Dynamic Econometric Models:

A. **Autoregressive Model**:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \beta_k Y_{t-k} + e_t$$

(With lagged dependent variable(s) on the RHS)

B. **Distributed-lag Model**:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \ldots + \beta_k X_{t-k} + e_t$$

(Without lagged dependent variables on the RHS)

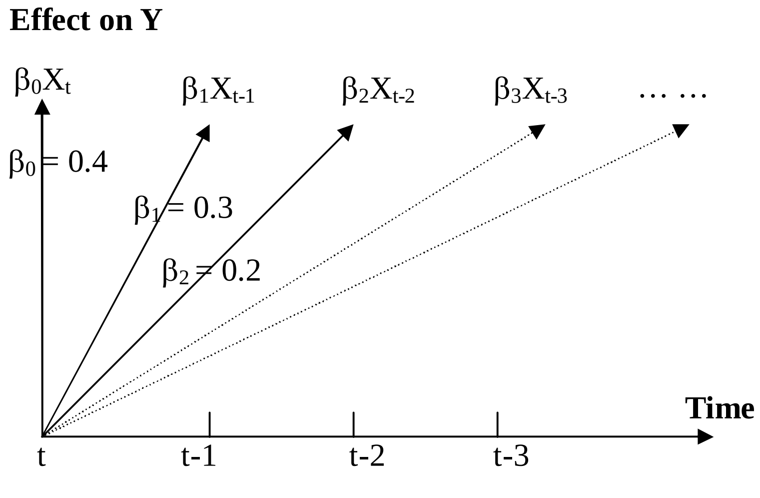
Where β_0 is known as the short run multiplier, or impact multiplier, because it gives the change in the mean value of Y following a unit change of X in the same time period. If the change of X is maintained at the same level thereafter, then, $(\beta_0 + \beta_1)$ gives the change in the mean value of Y in the next period, $(\beta_0 + \beta_1 + \beta_2)$ in the following period, and so on. These partial sums are called interim, or intermediate, multiplier. Finally, after k periods, that is

$\sum_{i=0}^{k} \beta_i = \beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k =$ B, therefore $\Sigma\beta_i$ is called the long run multiplier or total multiplier, or distributed-lag multiplier. If define the standardized $\beta_i^* = \beta_i / \Sigma\beta_i$,, then it gives the proportion of the long run, or total, impact felt by a certain period of time. In order for the distributed lag model to make sense, the lag coefficients must tend to zero as $k \to \infty$. This is not to say that β_2 is smaller than β_1; it only means that the impact of X_{t-k} on Y must eventually become small as k gets large.

For example: a consumption function regression is written as

$$Y = \alpha + 0.4X_t + 0.3\,X_{t-1} + 0.2\,X_{t-2} + 0.1X_{t-3} \ldots + e_t$$

Then the effect of a unit change of X at time t on Y and its subsequent time periods can be shown as the follow diagram:



The multiplier "round" of the Investment injection \$160 (k=1/(1-mpc))

Period	ΔI	ΔY	ΔC (if mpc = 0.5)
1	**\$160**	**\$160**	\$80
2		80	40
3		40	20
4		20	10
5		10	5
6		5	2.5
7		1.25	
		…	
1 => ∞		160	160

Other Examples: Money creation process, inflation process due to money supply, productivity growth due to expenditure or investment.

Period	New Deposits	New Loans	Required Reserve (10%)
1	\$1000	\$900	\$100
2	900	810	90
3	810	729	81
4	729	656.1	72.9
…			
Total	\$10,000	\$9,000	\$1,000

In general: Suppose the distributed-lag model is

$$\mathbf{Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \ldots + \beta_k X_{t-k} + e_t}$$

The basic idea: The long run responses of Y to a change in X are different from the immediate and short-run responses. And suppose the expect value in different periods is A. Then $E(X_t) = E(X_{t-1}) = E(X_{t-2}) = A$

$$\begin{aligned}\mathbf{E(Y_t)} &\mathbf{= \alpha + \beta_0 E(X_t) + \beta_1 E(Xt_{-1}) + \beta_2 E(Xt_{-2}) + .. + E(e_t)} \\ &\mathbf{= \alpha + \beta_0 A + \beta_1 A + \beta_2 A + \ldots + 0} \\ &\mathbf{= \alpha + \sum \beta_i A}\end{aligned}$$

This gives the constant long run corresponding to X = A, and E(Y) will remain at this level unit when X changes again.

