

Typing Rules

What are type systems made of?

- Type systems are formal systems
- Usually described using logical rules
 - Provide a *specification* for what is well-formed
 - Rather than an *implementation* for determining whether something is well-formed
- If we want a type *checker* we turn these rules into code
 - Can be easy or hard, depending on the type system

Type Relation

We are defining a *type judgment*

$$\Gamma \vdash e : \tau$$

Which means: with the type bindings in Γ , I can conclude that e has the type τ

Which implies that e is well-formed

Γ is the *type environment*: a map from $\langle id \rangle$ to τ (type)

E.g., [$x \leftarrow \text{boolean}$, $y \leftarrow \text{number}$]

Similar in spirit to a deferred substitution, but maps to types, not values

Inference rules

Judgments are typically defined as a set of inference rules

$$\frac{\mathbf{A} \quad \mathbf{B}}{\mathbf{C}}$$

This is a *rule*, which says: If I know A and B, then I can conclude C

- There could be 0 or more things above the bar

Terminology: A and B are *premises* C is the *conclusion*

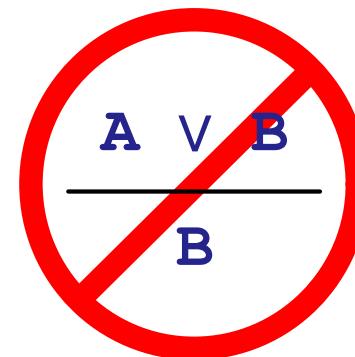
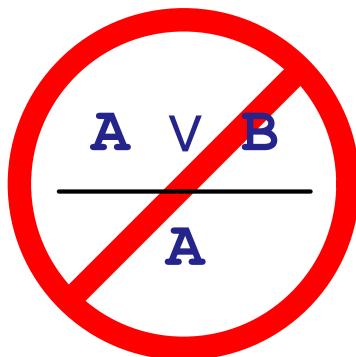
Inference Rule Example: Logical Or

These two rules *together* describe logical or

$$\begin{array}{c} \mathbf{A} \\ \hline \mathbf{A} \vee \mathbf{B} \end{array}$$

$$\begin{array}{c} \mathbf{B} \\ \hline \mathbf{A} \vee \mathbf{B} \end{array}$$

These two are false inferences



Type Rules

$\Gamma \vdash \text{<num>} : \text{number}$

$\Gamma \vdash \text{true} : \text{boolean}$

$\Gamma \vdash \text{false} : \text{boolean}$

(These have no premises)

$\Gamma \vdash e_1 : \text{number}$

$\Gamma \vdash e_2 : \text{number}$

$\Gamma \vdash \{+ e_1 e_2\} : \text{number}$

$1 : \text{number}$

$\text{true} : \text{boolean}$

$1 : \text{number}$

$2 : \text{number}$

$\{+ 1 2\} : \text{number}$

$1 : \text{number}$

$\text{false} : \text{boolean}$

$\{+ 1 \text{ false}\} : \text{no type}$

Type Rules

$\Gamma \vdash \text{<num>} : \text{number}$

$\Gamma \vdash \text{true} : \text{boolean}$

$\Gamma \vdash \text{false} : \text{boolean}$

(These have no premises)

