the Jeffreys prior.

Theorem: Proper reference prior

Assume $g : \Theta \to \mathbb{R}_+$ be such that

$$\int_\Theta J(\theta) g(\theta)^{1+1/\delta} d\theta < \infty, \qquad \int_\Theta J(\theta) g(\theta)^{1/\delta} d\theta < \infty.$$

Then the proper prior with p.d.f. $\pi^* \propto J \cdot g^{1/\delta}$ is the sensitivity–index–reference–prior over the ones such that their p.d.f. π verify $\pi \cdot g \in L^1(\Theta)$.

▶ This theorem states that the knowledge of the decay rates of J allows to build a proper reference prior.

Sensitivity Indices for design of experiments

What? A robust design of experiments for input-output models:

$$\mathbf{y} = (\mathbf{a}, \mathbf{z}), \text{ with } \mathbf{z} = h(\mathbf{a}, \theta) \text{ for a stochastic function } h.$$

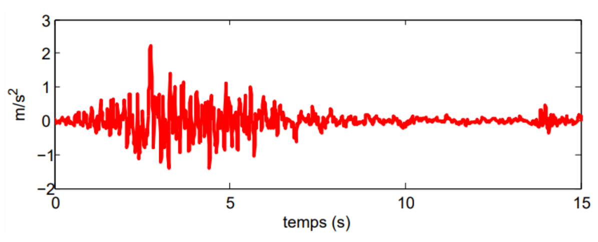
Why? For models of this type, careful choice of input data would improve the estimation.
How? With sensitivity indices, we maximize the information brought by the data to the posterior. We follow the idea of (Da Veiga 2015):

$$\text{maximize w.r.t. } a_{k+1}: \quad S(a_{k+1}) = \mathbb{E}[D(\mathbb{P}_{\theta|\mathbf{a},\mathbf{z}}||\mathbb{P}_{\theta|\mathbf{a},\mathbf{z},a_{k+1},z_{k+1}})]$$

PRACTICE

II/ Application to seismic fragility curves estimation

We seek to apply the above to the Bayesian estimation of seismic fragility curves, from a set of simulated or experimental data.

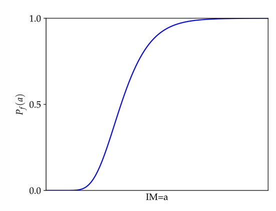


(1) Earthquake signals
input variable a

$\longrightarrow$

(2) Equipment responses
output variable z

$\longrightarrow$



(3) Fragility prediction
using Bayesian inference

(1) The ground-motion is summarized in a scalar: the Intensity Measure (IM). This is the input data a.
(2) The equipment fails or succeeds. The output data is binary: $z \in \{0, 1\}$.
(3) The log-normal model expresses $p(z|a,\theta) = \Phi\left(\frac{\log a - \log\alpha}{\beta}\right)$.

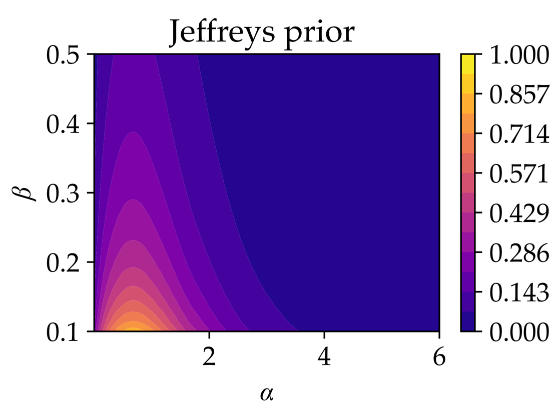
with $\theta = (\alpha, \beta)$, supposed to be a r.v. with distribution π: the prior.
The likelihood is given by $\ell_k(\mathbf{z}, \mathbf{a}|\theta) = \prod_{i=1}^k p(z_i|a_i,\theta)p(a_i)$, $(\mathbf{z}, \mathbf{a}) = (z_i, a_i)_{i=1}^k$.

Problem 1: Choice of the prior

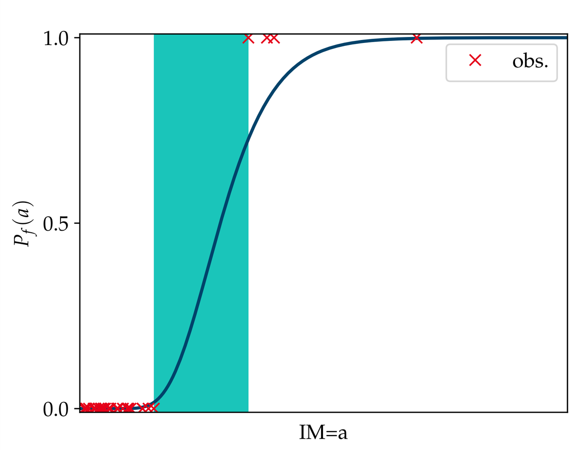
The data are limited due to the simulation cost of the studied equipment, therefore, the prior influences the estimations.
Also, nuclear safety requires a full auditability of the method, and so of all the choices involved.
The prior choice, thus, must be objective!

