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Forward and inverse modelling of eco-evolutionary dynamics in biological and economic systems

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Cover picture: Top: forest in Sorapiss, Dolomites, Italy. Bottom: New York City, USA. @ Luca
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Introduction

„ Nature loves to hide.

— Heraclitus (c.6th-5th century BCE)

1.1 Context

1.1.1 Biological and economic systems as complex adaptive systems

What are the similarities between biological and economic systems? Both are complex adaptive systems (CAS) ([Levin2002](#)), which are composed of heterogeneous entities structured at different levels of organizations, that interact in nonlinear ways and experience evolutionary processes. Interaction and evolutionary processes take many different forms and operate at different organizational level ([Levin1998](#)) (see ??). Interestingly, the variety of processes involved and their couplings do not necessarily lead to unpredictable, chaotic, or erratic structures and dynamics ([Olff2009](#)), but rather induce organised structural properties and behavior ([mitchell2009complexity](#)). In biological systems, those include patterns of species richness, where for instance montane regions are often associated with a disproportionately high number of species ([Rahbek2019](#)). In economic systems, those include the distribution of international income, where some countries have systematically developed much more rapidly than others ([ace moglu2001colonial](#)). A common direction on the research agenda in Biology and Economics is to comprehend the set of interaction and evolutionary processes that determine these emergent properties ([Nordbotten2018](#)), and how do they do so. In biological systems, the nature of the processes of interaction and evolution is identified, and the current challenge is to comprehend the mechanisms resulting from their couplings. In economic systems, we still do not exactly understand the nature of those processes, and how are they involved.

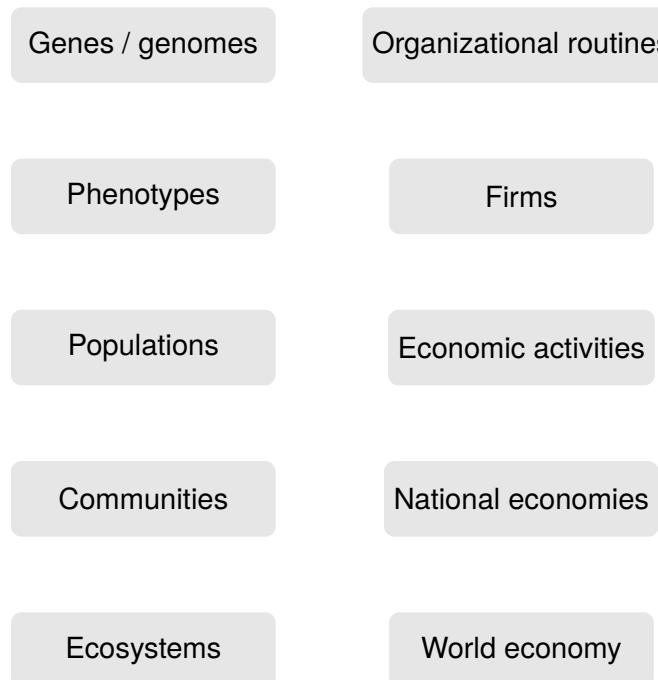


Fig. 1.1: Schematic diagram of proposed organisational levels in biological and economic systems. A is inspired from (Hendry+2016)

1.1.2 Ecological and evolutionary processes drive the dynamics of biological systems

In biological systems, interaction processes are generally designated as ecological processes, and involve fluxes of energy and matter across space and time, encompassing the processes of interaction between organisms (biotic interactions) and between organisms and their environment (abiotic interactions), and dispersal processes (movement of individual across space) (Vellend2010a). Evolutionary processes designate those processes responsible for the change of heritable characteristics (DNA, genes, phenotypes) over successive generations (Hall2013). The coupling between ecological and evolutionary processes is acknowledged since the very birth of the theory of Evolution, when Darwin realised a link between the different ecological opportunities across the Galápagos islands and the different beak shapes in the finches he found in each island (darwin2004origin), during his voyage on the Beagle. He reasoned that the variations in ecological opportunities lead to a differential in survival for certain phenotypes, which over time resulted in the evolution of different beak shapes. Since then, we know that ecological processes directly affect evolutionary response (Ezard2009). In the recent years, the idea that not only ecological processes can affect evolutionary response, but also that evolutionary

processes could affect ecological processes, has developed (XXX). Empirical studies have demonstrated that evolution can be rapid and occur on similar time scales as ecology (Hairston2005; Pelletier2009) and have quantifiable effects on ecological dynamics (Ezard2009), leading to feedbacks between ecological and evolutionary processes, so-called eco-evolutionary feedbacks (Pelletier2009; Schoener2011). Eco-evolutionary feedbacks involve situations where an ecological property influences evolutionary change, which then feeds back to an ecological property, or vice versa (Govaert2019a). Examples are feedbacks between population density (ecological property) and trait evolution (evolutionary change), which can lead to evolutionary branching through the effect of competition (Doebeli1999). Eco-evolutionary feedbacks are also involved in adaptation mechanisms (Doebeli1999), where species disperse and phenotypic variations allow to adapt to local environments (XXX). Those feedbacks may greatly influence the mechanisms driving the dynamics of ecosystems (Urban2016), but our understanding of their nature and effect is limited (Lion2022). In particular, eco-evolutionary feedbacks are expected to play a critical role in the evolution of the biosphere in the coming decades (Norberg2012), as ecosystems are being rapidly affected by anthropogenic pressure and with climate change (Ellis2011; Midgley2019). In order to mitigate the consequences of human development, it is of utmost urgency to better understand eco-evolutionary feedbacks (Norberg2012), and develop mechanistic models embedding this knowledge (Urban2016). This will in turn provide more reliable forecasts of ecosystem states (Clark2001), to help designing adequate management of ecosystem services (Urban2016).

1.1.3 Drivers of economic change

The processes that determine economic change is controversial in economics (Nelson2014). To explain economic development, mainstream economic theory (10.1093/cje/bet027) assumes that economic systems are in equilibrium, in the sense that the demand and supply of goods and services are balanced on all relevant markets. Firms are rational in maximizing profits by adapting to demand and supply, and the observed economic change is driven by exogenous forces, such as technological change (Romer1986). Evolutionary economics, promoted by the seminal work of Ref. (Nelson2014), criticizes this view and seeks to explain economic change by focusing on endogenous forces. Evolutionary economics suggests that interactions between firms and economic activities, and evolutionary processes acting upon them, are major processes contributing to economic change (Hodgson2019). For instance, firms or economic activities may interact positively or negatively (Wernerfelt1989; Pistorius2007Ozman2009; Saavedra2009a; Cohendet2018;

Menon2015), spread across space (RogersEverettM2003DoI; Zahra2000), and adapt (Cordes2006) or transform into new economic institutions (Freeman2002; Hodgson2004; Aldrich2008), affecting economic development at the regional and national scale. Because these processes are analogous to eco-evolutionary processes driving the dynamics of biological systems, a number of modelling approaches have borrowed concepts and methods from biology in the last decades, aiming at better understanding the processes underlying emergent properties in economic systems (Tacchella2018; Saavedra2009a; Scholl2020; Zhang2018; Modis1997; Saavedra2014; Farmer1999; Michalakelis2011; Marasco2016; Gatabazi2019; Cauwels56; Applegate2021; Suweis2015). For instance, (Saavedra2009a) has successfully used a model of mutualistic interaction to explain structural patterns in industrial cooperation. Also, (Scholl2020) uses the concepts of foodwebs and density dependence to explain market malfunctions and excess volatility in financial markets. However, those studies did not seek to understand how these processes may affect economic development. Recent modelling approaches developed in evolutionary biology may help to disentangle whether eco-evolutionary processes could explain differences in economic development across countries.

1.2 Modeling eco-evolutionary dynamics

1.2.1 Forward modelling of eco-evolutionary processes

The complex interplay between ecological and evolutionary processes, acting at different scales of time and space and organization, can hardly be studied with experimental approaches (Pontarp2019; Hagen2022). As such, a deductive approach, relying on forward modelling, has traditionally been put forward to advance our understanding of the mechanisms underlying (Brummitt2020). Along this approach, hypotheses about causal processes are embedded in a model, which forward integration generates emergent properties. Such emergent behavior may be seen as predictions from the consideration of the causal processes (May2004). The role of the modeller is to point at the mechanisms by which the properties emerge, disentangling the underlying interplay between the processes. In the early 1930s to 1940s, by formulating tractable mathematical models implementing the processes of reproduction, dispersal and mutations, the work of Fisher, Wright and Haldane has greatly contributed to the modern synthesis of evolutionary biology (huxley1942evolution), generally accepted as the basis of our current understanding of evolutionary dynamics. The mechanistic models commonly take the form of differential equations (DE), and express how the processes under investigation affect the rate of change of the population characteristics, such as the proportion

of a given allele. However, the requirement of tractable mathematical models (DEs that yield analytical solutions) has involved strong assumptions on the processes investigated, that are poorly representative of the complexity of eco-evolutionary feedbacks in nature (Govaert2019a). In particular, ecological scenarios have been strongly simplified, and did not take into account how evolution could affect population dynamics (Lion2022). As such, traditional mathematical models have omitted eco-evolutionary feedbacks and density dependence.

With the increase in computational capacity, novel modelling approaches relying on individual based models (IBMs) have appeared (deangelis2005individual). IBMs allow to capture processes acting at the individual level, requiring less simplifying assumptions than traditional mathematical models (deangelis2005individual). Capturing more realistic scenarios by allowing the forward integration of complex hypothesis, the lack of analytical tractability of IBMs may nonetheless occult the mechanisms underlying emergent properties (Lion2016; May2004). The recent development of mathematical techniques, such moment closure approximations (law1999moment; Gandhi2000; Nordbotten2020; Lion2016), adaptive dynamics theory (Metz1995), and probability theory (Champagnat2006), are generating novel pathways by filling the gap between IBMs and mathematical models. Analogous to renormalisation group analysis developed in quantum and statistical physics (Sayama), they form a toolbox to rigorously derive how emergent properties are influenced by processes operating at different organizational levels. As such, they allow an analytical underpinning to IBM simulations, and can generate a general understanding of the key mechanisms at stake (Lion2016).

The combination of numerical simulations and, e.g., adaptive dynamics theory, has successfully shed new lights on on the emergence of evolutionary branching under frequency-dependent selection (Dieckmann1999; Doebeli2003). An other example is the work of (Meszena1997; Debarre2013a; Mirrahimi2020), that has provided new insights on the effect of habitat heterogeneity on population dynamics. However, our current understanding of eco-evolutionary feedbacks omits potentially significant factors, such as the structuration of populations over complex spatial structures (XX) and highly dimensional phenotypic space (XXX).

The consideration of such details is important to advance our understanding, but raises challenging methodological issues. In particular, complex models may hinder the fundamental mechanisms underlying the emergence of a pattern. Also, the consideration of multiple traits leads to an increase in the dimensionality of the associated DE problem, which in turn leads to an exponential increase in computational cost (XXX). In order to better understand eco-evolutionary feedbacks, we need to investigate more realistic scenarios, which will, in turn, require the

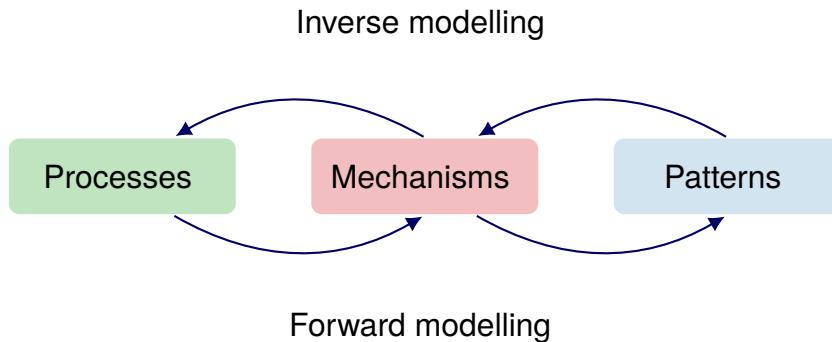


Fig. 1.2: Forward and inverse modelling approaches for the understanding of CAS. A forward modelling approach starts with assuming a set of processes, and seeks to understand how their interplay transforms in mechanisms that are associated with macroscopic patterns. An inverse modelling approach starts from empirical patterns (or observations) to deduce the underlying mechanisms and associated processes.

development of novel numerical methods that can cope with the extra computational cost.

1.2.2 Inverse modelling

While in forward modelling, observation data is uniquely considered at the end of the modelling cycle - when comparing predicted and empirical patterns to refine the baseline hypotheses -, inverse modelling integrates data at the start the modelling cycle (XXX)(??). Inverse modelling can take the form of parameter estimation (Schartau2017) or model selection (Johnson2004), both involving the use of inference methods to estimate, respectively, the most probable model parameter value, or the most probable model among candidates, given the data. Provided that they are inferred together with uncertainties, parameters can be interpreted to better understand the strengths and effects of the associated processes (Pontarp2019). For instance, (Higgins2010; Curtsdotter2019) infer and analyse the parameters of population dynamic models to understand processes involved in ecosystem functions. In model selection, the candidate models embed competing hypotheses about processes and mechanisms, and the relative support of each model given the data is used to discriminate (Johnson2004). For instance, (Skeels2022) shows that temperature-dependent evolutionary speed is the most likely mechanism to explain variations in biodiversity patterns, using inference methods to discriminate between alternative dynamic models embedding different evolutionary hypotheses. The inference methods used to calculate the parameter and model probabilities proceed by

defining a distance between the model simulation and the observation data (XXX), which relates to the probability of the parameters given the model and the data (**Schartau2017**). In parameter estimation, the most likely parameters are associated with the minimum distance to the data (XXX), and in model selection, the most likely models are associated with the minimum of the minimum model distance (XXX). Inference methods involve numerical methods to recover the model distance to the data (XXX), which success is dependent on the model formulation and its dynamical properties (XXX). Thus far, the use of inverse modelling for understanding eco-evolutionary dynamics has been limited because of a number of issues, including the high computational cost of the forward integration of eco-evolutionary models (**Fisher2018**), the large number of parameters involved (**Boyd2012**), and their strong nonlinearities (**Hastings1993; Huisman1999; Beninca2008**). Advances in the field of artificial intelligence could circumvent these issues, allowing to confront eco-evolutionary models with data (XXX).

1.2.3 Machine learning to leverage forward and inverse modelling

In the recent years, the field of artificial intelligence (AI) has made enormous progresses in computer vision (XXX) and natural language processing (XXX). At the backbone of this success are key computational techniques that could leverage the forward and inverse modelling of CAS. Computer vision and natural language processing rely on deep learning methods, that allow neural networks to learn abstract representation of mechanisms from large datasets (**LeCun2015**). These abstractions are hardly interpretable by humans (XXX), and their prediction ability is limited by the information contained in the training datasets. As such, neural networks cannot be used *per se* to gain scientific insights and extrapolate beyond observed trends (**Barnosky2012; Urban2016**). Nevertheless, their traditional applications and associated methods have been successfully derived in other scientific fields for this purpose (**Kashinath2021; Schneider2017; Yazdani2020; Rolnick2023**).

Neural networks have been used in forward modelling, to reduce the cost of the forward integration of climate models, by learning more efficient representations of physical mechanisms (XXX). They have also been used to approximate the solution of partial differential equations (PDEs) (**Sirignano2018dgm**), with the major advantage of approximating high dimensional problems at a lower computational cost than traditional methods. Underlying the training of neural network is the technique of backpropagation (XXX). This technique can be generalised to any scientific model against data (**Rackauckas2020**), with the potential to leverage inverse modelling techniques. As such, the derivation of AI techniques to investigate causal processes in CAS offers unique opportunities (**Frank2022**).

1.2.4 Programming languages

Combining ML techniques with scientific models requires computational environments that allow to easily develop scientific models, ensure simulation performance, and provide composability between ML and other scientific libraries (**Rackauckas2020**). Unfortunately, performance and composability are features that are poorly represented in mainstream programming languages used by the scientific community, such as Python, Matlab or R. Those languages are naturally attractive because they are dynamically typed (XX), allowing convenient development iterations. Nonetheless, prototypes written in Python, Matlab or R need to be rewritten in low level, compiled languages such as C, C++ or Fortran for speed and predictable mapping to hardware (**Perkel2019; Bezanson2017**). This conversion requires significant involvement, leading to a problem commonly designated as the "two language problem" (**Bezanson2017**). In order to circumvent issues of performance, most libraries in Python, Matlab or R rely on bindings with low level languages. For instance, the most used deep learning libraries in Python, TensorFlow and PyTorch, are internally written in C. However, bindings with low level languages come with major negative externalities, such as restricting the understandability of their internals to computer scientists – prohibiting potential development contributions from the scientific community –, and preventing the composability of libraries (XXX).

Julia is a programming language that was launched in 2012 to address the issue of the two-language problem (**Bezanson2017; Bezanson2018**). Julia was built over a type-specializing, just-in-time compiler, which makes it easy to generate highly performant programs, while preserving the essential features of Python, Matlab or R, such as dynamic typing and automatic memory management. Importantly, it relies on multiple dispatch, which allows to generate highly generic code with good performance. This permits to write libraries in pure Julia, guaranteeing productivity and composability. As such, the internals of any Julia library can be understood by non computer-scientists, who can further use his expertise to participate to its development. Many Julia libraries benefit from a high number of contributions of independent users (see, e.g., github.com/DifferentialEquations.jl). Multiple dispatch also allows to automatically generate the gradient of any Julia program without any modification (**ForwardDiff.jl; Zygote.jl**). This means that any scientific library in Julia, such as differential equation solvers, can be combined with deep learning tools, with unique opportunities for forward and inverse modelling problems (**Frank2022**). As such, Julia allows to prototype a program which is readily generic and can directly be shared to the research community. Further, by granting the composability of libraries, it allows to blend ML techniques with scientific models. This makes Julia a promising computational environment to accelerate research in CAS.

1.3 Thesis outline

In summary, while it is increasingly acknowledged that feedbacks between ecological and evolutionary processes play an important role in the dynamics of biological systems (**Pelletier2009; Urban2016**), our understanding of the mechanisms in which they are involved has been limited to simplified scenarios. Further, while analogous processes have been suggested to influence the dynamics of economic systems (**Hodgson2019**), a quantification of their effect is missing. Under increasing anthropogenic pressure, these research directions become essential (**Urban2016**), but raise challenging methodological issues. Here, I present novel forward and inverse modelling approaches to advance our understanding of eco-evolutionary dynamics in biological and economic systems, and utilise them to shed light on the underlying processes and resulting mechanisms.

In ??, I investigate how eco-evolutionary processes, in combination with complex habitat spatial structures, influence the trait distribution of biological populations. I proceed using a forward modelling approach, building a stochastic eco-evolutionary IBM where individuals are structured over a spatial graph, and experience the fundamental processes of reproduction, competition, mutation and migration. I seek to understand how those microscopic forces result in trait differentiation at the population level. I derive DE approximations of the IBM that, together with extensive numerical simulations, provide analytical insights into how the graph properties affect the population size and trait differentiation. In particular, I show that three main graph properties, measuring landscape connectivity, heterogeneity in connectivity, and habitat spatial auto-correlation, shape the trait differentiation of the biological population. These results establish mechanistic links between landscape features and the eco-evolutionary dynamics of biological populations.

In ??, I develop an inverse modelling method to estimate the parameters of highly nonlinear population dynamic models. The method is based on a machine learning framework and involves AI techniques together with a novel learning strategy. This learning strategy consists in training the model against mini-batches of data with short time horizon, which I analytically show to regularize the learning problem. I implement the ML framework in the Julia library **MiniBatchInference.jl**, and demonstrate through numerical experiments that it can efficiently estimate model parameters and provide statistical evidences for causal processes from noisy, incomplete and independent time series. Altogether, the proposed ML framework is a workhorse for inverse modelling and can elucidate mechanistic pathways in biological and economic systems.

In ??, I quantify the effect of eco-evolutionary processes on the dynamics of economic systems. I employ the ML framework developed in ?? to investigate how

alternative eco-evolutionary population dynamic models can explain the dynamics of economic activities in the richest 100 countries, relying on 59 year of economic data. The models embed the processes of ecological interactions between economic activities, spatial transfers, and economic activity transformations, which support is compared to a simple logistic growth model, taken as a null model. I find strong statistical evidence for positive interactions between national economic activities, and spatial transfers across countries. To my knowledge, this is the first study that provides quantitative evidences that similar processes may influence the dynamics of biological and economic systems.

In ??, I extend two recent methods to solve high dimensional PDEs, in order to handle non-local nonlinear terms. The first method relies on Picard iterations, while the second is based on machine learning and involves neural networks to approximate PDE solutions. The numerical difficulties arising due to the non-local term are avoided by using a plain vanilla Monte Carlo integration. I implement the methods in the Julia library **HighDimPDE.jl**, and evaluate their performance on high dimensional PDE models arising in physics and biology, including population dynamic eco-evolutionary models. For all models, the methods yield good results with short run times, offering the possibility to include more realism in future eco-evolutionary models.

2

Eco-evolutionary model on spatial graphs reveals how habitat structure affects phenotypic differentiation

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“ The Lord is subtle, but He is not malicious.

— Albert Einstein (1879-1955)

Differentiation mechanisms are influenced by the properties of the landscape over which individuals interact, disperse and evolve. Here, we investigate how habitat connectivity and habitat heterogeneity affect phenotypic differentiation by formulating a stochastic eco-evolutionary model where individuals are structured over a spatial graph. We combine analytical insights into the eco-evolutionary dynamics with numerical simulations to understand how the graph topology and the spatial distribution of habitat types affect differentiation. We show that not only low connectivity but also heterogeneity in connectivity promotes neutral differentiation, due to increased competition in highly connected vertices. Habitat assortativity, a measure of habitat spatial auto-correlation in graphs, additionally drives differentiation under habitat-dependent selection. While assortative graphs systematically amplify adaptive differentiation, they can foster or depress neutral differentiation depending on the migration regime. By formalising the eco-evolutionary and spatial dynamics of biological populations on graphs, our study establishes fundamental links between landscape features and phenotypic differentiation.

2.1 Introduction

Biodiversity results from differentiation processes influenced by the features of the landscape over which populations are distributed (**Hubbel2001**). The documentation of high levels of species diversity in mountain regions and riverine systems suggests that complex connectivity patterns and habitat heterogeneity foster differentiation (**Rahbek2019a; Ding2020a; Dias2014; Guegan1998**). However, hypotheses formulated based on empirical evidence should be complemented by mechanistic models to crystallise a causal understanding between processes and patterns (**Levin2002**). While the number of simulation studies is growing steadily (**Cabral2017**), such studies often lack a mathematical formalism to facilitate the interpretation of the model outcomes by providing an analytical underpinning to the simulation results (**Lion2016**).

Phenotypic differentiation processes emerge as a result of mutation, selection and migration and can be classified as neutral or adaptive (**Holderegger2006**). Neutral differentiation is initiated by the stochastic drift of local phenotypes when spatial isolation and limited dispersal create barriers to gene flow, allowing distinct phenotypes to emerge in spatially structured populations (**Slatkin1993a**). In contrast, adaptive differentiation results from heterogeneous selection, which promotes distinct, locally well-adapted phenotypes in populations occupying patches with different habitat conditions (**Dieckmann1999**). The evolution of neutral phenotypes and of adaptive phenotypes are not independent, as selective forces can indirectly select for those neutral phenotypes that happen to be linked to the fittest adaptive phenotypes, a mechanism called the hitchhiking effect (**Kaplan1989**). Moreover, selection can generate barriers to gene flow between populations in heterogeneous habitat landscapes (**Orsini2013; Wang2014**), a phenomenon coined isolation by environment, which can amplify neutral differentiation. How neutral processes, adaptive processes and their interplay are affected by landscape features is difficult to comprehend without a formalised mechanistic model (**GARANT2007**).

Models link patterns to processes (**Levin2002**), and the explicit representation of the landscape within an eco-evolutionary model can lead to a causal understanding of how landscape features shape differentiation. Spatial graphs provide a convenient mathematical representation of landscapes, where vertices represent suitable habitats hosting populations, and edges capture the connectivity between habitats (**Dale2010**). Under ecological dynamics, metapopulation models have been used to study the role of graph topology in the persistence and stability of metapopulation (**Holland2008; Gilarranz2012; Mari2014; Gravel2016**) and community diversity (**Carrara2012; Thompson2017; Suzuki2021**). Evolutionary mechanisms are nevertheless fundamental drivers of diversity, and should therefore be explicitly integrated into models (**Pelletier2009**). Evolutionary game theory explores how graph topology impacts the fixation probability and the fixation time of a mutated phenotype (**Tkadlec2019**). However, the framework does not consider the continuous accumulation of mutations, and is therefore not suited to addressing the emergence of phenotypic differentiation. By combining a metapopulation model with a model of neutral evolution, (**Economou2007; Economou2010**) investigated how graph topology affects neutral diversity. Their approach demonstrated the key role of topological properties in shaping diversity, and its predictions could be matched with empirical data from e.g. river basins

(**Muneepeerakul2008**). Nonetheless, diversity results from the combination of neutral and adaptive processes developing at the population level. A first principles modelling approach considering spatial graphs, but also building upon the elementary processes of ecological interactions, reproduction, mutation and migration may therefore be promising to investigate the emergence of diversity.

Stochastic models for structured populations, rooted in the microscopic description of individuals, offer a generic framework for modelling eco-evolutionary dynamics (**Champagnat2006; Bansaye2015**). In particular, these models can capture the interplay between population dynamics, spatial dynamics and phenotypic evolution, while providing a rigorous set-up for analytical investigation. By anchoring this modelling paradigm in a mathematical framework, the work of Champagnat et al. (**Champagnat2006**) generalises models of population genetics (**Burger2000**) (investigating the evolution of the frequencies of alleles) and quantitative genetics (**Slatkin1978; Lande1991; Nagylaki1994**) (investigating the evolution of phenotypic traits), which stimulated research into the link between spatial population structure and neutral differentiation. The framework embraces density-dependent selection, which could explain the emergence of phenotypic differentiation from competition processes (**Dieckmann1999**), and how spatial segregation can emerge as a byproduct of these adaptive processes along environmental gradients (**Doebeli2003**). Related models have addressed the effects of landscape dynamics and habitat heterogeneity on adaptive differentiation, providing mathematical insights into the dynamics (**Meszena1997; Aguilee2012; Debarre2013; Wickman2017; Polechova2018; Mirrahimi2020**). Because it accounts for finite population size, the baseline model of Champagnat et al. (**Champagnat2006**) can also capture neutral differentiation dynamics and therefore the coupling between neutral and adaptive processes (**Billiard2015; Anceschi2019**). Nonetheless, the aforementioned studies were not spatially explicit (**Billiard2015; Anceschi2019**) or they assumed regular spatial structures (regular graphs (**Meszena1997; Aguilee2012; Debarre2013; Mirrahimi2020**) or continuous space (**Doebeli2003; Wickman2017; Polechova2018**)), therefore not addressing the role of the spatial complexity of landscapes. A stochastic individual-based model using spatial graphs as a representation of the landscape could help formalise fundamental links between landscape features and phenotypic differentiation.

A key challenge is to understand how individual dynamics result in the emergence of differentiation in complex landscapes (**Manel2003**). Here, we investigate how complex connectivity patterns and habitat heterogeneity affect both neutral and adaptive phenotypic differentiation by constructing an individual-based model (IBM) that accounts for eco-evolutionary dynamics on spatial graphs. The individuals disperse between habitat patches and possess co-evolving neutral and adaptive traits. The finite size of local populations generates neutral differentiation by inducing a stochastic drift in the neutral trait evolution, while heterogeneous selection gives rise to adaptive differentiation. Macroscopic properties of the model are analytically tractable, and we obtain a deterministic approximation of population size and adaptive trait dynamics which connects the emerging patterns to the graph properties that generate them. However, neutral differentiation is stochastic by nature, which complicates its analytical underpinning. We therefore rely on numerical simulations of the IBM to measure the effect of graph topology on neutral differentiation. In the case where heterogeneous selection is absent, we investigate how graph topology

affects neutral differentiation. In the case of heterogeneous selection, we investigate how the graph topology, in combination with the spatial distribution of habitat types, affects levels of (i) adaptive and (ii) neutral differentiation. By combining analytical methods with numerical simulations, we expect to identify graph properties that determine the level of differentiation. Overall, our study establishes causal links between landscape properties and population differentiation and contributes to a fundamental understanding of how landscape features promote biodiversity.

2.2 Results

2.2.1 Eco-evolutionary model on spatial graphs

We establish an individual-based model (IBM) where individuals are structured over a trait space and a graph representing a landscape. For the sake of simplicity, we consider the case of asexual reproduction and haploid genetics (**Champagnat2006**). Individuals die, reproduce, mutate and migrate in a stochastic fashion, which together results in macroscopic properties. The formulation of the stochastic IBM allows an analytical description of the dynamics at the population level, which links emergent properties to the elementary processes that generate them.

The trait space $\mathcal{X} \subseteq \mathbb{R}^d$ is continuous and can be split into a neutral trait space \mathcal{U} and an adaptive trait space \mathcal{S} . We refer to neutral traits $u \in \mathcal{U}$ as traits that are not under selection, in contrast to adaptive traits $s \in \mathcal{S}$, which experience selection. The graph denoted by G is composed of a set of vertices $\{v_1, v_2, \dots, v_M\}$ that correspond to habitat patches (suitable geographical areas), and a set of edges that constrain the movement of individuals between the habitat patches. We use the original measure of genetic differentiation for quantitative traits Q_{ST} (standing for Q -statistics) in the case of haploid populations (**Lande1992; WHITLOCK2008**). We denote the neutral trait value of the k -th individual on v_i as $u_k^{(i)}$, the number of individuals on v_i as $N^{(i)}$, the mean neutral trait on v_i as $\bar{u}^{(i)}$, and the mean neutral trait in the metapopulation as \bar{u} . It follows that we quantify neutral differentiation $Q_{ST,u}$ as

$$Q_{ST,u} = \sigma_{B,u}^2 / (\sigma_{B,u}^2 + \sigma_{W,u}^2) \quad (2.1)$$

where $\sigma_{B,u}^2 = \mathbb{E} \left[\frac{1}{M} \sum_i (\bar{u}^{(i)} - \bar{u})^2 \right]$ denotes the expected neutral trait variance between the vertices and $\sigma_{W,u}^2 = \frac{1}{M} \sum_i \mathbb{E} \left[\frac{1}{N^{(i)}} \sum_k (u_k^{(i)} - \bar{u}^{(i)})^2 \right]$ denotes the average expected neutral trait variance within vertices. We similarly quantify adaptive differentiation $Q_{ST,s}$.

Following the Gillespie update rule (**Gillespie1976**), individuals with trait $x_k \in \mathcal{X}$ on vertex v_i are randomly selected to give birth at rate $b^{(i)}(x_k)$ and die at rate $d(N^{(i)}) = N^{(i)}/K$, where K is the local carrying capacity. The definition of d therefore captures competition, which is proportional to the number of individuals on a vertex and does not depend on the individuals' traits (we relax this assumption later on). The offspring resulting from a birth event inherits the parental traits, which can independently be affected by mutations with probability μ . A mutated trait differs from the parental trait by a random change

that follows a normal distribution with variance σ_μ^2 (corresponding to the continuum of alleles model (**Kimura1964**)). The offspring can further migrate to neighbouring vertices by executing a simple random walk on G with probability m . A schematic overview of the two different settings considered is provided in ???. Under the setting with no selection, individuals are only characterised by neutral traits so that $\mathcal{X} = \mathcal{U}$. For individuals on a vertex with trait $x_k \equiv u_k$ we define $b^{(i)}(x_k) \equiv b$, so that the birth rate is constant. This ensures that neutral traits do not provide any selective advantage. Under the setting with heterogeneous selection, each vertex of the graph v_i is labelled by a habitat type with environmental condition Θ_i that specifies the optimal adaptive trait value on v_i . It follows that, for individuals with traits $x_k = (u_k, s_k) \in \mathcal{U} \times \mathcal{S}$ on v_i , we define

$$b^{(i)}(x_k) \equiv b^{(i)}(s_k) = b(1 - p(s_k - \Theta_i)^2) \quad (2.2)$$

where p is the selection strength (**Mirrahimi2020**). This ensures that the maximum birth rate on v_i is attained for $s_k = \Theta_i$, which results in a differential advantage that acts as an evolutionary stabilising force. In the following we consider two habitat types denoted by \bullet and \circ with symmetric environmental conditions θ_\bullet and θ_\circ , so that $\Theta_i \in \{\theta_\bullet, \theta_\circ\}$ and $\theta_\circ = -\theta_\bullet = \theta$, where θ can be viewed as the habitat heterogeneity (**Mirrahimi2020**).

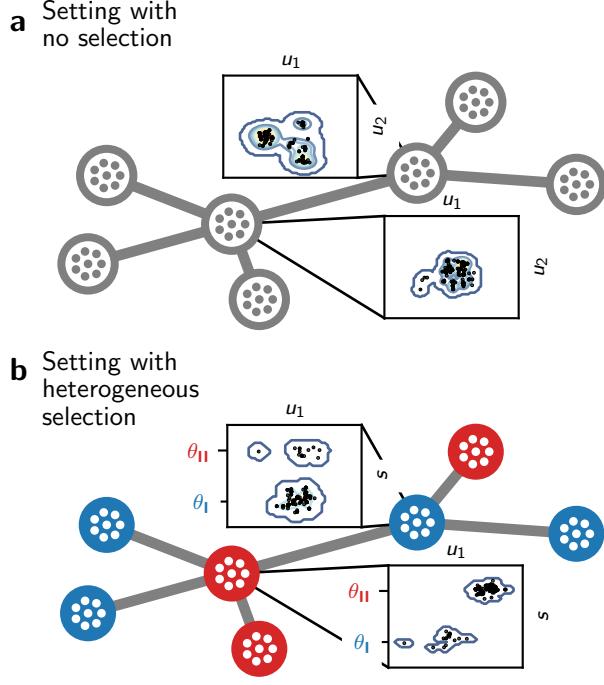


Fig. 2.1: Graphical representation of the structure of individuals in the eco-evolutionary model. (a) Setting with no selection, where individuals are characterised by a set of neutral traits $u \in \mathcal{U}$. The scatter plots represent a projection of the first two components of u for the individuals present on the designated vertices at time $t = 1000$, obtained from one simulation of the IBM. (b) Setting with heterogeneous selection. In this setting, individuals are additionally characterised by adaptive traits $s \in \mathcal{S}$. Blue vertices favour the optimal adaptive trait value θ_\bullet , while red vertices favour θ_\circ . The scatter plots represent a projection of the first component of u and s for the individuals present on the designated vertices at time $t = 1000$, obtained from one simulation. The majority of individuals are locally well-adapted and have an adaptive trait close to the optimal value, but some maladaptive individuals originating from neighbouring vertices are also present. $m = 0.05$.

2.2.2 Deterministic approximation of the population dynamics under no selection

The model can be formulated as a measure-valued point process ((Bansaye2015) and ??). Under this formalism, we demonstrate in the ?? how the population size and the trait dynamics show a deterministic behaviour when a stabilising force dampens the stochastic fluctuations. This makes it possible to express the dynamics of the macroscopic properties with deterministic differential equations, connecting emergent patterns to the processes that generate them. In particular, in the setting of no selection, competition stabilises

the population size fluctuations, and the dynamics can be considered deterministic and expressed as

$$\partial_t N_t^{(i)} = N_t^{(i)} \left[b(1 - m) - \frac{N_t^{(i)}}{K} \right] + mb \sum_{j \neq i} \frac{a_{i,j}}{d_j} N_t^{(j)} \quad (2.3)$$

where $A = (a_{i,j})_{1 \leq i,j \leq M}$ is the adjacency matrix of the graph G and $D = (d_1, d_2, \dots, d_M)$ is a vector containing the degree of each vertex (number of edges incident to the vertex). The first term on the right-hand side corresponds to logistic growth, which accounts for birth and death events of non-migrating individuals. The second term captures the gains due to migrations, which depend on the graph topology. Assuming that all vertices with the same degree have an equivalent position on the graph, corresponding to a mean field approach (see ??), one can obtain a closed-form solution from ?? (see ??), which shows that the average population size \bar{N} scales with $\langle \sqrt{k} \rangle^2 / \langle k \rangle$, where $\langle k \rangle$ is the average vertex degree and $\langle \sqrt{k} \rangle$ is the average square-rooted vertex degree. The quantity $\langle \sqrt{k} \rangle^2 / \langle k \rangle$, denoted as h_d , relates to the homogeneity in vertex degree of the graph and can therefore be viewed as a measure negatively associated with heterogeneity in connectivity. Simulations of the IBM illustrate that h_d can explain differences in population size for complex graph topologies with varying migration regimes (??a for graphs with $M = 7$ vertices and ??a for $M = 9$). This analytical result is connected to theoretical work on reaction diffusion processes (Colizza2007) and highlights that irregular graphs (graphs whose vertices do not have the same degree) result in unbalanced migration fluxes that affect the ecological balance between births and deaths. Highly connected vertices present an oversaturated carrying capacity ($N^{(i)} > bK$, see ??), increasing local competition and lowering total population size compared with regular graphs (??a). Because populations with small sizes experience more drift ((Burger2000) and ??), this result indicates that graph topology affects neutral differentiation not only through population isolation, but also by affecting population dynamics.

Nonetheless, the stochasticity of the processes at the individual level can propagate to the population level and substantially affect the macroscopic properties. In particular, neutral differentiation emerges from the stochastic fluctuations of the populations' neutral trait distribution. These fluctuations complicate an analytical underpinning of the dynamics, and in this case simulations of the IBM offer a straightforward approach to evaluate the level of neutral differentiation.

2.2.3 Effect of graph topology on neutral differentiation under no selection

We study a setting with no selection and investigate the effect of the graph topology on neutral differentiation. When migration is limited, individuals' traits are coherent on each vertex but stochastic drift at the population level generates neutral differentiation between the vertices. Migration attenuates neutral differentiation because it has a correlative effect on local trait distributions. Following (Economo2007; Carrara2012; Thompson2017), we expect that the intensity of the correlative effect depends on the average path length of the graph $\langle l \rangle$, defined as the average shortest path between all pairs of vertices (Bounova2012).

For a constant number of vertices, $\langle l \rangle$ is strictly related to the mean betweenness centrality and quantifies the graph connectivity (Bounova2012). High $\langle l \rangle$ implies low connectivity and a greater isolation of populations, and hence we expect that graphs with high $\langle l \rangle$ are associated with high differentiation levels. We consider various graphs with an identical number of vertices and run simulations of the IBM to obtain the neutral differentiation level $Q_{ST,u}$ attained after a time long enough to discard transient dynamics (see ??). We then interpret the discrepancies in $Q_{ST,u}$ across the simulations by relating them to the underlying graph topologies.

We observe strong differences in $Q_{ST,u}$ across graphs for varying m , and find that $\langle l \rangle$ explains at least 55% of the variation in $Q_{ST,u}$ across all graphs with $M = 7$ vertices for (??b). Nonetheless, some specific graphs, such as the star graph, present higher levels of $Q_{ST,u}$ than expected by their average path length. To explain this discrepancy, we explore the effect of homogeneity in vertex degree h_d , as we showed in ?? that it decreases population size, which should in turn increase $Q_{ST,u}$ by intensifying stochastic drift. We find that h_d explains 57% of the variation for low m (??c). However, the fit remains similar after correcting for differences in population size (see ??), indicating that irregular graphs structurally amplify the isolation of populations. Unbalanced migration fluxes lead central vertices to host more individuals than allowed by their carrying capacity. This causes increased competition that results in a higher death rate, so that migrants have a lower probability of further spreading their trait. Highly connected vertices therefore behave as bottlenecks, increasing the isolation of peripheral vertices and consequently amplifying $Q_{ST,u}$.

We then evaluate the concurrent effect of $\langle l \rangle$ and h_d on $Q_{ST,u}$ with a multivariate regression model that we fit independently for low and high migration regimes (??d). The multivariate regression model explains at least 70% of the variation in $Q_{ST,u}$ for the migration regimes considered and for graphs with $M = 7$ vertices (see ?? for details). Moreover, we find that $\langle l \rangle$ and h_d have akin contributions to neutral differentiation for low m , but the effect of $\langle l \rangle$ increases for higher migration regimes while the effect of h_d decreases. To ensure that these conclusions can be generalised to larger graphs, we conduct the same analysis on a subset of graphs with $M = 9$ vertices and find congruent results (??). In the absence of selection and with competitive interactions, graphs with a high average path length $\langle l \rangle$ and low homogeneity in vertex degree h_d , or similarly graphs with low connectivity and high heterogeneity in connectivity, show high levels of neutral differentiation.

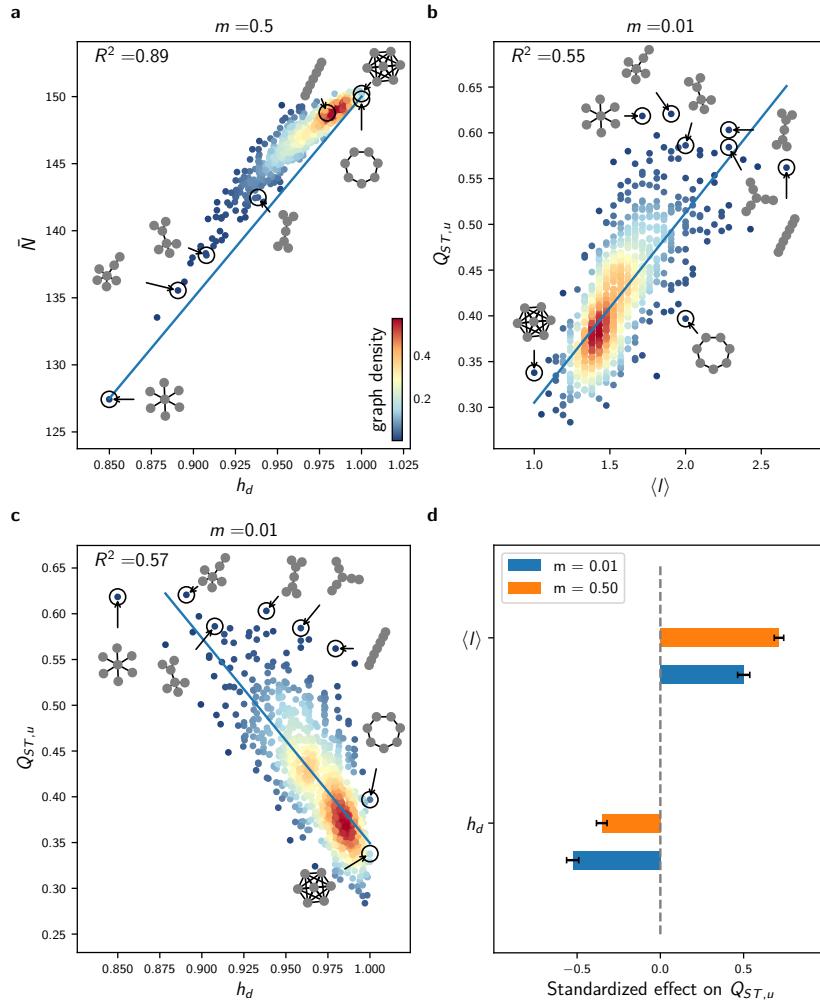


Fig. 2.2: Effect of $\langle l \rangle$ and h_d on average population size \bar{N} and neutral differentiation $Q_{ST,u}$ in the setting with no selection. (a) Response of \bar{N} to homogeneity in degree $h_d = \langle \sqrt{k} \rangle^2 / \langle k \rangle$ for all undirected connected graphs with $M = 7$ vertices and $m = 0.5$. (b) Response of $Q_{ST,u}$ to average path length $\langle l \rangle$ for similar simulations obtained with $m = 0.01$. (c) Response of $Q_{ST,u}$ to homogeneity in degree h_d for the same data. In (a), (b) and (c), each dot represents average results from 5 replicate simulations of the IBM, the colour scale corresponds to the proportion of the graphs with similar x and y axis values (graph density), and the blue line corresponds to a linear fit. (d) Standardized effect of h_d and $\langle l \rangle$ on $Q_{ST,u}$, obtained from multivariate regression models independently fitted on similar data obtained for $m = 0.01$ and $m = 0.5$. The contributions of $\langle l \rangle$ and h_d to $Q_{ST,u}$ are alike for low migration regimes. Error bars show 95% confidence intervals. Analogous results on graphs with $M = 9$ vertices are presented in ?? and all regression details can be found in ??.

2.2.4 Deterministic approximation of the population dynamics and adaptation under heterogeneous selection

We next consider heterogeneous selection and investigate the response of adaptive differentiation to the spatial distribution of habitat types, denoted as the Θ -spatial distribution. Adaptive differentiation emerges from local adaptation, but migration destabilises adaptation as a result of the influx of maladaptive migrants. We expect that higher connectivity between vertices of similar habitat type increases the level of adaptive differentiation, because it increases the proportion of well-adapted migrants. Local adaptation can be investigated by approximating the stochastic dynamics of the trait distribution with a deterministic partial differential equation (PDE). We demonstrate under mean field assumption how the deterministic approximation can be reduced to an equivalent two-habitat model. We analyse the reduced model with the theory of adaptive dynamics (**Meszena1997; Mirrahimi2020**) and find a critical migration threshold m^* that determines local adaptation. m^* depends on a quantity coined the habitat assortativity r_Θ , and we demonstrate with numerical simulations that r_Θ determines the overall adaptive differentiation level $Q_{ST,s}$ reached at steady state in the deterministic approximation.