$\Gamma \vdash e_1 : \text{number}$ $\Gamma \vdash e_2 : \text{number}$

$\Gamma \vdash \{+ e_1 e_2\} : \text{number}$

$1 : \text{number}$

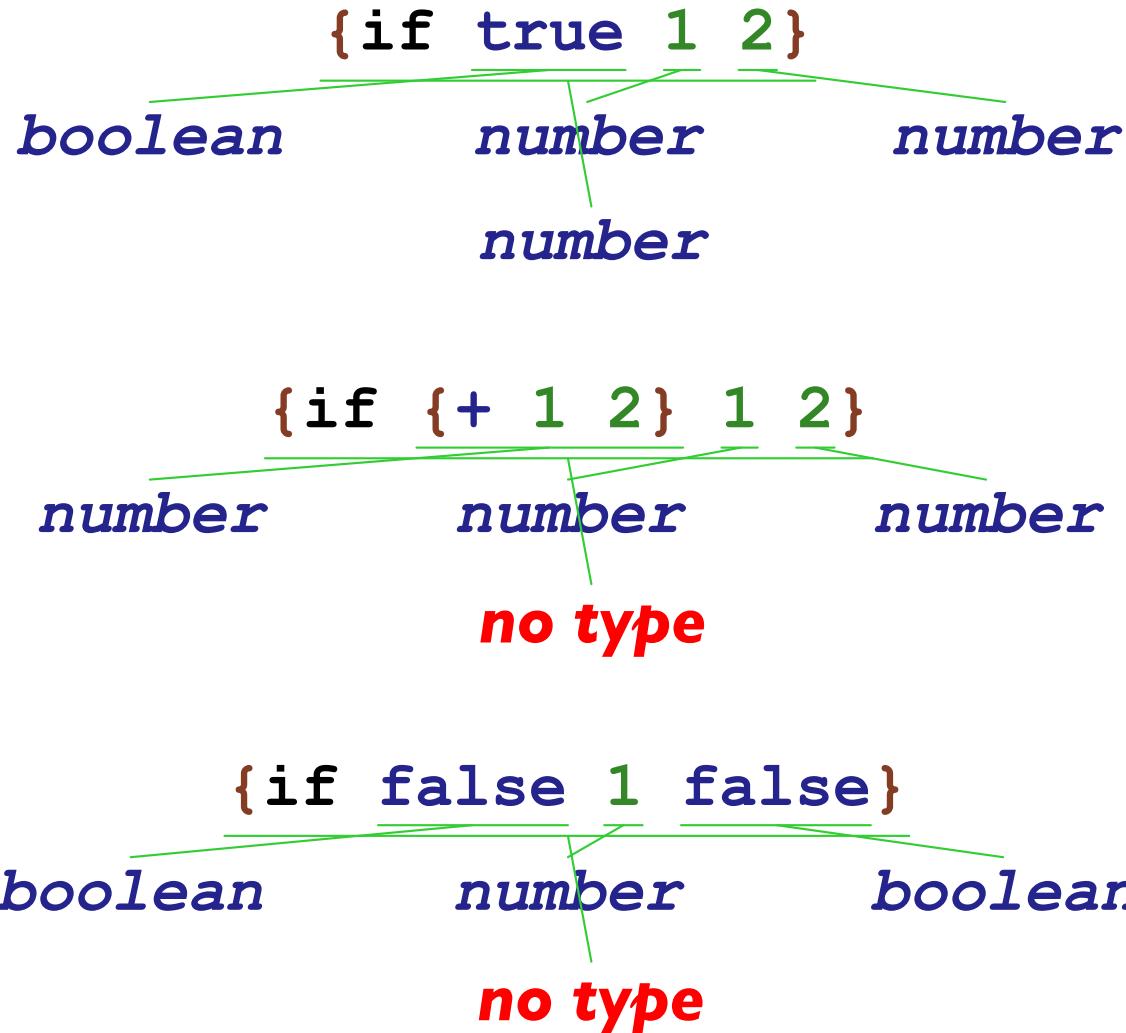
$2 : \text{number}$

$\{+ 1 2\} : \text{number}$

$3 : \text{number}$

$\{+ \{+ 1 2\} 3\} : \text{number}$

Types: Conditionals



Conditional Type Rules

$$\Gamma \vdash e_1 : \text{boolean}$$
$$\Gamma \vdash e_2 : \tau_0$$
$$\Gamma \vdash e_3 : \tau_0$$

$$\Gamma \vdash \{\text{if } e_1 \ e_2 \ e_3\} : \tau_0$$
$$\text{true}: \text{boolean}$$
$$1: \text{number}$$
$$2: \text{number}$$

$$\{\text{if true 1 2}\}: \text{number}$$
$$\{+ 1 2\}: \text{number}$$
$$1: \text{number}$$
$$2: \text{number}$$

$$\{\text{if } \{+ 1 2\} \ 1 \ 2\}: \text{no type}$$
$$\text{false}: \text{boolean}$$
$$1: \text{number}$$
$$\text{false}: \text{boolean}$$

$$\{\text{if false 1 false}\}: \text{no type}$$

Types: Variables and Functions

x : no type

{fun {x : boolean} x}

boolean

(boolean → boolean)

{fun {x : boolean} {if x 1 2}}

boolean

number

number

number

(boolean → number)

Variable and Function Type Rules

[... <id> $\leftarrow \tau$...] \vdash <id> : τ

$\Gamma[<\!\!id\!\!>\leftarrow \tau_1] \vdash e : \tau_2$

$\Gamma \vdash \{\text{fun } \{<\!\!id\!\!> : \tau_1\} \ e\} : (\tau_1 \rightarrow \tau_2)$