Suppose look for one period ahead (t+1), and $X_{t+1} = A+1$

$$\mathbf{E(Y_{t+1}) = \alpha + \beta_0 E(X_{t+1}) + \beta_1 E(X_t) + \beta_2 E(X_{t-1}) + \ldots + E(e_{t+1})}$$
$$\mathbf{= \alpha + \beta_0(A+1) + \beta_1 A + \beta_2 A + \ldots + 0}$$
$$\mathbf{= \alpha + \beta_0 + \beta_0 A + \beta_1 A + \beta_2 A + \ldots}$$
$$\mathbf{= \alpha + \beta_0 + \sum \beta_i A}$$
$$\mathbf{E(Y_{t+2}) = \alpha + \beta_0 E(X_{t+2}) + \beta_1 E(X_{t+1}) + \beta_2 E(X_t) + \ldots + E(e_{t+2})}$$
$$\mathbf{= \alpha + \beta_0(A+1) + \beta_1(A+1) + \beta_2 A + \ldots + 0}$$
$$\mathbf{= \alpha + \beta_0 + \beta_1 + \beta_0 A + \beta_1 A + \beta_2 A + \ldots}$$
$$\mathbf{= \alpha + \beta_0 + \beta_1 + \sum \beta_i A}$$
$$\mathbf{E(Y_{t+3}) = \alpha + \beta_0 + \beta_1 + \beta_2 + \sum \beta_i A}$$
$$\mathbf{E(Y_{t+j}) = \alpha + \beta_0 + \beta_1 + \beta_2 + \ldots + \beta_j + \sum \beta_i A}$$

For all periods after t+3, E(Y) remains unchanged at the level given by $E(Y_{t+3})$, given no further change in X. Thus, $E(Y) = \alpha + \sum \beta_i X$

Reasons for Lags:

- Psychological reason: due to habit or inertia nature, people will not react fully to changing factors, e.g. Income, price level, money supply etc.
- Information reason: because imperfect information makes people hesitate on their full response to changing factors.
- Institutional reason: people cannot react to change because of contractual obligation.

Ad Hoc Estimation of Distributed-Lag Models

Estimation method:

First regress Y_t on X_t, then regress Y_t on X_t and X_{t-1}, then regress Y_t on X_t, X_{t-1} and X_{t-2}, and so on.

$Y = \alpha + \beta_0 X_t$

$Y = \alpha + \beta_0 X_t + \beta_1 X_{t-1}$

$Y = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2}$

$Y = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \beta_3 X_{t-3}$

$Y = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \beta_3 X_{t-3} + \ldots$

… …

This sequential procedure stops when the regression coefficients of the lagged variables start becoming statistically insignificant and / or the coefficient of at least one of the variables change signs, which deviate from our expectation.

- Significant t-statistics of each coefficient
- The sign of β_i does not change
- Highest R^2 and $\overline{R}^2$
- Min. values of BIC and/or AIC

Drawbacks:

- No a priori guide as to what is the maximum length of the lag.
- If many lags are included, then fewer degree of freedom left, and it makes statistical inference somewhat shaky.
- Successive lags may suffer from multi-collinearity, which lead to imprecise estimation. (When $Cov(X_{t-i}, X_{t-j})$ is high).
- Need long enough data to construct the distributed-lag model.

I. Koyck Approach to Distributed-Lag Models (Geometrical lag Model):

$$Y_t = \alpha + \beta_0 X_t + \beta_0\lambda X_{t-1} + \beta_0\lambda^2 X_{t-2} + \ldots + e_t$$

or

$$Y_t = \alpha + \beta_0 \lambda^0 X_t + \beta_0 \lambda^1 X_{t-1} + \beta_0\lambda^2 X_{t-2} + \ldots + e_t$$

Where $\beta_i = \beta_0\lambda^i$ and $0 < \lambda < 1$

(A priori restriction on β_i by assumption that the β_i s follow a systematic pattern.)