Heterogeneous selection, captured by the dependence of the birth rate on Θ_i , generates a stabilising force that dampens the stochastic fluctuations of the adaptive trait distribution. The dynamics of the adaptive trait distribution consequently shows a deterministic behavior and we demonstrate in the ?? and ???? that the number of individuals on v_i with traits $s \in \Omega \subset \mathcal{S}$ can be approximated by the quantity $\int_\Omega n_t^{(i)}(s)ds$, where $n_t^{(i)}$ is a continuous function solution of the PDE

$$\begin{aligned} \partial_t n_t^{(i)}(s) = & n_t^{(i)}(s) \left[b^{(i)}(s)(1 - m) - \frac{1}{K} \int_S n_t^{(i)}(\mathbf{s})d\mathbf{s} \right] \\ & + m \sum_{j \neq i} b_j(s) \frac{a_{i,j}}{d_j} n_t^{(j)}(s) \\ & + \frac{1}{2} \mu \sigma_\mu^2 \Delta_s \left[b^{(i)}(s) n_t^{(i)}(s) \right] \end{aligned} \quad (2.4)$$

?? is similar to ??, except that it incorporates an additional term corresponding to mutation processes and that the birth rate is trait dependent. We show how ?? can be reduced to an equivalent two-habitat model under mean field assumption. The mean field approach differs slightly from the setting with no selection because vertices are labelled with Θ_i . Here we assume that vertices with similar habitat types have an equivalent position on the graph (see ?? for a graphical representation), so that all vertices with habitat type \bullet are characterised by the identical adaptive trait distribution that we denote by \bar{n}^\bullet , and are associated with the birth rate $b^\bullet(s) = b(1 - p(s - \theta_\bullet)^2)$. Let $P(\bullet, \bullet)$ denote the proportion of edges connecting a vertex v_i of type \bullet to a vertex v_j of type \bullet , and let $P(\bullet)$ denote the proportion of vertices v_i

of type \bullet . By further assuming that habitats are homogeneously distributed on the graph so that $P(\bullet) = P(\bullet) = \frac{1}{2}$, ?? transforms into

$$\begin{aligned}\partial_t \bar{n}_t^\bullet(s) &= \bar{n}_t^\bullet(s) \left[b^\bullet(s)(1-m) - \frac{1}{K} \int_S \bar{n}_t^\bullet(\mathbf{s}) d\mathbf{s} \right] + \frac{1}{2} \mu \sigma_\mu^2 (\Delta_s b^\bullet \bar{n}_t^\bullet)(s) \\ &\quad + \frac{m}{2} [(1-r_\Theta)b^\bullet(s)\bar{n}_t^\bullet(s) + (1+r_\Theta)b^\bullet(s)\bar{n}_t^\bullet(t)]\end{aligned}\quad (2.5)$$

(see ??), where we define

$$r_\Theta = 2(P(\bullet, \bullet) - P(\bullet, \bullet)) \quad (2.6)$$

as the habitat assortativity of the graph, which ranges from -1 to 1 . When $r_\Theta = -1$, all edges connect dissimilar habitat types (disassortative graph), while as r_Θ tends towards 1 the graph is composed of two clusters of vertices with identical habitat types (assortative graph). ?? can be analysed with the theory of adaptive dynamics (**Meszena1997**; **Debarre2013**; **Mirrahimi2020**), a mathematical framework that provides analytical insights by assuming a trait substitution process. Following this assumption, the mutation term in ?? is omitted and the phenotypic distribution results in a collection of discrete individual types that are gradually replaced by others until evolutionary stability is reached (see ?? and (**Meszena1997**; **Debarre2013**; **Mirrahimi2020**) for details). By applying the theory of adaptive dynamics, we find a critical migration rate m^*

$$m^* = \frac{1}{(1-r_\Theta)} \frac{4p\theta^2}{(1+3p\theta^2)} \quad (2.7)$$

so that when $m > m^*$, a single type of individual exists with adaptive trait $s^* = (\theta_\bullet + \theta_\bullet)/2 = 0$ in the steady state (see ?? for the derivation of ??). In this case, adaptive differentiation $Q_{ST,s}$ is nil and the average population size is given by $\bar{N} = bK(1-p\theta)^2$. In contrast, when $m = 0$ and/or $r_\Theta = 1$, all individuals are locally well-adapted with trait Θ_i on v_i , and it follows that the average population size is higher and equal to $\bar{N} = bK$, while adaptive differentiation is maximal and equal to $Q_{ST,s} = \text{Var}(\Theta)/(\text{Var}(\Theta)+0) = 1$. When $0 < m < m^*$, the coexistence of two types of individuals on each vertex v_i is predicted but the calculation of the trait values is more subtle. To understand the effect of m and r_Θ on the local trait distributions and on $Q_{ST,s}$, we therefore leave behind the adaptive dynamics framework and numerically solve ?? by including the mutation term. When $0 < m < m^*$, the local trait distributions are bimodal with peaks corresponding to the two types of individuals predicted by the adaptive dynamics. The highest peak corresponds to the well-adapted individuals, whose adaptation is destabilised by the influx of maladaptive migrants (??a). This phenomenon is dampened as r_Θ increases, since the proportion of maladaptive migrants is reduced in assortative graphs (??b). As a consequence, the habitat assortativity r_Θ increases the differentiation $Q_{ST,s}$ when $0 < m < m^*$ (??c). The simulations further confirm that the adaptive dynamics prediction given by ?? is still valid when the continuous accumulation of mutations is considered, so that for $m > m^*$ the local trait distributions obtained from ?? are unimodal and $Q_{ST,s}$ vanishes (??a,c). Our analysis of the mean field deterministic approximation ?? therefore demonstrates that assortative graphs present high levels of adaptive differentiation $Q_{ST,s}$. On the other hand, the analysis shows

that $Q_{ST,s}$ rapidly declines with increasing m on disassortative graphs, until $Q_{ST,s}$ vanishes when $m > m^*$.

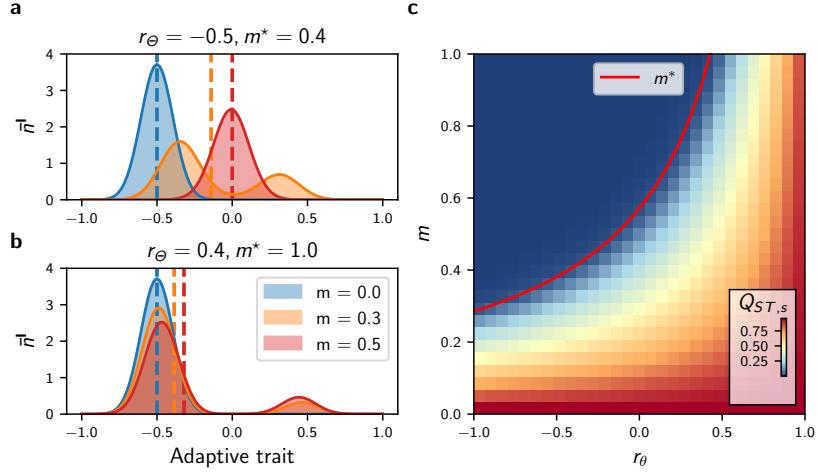


Fig. 2.3: Effect of habitat assortativity r_Θ and migration m on the local adaptive trait distribution \bar{n}^* and on the adaptive differentiation level $Q_{ST,s}$ under the mean field, deterministic approximation ???. (a) Effect of m and r_Θ on \bar{n}^* . Migration induces the apparition of maladaptive individuals (centred around $\theta_0 = 0.5$), which destabilise local adaptation by displacing the mean value of the well-adapted individuals (centred around $\theta_0 = -0.5$). Together with the decrease in local adaptation, migration causes a displacement of the mean value of the local trait distribution (represented by the vertical dashed lines), which decreases local population size and adaptive differentiation $Q_{ST,s}$. (b) Similar data for higher r_Θ . Increasing r_Θ increases population size and $Q_{ST,s}$. (c) Effect of r_Θ on $Q_{ST,s}$. The red line indicates the critical migration threshold m^* predicted by ??; $Q_{ST,s}$ vanishes when $m > m^*$.

2.2.5 Effect of graph topology on adaptive differentiation under heterogeneous selection

To generalise the conclusions drawn from the mean field deterministic approximation ??, we generate different Θ -spatial distributions for varying graph topology, and compare outputs of the IBM simulations with those of ?? (see ?? for the details of the simulations). For each combination of Θ -spatial distribution and graph, we compute the habitat assortativity r_Θ , since r_Θ can be generalised from ?? to any graph topology following the original definition of (Newman2003a) as

$$r_\Theta = \frac{\text{Cov}(\Theta_x, \Theta_\wedge)}{\sigma_{\Theta_x} \sigma_{\Theta_\wedge}} \quad (2.8)$$

where Θ_x and Θ_\wedge denote the sets of habitats found at the toe and tip of each directed vertex of graph V , and $\langle \Theta_x \rangle, \langle \Theta_\wedge \rangle$ and $\sigma_{\Theta_x}, \sigma_{\Theta_\wedge}$ denote their respective means and standard deviations (see ??). The mean field deterministic approximation ?? is in very good agreement with the IBM simulations for general graph ensembles at low migration regimes, and captures

the response of \bar{N} and $Q_{ST,s}$ to r_Θ (??). Nonetheless, under high migration regimes, higher levels of $Q_{ST,s}$ are observed in the stochastic simulations compared with the mean field deterministic approximation (??). We hypothesize that this reinforcement is generated by stochastic drift, which must become the main driver of differentiation when local adaptation is lost for $m > m^*$, and perform a multivariate regression analysis to investigate the additional effect of $\langle l \rangle$ and h_d on $Q_{ST,s}$. As expected, the analysis highlights that the effect of $\langle l \rangle$ and h_d are substantial and complement the effect of r_Θ for high m (??c for graphs with $M = 7$ vertices and ??a for $M = 9$), further explaining the discrepancies observed (see ??).

We extend our analyses to realistic landscapes with a continuum of habitat types by running simulations on graphs obtained from real spatial habitat datasets and by considering mean annual temperature as a proxy for habitat type (see ??). We also consider simulations accounting for trait-dependent competition to test whether our results hold under more complex ecological processes (see ?? for the implementation details and ?? for the results). The simulations are congruent and show that the effects of r_Θ , h_d and $\langle l \rangle$ are similar under these alternative settings, underlining the robustness of these metrics and the generality of our conclusions. Taken together, these results indicate that under sufficiently strong selection and sufficiently high habitat heterogeneity, adaptive differentiation $Q_{ST,s}$ is mainly driven by habitat assortativity r_Θ . Nonetheless, local adaptation is lost in disassortative graphs when $m > m^*$, such that $\langle l \rangle$ and h_d become complementary determinants of $Q_{ST,s}$ for high migration regimes.

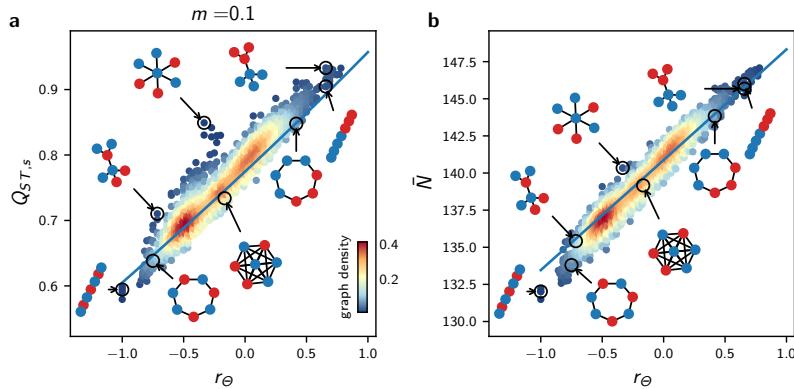


Fig. 2.4: Effect of habitat heterogeneity r_Θ on $Q_{ST,s}$ and average population size \bar{N} for general graph ensembles. (a) Effect of r_Θ on $Q_{ST,s}$ for all undirected connected graphs with $M = 7$ vertices and varying r_Θ , for $m = 0.1$. (b) Effect of r_Θ on average population size \bar{N} for the same simulations. In (a) and (b), each dot represents average results from 5 replicate simulations of the IBM, the colour scale corresponds to the proportion of the graphs with similar x and y axis values (graph density), and the blue lines correspond to results obtained from the mean field approximation ???. Insights from ?? are congruent with the IBM simulations for complex habitat connectivity patterns at low m . Similar results with $m = 0.5$ are presented in ??.

2.2.6 Effect of habitat assortativity on neutral differentiation under heterogeneous selection

We finally consider a setting with heterogeneous selection where individuals carry both neutral and adaptive traits. With distinct habitat types, selection promotes neutral differentiation by reducing the birth rate of maladaptive migrants, reinforcing the isolation of local populations. We have shown above that adaptive differentiation $Q_{ST,s}$ is driven by habitat assortativity r_Θ , so we expect r_Θ , together with the topological metrics found in the setting with no selection, to influence the level of neutral differentiation $Q_{ST,u}$. We first investigate how the response of $Q_{ST,u}$ to migration compares between the setting with no selection and the setting with heterogeneous selection for graphs with an identical topology. We then examine how the response compares between graphs with an identical topology but different r_Θ . We finally consider simulations on different graphs with varying r_Θ to assess the concurrent effect of $\langle l \rangle$, h_d and r_Θ on $Q_{ST,u}$.

Migration has a fitness cost because maladaptive migrants present lower fitness. Under an equivalent migration regime, migrants therefore have a lower probability of reproduction, increasing the populations' isolation compared with a setting without selection. Simulations with varying m on the complete graph confirm that selection in heterogeneous habitats reinforces $Q_{ST,u}$ compared with a setting without selection (??a). Nonetheless, previous results show that adaptive differentiation $Q_{ST,s}$ vanishes on a disassortative graph when $m > m^*$, implying that individuals become equally fit in all habitats. In this case, the isolation effect of heterogeneous selection is lost and $Q_{ST,u}$ reaches a similar level as in the setting with no selection for $m > m^*$ (??a), although $Q_{ST,u}$ is slightly higher in the setting with heterogeneous selection due to a lower population size ($\bar{N} = bK(1 - p\theta)$ vs. $\bar{N} = bK$, see section above and ??). This suggests that r_Θ reinforces $Q_{ST,u}$, as assortative graphs sustain higher levels of adaptive differentiation (????). Simulations on the path graph with varying Θ -spatial distribution support this conclusion for high migration regimes, but show the opposite relationship under low migration regimes, where the habitat assortativity r_Θ decreases $Q_{ST,u}$ (??b). Assortative graphs are composed of large clusters of vertices with similar habitats, within which migrants can circulate without fitness losses. Local neutral trait distributions become more correlated within these clusters, resulting in a decline in $Q_{ST,u}$ for assortative graphs compared with disassortative graphs. ??b therefore highlights the ambivalent effect of r_Θ on $Q_{ST,u}$. r_Θ reinforces $Q_{ST,u}$ by favouring adaptive differentiation, but also decreases $Q_{ST,u}$ by decreasing population isolation within clusters of vertices with the same habitat type.

We compare the effect of r_Θ on $Q_{ST,u}$ to the effect of the topology metrics $\langle l \rangle$ and h_d found in the setting with no selection using a multivariate regression analysis on simulation results obtained for different graphs with varying Θ -spatial distribution (??d for graphs with $M = 7$ vertices and ??b for $M = 9$). The multivariate model explains the discrepancies in $Q_{ST,u}$ across the simulations for low and high migration regimes (see ?? for details), and we find that r_Θ , $\langle l \rangle$ and h_d contribute similarly to neutral differentiation. Hence, the effects of r_Θ and the topology metrics $\langle l \rangle$ and h_d add up under heterogeneous selection. A change in sign of the standardized effect of r_Θ on $Q_{ST,s}$ for low and high migration regimes verifies that the ambivalent effect of r_Θ on $Q_{ST,u}$ found on the path graph holds for general graph

ensembles. Simulations with trait-dependent competition and simulations on realistic graphs with a continuum of habitat types equally confirm the ambivalent effect of r_Θ and further support the complementary effect of $\langle l \rangle$ and h_d on $Q_{ST,u}$ (see ??). $\langle l \rangle$ and h_d therefore drive neutral differentiation with and without heterogeneous selection. r_Θ becomes an additional determinant of neutral differentiation under heterogeneous selection. In contrast to the non-ambivalent, positive effect of habitat assortativity on adaptive differentiation, r_Θ can amplify or depress neutral differentiation depending on the migration regime considered.

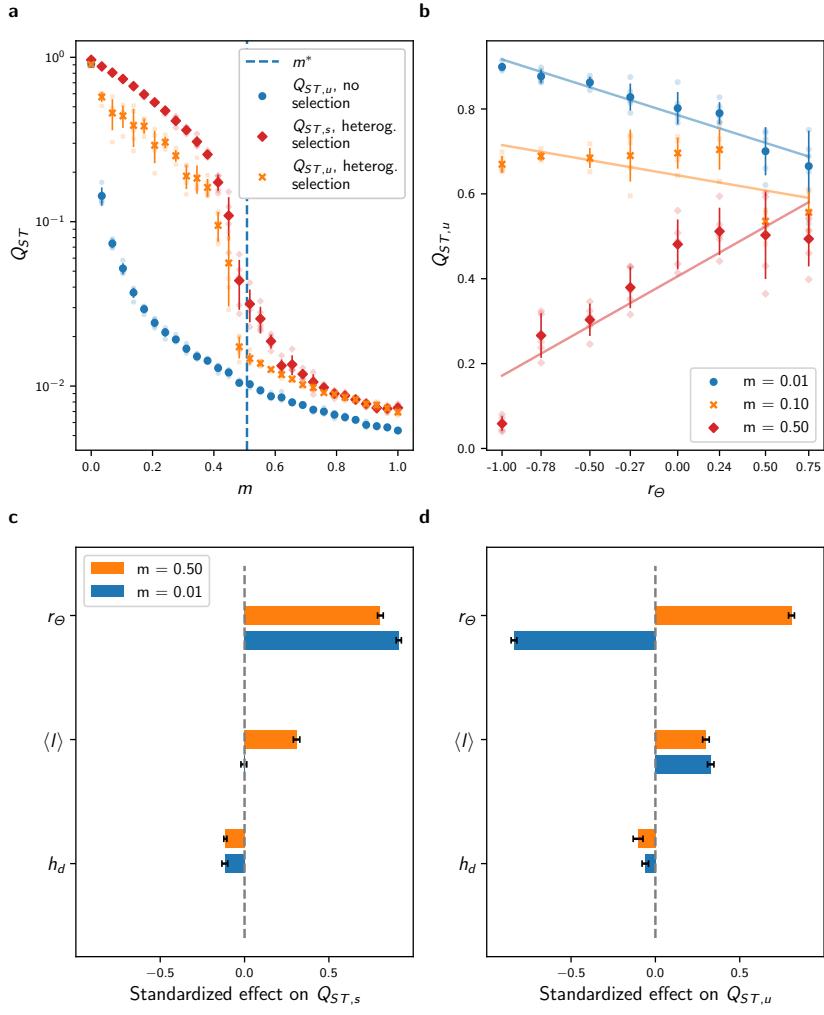


Fig. 2.5: Effect of r_Θ , $\langle I \rangle$ and h_d on $Q_{ST,s}$ and $Q_{ST,u}$ in the setting with heterogeneous selection. (a) Comparison of the response of $Q_{ST,u}$ to migration with the response of $Q_{ST,s}$ in the setting with no selection for the complete graph. The dashed vertical blue line corresponds to the critical migration regime m^* predicted by ???. Heterogeneous selection increases $Q_{ST,u}$ when $m < m^*$, but local adaptation is lost when $m > m^*$, and in this case $Q_{ST,u}$ reaches similar levels as $Q_{ST,s}$ in the setting with no selection. (b) Response of $Q_{ST,u}$ to r_Θ and migration for the path graph. r_Θ correlates positively with $Q_{ST,u}$ for high m , but correlates negatively for low m . In (a-b), each plain dot represents average results from 5 replicate simulations, the bars represent one standard deviation, and each fade dot represents a single replicate value. (c-d) Standardized effect of h_d , $\langle I \rangle$ and r_Θ on $Q_{ST,s}$ and $Q_{ST,u}$ obtained from a multivariate regression model independently fitted for low and high migration regimes on average results from 5 replicate simulations of the IBM on all undirected connected graphs with $M = 7$ vertices and varying r_Θ (see ??). The ambivalence of the effect of r_Θ on $Q_{ST,u}$ found for the path graph holds for general graph ensembles and adds up to that of $\langle I \rangle$ and h_d . Error bars show 95% confidence intervals. Analogous results on graphs with $M = 9$ vertices are presented in ?? and all regression details can be found in ??.

2.3 Discussion

Using analytical tools and simulations, we have built upon a graph representation of landscapes and a stochastic individual-based model to investigate how landscape features drive phenotypic differentiation. Our study is based on a first principles modelling approach (**Champagnat2006**) describing the stochastic dynamics of individuals and capturing the interplay between population dynamics, phenotypic evolution and spatial dynamics in heterogeneous habitats. In contrast to metacommunity models (**Holland2008; Gilarranz2012; Mari2014; Gravel2016; Carrara2012; Thompson2017; Suzuki2021**) and evolutionary metacommunity models (**Economou2007; Economou2010**), we have focused on differentiation at the population level. Quantitative genetics and population genetics studies have investigated the effect of topology on differentiation under the assumption of non-overlapping generations, constant population sizes and regular spatial structures (**Burger2000; Kimura1964; Lande1991; Nagylaki1994; Yeaman2011**). Generalising beyond these assumptions, our modelling framework accounts for population dynamics and includes competition and frequency-dependent selection. The systematic investigation of the effect of topology on differentiation over general graph ensembles and under different ecological settings shows that average path length $\langle l \rangle$, homogeneity in vertex degree h_d and habitat assortativity r_Θ contribute equally to differentiation. These results support correlative studies that have associated population differentiation (**Manel2003; McRae2007**) and species richness (**Liu2018; Dias2014; Rahbek2001; Kreft2007a; Davies2007; Veech2007; Guegan1998; Stein2014**) with a variety of metrics used as surrogates for connectivity, connectivity heterogeneity and habitat heterogeneity. To further our understanding of the origin of spatial biodiversity patterns, the contribution of landscape properties to discrepancies in population differentiation could be investigated at large scales by (i) using techniques to project real landscapes on graphs (see ??a–b); (ii) characterising the landscape features with $\langle l \rangle$, h_d and r_Θ ; and (iii) relating the obtained metrics maps to observation data. More generally, the proposed eco-evolutionary model on spatial graphs could be combined with approximate bayesian computation to estimate ecological, spatial and evolutionary processes of real populations from observation data, similarly to (**Lepers2021**). This approach might improve current inferential techniques based on models that do not account for competition nor heterogeneous selection (see e.g. (**Petkova2015**)). Overall, our results point to topology metrics that can connect spatial biodiversity patterns to the generating eco-evolutionary and spatial processes.

In the absence of selection, neutral differentiation is more pronounced on graphs with a high average path length $\langle l \rangle$, but is also negatively associated with homogeneity in degree h_d (??c–d). $\langle l \rangle$ generalises the concept of dimensionality in (**Kimura1964; Lande1991; Nagylaki1994**), where it is shown that differentiation is lower for two-dimensional grid graphs compared with path graphs. $\langle l \rangle$ also closely relates to the concept of resistance distance shown theoretically and empirically to drive genetic differentiation (**McRae2006; McRae2007**). At the species level, a similar effect of $\langle l \rangle$ on β -diversity (pairwise differences in species composition) has been reported with the graph metacommunity model of

(Carrara2012) and with the graph eco-evolutionary metacommunity model of (Economo2007). Accounting for population dynamics and specifically including competition processes, we have shown that not only $\langle l \rangle$ but also h_d affects neutral phenotypic differentiation (??c,d). Our model realistically assumes that population growth is limited by the local carrying capacity. The latter becomes saturated on highly connected vertices in irregular graphs, an effect that has been experimentally documented in microcosm experiments (Altermatt2018). As a consequence, central vertices behave as bottlenecks and amplify the isolation of peripheral vertices (Orsini2013). The role of h_d cannot be captured with classical metapopulation and quantitative genetics models or with models of evolutionary dynamics in graphs, as they assume constant population size. This behaviour should be prevalent in patchy landscapes where interspecific competition is high because of limiting resources. Our study highlights that heterogeneity in connectivity can reinforce differentiation patterns through the creation of unbalanced migration fluxes which affect ecological equilibrium.

Habitat assortativity r_Θ is a useful indicator for assessing how the spatial distribution of habitat types modulates local adaptation and adaptive differentiation in complex landscapes (Richardson2014). While adaptation has been extensively studied along environmental gradients (Slatkin1973; Slatkin1978; Kirkpatrick1997; Polechova2015; Polechova2018; AndradeRestrepo2019; Doebeli2003), landscapes can be patchy and it is unrealistic to assume regularity (Dale2010). Our model of heterogeneous selection on spatial graphs extends the two-habitat setting investigated in (Meszena1997; Yeaman2011; Debarre2013; Mirrahimi2020) and captures irregularity in connectivity between distinct habitats (Dale2010). Similarly to the aforementioned studies, we have found a critical migration regime m^* that dictates the possibility of adaptation. ?? indicates that m^* increases with increasing selection strength p and with increasing environmental heterogeneity θ , the latter playing a similar role as the slope of the environmental gradient in (Slatkin1973; Slatkin1978; Polechova2015; Polechova2018). Local adaptation would consequently be sustained under higher migration regimes following an increase in these parameters. Additionally, the critical migration regime m^* in ?? involves the habitat assortativity r_Θ , which must be regarded as a measure of habitat spatial auto-correlation based on the dispersal range of a species (Richardson2014). Our results indicate that for general habitat distributions, r_Θ is the main determinant of adaptive differentiation under sufficiently strong selection p and high habitat heterogeneity θ , irrespective of the graph topology (??c, ??a and ??). As p decreases, however, the effect of stochastic drift on $Q_{ST,s}$ should increase, and in this case the topology metrics $\langle l \rangle$ and h_d should become the most important determinants of $Q_{ST,s}$. Our results predict that in landscapes with heterogeneous habitats and where selection is strong, populations structured over assortative habitats are larger, support higher adaptive differentiation, and can be locally well-adapted even in the case where migration rates are high.

Spatial eco-evolutionary feedbacks in heterogeneous habitats can critically affect differentiation (Richardson2014). While most eco-evolutionary studies have investigated diversification by considering a unique adaptive trait (Doebeli2003; Kirkpatrick1997; Polechova2015; Polechova2018), distinguishing between neutral and adaptive processes is crucial (Holderegger2006) and our work underlines the distinct responses of neutral and adaptive differentiation to landscape features (??c vs. ??d). Our study builds

upon recent mathematical models that consider the co-evolution of neutral and adaptive traits (**Billiard2015**; **Anceschi2019**) and extends those works to a spatial context. Our work provides an analytical framework to the concept of isolation by environment (IBE) (**Orsini2013**), which has been suggested to be one of the most important mechanisms governing differentiation in nature (**Wang2014**). Heterogeneous selection leads to more isolation by modifying the fitness of migrants (**Polechova2018**), which further reduces gene flow (**Richardson2014**) and therefore affects the level of neutral differentiation (??a) (**GARANT2007**). Our work proposes a mechanism by which habitat assortativity, relative to the migration regime, controls the direction of the effect of habitat heterogeneity on differentiation (??d). Patchy, heterogeneous habitats can promote neutral differentiation as a result of selection that reduces effective migration (**Stein2014**). Nonetheless, adaptive differentiation decreases substantially when migration is high relative to the critical migration regime m^* . In this case, neutral differentiation should be higher in landscapes with more aggregated habitats (**Richardson2014**). Our study suggests that habitat assortativity must be considered for a complete understanding of differentiation in complex environments (**Stein2014**).

In conclusion, we have established how differentiation can emerge at the population level from eco-evolutionary feedbacks in complex landscapes by using an analytical description of micro-evolutionary processes explicitly accounting for spatial dynamics over graphs. Our study formalises how differentiation emerges from the interplay between spatial dynamics, the co-evolution of neutral and adaptive traits, and landscape properties. Connectivity and habitat assortativity emerge as core determinants of differentiation in spatial graphs. These results resonate with empirical findings and previous theoretical works. Our study further stresses that habitat assortativity can depress or foster neutral differentiation depending on the migration regime. Additionally, our work highlights that heterogeneity in connectivity is an equally strong determinant of differentiation because highly connected habitats behave as bottlenecks, increasing the isolation of peripheral habitats. The present approach offers a promising framework for studying complex adaptive systems, as it can elucidate how macroscopic properties emerge from microscopic processes acting upon agents structured over complex spatio-evolutionary structures.

2.4 Methods

2.4.1 Mean field approximation

In the setting with no selection, the mean field approach involves the assumption that all vertices having the same degree are equivalent. For this, let $P(k, k')$ denote the proportion of edges that map a vertex with degree k to a vertex with degree k' , and consider the average population size $\bar{N}_t^{(k)}$ in each vertex with degree k at time t . An individual has probability $P(k, k')/k'$ to migrate from a vertex with degree k' to a vertex with degree k . Viewing $a_{i,j}/d_j$ as

the probability that an individual on v_i chosen for migration moves to v_j , ?? then transforms into

$$\partial_t \bar{N}_t^{(k)} = \bar{N}_t^{(k)} \left[b(1-m) - \frac{\bar{N}_t^{(k)}}{K} \right] + mbk \sum_{k' \in V} \frac{P(k, k')}{k'} \bar{N}_t^{(k')} \quad (2.9)$$

Assuming uncorrelated graphs for which $P(k, k')/k' = P(k')/k'$, where $\langle k \rangle$ denotes the average degree of the graph (**Colizza2007**), yields

$$\partial_t \bar{N}_t^{(k)} = \bar{N}_t^{(k)} \left[b(1-m) - \frac{\bar{N}_t^{(k)}}{K} \right] + mb \frac{k}{\langle k \rangle} \bar{N}_t \quad (2.10)$$

where

$$\bar{N}_t = \sum_k P(k) \bar{N}_t^{(k)}. \quad (2.11)$$

When solving for the stationary state and setting $m = 1$, one obtains $\bar{N}^{(k)} = \sqrt{bK \frac{k}{\langle k \rangle}}$ from ???. Combining this with ?? yields

$$\bar{N} = bK \langle \sqrt{k} \rangle^2 / \langle k \rangle \quad (2.12)$$

In the setting with heterogeneous selection, the mean field approach involves the assumption that all vertices with a similar habitat are equivalent. In this case, an individual from a vertex of habitat type \bullet has the probability $P(\bullet, \bullet)/P(\bullet)$ of migrating to a vertex of type \bullet , and therefore ?? transforms into

$$\begin{aligned} \partial_t \bar{n}_t^\bullet(s) &= \bar{n}_t^\bullet(s) \left[b^\bullet(s)(1-m) - \frac{1}{K} \int_S \bar{n}_t^\bullet(\mathbf{s}) d\mathbf{s} \right] + \frac{1}{2} \mu \sigma_\mu^2 \Delta_s [b^\bullet(s) \bar{n}_t^\bullet(s)] \\ &\quad + m \sum_{i \in \{\bullet, \bullet\}} b_i(s) \frac{P(\bullet, i)}{P(i)} \bar{n}_t^i(s) \end{aligned} \quad (2.13)$$

Considering that $P(\bullet) = P(\bullet) = \frac{1}{2}$ (habitats are equally distributed), $P(\bullet, \bullet) + P(\bullet, \bullet) = P(\bullet)$ (sum of conditional expectations), and $r_\Theta = 2(P(\bullet, \bullet) - P(\bullet, \bullet))$??, one obtains

$$P(\bullet, \bullet) = \frac{1}{4}(1 - r_\Theta) \quad \text{and} \quad P(\bullet, \bullet) = \frac{1}{4}(1 + r_\Theta) \quad (2.14)$$

Combining ?? with ?? yields ???. We show in the ?? how one can derive ?? from the general definition of assortativity given in ?? and initially introduced in (**Newman2003a**).

2.4.2 Adaptive dynamics on graphs

The adaptive dynamics theory considers a monomorphic population that evolves following a trait substitution process (**Meszena1997**). Accordingly, the trait s of the monomorphic metapopulation evolves gradually along the direction given by its fitness gradient, until

it reaches a singular strategy s^* for which the fitness gradient vanishes. By omitting the mutation term, ?? can be written in the matrix form

$$\partial_t \bar{\mathbf{n}}_t(s) = M(s, \bar{\mathbf{N}}_t) \bar{\mathbf{n}}_t(s) \quad (2.15)$$

where $\bar{\mathbf{n}}_t = (\bar{n}_t^\bullet, \bar{n}_t^\Theta)$ and $\bar{\mathbf{N}}_t = (\bar{N}_t^\bullet, \bar{N}_t^\Theta)$ are the vectors containing the population densities and the population size on each habitat type, and

$$M(s, \bar{\mathbf{N}}) = \begin{bmatrix} \mathfrak{r}^\bullet(s, \bar{N}^\bullet) & \frac{m}{2}(1 - r_\Theta)b^\bullet(s) \\ \frac{m}{2}(1 - r_\Theta)b^\bullet(s) & \mathfrak{r}^\bullet(s, \bar{N}^\bullet) \end{bmatrix} \quad (2.16)$$

is the so-called projection matrix (**Meszena1997**), with $\mathfrak{r}^\bullet(s, \bar{N}^\bullet) = b^\bullet(s)(1 + \frac{m}{2}(r_\Theta - 1)) - \bar{N}^\bullet/K$. The overall fitness of individuals with trait s is the leading eigenvalue of M , which we denote with $\lambda(s, \bar{\mathbf{N}})$. We obtain the singular strategy s^* by setting the fitness gradient $\frac{\partial \lambda}{\partial s}(s, \bar{\mathbf{N}}) = 0$, from which we further obtain the demographic equilibrium $\bar{\mathbf{N}}^{s^*}$. Because of symmetries, we must have $\bar{N}^{\bullet, s^*} = \bar{N}^{\Theta, s^*}$ and $s^* = \frac{\theta_\bullet + \theta_\Theta}{2} = 0$, such that $\bar{N}^{\bullet, s^*} = \bar{N}^{\Theta, s^*} = bK(1 - p\theta^2)$. s^* is said to be evolutionary stable if no mutants can invade, i.e. if s^* locally maximises the fitness of a mutant with trait y in the resident population with trait s^* , given by $\lambda(y, \bar{\mathbf{N}}^{s^*})$ (see (**Meszena1997**) for details). One can show that $\left[\frac{\partial \lambda}{\partial y}(y, \bar{\mathbf{N}}^{s^*}) \right]_{y=s^*} = 0$ and the condition for evolutionary stability becomes $\left[\frac{\partial^2 \lambda}{\partial y^2}(y, \bar{\mathbf{N}}^{s^*}) \right]_{y=s^*} < 0$. We compute and simplify this inequality through computer algebra (see Mathematica notebook provided in the simulation code), which leads to ??.

2.4.3 Numerical simulations

The model was implemented in a multi-purpose Julia package called `EvoId.jl`, available at <https://github.com/vboussange/EvoId.jl>. For each result presented, $b = 1$, local carrying capacity $K = 150$, selection strength $p = 1$, mutation rate $\mu = 0.1$, mutation range $\sigma_\mu = 5 \cdot 10^{-2}$, and total time span $t = 1000$. This parameter choice made it possible to discard transient dynamics while obtaining results in a reasonable computational time (see ??). In settings (1) and (2), we ran simulations on all of the 853 undirected connected graphs with $M = 7$ vertices and on 1126 of the 261,080 undirected connected graphs with $M = 9$ vertices, listed at <http://oeis.org/A001349>. Graphs with $M = 9$ vertices were selected with a stratified sampling method: we randomly sampled without replacement a maximum of 50 graphs for each class of graphs with an equal number of vertices. For the setting with heterogeneous selection, we generated the labeled graphs by randomly generating Θ -spatial distributions, and by using a stratified sampling strategy to select without replacement at most 3 and 2 Θ -spatial distributions corresponding to the quartiles of the r_Θ values obtained, respectively for graphs with $M = 7$ and $M = 9$ vertices. This sampling strategy allowed to obtain a uniform distribution of the topology metrics investigated in the study, and therefore permitted to correctly represent the population of graphs to investigate their effect on differentiation. We then computed $Q_{ST,u}$ and $Q_{ST,s}$, which we further averaged over the last time steps and across the replicates. Since the dynamics of $Q_{ST,u}$ is characterised by large quadratic variations, we simulated individuals with $d = 300$ neutral traits, where

each trait can independently be affected by mutations. $Q_{ST,u}$ values presented were then obtained from the average $Q_{ST,u}$ for each trait. This reduced the variance of the numerical simulations and is also biologically meaningful because populations are characterised by many traits, most of which are neutral (**Holderegger2006**). As initial conditions, MK individuals were homogeneously distributed over all of the vertices, with traits centred on 0 and with standard deviation σ_μ . Graph metrics used for the meta-analysis were calculated using the **LightGraphs.jl** library (**Bromberger2017**). We numerically solved the PDEs with a finite difference scheme using **DifferentialEquations.jl** (**Rackauckas2017**), ensuring that the domain was large enough to avoid border effects.

2.4.4 Statistics and reproducibility

Statistical analyses were conducted in Julia using **StatsKit.jl**. All simulations can be exactly reproduced from the code available at <https://github.com/vboussange/differentiation-in-spatial-graphs>.

2.5 Data availability

The data underlying our figures is available at <https://github.com/vboussange/differentiation-in-spatial-graphs>.

2.6 Code availability

The simulation code is available at <https://github.com/vboussange/differentiation-in-spatial-graphs>.

2.7 Acknowledgements

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2.A Supplementary Note

2.A.1 Mathematical construction of the model

The model is a measure-valued point process (**Bansaye2015**), so that individuals are represented as dirac functions $\delta_{x_k^{(i)}}$, where $x_k^{(i)} \in \mathcal{X}$ corresponds to the traits' value of

individual k located on vertex v_i . Under this formalism, the population on v_i is represented as a sum of dirac functions $\nu^{(i)} = \sum_k^{N^{(i)}} \delta_{x_k^{(i)}}$, where $N^{(i)}$ is the local population size. It follows that the time variation of the process can be described by the so-called infinitesimal generator L , defined for all real valued functions ϕ as

$$L\phi(\nu_t^{(i)}) = \partial_t \mathbb{E} [\phi(\nu_t^{(i)})] \quad (\text{S2.17})$$

(see (Linke2015) for an introduction to infinitesimal generators). ?? provides the expected time variation at time t of e.g. the population size by choosing $\phi(\nu_t^{(i)}) = \int_{\mathcal{X}} \nu_t^{(i)}(dx)$. Recall that we use $b^{(i)}$ to denote the birth rate on vertex v_i , d for the death rate, μ for the mutation probability, m for the migration probability, $\mathcal{M}(x, y) = \frac{1}{\sqrt{2\pi\sigma_\mu}} \exp\left(\frac{-||x-y||^2}{2\sigma_\mu}\right)$ for the mutation kernel, K for the local carrying capacity, $A = (a_{i,j})_{1 \leq i,j \leq M}$ for the adjacency matrix of the graph G , and $D = (d_1, d_2, \dots, d_M)$ for the vector containing the degree of each vertex. In order to explicitly write the generator L , let us recall that five events of different natures can alter the number of individuals with trait x on vertex v_i :

- an individual on v_i with trait x can give birth to an offspring that does not experience mutations nor migration, at rate $(1 - \mu)(1 - m)b^{(i)}(x)$,
- an individual on v_i with trait y can give birth to an offspring with mutated trait x that does not experience migration, at rate $\mu(1 - m)\mathcal{M}(x, y)b^{(i)}(y)$,
- an individual on v_i with trait x can die, at rate $d(N^{(i)}) = \frac{N^{(i)}}{K} = \frac{1}{K} \int_{\mathcal{X}} \nu_t^{(i)}(dx)$,
- an individual on v_j with trait x can give birth to an offspring that does not experience mutations and migrates to v_i , at rate $\frac{a_{i,j}}{d_j}(1 - \mu)m b^{(j)}(x)$,
- an individual on v_j with trait y can give birth to an offspring with mutated trait x that migrates to v_i , at rate $\frac{a_{i,j}}{d_j} \mu m \mathcal{M}(x, y) b^{(j)}(y)$.

Summing over all all individuals and all vertices yields

$$\begin{aligned} L\phi(\nu_t^{(i)}) &= \int_{\mathcal{X}} \left\{ b^{(i)}(\mathbf{x})(1 - \mu)(1 - m)(\phi(\nu_t^{(i)} + \delta_{\mathbf{x}}) - \phi(\nu_t^{(i)})) \right\} \nu_t^{(i)}(d\mathbf{x}) \\ &\quad \text{births w/o mutations, w/o migrations} \\ &+ \int_{\mathcal{X}} \left\{ \mu(1 - m) \int_{\mathcal{X}} b^{(i)}(y)(\phi(\nu_t^{(i)} + \delta_y) - \phi(\nu_t^{(i)})) \mathcal{M}(\mathbf{x}, y) dy \right\} \nu_t^{(i)}(d\mathbf{x}) \\ &\quad \text{births w/ mutations, w/o migrations} \\ &+ \iint_{\mathcal{X}} \left\{ \frac{1}{K} (\phi(\nu_t^{(i)} - \delta_{\mathbf{x}}) - \phi(\nu_t^{(i)})) \nu_t^{(i)}(dy) \nu_t^{(i)}(dx) \right\} \\ &\quad \text{deaths} \\ &+ \sum_{j \neq i} \frac{a_{i,j}}{d_j} \int_{\mathcal{X}} \mu m \left\{ \int_{\mathcal{X}} b^{(j)}(y)(\phi(\nu^{(j)} + \delta_{\mathbf{x}}) - \phi(\nu^{(j)})) \mathcal{M}(\mathbf{x}, y) dy \right\} \nu_t^{(j)}(d\mathbf{x}) \\ &\quad \text{migrations w/ mutations} \\ &+ \sum_{j \neq i} \frac{a_{i,j}}{d_j} \int_{\mathcal{X}} \left\{ b^{(j)}(\mathbf{x})(1 - \mu)m(\phi(\nu^{(j)} + \delta_{\mathbf{x}}) - \phi(\nu^{(j)})) \right\} \nu_t^{(j)}(d\mathbf{x}). \quad (\text{S2.18}) \\ &\quad \text{migrations w/o mutations} \end{aligned}$$

Taking expectations in ??, one can obtain an equation for the mean trajectory of the quantity of interest, $\mathbb{E} [\phi(\nu_t^{(i)})]$. Nonetheless, ?? involves an integral with respect to $\nu_t^{(i)}(dx)\nu_t^{(i)}(dy)$, making it impossible to obtain an explicit solution. It is therefore unclear whether one can gain insight into the stochastic dynamics from ?? without simplifying assumptions. We refer to (**Champagnat2006**) for a detailed discussion on the topic.

2.A.2 Deterministic approximation

One strategy to overcome the difficulties encountered above is to assimilate the process to its mean trajectory, assuming that $\mathbb{E} [\nu_t^{(i)}] \approx \nu_t^{(i)}$ and further approximating $\nu_t^{(i)}$ with a continuous deterministic function $n_t^{(i)}$. Such strategy inherently neglects the stochasticity of the process, which is reasonable provided that a force dampens the stochastic fluctuations of the quantity of interest.

Setting with no selection

Consider a setting with no selection and recall that in this setting where $x \equiv u \in \mathcal{X} = \mathcal{U}$ we define

$$b^{(i)}(x) \equiv b \quad (\text{S2.19})$$

By applying the strategy mentioned above and choosing $\phi(n_t^{(i)}) = \int_{\mathcal{X}} n_t^{(i)}(x) dx$, ?? transforms into the deterministic approximation of the population size dynamics given in the main-text by

$$\partial_t N_t^{(i)} = N_t^{(i)} \left[b(1-m) - \frac{N_t^{(i)}}{K} \right] + mb \sum_{j \neq i} \frac{a_{i,j}}{d_j} N_t^{(j)}. \quad (\text{S2.20})$$

Competition stabilises the population size dynamics, which behaves deterministically. This is supported by ??a, which shows how ?? accurately describes the population size for varying migration regimes. Nonetheless, stochastic fluctuations drive the dynamics of the neutral trait distribution. Attempting to characterise the neutral trait distribution with the same strategy, this time setting $\phi(n_t^{(i)}) = n_t^{(i)}(u)$, yields

$$\begin{aligned} \partial_t n_t^{(i)}(u) &= n_t^{(i)}(u) \left[b(1-m)(1-\mu) - \frac{1}{K} \int_{\mathcal{U}} n_t^{(i)}(\mathbf{u}) d\mathbf{u} \right] \\ &\quad + (1-m)\mu b \int_{\mathcal{U}} n_t^{(i)}(\mathbf{u}) \mathcal{M}(u, \mathbf{u}) d\mathbf{u} \\ &\quad + m\mu b \sum_{j \neq i} \frac{a_{i,j}}{d_j} \int_{\mathcal{U}} n_t^{(j)}(u) \mathcal{M}(u, \mathbf{u}) d\mathbf{u} \\ &\quad + m(1-\mu)b \sum_{j \neq i} \frac{a_{i,j}}{d_j} b n_t^{(j)}(u). \end{aligned} \quad (\text{S2.21})$$

Solving for ??, one can show that the variance of $n_t^{(i)}$ continuously grows in time (see ??) and tends to infinity as time goes to infinity, which is an unrealistic behaviour considering finite populations. Intuitively, this reflects the fact that no stabilising force acts on the neutral trait distribution, such that random fluctuations play a major role in driving the dynamics of the stochastic process. ?? shows how IBM trajectories significantly differ from ??, and ?? illustrates how diversity metrics obtained from ?? do not match those obtained from simulations of the IBM.

Setting with heterogeneous selection

In contrast to the neutral trait dynamics, the adaptive distribution can successfully be approximated by a deterministic description because selection pressure acts as a stabilising force and stabilises the populations' adaptive trait, dampening the stochastic fluctuations. Consider the setting with heterogeneous selection and recall that in this setting where $x \equiv (s, u) \in \mathcal{X} = \mathcal{S} \times \mathcal{U}$ we define

$$b^{(i)}(x) \equiv b(1 - p(s - \theta_i)^2). \quad (\text{S2.22})$$

By applying the same strategy as above to characterise the adaptive trait distribution $n_t^{(i)}(s)$ by choosing $\phi(n_t^{(i)}) = n_t^{(i)}(s) \equiv \int_{\mathcal{U}} n_t^{(i)}(u, s) du$, ?? transforms into

$$\begin{aligned} \partial_t n_t^{(i)}(s) &= n_t^{(i)}(s) \left[b^{(i)}(s)(1-m)(1-\mu) - \frac{1}{K} \int_{\mathcal{S}} n_t^{(i)}(\mathbf{s}) d\mathbf{s} \right] \\ &\quad + (1-m)\mu \int_{\mathcal{S}} b^{(i)}(\mathbf{s}) n_t^{(i)}(\mathbf{s}) \mathcal{M}(\mathbf{s}, s) d\mathbf{s} \\ &\quad + m\mu \sum_{j \neq i} \frac{a_{i,j}}{d_j} \int_{\mathbb{R}} b^{(j)}(\mathbf{s}) n_t^{(j)}(s) \mathcal{M}(\mathbf{s}, s) d\mathbf{s} \\ &\quad + m(1-\mu) \sum_{j \neq i} \frac{a_{i,j}}{d_j} b^{(j)}(s) n_t^{(j)}(s). \end{aligned} \quad (\text{S2.23})$$

Assuming that the variance of the mutation kernel is small, one can use a diffusion approximation for the mutation term (**Kimura1965**; **Debarre2013**; **Mirrahimi2020**)

$$\int_{\mathcal{S}} b^{(i)}(\mathbf{s}) n_t^{(i)}(\mathbf{s}) \mathcal{M}(\mathbf{s}, s) d\mathbf{s} = b^{(i)}(s, t) n_t^{(i)}(s) + \frac{1}{2} \sigma_{\mu}^2 \Delta_s (b^{(i)} n_t^{(i)})(s). \quad (\text{S2.24})$$

Neglecting the terms in $m\mu$, we obtain

$$\begin{aligned} \partial_t n_t^{(i)}(s) &= n_t^{(i)}(s) \left[b^{(i)}(s, t)(1-m-\mu) - \frac{1}{K} \int_{\mathcal{S}} n_t^{(i)}(\mathbf{s}) d\mathbf{s} \right] \\ &\quad + \mu \left[b^{(i)}(s, t) n_t^{(i)}(s) + \frac{1}{2} \sigma_{\mu}^2 \Delta_s (b^{(i)} n_t^{(i)})(s) \right] \\ &\quad + m \sum_{j \neq i} b^{(j)}(s, t) n_t^{(j)}(s) a_{i,j} \end{aligned} \quad (\text{S2.25})$$

which, after rearranging terms, yields the elegant deterministic approximation of the adaptive trait dynamics

$$\begin{aligned}\partial_t n_t^{(i)}(s) &= n_t^{(i)}(s) \left[b^{(i)}(s)(1-m) - \frac{1}{K} \int_{\mathcal{S}} n_t^{(i)}(\mathbf{s}) d\mathbf{s} \right] \\ &\quad + m \sum_{j \neq i} b^{(j)}(s) \frac{a_{i,j}}{d_j} n_t^{(j)}(s) \\ &\quad + \frac{1}{2} \mu \sigma_\mu^2 \Delta_s \left[b^{(i)}(s) n_t^{(i)}(s) \right].\end{aligned}\tag{S2.26}$$

Setting $m = 0$ (**Mirrahimi2020**) shows that ?? admits a stationary solution that is Gaussian, with variance $\sqrt{\mu} \sigma_\mu^2 / \sqrt{p}$. Therefore, the variance of the adaptive trait distribution stabilises to a finite value. Intuitively, this reflects the fact that the random fluctuations of the adaptive trait distribution are dampened by the stabilising force of selection. Provided that the selection strength p is large enough, ?? is a good approximation of the adaptive trait distribution obtained from the stochastic process. ?? shows how IBM trajectories are similar to the ones obtained from ??, and ?? illustrates how diversity metrics obtained from ?? match those obtained from simulations of the IBM.