$\emptyset \vdash x : \text{no type}$

[x \leftarrow boolean] \vdash x:boolean

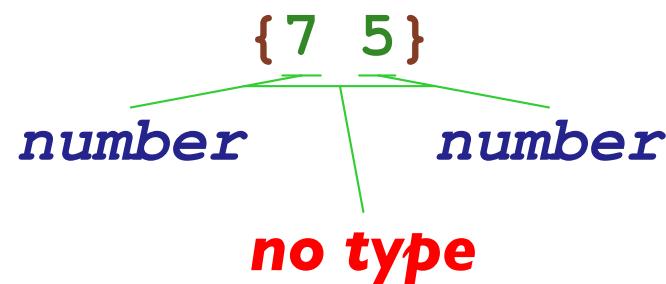
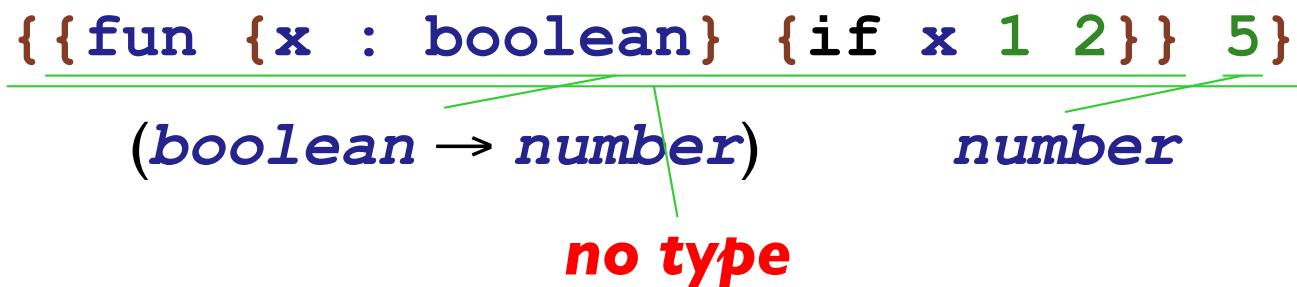
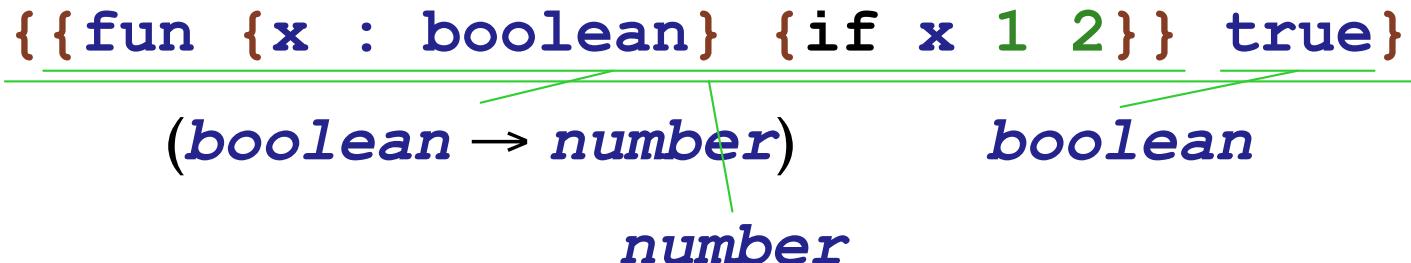
$\emptyset \vdash \{\text{fun } \{x : \text{boolean}\} \ x\} : (\text{boolean} \rightarrow \text{boolean})$

[x \leftarrow boolean] \vdash x:boolean [x \leftarrow boolean] \vdash 1:number [x \leftarrow boolean] \vdash 2:number

[x \leftarrow boolean] \vdash {if x 1 2}:number

$\emptyset \vdash \{\text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{boolean} \rightarrow \text{number})$

Types: Function Calls



Function Call Type Rule

$$\frac{\Gamma \vdash e_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \{e_1 \ e_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\} : (\text{boolean} \rightarrow \text{number}) \quad \emptyset \vdash \text{true} : \text{boolean}}{\emptyset \vdash \{\{\text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{number}}$$

$$\frac{\emptyset \vdash \text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\} : (\text{boolean} \rightarrow \text{number}) \quad \emptyset \vdash 5 : \text{number}}{\emptyset \vdash \{\{\text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

$$\emptyset \vdash 7 : \text{number} \quad \emptyset \vdash 5 : \text{number}$$

$$\emptyset \vdash \{7 \ 5\} : \text{no type}$$

Types: Multiple Arguments

{fun {x : number y : number} {+ x y}}

number
number
number

(number number → number)

{ {fun {x : number y : number} {+ x y}} 5 6}

(number number → number) number number
 number

{ {fun {x : number y : number} {+ x y}} 5 }

(number number → number) number
 no type

Revised Function and Call Rules

$$\frac{\Gamma[\langle \text{id} \rangle_1 \leftarrow \tau_1 \dots \langle \text{id} \rangle_n \leftarrow \tau_n] \vdash e : \tau_0}{\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle_1 : \tau_1 \dots \langle \text{id} \rangle_n : \tau_n \} \ e \} : (\tau_1 \dots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash e_0 : (\tau_1 \dots \tau_n \rightarrow \tau_0) \quad \Gamma \vdash e_1 : \tau_1 \quad \dots \quad \Gamma \vdash e_n : \tau_n}{\Gamma \vdash \{ e_0 \ e_1 \ \dots \ e_n \} : \tau_0}$$