λ	β_0	β_1	β_2	β_3	β_4	B_5	β_6	β_7	…
0.8	β_0	0.8	0.64	0.512	0.409	0.327	0.262	0.209	….
0.3	β_0	0.3	0.09	0.027	0.0081	0.0024	0.0007	0.0002	…

$$\sum_{k=0}^{\infty} \beta_k = \beta_0(1 + \lambda + \lambda^2 + \lambda^3 + \ldots) = \beta_0\left(\frac{1}{1-\lambda}\right)$$ which is the long run multiplier

Then

$$Y_{t-1} = \alpha + \beta_0 X_{t-1} + \beta_0\lambda X_{t-2} + \beta_0\lambda^2 X_{t-3} + \ldots + e_{t-1}$$

$$\lambda Y_{t-1} = \lambda\alpha + \beta_0 \lambda X_{t-1} + \beta_0\lambda^2 X_{t-2} + \beta_0\lambda^3 X_{t-3} + \ldots + \lambda e_{t-1}$$

and

$$(Y_t - \lambda Y_{t-1}) = (\alpha - \lambda\alpha) + + \beta_0 X_t + (\cancel{\beta_0\lambda X_{t-1}} - \cancel{\beta_0 \lambda X_{t-1}}) + (\cancel{\beta_0\lambda^2 X_{t-2}} - \cancel{\beta_0\lambda^2 X_{t-2}})$$
$$+ (\cancel{\beta_0\lambda^3 X_{t-3}} - \cancel{\beta_0\lambda^3 X_{t-3}}) + \ldots + (e_t - \lambda e_{t-1})$$

By Koyck transformation (from a distributed-lag model transformed into an autoregressive model):

$$Y_t - \lambda Y_{t-1} = \alpha(1-\lambda) + \beta_0 X_t + (e_t - \lambda e_{t-1})$$

$\Rightarrow$ $Y_t = \alpha(1-\lambda) + \beta_0 X_t + \lambda Y_{t-1} + v_t$ where $v_t \sim iid(0, \sigma^2)$

$$\boxed{\Rightarrow\ Y_t = \theta_0 + \theta_1 X_t + \lambda Y_{t-1} + v_t}$$

The Y_{t-1} is a short run dynamic term and is built into the autoregressive model. The important of this autoregressive model gives the long-run multiplier that implied by the distributed lags model. The long-run multiplier can be obtained from the autoregressive model by calculating $\theta_0(1/(1-\lambda))$.

II. Rationalization of the Koyck Model: Adaptive Expectation Model

$\mathbf{Y_t = \beta_0 + \beta_1 X^*_t + e_t}$ (Long run function, and X* is unobserved expected level)

Go back one period and multiply $(1-\gamma)$, it becomes:

$\mathbf{(1-\gamma)Y_{t-1} = (1-\gamma)\beta_0 + \beta_1(1-\gamma)X^*_{t-1} + (1-\gamma)e_{t-1}}$ (to be subtracted later)

By postulating:

$\mathbf{X^*_t - X^*_{t-1} = \gamma(X_t - X^*_{t-1})}$ where $0 < \gamma < 1$ and γ is the coefficient of expectation

$\mathbf{X^*_t = \gamma X_t - + (1-\gamma) X^*_{t-1}}$

$$\mathbf{Y_t = \beta_0 + \beta_1 \{ \gamma X_t - + (1-\gamma) X^*_{t-1}\} + e_t}$$

$\Rightarrow$ $\mathbf{Y_t = \beta_0 + \beta_1 \gamma X_t - + \beta_1(1-\gamma) X^*_{t-1} + e_t}$

By subtracting:

$$\mathbf{Y_t - (1-\gamma)Y_{t-1} = \gamma\beta_0 + \beta_1\gamma X_t + [e_t - (1-\gamma)e_{t-1}]}$$

$\Rightarrow$ $\mathbf{Y_t = \gamma\beta_0 + \beta_1\gamma X_t + (1-\gamma)Y_{t-1} + v_t}$ (short run dynamic function)

$\boxed{\Rightarrow \mathbf{Y_t = \theta_0 + \theta_1 X_t + \theta_2 Y_{t-1} + v_t}}$

A short run dynamic function is used to derive for long run function.