2.A.3 Trait-dependent competition

To test whether the effects of the metrics hold under more complex ecological processes, we designed an extra experiment considering heterogeneous selection and adaptive trait-dependent competition, where the death rate of individuals on v_i with traits $x_k^{(i)} = (u_k^{(i)}, s_k^{(i)}) \in \mathcal{U} \times \mathcal{S}$ is given by

$$d(x_k^{(i)}, \nu^{(i)}) = \frac{1}{K} \int_{\mathcal{S}} \exp\left(-\frac{(s_k^{(i)} - \mathbf{s})^2}{2\sigma_\alpha^2}\right) \nu^{(i)}(\mathbf{s})\tag{S2.27}$$

where σ_α is the competition bandwidth. This competition kernel tends to increase the population size, as it decreases the overall competition. The adaptive dynamics theory predicts that when $m = 0$, competition promotes two distinct types of individuals at either side of the adaptive trait optimum for a competition bandwidth $\sigma_\alpha < 1/\sqrt{2p}$, while a single type is observed when $\sigma_\alpha > 1/\sqrt{2p}$ (**DoebeliMichael2011Ad**). We performed simulations in both cases for graphs with $M = 7$ vertices and show results of the multivariate regression analyses in ?. The analyses demonstrate that the trends reported in the main manuscript remain unchanged in both cases.

2.A.4 Derivation of the habitat assortativity metric r_Θ in binary environments

We demonstrate here how the habitat assortativity r_Θ relates to the conditional probability of habitats being connected, and we show how r_Θ simplifies under mean field assumption.

Following the original definition of (Newman2003a), habitat assortativity r_Θ is defined as the Pearson correlation of environmental conditions θ at either ends of the vertices V of graph G , that is

$$r_\Theta = \frac{\text{Cov}(\Theta_x, \Theta_\wedge)}{\sqrt{\text{Var}(\Theta_x)\text{Var}(\Theta_\wedge)}} = \frac{\langle\Theta_x\Theta_\wedge\rangle - \langle\Theta_x\rangle\langle\Theta_\wedge\rangle}{\sqrt{(\langle\Theta_x^2\rangle - \langle\Theta_x\rangle^2)(\langle\Theta_\wedge^2\rangle - \langle\Theta_\wedge\rangle^2)}} \quad (\text{S2.28})$$

where Θ_x and Θ_\wedge denote the sets of environmental conditions found at the toe and tip of each directed vertex of graph V , and $\langle\Theta_x\rangle$ and $\langle\Theta_\wedge\rangle$ denote their respective mean values.

Let $P(\bullet, \bullet)$ be the proportion of edges that connect a vertex of habitat type \bullet to a vertex of habitat type \bullet . One can also view $P(\bullet, \bullet)$ as the conditional probability that a vertex of type \bullet is connected to a vertex of type \bullet . Let $P(\bullet)$ denote the proportion of vertices that are of type \bullet . First observe that for undirected graphs, one has $\langle\Theta_x\rangle = \langle\Theta_\wedge\rangle$ and $\langle\Theta_x^2\rangle = \langle\Theta_\wedge^2\rangle$. Assuming that habitats are symmetric and binary, it follows that $\theta_{\bullet|} = -\theta_{\bullet|}$. Then

$$\begin{aligned} \langle\Theta_x\Theta_\wedge\rangle &= P(\bullet, \bullet)\theta_{\bullet|}^2 + P(\bullet, \bullet)\theta_{\bullet|}^2 + [P(\bullet, \bullet) + P(\bullet, \bullet)]\theta_{\bullet|}\theta_{\bullet|} \\ &= \theta_{\bullet|}^2 (P(\bullet, \bullet) + P(\bullet, \bullet) - [P(\bullet, \bullet) + P(\bullet, \bullet)]), \end{aligned} \quad (\text{S2.29})$$

$$\begin{aligned} \langle\Theta_x\rangle &= P(\bullet)\theta_{\bullet|} + P(\bullet)\theta_{\bullet|} \\ &= \theta_{\bullet|} [P(\bullet) - P(\bullet)], \end{aligned} \quad (\text{S2.30})$$

$$\begin{aligned} \langle\Theta_x^2\rangle &= P(\bullet)\theta_{\bullet|}^2 + P(\bullet)\theta_{\bullet|}^2 \\ &= \theta_{\bullet|}^2 [P(\bullet) + P(\bullet)] \\ &= \theta_{\bullet|}^2. \end{aligned} \quad (\text{S2.31})$$

Combining ??, ?? and ?? with ?? one gets

$$\begin{aligned} r_\Theta &= \frac{\langle\Theta_x\Theta_\wedge\rangle - \langle\Theta_x\rangle\langle\Theta_\wedge\rangle}{\langle\Theta_x^2\rangle - \langle\Theta_x\rangle^2} \\ &= \frac{P(\bullet, \bullet) + P(\bullet, \bullet) - [P(\bullet, \bullet) + P(\bullet, \bullet)] - (P(\bullet) - P(\bullet))^2}{P(\bullet) + P(\bullet) - (P(\bullet) - P(\bullet))^2} \\ &= \frac{P(\bullet, \bullet) + P(\bullet, \bullet) - [P(\bullet, \bullet) + P(\bullet, \bullet)] - (P(\bullet) - P(\bullet))^2}{1 - (P(\bullet) - P(\bullet))^2}. \end{aligned} \quad (\text{S2.32})$$

Assuming that habitats are homogeneously distributed, we have $P(\bullet) = P(\bullet) = \frac{1}{2}$ and thus we obtain

$$r_\Theta = P(\bullet, \bullet) + P(\bullet, \bullet) - [P(\bullet, \bullet) + P(\bullet, \bullet)]. \quad (\text{S2.33})$$

The mean field approximation involves the assumption that all vertices with similar habitats are equivalent in terms of their connections with other habitats, so that $P(\bullet, \bullet) = P(\bullet, \bullet)$ and $P(\bullet, \bullet) = P(\bullet, \bullet)$, which yields $r_\Theta = 2(P(\bullet|, \bullet|) - P(\bullet|, \bullet|))$.

2.B Supplementary Figures

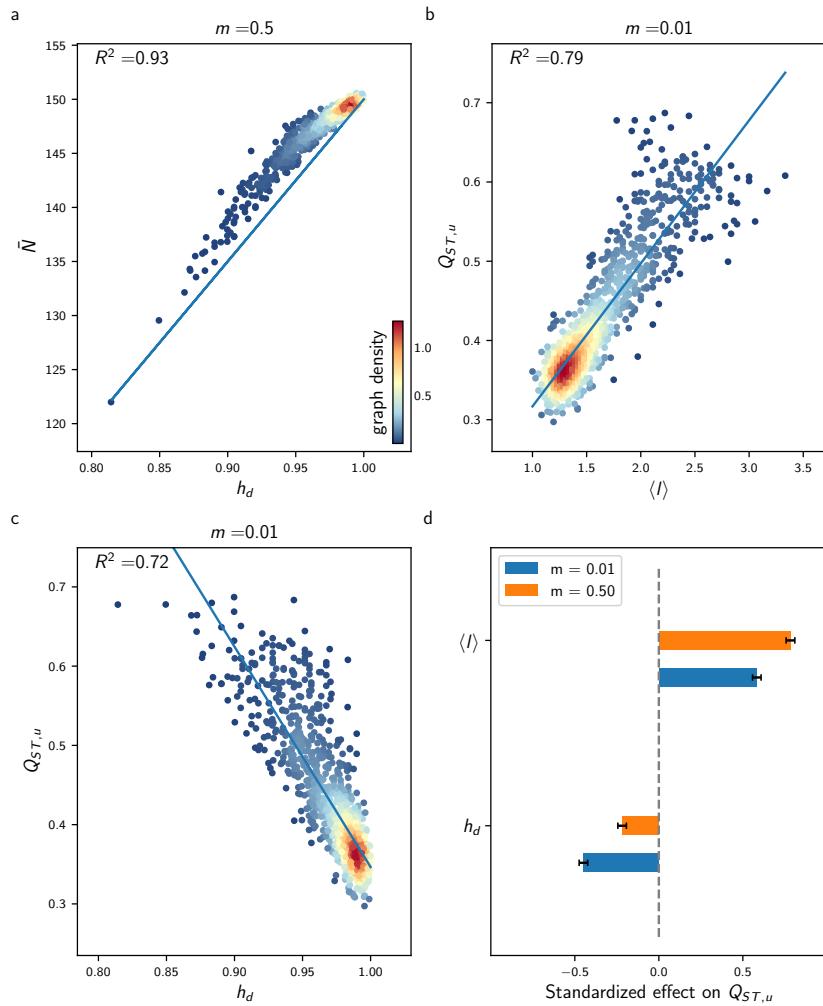


Fig. S2.6: Effect of $\langle l \rangle$ and h_d on average population size \bar{N} and neutral differentiation $Q_{ST,u}$ under the setting with no selection, analogous to ?? but for 1126 of the 261,080 undirected connected graphs with $M = 9$ vertices.

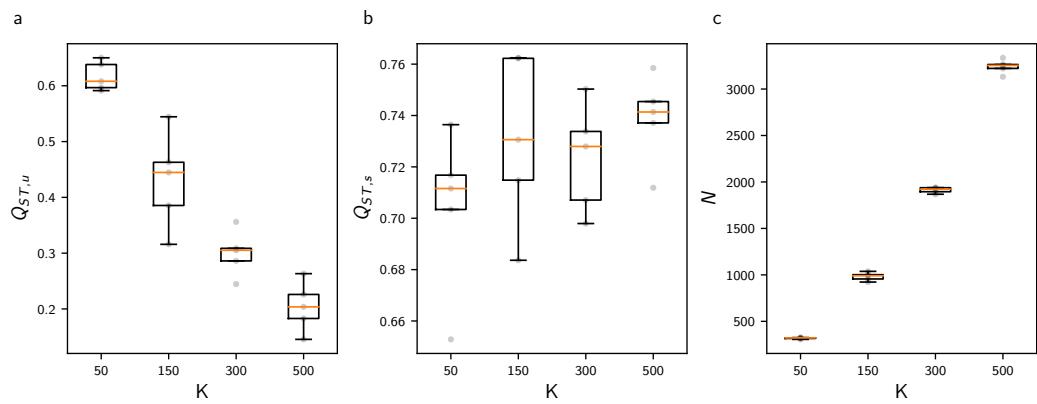


Fig. S2.7: Effect of the carrying capacity K on $Q_{ST,u}$, $Q_{ST,s}$ and metapopulation size N for the line graph with $M = 7$ vertices for $m = 0.1$. Decreasing K increases $Q_{ST,u}$ as it favours drift, but it does not influence $Q_{ST,s}$. Each boxplot is based on 5 replicate simulations of the IBM, and fade dots represent single values for each replicate.

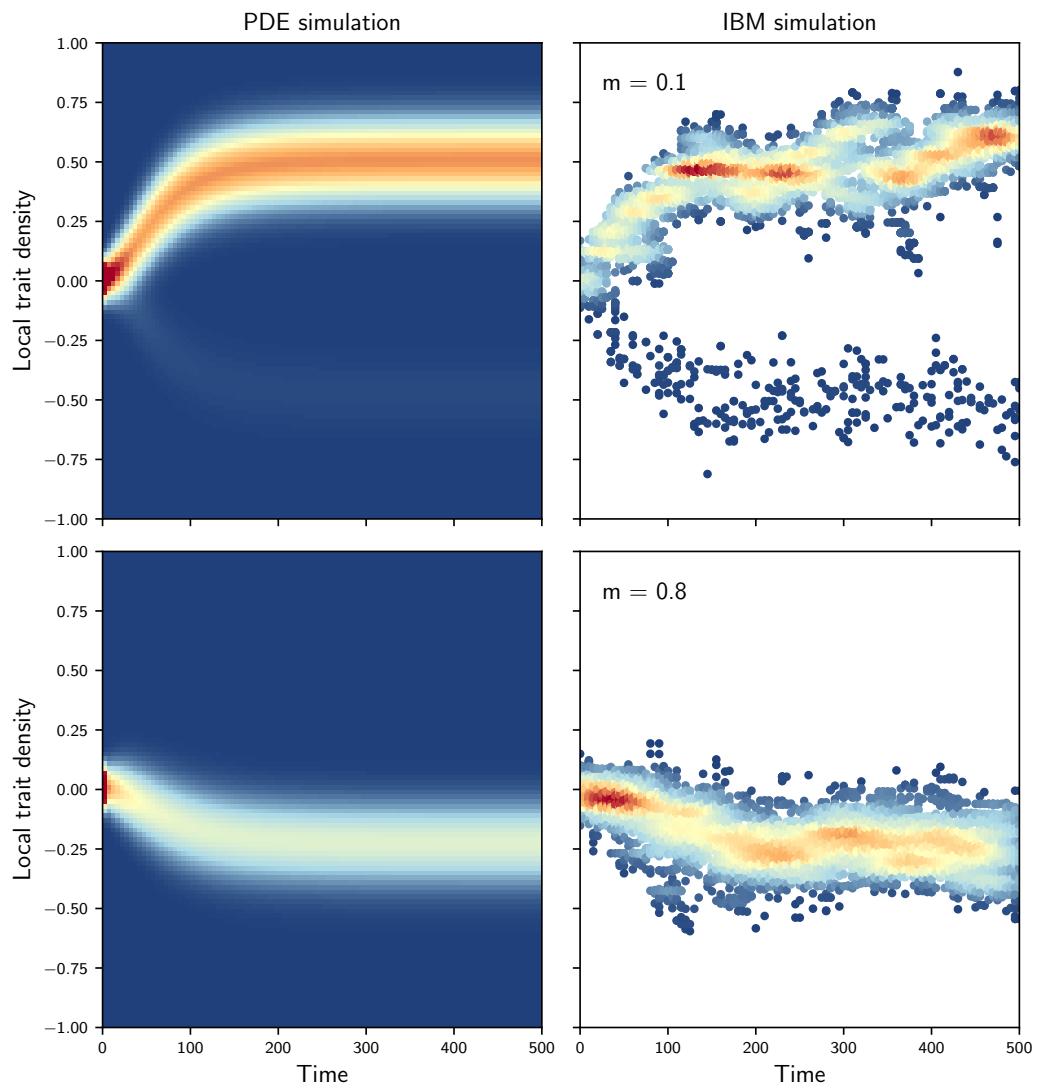


Fig. S2.8: Comparison of the adaptive trait density on one vertex obtained from ?? (left) and from the IBM simulations (right) in the setting with heterogeneous selection, for the star graph with $M = 7$ vertices. The densities obtained from ?? and from the IBM are qualitatively similar.

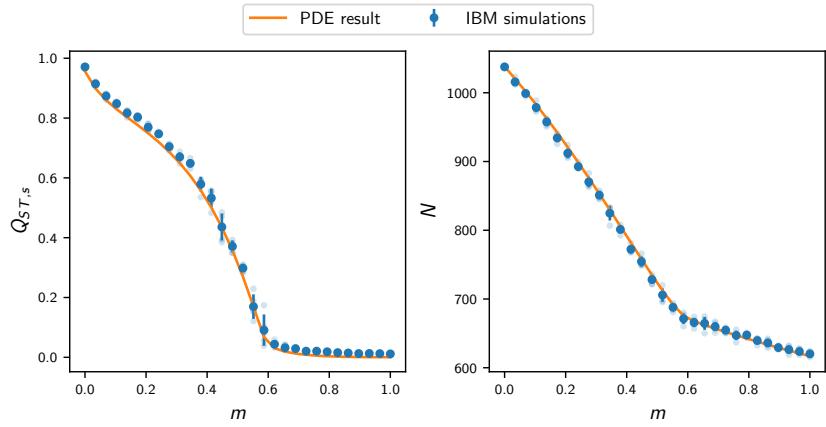


Fig. S2.9: Comparison of $Q_{ST,s}$ and N obtained from the deterministic approximation ?? and from IBM simulations in the setting with heterogeneous selection, on the star graph with $M = 7$ vertices. $Q_{ST,s}$ and population size obtained from ?? closely match the IBM simulations. Each plain dot represents average results from 5 replicate simulations of the IBM, bars represent one standard deviation, and each fade dot represents a single replicate value.

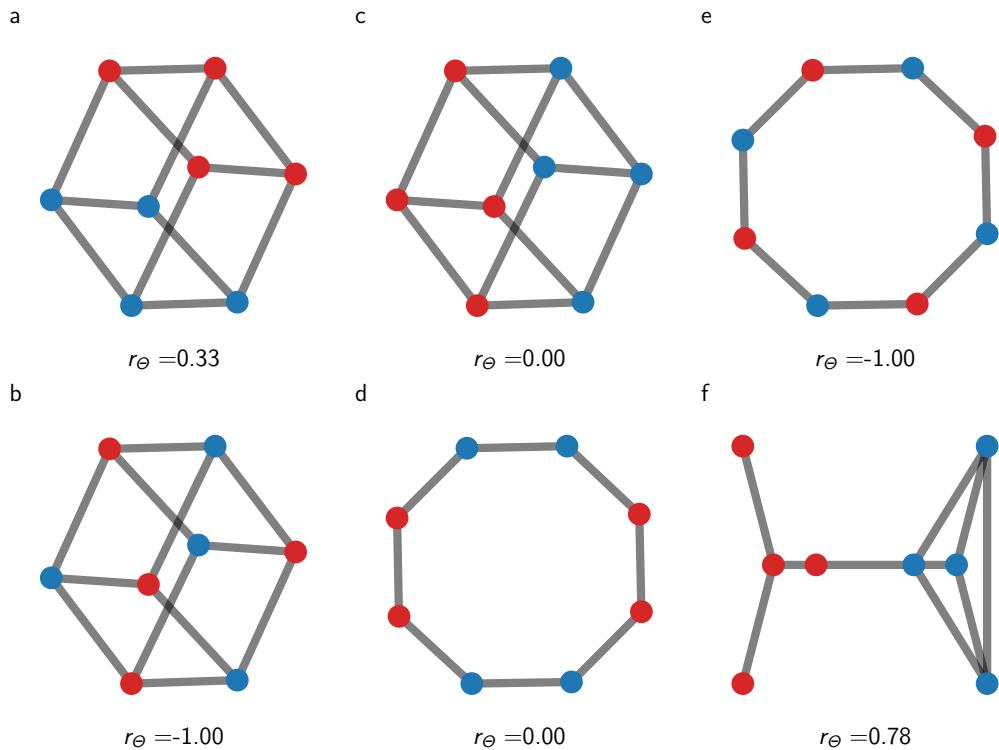


Fig. S2.10: Graphs with spatial distribution of habitat types corresponding to different habitat assortativity r_Θ . Graphs (ad) can be described exactly with a mean field approach, as blue and red vertices have an equivalent position on the graph.

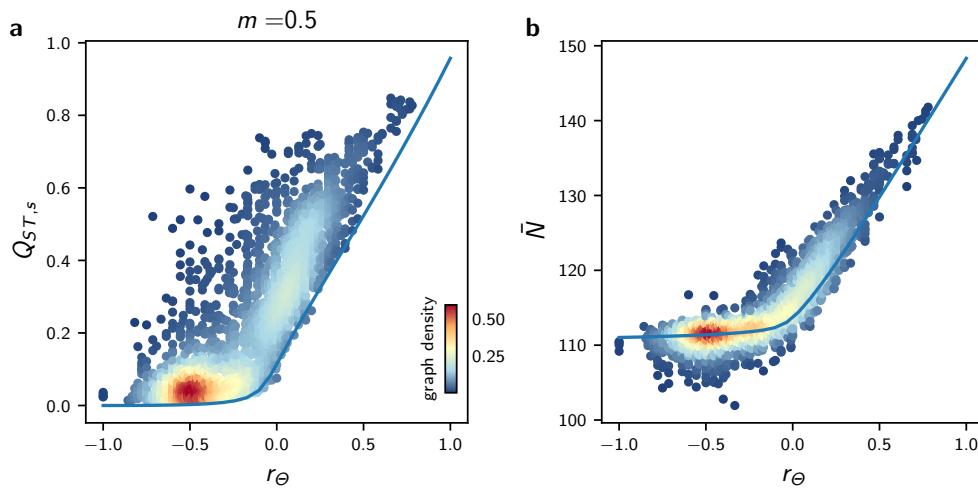


Fig. S2.11: Effects of habitat heterogeneity r_Θ on $Q_{ST,s}$ and average population size \bar{N} for all undirected connected graphs with $M = 7$ vertices and varying r_Θ , obtained for similar simulations to those in ?? with $m = 0.5$. In (a) and (b), each dot represents average results from 5 replicate simulations of the IBM, the colour scale corresponds to the proportion of the graph with similar x and y axis values (graph density), and the blue lines correspond to results obtained from the mean field, deterministic approximation ???. Deviations from the mean field, deterministic approximation ?? can be explained by differences in $\langle l \rangle$ and h_d between the graphs.

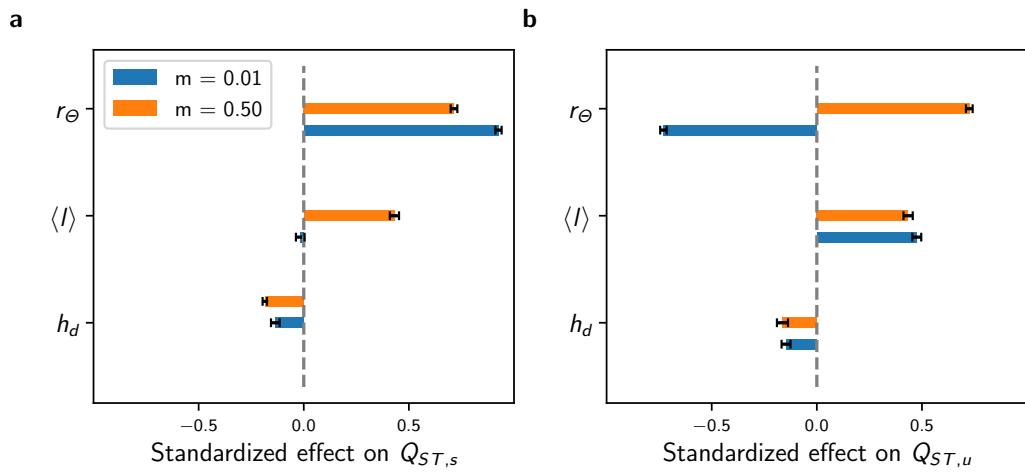


Fig. S2.12: Standardized effects of h_d , $\langle l \rangle$ and r_Θ on $Q_{ST,s}$ and $Q_{ST,u}$ obtained from multivariate regression models independently fitted for low and high migration regimes on average results from 5 replicate simulations of the IBM, analogous ??c-d but for 1126 of the 261,080 undirected connected graphs with $M = 9$ vertices and varying r_Θ (see ?? for details). Error bars show 95% confidence intervals.

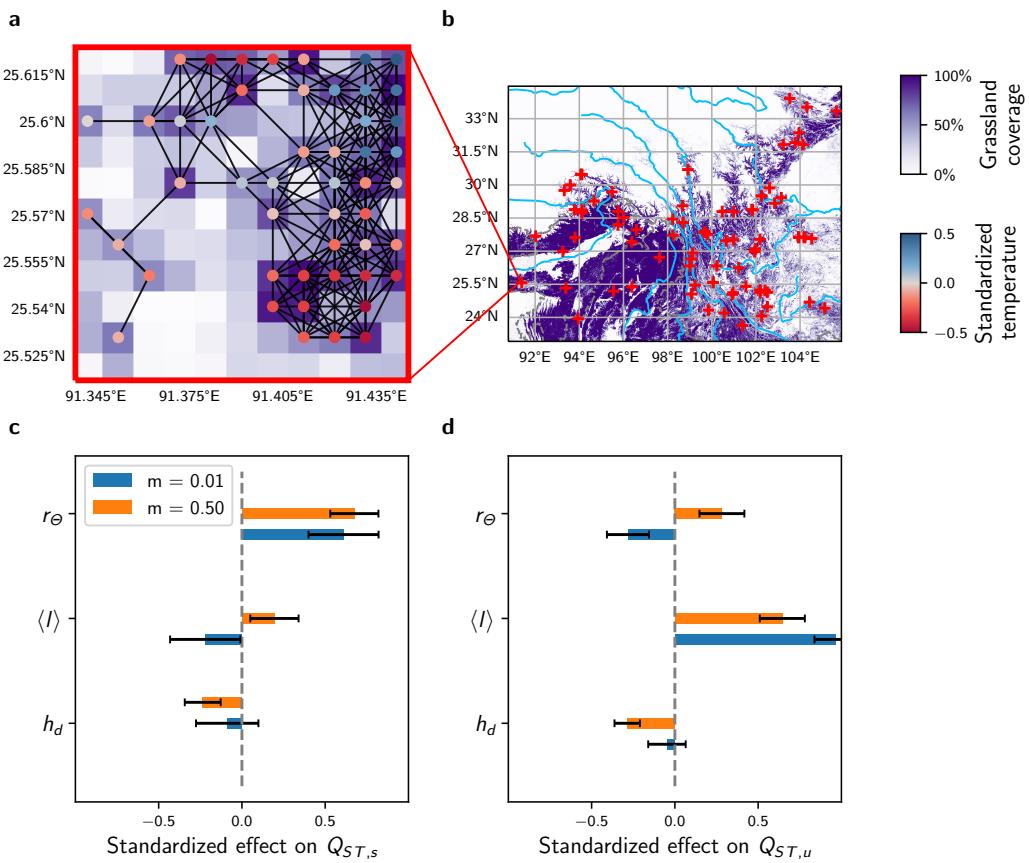


Fig. S2.13: Simulations on graphs with $M = 49$ vertices obtained from real spatial habitat datasets, in the setting with heterogeneous selection. The region from where graphs are obtained is centred on the Hengduan Mountains in Southwest China, one of the most species-rich temperate mountain biota globally (Ding2020a). (a) Graphical representation of a geographical area of size $0.11^\circ \times 0.11^\circ$. To create the graph, we considered biological populations living in grasslands, and used the dataset provided in (Jung2020) containing global grassland coverage at 0.01° resolution. We assigned a vertex to a geographical area of size $0.01^\circ \times 0.01^\circ$ if its grassland coverage was above a threshold arbitrarily set to 50%. We further assumed that two vertices were connected if their euclidean distance was below a certain dispersal range, which we let vary from 1 to 2.5 km. Local annual average temperature was considered as the value that captures environmental conditions at each vertex. Temperature data was obtained from the CHELSA dataset (Karger2017). (b) Grassland coverage for the considered region. Blue lines correspond to rivers and dashed grey lines correspond to country borders. Red crosses indicate the locations of the 83 graphs sampled for the simulations used in (c-d). (c-d) Standardized effects of h_d , $\langle l \rangle$ and r_Θ on $Q_{ST,s}$ and $Q_{ST,u}$ obtained from multivariate regression models independently fitted for low and high migration regimes to average results from 5 replicate simulations of the IBM on the 83 graphs which location is illustrated in (c) (see ?? for simulation details). Error bars show 95% confidence intervals.

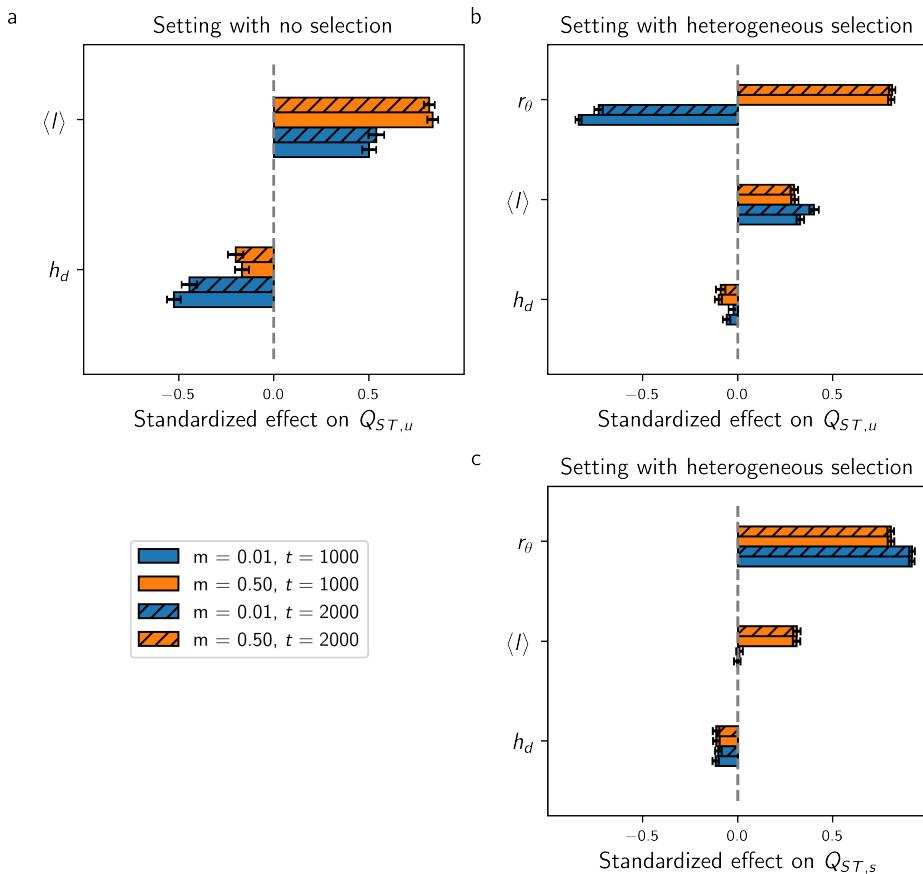


Fig. S2.14: Standardized effects of h_d , $\langle l \rangle$ and r_Θ on $Q_{ST,u}$ in the setting with no selection and in the setting with heterogeneous selection for the time horizons $t = 1000$ and $t = 2000$, obtained from multivariate regression models independently fitted for low and high migration regimes to average results from 5 replicate simulations of the IBM on all undirected connected graphs with $M = 7$ vertices and varying r_Θ (see ?? for details). (a–c) illustrate that the effects of the topology metrics on $Q_{ST,u}$ and $Q_{ST,s}$ remain constant for $t > 1000$ in both the settings without selection and with heterogeneous selection. Error bars show 95% confidence intervals.

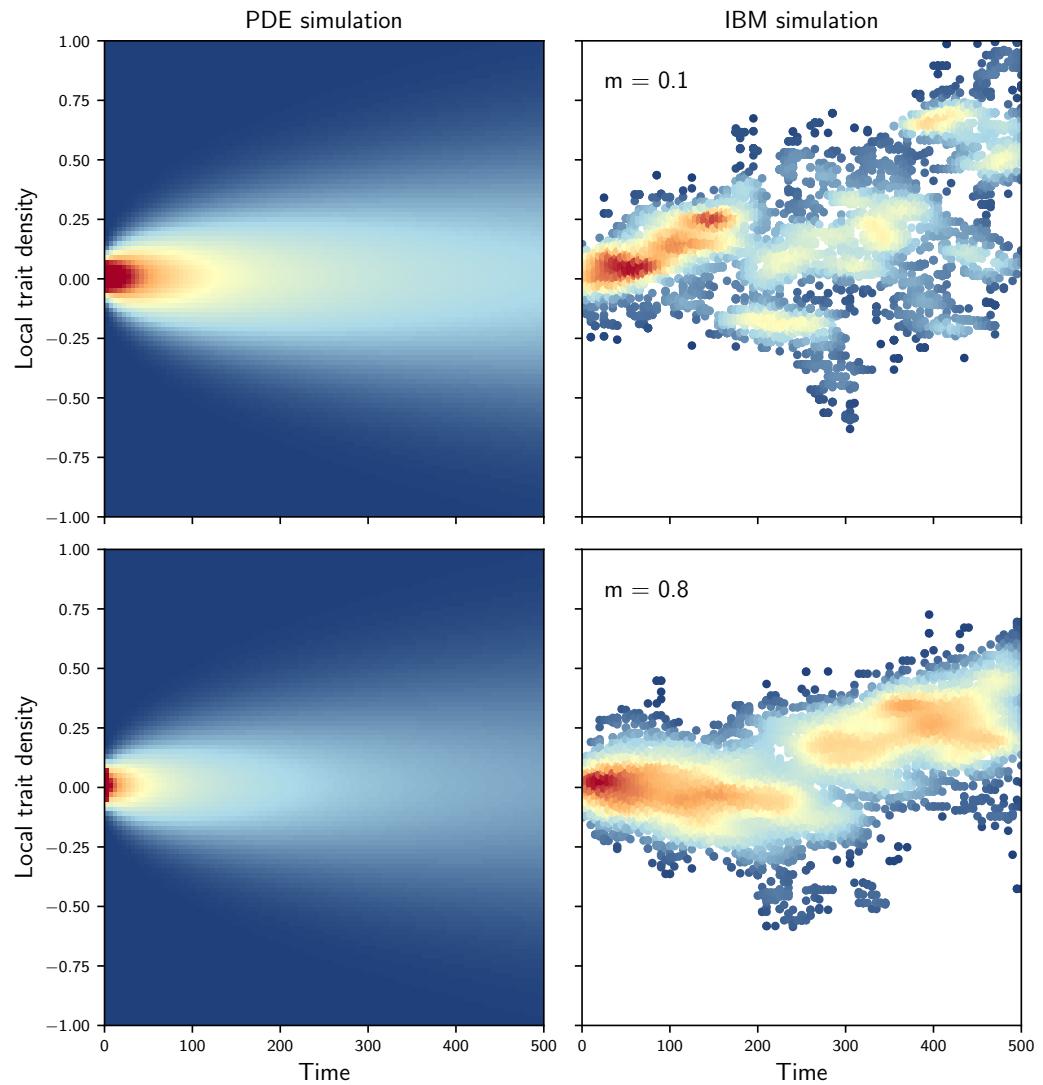


Fig. S2.15: Comparison of the neutral trait density on one vertex obtained from ?? (left) and from the IBM simulations (right) in the setting with no selection, for the chain graph. The densities obtained from ?? and from the IBM are dissimilar.

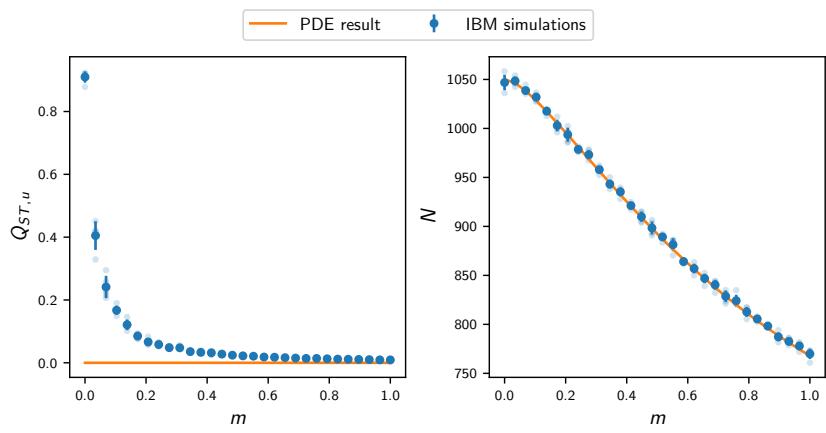


Fig. S2.16: Comparison of results obtained from the deterministic approximations $Q_{ST,u}$ and from IBM simulations in the setting with no selection, on the star graph with $M = 7$ vertices. While $Q_{ST,u}$ can capture population size, N is not able to capture $Q_{ST,u}$. Each plain dot represents average results from 5 replicate simulations, bars represent one standard deviation, and each fade dot represents a single replicate value.

2.C Supplementary Tables

	$Q_{ST,u}$				$Q_{ST,u} - bN$	
m	0.01	0.50	0.01	0.50	0.01	0.50
(Intercept)	0.000 (0.023)	-0.000 (0.017)	-0.000 (0.023)	-0.000 (0.025)	-0.000 (0.023)	-0.000 (0.028)
$\langle l \rangle$	0.739*** (0.023)	0.872*** (0.017)				
h_d			-0.753*** (0.023)	-0.674*** (0.025)	-0.753*** (0.023)	-0.143*** (0.028)
Number of sim.	853	853	853	853	853	853
R^2	0.546	0.760	0.567	0.454	0.567	0.030

Tab. S2.1: Linear regression model coefficients for the effect of topology metrics on $Q_{ST,u}$ in the setting with no selection, based on all graphs with $M = 7$ vertices. *** $P < 0.001$

	$M = 7$		$M = 9$	
	$Q_{ST,u}$			
m	0.01	0.50	0.01	0.50
(Intercept)	-0.000 (0.017)	-0.000 (0.013)	0.000 (0.009)	-0.000 (0.010)
h_d	-0.527*** (0.019)	-0.352*** (0.014)	-0.449*** (0.013)	-0.218*** (0.013)
$\langle l \rangle$	0.500*** (0.019)	0.712*** (0.014)	0.583*** (0.013)	0.784*** (0.013)
Number of sim.	853	853	1,126	1,126
R^2	0.766	0.858	0.899	0.896

Tab. S2.2: Multivariate linear regression model coefficients for the effect of topology metrics on $Q_{ST,u}$ in the setting with no selection. *** $P < 0.001$

	$M = 7$				$M = 9$			
	$Q_{ST,s}$		$Q_{ST,u}$		$Q_{ST,s}$		$Q_{ST,u}$	
m	0.01	0.50	0.01	0.50	0.01	0.50	0.01	0.50
(Intercept)	-0.000 (0.008)	-0.000 (0.009)	-0.000 (0.009)	-0.000 (0.009)	0.000 (0.008)	0.000 (0.008)	0.000 (0.008)	0.000 (0.008)
h_d	-0.117*** (0.009)	-0.114*** (0.010)	-0.060*** (0.010)	-0.102*** (0.010)	-0.135*** (0.010)	-0.185*** (0.011)	-0.146*** (0.011)	-0.164*** (0.011)
$\langle l \rangle$	-0.004 (0.009)	0.308*** (0.010)	0.328*** (0.010)	0.300*** (0.010)	-0.017 (0.010)	0.431*** (0.011)	0.475*** (0.011)	0.434*** (0.011)
r_Θ	0.914*** (0.008)	0.805*** (0.009)	-0.838*** (0.009)	0.807*** (0.009)	0.926*** (0.008)	0.715*** (0.008)	-0.730*** (0.008)	0.725*** (0.008)
Number of sim.	2,548	2,548	2,548	2,548	2,250	2,250	2,250	2,250
R^2	0.845	0.808	0.808	0.799	0.870	0.853	0.862	0.851

Tab. S2.3: Multivariate linear regression model coefficients for the effect of the topology metrics on $Q_{ST,u}$ and $Q_{ST,s}$ in the setting with heterogeneous selection.

*** $P < 0.001$

	$Q_{ST,s}$		$Q_{ST,u}$	
m	0.1	0.50	0.1	0.50
(Intercept)	-0.000 (0.093)	-0.000 (0.064)	0.000 (0.056)	-0.000 (0.059)
h_d	-0.088 (0.094)	-0.235*** (0.065)	-0.048 (0.057)	-0.286*** (0.060)
$\langle l \rangle$	-0.220* (0.106)	0.195** (0.073)	0.965*** (0.064)	0.645*** (0.068)
r_Θ	0.610*** (0.106)	0.675*** (0.073)	-0.282*** (0.063)	0.282*** (0.068)
Number of sim.	83	83	83	83
R^2	0.313	0.675	0.752	0.717

Tab. S2.4: Multivariate linear regression model coefficients for the effect of topology metrics on $Q_{ST,u}$ and $Q_{ST,s}$ on real graphs with $M = 49$ vertices in the setting with heterogeneous selection. * $P < 0.05$, ** $P < 0.01$, *** $P < 0.001$

	$\sigma_a = 0.5 < 1/\sqrt{2p}$				$\sigma_a = 1 > 1/\sqrt{2p}$			
	$Q_{ST,s}$		$Q_{ST,u}$		$Q_{ST,s}$		$Q_{ST,u}$	
m	0.05	0.50	0.05	0.50	0.05	0.50	0.05	0.50
(Intercept)	0.000 (0.005)	-0.000 (0.010)	-0.000 (0.011)	-0.000 (0.010)	0.000 (0.004)	-0.000 (0.008)	0.000 (0.012)	-0.000 (0.007)
h_d	-0.228*** (0.006)	-0.118*** (0.011)	-0.171*** (0.012)	-0.169*** (0.012)	-0.166*** (0.004)	-0.128*** (0.009)	-0.178*** (0.013)	-0.139*** (0.008)
$\langle l \rangle$	0.084*** (0.006)	0.373*** (0.011)	0.461*** (0.012)	0.573*** (0.012)	0.002 (0.004)	0.296*** (0.009)	0.483*** (0.013)	0.286*** (0.008)
r_Θ	0.922*** (0.005)	0.741*** (0.010)	-0.657*** (0.011)	0.508*** (0.010)	0.967*** (0.004)	0.816*** (0.008)	-0.585*** (0.012)	0.837*** (0.007)
Number of sim.	2,548	2,548	2,548	2,548	2,548	2,548	2,548	2,548
R^2	0.934	0.768	0.716	0.732	0.962	0.828	0.659	0.861

Tab. S2.5: Multivariate linear regression model coefficients for the effect of topology metrics on $Q_{ST,u}$ and $Q_{ST,s}$ in the setting of trait-dependent competition and heterogeneous selection (??), based on all graphs with $M = 7$ vertices.

*** $P < 0.001$

3

Mini-batching ecological data to improve ecosystem models with machine learning

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” *Les données pertinentes détiennent les réponses.*

— French anagram

Ecosystems are involved in global biogeochemical cycles that regulate climate and provide essential services to human societies. Mechanistic models are required to describe ecosystem dynamics and anticipate their response to anthropogenic pressure, but their adoption has been limited in practice because of issues with parameter identification and because of model inaccuracies. While observations could be used to directly estimate parameters and improve models, model nonlinearities as well as shallow, incomplete and noisy datasets complicate this process. Here, we propose a machine learning (ML) framework relying on a mini-batch method combined with automatic differentiation and state-of-the-art optimizers. By splitting the data into mini-batches with a short time horizon, we show both analytically and numerically that the mini-batch method regularizes the learning problem. When combined with the proposed numerical implementation, the resulting ML framework can efficiently learn the parameters of complex dynamical models and is a workhorse for model selection. We evaluate the performance of the ML framework in recovering the dynamics of a simulated food-web. We show that it can efficiently learn from noisy, incomplete and independent time series, accurately estimating the model parameters and providing reliable short-term forecasts. We further show that the ML framework can provide statistical support for the true generating model among several candidates. In summary, the proposed ML framework can efficiently learn from data and elucidate mechanistic pathways to improve our understanding and predictions of ecosystem dynamics.

3.1 Introduction

Ecosystems are complex systems involving many interacting functional entities which together play a major role in regulating global biogeochemical cycles (**Bonan2008**) and delivering essential services to humans (**Kremen2005**). Ecosystems currently face intense disruption from anthropogenic pressure, through pollution and land use (**Doney2010; Ellis2011**), and from climate change (**Midgley2019**). In order to anticipate the responses of ecosystems to these disruptions, models that can extrapolate ecological dynamics beyond observations are required (**Boyd2012**). A major challenge is that the processes driving ecological dynamics are nonlinear, resulting in complex responses and feedbacks (**Scheffer2001**). Nonlinearity greatly affects the capacity of modelling approaches that do not incorporate specific biological knowledge to reliably project current trends into the future (**Barnosky2012**). For instance, while methods based on statistical descriptions (**Deneu2021**) and nonparametric methods (**Ye2015; Ye2016; Deyle2016**) have adequate interpolation capabilities, they are ill-suited for extrapolating beyond observed trends (**Barnosky2012; Urban2016**). In contrast, mechanistic ecosystem models integrate constraints on the expected dynamics by explicitly modelling interactions, feedback loops and dependencies between ecosystem components (**Geary2020**). While this should ensure a more robust forecast under large disruptions (**Norberg2012**), ecosystems models suffer in practice from parametrization issues, i.e. inaccuracies in the mathematical formulation of the processes and issues in identifying the correct parameter values (**DeAngelis2015**). These drawbacks have limited their broad adoption (**Urban2016**). Learning the parametrization of ecosystem models from observation data by blending specific biological knowledge and ML methods could improve our representation of ecosystem processes and help us to anticipate ecosystem responses to global changes.

The parametrization of ecosystem models can be indirectly learnt from observations by calibrating the parameters from the data collectively using inference methods. These methods proceed by maximizing the posterior probability of the parameters given the observations, but their success is subject to a number of issues, some of which specifically relating to ecosystem model properties. Among these issues, the exploration of the posterior landscape demands repeated model simulations, but ecosystem models are usually associated with a high computational cost that limits the number of possible runs (**Fisher2018**). Additionally, the complexity of processes requires a large number of parameters. Due to the curse of dimensionality (**Boyd2012**), this complicates the exploration of the posterior distribution and can further leave many parameters poorly constrained (**Gutenkunst2007**). The limited availability of observations, which are usually composed of multiple partial short-term time series (**Scheiter2013; Schartau2017**), accentuates the lack of parameter constraints. Moreover, ecosystem dynamics can be strongly dependent on the initial conditions (ICs) and show a chaotic behaviour (**Hastings1993; Huisman1999; Beninca2008**), or can be associated with a large panel of dynamics depending on the parameter values. In such case, small perturbations of the ICs or parameter values lead to large divergences in the model outcomes, causing numerical problems in finding the most probable parameters (**DeAngelis2015**). Last but not least, in contrast to fields such as climate and weather modelling, the derivation of fundamental processes regulating ecosystems is far from being established (**Gehlen2015**;

Schartau2017; Purves2013), resulting in inaccurate mechanistic pathways and uncertain mathematical formulations (**Gentleman2003**), limiting extrapolation to unseen data. To summarize, the parametrization of ecosystem models requires inference methods that are robust despite the models' complexity, the limited observation data, and the inaccurate description of ecological processes.

A variety of data assimilation and ML methods are increasingly being used to parametrize ecosystem models. Bayesian inference with Markov Chain Monte Carlo methods, used in (**Lignell2013; Higgins2010; Xu2006; Fiechter2013; Rosenbaum2019**), offer the advantage of quantifying uncertainties by inferring the full posterior probability distribution of the unknown parameters. This is achieved by a global exploration of the parameter space, which makes Bayesian methods computationally expensive and particularly prone to the curse of dimensionality (**Gosh2021**). Simulated annealing (**Matear1995**), genetic algorithms (**Ward2010**), and sequential methods such as extended Kalman filtering and ensemble Kalman methods (**Bertino2003; Doron2013; Gharamti2017**) have been used as alternatives, but are similarly subject to the curse of dimensionality and demand a large number of model evaluations. Variational methods rely on the model adjoint, i.e. the model sensitivity to the parameters, to explore more efficiently the parameter space, iteratively updating the parameter estimates using the gradient of the posterior landscape. Such methods therefore demand less evaluation (**Schartau2017**), which explains their wide adoption in the field of artificial intelligence to train highly parametrized neural networks (up to the order of 10^8 parameters (**NIPS2017_3f5ee243**)) and their use in calibrating marine ecosystem models (**Fennel2001; Spitz1998; Xiao2014; Pelc2012**) (see (**Schartau2017**) for a review) and terrestrial ecosystem models (**Zhu2015; DeLong2014; Curtsdotter2019**). However, as the complex dynamics of ecosystem models tend to be associated with rugged posterior landscapes, variational methods are prone to converging to local minima, making variational methods very sensitive to the choice of initial model parameters (**Gabor2015; Schartau2017**). Ecosystem models are specified as differential equations that depend not only on parameters but also on ICs. The state-dependency of ecosystems means that neglecting the estimation of initial ICs might compromise the correct fitting of the parameters and the forecast skill (**Lignell2013**). However, few of the aforementioned studies have addressed the problem of IC estimation (but see (**Pelc2012**)). Finally, the numerical implementation of variational methods is also challenging, as the model adjoint is difficult to obtain and maintain as the model is modified (**Lawson1995; Pelc2012; Gharamti2017**). Novel methods for model parametrization are emerging, thanks to advances in the field of artificial intelligence (**Willard2020; Kashinath2021; Alber2019; Peng2021**), providing new opportunities to better address these issues.