Our solution

- ▶ We computed the Jeffreys prior in the context of fragility curves (Van Biesbroeck et al. 2023a). We derive from its decay rates the proper reference prior based on sensitivity indices.
- ▶ The prior is approximated from its decay rates in the same fashion as in (Gu et al. 2018) for fast computation and fast generation of *a posteriori* estimates.



Problem 2: Degenerate samples of the data



When the observed data are such that failures and non-failures are discriminated w.r.t. their IM, the likelihood decays slowly as a function of β, leading to improper posterior with Jeffreys prior. We say the data are "degenerate".

$$\beta \to 0 \longleftrightarrow \beta \to \infty$$
$$\frac{D}{\beta} \;\; D' \longleftrightarrow \frac{C}{\beta^3} \;\; C''$$

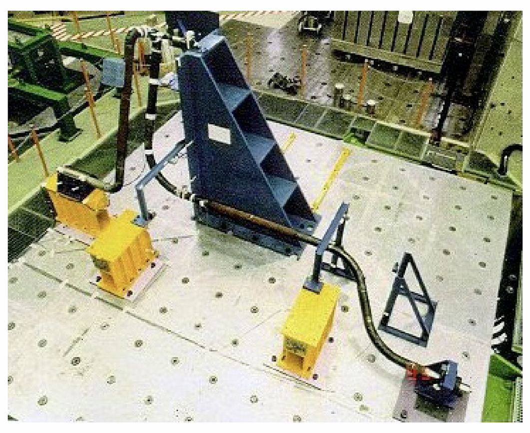
Above are the decay rates of the likelihood and of the Jeffreys prior w.r.t. β when the data are degenerate.

▶ This phenomenon happens more likely when the data is sparse, when the correlation between the IM and the structure's response is high, or when failures are rare.

Our solution

We carry out experimental design based on the maximization of sensitivity indices, in order to choose the right IM for robust estimation.

A case study

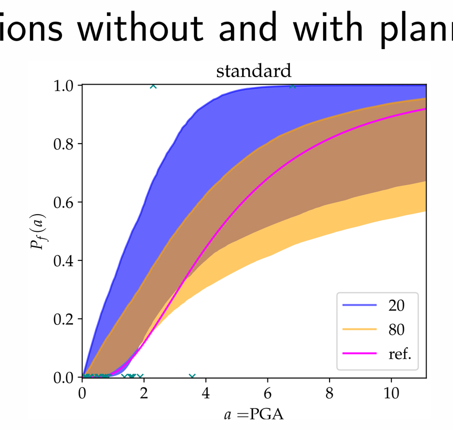


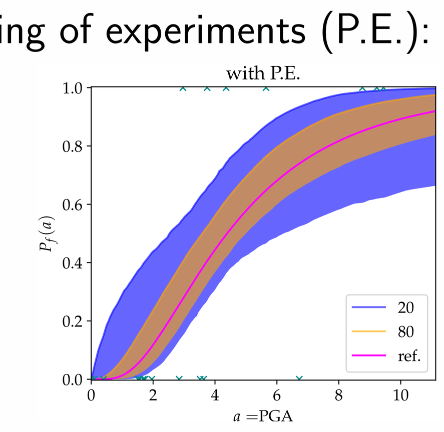
Piping system mock-up of a nuclear reactor

- Equipment submitted to seismic signals. Real experiments on a shaking table. For the case study: numerical simulations (10^4 computations carried out for benchmark).
- Failure of the equipment defined as the excessive rotation of the first elbow.

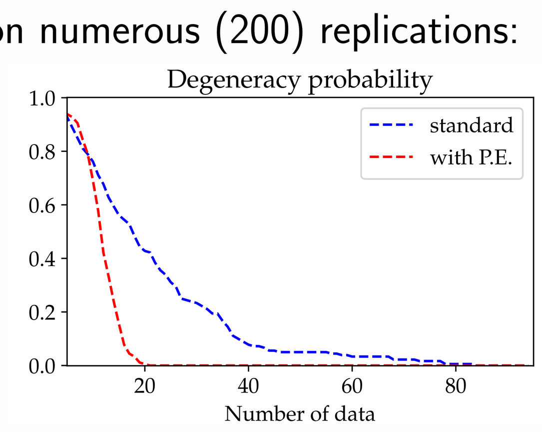
Results

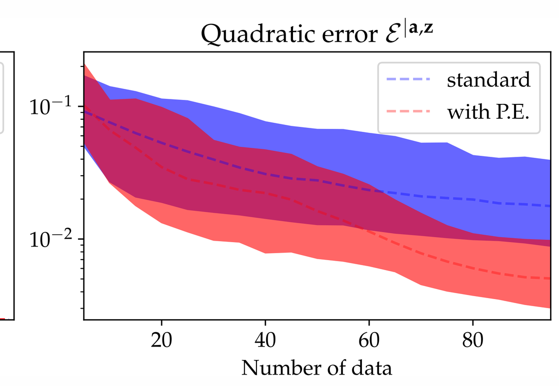
▶ Example of estimations without and with planning of experiments (P.E.):





▶ Performances on numerous (200) replications:





The results demonstrate the robustness and the reliability of the method.

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