III. Rationalization of the Koyck Model: <u>Partial Adjustment Model</u>

$\mathbf{Y^*_t = \beta_0 + \beta_1 X_t + e_t}$ (Long run function and Y* is unobserved and desired level)

By

$\mathbf{Y_t - Y_{t-1} = \delta(Y^*_t - Y_{t-1})}$ where $0 < \delta < 1$ and the δ is the coefficient of adjustment

➔ $\mathbf{Y_t = \delta Y^*_t + (1-\delta) Yt_{-1}}$

By substituting:

$$\mathbf{Y_t = \delta\,[\beta_0 + \beta_1 X_t + e_t] + (1-\delta) Y_{t-1}}$$

➔ $\mathbf{Y_t = \delta\beta_0 + \delta\beta_1 X_t + (1-\delta) Y_{t-1} + \delta e_t}$ (short run function)

➔ $\boxed{\mathbf{Y_t = \theta_0 + \theta_1 X_t + (1-\delta) Y_{t-1} + v_t}}$ where $v_t \sim \text{iid}(0, \sigma^2)$

Use the short run function to derive long run function.

IV. Combination of Adaptive Expectation and Partial adjustment

$\mathbf{Y^*_t = \beta_0 + \beta_1 X^*_t + e_t}$ where Y* and X* are the unobserved and desired level

Since the postulations of adaptive expectation and partial adjustment are

$\mathbf{Y_t - Y_{t-1} = \delta(Y^*_t - Y_{t-1})}$; ➔ $\mathbf{Y_t = \delta Y^*_t + (1-\delta) Yt_{-1}}$

$\mathbf{X^*_t - X^*_{t-1} = \gamma\,(X_t - X^*_{t-1})}$; ➔ $\mathbf{X^*_t = \gamma X_t - + (1-\gamma)\, X^*_{t-1}}$

➔ $\mathbf{Y_t = \delta[\beta_0 + \beta_1 X^*_t + e_t] + (1-\delta) Y_{t-1}}$

Go back one period and multiply $(1-\gamma)$ on both side, it becomes:

$\mathbf{(1-\gamma) Y_{t-1} = \delta(1-\gamma)[\beta_0 + \beta_1 X^*_{t-1} + e_{t-1}] + (1-\delta)(1-\gamma) Y_{t-2}}$

$\mathbf{(1-\gamma) Y_{t-1} = \delta(1-\gamma)\beta_0 + \delta(1-\gamma)\beta_1 X^*_{t-1} + (1-\delta)(1-\gamma) Y_{t-2} + \delta(1-\gamma) e_{t-1}}$

(to be subtracted later)

➔ $\mathbf{Y_t = \delta\beta_0 + \delta\beta_1 X^*_t + (1-\delta) Y_{t-1} + \delta e_t}$

➔ $\mathbf{Y_t = \delta\beta_0 + \delta\beta_1[\gamma X_t - + (1-\gamma)\, X^*_{t-1}] + (1-\delta) Y_{t-1} + \delta e_t}$

$\Rightarrow$ $Y_t = \delta\beta_0 + \delta\gamma\beta_1 X_t$ -+ $\delta(1-\gamma)\beta_1 X^*_{t-1} + (1-\delta)Y_{t-1} + \delta e_t$

By subtracting:

$$Y_t - (1-\gamma)Y_{t-1} = \delta\gamma\beta_0 + \delta\gamma\beta_1 X_t + (1-\delta)Y_{t-1} - (1-\delta)(1-\gamma)Y_{t-2} + \delta e_t - \delta(1-\gamma)e_{t-1}$$

$$\Rightarrow Y_t = \delta\gamma\beta_0 + \delta\gamma\beta_1 X_t + \{(1-\delta) + (1-\gamma)\}Y_{t-1} - (1-\delta)(1-\gamma)Y_{t-2} + \{\delta e_t - \delta(1-\gamma)e_{t-1}\}$$

$$\boxed{\Rightarrow Y_t = \alpha_0 + \alpha_1 X_t + \alpha_2 Y_{t-1} - \alpha_3 Y_{t-2} + v_t}$$

Detecting Autocorrelation in Autoregressive Models: Durbin-*h* test:

$$h = (1 - \frac{DW}{2})\sqrt{\frac{n}{1 - n \times Var(\alpha_2)}}$$

Where var(α_2) = Variance of coefficient of the lagged dependent variable, Y_{t-1}

If h >1.96, reject Ho. $\Rightarrow$ There is positive first order autocorrelation

If h <-1.96, reject Ho. $\Rightarrow$ There is negative first order autocorrelation

If –1.96 <h <1.96, do not reject Ho. $\Rightarrow$ There is no first order autocorrelation

In general, **Lagrange Multiplier** test is preferred to the ***h***-test because it comparatively suits for large as well as small sample size.