Here, we propose a ML framework relying on a mini-batch method inspired by multiple shooting methods (**Pisarenko2004**) and on automatic differentiation and state-of-the-art optimizers to efficiently learn the parametrization of ecosystem models from observation data. The mini-batch method divides the training problem into mini-batches with a short time horizon. We show analytically how this learning strategy regularizes the ill-behaviour of the loss function arising from the strong nonlinearities of ecosystem models. We implement the mini-batch method in the software ecosystem SciML (**Rackauckas2020a**), which provides advanced optimizers and allows the automatic generation of efficient and accurate model

adjoints, leading to excellent performance. The resulting ML framework makes it possible to efficiently combine the information contained in short, independent time series, and is a workhorse for performing model selection and improving model accuracy. We evaluate the performance of the ML framework in recovering the chaotic dynamics of simulated food webs. We assume a perfect-model setting and test the capacity of the ML framework to recover the true parameters and provide forecasts based on noisy and incomplete observations, and we explore its efficiency in combining the information from multiple time series. Additionally, we investigate whether the ML framework can recover the most appropriate model structure among candidate models. By blending biological knowledge and ML methods, the proposed ML framework is interpretable and data-efficient, and it facilitates mechanism discovery. The proposed approach thus shows promise in improving our ability to understand and forecast ecosystem dynamics.

3.2 Machine learning framework for ecosystem models

3.2.1 Ecosystem model parametrization as a learning problem

Ecosystem models

Ecosystem models generally consist of a system of ordinary differential equations (ODEs) of the form

$$\begin{aligned}\dot{x}(t) &= f(t, x(t), p) \\ x(0) &= x_0 \\ y(t) &= h(x(t)) + \epsilon(t)\end{aligned}\tag{3.1}$$

where $x(t) \in \mathbb{R}^m$ is a vector of state variables that might represent species abundance, resources availability or functional group biomass, $y(t) \in \mathbb{R}^d$ is a vector of observables that contains a subset or aggregates of the state variables, and $p \in \mathbb{R}^q$ is the model parameter vector. h is a function that maps the state variables to the observables, and we assume that the observables are contaminated with a white noise ϵ of Gaussian type, with zero mean and variance–covariance matrix Σ_y . Denoting by $\theta = (x_0, p)$ the vector containing the ICs and the parameters, the model may be viewed as a map \mathcal{M} parametrized by time t that takes the parameters θ to the state variables x

$$\begin{aligned}\mathcal{M}(t, \theta) &= x(t) \\ &= \int_0^t f(s, x(s), p) ds + x_0\end{aligned}\tag{3.2}$$

Inverse modelling

Taking expectations over the noise realizations yields $\mathbb{E}[y(t)] = h(\mathcal{M}(t, \theta))$, and it follows that the conditional likelihood of each observation $y_k \equiv y(t_k)$, given the parameters θ and the model \mathcal{M} denoted by $p(y_k | \theta, \mathcal{M})$, follows the distribution of the residuals $\epsilon_k \equiv \epsilon(t_k) = y(t_k) - h(\mathcal{M}(t_k, \theta))$, which corresponds to the multivariate normal distribution

$\mathcal{N}_{0, \Sigma_y}$. Following a Bayesian approach, the parametrization of the ecosystem model can be performed on the basis of the parameter and model posterior probability $p(\theta, \mathcal{M}|\mathbf{y}_{1:K})$, i.e. the conditional probability density of the parameter values θ and the model \mathcal{M} given the data, given by

$$p(\theta, \mathcal{M}|\mathbf{y}_{1:K}) \propto p(\mathbf{y}_{1:K}|\theta, \mathcal{M})p(\theta, \mathcal{M}) \quad (3.3)$$

where $\mathbf{y}_{1:K} = (y_1, \dots, y_K)$, $p(\mathbf{y}_{1:K}|\theta, \mathcal{M})$ is the product of the conditional likelihood of each observation y_k

$$\begin{aligned} p(\mathbf{y}_{1:K}|\theta, \mathcal{M}) &= \prod_{i=1}^K p(y_i|\theta, \mathcal{M}) \\ &= \prod_{k=1}^K \frac{1}{\sqrt{(2\pi)^d |\Sigma_y|}} \exp\left(-\frac{1}{2} \epsilon_k^T \Sigma_y^{-1} \epsilon_k\right) \end{aligned} \quad (3.4)$$

and $p(\theta, \mathcal{M})$ is the prior distribution of the model and its associated parameter values. The model \mathcal{M} is included in the probabilistic quantities in order to accommodate multiple candidate models (see ??).

A variational method to obtain a Bayesian estimate of θ involves maximizing $p(\theta, \mathcal{M}|\mathbf{y}_{1:k})$ to obtain the maximum a posteriori (MAP) estimator (**Bocquet2019**), which is equivalent to a maximum likelihood approach under a uniform prior distribution of the parameters, i.e. when no prior information on the parameter values is used (**Schartau2017**). Observing that maximizing $p(\theta, \mathcal{M}|\mathbf{y}_{1:K})$ is equivalent to minimizing $-\log p(\theta|\mathbf{y}_{1:K}, \mathcal{M})$ and assuming a normal prior distribution of the parameters $\mathcal{N}_{p_b, \Sigma_p}$, one can obtain the MAP $\hat{\theta}$

$$\hat{\theta} = \arg \min_{\theta} L_{\mathcal{M}}(\theta) \quad (3.5)$$

where

$$L_{\mathcal{M}}(\theta) = \frac{1}{2} \left[\sum_{k=1}^{K-1} \|y_k - h(\mathcal{M}(t_k, \theta))\|_{\Sigma_y}^2 + \|p - p_b\|_{\Sigma_p}^2 \right] \quad (3.6)$$

(**Schneider2017; Raue2009**) and where we use the notation $\|y\|_{\Sigma}^2 = y \Sigma^{-1} y^T$. ?? is similar to a traditional least squares function commonly used in regression, where the second summand is the analogue of a regularization term for the weights and biases of e.g. a neural network.

Gradient-based optimizers can then be used to efficiently obtain $\hat{\theta}$ in ??, iteratively updating the parameter vector θ_m given the gradient of the loss function, denoted by $\nabla_{\theta} L_{\mathcal{M}}$, to navigate the surface defined by $L_{\mathcal{M}}$ with the aim to find the global minimum where $\nabla_{\theta} L_{\mathcal{M}}(\hat{\theta}) = 0$. As an example, the plain vanilla gradient descent algorithm is given by

$$\theta_{m+1} = \theta_m - \gamma \nabla_{\theta} L_{\mathcal{M}}(\theta_m) \quad (3.7)$$

where γ is the learning rate. Other gradient-based algorithms, such as the ADAM optimizer used in the section below, employ more advanced updating strategies to avoid convergence to local minima but stay in the spirit of ??.

Information indigestion

A naive minimization of $L_{\mathcal{M}}(\theta)$ with gradient-based methods is likely to fail, as its associated surface cannot be navigated properly. As illustrated in ??A, the loss surface associated with models characterized by complex dynamics consists of multiple local minima, which cause problems of convergence in efforts to reach the global minimum (depicted by the orange curve in ??B). Furthermore, in a neighbourhood of the global minimum, the gradient of the loss function is very large (a "ravine" with almost vertical walls), leading the optimizer to overshoot the true parameter values (depicted by the green curve in ??B). In ?? we show in a general setting that these problematic features arise from the dynamical properties of ecosystem models: when the dynamics are chaotic or exhibit a limit cycle (as is often the case for ecological dynamics (Bjornstad2001; Hastings1993; Huisman1999; Beninca2008)), the dynamical trajectories exhibit high sensitivity to the model parameters and ICs. This means that a small modification of the parameters or ICs leads to large divergences over time. The prevalence of large deviations causes discontinuities, appearing as many sub-optimal local minima on the loss surface. Moreover, the true minimum can only be found in a narrow ravine that becomes narrower as the number of data points increases. Such surfaces are hardly navigable with gradient descent methods, but since the behaviour of $L_{\mathcal{M}}(\theta)$ critically depends on the time horizon, we reformulate ?? in the following section by splitting the time series into mini-batches with a short time horizon.

3.2.2 ML framework for ecosystem models

Description of the mini-batch method

We propose a mini-batch method that splits the data into mini-batches with a short time horizon. Under perturbed parameters and ICs, chaotic or limit cycle dynamics only diverge after some characteristic simulation time; by splitting the time series into small mini-batches, discontinuities that cause the poor navigability of the landscape are therefore avoided. By averaging the associated losses during the training, the mini-batch method regularizes the loss function and makes it possible to combine the information contained in independent time series.

Consider the availability of S independent time series, where each time series $s \in \{1, \dots, S\}$ contains $K^{(s)}$ observations $\mathbf{y}_{1:K^{(s)}}$. To improve the ill-behaviour of ??, we split each time series into $M^{(s)}$ mini-batches, each of which defines a loss denoted by $L_{\mathcal{M}}^{(s,m)}$. Averaging the losses $L_{\mathcal{M}}^{(s,m)}$ leads to a reformulation of the loss function in ??, yielding

$$\begin{aligned} L_{\mathcal{M}}^*(\theta) &= \frac{1}{S} \sum_{s=1}^S \frac{1}{M^{(s)}} \sum_{m=0}^{M^{(s)}-1} L_{\mathcal{M}}^{(s,m)}(\theta) \\ &= \frac{1}{S} \sum_{s=1}^S \frac{1}{M^{(s)}} \sum_{m=0}^{M^{(s)}-1} \left[\frac{1}{K^{(s,m)}} \sum_{k=1}^{K^{(s,m)}} \|y_{k+mK^{(s)}/M^{(s)}}^{(s)} - h\left(\mathcal{M}(t_{k+mK^{(s)}/M^{(s)}}, p, x_0^{(s,m)})\right)\|_{\Sigma_y} \right. \\ &\quad \left. + \|y_{mK^{(s)}/M^{(s)}}^{(s)} - h(x_0^{(s,m)})\|_{\Sigma_{x_0}} + \|p_b - p\|_{\Sigma_p} \right] \end{aligned} \tag{3.8}$$

where $\theta = (p, x_0^{(1,1)}, \dots, x_0^{(1,M_1)}, x_0^{(2,1)}, \dots, x_0^{(S,M_S)})$ consists of the augmented parameter vector containing $\sum_{s=1}^S [M^{(s)} - 1]$ additional ICs to be estimated and denoted by $x_0^{(s,m)}$,

$$K^{(s,m)} = \begin{cases} K^{(s)}/M^{(s)} + 1 & : m < M \\ K^{(s)}/M^{(s)} & : m = M \end{cases}$$

indicates the number of points to include in the m th batch, and p_b designates the prior knowledge of the parameter values. We refer to ??C for a graphical representation of ???. In contrast to ??, the ICs for each batch are constrained by an extra term with weight $\Sigma_{x_0}^{-1}$, which is needed in practice for better convergence. The ICs $x_0^{(s,m)}$ are nuisance parameters that augment the dimensionality of the parameter space as they are inferred for each batch. Nonetheless, the efficiency of gradient-based methods, together with the suggested numerical implementation detailed in the section below, largely circumvent the additional cost associated with the augmented dimensionality. $K^{(s,m)}$ expresses the overlap between each batch of data for $m < M^{(s)}$ and ensures that all the available information is utilized to constrain the parameter vector p . By smoothing the ruggedness of the loss surface (orange curve in ??A), the proposed mini-batch method yields an improved navigability (orange curve in ??B). We show analytically in ?? that using $L_{\mathcal{M}}^*$ yields a more navigable loss surface than if $L_{\mathcal{M}}(\theta)$ is used in ??.

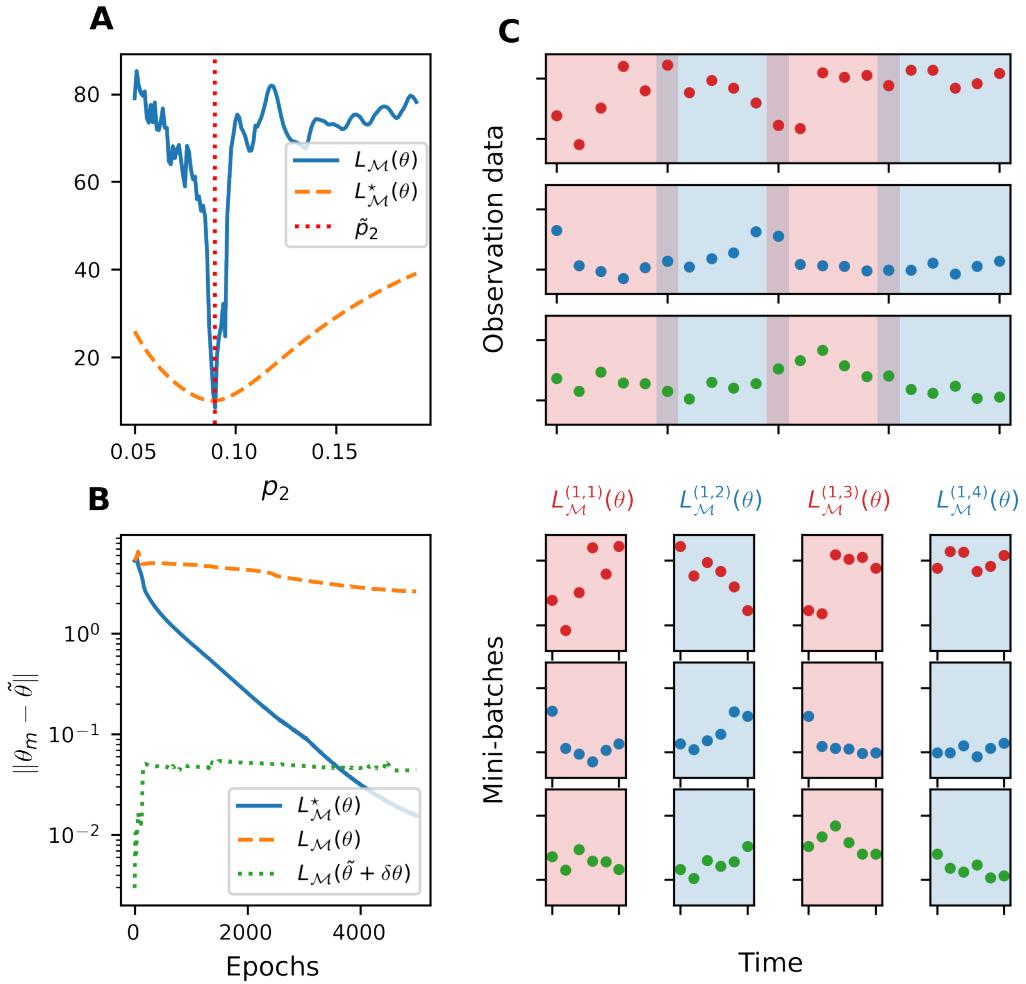


Fig. 3.1: Illustration of the mini-batch method. **A** Characteristic features of the naive loss function $L_{\mathcal{M}}(\theta)$ and the mini-batch loss function $L_{\mathcal{M}}^*(\theta)$. The blue and orange lines correspond to cross sections of $L_{\mathcal{M}}(\theta)$ and $L_{\mathcal{M}}^*(\theta)$, respectively, obtained from the ecosystem model presented in the section ???. While the cross section of $L_{\mathcal{M}}(\theta)$ presents many local minima and a very large gradient in the neighbourhood of the true parameter, which renders the navigability of the loss surface difficult, the cross section of $L_{\mathcal{M}}^*(\theta)$ is smooth and shows a single minimum, illustrating the regularization induced by the mini-batch method. **B** Convergence of gradient descent algorithms applied to $L_{\mathcal{M}}$ and $L_{\mathcal{M}}^*$. The blue, orange and green lines correspond to the loss function evaluated against the epochs (number of parameter updates) using $L_{\mathcal{M}}$, $L_{\mathcal{M}}$ starting from a parameter value close to the true parameters, and $L_{\mathcal{M}}^*$, respectively. The ill-behaviour of $L_{\mathcal{M}}$ leads to the convergence to a local minimum, while $L_{\mathcal{M}}^*$ is associated with a smooth convergence to the true parameters. **C** Graphical representation of the proposed mini-batch method. To improve the navigability of the posterior landscape, the algorithm splits the time series into mini-batches with short time horizons (blue and red portions of the time series). Since mini-batches are treated independently, the method naturally extends to independent time series.

Numerical implementation of the ML framework

The choice of the optimizer and the correct calculation of the model sensitivity to the parameters and ICs, upon which the gradient of the loss function $\nabla L_{\mathcal{M}}^*(\theta)$ depends, play an essential role in the success of the minimization of the loss function and the subsequent correct estimation of the MAP. To accelerate the learning process and make it more robust, we propose to combine the mini-batch method with modern optimizers and automatic differentiation. Building upon the software ecosystem SciML (Rackauckas2020a), we use **DifferentialEquations.jl** for the forward integration of the ecosystem model, as it provides highly efficient ODE solvers (Rackauckas2017). **DifferentialEquations.jl** is additionally compatible with automatic differentiation and includes an extensive set of sensitivity analysis methods (Ma2021), enabling the automatic generation of the model sensitivity to the parameters and ICs and guaranteeing their accuracy. This automatic generation greatly reduces the effort and potential errors associated with the adjoint code construction, enabling continuous development of the models. The accuracy is also an essential feature, as the model sensitivities are critically involved in the minimization of ?? and their inaccuracies can compromise the convergence of the gradient-based optimizers (Gholaminejad2019). The interoperability of the SciML ecosystem further makes it possible to benefit from the tooling of the deep learning library **Flux.jl** and the nonlinear optimization library **Optim.jl** (KMogensen2018), providing state-of-the-art optimizers that are computationally efficient and well suited for highly parameterized models (Ruder2016). We use the adaptive, momentum-based Adam optimizer (Kingma2014) to converge in the basin of attraction of the true parameters, which we substitute with the limited memory BroydenFletcherGoldfarbShanno optimizer (L-BFGS) (Liu1989) for the final training epochs to ensure faster and more accurate convergence.

The reformulation of the learning problem in ??, together with the numerical implementation suggested above, define the proposed ML framework, which we benchmark with a concrete case scenario in the next section.

3.3 Simulated food-web model as a case study

We evaluate the performance of the ML framework by considering a food-web ecosystem composed of three functional compartments including a resource, consumers and predators. We use a reference model to generate the observation data and first assume a perfect-model setting, evaluating the performance of the ML framework in parametrizing the reference model for different noise levels, with incomplete observations, and with an increasing number of independent time series. Second, we relax the perfect-model assumption by considering two plausible candidate models capturing contrasting hypotheses regarding the ecological processes, and test whether the ML framework can provide support for the true generating model by combining it with information-based model selection.

3.3.1 Three-compartment food-web ecosystem

We use a reference food-web model investigated in (**Hastings1991**; **McCann1994a**; **McCann1994**; **Klebanoff1994**) where a resource R is eaten by consumers C , which in turn are fed upon by predators P (model \mathcal{M}_1 in ??). We further consider an "omnivory variant" of the reference model introduced in (**McCann1997**), where predators are omnivorous and can feed upon the resource with a determined strength ω (model \mathcal{M}_2 in ??). These models generate fluctuations that resemble the behaviour of observed ecological time series (**Bjornstad2001**), they produce chaotic dynamics that are notoriously challenging to forecast for a wide range of realistic parameters (**Post2000**), and they have been used as benchmarks for proposed ecosystem forecasting methods (see (**Perretti2013**; **Deyle2016**; **Ye2016**)).

After nondimensionalization, the three-compartment model and the omnivory variant comprise a total of six and nine parameters, respectively: the mass-specific metabolic rate of consumers and predators x_C and x_P , the ingestion rate per unit metabolic rate of consumers and predators y_C and y_P (decomposed into y_{PC} , y_{PR} for the omnivory variant), the half saturation densities for the type II functional responses of the consumers and predators R_0 (decomposed into R_0 , R_{02} for the omnivory variant) and C_0 , and the omnivory strength ω for the omnivory variant (see ?? for the ODE details). Time is nondimensionalized by the resource growth rate and set to the biologically realistic value of 100% biomass increase per day, so that one unit of time corresponds to one day. The parameters in the simulations are set to the biologically realistic values proposed by (**McCann1994**; **McCann1997**), which additionally ensure that the dynamics of the system are chaotic or show oscillations (see ?? for details).

We generate the observation data by sampling the simulated ecosystem dynamics and by contaminating the samples with noise. The noise variance–covariance matrix Σ_y is set to be diagonal, with entries that are proportional to the sample variances of the observables

$$\text{diag } \Sigma_y = r^2 [\text{Var}(\tilde{y}_1), \dots, \text{Var}(\tilde{y}_d)] \quad (3.9)$$

where r indicates the noise level and \tilde{y} corresponds to the noiseless data generated with the true parameter values $\tilde{\theta}$. We sample the simulated dynamics after a long burn-in time ($t > 500$) to ensure that transient dynamics are not observed. A visual representation of the generated data is displayed in ??B. We assume a uniform distribution of the parameter priors, and randomly draw initial parameter estimates from a uniform distribution so that the initial parameter estimates follow $\mathcal{U}(0, 2\tilde{\theta})$.

We consider two different settings: one where all compartment abundances are observable, i.e. the observing system map h is the identity, and one where only predator and consumer abundances are available, i.e. discarding the resource abundance data. For both settings, structural identifiability is tested with the Julia library **StructuralIdentifiability.jl** (**Dong2021**) and verified globally. This means that in theory, the unique observation of predator and consumer abundances carries the information required for a complete characterization of all the model parameters.

For the meta-parameters of the ML framework, we set $\Sigma_{x_0} = \frac{M^{(1)}}{K^{(1)}} \Sigma_y$, we use Adam with $\gamma_m = \mathbb{1}_{[0,2000]}(m)10^{-1} + \mathbb{1}_{[0,2000]}(m)10^{-2} + \mathbb{1}_{[0,2000]}(m)10^{-3}$, $\beta_1 = 0.9$ and $\beta_2 = 0.999$ for

the first 6000 epochs, where γ_m corresponds to the learning rate of the m th epoch, and we use L-BFGS for the last 200 epochs.

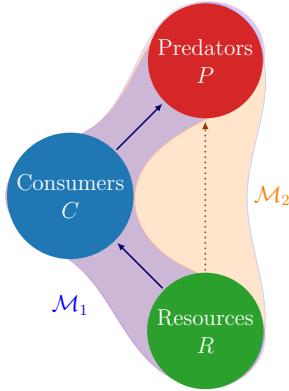


Fig. 3.2: Reference food web systems considered. In ??, the blue model \mathcal{M}_1 from (Hastings1991) is considered, where a resource is eaten by consumers, which are themselves eaten by predators. In ??, the orange model \mathcal{M}_2 , corresponding to the omnivory variant introduced in (McCann1997), is also considered.

3.3.2 Parameter learning in a perfect-model setting

We generate time series from the reference food-web model (blue model in ??) under varying noise levels and for both settings with total and partial observations, sampling from the model simulations every four days (see ??C for an illustration of the generated data). We then apply the ML method to the generated data, with a focus on evaluating its performance in recovering the true parameters \tilde{p} and on its forecast skill.

We evaluate the performance in recovering the true parameters with two metrics, namely the coefficient of determination between the true parameters \tilde{x}_P and the estimated parameters \hat{x}_P , denoted by $R_{x_P}^2$, and the relative parameter error for the ensemble of training simulations, denoted by $|\tilde{p} - \hat{p}|/\tilde{p}$ and calculated as the median relative parameter error across the six estimated parameters. To evaluate the out-of-sample forecast skill, we simulate the model beyond the training time span by using the estimated ICs of the last batch of each independent time series. Further, we quantify the forecast skill, denoted by ρ^2 , by computing the mean squared correlation across all the independent time series between the prey abundance generated with the true parameter $\tilde{\theta}$ and the predicted prey abundance. We obtain summary statistics of the metrics by varying the critical parameter value x_P , generating a total of 50 simulations for each noise level and setting considered. While only x_P is varied, all the parameters together with the ICs are collectively fitted.

Robustness of the ML framework against noise and incomplete observations

We set the number of time series to $S = 1$, the time series length to $K = 80$, and the number of mini-batches to $M = 8$. We investigate the ML framework performance against observational noise and under the setting with complete or partial observations.

In the complete observation setting, the ML framework can very accurately recover the true parameter values under a moderate observational noise, with mean $|(\hat{p} - \tilde{p})/\tilde{p}| = 6\%$ and $R_{xp}^2 = 0.99$ for 20% observational noise ($r = 0.2$; see ??A, red dots). In the partial observation setting, fair results are also obtained with $|(\hat{p} - \tilde{p})/\tilde{p}| = 16\%$ and $R_{xp}^2 = 0.89$ (??A, blue triangles). On top of being accurate, the ML framework further shows a very short inference time, i.e. 36 seconds in the complete observation setting and 34 seconds in the partial observation setting (see ?? for details). To investigate systematically how the ML framework accommodates different levels of observational noise, we vary the noise level from $r = 0.$ to $r = 1$ and calculate $|(\hat{p} - \tilde{p})/\tilde{p}|$. Results reported in ??B show that the ML framework handles observational noise well and the response of the logarithm of $|(\hat{p} - \tilde{p})/\tilde{p}|$ to r only differs by a constant factor between the complete and partial observation settings.

Given that the food-web dynamics are chaotic (??), excellent performance in parameter estimation might not be sufficient to provide accurate forecasts. We therefore also test the ML algorithm by evaluating how the forecast skill ρ^2 is affected by the noise level under complete or partial observations. Results for the complete and partial observation settings reported in ??C show good forecast skill under moderate observational noise, where ρ^2 linearly decreases with the time horizon considered.

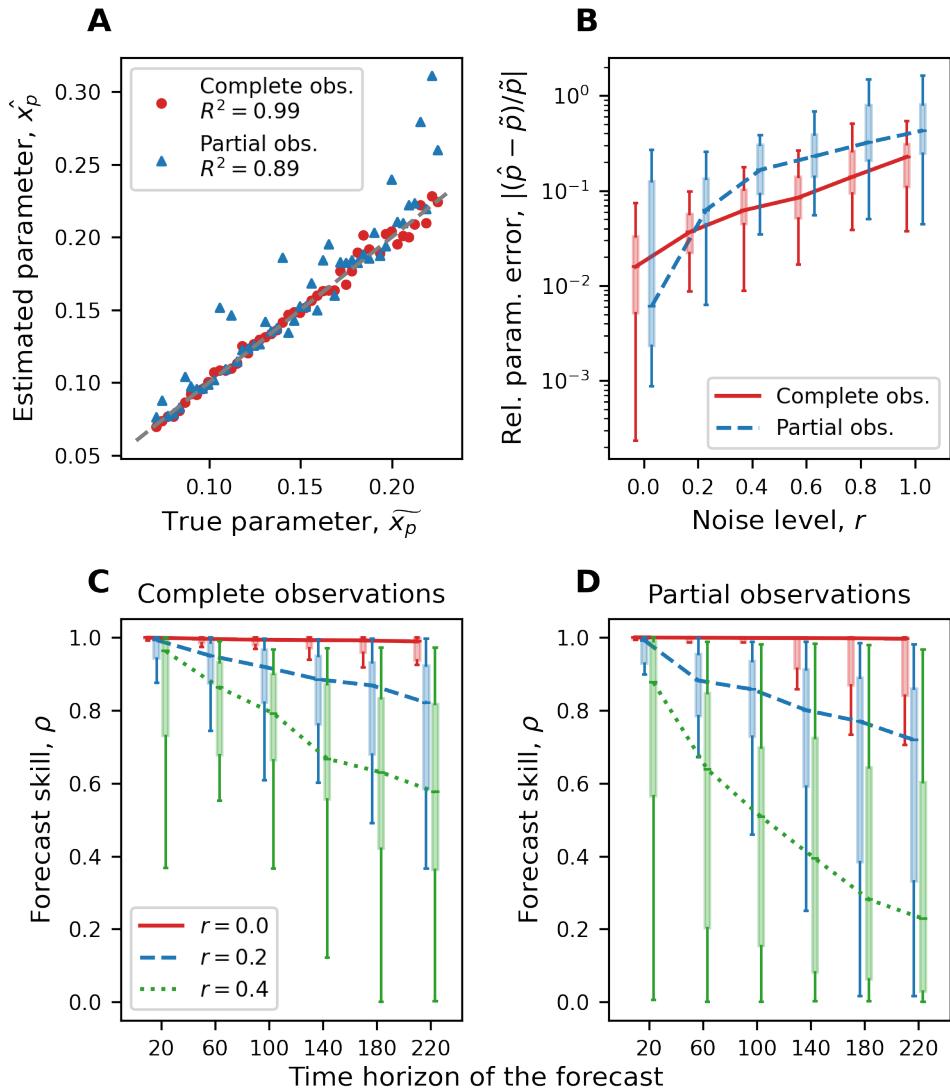


Fig. 3.3: Performance of the proposed ML framework for varying noise levels, under the complete and the partial observation setting. A True parameter \tilde{x}_P against estimated parameter \hat{x}_P for $r = 0.2$ under the complete and the partial observation setting. Although the parameter estimation is more accurate when complete observations of abundance are available, parameters show a fair fit when estimated with partial observations. B Relative parameter error $|(\hat{p} - \bar{p})/\bar{p}|$ for varying noise levels under the setting of complete or partial observations. B supports the above observation for varying noise levels. C Forecast skill ρ^2 of the trained model under the complete observation setting. D Analogous data under the partial observation setting. In A–D, the batch size is set to $m = 6$.

Capability of the ML framework to harness multiple time series

We further investigate the ability of the ML framework to process and combine information from independent datasets. We reduce the time horizon of the observation data by setting the time series length to $K = 12$, generate two datasets comprising $S = 1$ and $S = 6$ independent time series, respectively, – obtained from independent ICs – and set the number of batches for each time series s to $M^{(s)} = 2$. In both the complete and partial observation settings, we find that the relative parameter error $|(\hat{p} - \tilde{p})/\tilde{p}|$ is consistently lower in the simulations with a larger number of time series (??A). The forecast skill is also consistently improved as more independent time series are processed (??B), and the forecast skill for long-term predictions considerably increases. These results confirm the robustness of the ML framework against noise and partial observations, and show that the ML framework can efficiently harness the information from disparate observation datasets.

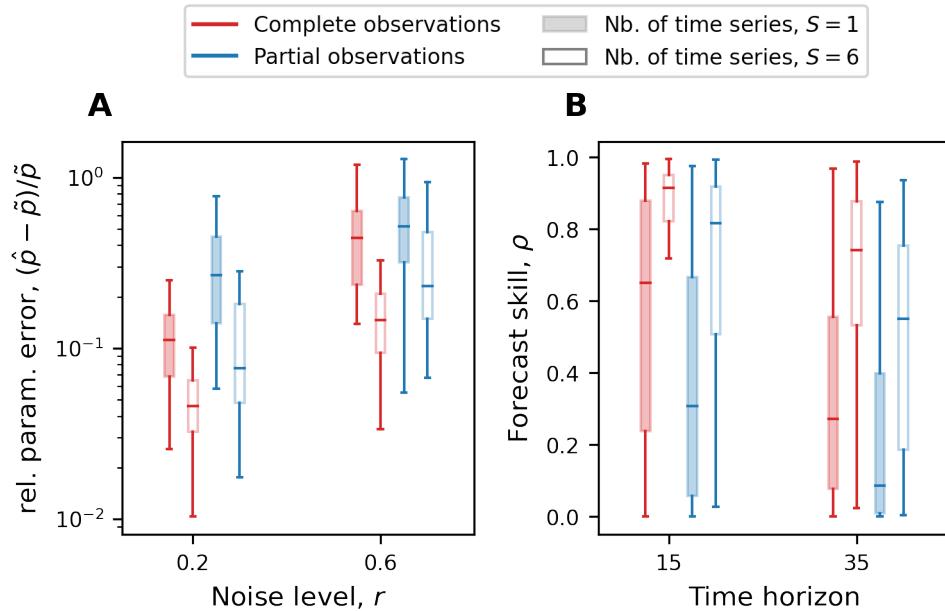


Fig. 3.4: Performance of the ML framework in processing and combining the information of multiple independent data sets. **A** Relative parameter error $|(\hat{p} - \tilde{p})/\tilde{p}|$ for different numbers of time series and levels of noise, under the complete and the partial observation setting. **B** Forecast skill for different numbers of time series and time horizons of the forecasts, under the complete and the partial observation setting. $r = 0.1$. In **A–B**, $|(\hat{p} - \tilde{p})/\tilde{p}|$ decreases while ρ^2 increases as the number of time series processed increases, demonstrating the capacity of the ML framework to process and combine the information from independent time series. In **A–B**, each box plot corresponds to 100 independent simulations where \tilde{x}_P varies $\tilde{x}_P \in [0.071, 0.225]$, and the batch size is set to $m = 6$.

3.3.3 Elucidating mechanistic pathways

Finally, we relax the perfect-model assumption and investigate whether the ML framework can provide statistical support for the true generating model among several candidates with information-based model selection. Specifically, we investigate whether the ML framework can detect omnivory from single observations of time series. We generate multiple observation datasets from the omnivory variant model \mathcal{M}_2 for different omnivory strengths ω and noise levels r . We consider both the standard model \mathcal{M}_1 and the omnivory variant model \mathcal{M}_2 as two plausible candidate models (see ?? for a graphical illustration of the models). We use the Akaike information criterion (AIC) to select the model with the strongest support in relation to the data (Mangan2017). In the specific case of our framework, we calculate the AIC as

$$\text{AIC}_{\mathcal{M}_i} = -2 \ln(p(\hat{\theta}, M_i | \mathbf{y}_{1:K})) + 2k_{\mathcal{M}_i} \quad (3.10)$$

where $p(\hat{\theta}, \mathcal{M}_i | \mathbf{y}_{1:K})$ corresponds to the maximum value of the likelihood of the model \mathcal{M}_i given the data, and $k_{\mathcal{M}_i}$ is the number of parameters in the model \mathcal{M}_i . The AIC ranks the most probable models by penalizing complexity to balance information loss and parsimony, where candidate models with the lowest scores are ranked as the most likely. We consider the Akaike weights $w_{\mathcal{M}_i} = \frac{\exp(-\Delta\text{AIC}_{\mathcal{M}_i}/2)}{\sum_j \exp(-\Delta\text{AIC}_{\mathcal{M}_j}/2)}$, where $\Delta\text{AIC}_{\mathcal{M}_i} = \text{AIC}_{\mathcal{M}_i} - \min_j \text{AIC}_{\mathcal{M}_j}$, which can be directly interpreted as the probability that \mathcal{M}_i is the most appropriate model given the data (see (Burnham2002)). We expect that the Akaike weights provide support for the generating model \mathcal{M}_2 only across values where $\omega > 0$, as \mathcal{M}_1 is equivalent to \mathcal{M}_2 when $\omega = 0$ and \mathcal{M}_2 is penalized by its three additional parameters.

In the complete observation setting, for moderate observational noise ($r = 0.1$) we find that \mathcal{M}_1 is given strong support for $\omega < 0.07$ ($w_{\mathcal{M}_1} > 98\%$) and that \mathcal{M}_2 is favored for $\omega > 0.08$ ($w_{\mathcal{M}_2} > 99\%$), providing overall strong support for the true model over a large range of ω values (??A). As the observational noise increases the support strength naturally decreases, leading to an increased range of ω values where the simplest model \mathcal{M}_1 is favored or where no model is given strong support (??B-C). On the other hand, in the partial observation setting, the lack of data prevents the correct estimation of the omnivory variant model parameters, leading model \mathcal{M}_1 to be supported for an even larger range of ω values (??).

Overall, the ML framework provides statistical support for the model embedding the most appropriate hypotheses given the available data. With appropriate data, the proposed ML framework can therefore elucidate mechanistic pathways and infer ecological processes by utilizing information-based model selection.

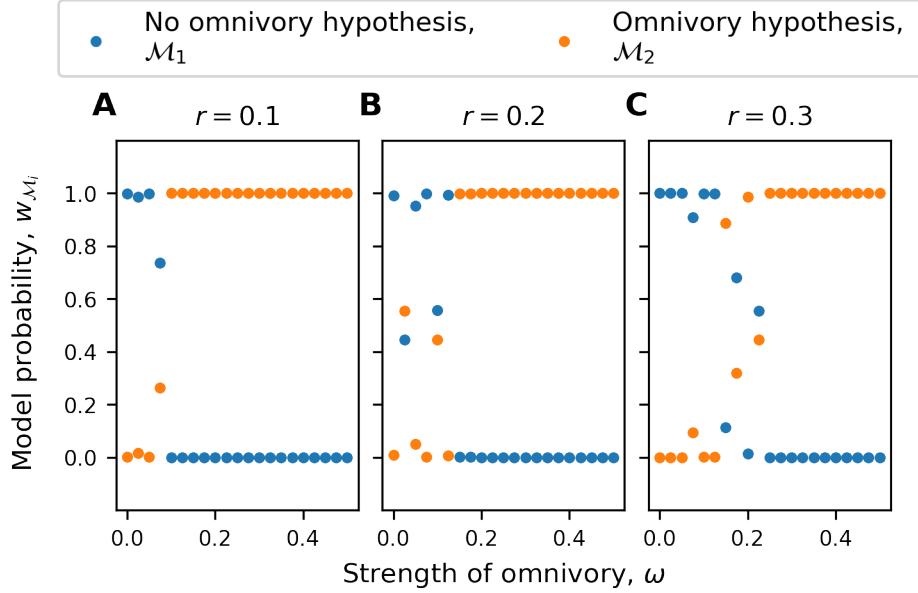


Fig. 3.5: Performance of the ML framework in supporting the predator omnivory hypothesis in a food web. A–C Hypothesis testing for levels of noise $r = 0.1, 0.2, 0.3$. Blue dots correspond to $w_{\mathcal{M}_1}$, the Akaike weights of the simple food-web model, and orange dots correspond to $w_{\mathcal{M}_2} = 1 - w_{\mathcal{M}_1}$, the Akaike weights of the omnivory model, which can be interpreted as model probabilities. A–C indicate that the ML framework can detect omnivory, as the omnivory model \mathcal{M}_2 is given strong support ($w_{\mathcal{M}_2} > 99\%$) for most of the w range investigated.

3.4 Discussion

We propose a ML framework combining a mini-batch method inspired by multiple shooting methods (Pisarenko2004) with automatic differentiation (Rackauckas2020a) and state-of-the-art variational optimizers (Kingma2014) to efficiently and accurately parametrize complex dynamical models. We show formally that splitting the data into mini-batches with a short time horizon regularizes the loss function associated with dynamical models characterized by complex dynamics, such as chaotic dynamics and limit cycles (??). We demonstrate numerically that this reformulation ensures the success of gradient-based optimizers to parametrize ecosystem models (??????). This mini-batch method is also relevant beyond variational methods and applies to any inferential method navigating the posterior landscape, such as evolutionary algorithms (wilke2001evolution; Rodriguez-Fernandez2006) or Markov Chain Monte Carlo methods (Lignell2013; Higgins2010; Xu2006; Fiechter2013; Rosenbaum2019). The proposed approach is particularly relevant for the parametrization of ecosystem models incorporating realistic ecological and adaptive mechanisms (Urban2016), which are generally associated with strong nonlinearities due to the complexity of processes linking interacting ecological compartments (Bjornstad2001; Hastings1993; Huisman1999; Beninca2008). It further integrates the practical constraints of available ecological datasets (Dornelas2018), accommodating incomplete, noisy, shallow and independent observation data. Overall, the ML framework successfully blends ML methods with mechanistic

ecosystem models to learn from ecological time series, and it could therefore improve our quantitative understanding of ecosystem dynamics and help to anticipate their responses to global changes (**Urban2016**).

Our work contributes to the ongoing effort to better assimilate observational data into mechanistic models (**Schartau2017; Raissi2019; Kashinath2021**), with a specific focus on the parametrization of ecosystem models with strong nonlinearities. Recently, (**Yazdani2020**) proposed an alternative framework dubbed "systems biology informed deep learning", where a neural network is fitted to the data and the additional mechanistic model constraints are integrated. This alternative framework extends previous colocation methods (**Ramsay2007; Cao2008**) and has the advantage of being able to parametrize stochastic models. As it requires the selection of a neural network architecture and a "goodness of fit" parameter, it nevertheless imposes an additional layer of complexity, which might negatively affect the model parametrization (**Yazdani2020**). In contrast, the ML framework proposed here trains the model directly against data, using automatic differentiation and sensitivity analysis in order to apply variational optimizers directly to the model simulations. This makes it possible to bypass the use of neural networks, rendering the parametrization process simpler and more amenable to model selection (**Ramsay2007**).

By integrating the practical constraints imposed by ecological datasets, the ML framework can learn from short time series with partial and noisy observations (????). Local ecosystem surveys, such as marine trawling surveys or local terrestrial surveys ((**Pinsky2013; Dornelas2018; Burrows2019**) and references therein), provide time series that are generally shallow in time but composed of many replicates (**Hsieh2008; Clark2015**), in part due to the practical difficulties of long-term monitoring (**Ye2016**). Our results show that the inclusion of multiple independent time series in the training dataset reduces the error in the parameter estimates and increases the forecast skill (??). This indicates that the proposed ML framework could, in practice, efficiently harness the information available in current ecological datasets. Instead of directly comparing simulated and observed data, matching time-averaged statistics between observations and simulations (e.g. means and covariances) could further yield an improved assimilation of observations from diverse data sources, such as global observations of productivity from satellites and local surveys, as proposed for climate models (**Schneider2017**). Overall, the proposed ML framework accommodates the specificities of current ecological time series and can improve the assimilation of ecological data into mechanistic ecosystem models.

Our work can help elucidate mechanistic pathways by contrasting hypotheses embedded in model variants. Using information-criterion-based model selection, we demonstrate with a case study that the ML framework is able to provide statistical support for the true generating model among two different candidates (??). Importantly, the ML framework can perform model selection on complex models, incorporating key mechanisms such as trait–species interactions, evolutionary potential and responses to environmental conditions, which have been shown to be important in mediating ecosystem dynamics and must be refined in models to improve predictive accuracy (**Urban2016**). The ML framework can therefore lead to the improvement of current ecosystem models and knowledge, which is crucially needed given that key ecological processes are only partially described in most ecosystem models (**Schartau2017**). AIC can also be used to ensure the interpretability of

the model parameters, and should be preferred to estimating the parameter uncertainty through e.g. the Cramer Rao inequality (**Burnham2002**): by favouring models with less complexity, model selection techniques disqualify uninformative parameters to ensure interpretability (**Burnham2002**). This has the extra benefit of reducing the dimensionality of the parameter space, hence improving the estimation of other parameters. Following recent novel approaches to investigate ecological hypotheses (**Curtsdotter2019**), our method contributes to the development of a process understanding of ecosystem functions and provides a path forward to better link ecological theory and data.

The proposed approach still presents a set of limitations, which might hamper its success under specific situations. First, while the use of mini-batches smooths the loss surface and ensures better convergence, it also flattens the loss surface around the true parameter value, which consequently deteriorates the precision of the inferred parameters because the loss function takes similar values in an extended neighbourhood of the true parameters. To circumvent this issue, iterative training can be performed, where the learning is initiated by a short batch length $K^{(s)}$ to identify the region with the most probable parameters, and in subsequent iterations the batch length is increased to improve the precision of the inference. Iterative training could also improve the lack of statistical support obtained in hypothesis testing experiments (see simulations in **????** where none of the models is given statistical support), as it would increase differences in likelihood for parameter values around the neighbourhood of the true parameter values. Second, our results highlight that the data might not provide enough constraints for a correct parametrization (??, partial observation setting and $S = 1$). Pre-experimental analyses with simulated synthetic data might therefore be required to design the sampling protocol and campaign to ensure an adequate sampling effort (**Banks2017; Laubmeier2018**). Third, while in ?? it is assumed that the parameter values are the same across the time series, strong regional variability might also be observed among the spatially replicated data, causing parameter values to vary across the replicates. The knowledge of this variability could motivate partial pooling (**Beaumont2010**) or parametrization of the parameters in terms of environmental conditions (**Pahlow2008**), to account for the independence of the parameter values across the replicates. Finally, while the proposed ML framework greatly improves convergence, it could still be that – even with a large amount of data – poor initial parameter estimates, a large number of free parameters, or high noise levels prevent convergence to the true minimum. Performing multiple runs with varying initial parameter estimates can ensure that the maximum a priori estimate is reliable. If this is not the case, stochasticity could further be introduced within the ML framework to prevent the convergence to local minima, where only a subset of mini-batches are fitted at each epoch (**bottou2012stochastic**).

3.5 Conclusion

We proposed a ML framework based on a mini-batch method combined with automatic differentiation and state-of-the-art optimizers to estimate the parameters and improve the forecast skill of complex ecosystem models from observation data. The ML framework was benchmarked with a realistic ecosystem model characterized by strong nonlinearities and

delivered excellent performance, accommodating the practical constraints imposed by the quality and availability of ecological datasets. Our experiments have further illustrated the ability of the ML framework to discriminate between several candidate models, enabling the testing of ecological theories against data and the improvement of current mechanistic models. Given the increasing number of ecological datasets following the development of monitoring technologies such as environmental DNA (**Ruppert2019**), remote sensing (**Jetz2019**), bioacoustics (**Aide2013**), and citizen observations (**GBIF**), the proposed ML framework opens up new opportunities for the quantitative investigation of current ecosystem functions (**Curtsdotter2019**) and the prediction of ecosystem responses to increasing disruptions (**Urban2016**).

3.6 Code availability

The ML framework is implemented in the multi-purpose Julia package **MiniBatchInference.jl** available at <https://github.com/vboussange/MiniBatchInference.jl>, and the simulation code is available at <https://github.com/vboussange/mini-batching-ecological-data>.

3.7 Acknowledgements

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3.A Supplementary Information

We show that the loss function $L_{\mathcal{M}}(\theta)$ in the main manuscript in ?? is ill-behaved for models with complex dynamics when the time horizon is large. We proceed by first analysing the dynamics of models with complex dynamics showing chaotic behaviour or limit cycles, and approximate the divergence of perturbed dynamical trajectories. We then show that the divergence in dynamics translates into a loss function whose surface is rugged in most of the parameter space, and that the gradient of the loss function around the true parameters becomes exponentially steeper with time. We conclude by formally discussing how the proposed mini-batch method regularizes the ill-behaviour of the loss function.

3.A.1 Dynamics under perturbations

Perturbed initial conditions

Consider the trajectory of the state variables

$$\begin{aligned} x(t) &= \mathcal{M}(t, p, x_0) \\ &= \int_0^t f(s, x(s), p) ds + x_0 \end{aligned} \tag{S3.11}$$

and consider the perturbed trajectory

$$x_{\delta x_0}(t) = \mathcal{M}(t, p, x_0 + \delta x_0) \quad (\text{S3.12})$$

whose initial conditions (ICs) x_0 are perturbed by δx_0 . Assuming that the system is chaotic and that δx_0 is small, the distance between the perturbed trajectory and the original one grows as

$$\|x(t) - x_{\delta x_0}(t)\| \sim e^{\lambda t} \delta x_0 \quad (\text{S3.13})$$

where λ is the largest Lyapunov exponent of the system (**strogatz2018nonlinear**). After enough time, the trajectories diverge so much that they effectively become independent samples of the phasespace: the trajectories forget their ICs, and ergodic theory ensures that the positions of $x(t)$ and $x_{\delta x_0}(t)$ are better described by a random variable \mathcal{R} with probability density given by the density of orbits in the chaotic attractor, the so-called invariant measure of the chaotic attractor (**jost2005dynamical**). The distance between $x(t)$ and $x_{\delta x_0}(t)$ can therefore be described as

$$\|x(t) - x_{\delta x_0}(t)\| \sim \|\mathcal{R}_1 - \mathcal{R}_2\| \quad (\text{S3.14})$$

Considering the observation function h , it follows that

$$\begin{aligned} \|h(x(t)) - h(x_{\delta x_0}(t))\| &\sim \left. \frac{\partial h}{\partial x} \right|_{x(t)} e^{\lambda t} \delta x_0 \text{ for } t \ll \frac{1}{\lambda} \\ \|h(x(t)) - h(x_{\delta x_0}(t))\| &\sim \|h(\mathcal{R}_1) - h(\mathcal{R}_2)\| \text{ for } t \gg \frac{1}{\lambda} \end{aligned} \quad (\text{S3.15})$$

(see ?? for an illustration of the divergence behaviour over time).

Perturbed model parameters

Consider now a trajectory $x_{\delta p}$ with a small perturbation of the parameters δp . From ????, it follows that

$$\begin{aligned} \dot{x}_{\delta p}(t) &= f(t, x_{\delta p}(t), p + \delta p) \\ &\sim f(t, x_{\delta p}(t), p) + \frac{\partial f(t, x_{\delta p}(t), p)}{\partial p} \delta p \\ &\sim f(t, x(t), p) + \frac{\partial f(t, x_{\delta p}(t), p)}{\partial p} \delta p + \frac{\partial f(t, x(t), p)}{\partial x(t)} \frac{\partial x(t)}{\partial p} \delta p \end{aligned} \quad (\text{S3.16})$$

which is dominated by the first term under small values of δp , and is thus subject to chaotic dynamics. Similar to a perturbation of the ICs, the small perturbation δp generates a divergence in the dynamical trajectories that grows exponentially until they become uncorrelated. For small δp , the distance between the true and deviated trajectory can therefore be approximated as

$$\|x(t) - x_{\delta p}(t)\| \sim \int_0^t e^{\lambda(t-s)} \|x(s) - x_{\delta p}(s)\| ds \sim e^{\lambda t} g(\delta p) \quad (\text{S3.17})$$

where $g(\delta p) = \left\| \frac{\partial f(t, x_{\delta p}(t), p)}{\partial p} + \frac{\partial f(t, x(t), p)}{\partial x(t)} \frac{\partial x(t)}{\partial p} \right\|$ gives the scale of the divergence between the two trajectories as a function of δp . Similar to a perturbation of the ICs, the difference in trajectories grows to the point where the trajectories become effectively independent after a long time. Hence, it follows that

$$\begin{aligned}\|h(x(t)) - h(x_{\delta p}(t))\| &\sim \left. \frac{\partial h}{\partial x} \right|_{x(t)} e^{\lambda t} g(\delta p) \text{ for } t \ll \frac{1}{\lambda} \\ \|h(x(t)) - h(x_{\delta p}(t))\| &\sim \|h(\mathcal{R}_1) - h(\mathcal{R}_2)\| \text{ for } t \gg \frac{1}{\lambda}\end{aligned}\quad (\text{S3.18})$$

(see ?? for an illustration of the divergence behaviour over time).

In the following section, we call the first divergence regime the informative divergence regime, where the loss grows with the distance to the true parameters, and we call the second divergence regime the mixed divergence regime, where the loss is dominated by the random-like behaviour. Given that δp and δx_0 behave similarly, we employ θ and $\delta\theta$ to encompass both perturbations of parameters and ICs, and denote by $g(\delta\theta)$ the function that gives the scale of the divergence in trajectories for both $g(\delta p)$ and $\|\delta x_0\|$.

Transition in the parameter space between the informative and the mixed regime

For a fixed time horizon t , and depending on the shape of the chaotic attractor, the magnitude of the perturbation determines the divergence regime. If the perturbation is small, the trajectories will be aligned, but for large perturbations they will effectively become two independent trajectories.

The transition between the two regimes can be studied by noting that the informative divergence should remain in the same order of magnitude as the mixed divergence. The reason is that the expected value of the squared divergence between two trajectories $x(t)$ and $x_{\delta\theta}(t)$ for large t is

$$\mathbb{E} [\|h(\mathcal{R}_1) - h(\mathcal{R}_2)\|^2] = 2\text{Var}[h(\mathcal{R})] \lesssim \max |h(\mathcal{R}) - \mathbb{E}[h(\mathcal{R})]|^2 \quad (\text{S3.19})$$

meaning that the expected value of the squared divergence in the mixed regime is in the same order of magnitude as the maximum distance within the phasespace. On the other hand, the divergence of any two trajectories in the chaotic attractor cannot be larger than the maximum distance between two points in the attractor, which is itself bounded through the triangle inequality as $2 \max |h(\mathcal{R}) - \mathbb{E}[h(\mathcal{R})]|$.

Since the growth of the informative regime has to remain in the same order of magnitude as the mixed regime, at the regime transition we must have

$$e^{\lambda t} g(\delta\theta) \sim \mathbb{E} [\|h(\mathcal{R})\|] \quad (\text{S3.20})$$

?? implies that, for a given time horizon t , the magnitude of the critical perturbation $\delta\theta^*$ associated with the regime transition satisfies

$$\|\delta\theta^*\| \sim e^{-\lambda t} \quad (\text{S3.21})$$

Limit cycles

While in the section above a chaotic system was assumed to provide an approximation for the divergence of the trajectories, a similar approximation applies for systems characterized by limit cycles. Considering a system $x(t)$ with a limit cycle characterized by the phase ωt with frequency ω , i.e.

$$x(t) = \mathfrak{f}(\phi(t)) = \mathfrak{f}(\omega t) \mod 2\pi \quad (\text{S3.22})$$

a perturbation of the parameters δp might lead to a perturbed frequency $\delta\omega$, further leading to a difference in phases

$$\delta\phi(t) = \phi(t) - \phi_{\delta\omega}(t) = \delta\omega t \mod 2\pi \quad (\text{S3.23})$$

For $t \lesssim \frac{1}{\delta\omega}$, $\delta\phi(t)$ grows linearly with $\delta\omega$, but once $t \gg \frac{1}{\delta\omega}$, the change of phase $\delta\phi(t)$ is affected by the modulo operation. As this operation is nonlinear, a small random perturbation $\delta\omega$ results in a random uniform phase over the interval $[0, 2\phi]$. For a large time horizon, $x_{\delta\omega}$ is thus uniformly spread over the circular line given by the phasespace of the dynamical system. Hence, the approximation in ?? applies for cyclic dynamics, except that the initial divergence is linear rather than exponential, and that in contrast to ??, a change affecting the initial position will not grow over time.

3.A.2 Consequences for the shape of the loss surface

The approximation of the divergence of trajectories in ????, together with the transition boundary determined by ??, can be used to characterize the surface associated with the loss function.

Omitting the term corresponding to the priors and the variance–covariance matrix Σ_y for simplicity, the loss function presented in the main text is expressed as

$$\begin{aligned} L_{\mathcal{M}}(\theta) &= \frac{1}{K} \sum_{k=1}^K \|y_k - h(\mathcal{M}(t_k, \theta))\|^2 \\ &= \frac{1}{K} \sum_{k=1}^K \|h(\tilde{x}(t_k)) + \epsilon(t_k) - h(\mathcal{M}(t_k, \theta))\|^2 \end{aligned} \quad (\text{S3.24})$$

where the parameter vector θ is decomposed into the model parameter vector p and the ICs x_0, y_k correspond to the observations, \tilde{x} corresponds to the true trajectory and $\epsilon(t)$ is the observational noise. As the noise is independent of the dynamics, it is uncorrelated with $h(\tilde{x}(t_k)) - \mathcal{M}(t_k, \theta)$, meaning that the loss can be split in expectation

$$\mathbb{E}[L_{\mathcal{M}}(\theta)] = \frac{1}{K} \sum_{k=1}^K \|h(\tilde{x}(t_k)) - h(\mathcal{M}(t_k, \theta))\|^2 + \text{Var}[\epsilon] \quad (\text{S3.25})$$

where the noise term ($\text{Var}[\epsilon]$) is independent from the parameters θ . Assuming that $\theta = \tilde{\theta} + \delta\theta$, where $\tilde{\theta}$ correspond to the true parameters and ICs, every term in the sum corresponds

to a squared distance between the true trajectory $\tilde{x}(t)$ and a perturbed trajectory $\tilde{x}_{\delta\theta}(t)$. Using [???????](#) we obtain the loss function approximation

$$L_{\mathcal{M}}(\theta) \sim \sum_{k=1}^{\min\{K, K^*\}} e^{2\lambda t_k} g^2(\delta\theta) + \sum_{k=\min\{K, K^*\}+1}^K \text{Var}[h(\mathcal{R})] + \text{Var}[\epsilon] \quad (\text{S3.26})$$

where K^* is the observation index corresponding to the time horizon t_{K^*} , where the transition between the informative and the mixed regime happens for the perturbation $\delta\theta$, obtained from [??](#).

The distribution of the observation times and the magnitude of the perturbation $\delta\theta$ determine whether the loss is dominated by the informative or by the mixed divergence regime. For a fixed perturbation $\delta\theta$, assuming that the observations y_k are uniformly distributed over the time interval $[0, t]$, [??](#) yields for $t \ll \frac{\log(|\delta\theta|)}{\lambda}$ that the loss $L_{\mathcal{M}}$ is dominated by the informative divergence regime, whereas if $t \gg \frac{\log(|\delta\theta|)}{\lambda}$ the loss is dominated by the mixed divergences. In the region where the loss is dominated by mixed divergences, the loss $L_{\mathcal{M}}$ has an expected value of order $\mathcal{O}(\text{Var}[h(\mathcal{R})] + \text{Var}[\epsilon])$ and does not grow monotonically with $\delta\theta$. It corresponds to a "random-like" surface populated with local minima, and is consequently characterized by an uninformative gradient preventing local optimizers from converging to the true parameters $\tilde{\theta}$ (see [??B](#), orange dashed curve). On the other hand, in the region where the divergence in trajectories belongs to the informative regime, the loss is convex and grows with $\delta\theta$. Its associated gradient $\nabla_{\theta} L_{\mathcal{M}}(\theta) \sim \sum_{k=1}^K e^{2\lambda t_k} \nabla_{\theta} g^2(\delta\theta)$ consequently contains relevant information for the use of variational optimizers, but the loss surface becomes exponentially steeper as the time horizon increases. As shown by the green dotted curve in [??B](#), this large gradient in the vicinity of the optimal parameters is likely to lead gradient-based optimizers to overshoot and not converge to the true parameters $\tilde{\theta}$. [??](#) further indicates that the volume of the region in the parameter space where the loss is informative shrinks exponentially as the time horizon increases, implying that for large time horizons, the uninformative region is predominant.

3.A.3 Regularizing the loss surface with mini-batches

To prevent the situation with a mixed divergence regime and to decrease the gradient in a vicinity of the true parameters, we reformulate the loss function as the average of loss functions defined over mini-batches of short time horizons. In the following section, we compare the properties of the naive loss function $L_{\mathcal{M}}$ in [??](#) with the mini-batch loss function $L_{\mathcal{M}}^*$ proposed in [??](#) in the main manuscript, and further discuss the limitations of the method in the presence of noise.

Omitting the term corresponding to the priors and ICs and the noise shape Σ_y , and assuming a single time series for simplicity, the mini-batch loss function presented in the main text can be expressed as

$$L_{\mathcal{M}}^*(\theta) = \frac{1}{M} \sum_{m=0}^{M-1} \frac{1}{K} \sum_{k=1}^{K^{(m)}} \|y_{k+mK/M} - h\left(\mathcal{M}(t_{k+mK/M}, p, x_0^{(m)})\right)\|^2 \quad (\text{S3.27})$$

where M is the number of mini-batches, $x_0^{(m)}$ corresponds to the ICs for mini-batch m inferred at time $t_{mK/M}$, and $K^{(m)} = \begin{cases} K/M + 1 & : m < M \\ K/M & : m = M \end{cases}$ is the number of data points in the m th batch. The loss function can be split in expectation as

$$\mathbb{E}[L_M^*(\theta)] = \frac{1}{M} \sum_{m=0}^{M-1} \frac{1}{K} \sum_{k=1}^{K^{(m)}} \|h(\tilde{x}(t_{k+mK/M})) - \mathcal{M}(t_{k+mK/M}, p, x_0^{(m)})\|^2 + \text{Var}[\epsilon] \quad (\text{S3.28})$$

Assuming that $x_0^{(m)} = \tilde{x}(t_{mK/M}) + \delta x_0^{(m)}$ when $\delta x_0^{(m)}$ is small, and assuming that the observation times are regularly spaced so that $t_{k+1} - t_k = \Delta t$, the time length of simulated trajectories (i.e. the time elapsed between the time when the perturbation is applied and the end time of the simulated trajectory) is divided by the number of mini-batches, in comparison to the time length of the simulated trajectory in ?? ($t = t_{K/M+1} - t_0 = K/M\Delta t$ in ??, in comparison to $t = K\Delta t$ in ??). Further assuming that $K\Delta t \gg \frac{\log(|\delta\theta|)}{\lambda}$ and choosing the number of mini-batches M so that $K/M\Delta t \ll \frac{\log(|\delta\theta|)}{\lambda}$, we apply the approximation ??, which leads to

$$\begin{aligned} L_M(\theta) &\sim \text{Var}[\epsilon] + \text{Var}[h(\mathcal{R})] \\ L_M^*(\theta) &\sim \text{Var}[\epsilon] + g^2(\delta\theta) \sum_{m=0}^{M-1} e^{2\lambda(K/M\Delta t)} \end{aligned} \quad (\text{S3.29})$$

While L_M is dominated by the mixed regime, L_M^* is dominated by the informative regime because the simulation time remains small, permitting the successful use of variational optimizers.

The number of mini-batches M should be determined by considering the dynamical behaviour of the system and the level of noise in the observation, because a large number of mini-batches M smooths out the loss surface but also entails more sensitivity to the level of noise. Indeed, in $L_M(\theta)^*$ the relative effect of the second term corresponding to the observational noise ϵ increases when the number of mini-batches M increases. The value of M should therefore be chosen wisely to balance the benefits of mini-batches, i.e. widening the region of the parameter space where L_M is well behaved and reducing the overshooting problem, and their cons, i.e. their tendency to increase the importance of noise.

3.B Three-compartment food-web models

3.B.1 Reference food-web model

We used the three-species chaotic food-web model from (Hastings1991), formulated as

$$\begin{aligned} \frac{d}{dt}R &= R(1 - R) - x_C y_C \frac{CR}{R + R_0} \\ \frac{d}{dt}C &= x_C C \left[-1 + x_C \frac{R}{R + R_0} \right] - x_P y_P \frac{PC}{C + C_0} \\ \frac{d}{dt}P &= x_P P \left[-1 + y_P \frac{C}{C + C_0} \right], \end{aligned} \quad (\text{S3.30})$$

with the biologically realistic parameter values $x_C = 0.4$, $0.071 \leq x_P \leq 0.225$, $y_C = 2.01$, $y_P = 5$, $R_0 = 0.16129$, and $C_0 = 0.5$ (**McCann1994**). The dynamics of the system are chaotic for this set of parameter values.

3.B.2 Omnivory variant food-web model

We used the three-species food-web model from (**McCann1997**), formulated as

$$\begin{aligned} \frac{d}{dt}R &= R(1 - R) - x_C x_C \frac{CR}{R + R_0} - \omega x_P y_{PR} \frac{PR}{R_{02} + (1 - \omega)C + \omega R} \\ \frac{d}{dt}C &= x_C C \left[-1 + x_C \frac{R}{R + R_0} \right] - (1 - \omega) x_P y_{PC} \frac{PC}{\omega R + (1 - \omega)C + C_0} \\ \frac{d}{dt}P &= x_P P \left[-1 + (1 - \omega) x_P y_{PC} \frac{C}{\omega R + (1 - \omega)C + C_0} \right. \\ &\quad \left. + \omega x_P y_{PR} \frac{R}{\omega R + (1 - \omega)C + R_{02}} \right] \end{aligned} \quad (\text{S3.31})$$

with the biologically realistic parameter values $x_C = 0.4$, $x_P = 0.08$, $y_C = 2.009$, $y_{PR} = 2$, $y_{PC} = 5$, $R_0 = 0.16129$, $C_0 = 0.5$, and $0 \leq \omega \leq 0.5$. For this set of parameter values, the dynamics of the system are chaotic for $\omega \lesssim 0.20$, consist of a limit cycle for $0.20 \lesssim \omega \lesssim 0.35$, and consist of damped oscillations for $0.35 \lesssim \omega$.

3.C Supplementary Figures

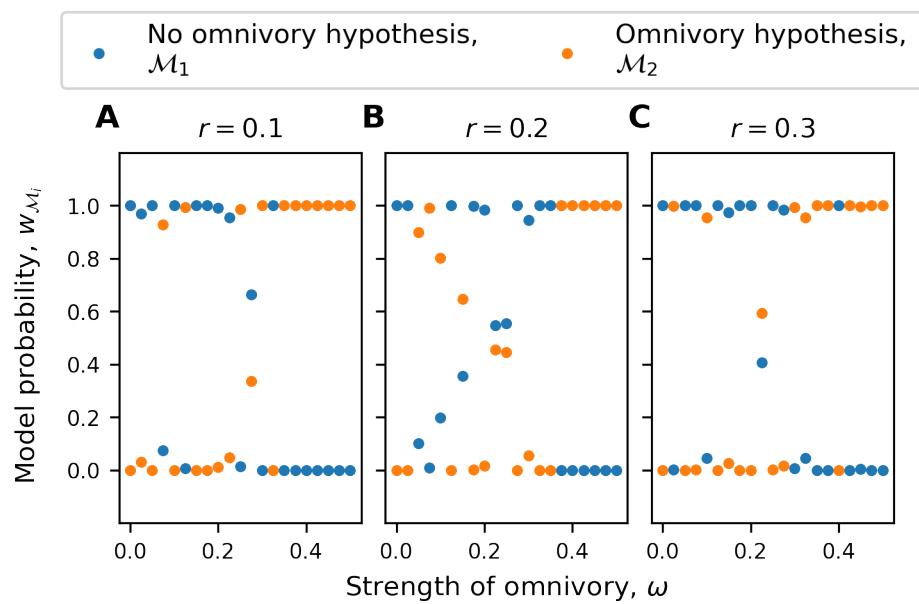


Fig. S3.6: Performance of the ML framework in supporting the predator omnivory hypothesis in a food web for the partial observation setting. In A, B and C where $r = 0.1, 0.2, 0.3$, the lack of data prevents the correct estimation of the omnivory variant model parameters, leading model M_1 to be supported for a wider range of ω in contrast to the complete observation setting (??A).

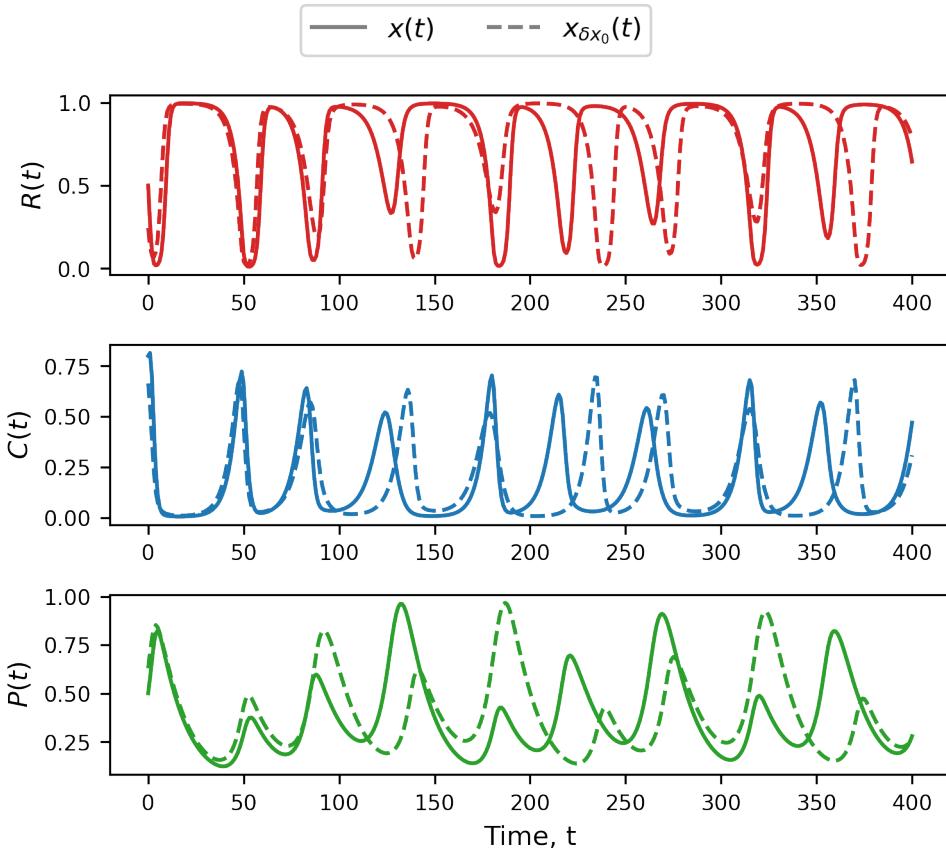


Fig. S3.7: Divergence between the trajectory $x(t)$ and a perturbed trajectory $x_{\delta x_0}(t)$, obtained from the reference food-web model from Hastings1991 and detailed in ???. For $t \lesssim 100$, $x(t)$ and $x_{\delta x_0}(t)$ are correlated and the divergence regime is informative, but for $t \gtrsim 100$ the trajectories become essentially uncorrelated, corresponding to the mixed divergence regime.

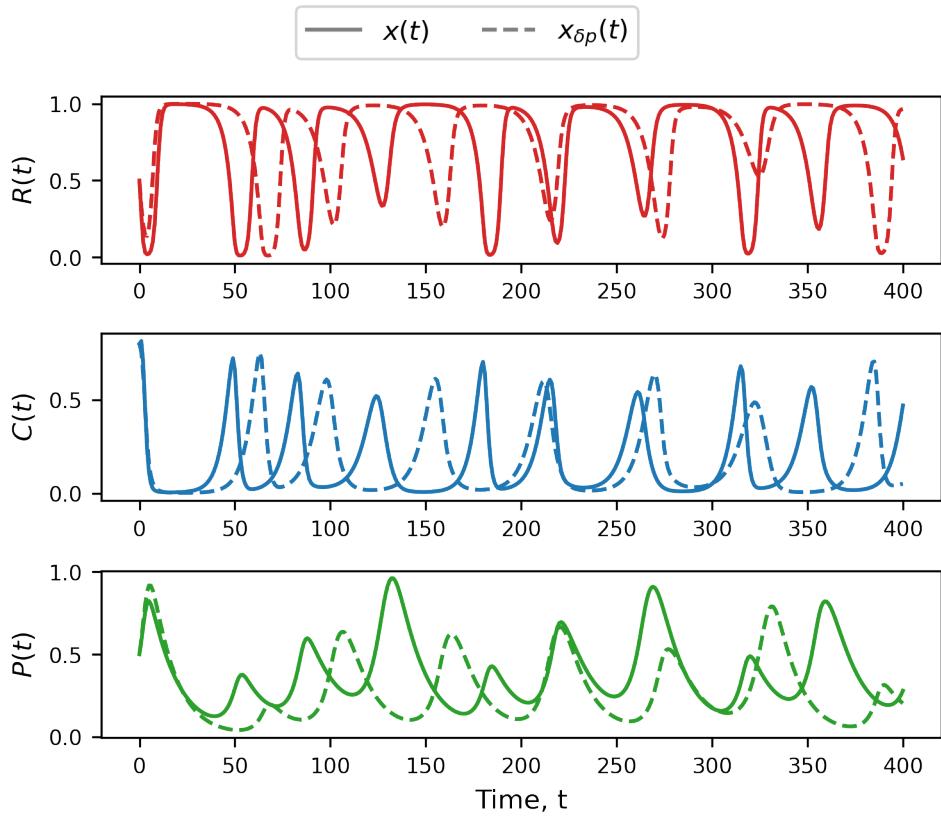


Fig. S3.8: Divergence between the trajectory $x(t)$ and a perturbed trajectory $x_{\delta p}(t)$, obtained from the reference food-web model from Hastings1991 and detailed in ???. For $t \lesssim 40$, $x(t)$ and $x_{\delta p}(t)$ are correlated and the divergence regime is informative, but for $t \gtrsim 40$ the trajectories become essentially uncorrelated, corresponding to the mixed divergence regime.

3.D Supplementary Tables

Setting	Median simulation time	Mean simulation time	Std. simulation time
Complete observations	35.9977436	39.5980174	20.5789433
Partial observations	34.1293998	39.1896534	21.8191709

Tab. S3.1: Simulation time for the complete and partial observation settings.

4

Analogous processes to ecological interactions and dispersal shape the dynamics of economic activities

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” *Thus, dare! your heritage, your acquisition,
Stories, lessons from the mouths of your fathers,
Laws and customs, names of the ancient Gods,
Forget them with hardness to raise your eyes
like newborns, on divine nature.*

— Friedrich Hölderlin (1770-1843)
The Death of Empedocles (1st version)

The fundamental processes of ecological interactions, dispersal and mutations shape the dynamics of biological systems, and analogous processes involving economic interactions, spatial transfers and economic transformations have been proposed to explain economic change. This hypothesis is compelling because it explains economic change through endogenous mechanisms, but it has not yet been systematically tested. Building upon an analogy between economic activities and biological functional groups, we investigate whether such processes can explain the temporal dynamics of 9 elemental economic activities at the national level. We rely on an inverse modelling technique together with 59 years of economic data to estimate the statistical support of each process, embedded in population dynamic models. The models couple the dynamics of individual economic activities by accounting, respectively, for processes involving positive and negative interactions between economic activities, spatial transfers, and economic activity transformations, which support we compare against a logistic growth model taken as a null model. Over 77 countries considered, we find that the null model is rejected in 45 countries against the alternative models, and obtain a strong support for the models capturing positive interactions between national economic activities and spatial transfers across countries. These results suggest that processes akin to those displayed in biological ecosystems play a significant role on the dynamics of economic systems at the national level. Paralleling variations across different ecosystems types, the strength-of-evidence varies across countries for each model, and might underlie differences in institutional contexts and historical contingencies. Overall, our study provides a new quantitative, biologically inspired framework to study the forces shaping economic change.

4.1 Introduction

The fields of evolutionary biology and economics have mutually exchanged ideas for the past two centuries (**Dopfer2007**), and analogies between processes in biology and economics have been increasingly used during the 20th century to explain economic development (**Ruth1996**). A pioneer was Schumpeter, who notoriously proposed that economic dynamics is driven by innovations that transform the economy – so called periods of "creative destruction" (**schumpeter2017theory**) – similar to the punctuated equilibrium changes observed throughout the development of life on Earth (**gould1972**). The analogy with biology was further developed within the field of evolutionary economics (**Hodgson2019**), largely promoted by the seminal work of **nelson1985evolutionary**. The premise of evolutionary economics considers habits, customs, and organizational routines as "replicators", i.e. atomic units playing the role of genes and defining as a whole the cohesive identity of an economic entity (**Hodgson2019**). Under this framework, firm business strategies determine how they transform commodities and knowledge into new knowledge, new technologies or other industrial products with more added value. The replicators define the fitness of an economic entity within a given economic context, which grows in terms of assets and human capital and survive within an ecosystem of entities (**Hodgson2002**), continuously adapting and experiencing evolutionary processes (**Veblen1898**). While biological analogies have resulted in useful insights on the plausible drivers of economic growth (**Dopfer2007**), the proposed pathways have mostly been qualitatively investigated. Computational tools have recently allowed to test biological hypotheses against data to gain a quantitative understanding on the ecological and evolutionary processes shaping the dynamics of biological systems (**Pontarp2019; Boussange2022; Boussange2022a**). These tools could leverage the qualitative insights gained from biological analogies and provide a quantitative framework to investigate the forces shaping economic development.

Interactions between biological organisms, movement of individuals across space via dispersal, and mutation of phenotypes, represent fundamental processes that drive the dynamics of ecosystems (**Vellend2010**), and similar eco-evolutionary processes may shape the dynamic of economic systems. Analogously to biological organisms, economic entities interact in a mutualistic or competitive fashion (**Pistorius1997**). While biological organisms engage in negative interactions when e.g. competing for similar resources (**GRIME1973**), economic entities have negative effects on each other when relying on workforce with similar knowledge or attracting similar potential investors (**Wernerfelt1989**). Positive interactions are observed between biological organisms with e.g. the accumulation of nutrients, the provision of shade, or the protection from herbivores (**Wernerfelt1989; Callaway2002**). In parallel, positive effects between economic entities appear when they are connected through supply chains (**Ozman2009; Saavedra2009a**), or because of agglomeration externalities **VanDerPanne2004**, such as knowledge spillovers (**Menon2015**) and the attraction of elite workers (**Cohendet2018**). Colonization and migration processes play a major role in the development of ecosystems, where dispersal mechanisms initiate ecological succession with the settlement and growth of new species (**Leibold2004**). Similarly, economic processes contribute to spread knowledge and organizational routines across space. Those processes involve international business expansions (**Zahra2000; Zahra2000**), labour mobility

(**Boschma2008**), and the diffusion of innovations (**RogersEverettM2003DoI**). Finally, evolutionary processes allow the adaptation of organisms to changing environmental conditions (**Bell2017**), while in economic systems, variations in organizational routines allow economic entities to adapt to economic contexts (**Cordes2006**) and transform into new economic entities (**Freeman2002; Hodgson2004; Aldrich2008**). While economic interactions, spatial transfers and economic activity transformations are regularly documented in evolutionary economics, the significance of these processes on long-term economic development has been seldom quantified using economic data.

Focusing on endogenous forces and borrowing concepts and methods from biology, a number of modelling approaches have broken with the tradition of standard economic modelling and approaches to quantitatively investigate the processes driving economic patterns (**Tacchella2018**). For instance, **Saavedra2009a** investigated the effect of cooperation within firms with a model of consumerresource interactions. **Scholl2020** developed a theory of market ecology to interpret market phenomena and predict market behavior, where financial trading strategies are the analogous to biological species. Lotka-Volterra models were used in the prediction of technology evolution (**Zhang2018**), to evaluate product and firm competition (**Modis1997; Saavedra2014**), to understand the drivers of market share dynamics (**Farmer1999; Michalakelis2011; Marasco2016; Gatabazi2019**) and to estimate the value of firms (**Cauwels56**). **Applegate2021** investigated an ecological model of competition-colonization dynamics to understand the distribution of firm sizes, and **Suweis2015** used a population dynamic model to evaluate the effect of international trade on global food security. The availability of fine grained datasets of economic activities, together with dimensionality reduction techniques, have provided insights into the endogenous processes shaping economic development (**Mealy2019; Hidalgo2021**). In particular, the detailed global trade data compiled by United Nations statistical Division and cleaned by **Hidalgo2021** consists in a time series of 59 years of economic activity that can be combined with inverse modelling techniques to learn about the processes influencing the long-term dynamics of economic systems.

Here, we investigate whether eco-evolutionary processes can quantitatively explain economic growth using an inverse modelling technique together with 59 years of data on economic activity. Relating the temporal evolution of the capital of economic activities to the temporal evolution of the biomass of functional groups within an ecosystem, we quantitatively assess the effect of eco-evolutionary processes on the long-term development of national economies. Functional groups, as the aggregation of species sharing similar characteristics and having similar functions within the ecosystem, are the required elemental units to model ecosystem dynamics. Analogously, as the aggregation of firms with similar output nature (**Applegate2021**), we consider 9 economic activities to form the elemental units of economic systems, and conduct our investigation by modelling their temporal dynamics. We consider a null model implementing the fundamental processes of self-replication and self-limitation (\mathcal{M}_{null}), where no couplings between economic activities are captured, which we contrast to alternative dynamic models that capture negative and positive ecological interactions ($\mathcal{M}_{\alpha^-}, \mathcal{M}_{\alpha^+}$), spatial transfers (\mathcal{M}_δ), and economic activity transformations (\mathcal{M}_μ). We use the mini-batching machine learning technique of **Boussange2022a** to estimate the maximum likelihood of each model for 100 countries, and use a model selection

technique to evaluate the statistical support of each alternative model, based on a data set of national exports from 1962 to 2020, taken as a proxy for the temporal development of the capital of economic activities. We first show that the machine learning method, together with the model selection procedure, can provide support for the generating processes in a controlled experiment. Further applying the model selection procedure to the empirical data, we find evidences for eco-evolutionary processes, and observe differences in model support across countries, which may reflect idiosyncrasies of economic development. By adopting a biological perspective on economic dynamics, our study sets the basis for an understanding of the endogenous forces determining economic growth, and proposes a complementary perspective to mainstream economic theory ([10.1093/cje/bet027](https://doi.org/10.1093/cje/bet027)) on the drivers of economic dynamics.

4.2 Methods

4.2.1 Eco-evolutionary model to characterise the dynamics of economic activities

We derive a general population dynamic model where the dynamics of national economic activities is driven self-replication, self-limitation, interactions with other activities, spatial transfers between countries, and activity transformations within the country. We present the general model in the following, which we further split into alternative sub-models to test the support of each process.

In the general population dynamic model that includes all the processes investigated, we consider that the rate of change of the size of an economic activity i in the country c , denoted as $n_i^{(c)}$, follows

$$\begin{aligned} \frac{d}{dt}n_i^{(c)}(t) = & r_i^{(c)}n_i^{(c)}(t) \left(1 - b_i^{(c)}n_i^{(c)}(t) + \sum_{j \neq i}^{N^{(c)}} \alpha_{i,j}^{(c)}n_j^{(c)}(t) \right) + \sum_l^M \delta_i^{(l,c)} \left(n_i^{(l)}(t) - n_i^{(c)}(t) \right) \\ & + \sum_j^{N^{(c)}} \mu_{j,i}^{(c)} \left(n_j^{(c)}(t) - n_i^{(c)}(t) \right), \end{aligned} \quad (4.1)$$

where $N^{(c)}$ is the number of economic activities considered in country c and M is the number of countries considered. In ??, the first summand corresponds to a Lotka Volterra model for ecological communities (see, e.g., ([Bunin2017](#); [Scheffer2006a](#); [Case1990](#))), where $r_i^{(c)}$ is the growth rate, involved in the self-replication, and $b_i^{(c)}$ is the self-interaction coefficient, involved in the self-limitation. $b_i^{(c)}$ can also be interpreted as the inverse of the country's carrying capacity for the activity. The individual dynamics of economic activities are altered by interactions, where $\alpha_{i,j}^{(c)}$ captures the interaction between activity i and activity j at location c . The interaction between i and j is mutualistic if $\alpha_{i,j}^{(c)} > 0$, and competitive when $\alpha_{i,j}^{(c)} < 0$, altering positively or negatively the rate of change of the activity i by the presence

of other activities. The second summand accounts for spatial transfers across locations (see, e.g., (Tilman1994b)), where $\delta_i^{(l,c)}$ is the rate of spatial transfer for activity i between locations l and c . The last summand corresponds to economic activity transformations within country c , where $\mu_{j,i}^{(c)}$ is the rate of transfer dictating how fast activity j transforms into activity i . In the sense of the quasi-species model (eigen1988molecular), this term can be interpreted as accounting for evolutionary processes, and can contribute to the development of an activity in the presence of other activities. A graphical representation of the model is given in ??.

?? involves a high number of independent parameters to fit (i.e., $N^{(c)}(1 + M + 2N^{(c)})$ parameters for each country), which can be reduced for the sake of parsimony under mean field assumptions. Specifically, we assume that all economic activities interact similarly, so that $\alpha_{i,j}^{(c)} = \alpha^{(c)}$. We assume that activity transformations are symmetric and occur at similar rates for all activities, so that $\mu_{i,j}^{(c)} = \mu^{(c)}$, and we assume that spatial transfers occur at similar rates for all activities and all countries, so that $\delta_i^{(l,c)} = \delta^{(c)}$. These mean field assumptions reduce the number of parameters to $2N + 3$. In summary, the parameters $r_i^{(c)}$ and $b_i^{(c)}$ are assumed activity- and country-dependent, and $\alpha^{(c)}$, $\mu^{(c)}$, and $\delta^{(c)}$ are assumed country-dependent. We further assume that the values of these parameters are determined by the activity's characteristics and the country's institutional system (comprising e.g. the taxation regime, the system of innovation, the legal system, the intellectual property right and the socio-cultural background of the society) and resources (comprising e.g. labour force, knowledge capital, agricultural resources, mineral resources and energy resources), but that they do not change through time.

To investigate how the data supports each of the process embedded in ??, we further decompose the model into five different sub-models, denote by \mathcal{M}_{null} , \mathcal{M}_{α^+} , \mathcal{M}_{α^-} , \mathcal{M}_δ , and \mathcal{M}_μ , as follows:

$$\mathcal{M}_{null} : \quad \frac{d}{dt}n_i^{(c)}(t) = r_i^{(c)}n_i^{(c)}(t)(1 - b_i^{(c)}n_i^{(c)}(t)) \quad (4.2)$$

$$\mathcal{M}_{\alpha^+} : \quad \frac{d}{dt}n_i^{(c)}(t) = r_i^{(c)}n_i^{(c)}(t) \left(1 - b_i^{(c)}n_i^{(c)}(t) + \alpha^{(c)} \sum_{j \neq i} n_j^{(c)}(t) \right), \quad \alpha > 0 \quad (4.3)$$

$$\mathcal{M}_{\alpha^-} : \quad \frac{d}{dt}n_i^{(c)}(t) = r_i^{(c)}n_i^{(c)}(t) \left(1 - b_i^{(c)}n_i^{(c)}(t) + \alpha^{(c)} \sum_{j \neq i} n_j^{(c)}(t) \right), \quad \alpha < 0 \quad (4.4)$$

$$\mathcal{M}_\delta : \quad \frac{d}{dt}n_i^{(c)}(t) = r_i^{(c)}n_i^{(c)}(t)(1 - b_i^{(c)}n_i^{(c)}(t)) + \delta^{(c)} \left(\overline{n_i^{(c)}}(t) - n_i^{(c)}(t) \right) \quad (4.5)$$

$$\mathcal{M}_\mu : \quad \frac{d}{dt}n_i^{(c)}(t) = r_i^{(c)}n_i^{(c)}(t)(1 - b_i^{(c)}n_i^{(c)}(t)) + \mu^{(c)} \sum_j^{N^{(c)}} \left(n_j^{(c)}(t) - n_i^{(c)}(t) \right) \quad (4.6)$$

where

$$\overline{n_i^{(c)}} = \frac{1}{M-1} \sum_{l \neq c}^M n_l^{(l)} \quad (4.7)$$

accounts for the capital of activity i at the global level, excluding country c . The term $\overline{n_i^{(c)}}$ appearing in ??? arises from the mean field assumption used for spatial transfers, where

$$\begin{aligned}
\sum_l^M \delta_i^{(l,c)} \left(n_i^{(l)}(t) - n_i^{(c)}(t) \right) &= \delta^{(c)} \sum_l^M \left(n_i^{(l)}(t) - n_i^{(c)}(t) \right) \\
&= \delta^{(c)} \sum_{l \neq c}^M \left(n_i^{(l)}(t) - n_i^{(c)}(t) \right) \\
&= \delta^{(c)} \left(\sum_{l \neq c}^M n_i^{(l)}(t) - (M-1)n_i^{(c)} \right) \\
&\propto \delta^{(c)} \left(\overline{n_i^{(c)}} - n_i^{(c)} \right).
\end{aligned} \tag{4.8}$$

The simplest model \mathcal{M}_{null} embeds self-replication and self-limitation without any further coupling force acting upon economic activities, and is considered as the null model. The alternative models $\mathcal{M}_{\alpha+}$, $\mathcal{M}_{\alpha-}$, \mathcal{M}_δ , and \mathcal{M}_μ capture self-replication and self-limitation together with one of the additional eco-evolutionary process investigated. Because we find inconsistent maximum likelihood estimations in the controlled experiment detailed in the ?? for models incorporating combinations of the processes in ????????, we do not test more complex models. For \mathcal{M}_δ , in ??, we assume that spatial transfers do not affect the dynamics of $n_i^{(c)}$. As such, the dynamics of $n_i^{(c)}$ and $\overline{n_i^{(c)}}$ is decoupled, and we use linear interpolations of the data to model $\overline{n_i^{(c)}}$. This assumption allows to divide by two the number of state variables to simulate in each country, greatly improving the computational complexity of the maximum likelihood estimation procedure. In the following, we designate by $\mathcal{M}(t, \theta^{(c)})$ the vector of economic activity capitals predicted by model \mathcal{M} at time t for country c with the parameter vector $\theta^{(c)}$, that includes the growth rates $r_i^{(c)}$, the self limitation rates $b_i^{(c)}$, the specific model parameters $\alpha^{(c)}$, $\mu^{(c)}$, $\delta^{(c)}$, and the initial condition vector $n^{(c)}(t_0)$ for country c .

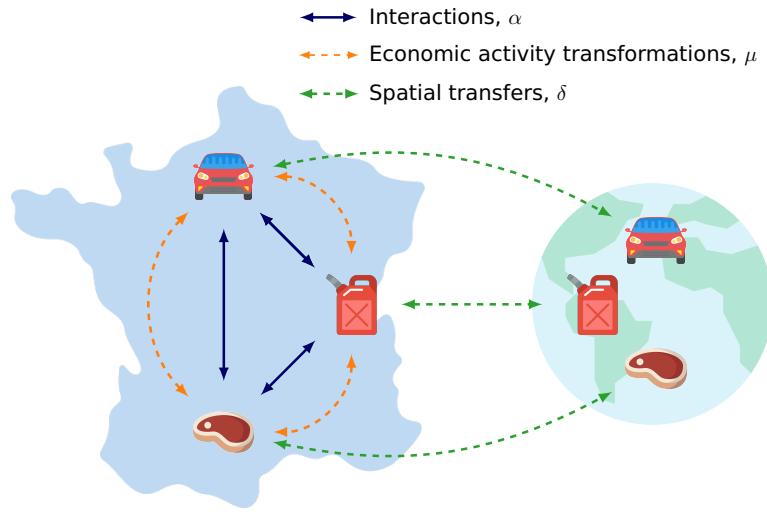


Fig. 4.1: General population dynamic model used to characterise the dynamics of economic activities, capturing ecological interactions, spatial transfers and economic activity transformations. For each country c , each one of the 9 economic activities considered is assumed to be driven by replication and self-limitation processes, characterised respectively by a growth rate $r_i^{(c)}$ and a self-limitation coefficient $b_i^{(c)}$, which values are assumed to be country-dependent and determined by the institutional context and the resources available (in terms of human capital, knowledge capital and natural resources). In addition, interactions between activities within each country are determined by $\alpha^{(c)}$ and can be mutualistic ($\alpha^{(c)} > 0$) or competitive ($\alpha^{(c)} < 0$). Due to spatial transfers, the capital of an economic sector can experience positive (resp. negative) fluxes of capital when it is lower (resp. higher) than the average global capital for the economic sector at time t , arising at rate $\delta^{(c)}$. Due to economic activity transformations, the capital of an economic sector can experience positive (resp. negative) fluxes of capital when the capital it is lower (resp. higher) than the capital of other economic sectors within the country at time t , arising at rate $\mu^{(c)}$. The parameters $r_i^{(c)}$, $b_i^{(c)}$, $\alpha^{(c)}$, $\delta^{(c)}$ and $\mu^{(c)}$ that best fit the empirical data are estimated in each country for each sub-model in ??????????, to investigate whether the proposed processes have a significant effect on the temporal evolution of the capital of economic activities. This diagram has been designed using resources from Flaticon.com.

4.2.2 Empirical data and model likelihood

We consider time series of global trade data as a proxy for the evolution of the capital of economic activities through time. Export data is synthetic of the capital of an economic activity within a country because the more it is exporting, the more it is competing in the global market, and therefore the larger it is in terms of capital (**Tacchella2018**). We specifically use the database compiled by the United Nations Statistical Division COMTRADE, categorized in the Standard International Trade Classification (SITC, revision 2) at the 1 digit level, covering 11 categories of economic activities in 249 countries from 1962 to 2020. Because of their low quality (**Hidalgo2021**), we discard the "Unspecified" and "Services" category, resulting in 9 economic categories to model (see ?? for details). As shown in the ??, this large number of data points contains the information necessary to recover the plausible generating processes. Export values for activity i in country c , denoted by $X_i^{(c)}(t)$, are discounted by the national population at time t , denoted by $P^{(c)}(t)$, in order to compare the economic variables across time and across countries. This leads to expressing the observation data for the capital of activity i in country c as $y_i^{(c)}(t) = X_i^{(c)}(t)/P^{(c)}(t)$. In each country, we only consider economic activities which have sustained a significant size relative to the world trends for at least 4 years (revealed comparative advantage > 1 , see (**Hidalgo2021**)). We assume that the observation data is contaminated by a white noise with log-normal distribution ϵ with zero mean and a variance-covariance matrix $\Sigma = \sigma^2 I$ where σ is the noise level, which is a reasonable error model for population dynamics (**Schartau2017**), so that $y_i^{(c)}(t) = n_i^{(c)}(t) \exp(\epsilon_i^{(c)}(t))$. This leads to expressing the likelihood of model \mathcal{M} in country c , denoted by $\mathcal{L}(\theta_{\mathcal{M}}^{(c)} | \mathbf{y}^{(c)}, \mathcal{M})$, as

$$\mathcal{L}(\theta_{\mathcal{M}}^{(c)} | \mathbf{y}^{(c)}, \mathcal{M}) = \prod_{j=1}^{T^{(c)}} \frac{1}{\sqrt{(2\pi)^{N^{(c)}} |\Sigma| y_j^{(c)}(t_j)}} \exp\left(-\frac{1}{2} [\tilde{d}^{(c)}(t_j)]^{T^{(c)}} \Sigma^{-1} [\tilde{d}^{(c)}(t_j)]\right) \quad (4.9)$$

where $\theta_{\mathcal{M}}^{(c)}$ refers to the parameters and the initial conditions for model \mathcal{M} , $\mathbf{y}^{(c)} = (y^{(c)}(t_1), \dots, y^{(c)}(t_T))$ designate the $T^{(c)}$ time points of economic data available for country c , $y^{(c)}(t) = (y_1(t), \dots, y_N^{(c)}(t))$ is the vector of economic activity capital at time t , and $\tilde{d}^{(c)}(t_j) = \ln(y^{(c)}(t_j)) - \ln(\mathcal{M}(t_j, \theta_{\mathcal{M}}^{(c)}))$. We detail in the next section how do we obtain the maximum likelihood estimate $\hat{\theta}_{\mathcal{M}}^{(c)}$ that maximises ?? given $\mathbf{y}^{(c)}$ for each model \mathcal{M} and country c , further used for selecting the most probable model given the data (**Burnham2002**). In the following, we drop the indices $^{(c)}$ for clarification.