The steps of Lagrange Multiplier Test:

(1) Obtain the residuals from the estimated equation:

$$\hat{v}_t = Y_t - \hat{Y}_t = Y_t - \hat{\alpha}_0 - \hat{\alpha}_1 X_t - \hat{\alpha}_2 Y_{t-1} - \hat{\alpha}_3 Y_{t-2}$$

(2) Run an auxiliary regression:

$$\hat{v}_t = \alpha_0' - \alpha_1' X_t - \alpha_2' Y_{t-1} - \alpha_3' Y_{t-2} + \beta\, v_{t-1} + \varepsilon_t$$

(3) Test the null hypothesis that β=0 with the following test statistic:

$$\mathbf{LM = nR^2} \quad \sim \quad \chi^2$$

LM has a chi-square distribution with degrees of freedom equal to the number of restrictions in the null hypothesis (in this case is one). If LM is greater the critical chi-square value, then we reject the null hypothesis and conclude that there is indeed autocorrelation exist in the original autoregressive equation.

V. Almon Approach to Distributed-Lag Models:

- (Polynomial Distributed-Lag model)

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \ldots + \beta_k X_{t-k} + e_t$$

$$\Rightarrow \quad Y_t = \alpha + \Sigma\beta_i X_{t-i} + e_t \quad \text{Where } i = 0 \text{ to } k$$

Suppose the following model have only two-lagged terms of X_t, such as

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + e_t$$

Almon assumes that β_i can be approximated by a suitable-degree polynomial in i which is the length of lags. For example, if there is a second degree of polynomial of β_i:
i.e., $\beta_i = a_0 + a_1 i + a_2 i^2$

Then $$Y_t = \alpha + \Sigma(a_0 + a_1 i + a_2 i^2) X_{t-i} + e_t$$

$$Y_t = \alpha + a_0 \Sigma X_{t-i} + a_1 \Sigma\, i X_{t-i} + a_2 \Sigma\, i^2 X_{t-i} + e_t$$

After transforming the variables, the distributed-lag model becomes:

$$\boxed{Y_t = \alpha + a_0 Z_{0t} + a_1 Z_{1t} + a_2 Z_{2t} + e_t}$$

Where
$$Z_{0t} = \Sigma\, X_{t-i} = X_t + X_{t-1} + X_{t-2}$$
$$Z_{1t} = \Sigma\, i X_{t-i} = X_{t-1} + 2X_{t-2}$$
$$Z_{2t} = \Sigma\, i^2 X_{t-i} = X_{t-1} + 4X_{t-2}$$

After transforming the Z_{0t}, Z_{1t}, Z_{2t}, run the regression on Yt against on all Z_{it} to find the estimated coefficient values of $\hat{a}_0$, $\hat{a}_1$, $\hat{a}_3$, then deduces the estimated values of $\hat{\beta}_i$'s for the original postulated model.

For example, (i) if $\beta_i = a_0 + a_1 i + a_2 i^2$, it is a ***second*** degree of polynomial

$\beta_0 = a_0$

$\beta_1 = a_0 + a_1$

$\beta_2 = a_0 + 2a_1 + 4a_2$

$\beta_3 = a_0 + 3a_1 + 9a_2$

$\ldots$

$\beta_k = a_0 + k a_1 + k^2 a_2$

In general, if there are k lagged terms of X and the degree of the polynomial is 2, then

$$\beta_k = a_0 + k a_1 + k^2 a_2$$

For example, (ii) if $\beta_i = a_0 + a_1 i + a_2 i^2 + a_3 i^3$, it is a ***third*** degree of polynomial