4.2.3 Maximum likelihood estimation

In order to obtain the maximum likelihood of each model given the data, we employ the machine learning framework detailed (**Boussange2022a**), which is based on a variational optimisation method and a learning strategy coined "mini-batching". The likelihood of model \mathcal{M}_i is maximised by training \mathcal{M}_i against mini-batches of data comprising only $K < T^{(c)}$

data points from the full time series, where both the parameters and initial conditions are estimated. The mini-batching strategy ensures the convergence of the optimization towards the maximum likelihood estimate, provided that the choice of K is appropriate given the data and the model investigated (**Boussange2022a**). A large K might induce convergence towards a local minimum, while a low K might flatten the likelihood landscape, where all models would be provided equal support. In the controlled experiments in ??, we find that $K = 20$ (models are trained against batches of 20 years) ensures a good convergence, while reasonably discriminating the models (??). We therefore use $K = 20$ throughout all the experiments detailed in the manuscript, and discard countries where data is available for less than $K = 20$ years. The library **MiniBatchInference.jl** (**Boussange2022a**) is employed for the numerical implementation, with the gradient descent optimisers ADAM (**Kingma2014**) during the first 800 epochs to converge in the basin of attraction of the maximum likelihood estimate, which we substitute for the BroydenFletcherGoldfarbShanno optimizer (BFGS) (**fletcher2013practical**) for the final 800 training epochs to ensure faster and more accurate convergence. As a cross checking procedure, for each country and each model we perform 5 optimization runs with different initial parameters, where $r_i, b_i, \alpha, \mu, \delta$ are drawn from the uniform random distribution $\mathcal{U}_{[0.05, 0.15]}$, $\mathcal{U}_{[0.5, 1.5]}$, $\mathcal{U}_{[0.5, 1.5]}$, $\mathcal{U}_{[0.0005, 0.0015]}$ and $\mathcal{U}_{[0.0005, 0.0015]}$ respectively. We then take the best run among the 5 optimization runs, making sure that the likelihood estimates of each run are similar in value.

4.3 Results

4.3.1 Validation with synthetic data

We first investigate whether the inverse modelling technique of **Boussange2022a** together with the proposed sub-models in ?????????? can detect signatures of eco-evolutionary processes in a controlled experiment, using synthetic datasets. We proceed by generating multiple observation datasets from the models \mathcal{M}_{α^-} , \mathcal{M}_{α^+} , \mathcal{M}_δ and \mathcal{M}_μ with realistic r_i and b_i parameters and with different values for the parameters α , μ , and δ and with different values for the noise level σ . We consider \mathcal{M}_{null} , \mathcal{M}_{α^-} , \mathcal{M}_{α^+} , \mathcal{M}_δ and \mathcal{M}_μ as equally plausible candidate models for each generated datasets, and apply the maximum likelihood estimation method detailed in the ?? to obtain the maximum likelihood $\mathcal{L}(\hat{\theta}, \mathcal{M}|\mathbf{y})$ of each model for each generated dataset. We then use the Bayesian information criterion (BIC) to select the model with the strongest strength-of-evidence in relation to the data (**Mangan2017**), calculating the BIC for model \mathcal{M}_i as $BIC_{\mathcal{M}_i} = -2 \ln(\mathcal{L}(\hat{\theta}, \mathcal{M}_i|\mathbf{y})) + k_{\mathcal{M}} \ln(NT)$ where N is the number of activities in country c , T is the number of time points in the time series considered (NT therefore being the number of data points for country c) and $k_{\mathcal{M}}$ is the number of parameters in the model \mathcal{M} for country c . The BIC ranks the most probable models by penalizing complexity to balance information loss and parsimony, where candidate models with the lowest scores are ranked as the most likely (**Mangan2017**). We consider the relative BIC score $\Delta BIC_{\mathcal{M}_i}$, which allows a strength-of-evidence comparison across models and is calculated as $\Delta BIC_{\mathcal{M}_i} = BIC_{\mathcal{M}_i} - \min_j BIC_{\mathcal{M}_j}$. We expect the ΔBIC scores to only provide support for the true generating model when the process considered has a

significant effect on the observed dynamics. Under realistic observational noise ($\sigma = 0.2$), we find overall strong support for the true models (i.e., when \mathcal{M}_i is the true generating model, $\Delta\text{BIC}_{\mathcal{M}_i} = 0$ and $\Delta\text{BIC}_{\mathcal{M}_{j,j \neq i}} > 10$, see ??A-C) when the values of the parameters α , μ and δ exceed a certain threshold. Moreover, the more extreme the value of α , μ , and δ , the more support is given to the true generating model ??A-C. In contrast, when the signature of the underlying process on the dynamics is not sufficiently strong, the null model \mathcal{M}_{null} is given the most strength-of-evidence ($\Delta\text{BIC}_{\mathcal{M}_{null}} = 0$). These results hold for varying noise levels (see ?? for $\sigma = 0.3$), indicating that the models together with the maximum likelihood estimation method are well adapted to investigate the influence of eco-evolutionary processes with the dataset considered. Based upon this experiment and classical model selection criterion (Burnham2002), we decide to accept the hypothesis that the null model is the best model given the data when $\Delta\text{BIC}_{\mathcal{M}_{null}} \leq 10$. In addition, when $\text{BIC}_{\mathcal{M}_i} - \text{BIC}_{\mathcal{M}_{null}} < -10$ we conclude that \mathcal{M}_i is supported against \mathcal{M}_{null} , and we conclude that \mathcal{M}_i is the best model given the data if $\Delta\text{BIC}_{\mathcal{M}_i} = 0$ and if for all other model $\mathcal{M}_{j,j \neq i}$, we have that $\Delta\text{BIC}_{\mathcal{M}_{j,j \neq i}} > 10$. Overall, the proposed eco-evolutionary model together with the maximum likelihood estimation leads to high discrimination ability when combined with the BIC-based model selection procedure, and can provide strength-of-evidence for eco-evolutionary processes that may shape the dynamics of economic systems.

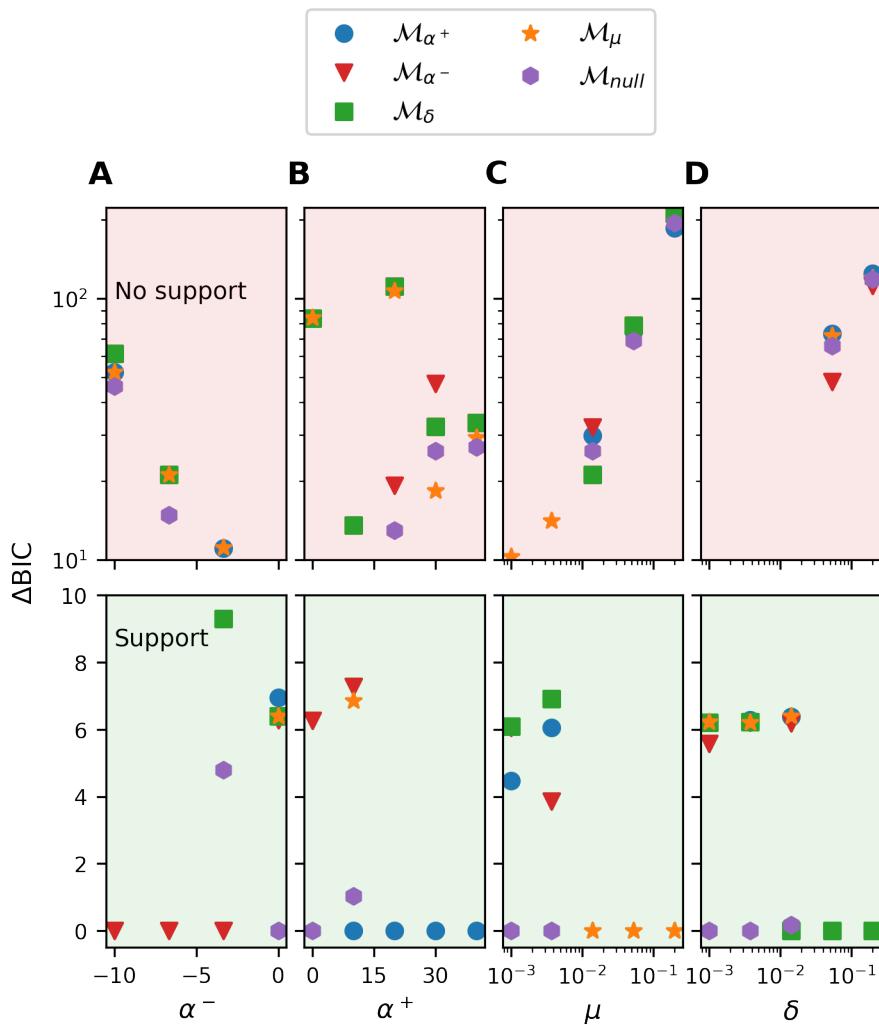


Fig. 4.2: Validation of the model selection procedure. In A and B we generate data with models M_{α^-} and M_{α^+} with varying α , in C we generate data with model M_{δ} with varying δ and in D we generate data with model M_{μ} with varying μ . We then contaminate the generated data with realistic noise ($\sigma = 0.2$), and seek for the maximum likelihood of all models M_{null} , M_{α^-} , M_{α^+} , M_{δ} and M_{μ} . The maximum likelihood of each model is then used to compute difference in Bayesian Information Criterion for each model, ΔBIC , which comparison provides a strength-of-evidence for each model. A-D show that the model selection procedure is valid, since only the true models are placed in the "Strong support" category when no other model is given support (equivalently, when all other model are placed in the "No support" category).

4.3.2 Discrepancies in dynamical regime across countries

We apply the maximum likelihood estimation method detailed in the ?? to obtain the maximum likelihood of each model for the world top 100 countries with the highest GDP per capita, as of 2020 (see list in ??). We first investigate the quality of best fit, and further detail the results for each model in the following section. In the following, we refer to countries by their ISO 3166-1 reference code, which associated plain names are given in ?. Details of the numerical simulations are provided in ?. Over the 100 countries investigated, only 78 countries have sufficiently long time series to be included in the analyses (countries with at least 20 years of data, see ??), while 77 countries show consistent maximum likelihood estimates (best model coefficient of determination $R^2 > 0$). Among the 77 countries, we find that the best models have a good fit to the data, and capture well the long term growth of economic activities (median explained variance $R^2 = 0.940$, see ?? and ??????? for graphical illustrations of the fit). While this demonstrates the relevance of the proposed models, we observe discrepancies across countries in how well the best models perform (standard deviation of explained variance $\text{std}(R^2) = 0.074$, see ??D for a graphical illustration of the variance in terms of the model loglikelihoods). This discrepancy is explained by the number of data points available in each country, as it differs among countries and since more data points yield a higher model loglikelihood ($\beta = 0.928 \pm 0.046$, $p < 0.001$, see ??). We additionally observe a positive effect of the GDP on the residuals ($\beta = 0.693 \pm 0.187$, $p < 0.001$, see ?? and ??D), indicating that rich economies have a more deterministic behavior than poorer ones. The residuals are further negatively related to the number of economic activities ($\beta = -0.230 \pm 0.071$, $p < 0.01$, see ??), indicating the models investigated might be less appropriate for qualifying the dynamics of diversified economies.

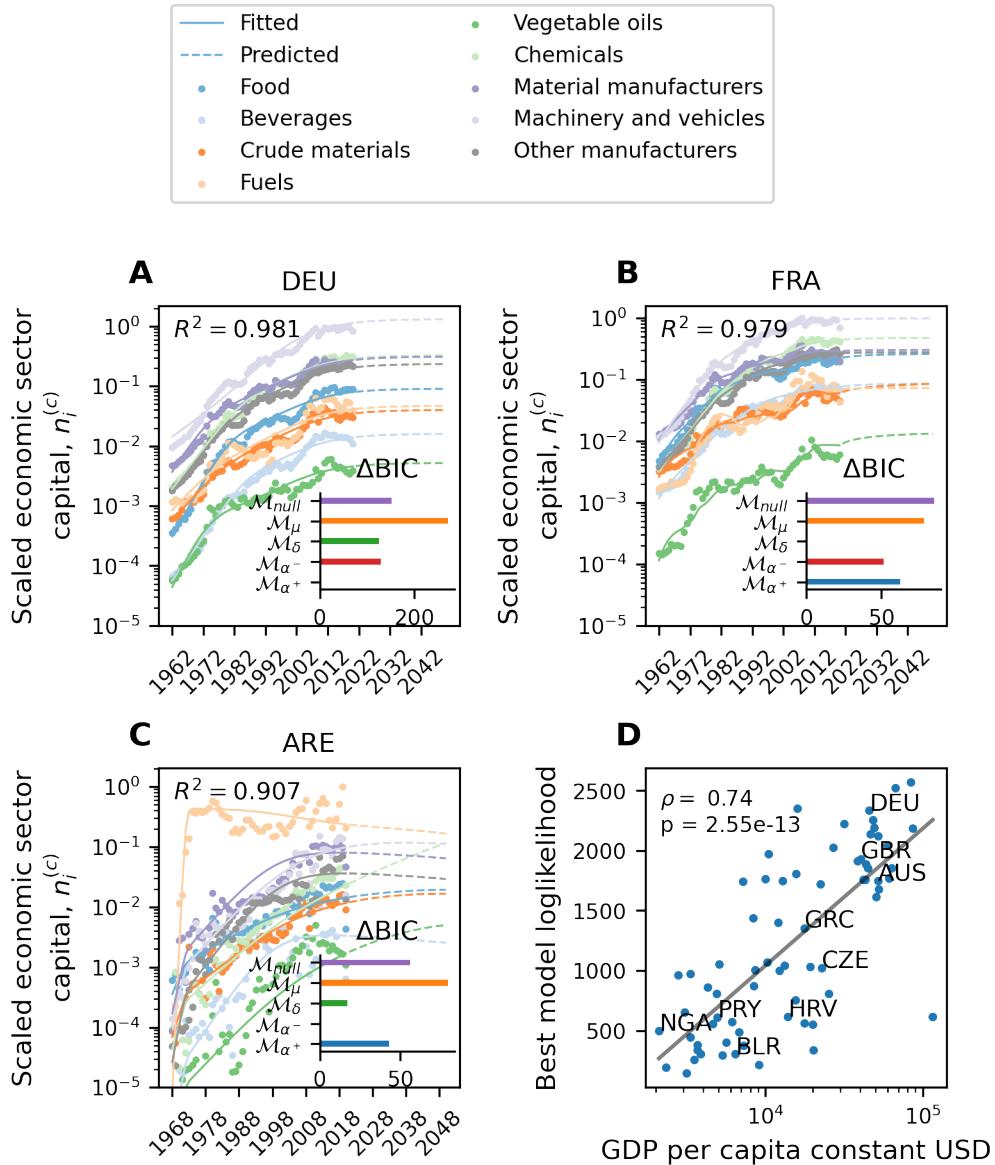


Fig. 4.3: Best model fits and associated predictions for DEU, FRA and ARE, and effect of GDP on best model loglikelihoods. A-C Best model fits and associated predictions for DEU, FRA and ARE. Scatter dots represent the empirical data, plain lines correspond to the best model fits, and dotted lines correspond to the best model predictions. The inset plots provide the ΔBIC for each country. D Effect of GDP on model loglikelihood. The plain line corresponds to a linear fit.

4.3.3 Evidence for eco-evolutionary processes

Applying the model selection procedure detailed in the controlled experiment, we investigate in details the effect of eco-evolutionary processes on the dynamics of the countries investigated. We find that the null model is rejected in 45 of the 77 countries, in favor of alternative models (??B). Among the alternative models, \mathcal{M}_{α^+} , capturing positive interactions between economic activities, is the most frequently supported (??B,C), and is given considerably more support against \mathcal{M}_{null} than \mathcal{M}_{α^-} and \mathcal{M}_μ (??A,D). The model with spatial transfers, \mathcal{M}_δ , is the second most supported (??B,C) and is given the most support against \mathcal{M}_{null} (??A,D). The models capturing negative interactions, \mathcal{M}_{α^-} , and capturing economic activity transformations, \mathcal{M}_μ , are the least supported and given the least strength-of-evidence (??B,D), but are still ranked as most supported models in some countries (??A).

In contrast to \mathcal{M}_{α^-} , \mathcal{M}_δ , \mathcal{M}_μ and \mathcal{M}_{α^+} , \mathcal{M}_δ , \mathcal{M}_μ , which implement complementary processes, \mathcal{M}_{α^-} and \mathcal{M}_{α^+} are structurally antagonistic, but are nonetheless equally supported in some countries (see, e.g., FIN and FRA in ??A). This ambiguity highlights that the success of each model arises from their ability to capture different features of the empirical data. For instance, \mathcal{M}_{α^-} is the only model capturing capital decrease (see, e.g., ARE in ???? or very low capital growth (see, e.g., HRV in ??)). \mathcal{M}_{α^+} captures well periods of exponential economic growth followed by a growth slow down, and correlation patterns in growth between activities (see, e.g., DEU and HUN in ????). We find that this dynamics is associated with an increase in the self limitation coefficients b_i , that compensates the beneficial effect of positive interactions (??B). In contrast, the model \mathcal{M}_{null} tends to overestimate the capital size for such dynamics. \mathcal{M}_δ is the only model reproducing the oscillations observed in the empirical data (see e.g. FRA in ?? and BEL, GBR in ??), where oscillations arise from the term $\overline{n^{(c)}}$ through mismatches between the local capital size and the global capital size of a given economic activity. \mathcal{M}_δ tends to overestimate the growth rate coefficients r_i against other models (??A), due to a source-sink dynamics. Indeed, the dispersal term acts as capital sink for most economic activities in most countries, where only a handful of under-developed activities, relative to global trends in the world economy, benefit from source dynamics (see, e.g., KOR in ??C). Characteristic trends of \mathcal{M}_μ are difficult to observe given the low number of cases where it is given statistical support.

Overall, each model is supported against \mathcal{M}_{null} in some countries, suggesting that interactions between activities, spatial transfers, and economic activity transformations are plausible processes that may considerably affect the development of economic systems. Nonetheless, positive interactions and spatial transfers stand out as the most supported processes, both in term of the number of countries where they are given support, and in term of the strength-of-evidence against \mathcal{M}_{null} .

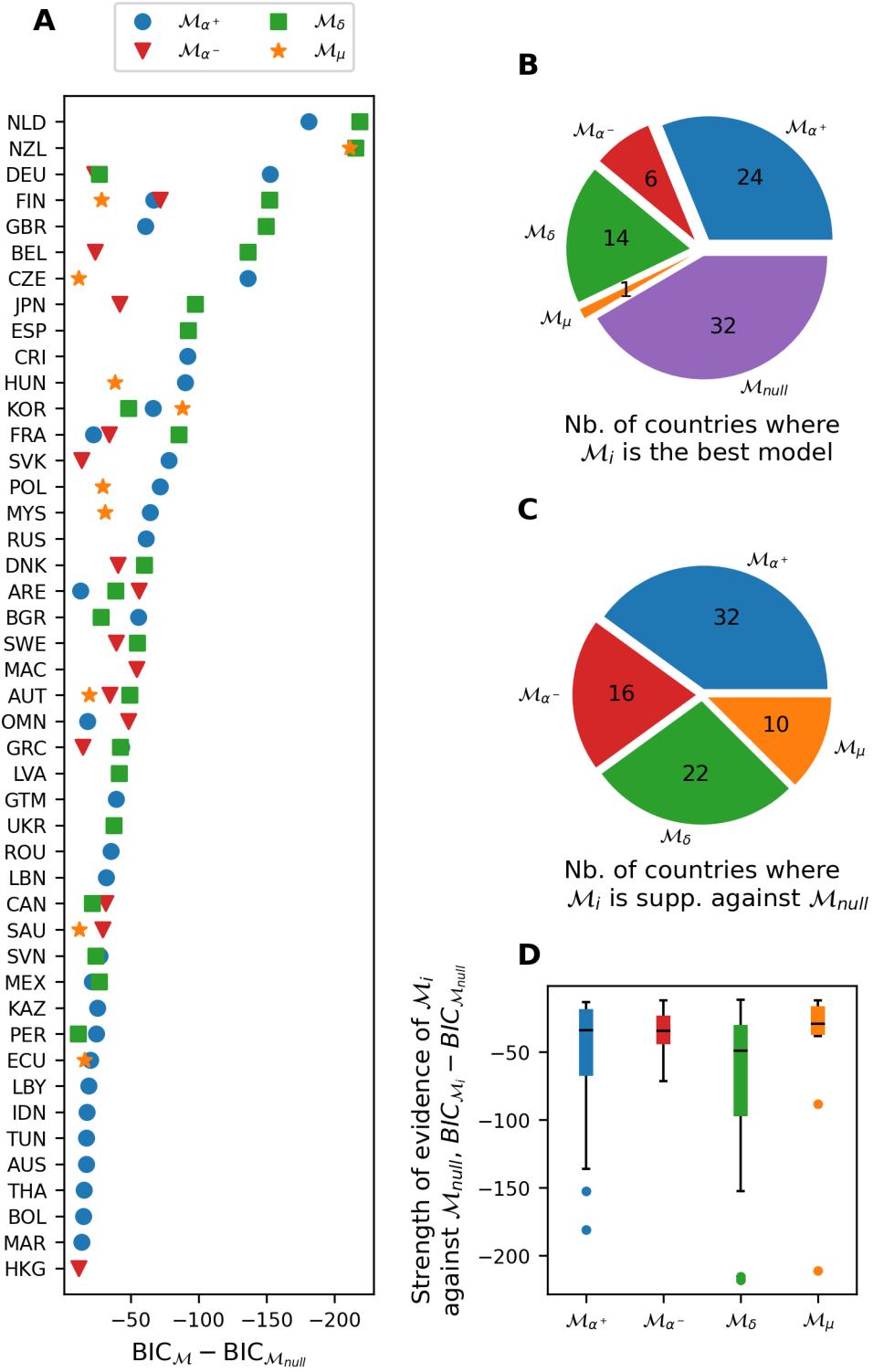


Fig. 4.4: Statistical support based on Bayesian Information Criterion for the population dynamic models \mathcal{M}_{α^-} , \mathcal{M}_{α^+} , \mathcal{M}_δ and \mathcal{M}_μ , implementing eco-evolutionary processes. A Strength-of-evidence comparison for \mathcal{M}_{α^-} , \mathcal{M}_{α^+} , \mathcal{M}_δ and \mathcal{M}_μ for each country, for countries where $\mathcal{M}_{\text{null}}$ is rejected, sorted by maximum strength-of-evidence. B Number of countries where models $\mathcal{M}_{\text{null}}$, \mathcal{M}_{α^+} , \mathcal{M}_{α^-} , \mathcal{M}_δ , and \mathcal{M}_μ are ranked as best models. C Number of countries where models \mathcal{M}_{α^-} , \mathcal{M}_{α^+} , \mathcal{M}_δ and \mathcal{M}_μ are accepted against $\mathcal{M}_{\text{null}}$. D Global strength-of-evidence comparison for \mathcal{M}_{α^-} , \mathcal{M}_{α^+} , \mathcal{M}_δ , and \mathcal{M}_μ against $\mathcal{M}_{\text{null}}$. A-D show that \mathcal{M}_{α^+} and \mathcal{M}_δ stand out as the most supported models, both in term of number of countries where they are given support, and in term of strength-of-evidence.

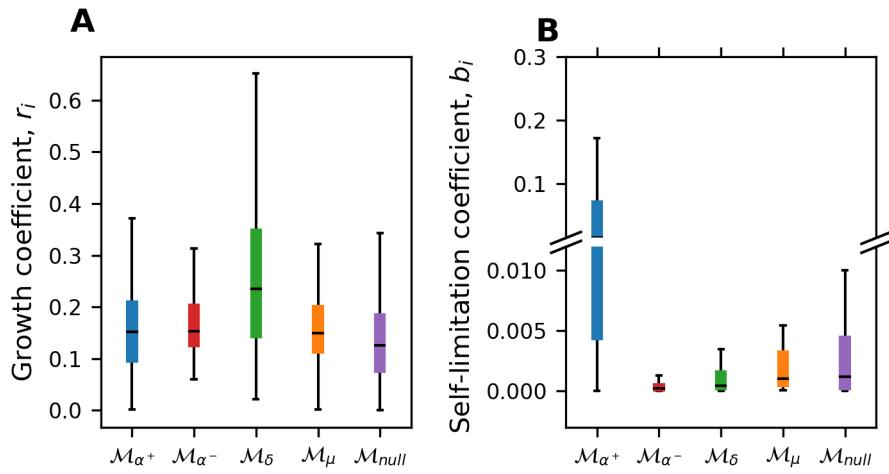


Fig. 4.5: Differences in growth rates r_i and coefficients of self-limitation b_i between M_{α^-} , M_{α^+} , M_δ , M_μ , and M_{null} . In A-B, we aggregate the parameters r_i and b_i across all countries and all economic activities. A shows that M_δ overestimates r_i against the other models. This is because for dispersal rate $\delta > 0$, most economic activities experience a negative capital flux, and consequently need to grow more to compensate for the losses. B shows that M_{α^+} tends to overestimate b_i relative to the other models, which negative effect on growth dynamics compensates the beneficial effect of positive interactions.

4.4 Discussion

Our results indicate that dynamic models embedding eco-evolutionary processes, commonly used to characterise the dynamics of biological populations (**Bunin2017; Scheffer2006a; Case1990; Tilman1994b; eigen1988molecular**), can appropriately characterize the dynamics of economic activities (??). Together with state of the art computational methods and 59 years of economic data, the models provide a framework to assess the effect of eco-evolutionary processes on the long-term development of national economies. Over the 77 countries investigated, the null model – embedding replication and self limitation processes only – was rejected 45 times in favor of alternative eco-evolutionary models (??B), and each of the alternative model were supported against the null model in some countries. Hence, eco-evolutionary models, that couple the individual dynamics of economic activities, provide a better fit to the economic dynamics at the country level, suggesting that economic development carries the effect of the inter-dependency between economic activities. The models capturing positive interactions between economic activities ($\mathcal{M}_{\alpha+}$) and spatial transfers (\mathcal{M}_δ) were overall given more strength-of-evidence, providing a quantitative support that these processes are central in the development of economic systems. Together, our study suggests that dynamic models inspired from biology are well suited to characterise economic dynamics.

Economic activities sustain the development of each other (??B-D), suggesting that economic diversity promotes economic development (**Saviotti2020**). In our models, the model with positive interactions between economic activities was the most frequently supported model (??B-C), outlining that positive interactions may be ubiquitous in economic systems. Maximum likelihood estimates for $\mathcal{M}_{\alpha+}$ attribute the growth of economic activities to positive interactions rather than to their endogenous growth, since $\mathcal{M}_{\alpha+}$ is associated with high self-limitation coefficients (??B). Positive interactions may arise from a variety of sub-processes acting upon firms, including business interactions through supply chains **Ozman2009; Saavedra2009a** and agglomeration externalities **VanDerPanne2004**. Our results therefore suggest that their effect scales up and at the national level and result in economic activities sustaining each other through positive feedback.

The dynamics of economic activities seems also to be greatly affected by spatial transfers (??B,D), which may arise from transfers of knowledge and organisational routines between countries **Hodgson2019** through different underlying economic processes (**Zahra2000; Zahra2000; RogersEverettM2003DoI; Boschma2008**). Our results show a strong support in favor of spatial transfers (??D), and economic processes such as business international expansions (**Zahra2000**), labour mobility (**Zahra2000**), and the diffusion of innovations (**Boschma2008**) may underly this support. However, \mathcal{M}_δ is less frequently supported against the null model than model $\mathcal{M}_{\alpha+}$ (??C). This may reflect that some countries are overall more akin to the underlying spatial economic processes, possibly due to varying cognitive, organizational, social, institutional and geographic proximities with leading economies (**Boschma2005**). Hence, spatial processes involved in the transfer of knowledge

and organizational routines have an important effect on economic change at the national level.

The strong discrepancies in model supports observed across countries indicate strong idiosyncratic processes, that can relate to the countries' institutional context (**Acemoglu2005**) or historical contingencies (**Silverberg2005a**). We found that developing economies have a distinct, less predictable dynamical behavior, as the likelihood of the best models showed a negative relationship with country GDP (??D and ??). In line with our findings, (**Cristelli2015**) also reports a difference in dynamical regime between mature and developing economies, where the GDP dynamics of developing economies is found less predictable than that of rich countries. Those lower model fits indicate that singular processes may importantly influence the dynamics of developing countries. Also, while we found a statistical advantage for \mathcal{M}_{α^+} and \mathcal{M}_δ over the alternative models (??B-D), the support and strength-of-evidence for each model strongly varied across countries (??A), without emergent pattern relating the strength-of-evidence to macroeconomic characteristics such as, e.g., country GDP per capita. This idiosyncrasy may be due to the oversimplicity of the model investigated: for instance, while \mathcal{M}_{α^+} considers equal strength of positive interactions between pairs of economic sectors (??), it could be that interaction strength is pair specific, implying that the likelihood for the positive interaction hypothesis is underestimated in countries with many economic activities. An other explanation is that this discrepancy is inherent to the idiosyncrasy of country histories and characteristics. Together, the discrepancies in model fits and model supports invite to refine the details of the processes investigated, in order to better link them with macroeconomic patterns.

For the sake of parsimony, the population dynamic models proposed in ?????????? rely on a set of assumptions, which could be further relaxed (??) to study eco-evolutionary processes acting upon economic activities in more details. The models investigated rely on the mean field assumptions that economic activities interact with equal strengths, and proceed to spatial transfers across countries and capital transfer between each other with equal rates. Nonetheless, biological organisms interact through interaction networks (**Bascompte2003**), and structured link have been also documented in economic networks (**C.A.HidalgoB.Klinger; Bustos2012; Saavedra2009a; Schweitzer2009; Giuliani2007**). For instance, **C.A.HidalgoB.Klinger** suggests that economic activities are related to one another through a network of relatedness, which could be integrated in \mathcal{M}_{α^-} , \mathcal{M}_{α^+} and \mathcal{M}_μ , weighting the interactions and transfers by the relatedness of two activities. Similarly, spatial transfers are likely to depend on proximity metrics between countries (**Boschma2005**). Model developments will be required to assess the importance of economic activity relatedness and country proximity on economic dynamics. Moreover, the strength and directionality of interactions, spatial transfers and economic activity transformations may temporally change throughout the development of economic activities (**Pistorius1997**), while we assumed that the parameters are constant through time. Exploring more complex eco-evolutionary models will demand, however, richer time series, in order to extract the extra information needed to constrain additional parameters.

In conclusion, our results foster a biologically inspired approach to understand the mechanisms shaping the endogenous dynamics of economic systems. Using population dynamic models and temporal economic data, our study demonstrates that positive interactions

between economic activities and spatial transfers considerably influence the dynamics of economic activities at the national, and may be fundamental drivers of economic change. The quantitative paradigm to study economic systems has mainly relied on models inspired from mechanics, which assume a world in equilibrium (**sornette2014physics**). Eco-evolutionary biology, focusing on nonlinear dynamical processes and emergence, seems to be a more appropriate paradigm to study the fundamental forces shaping the dynamics of economic systems. It can provide quantitative models to test fundamental hypotheses proposed in the field of evolutionary economics and geography economics, and suggest new ones. We call this promising research venue "econobiology".

4.5 Code availability

The simulation code is available on request.

4.6 Acknowledgements

L.P. and V.B. were supported by the SNF grant 310030E_205556.

4.A Supplementary Figures

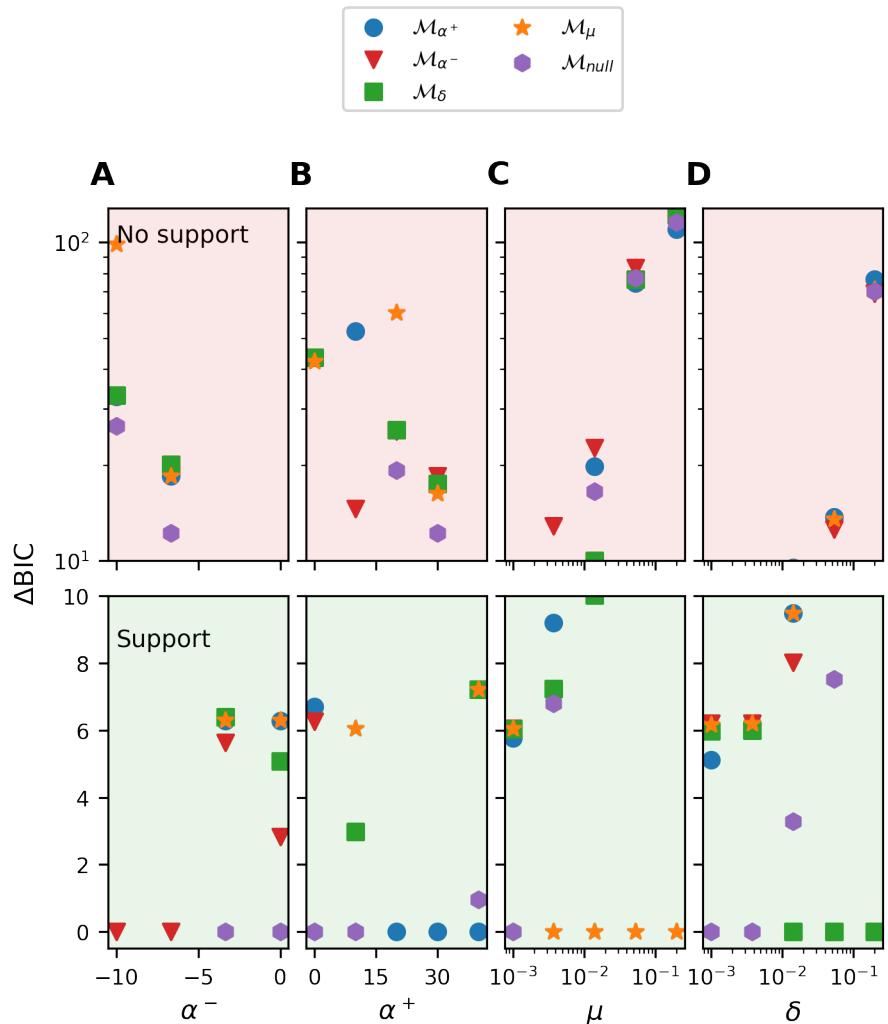


Fig. S4.6: Validation of the model selection procedure. The figure is analogous to ?? but with $\sigma = 0.3$.

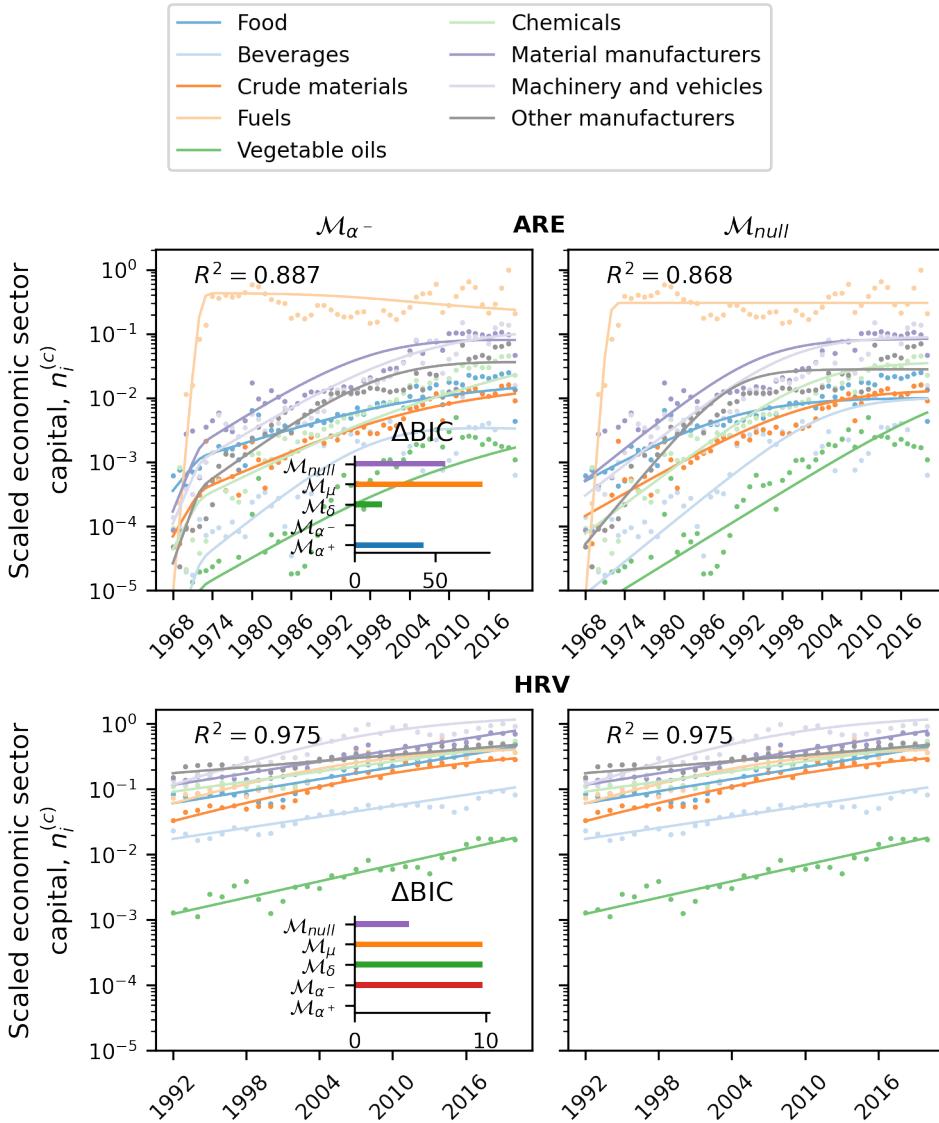


Fig. S4.7: Fits of model \mathcal{M}_{α^-} for ARE and HRV, where \mathcal{M}_{α^-} is given support against \mathcal{M}_{null} .

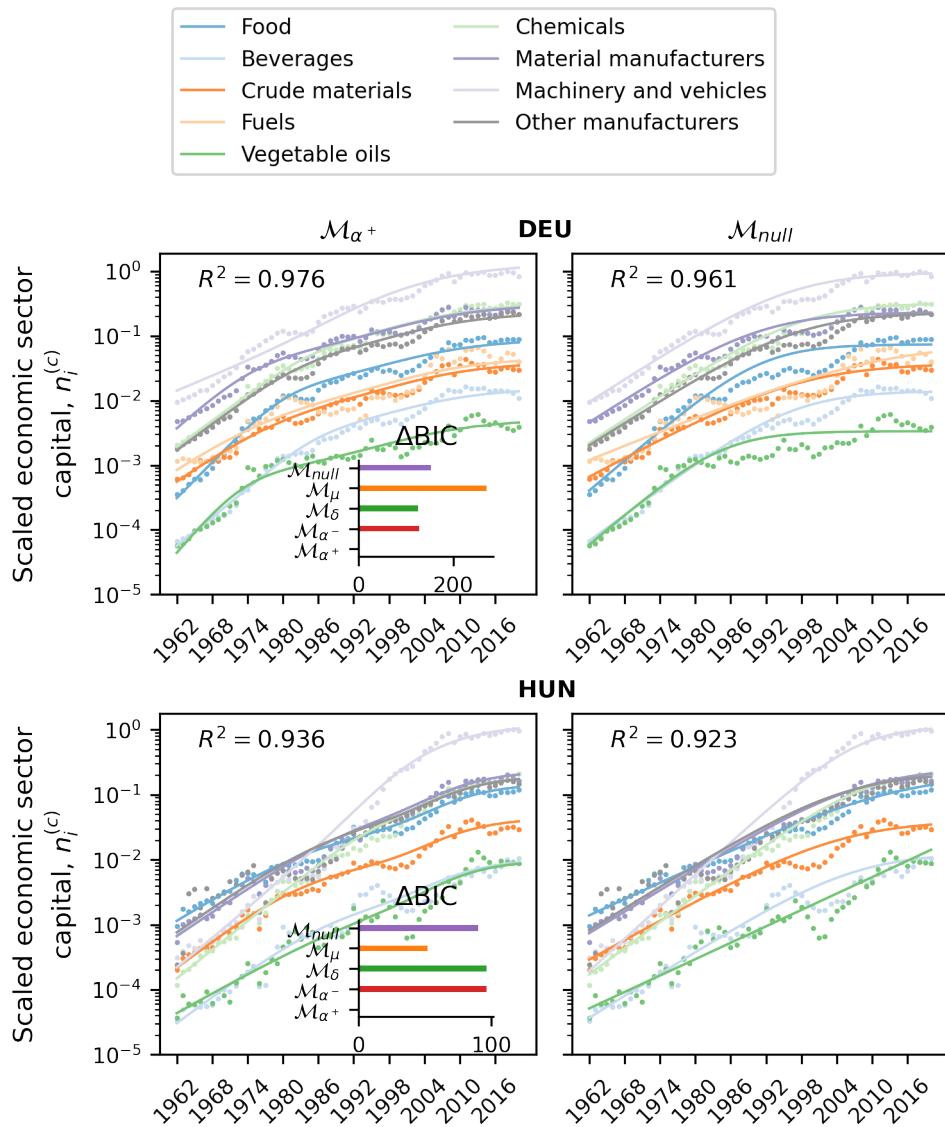


Fig. S4.8: Fits of model \mathcal{M}_{α^+} for DEU and HUN, where \mathcal{M}_{α^+} is given support against $\mathcal{M}_{\text{null}}$.

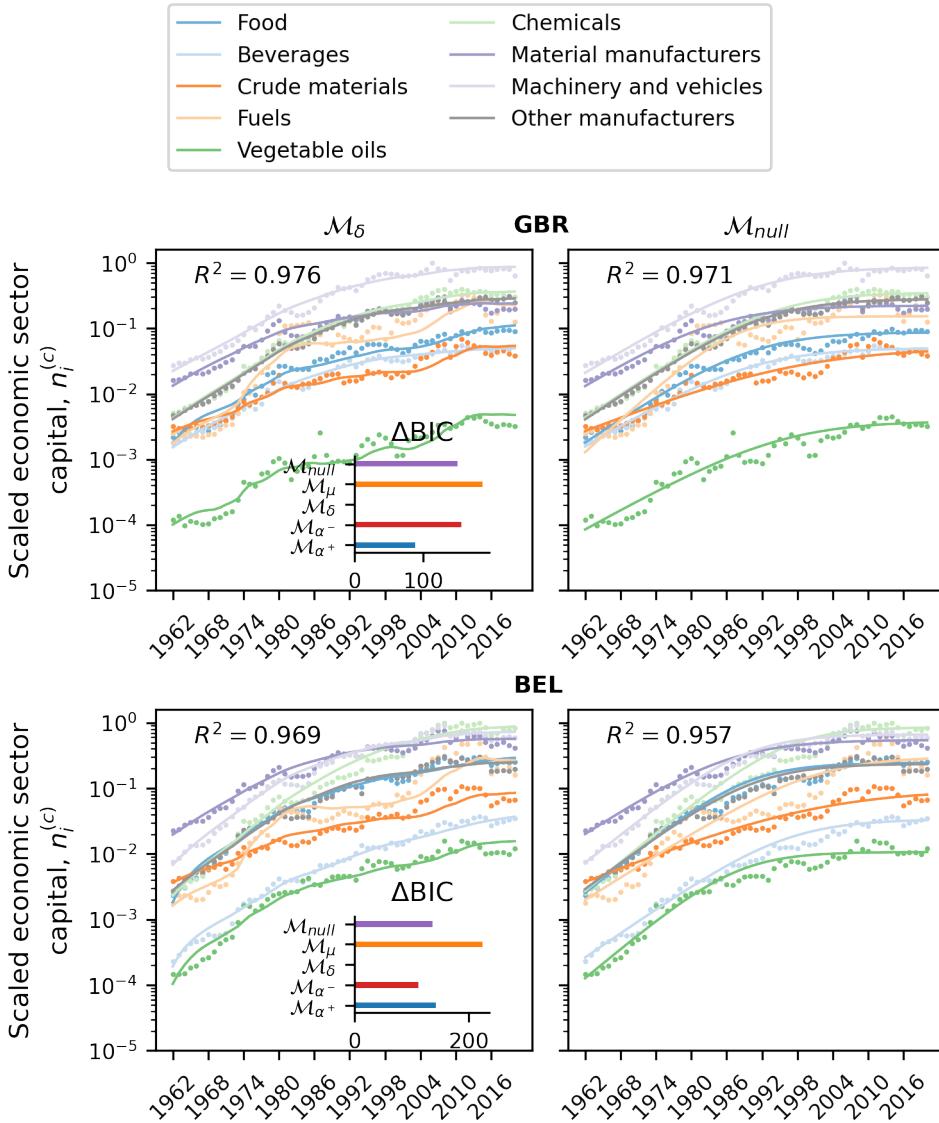


Fig. S4.9: Fits of model \mathcal{M}_δ for GBR and BEL, where \mathcal{M}_δ is given support against \mathcal{M}_{null} .

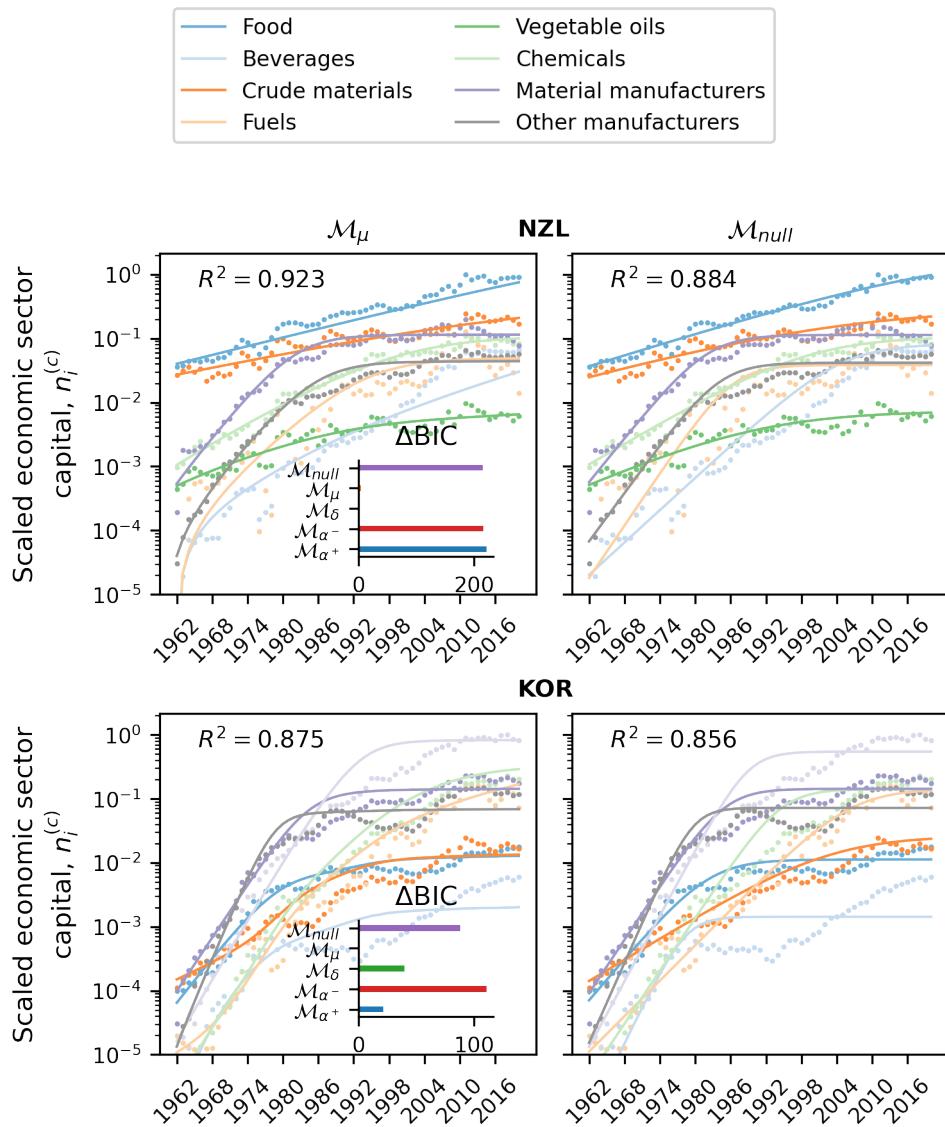


Fig. S4.10: Fits of model \mathcal{M}_{μ} for NZL and KOR, where \mathcal{M}_{μ} is given support against \mathcal{M}_{μ}^{null} .