$\beta_0 = a_0$

$\beta_1 = a_0 + a_1 + a_2 + a_3$

$\beta_2 = a_0 + 2a_1 + 4a_2 + 8a_2$

$\beta_3 = a_0 + 3a_1 + 9a_2 + 27a_3$

…

$\beta_k = a_0 + k a_1 + k^2 a_2 + k^3 a_3.$

In general, if there are k lagged terms of X and the degree of the polynomial is 3, then

$$\beta_k = a_0 + k a_1 + k^2 a_2 + k^3 a_3$$

Then $$Y_t = \alpha + \Sigma(a_0 + a_1 i + a_2 i^2 + i^3 a_3) X_{t-i} + e_t$$

$$Y_t = \alpha + a_0 \Sigma X_{t-i} + a_1 \Sigma\, i X_{t-i} + a_2 \Sigma\, i^2 X_{t-i} + a_3 \Sigma\, i^3 X_{t-i} + e_t$$

After transforming the variables, the distributed-lag model becomes:

$$\boxed{Y_t = \alpha + a_0 Z_{0t} + a_1 Z_{1t} + a_2 Z_{2t} + a_3 Z_{3t} + e_t}$$

Where $Z_{0t} = \Sigma X_{t-i} = X_t + X_{t-1} + X_{t-2} + X_{t-3}$

$Z_{1t} = \Sigma\, i X_{t-i} = X_{t-1} + 2X_{t-2} + 3X_{t-3}$

$Z_{2t} = \Sigma\, i^2 X_{t-i} = X_{t-1} + 4X_{t-2} + 9X_{t-3}$

$Z_{3t} = \Sigma\, i^3 X_{t-i} = X_{t-1} + 8X_{t-2} + 27X_{t-3}$

For example, (iii) if $\beta_i = a_0 + a_1 i + a_2 i^2 + a_3 i^3 + a_4 i^4$, it is a ***fourth*** degree of polynomial

$\beta_0 = a_0$

$\beta_1 = a_0 + a_1 + a_2 + a_3 + a_4$

$\beta_2 = a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4$

$\beta_3 = a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4$

$\beta_4 = a_0 + 4a_1 + 16a_2 + 64a_3 + 256a_4$

In general, if there are k lagged terms of X and the degree of the polynomial is 4, then

$$\beta_k = a_0 + k a_1 + k^2 a_2 + k^3 a_3 + k^4 a_4$$

Then $Y_t = \alpha + \Sigma(a_0 + a_1 i + a_2 i^2 + i^3 a_3 + i^4 a_4) X_{t-i} + e_t$

$$Y_t = \alpha + a_0 \Sigma X_{t-i} + a_1 \Sigma\, i X_{t-i} + a_2 \Sigma\, i^2 X_{t-i} + a_3 \Sigma\, i^3 X_{t-i} + a_4 \Sigma i^4 X_{t-i} + e_t$$

After transforming the variables, the distributed-lag model becomes:

$$\boxed{Y_t = \alpha + a_0 Z_{0t} + a_1 Z_{1t} + a_2 Z_{2t} + a_3 Z_{3t} + a_4 Z_{4t} + e_t}$$

Where $Z_{0t} = \Sigma\, X_{t-i} = X_t + X_{t-1} + X_{t-2} + X_{t-3} + X_{t-4}$

$Z_{1t} = \Sigma\, i X_{t-i} = X_{t-1} + 2X_{t-2} + 3X_{t-3} + 4X_{t-4}$

$Z_{2t} = \Sigma\, i^2 X_{t-i} = X_{t-1} + 4X_{t-2} + 9X_{t-3} + 16X_{t-4}$

$Z_{3t} = \Sigma\, i^3 X_{t-i} = X_{t-1} + 8X_{t-2} + 27X_{t-3} + 64X_{t-4}$

$Z_{4t} = \Sigma\, i^4 X_{t-i} = X_{t-1} + 16X_{t-2} + 81X_{t-3} + 256X_{t-4}$

Causality in Economics: Granger Test

The distributed lags model can be applied to test the direction of causality in economic relationship. Such test is useful when we know the two economic variables are related but we don't know which variable causes the other to move.