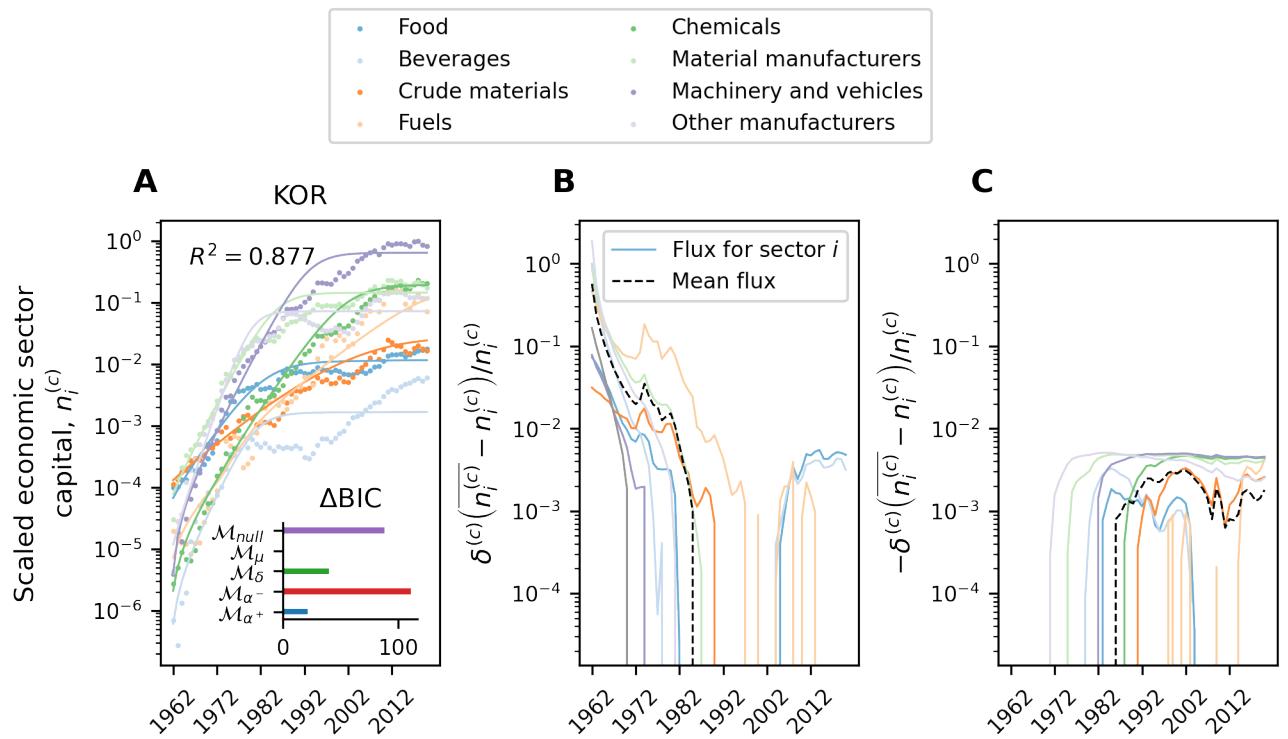


Fig. S4.11: Contributions of fluxes due to spatial transfers on growth dynamics. A shows the growth dynamics for KOR, together with the best model fit. B-C show the positive and negative fluxes, respectively, due to spatial transfers for each economic activity, relative to the capital size of economic activity. The contributions of spatial transfers to growth decreases over time.

4.B Supplementary Tables

Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
ARE	\mathcal{M}_{α^+}	1.2527837	0.0443378	0.8985771	2194.9947804	-331.4965062
ARE	\mathcal{M}_{α^-}	1.1323544	0.0001987	0.9070505	2209.3385519	-374.6811568
ARE	\mathcal{M}_δ	1.1547997	0.0018924	0.9037398	2204.4213038	-357.3571273
ARE	\mathcal{M}_μ	1.3237895	0.0096762	0.8908284	2175.1338748	-295.0532417
ARE	\mathcal{M}_{null}	1.3210473	0.0185026	0.8945233	2184.4961044	-318.3014521
ARG	\mathcal{M}_{α^+}	0.5362716	0.0000000	0.9126885	873.2783542	-479.0901860
ARG	\mathcal{M}_{α^-}	0.5260177	0.0000000	0.9148312	877.0708569	-486.6684522
ARG	\mathcal{M}_δ	0.5347459	0.0882225	0.9127092	873.5096285	-479.1622984
ARG	\mathcal{M}_μ	0.5362716	0.0000000	0.9126884	873.2790114	-479.0898098
ARG	\mathcal{M}_{null}	0.5362716	0.0000000	0.9126885	873.2797678	-484.8104788
AUS	\mathcal{M}_{α^+}	0.2493686	0.0259488	0.9701803	1746.6372258	-1152.0686387
AUS	\mathcal{M}_{α^-}	0.2639190	0.0000105	0.9687099	1734.9134560	-1128.5804718
AUS	\mathcal{M}_δ	0.3009456	0.0013734	0.9620972	1683.0297057	-1035.0200199
AUS	\mathcal{M}_μ	0.2639179	0.0000000	0.9687119	1734.9252830	-1128.6123545
AUS	\mathcal{M}_{null}	0.2639178	0.0000001	0.9687114	1734.9245624	-1134.7946683
AUT	\mathcal{M}_{α^+}	0.2068390	0.0001061	0.9818722	2225.2530696	-1414.8850719
AUT	\mathcal{M}_{α^-}	0.1927354	0.0000609	0.9831719	2245.3364190	-1455.7289956
AUT	\mathcal{M}_δ	0.1858071	0.0000049	0.9836237	2253.3349649	-1470.6709565
AUT	\mathcal{M}_μ	0.1967636	0.0000009	0.9827050	2238.1320372	-1440.7057723
AUT	\mathcal{M}_{null}	0.2068390	0.0000001	0.9818725	2225.2566207	-1421.2047025
BEL	\mathcal{M}_{α^+}	0.1723450	0.0000000	0.9732590	1784.0358726	-1495.7959922
BEL	\mathcal{M}_{α^-}	0.1648341	0.0000001	0.9746902	1798.8747676	-1525.9943707
BEL	\mathcal{M}_δ	0.1324942	0.2790846	0.9793798	1854.1829608	-1638.4943651
BEL	\mathcal{M}_μ	0.1968834	0.0000000	0.9689876	1743.5335080	-1414.4390435
BEL	\mathcal{M}_{null}	0.1723450	0.0000000	0.9732588	1784.0363416	-1502.0995394
BGR	\mathcal{M}_{α^+}	0.7781416	0.0479922	0.8288290	1764.4641596	-649.6618052
BGR	\mathcal{M}_{α^-}	0.8838256	0.0000000	0.8083396	1732.3309835	-587.5907358
BGR	\mathcal{M}_δ	0.8266382	0.0009712	0.8200675	1750.0324119	-622.2562137
BGR	\mathcal{M}_μ	0.8597119	0.0000000	0.8124862	1737.9191192	-599.5986287
BGR	\mathcal{M}_{null}	0.8838256	0.0000922	0.8083396	1732.3319709	-593.8988785
BHR	\mathcal{M}_{α^+}	1.6567106	0.0171776	0.8702070	1089.2193585	55.6414398
BHR	\mathcal{M}_{α^-}	1.5961899	0.0523684	0.8790107	1092.1370550	33.5162361
BHR	\mathcal{M}_δ	1.6725460	0.0380687	0.8704729	1089.9680617	54.9955669
BHR	\mathcal{M}_μ	1.6588562	0.0042337	0.8710776	1090.6511513	53.5213442
BHR	\mathcal{M}_{null}	1.6091897	0.0288135	0.8782411	1100.0620639	29.7611037
BLR	\mathcal{M}_{α^+}	0.2314568	0.0000000	0.7546141	302.7120092	-265.0157361
BLR	\mathcal{M}_{α^-}	0.2310920	0.0000000	0.7559656	303.1611263	-265.8441406
BLR	\mathcal{M}_δ	0.2312815	0.0000001	0.7550474	302.8040002	-265.2808168

	Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
	BLR	\mathcal{M}_μ	0.2314569	0.0000000	0.7546081	302.7103969	-265.0120688
	BLR	\mathcal{M}_{null}	0.2314568	0.0000000	0.7546151	302.7104437	-270.0269676
	BOL	\mathcal{M}_{α^+}	0.3539741	0.5305266	0.9268044	146.3653765	-107.0340735
	BOL	\mathcal{M}_{α^-}	0.4496981	0.0000247	0.9048095	135.9258779	-87.5909551
	BOL	\mathcal{M}_δ	0.4209790	0.0000005	0.9104235	138.0751496	-92.0891975
	BOL	\mathcal{M}_μ	0.4205466	0.0000066	0.9104306	138.0818576	-92.0950317
	BOL	\mathcal{M}_{null}	0.4496985	0.0000003	0.9048066	135.9237089	-91.8927180
	BRA	\mathcal{M}_{α^+}	0.3172497	0.0000001	0.9440110	488.0802898	-388.2378022
	BRA	\mathcal{M}_{α^-}	0.3249407	0.0000000	0.9424996	485.4392675	-383.3634446
	BRA	\mathcal{M}_δ	0.3249407	0.0058828	0.9424995	485.4379807	-383.3630892
	BRA	\mathcal{M}_μ	0.3206030	0.0000000	0.9433234	486.8511429	-386.0040451
	BRA	\mathcal{M}_{null}	0.3249407	0.0000000	0.9424990	485.4350185	-388.5710346
	CAN	\mathcal{M}_{α^+}	0.1420790	0.0000000	0.9813800	1740.6230285	-1592.8111504
	CAN	\mathcal{M}_{α^-}	0.1348187	0.0000000	0.9826218	1759.3484402	-1630.7054690
	CAN	\mathcal{M}_δ	0.1345303	0.0002367	0.9823122	1755.3549921	-1621.0102338
	CAN	\mathcal{M}_μ	0.1420790	0.0000000	0.9813802	1740.6331982	-1592.8186471
	CAN	\mathcal{M}_{null}	0.1420790	0.0000000	0.9813800	1740.6275036	-1599.1212257
	CHE	\mathcal{M}_{α^+}	0.2379502	0.0002883	0.9792924	2188.2874195	-1290.4488236
	CHE	\mathcal{M}_{α^-}	0.2381246	0.0725138	0.9792943	2188.2300435	-1290.5016663
	CHE	\mathcal{M}_δ	0.2686683	0.0343412	0.9764831	2154.3432229	-1220.6077504
	CHE	\mathcal{M}_μ	0.2331059	0.0096259	0.9796038	2192.4685594	-1298.7697313
	CHE	\mathcal{M}_{null}	0.2381246	0.1101355	0.9792943	2188.2241859	-1296.8080908
	CHL	\mathcal{M}_{α^+}	0.5874845	0.0004616	0.9580845	1049.6437073	-446.2645519
	CHL	\mathcal{M}_{α^-}	0.6179149	0.0000022	0.9565743	1043.9289736	-435.4684853
	CHL	\mathcal{M}_δ	0.6179150	0.0000543	0.9565743	1043.9278574	-435.4687821
	CHL	\mathcal{M}_μ	0.6179149	0.0000583	0.9565745	1043.9242279	-435.4695969
	CHL	\mathcal{M}_{null}	0.6179149	0.0000344	0.9565743	1043.9283281	-441.1885246
	COL	\mathcal{M}_{α^+}	0.4490345	0.0000000	0.7782291	295.7157945	-221.5722013
	COL	\mathcal{M}_{α^-}	0.4690859	0.0000000	0.7690281	293.0967240	-216.6127614
	COL	\mathcal{M}_δ	0.4690859	0.0000000	0.7690293	293.0990791	-216.6134084
	COL	\mathcal{M}_μ	0.4690859	0.0000000	0.7690286	293.0969254	-216.6130390
	COL	\mathcal{M}_{null}	0.4690859	0.0000000	0.7690282	293.0970732	-221.4168595
	CRI	\mathcal{M}_{α^+}	2.1618680	0.1028749	0.8450607	1399.4320760	-59.4886382
	CRI	\mathcal{M}_{α^-}	3.0070923	0.0161348	0.7967607	1355.2447368	38.1976757
	CRI	\mathcal{M}_δ	3.2498039	0.0035771	0.7495929	1312.1428722	113.3310469
	CRI	\mathcal{M}_μ	2.6857576	0.0535064	0.7957142	1349.6977712	40.0465823
	CRI	\mathcal{M}_{null}	3.0070923	0.0000000	0.7967602	1355.2396475	32.3123594
	CUB	\mathcal{M}_{α^+}	0.9908768	0.0000001	0.7215276	478.2930704	-171.2727897
	CUB	\mathcal{M}_{α^-}	0.8902225	0.0000001	0.7426569	485.8364570	-185.7130367
	CUB	\mathcal{M}_δ	0.9908767	0.1727308	0.7215307	478.2948060	-171.2747793
	CUB	\mathcal{M}_μ	0.9585458	0.0000020	0.7316088	481.3950009	-178.0205603
	CUB	\mathcal{M}_{null}	0.9908762	0.0000003	0.7215403	478.3330403	-176.4906120
	CZE	\mathcal{M}_{α^+}	0.0580083	0.0054248	0.9869966	1018.8726966	-778.2751708

Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
CZE	\mathcal{M}_{α^-}	0.0904964	0.0069555	0.9776120	947.2785419	-636.4698424
CZE	\mathcal{M}_δ	0.0898238	0.0017040	0.9775473	945.9773418	-635.7157255
CZE	\mathcal{M}_μ	0.0886011	0.0000000	0.9790624	955.8358104	-653.9506041
CZE	\mathcal{M}_{null}	0.0904964	0.0000000	0.9776120	947.2785787	-642.0340782
DEU	\mathcal{M}_{α^+}	0.1477448	0.2011817	0.9809597	2336.7255710	-1555.4873746
DEU	\mathcal{M}_{α^-}	0.1947722	0.0000414	0.9759068	2272.0947772	-1426.2671483
DEU	\mathcal{M}_δ	0.1853032	0.0018003	0.9760610	2275.9987490	-1429.7922153
DEU	\mathcal{M}_μ	0.2447523	0.1317466	0.9688203	2202.4488643	-1284.7143596
DEU	\mathcal{M}_{null}	0.2020155	0.0000000	0.9745613	2257.6743661	-1402.7410593
DNK	\mathcal{M}_{α^+}	0.1199929	0.0000000	0.9750129	1737.6862271	-1681.0788644
DNK	\mathcal{M}_{α^-}	0.1114577	0.0012059	0.9770632	1761.0406979	-1728.0820104
DNK	\mathcal{M}_δ	0.1067551	0.0075976	0.9778510	1770.9719428	-1747.2704361
DNK	\mathcal{M}_μ	0.1199929	0.0001201	0.9750130	1737.6879470	-1681.0794476
DNK	\mathcal{M}_{null}	0.1199929	0.0000000	0.9750128	1737.6829698	-1687.3839457
DOM	\mathcal{M}_{α^+}	0.6961046	0.0003898	0.9155814	373.7708776	-261.0889773
DOM	\mathcal{M}_{α^-}	0.7001650	0.0000000	0.9152206	373.2591147	-260.3086751
DOM	\mathcal{M}_δ	0.7001650	0.0000001	0.9152197	373.2484394	-260.3065475
DOM	\mathcal{M}_μ	0.7001650	0.0000000	0.9152199	373.2546911	-260.3071059
DOM	\mathcal{M}_{null}	0.7001650	0.0000000	0.9152205	373.2634442	-265.5179004
DZA	\mathcal{M}_{α^+}	1.2993726	0.0001052	0.9607808	442.7559336	-90.4793137
DZA	\mathcal{M}_{α^-}	1.2997836	0.0000901	0.9607793	442.6671504	-90.4747253
DZA	\mathcal{M}_δ	1.2996518	0.0001250	0.9607617	442.6369237	-90.4198633
DZA	\mathcal{M}_μ	1.2996155	0.0000682	0.9607943	442.7400231	-90.5214100
DZA	\mathcal{M}_{null}	1.2996035	0.0000559	0.9607710	442.4031491	-95.2526666
ECU	\mathcal{M}_{α^+}	1.5337628	0.0523325	0.9297865	401.2774868	-67.1390744
ECU	\mathcal{M}_{α^-}	1.6887306	0.0000045	0.9209043	384.0521122	-47.8421360
ECU	\mathcal{M}_δ	1.5963688	0.0266030	0.9251918	390.1548236	-56.8705370
ECU	\mathcal{M}_μ	1.5439082	0.0000015	0.9279190	392.9889600	-62.8867317
ECU	\mathcal{M}_{null}	1.7791235	0.0279990	0.9178951	382.7367272	-46.8807035
ESP	\mathcal{M}_{α^+}	0.1856080	0.0000028	0.9422837	1975.7328569	-1434.0717545
ESP	\mathcal{M}_{α^-}	0.1855373	0.0076910	0.9424047	1976.2639184	-1435.2236056
ESP	\mathcal{M}_δ	0.1548412	0.0000287	0.9517866	2025.4954451	-1532.8383743
ESP	\mathcal{M}_μ	0.1856080	0.0000000	0.9422840	1975.7359458	-1434.0745080
ESP	\mathcal{M}_{null}	0.1856081	0.0000000	0.9422835	1975.7284854	-1440.3779295
FIN	\mathcal{M}_{α^+}	0.3280622	0.0797648	0.9728173	2152.1380163	-1126.4399534
FIN	\mathcal{M}_{α^-}	0.3360629	0.0000399	0.9730371	2153.3108657	-1130.8958714
FIN	\mathcal{M}_δ	0.2788228	0.0981958	0.9767311	2193.0232499	-1211.7880069
FIN	\mathcal{M}_μ	0.3539563	0.0007086	0.9708473	2130.2102048	-1088.0271204
FIN	\mathcal{M}_{null}	0.3797915	0.0000000	0.9689312	2113.4406174	-1059.3878652
FRA	\mathcal{M}_{α^+}	0.1373927	0.0299971	0.9764265	1880.1804428	-1599.3274779
FRA	\mathcal{M}_{α^-}	0.1386926	0.0000073	0.9769100	1886.0075800	-1610.7067588
FRA	\mathcal{M}_δ	0.1198313	0.0474908	0.9789739	1914.1483197	-1662.1121244
FRA	\mathcal{M}_μ	0.1421161	0.1610016	0.9757525	1872.4761689	-1583.8529740

	Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
FRA	\mathcal{M}_{null}	0.1466084	0.0000000	0.9751495	1865.9199173	-1576.6741035	
GBR	\mathcal{M}_{α^+}	0.1669051	0.1074419	0.9775211	1884.4171476	-1484.4759333	
GBR	\mathcal{M}_{α^-}	0.1938987	0.0000000	0.9746001	1850.8502461	-1417.4039255	
GBR	\mathcal{M}_δ	0.1413371	0.0012231	0.9808639	1931.1521839	-1572.8654147	
GBR	\mathcal{M}_μ	0.1999852	0.0000000	0.9731472	1835.8110921	-1386.8661426	
GBR	\mathcal{M}_{null}	0.1903921	0.0000000	0.9745839	1850.8009895	-1423.3628352	
GHA	\mathcal{M}_{α^+}	0.3490916	0.0000916	0.7666365	195.2026830	-249.8849106	
GHA	\mathcal{M}_{α^-}	0.3531896	0.0000000	0.7657012	194.8203537	-249.3969268	
GHA	\mathcal{M}_δ	0.3076765	0.0053948	0.7871236	200.1576962	-261.0949413	
GHA	\mathcal{M}_μ	0.3531896	0.0000000	0.7657012	194.8204784	-249.3969083	
GHA	\mathcal{M}_{null}	0.3531896	0.0000000	0.7657012	194.8205084	-254.2009194	
GRC	\mathcal{M}_{α^+}	0.4896514	0.0161565	0.8280443	1354.6952066	-793.6899983	
GRC	\mathcal{M}_{α^-}	0.5290440	0.0000000	0.8177233	1337.4385948	-765.2452269	
GRC	\mathcal{M}_δ	0.4874272	0.0273553	0.8276619	1353.0776742	-792.6062212	
GRC	\mathcal{M}_μ	0.5490437	0.0000000	0.8073630	1324.7477342	-738.2677202	
GRC	\mathcal{M}_{null}	0.5412571	0.0000001	0.8096391	1327.2211290	-750.2581772	
GTM	\mathcal{M}_{α^+}	1.3642600	0.0080365	0.9021763	558.2459476	-159.0748496	
GTM	\mathcal{M}_{α^-}	1.5874596	0.0000000	0.8823431	535.2421777	-114.0311039	
GTM	\mathcal{M}_δ	1.5874598	0.0000000	0.8823458	535.2459962	-114.0367881	
GTM	\mathcal{M}_μ	1.6555121	0.0000000	0.8757806	528.0917573	-100.7875649	
GTM	\mathcal{M}_{null}	1.5874596	0.0000000	0.8823431	535.2423513	-119.5283970	
HKG	\mathcal{M}_{α^+}	0.2509271	0.0014428	0.9787234	2126.7218635	-1260.9135186	
HKG	\mathcal{M}_{α^-}	0.2445060	0.0000013	0.9794208	2135.6830150	-1279.2081293	
HKG	\mathcal{M}_δ	0.2517358	0.1089110	0.9785874	2124.9567552	-1257.4134379	
HKG	\mathcal{M}_μ	0.2509189	0.2308280	0.9787245	2126.7325395	-1260.9402382	
HKG	\mathcal{M}_{null}	0.2509189	0.0014568	0.9787245	2126.7360861	-1267.2474677	
HRV	\mathcal{M}_{α^+}	0.0659912	0.0001262	0.9787833	618.7632652	-772.8094871	
HRV	\mathcal{M}_{α^-}	0.0696456	0.0000000	0.9780032	613.7143315	-763.0604576	
HRV	\mathcal{M}_δ	0.0696456	0.0000271	0.9780032	613.7143075	-763.0603906	
HRV	\mathcal{M}_μ	0.0696456	0.0000000	0.9780032	613.7143096	-763.0604842	
HRV	\mathcal{M}_{null}	0.0696456	0.0000345	0.9780032	613.7143409	-768.6588048	
HUN	\mathcal{M}_{α^+}	0.2514814	0.0000000	0.9582531	2350.3148004	-1119.3586592	
HUN	\mathcal{M}_{α^-}	0.3220814	0.0000345	0.9491220	2303.4750542	-1022.8291131	
HUN	\mathcal{M}_δ	0.3220814	0.0000002	0.9491218	2303.4782840	-1022.8276039	
HUN	\mathcal{M}_μ	0.2834268	0.0000000	0.9535596	2324.8152044	-1067.3653028	
HUN	\mathcal{M}_{null}	0.3220814	0.0000000	0.9491220	2303.4784333	-1029.0201300	
IDN	\mathcal{M}_{α^+}	0.5192693	0.0000000	0.9146493	305.0747698	-197.7229477	
IDN	\mathcal{M}_{α^-}	0.6211057	0.0000000	0.8971687	292.8324976	-174.9917751	
IDN	\mathcal{M}_δ	0.5703114	0.0003749	0.9049279	298.3149198	-184.5632250	
IDN	\mathcal{M}_μ	0.6211057	0.0000000	0.8971704	292.8323470	-174.9937641	
IDN	\mathcal{M}_{null}	0.6211057	0.0000000	0.8971695	292.8317590	-179.7967128	
IRL	\mathcal{M}_{α^+}	0.2088275	0.0143105	0.9772782	2569.1155517	-1359.9304403	
IRL	\mathcal{M}_{α^-}	0.2073867	0.0000036	0.9774790	2571.5504173	-1364.8037711	

Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
IRL	\mathcal{M}_δ	0.2090907	0.0200323	0.9772362	2568.5462746	-1358.9156430
IRL	\mathcal{M}_μ	0.2088306	0.0352422	0.9772776	2569.1090642	-1359.9153338
IRL	\mathcal{M}_{null}	0.2088274	0.0144979	0.9772784	2569.1199148	-1366.2417945
ISR	\mathcal{M}_{α^+}	0.4160179	0.0062564	0.9709201	1893.2428702	-1010.4705652
ISR	\mathcal{M}_{α^-}	0.4349423	0.0000136	0.9701674	1885.8473780	-996.4417954
ISR	\mathcal{M}_δ	0.4339349	0.0008507	0.9698446	1882.2493491	-990.5338945
ISR	\mathcal{M}_μ	0.4349423	0.0000322	0.9701674	1885.8468109	-996.4418756
ISR	\mathcal{M}_{null}	0.4349423	0.0000111	0.9701674	1885.8485980	-1002.7501047
JOR	\mathcal{M}_{α^+}	1.3140447	0.0000227	0.8158654	859.9004778	-213.5884682
JOR	\mathcal{M}_{α^-}	1.3883468	0.0000158	0.8112079	854.4614037	-204.7457457
JOR	\mathcal{M}_δ	1.3140447	0.0010034	0.8158620	859.9018193	-213.5819671
JOR	\mathcal{M}_μ	1.3044454	0.0040242	0.8169297	860.5974454	-215.6406373
JOR	\mathcal{M}_{null}	1.3140446	0.0000000	0.8158654	859.9040348	-219.4577945
JPN	\mathcal{M}_{α^+}	0.0814135	0.0000000	0.9672378	794.7905570	-850.5394247
JPN	\mathcal{M}_{α^-}	0.0683921	0.0113106	0.9730056	818.4672910	-897.7889660
JPN	\mathcal{M}_δ	0.0513784	0.0000323	0.9785380	846.1711542	-953.7491905
JPN	\mathcal{M}_μ	0.0814137	0.0000001	0.9672373	794.7938193	-850.5360642
JPN	\mathcal{M}_{null}	0.0814135	0.0000002	0.9672371	794.7868478	-856.0313369
KAZ	\mathcal{M}_{α^+}	0.1865516	0.0000476	0.8295751	214.6552620	-174.1895588
KAZ	\mathcal{M}_{α^-}	0.2496131	0.0000000	0.7623985	199.8289770	-144.2825364
KAZ	\mathcal{M}_δ	0.2390774	0.0000000	0.7721660	201.6233048	-148.0605220
KAZ	\mathcal{M}_μ	0.2362297	0.0000001	0.7769984	202.2588602	-149.9899724
KAZ	\mathcal{M}_{null}	0.2496130	0.0000001	0.7624011	199.8317347	-148.7833403
KOR	\mathcal{M}_{α^+}	0.6920979	0.0649634	0.9155851	2216.2272046	-646.0307164
KOR	\mathcal{M}_{α^-}	0.8387203	0.0000073	0.8985858	2168.9233801	-556.4976910
KOR	\mathcal{M}_δ	0.6985206	0.0000000	0.9123594	2202.6982338	-627.7305826
KOR	\mathcal{M}_μ	0.6614961	0.2464106	0.9192266	2224.9307739	-667.5495675
KOR	\mathcal{M}_{null}	0.7946315	0.0000000	0.9020007	2176.7551440	-579.4032059
LBN	\mathcal{M}_{α^+}	1.3016604	0.0240624	0.8243273	807.8076783	-327.5331892
LBN	\mathcal{M}_{α^-}	1.4386036	0.0000001	0.8073042	782.3615142	-282.3978581
LBN	\mathcal{M}_δ	1.4380388	0.0005367	0.8117306	789.9437981	-293.7384027
LBN	\mathcal{M}_μ	1.4320212	0.0000001	0.8135593	792.1142739	-298.5017188
LBN	\mathcal{M}_{null}	1.4416777	0.0000000	0.8100418	788.0379102	-295.5709325
LBY	\mathcal{M}_{α^+}	2.2780600	0.0000128	0.9267840	378.4283856	-11.0745573
LBY	\mathcal{M}_{α^-}	2.8894278	0.0000473	0.9079340	364.6708243	16.8746157
LBY	\mathcal{M}_δ	3.5313144	0.0002249	0.8853192	349.5494084	43.6715402
LBY	\mathcal{M}_μ	3.4722522	0.0000224	0.8868652	350.6939375	42.0157078
LBY	\mathcal{M}_{null}	2.7553149	0.0000477	0.9107669	365.7889379	8.2576748
LKA	\mathcal{M}_{α^+}	1.5941187	0.0003071	0.8115942	336.9975384	-92.8885083
LKA	\mathcal{M}_{α^-}	1.6362543	0.0000000	0.8102917	337.0514707	-91.6277204
LKA	\mathcal{M}_δ	1.6322208	0.0022002	0.8084860	335.3889986	-89.8940407
LKA	\mathcal{M}_μ	1.6362543	0.0025755	0.8102917	337.0514702	-91.6277204
LKA	\mathcal{M}_{null}	1.6362543	0.0000000	0.8102917	337.0514708	-96.8372065

	Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
LTU	\mathcal{M}_{α^+}	0.2792254	0.0000068	0.9197774	550.3188221	-359.2088757	
LTU	\mathcal{M}_{α^-}	0.2792263	0.0011184	0.9197673	550.3111851	-359.1746619	
LTU	\mathcal{M}_δ	0.2539895	0.0000002	0.9241161	555.7579934	-374.2208641	
LTU	\mathcal{M}_μ	0.2554806	0.0000000	0.9240155	556.4417023	-373.8632704	
LTU	\mathcal{M}_{null}	0.2792264	0.0000000	0.9197682	550.3064726	-364.7762372	
LUX	\mathcal{M}_{α^+}	0.1687301	0.0027144	0.9756622	606.0315078	-387.5202145	
LUX	\mathcal{M}_{α^-}	0.1556444	0.0423426	0.9776990	616.1184100	-405.6120311	
LUX	\mathcal{M}_δ	0.1556444	0.0037217	0.9776974	616.1053905	-405.5967384	
LUX	\mathcal{M}_μ	0.1556444	0.0514556	0.9776986	616.1152846	-405.6079128	
LUX	\mathcal{M}_{null}	0.1556444	0.0000000	0.9776985	616.1172377	-410.9401680	
IVA	\mathcal{M}_{α^+}	0.2953602	0.0000000	0.9317351	543.8952197	-366.6813027	
IVA	\mathcal{M}_{α^-}	0.2877696	0.0000698	0.9322366	544.6567294	-368.6722851	
IVA	\mathcal{M}_δ	0.2280066	0.0002703	0.9426518	563.8282263	-413.7302090	
IVA	\mathcal{M}_μ	0.2682572	0.0000000	0.9354481	550.4990299	-381.7817245	
IVA	\mathcal{M}_{null}	0.2953602	0.0000000	0.9317370	543.8963214	-372.2873909	
MAC	\mathcal{M}_{α^+}	1.1294488	0.0004599	0.9315991	1531.4525516	-289.1433884	
MAC	\mathcal{M}_{α^-}	0.9767320	0.0000041	0.9406784	1567.2754683	-349.5263422	
MAC	\mathcal{M}_δ	1.1294497	0.0051249	0.9315974	1531.4577652	-289.1325538	
MAC	\mathcal{M}_μ	1.1294485	0.0005140	0.9315978	1531.4587640	-289.1354956	
MAC	\mathcal{M}_{null}	1.1294486	0.0005148	0.9315990	1531.4584828	-295.1922572	
MAR	\mathcal{M}_{α^+}	0.8537848	0.0234547	0.9349568	650.3604985	-264.3866710	
MAR	\mathcal{M}_{α^-}	0.9295200	0.0000018	0.9296139	640.4149824	-245.1246477	
MAR	\mathcal{M}_δ	0.9016048	0.0040674	0.9319661	643.9018432	-253.4178828	
MAR	\mathcal{M}_μ	0.9173997	0.0000204	0.9310205	642.7052567	-250.0499976	
MAR	\mathcal{M}_{null}	0.9295200	0.0000035	0.9296140	640.4143693	-250.6218776	
MEX	\mathcal{M}_{α^+}	0.4508260	0.0000993	0.9151172	1435.2035699	-631.2102997	
MEX	\mathcal{M}_{α^-}	0.4876843	0.0000001	0.9084882	1420.6198791	-603.6884578	
MEX	\mathcal{M}_δ	0.4392520	0.0149692	0.9163000	1437.2548352	-636.3463466	
MEX	\mathcal{M}_μ	0.4866618	0.0092329	0.9082319	1420.0192358	-602.6647468	
MEX	\mathcal{M}_{null}	0.4876843	0.0000001	0.9084879	1420.6159465	-609.5898333	
MYS	\mathcal{M}_{α^+}	0.2895046	0.0785445	0.9539404	1973.5958108	-1124.8476489	
MYS	\mathcal{M}_{α^-}	0.3399726	0.0000000	0.9474011	1937.0418339	-1054.3528091	
MYS	\mathcal{M}_δ	0.3399726	0.0000000	0.9474014	1937.0487616	-1054.3553106	
MYS	\mathcal{M}_μ	0.3169605	0.0000000	0.9509908	1955.6429441	-1091.8872434	
MYS	\mathcal{M}_{null}	0.3399726	0.0000000	0.9474012	1937.0402581	-1060.6285880	
NGA	\mathcal{M}_{α^+}	2.7990310	0.0000239	0.9395854	495.7796031	11.7048325	
NGA	\mathcal{M}_{α^-}	2.7992006	0.0000615	0.9395758	495.6501030	11.7242375	
NGA	\mathcal{M}_δ	2.6256522	0.0000111	0.9431462	498.4576558	4.2936995	
NGA	\mathcal{M}_μ	2.7268460	0.0000217	0.9411712	496.2771356	8.4598284	
NGA	\mathcal{M}_{null}	2.7990746	0.0000159	0.9395804	495.7132708	6.9110458	
NLD	\mathcal{M}_{α^+}	0.1466304	0.0000001	0.9582107	1657.4935262	-1562.8725186	
NLD	\mathcal{M}_{α^-}	0.2092032	0.0000000	0.9418981	1567.4351623	-1381.9447140	
NLD	\mathcal{M}_δ	0.1340786	0.0351018	0.9609515	1678.9038968	-1600.1142643	

	Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
	NLD	\mathcal{M}_μ	0.2097818	0.0272475	0.9412098	1564.2116085	-1375.4789279
	NLD	\mathcal{M}_{null}	0.2097818	0.0000000	0.9412096	1564.2134536	-1381.7853394
	NOR	\mathcal{M}_{α^+}	0.2399445	0.0130421	0.9780959	2521.5074784	-1298.8700700
	NOR	\mathcal{M}_{α^-}	0.2390888	0.0016231	0.9781834	2522.6463097	-1301.0687029
	NOR	\mathcal{M}_δ	0.2400056	0.0128037	0.9780893	2521.4101099	-1298.7067950
	NOR	\mathcal{M}_μ	0.2399438	0.0130562	0.9780959	2521.5037000	-1298.8707092
	NOR	\mathcal{M}_{null}	0.2398829	0.0193799	0.9780996	2521.5638888	-1305.2721046
	NZL	\mathcal{M}_{α^+}	0.9073281	0.0037365	0.9129915	1654.8440041	-567.6419686
	NZL	\mathcal{M}_{α^-}	0.8772374	0.0000059	0.9140447	1656.6103203	-573.5852855
	NZL	\mathcal{M}_δ	0.4805584	0.0203982	0.9447784	1759.6846130	-789.5124824
	NZL	\mathcal{M}_μ	0.4917836	0.0120060	0.9442763	1758.4536220	-785.0947066
	NZL	\mathcal{M}_{null}	0.9073281	0.0000000	0.9129914	1654.8396788	-573.8317817
	OMN	\mathcal{M}_{α^+}	0.7889506	0.0396981	0.9139409	1112.8720701	-153.1574333
	OMN	\mathcal{M}_{α^-}	0.6700664	0.0860673	0.9239838	1120.4887405	-183.5588674
	OMN	\mathcal{M}_δ	0.8933917	0.0678704	0.9031544	1102.8136604	-124.2268460
	OMN	\mathcal{M}_μ	0.8791464	0.0000000	0.9020953	1095.4922578	-121.5620907
	OMN	\mathcal{M}_{null}	0.8520861	0.0006775	0.9052430	1100.9144521	-135.0695316
	PAN	\mathcal{M}_{α^+}	1.1143223	0.0000000	0.8199065	998.7151866	-355.1878253
	PAN	\mathcal{M}_{α^-}	1.1149436	0.0000000	0.8201169	998.8693793	-355.6869349
	PAN	\mathcal{M}_δ	1.1100368	0.0002924	0.8199278	998.2834172	-355.2383815
	PAN	\mathcal{M}_μ	1.1149435	0.0000000	0.8201169	998.8720122	-355.6869611
	PAN	\mathcal{M}_{null}	1.1149436	0.0000000	0.8201166	998.8746400	-361.7432027
	PER	\mathcal{M}_{α^+}	0.8254032	0.0311618	0.8826033	573.8679392	-274.9069780
	PER	\mathcal{M}_{α^-}	0.9361490	0.0000000	0.8671464	559.4016095	-244.7267453
	PER	\mathcal{M}_δ	0.8882912	0.0000974	0.8760467	567.2656212	-261.6465567
	PER	\mathcal{M}_μ	0.9055167	0.0000000	0.8725708	564.1687729	-254.8983509
	PER	\mathcal{M}_{null}	0.9361490	0.0000000	0.8671464	559.4031274	-250.2239393
	POL	\mathcal{M}_{α^+}	0.2535426	0.0000000	0.9438326	1807.0469575	-1001.0879697
	POL	\mathcal{M}_{α^-}	0.3163218	0.0000000	0.9326013	1768.7678021	-923.2501964
	POL	\mathcal{M}_δ	0.3162856	0.0000403	0.9326143	1768.7760416	-923.3322541
	POL	\mathcal{M}_μ	0.2782767	0.0000000	0.9379993	1785.4443202	-958.8962784
	POL	\mathcal{M}_{null}	0.3163218	0.0000000	0.9326025	1768.7577185	-929.3142565
	PRT	\mathcal{M}_{α^+}	0.3321047	0.0000076	0.9133453	1726.5468270	-1093.0105775
	PRT	\mathcal{M}_{α^-}	0.3453640	0.0127740	0.9112486	1719.7441003	-1079.8850541
	PRT	\mathcal{M}_δ	0.3369898	0.0000004	0.9136215	1727.2952235	-1094.7635003
	PRT	\mathcal{M}_μ	0.3417620	0.0000000	0.9114441	1720.4004837	-1081.0956205
	PRT	\mathcal{M}_{null}	0.3453640	0.0000000	0.9112488	1719.7422418	-1086.1944995
	PRY	\mathcal{M}_{α^+}	0.7911077	0.0000299	0.8628434	618.4255027	-283.9049804
	PRY	\mathcal{M}_{α^-}	0.8457188	0.0000000	0.8541465	610.2534769	-268.9040302
	PRY	\mathcal{M}_δ	0.8457189	0.0000006	0.8541473	610.2555041	-268.9052712
	PRY	\mathcal{M}_μ	0.8457188	0.0000003	0.8541469	610.2551282	-268.9046708
	PRY	\mathcal{M}_{null}	0.8457188	0.0000007	0.8541468	610.2536061	-274.4016218
	QAT	\mathcal{M}_{α^+}	1.9266798	0.0038751	0.9118724	1618.1838222	-63.5800421

	Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
	QAT	\mathcal{M}_{α^-}	2.0139147	0.0000255	0.9086920	1608.4377172	-51.8806944
	QAT	\mathcal{M}_δ	1.9353840	0.0001128	0.9115033	1615.0203438	-62.2009448
	QAT	\mathcal{M}_μ	1.9353839	0.0226246	0.9115036	1615.0306966	-62.2018858
	QAT	\mathcal{M}_{null}	1.9353843	0.0000104	0.9115025	1615.0092877	-67.9971407
	ROU	\mathcal{M}_{α^+}	0.8900993	0.0290867	0.8907507	1747.7760599	-447.7210801
	ROU	\mathcal{M}_{α^-}	1.0376017	0.0000015	0.8796222	1727.7138904	-406.3009748
	ROU	\mathcal{M}_δ	1.0376021	0.0000221	0.8796235	1727.7104815	-406.3056340
	ROU	\mathcal{M}_μ	1.0008829	0.0125723	0.8786329	1723.9683488	-402.8060015
	ROU	\mathcal{M}_{null}	1.0376017	0.0000001	0.8796230	1727.7051904	-412.3607662
	RUS	\mathcal{M}_{α^+}	0.6174841	0.0000000	0.9566539	1070.3858708	-349.4521333
	RUS	\mathcal{M}_{α^-}	0.8291502	0.0000002	0.9429638	1037.3556155	-282.4815040
	RUS	\mathcal{M}_δ	0.8291506	0.0000000	0.9429651	1037.3633751	-282.4866992
	RUS	\mathcal{M}_μ	0.8194430	0.0000824	0.9438272	1038.7924091	-286.2030003
	RUS	\mathcal{M}_{null}	0.8291502	0.0000000	0.9429641	1037.3532266	-287.9799069
	SAU	\mathcal{M}_{α^+}	1.4447598	0.0000015	0.9319666	318.2772364	-63.7460639
	SAU	\mathcal{M}_{α^-}	1.1467943	0.0000016	0.9491154	335.8594189	-98.0178968
	SAU	\mathcal{M}_δ	1.4317991	0.0000000	0.9331044	319.2716499	-65.7362927
	SAU	\mathcal{M}_μ	1.3103942	0.0000004	0.9409868	327.3080046	-80.5301406
	SAU	\mathcal{M}_{null}	1.4447565	0.0000027	0.9319483	318.1644064	-68.4849842
	SGP	\mathcal{M}_{α^+}	0.4390113	0.0002142	0.9326026	2040.0506241	-979.1514085
	SGP	\mathcal{M}_{α^-}	0.4389674	0.1024984	0.9330349	2041.7687561	-982.6841384
	SGP	\mathcal{M}_δ	0.4381149	0.0988135	0.9327193	2040.8073272	-980.1034145
	SGP	\mathcal{M}_μ	0.4390113	0.0001842	0.9326026	2040.0476396	-979.1513242
	SGP	\mathcal{M}_{null}	0.4390113	0.1632410	0.9326026	2040.0510546	-985.4594657
	SVK	\mathcal{M}_{α^+}	0.0897265	0.2508679	0.9841495	1031.5427644	-671.5084754
	SVK	\mathcal{M}_{α^-}	0.1076807	0.0000152	0.9797558	999.7833418	-607.6491798
	SVK	\mathcal{M}_δ	0.1297093	0.0563428	0.9763511	979.6865381	-567.0771140
	SVK	\mathcal{M}_μ	0.1164278	0.0012728	0.9782005	991.2554924	-588.3305464
	SVK	\mathcal{M}_{null}	0.1135388	0.0000000	0.9781674	991.0807478	-593.4997133
	SVN	\mathcal{M}_{α^+}	0.1106088	0.0196362	0.9794616	807.4033352	-625.2063291
	SVN	\mathcal{M}_{α^-}	0.1204472	0.0001807	0.9768129	791.5610114	-592.4550281
	SVN	\mathcal{M}_δ	0.1060602	0.0012029	0.9792287	805.5613897	-622.1622849
	SVN	\mathcal{M}_μ	0.1198190	0.0006017	0.9769728	792.4983162	-594.3233439
	SVN	\mathcal{M}_{null}	0.1204460	0.0001852	0.9768135	791.5636796	-598.0604505
	SWE	\mathcal{M}_{α^+}	0.1794411	0.0000246	0.9827111	2093.2525551	-1458.8338866
	SWE	\mathcal{M}_{α^-}	0.1679885	0.0011001	0.9840927	2115.9414425	-1504.5572179
	SWE	\mathcal{M}_δ	0.1604230	0.0619388	0.9845382	2124.0345953	-1520.1547048
	SWE	\mathcal{M}_μ	0.1773051	0.0333840	0.9828882	2095.9941251	-1464.4867308
	SWE	\mathcal{M}_{null}	0.1794411	0.0000000	0.9827112	2093.2494231	-1465.1444556
	THA	\mathcal{M}_{α^+}	0.3214689	0.0000002	0.9515592	1740.3783079	-881.8617066
	THA	\mathcal{M}_{α^-}	0.3563113	0.0000219	0.9490399	1729.7252912	-860.2127021
	THA	\mathcal{M}_δ	0.3563231	0.0000376	0.9490292	1729.6876773	-860.1227083
	THA	\mathcal{M}_μ	0.3563111	0.0000166	0.9490411	1729.7307152	-860.2231110

Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
THA	\mathcal{M}_{null}	0.3563111	0.0000002	0.9490411	1729.7276038	-866.2793417
TKM	\mathcal{M}_{α^+}	0.6120721	0.0002175	0.8484762	110.4929157	-52.1194340
TKM	\mathcal{M}_{α^-}	0.6125306	0.0002738	0.8486140	110.5452317	-52.1740147
TKM	\mathcal{M}_δ	0.6125441	0.0000282	0.8483613	110.4933587	-52.0739457
TKM	\mathcal{M}_μ	0.6125429	0.0000481	0.8483465	110.4568505	-52.0681256
TKM	\mathcal{M}_{null}	0.6125373	0.0000492	0.8483298	110.4627760	-56.1558440
TUN	\mathcal{M}_{α^+}	0.6040435	0.0519402	0.8800402	970.7830183	-525.8558531
TUN	\mathcal{M}_{α^-}	0.6553858	0.0000039	0.8721401	959.5907734	-502.5129657
TUN	\mathcal{M}_δ	0.6413280	0.0134767	0.8740118	962.3288209	-507.9103306
TUN	\mathcal{M}_μ	0.6486814	0.0000013	0.8731389	960.7741688	-505.3833771
TUN	\mathcal{M}_{null}	0.6553859	0.0000000	0.8721396	959.5919379	-508.4144076
TUR	\mathcal{M}_{α^+}	0.5613733	0.0036904	0.9221724	1006.5304764	-455.9797216
TUR	\mathcal{M}_{α^-}	0.5743223	0.0000163	0.9212163	1004.4008375	-452.2555678
TUR	\mathcal{M}_δ	0.5537710	0.0000113	0.9241992	1010.4413593	-464.0276218
TUR	\mathcal{M}_μ	0.5564595	0.0000000	0.9225319	1006.5641070	-457.3917962
TUR	\mathcal{M}_{null}	0.5743230	0.0000163	0.9212166	1004.4010786	-457.9771897
UKR	\mathcal{M}_{α^+}	0.2824879	0.0000003	0.8475939	237.3301802	-170.0780074
UKR	\mathcal{M}_{α^-}	0.2448255	0.0000002	0.8612119	241.8217522	-181.3100499
UKR	\mathcal{M}_δ	0.1863009	0.0000001	0.8928971	257.3043951	-212.4090202
UKR	\mathcal{M}_μ	0.2496387	0.0000212	0.8602308	242.1071349	-180.4647504
UKR	\mathcal{M}_{null}	0.2824878	0.0000003	0.8476001	237.3437052	-174.8704237
URY	\mathcal{M}_{α^+}	0.3707057	0.0004125	0.9539609	760.9348292	-446.7383706
URY	\mathcal{M}_{α^-}	0.3801091	0.0000909	0.9515936	754.4475193	-433.2002436
URY	\mathcal{M}_δ	0.3801091	0.0739791	0.9515934	754.4467727	-433.1992421
URY	\mathcal{M}_μ	0.4148870	0.0000000	0.9462801	739.8061158	-405.0794658
URY	\mathcal{M}_{null}	0.3801091	0.0000489	0.9515934	754.4502665	-438.7974684
USA	\mathcal{M}_{α^+}	0.1668963	0.0055876	0.9697761	1854.4974683	-1504.7078129
USA	\mathcal{M}_{α^-}	0.1691152	0.0000027	0.9698283	1854.8253141	-1505.6582104
USA	\mathcal{M}_δ	0.1869925	0.0006049	0.9658160	1820.5736973	-1437.1131786
USA	\mathcal{M}_μ	0.1691152	0.0211774	0.9698284	1854.8269747	-1505.6591624
USA	\mathcal{M}_{null}	0.1691152	0.0221088	0.9698284	1854.8267976	-1511.9671769
VEN	\mathcal{M}_{α^+}	0.7248894	0.0302883	0.9314021	368.7917156	-158.4962726
VEN	\mathcal{M}_{α^-}	0.6719343	0.0000000	0.9369305	373.8565502	-168.7471573
VEN	\mathcal{M}_δ	0.7248895	0.0000351	0.9314017	368.7906482	-158.4955756
VEN	\mathcal{M}_μ	0.7248898	0.0371293	0.9314029	368.7920561	-158.4976122
VEN	\mathcal{M}_{null}	0.7248892	0.0000001	0.9314027	368.7897007	-163.3013450
VNM	\mathcal{M}_{α^+}	2.3967138	0.0012084	0.5717167	960.8749519	-14.6593163
VNM	\mathcal{M}_{α^-}	2.4133088	0.0000400	0.5678732	960.0251253	-13.0243703
VNM	\mathcal{M}_δ	2.4132716	0.0000647	0.5679233	959.9523137	-13.0455685
VNM	\mathcal{M}_μ	2.2242733	0.0000015	0.5854469	961.8716340	-20.6221612
VNM	\mathcal{M}_{null}	2.4132930	0.0000029	0.5678849	959.9594100	-18.2388068
ZAF	\mathcal{M}_{α^+}	0.4738056	0.0014702	0.9471435	1054.7190010	-621.0423136
ZAF	\mathcal{M}_{α^-}	0.4936819	0.0000042	0.9466293	1051.1700514	-617.4985641

Country	Model	Loss	Rel. std. loss	R2	Log-likelihood	ΔBIC
ZAF	\mathcal{M}_δ	0.4936820	0.0001680	0.9466294	1051.1688630	-617.4996371
ZAF	\mathcal{M}_μ	0.4936819	0.0000044	0.9466295	1051.1689693	-617.4999436
ZAF	\mathcal{M}_{null}	0.4936819	0.0001973	0.9466293	1051.1698206	-623.4012024

Tab. S4.2: **Results of the numerical simulations.** Numerical values for each model and each country correspond to the optimization run with the minimum loss value. "Rel. std. loss" corresponds to the standard deviation of the loss values obtained for the five independent runs, divided by the mean loss value. Low "Rel. std. loss" indicates that the optimization runs have converged towards the same maximum likelihood.