The Granger causality test assumes that the information relevant to the prediction of the respective variables, e.g. Y(GNP) and X(Money), is contained solely in the time series data on these variables.

$$Y_t = \mu + \sum_{i=1}^{k} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j Y_{t-j} + e_{1t}$$

$$X_t = \mu' + \sum_{i=1}^{m} \lambda_i X_{t-i} + \sum_{j=1}^{n} \delta_j Y_{t-j} + e_{2t}$$

Four different situations:

1. **Unidirectional causality** from M to GNP when $\boldsymbol{\Sigma\alpha_i \neq 0}$ and $\boldsymbol{\Sigma\delta_j = 0}$.
   (i.e., X → Y)
   The t-statistics of α_i are significant
   The t-statistics of δ_i are insignificant

2. **Unidirectional causality** from GNP to M when $\boldsymbol{\Sigma\alpha_i = 0}$ and $\boldsymbol{\Sigma\delta_j \neq 0}$.
   (i.e., Y → X)
   The t-statistics of α_i are insignificant
   The t-statistics of δ_i are significant

3. **Feedback or Bilateral causality** from M to GNP and GNP to M when $\boldsymbol{\Sigma\alpha_i \neq 0}$ and $\boldsymbol{\Sigma\delta_j \neq 0}$
   (i.e., X ←→ Y)
   The t-statistics of α_i are significant
   The t-statistics of δ_i are significant

4. **Independent** between GNP and M when $\boldsymbol{\Sigma\alpha_i = 0}$ and $\boldsymbol{\Sigma\delta_j = 0}$
   (i.e., X ←/→ Y)
   The t-statistics of α_i are insignificant
   The t-statistics of δ_i are insignificant

Steps for Granger Causality Test:

1. Ho: $\Sigma\boldsymbol{\alpha}_{\mathbf{i}} = 0$ (i.e. X does not cause Y) and H_1: $\Sigma\boldsymbol{\alpha}_{\mathbf{i}} \neq 0$
2. Run the restricted regression: Regress current Y on all lagged Y terms and (other variables if necessary), but do not include the lagged X variables in this regression.

   $$Y_t = \mu' + \Sigma\boldsymbol{\beta}_{\mathbf{j}}Y_{t-j} + e'_{1t}$$

   And obtain the $\mathbf{RSS_R}$.
3. Then run the unrestricted regression

   $$Y_t = \mu + \Sigma\boldsymbol{\alpha}_{\mathbf{i}}X_{t-i} + \Sigma\boldsymbol{\beta}_{\mathbf{j}}Y_{t-j} + e_{1t}$$

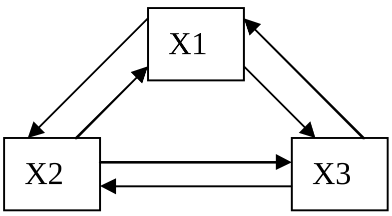
   And obtain the $\mathbf{RSS_{UR}}$.
4. $\mathbf{F^* = [(RSS_R - RSS_{UR})/m] / [RSS_{UR} /(n-k)]}$ where m is the number of lagged X terms and k is the number of parameters estimated in unrestricted regression.
5. If $F^* > F^c$ (critical value), then reject the Ho. That is X causes Y.

   If $F^* < F^c$ (critical value), then not reject the Ho. That is X does not cause Y.
6. For testing GNP causes M, just simply repeats the above five steps.

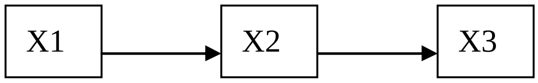
The limitations of Granger causality:

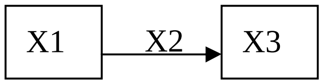
1. The number of lags includes in the unrestricted relationship can affect the level of significance of F. For example, with small samples, the choice of one lag of m versus longer lags (e.g., using m of 4 versus m of 8) affects the F-test.
2. In general, there is no good way to determine the lag length used for independent variables.
3. Econometricians have been able to show the Granger test can yield conflicting results.
4. Granger causality test cannot always proof of causality, it is confirmation of the direction of influence. It does not address the issues of causal closure. That is, there may be specification errors in the relationship.

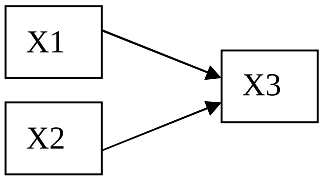
Example on three-variable causality test:

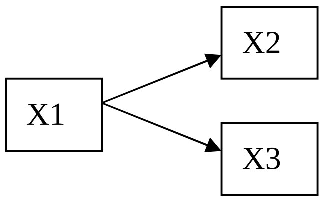


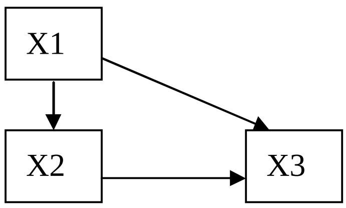
Some possible trivariate causality relationships:

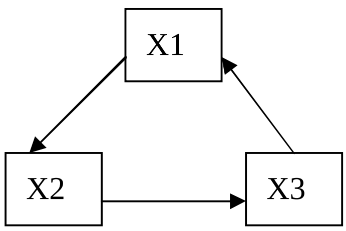












Empirical study:

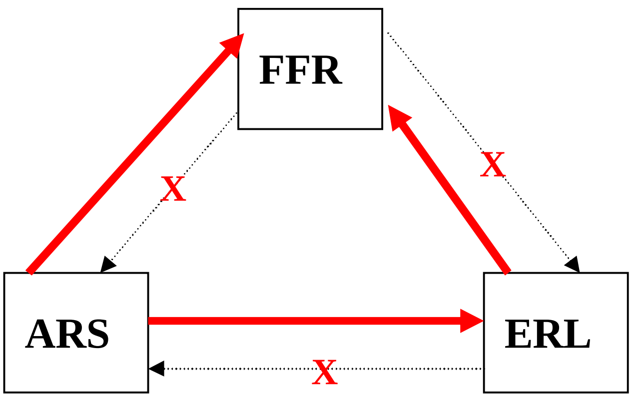
In order to test the international interest rate transmissions, we use the three market 1-month interest rates (i.e., US's Fed Rate (FFR), London's Euro-Dollar Rate (ERL), and Singapore's Asian-Dollar Rate (ARS)) to identify the transmission direction. The data are obtained from DataStream from 1980.01 to 2000.08

The Results of Granger Causality test:

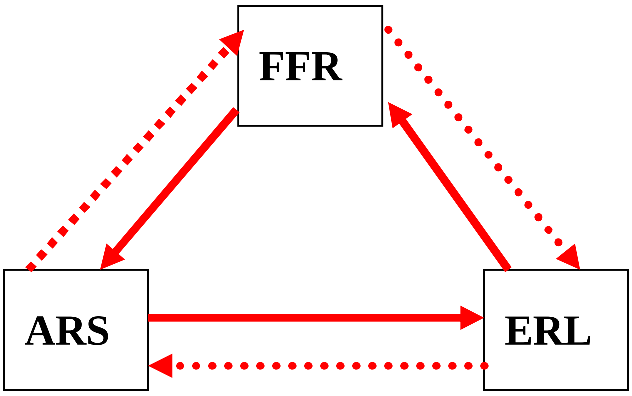
H_0 lags	ERL /→ ARS	ARS /→ERL	FFR /→ARS	ARS /→FFR	FFR /→ERL	ERL /→FFR	**Critical F^c**
1	7.211*	140.092*	**0.798**	78.287*	**0.129**	12.161*	3.84
2	**0.523**	53.714*	**0.552**	37.702*	**1.305**	10.341*	3.00
3	**0.160**	27.314*	**2.452**	15.425*	3.865*	5.576*	2.60
4	5.734*	26.270*	4.796*	12.935*	2.750*	4.523*	2.37
5	2.450*	25.471*	6.927*	8.600*	7.240*	2.310*	2.21
6	4.666*	26.277*	7.792*	8.447*	8.119*	3.233*	2.10
7	12.770*	26.826*	9.326*	6.637*	4.697*	3.641*	2.01
8							1.94
9	10.056*	25.149*	9.711*	8.598*	7.938*	4.893*	1.88
10							1.83
11							1.79
12	7.259*	15.913*	8.620*	7.678*	6.783*	4.707*	1.75

Note: "*" means the statistics are larger than critical values, thus the H_0 is rejected.

From the 2 to 3-lags result, we observe the transmission relationships are as following:



From the 4-lags result, we observe the transmission relationships of the three markets are as following:



The Granger(1969) approach to the question of "whether X causes Y" is applied to see how much of the current Y can be explained by the past values of Y and then to see whether adding lagged values of X can improve the explanation.

"Y is said to be Granger-caused by X" if X helps in the prediction of Y, or equivalently if the coefficients of the lagged Xs are statistically significant, however, it is important to note that the statement "X Granger causes Y" does not imply that Y is the effect or the results of X.