Country Code	Country Name	Country Code	Country Name
AGO	Angola	LBN	Lebanon
ARE	United Arab Emirates	LBY	Libya
ARG	Argentina	LKA	Sri Lanka
AUS	Australia	LTU	Lithuania
AUT	Austria	LUX	Luxembourg
AZE	Azerbaijan	IVA	Latvia
BEL	Belgium	MAC	Macao SAR, China
BGD	Bangladesh	MAR	Morocco
BGR	Bulgaria	MEX	Mexico
BHR	Bahrain	MMR	Myanmar
BLR	Belarus	MYS	Malaysia
BOL	Bolivia	NGA	Nigeria
BRA	Brazil	NLD	Netherlands
CAN	Canada	NOR	Norway
CHE	Switzerland	NZL	New Zealand
CHL	Chile	OMN	Oman
CHN	China	PAK	Pakistan
COL	Colombia	PAN	Panama
CRI	Costa Rica	PER	Peru
CUB	Cuba	POL	Poland
CZE	Czech Republic	PRI	Puerto Rico
DEU	Germany	PRT	Portugal
DNK	Denmark	PRY	Paraguay
DOM	Dominican Republic	QAT	Qatar
DZA	Algeria	ROU	Romania
ECU	Ecuador	RUS	Russian Federation
EGY	Egypt, Arab Rep.	SAU	Saudi Arabia
ESP	Spain	SDN	Sudan
ETH	Ethiopia	SGP	Singapore
FIN	Finland	SRB	Serbia
FRA	France	SVK	Slovak Republic
GBR	United Kingdom	SVN	Slovenia
GHA	Ghana	SWE	Sweden
GRC	Greece	SYR	Syrian Arab Republic
GTM	Guatemala	THA	Thailand
HKG	Hong Kong SAR, China	TKM	Turkmenistan
HRV	Croatia	TUN	Tunisia
HUN	Hungary	TUR	Turkey
IDN	Indonesia	TZA	Tanzania
IND	India	UKR	Ukraine
IRL	Ireland	URY	Uruguay
IRN	Iran, Islamic Rep.	USA	United States
IRQ	Iraq	UZB	Uzbekistan
ISR	Israel	VEN	Venezuela, RB
JOR	Jordan	VNM	Vietnam
JPN	Japan	WLD	World
KAZ	Kazakhstan	YEM	Yemen, Rep.
KEN	Kenya	ZAF	South Africa
KOR	Korea, Rep.		

Tab. S4.1: ISO 3166-1 country codes for each of the 100 country investigated.

	Loglikelihood	Loglikelihood residuals
(Intercept)	0.000 (0.045)	1.495** (0.474)
NT.	0.928*** (0.046)	
N		-0.230** (0.071)
log(GDP)		0.693*** (0.187)
Number of countries	69	69
R^2	0.861	0.174

Tab. S4.3: Linear regression model coefficients for the effect of the number of data points (NT) on the best model loglikelihood, and linear regression model coefficients for the effect of GDP and number of economic activities (N) on the residuals. *** $p < 0.001$

Deep learning approximations for non-local nonlinear PDEs with Neumann boundary conditions

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” *The key to growth is the introduction of higher dimensions of consciousness into our awareness.*

— Lao Tzu (c. 4th century BCE)

Nonlinear partial differential equations (PDEs) are used to model dynamical processes in a large number of scientific fields, ranging from finance to biology. In many applications standard local models are not sufficient to accurately account for certain non-local phenomena such as, e.g., interactions at a distance. In order to properly capture these phenomena non-local nonlinear PDE models are frequently employed in the literature. In this article we propose two numerical methods based on machine learning and on Picard iterations, respectively, to approximately solve non-local nonlinear PDEs. The proposed machine learning-based method is an extended variant of a deep learning-based splitting-up type approximation method previously introduced in the literature and utilizes neural networks to provide approximate solutions on a subset of the spatial domain of the solution. The Picard iterations-based method is an extended variant of the so-called *full history recursive multilevel Picard* approximation scheme previously introduced in the literature and provides an approximate solution for a single point of the domain. Both methods are mesh-free and allow non-local nonlinear PDEs with Neumann boundary conditions to be solved in high dimensions. In the two methods, the numerical difficulties arising due to the dimensionality of the PDEs are avoided by (i) using the correspondence between the expected trajectory of reflected stochastic processes and the solution of PDEs (given by the Feynman–Kac formula) and by (ii) using a plain vanilla Monte Carlo integration to handle the non-local term. We evaluate the performance of the two methods on five different PDEs arising in physics and biology. In all cases, the methods yield good results in up to 10 dimensions with short run times. Our work extends recently developed methods to overcome the curse of dimensionality in solving PDEs.

5.1 Introduction

In this article, we derive numerical schemes to approximately solve high-dimensional non-local nonlinear partial differential equations (PDEs) with Neumann boundary conditions. Such PDEs have been used to describe a variety of processes in physics, engineering, finance, and biology, but can generally not be solved analytically, requiring numerical methods to provide approximate solutions. However, traditional numerical methods are for the most part computationally infeasible for high-dimensional problems, calling for the development of novel approximation methods.

The need for solving non-local nonlinear PDEs has been expressed in various fields as they provide a more general description of the dynamical systems than their local counterparts ([Kavallaris2018](#); [delia2020numerical](#); [Sunderasan2020](#)). In physics and engineering, non-local nonlinear PDEs are found, e.g., in models of Ohmic heating production ([Lacey1995](#)), in the investigation of the fully turbulent behavior of real flows ([Caglioti1992](#)), in phase field models allowing non-local interactions ([Barone1971](#); [Gajewski2003](#); [Coleman1994](#); [Hairer2016](#)), or in phase transition models with conservation of mass ([RUBINSTEIN1992](#); [Stoleriu2011](#)); see [Kavallaris2018](#) for further references. In finance, non-local PDEs are used, e.g., in jump-diffusion models for the pricing of derivatives where the dynamics of stock prices are described by stochastic processes experiencing large jumps ([Merton1976](#); [Chan1999](#); [Kou2000](#); [Abergel2010](#); [Benth2001](#); [Sunderasan2020](#); [Cruz2020](#); [Tankov2003](#)). Penalty methods for pricing American put options such as in Kou's jump-diffusion model ([Huang2013](#); [Gan2020](#)), considering large investors where the agent policy affects the assets prices ([Amadori2003](#); [Abergel2010](#)), or considering default risks ([Pham2009](#); [Henry-Labordere2012](#)) can further introduce nonlinear terms in non-local PDEs. In economics, non-local nonlinear PDEs appear, e.g., in evolutionary game theory with the so-called replicator-mutator equation capturing continuous strategy spaces ([Oechssler2001](#); [Kavallaris2017](#); [Hamel2020](#); [Alfaro2016](#); [Alfaro2019](#)) or in growth models where consumption is non-local ([Banerjee2021](#)). In biology, non-local nonlinear PDEs are used, e.g., to model processes determining the interaction and evolution of organisms. Examples include models of morphogenesis and cancer evolution ([Lorz2013](#); [Chen2020](#); [Villa2021](#)), models of gene regulatory networks ([Pajaro2017](#)), population genetics models with the non-local Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher–KPP) equations ([FISHER1937](#); [Hamel2001](#); [Bian2017](#); [Perthame2007](#); [Berestycki2009b](#); [Houchmandzadeh2017](#); [Wang2021](#)), and quantitative genetics models where populations are structured on a phenotypic and/or a geographical space ([Burger1994](#); [Genieys2006a](#); [Berestycki2016](#); [Nordbotten2016](#); [Nordbotten2018](#); [Roques2016](#); [Doebeli2010](#); [Nordbotten2020](#)). In such models, Neumann boundary conditions are used, e.g., to model the effect of the borders of the geographical domain on the movement of the organisms.

Real world systems such as those just mentioned may be of considerable complexity and accurately capturing the dynamics of these systems may require models of high dimensionality ([Doebeli2010](#)), leading to complications in obtaining numerical approximations. For example, the number of dimensions of the PDEs may correspond in finance to the number of financial assets (such as stocks, commodities, exchange rates, and interest rates) in the involved portfolio; in evolutionary dynamics, to the dimension of the strategy space;

and in biology, to the number of genes modelled (**Pajaro2017**) or to the dimension of the geographical or the phenotypic space over which the organisms are structured. Standard approximation methods for PDEs such as finite difference approximation methods, finite element methods, spectral Galerkin approximation methods, and sparse grid approximation methods all suffer from the so called *curse of dimensionality* (**Bellman1957**), meaning that their computational costs increase exponentially in the number of dimensions of the PDE under consideration.

Numerical methods exploiting stochastic representations of the solutions of PDEs can in some cases overcome the curse of dimensionality. Specifically, simple Monte Carlo averages of the associated stochastic processes have been proposed a long time ago to solve high-dimensional linear PDEs, such as, e.g., Black–Scholes and Kolmogorov PDEs (**metropolis1949monte; bauer1958monte**). Recently, two novel classes of methods have proved successful in dealing with high-dimensional nonlinear PDEs, namely deep learning-based and full history recursive multilevel Picard approximation methods (in the following we will abbreviate *full history recursive multilevel Picard* by MLP). The explosive success of deep learning in recent years across a wide range of applications (**LeCun2015**) has inspired a variety of neural network-based approximation methods for high-dimensional PDEs; see **Beck2020** for an overview. One class of such methods is based on reformulating the PDE as a stochastic learning problem through the Feynman–Kac formula (**EHanJentzen2017; Han2018; beck2018solving**). In particular, the *deep splitting* scheme introduced in (**Beck2019**) relies on splitting the differential operator into a linear and a non-linear part and in that sense belongs to the class of splitting-up methods (**cox2013pathwise; gyongy2003splitting; hochbruck2005explicit**). The PDE approximation problem is then decomposed along the time axis into a sequence of separate learning problems. The deep splitting approximation scheme has proved capable of computing reasonable approximations to the solutions of nonlinear PDEs in up to 10000 dimensions. On the other hand, the MLP approximation method, introduced in (**Weinan2019; Hutzenthaler2020; Weinan2021**), utilizes the Feynman–Kac formula to reformulate the PDE problem as a fixed point equation. It further reduces the complexity of the numerical approximation of the time integral through a multilevel Monte Carlo approach. However, neither the deep splitting nor the MLP method can, until now, account for non-localness and Neumann boundary conditions.

The goal of this article is to overcome these limitations and thus we generalize the deep splitting method and the MLP approximation method to approximately solve non-local nonlinear PDEs with Neumann boundary conditions. We handle the non-local term by a plain vanilla Monte Carlo integration and address Neumann boundary conditions by constructing reflected stochastic processes. While the MLP method can, in one run, only provide an approximate solution at a single point $x \in D$ of the spatial domain $D \subseteq \mathbb{R}^d$ where $d \in \mathbb{N} = \{1, 2, \dots\}$, the machine learning-based method can in principle provide an approximate solution on a full subset of the spatial domain D (however, cf., e.g., (**heinrich1998monte; heinrich1999monte; grohs2021proof**) for results on limitations on the performance of such approximation schemes). We use both methods to solve five non-local nonlinear PDEs arising in models from biology and physics and cross-validate the results of the simulations. We manage to solve the non-local nonlinear PDEs with reasonable accuracy in up to 10 dimensions.

For an account of classical numerical methods for solving non-local PDEs, such as finite differences, finite elements, and spectral methods, we refer the reader to the recent survey ([delia2020numerical](#)). Several machine-learning based schemes for solving non-local PDEs can also be found in the literature. In particular, the *physics-informed neural network* and *deep Galerkin* approaches ([raissi2019physics](#); [Sirignano2018dgm](#)), based on representing an approximation of the whole solution of the PDE as a neural network and using automatic differentiation to do a least-squares minimization of the residual of the PDE, have been extended to fractional PDEs and other non-local PDEs ([pang2019fpinns](#); [lu2021deepxde](#); [guo2022monte](#); [alaradi2019extensions](#); [yuan2022apinn](#)). While some of these approaches use classical methods susceptible to the curse of dimensionality for the non-local part ([pang2019fpinns](#); [lu2021deepxde](#)), mesh-free methods suitable for high-dimensional problems have also been investigated ([guo2022monte](#); [alaradi2019extensions](#); [yuan2022apinn](#)).

The literature also contains approaches that are more closely related to the machine learning-based algorithm presented here. Frey & Köck ([frey2021deep](#); [frey2022deep](#)) propose an approximation method for non-local semilinear parabolic PDEs with Dirichlet boundary conditions based on and extending the deep splitting method in ([Beck2019](#)) and carry out numerical simulations for example PDEs in up to 4 dimensions. Castro ([castro2021deep](#)) proposes a numerical scheme for approximately solving non-local nonlinear PDEs based on ([hure2020deep](#)) and proves convergence results for this scheme. Finally, Gonon & Schwab ([gonon2021deep](#)) provide theoretical results showing that neural networks with ReLU activation functions have sufficient expressive power to approximate solutions of certain high-dimensional non-local linear PDEs without the curse of dimensionality.

There is a more extensive literature on machine learning-based methods for approximately solving standard PDEs without non-local terms but with various boundary conditions, going back to early works by [lagaris1998artificial](#); [lagaris2000neural](#) (see also [mcfall2009artificial](#)), which employed a grid-based method based on least-squares minimization of the residual and shallow neural networks to solve low-dimensional ODEs and PDEs with Dirichlet, Neumann, and mixed boundary conditions. More recently, approximation methods for PDEs with Neumann (and other) boundary conditions have been proposed using, e.g., physics-informed neural networks ([lu2021deepxde](#); [sukumar2022exact](#); [WangPerdikaris2020](#)), the *deep Ritz* method (based on a variational formulation of certain elliptic PDEs) ([e2018deep](#); [liao2021deep](#); [chen2020comparison](#)), or adversarial networks ([zang2020weak](#)).

The remainder of this article is organized as follows. ?? discusses a special case of the proposed machine learning-based method, in order to provide a readily comprehensible exposition of the key ideas of the method. ?? discusses the general case, which is flexible enough to cover a larger class of PDEs and to allow more sophisticated optimization methods. ?? presents our extension of the MLP approximation method to non-local nonlinear PDEs, which we use to obtain reference solutions in ???. ?? provides numerical simulations for five concrete examples of (non-local) nonlinear PDEs.

5.2 Machine learning-based approximation method in a special case

In this section, we present in ?? in ?? below a simplified version of our general machine learning-based algorithm for approximating solutions of non-local nonlinear PDEs with Neumann boundary conditions proposed in ?? below. This simplified version applies to a smaller class of non-local heat PDEs, specified in ?? below. In ?? we introduce some notation related to the reflection of straight lines on the boundaries of a suitable subset $D \subseteq \mathbb{R}^d$ where $d \in \mathbb{N}$, which will be used to describe time-discrete reflected stochastic processes that are employed in our approximations throughout the rest of the article. The simplified algorithm described in ?? below is limited to using neural networks of a particular architecture that are trained using plain vanilla stochastic gradient descent, whereas the full version proposed in ?? in ?? below is formulated in such a way that it encompasses a wide array of neural network architectures and more sophisticated training methods, in particular Adam optimization, minibatches, and batch normalization. Stripping away some of these more intricate aspects of the full algorithm is intended to exhibit more acutely the central ideas in the proposed approximation method.

The simplified algorithm described in this section as well as the more general version proposed in ?? in ?? below are based on the deep splitting method introduced in **Beck2019**, which combines operator splitting with a previous deep learning-based approximation method for Kolmogorov PDEs (**beck2018solving**); see also **Beck2020** for an exposition of these methods.

5.2.1 Partial differential equations (PDEs) under consideration

Let $T \in (0, \infty)$, $d \in \mathbb{N}$, let $\mathbb{D} \subseteq \mathbb{R}^d$ be a closed set with sufficiently smooth boundary $\partial_{\mathbb{D}}$, let $\mathbf{n}: \partial_{\mathbb{D}} \rightarrow \mathbb{R}^d$ be an outer unit normal vector field associated to \mathbb{D} , let $g \in C(D, \mathbb{R})$, let $\nu_x: \mathcal{B}(\mathbb{D}) \rightarrow [0, 1]$, $x \in \mathbb{D}$, be probability measures, let $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable, let $u = (u(t, x))_{(t,x) \in [0,T] \times \mathbb{D}} \in C^{1,2}([0, T] \times \mathbb{D}, \mathbb{R})$ have at most polynomially growing partial derivatives, assume¹ for every $t \in (0, T]$, $x \in \partial_{\mathbb{D}}$ that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$, and assume for every $t \in [0, T]$, $x \in \mathbb{D}$ that $u(0, x) = g(x)$, $\int_{\mathbb{D}} |f(u(t, x), u(t, \mathbf{x}))| \nu_x(d\mathbf{x}) < \infty$, and

$$(\frac{\partial}{\partial t} u)(t, x) = (\Delta_x u)(t, x) + \int_{\mathbb{D}} f(u(t, x), u(t, \mathbf{x})) \nu_x(d\mathbf{x}). \quad (5.1)$$

Our goal in this section is to approximately calculate under suitable hypotheses the solution $u: [0, T] \times \mathbb{D} \rightarrow \mathbb{R}$ of the PDE in ??.

¹Throughout this article we denote by $\langle \cdot, \cdot \rangle: (\bigcup_{n \in \mathbb{N}} (\mathbb{R}^n \times \mathbb{R}^n)) \rightarrow \mathbb{R}$ and $\|\cdot\|: (\bigcup_{n \in \mathbb{N}} \mathbb{R}^n) \rightarrow \mathbb{R}$ the functions which satisfy for all $n \in \mathbb{N}$, $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ that $\langle v, w \rangle = \sum_{i=1}^n v_i w_i$ and $\|v\| = \sqrt{\langle v, v \rangle} = [\sum_{i=1}^n |v_i|^2]^{1/2}$.

5.2.2 Reflection principle for the simulation of time discrete reflected processes

Framework 5.2.1 (Reflection principle for the simulation of time discrete reflected processes). Let $d \in \mathbb{N}$, let $\mathbb{D} \subseteq \mathbb{R}^d$ be a closed set with sufficiently smooth boundary $\partial_{\mathbb{D}}$, let $\mathbf{n}: \partial_{\mathbb{D}} \rightarrow \mathbb{R}^d$ be a suitable outer unit normal vector field associated to \mathbb{D} , let $\mathbf{c}: (\mathbb{R}^d)^2 \rightarrow \mathbb{R}^d$ satisfy for every $a, b \in \mathbb{R}^d$ that

$$\mathbf{c}(a, b) = a + [\inf(\{r \in [0, 1]: a + r(b - a) \notin \mathbb{D}\} \cup \{1\})](b - a), \quad (5.2)$$

let $\mathcal{R}: (\mathbb{R}^d)^2 \rightarrow (\mathbb{R}^d)^2$ satisfy for every $a, b \in \mathbb{R}^d$ that

$$\mathcal{R}(a, b) = \begin{cases} (a, b) & : \mathbf{c}(a, b) = a \\ (\mathbf{c}(a, b), b - 2\mathbf{n}(\mathbf{c}(a, b))\langle b - \mathbf{c}(a, b), \mathbf{n}(\mathbf{c}(a, b)) \rangle) & : \mathbf{c}(a, b) \notin \{a, b\} \\ (b, b) & : \mathbf{c}(a, b) = b, \end{cases} \quad (5.3)$$

let $P: (\mathbb{R}^d)^2 \rightarrow \mathbb{R}^d$ satisfy for every $a, b \in \mathbb{R}^d$ that $P(a, b) = b$, let $\mathcal{R}_n: (\mathbb{R}^d)^2 \rightarrow (\mathbb{R}^d)^2$, $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, satisfy for every $n \in \mathbb{N}_0$, $x, y \in \mathbb{R}^d$ that $\mathcal{R}_0(x, y) = (x, y)$ and $\mathcal{R}_{n+1}(x, y) = \mathcal{R}(\mathcal{R}_n(x, y))$, and let $R: (\mathbb{R}^d)^2 \rightarrow \mathbb{R}^d$ satisfy for every $x, y \in \mathbb{R}^d$ that

$$R(x, y) = \lim_{n \rightarrow \infty} P(\mathcal{R}_n(x, y)). \quad (5.4)$$

5.2.3 Description of the proposed approximation method in a special case

Framework 5.2.2 (Special case of the machine learning-based approximation method). Assume ??, let $T, \gamma \in (0, \infty)$, $N, M, K \in \mathbb{N}$, $g \in C^2(\mathbb{R}^d, \mathbb{R})$, $\mathfrak{d}, \mathfrak{h} \in \mathbb{N} \setminus \{1\}$, $t_0, t_1, \dots, t_N \in [0, T]$ satisfy $\mathfrak{d} = \mathfrak{h}(N + 1)d(d + 1)$ and

$$0 = t_0 < t_1 < \dots < t_N = T, \quad (5.5)$$

let $\tau_0, \tau_1, \dots, \tau_N \in [0, T]$ satisfy for every $n \in \{0, 1, \dots, N\}$ that $\tau_n = T - t_{N-n}$, let $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space, let $\xi^m: \Omega \rightarrow \mathbb{R}^d$, $m \in \mathbb{N}$, be i.i.d. $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$ -measurable random variables, let $W^m: [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $m \in \mathbb{N}$, be i.i.d. standard $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motions, for every $m \in \mathbb{N}$ let $\mathcal{Y}^m: \{0, 1, \dots, N\} \times \Omega \rightarrow \mathbb{R}^d$ be the stochastic process which satisfies for every $n \in \{0, 1, \dots, N-1\}$ that $\mathcal{Y}_0^m = \xi^m$ and

$$\mathcal{Y}_{n+1}^m = R(\mathcal{Y}_n^m, \mathcal{Y}_n^m + \sqrt{2}(W_{\tau_{n+1}}^m - W_{\tau_n}^m)), \quad (5.6)$$

let $\mathcal{L}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfy for every $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ that

$$\mathcal{L}(x) = \left(\frac{\exp(x_1)}{\exp(x_1) + 1}, \dots, \frac{\exp(x_d)}{\exp(x_d) + 1} \right), \quad (5.7)$$

for every $\theta = (\theta_1, \dots, \theta_{\mathfrak{d}}) \in \mathbb{R}^{\mathfrak{d}}$, $k, l, v \in \mathbb{N}$ with $v + l(k+1) \leq \mathfrak{d}$ let $A_{k,l}^{\theta,v}: \mathbb{R}^k \rightarrow \mathbb{R}^l$ satisfy for every $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ that

$$A_{k,l}^{\theta,v}(x) = \left(\theta_{v+kl+1} + \left[\sum_{i=1}^k x_i \theta_{v+i} \right], \dots, \theta_{v+kl+l} + \left[\sum_{i=1}^k x_i \theta_{v+(l-1)k+i} \right] \right), \quad (5.8)$$

let $\mathbb{V}_n: \mathbb{R}^{\mathfrak{d}} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $n \in \{0, 1, \dots, N\}$, satisfy for every $n \in \{1, 2, \dots, N\}$, $\theta \in \mathbb{R}^{\mathfrak{d}}$, $x \in \mathbb{R}^d$ that $\mathbb{V}_0(\theta, x) = g(x)$ and

$$\begin{aligned} \mathbb{V}_n(\theta, x) &= \\ &\left(A_{d,1}^{\theta,(\mathfrak{h}n+\mathfrak{h}-1)d(d+1)} \circ \mathcal{L} \circ A_{d,d}^{\theta,(\mathfrak{h}n+\mathfrak{h}-2)d(d+1)} \circ \dots \circ \mathcal{L} \circ A_{d,d}^{\theta,(\mathfrak{h}n+1)d(d+1)} \circ \mathcal{L} \circ A_{d,d}^{\theta,\mathfrak{h}nd(d+1)} \right)(x), \end{aligned} \quad (5.9)$$

let $\nu_x: \mathcal{B}(\mathbb{D}) \rightarrow [0, 1]$, $x \in \mathbb{D}$, be probability measures, for every $x \in \mathbb{D}$ let $Z_{x,k}^{n,m}: \Omega \rightarrow \mathbb{D}$, $k, n, m \in \mathbb{N}$, be i.i.d. random variables which satisfy for every $A \in \mathcal{B}(\mathbb{D})$ that $\mathbb{P}(Z_{x,1}^{1,1} \in A) = \nu_x(A)$, let $\Theta^n: \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^{\mathfrak{d}}$, $n \in \{0, 1, \dots, N\}$, be stochastic processes, for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$ let $\phi^{n,m}: \mathbb{R}^{\mathfrak{d}} \times \Omega \rightarrow \mathbb{R}$ satisfy for every $\theta \in \mathbb{R}^{\mathfrak{d}}$, $\omega \in \Omega$ that

$$\begin{aligned} \phi^{n,m}(\theta, \omega) &= \left[\mathbb{V}_n(\theta, \mathcal{Y}_{N-n}^m(\omega)) - \mathbb{V}_{n-1}(\Theta_M^{n-1}(\omega), \mathcal{Y}_{N-n+1}^m(\omega)) \right. \\ &\quad \left. - \frac{(t_n - t_{n-1})}{K} \left[\sum_{k=1}^K f(\mathbb{V}_{n-1}(\Theta_M^{n-1}(\omega), \mathcal{Y}_{N-n+1}^m(\omega)), \mathbb{V}_{n-1}(\Theta_M^{n-1}(\omega), Z_{\mathcal{Y}_{N-n+1}^m(\omega), k}^{n,m}(\omega))) \right] \right]^2, \end{aligned} \quad (5.10)$$

for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$ let $\Phi^{n,m}: \mathbb{R}^{\mathfrak{d}} \times \Omega \rightarrow \mathbb{R}^{\mathfrak{d}}$ satisfy for every $\theta \in \mathbb{R}^{\mathfrak{d}}$, $\omega \in \Omega$ that $\Phi^{n,m}(\theta, \omega) = (\nabla_\theta \phi^{n,m})(\theta, \omega)$, and assume for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$ that

$$\Theta_m^n = \Theta_{m-1}^n - \gamma \Phi^{n,m}(\Theta_{m-1}^n). \quad (5.11)$$

As indicated in ?? above, the algorithm described in ?? computes an approximation for a solution of the PDE in ??, i.e., a function $u \in C^{1,2}([0, T] \times D, \mathbb{R})$ which has at most polynomially growing derivatives, which satisfies for every $t \in (0, T]$, $x \in \partial_D$ that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$ and which satisfies for every $t \in [0, T]$, $x \in \mathbb{D}$ that $u(0, x) = g(x)$, $\int_{\mathbb{D}} |f(u(t, x), u(t, \mathbf{x}))| \nu_x(d\mathbf{x}) < \infty$, and

$$(\frac{\partial}{\partial t} u)(t, x) = (\Delta_x u)(t, x) + \int_{\mathbb{D}} f(u(t, x), u(t, \mathbf{x})) \nu_x(d\mathbf{x}). \quad (5.12)$$

Let us now add some explanatory comments on the objects and notations employed in ?? above. The algorithm in ?? decomposes the time interval $[0, T]$ into N subintervals at the times $t_0, t_1, t_2, \dots, t_N \in [0, T]$ (cf. ??). For every $n \in \{1, 2, \dots, N\}$ we aim to approximate the function $\mathbb{R}^d \ni x \mapsto u(t_n, x) \in \mathbb{R}$ by a suitable (realization function of a) fully-connected feedforward neural network. Each of these neural networks is an alternating composition of $\mathfrak{h} - 1$ affine linear functions from \mathbb{R}^d to \mathbb{R}^d (where we think of $\mathfrak{h} \in \mathbb{N} \setminus \{1\}$ as the *length* or *depth* of the neural network), $\mathfrak{h} - 1$ instances of a d -dimensional version of the standard logistic function and finally an affine linear function from \mathbb{R}^d to \mathbb{R} . Every such neural network can be specified by means of $(\mathfrak{h} - 1)(d^2 + d) + d + 1 \leq \mathfrak{h}d(d + 1)$ real parameters and so $N + 1$ of these neural networks can be specified by a parameter vector of length

$\mathfrak{d} = \mathfrak{h}(N+1)d(d+1) \in \mathbb{N}$. Note that $\mathcal{L}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ in ?? above denotes the d -dimensional version of the standard logistic function (cf. ??) and for every $k, l, v \in \mathbb{N}$, $\theta \in \mathbb{R}^{\mathfrak{d}}$ with $v + kl + l \leq \mathfrak{d}$ the function $A_{k,l}^{\theta,v}: \mathbb{R}^k \rightarrow \mathbb{R}^l$ in ?? denotes an affine linear function specified by means of the parameters $v+1, v+2, \dots, v+kl+l$ (cf. ??). Furthermore, observe that for every $n \in \{1, 2, \dots, N\}$, $\theta \in \mathbb{R}^{\mathfrak{d}}$ the function

$$\mathbb{R}^d \ni x \mapsto \mathbb{V}_n(\theta, x) \in \mathbb{R} \quad (5.13)$$

denotes a neural network specified by means of the parameters $\mathfrak{h}nd(d+1)+1, \mathfrak{h}nd(d+1)+2, \dots, (\mathfrak{h}n+\mathfrak{h}-1)d(d+1)+d+1$.

The goal of the optimization algorithm in ?? above is to find a suitable parameter vector $\theta \in \mathbb{R}^{\mathfrak{d}}$ such that for every $n \in \{1, 2, \dots, N\}$ the neural network $\mathbb{R}^d \ni x \mapsto \mathbb{V}_n(\theta, x) \in \mathbb{R}$ is a good approximation for the solution $\mathbb{R}^d \ni x \mapsto u(t_n, x) \in \mathbb{R}$ to the PDE in ?? at time t_n . This is done by performing successively for each $n \in \{1, 2, \dots, N\}$ a plain vanilla stochastic gradient descent (SGD) optimization on a suitable loss function (cf. ??).

Observe that for every $n \in \{1, 2, \dots, N\}$ the stochastic process $\Theta^n: \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^{\mathfrak{d}}$ describes the successive estimates computed by the SGD algorithm for the parameter vector that represents (via $\mathbb{V}_n: \mathbb{R}^{\mathfrak{d}} \times \mathbb{R}^d \rightarrow \mathbb{R}$) a suitable approximation to the solution $\mathbb{R}^d \ni x \mapsto u(t_n, x) \in \mathbb{R}$ of the PDE in ?? at time t_n . Next note that $M \in \mathbb{N}$ in ?? above denotes the number of gradient descent steps taken for each $n \in \{1, 2, \dots, N\}$ and that $\gamma \in (0, \infty)$ denotes the learning rate employed in the SGD algorithm. Moreover, observe that for every $n \in \{1, 2, \dots, N\}$, $m \in \{1, 2, \dots, M\}$ the function $\phi^{n,m}: \mathbb{R}^{\mathfrak{d}} \times \Omega \rightarrow \mathbb{R}$ denotes the loss function employed in the m th gradient descent step during the approximation of the solution of the PDE in ?? at time t_n (cf. ??). The loss functions employ a family of i.i.d. time-discrete stochastic processes $\mathcal{Y}^m: \{0, 1, \dots, N\} \times \Omega \rightarrow \mathbb{R}^d$, $m \in \mathbb{N}$, which we think of as discretizations of suitable reflected Brownian motions (cf. ??). In addition, for every $n \in \{1, 2, \dots, N\}$, $m \in \{1, 2, \dots, M\}$, $x \in D$ the loss function $\phi^{n,m}: \mathbb{R}^{\mathfrak{d}} \times \Omega \rightarrow \mathbb{R}$ employs a family of i.i.d. random variables $Z_{x,k}^{n,m}: \Omega \rightarrow D$, $k \in \mathbb{N}$, which are used for the Monte Carlo approximation of the non-local term in the PDE in ?? whose solution we are trying to approximate. The number of samples used in these Monte Carlo approximations is denoted by $K \in \mathbb{N}$ in ?? above.

Finally, for sufficiently large $N, M, K \in \mathbb{N}$ and sufficiently small $\gamma \in (0, \infty)$ the algorithm in ?? above yields for every $n \in \{1, 2, \dots, N\}$ a (random) parameter vector $\Theta_M^n: \Omega \rightarrow \mathbb{R}^{\mathfrak{d}}$ which represents a function $\mathbb{R}^d \times \Omega \ni (x, \omega) \mapsto \mathbb{V}_n(\Theta_M^n(\omega), x) \in \mathbb{R}$ that we think of as providing for every $x \in \mathbb{D}$ a suitable approximation

$$\mathbb{V}_n(\Theta_M^n, x) \approx u(t_n, x). \quad (5.14)$$

5.3 Machine learning-based approximation method in the general case

In this section we describe in ?? in ?? below the full version of our deep learning-based method for approximating solutions of non-local nonlinear PDEs with Neumann boundary

conditions (see ?? for a description of the class of PDEs our approximation method applies to), which generalizes the algorithm introduced in ?? in ?? above and which we apply in ?? below to several examples of non-local nonlinear PDEs.

5.3.1 PDEs under consideration

Let $T \in (0, \infty)$, $d \in \mathbb{N}$, let $\mathbb{D} \subseteq \mathbb{R}^d$ be a closed set with sufficiently smooth boundary $\partial_{\mathbb{D}}$, let $\mathbf{n}: \partial_{\mathbb{D}} \rightarrow \mathbb{R}^d$ be an outer unit normal vector field associated to \mathbb{D} , let $g: \mathbb{D} \rightarrow \mathbb{R}$, $\mu: \mathbb{D} \rightarrow \mathbb{R}^d$, and $\sigma: \mathbb{D} \rightarrow \mathbb{R}^{d \times d}$ be continuous, let $\nu_x: \mathcal{B}(\mathbb{D}) \rightarrow [0, 1]$, $x \in \mathbb{D}$, be probability measures, let $f: [0, T] \times \mathbb{D} \times \mathbb{D} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable, let $u = (u(t, x))_{(t,x) \in [0,T] \times \mathbb{D}} \in C^{1,2}([0, T] \times \mathbb{D}, \mathbb{R})$ have at most polynomially growing partial derivatives, assume for every $t \in [0, T]$, $x \in \partial_{\mathbb{D}}$ that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$, and assume for every $t \in [0, T]$, $x \in \mathbb{D}$ that $u(0, x) = g(x)$, $\int_{\mathbb{D}} |f(t, x, \mathbf{x}, u(t, x), u(t, \mathbf{x}))| \nu_x(d\mathbf{x}) < \infty$, and

$$\begin{aligned} \left(\frac{\partial}{\partial t} u \right)(t, x) &= \int_{\mathbb{D}} f(t, x, \mathbf{x}, u(t, x), u(t, \mathbf{x})) \nu_x(d\mathbf{x}) \\ &\quad + \langle \mu(x), (\nabla_x u)(t, x) \rangle + \frac{1}{2} \operatorname{Trace}(\sigma(x)[\sigma(x)]^*(\operatorname{Hess}_x u)(t, x)). \end{aligned} \quad (5.15)$$

Our goal is to approximately calculate under suitable hypotheses the solution $u: [0, T] \times \mathbb{D} \rightarrow \mathbb{R}$ of the PDE in ??).

5.3.2 Description of the proposed approximation method in the general case

Framework 5.3.1 (General case of the machine learning-based approximation method). Assume ??, let $T \in (0, \infty)$, $N, \varrho, \mathfrak{d}, \varsigma \in \mathbb{N}$, $(M_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}$, $(K_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$, $(J_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}$, $t_0, t_1, \dots, t_N \in [0, T]$ satisfy

$$0 = t_0 < t_1 < \dots < t_N = T, \quad (5.16)$$

let $\tau_0, \tau_1, \dots, \tau_N \in [0, T]$ satisfy for every $n \in \{0, 1, \dots, N\}$ that $\tau_n = T - t_{N-n}$, let $\nu_x: \mathcal{B}(\mathbb{D}) \rightarrow [0, 1]$, $x \in \mathbb{D}$, be probability measures, for every $x \in \mathbb{D}$ let $Z_{x,k}^{n,m,j}: \Omega \rightarrow \mathbb{D}$, $k, n, m, j \in \mathbb{N}$, be i.i.d. random variables which satisfy for every $A \in \mathcal{B}(\mathbb{D})$ that $\mathbb{P}(Z_{x,1}^{1,1,1} \in A) = \nu_x(A)$, let $f: [0, T] \times \mathbb{D} \times \mathbb{D} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space, for every $n \in \{1, 2, \dots, N\}$ let $W^{n,m,j}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $m, j \in \mathbb{N}$, be i.i.d. standard $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motions, for every $n \in \{1, 2, \dots, N\}$ let $\xi^{n,m,j}: \Omega \rightarrow \mathbb{R}^d$, $m, j \in \mathbb{N}$, be i.i.d. $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$ -measurable random variables, let $H: [0, T]^2 \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a function, for every $j \in \mathbb{N}$, $s \in \mathbb{R}^\varsigma$, $n \in \{0, 1, \dots, N\}$ let $\mathbb{V}_n^{j,s}: \mathbb{R}^\mathfrak{d} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a function, for every $n \in \{1, 2, \dots, N\}$, $m, j \in \mathbb{N}$ let $\mathcal{Y}^{n,m,j}: \{0, 1, \dots, N\} \times \Omega \rightarrow \mathbb{R}^d$ be a stochastic process which satisfies for every $k \in \{0, 1, \dots, N-1\}$ that $\mathcal{Y}_0^{n,m,j} = \xi^{n,m,j}$ and

$$\mathcal{Y}_{k+1}^{n,m,j} = H(\tau_{k+1}, \tau_k, \mathcal{Y}_k^{n,m,j}, W_{\tau_{k+1}}^{n,m,j} - W_{\tau_k}^{n,m,j}), \quad (5.17)$$

let $\Theta^n: \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^d$, $n \in \{0, 1, \dots, N\}$, be stochastic processes, for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$, $\mathbf{s} \in \mathbb{R}^c$ let $\phi^{n,m,\mathbf{s}}: \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ satisfy for every $\theta \in \mathbb{R}^d$, $\omega \in \Omega$ that

$$\begin{aligned} \phi^{n,m,\mathbf{s}}(\theta, \omega) &= \frac{1}{J_m} \sum_{j=1}^{J_m} \left[\mathbb{V}_n^{j,\mathbf{s}}(\theta, \mathcal{Y}_{N-n}^{n,m,j}(\omega)) - \mathbb{V}_{n-1}^{j,\mathbf{s}}(\Theta_{M_{n-1}}^{n-1}(\omega), \mathcal{Y}_{N-n+1}^{n,m,j}(\omega)) \right. \\ &\quad \left. - \frac{(t_n - t_{n-1})}{K_n} \left[\sum_{k=1}^{K_n} f(t_{n-1}, \mathcal{Y}_{N-n+1}^{n,m,j}(\omega), Z_{\mathcal{Y}_{N-n+1}^{n,m,j}(\omega), k}^{n,m,j}(\omega), \right. \right. \\ &\quad \left. \left. \mathbb{V}_{n-1}^{j,\mathbf{s}}(\Theta_{M_{n-1}}^{n-1}(\omega), \mathcal{Y}_{N-n+1}^{n,m,j}(\omega)), \mathbb{V}_{n-1}^{j,\mathbf{s}}(\Theta_{M_{n-1}}^{n-1}(\omega), Z_{\mathcal{Y}_{N-n+1}^{n,m,j}(\omega), k}^{n,m,j}(\omega)) \right] \right]^2, \end{aligned} \quad (5.18)$$

for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$, $\mathbf{s} \in \mathbb{R}^c$ let $\Phi^{n,m,\mathbf{s}}: \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ satisfy for every $\omega \in \Omega$, $\theta \in \{\vartheta \in \mathbb{R}^d: (\mathbb{R}^d \ni \eta \mapsto \phi^{n,m,\mathbf{s}}(\eta, \omega) \in \mathbb{R})$ is differentiable at $\vartheta\}$ that

$$\Phi^{n,m,\mathbf{s}}(\theta, \omega) = (\nabla_\theta \phi^{n,m,\mathbf{s}})(\theta, \omega), \quad (5.19)$$

let $\mathcal{S}^n: \mathbb{R}^c \times \mathbb{R}^d \times (\mathbb{R}^d)^{\{0,1,\dots,N\} \times \mathbb{N}} \rightarrow \mathbb{R}^c$, $n \in \{1, 2, \dots, N\}$, be functions, for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$ let $\psi_m^n: \mathbb{R}^e \rightarrow \mathbb{R}^d$ and $\Psi_m^n: \mathbb{R}^e \times \mathbb{R}^d \rightarrow \mathbb{R}^e$ be functions, and for every $n \in \{1, 2, \dots, N\}$ let $\mathbb{S}^n: \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^c$ and $\Xi^n: \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^e$ be stochastic processes which satisfy for every $m \in \mathbb{N}$ that

$$\mathbb{S}_m^n = \mathcal{S}^n(\mathbb{S}_{m-1}^n, \Theta_{m-1}^n, (\mathcal{Y}_k^{n,m,i})_{(k,i) \in \{0,1,\dots,N\} \times \mathbb{N}}), \quad (5.20)$$

$$\Xi_m^n = \Psi_m^n(\Xi_{m-1}^n, \Phi^{n,m,\mathbb{S}_m^n}(\Theta_{m-1}^n)), \quad \text{and} \quad \Theta_m^n = \Theta_{m-1}^n - \psi_m^n(\Xi_m^n). \quad (5.21)$$

In the setting of ?? above we think under suitable hypotheses for sufficiently large $N \in \mathbb{N}$, sufficiently large $(M_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}$, sufficiently large $(K_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$, every $n \in \{0, 1, \dots, N\}$, and every $x \in \mathbb{D}$ of $\mathbb{V}_n^{1,\mathbb{S}_{M_n}^n}(\Theta_{M_n}^n, x): \Omega \rightarrow \mathbb{R}$ as a suitable approximation

$$\mathbb{V}_n^{1,\mathbb{S}_{M_n}^n}(\Theta_{M_n}^n, x) \approx u(t_n, x) \quad (5.22)$$

of $u(t_n, x)$ where $u = (u(t, x))_{(t,x) \in [0,T] \times \mathbb{R}^d} \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ is a function with at most polynomially growing derivatives which satisfies for every $t \in (0, T]$, $x \in \partial\mathbb{D}$ that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$ and which satisfies for every $t \in [0, T]$, $x \in \mathbb{D}$ that $u(0, x) = g(x)$, $\int_{\mathbb{D}} |f(t, x, \mathbf{x}, u(t, x), u(t, \mathbf{x}))| \nu_x(d\mathbf{x}) < \infty$, and

$$\begin{aligned} \left(\frac{\partial}{\partial t} u \right)(t, x) &= \int_{\mathbb{D}} f(t, x, \mathbf{x}, u(t, x), u(t, \mathbf{x})) \nu_x(d\mathbf{x}) \\ &\quad + \langle \mu(x), (\nabla_x u)(t, x) \rangle + \frac{1}{2} \operatorname{Trace}(\sigma(x)[\sigma(x)]^*(\operatorname{Hess}_x u)(t, x)) \end{aligned} \quad (5.23)$$

(cf. ??)). Compared to the simplified algorithm in ?? above, the major new elements introduced in ?? are the following:

- (a) The numbers of gradient descent steps taken to compute approximations for the solution of the PDE at the times t_n , $n \in \{1, 2, \dots, N\}$, are allowed to vary with n , and so are specified by a sequence $(M_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}$ in ?? above.

- (b) The numbers of samples used for the Monte Carlo approximation of the non-local term in the approximation for the solution of the PDE at the times t_n , $n \in \{1, 2, \dots, N\}$, are allowed to vary with n , and so are specified by a sequence $(K_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}$ in ?? above.
- (c) The approximating functions $\mathbb{V}_n^{j,s}$, $(j, s, n) \in \mathbb{N} \times \mathbb{R}^s \times \{0, 1, \dots, N\}$, in ?? above are not specified concretely in order to allow for a variety of neural network architectures. For the concrete choice of these functions employed in our numerical simulations, we refer the reader to ??.
- (d) For every $m \in \{1, 2, \dots, M\}$ the loss function used in the m th gradient descent step may be computed using a minibatch of samples instead of just one sample (cf. ??). The sizes of these minibatches are specified by a sequence $(J_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}$.
- (e) Compared to ?? above, the more general form of the PDEs considered in this section (cf. ??) requires more flexibility in the definition of the time-discrete stochastic processes $\mathcal{Y}^{n,m,j}: \{0, 1, \dots, N\} \times \Omega \rightarrow \mathbb{R}^d$, $(n, m, j) \in \{1, 2, \dots, N\} \times \mathbb{N} \times \mathbb{N}$, which are specified in ?? above in terms of the Brownian motions $W^{n,m,j}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $(n, m, j) \in \{1, 2, \dots, N\} \times \mathbb{N} \times \mathbb{N}$, via a function $H: [0, T]^2 \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ (cf. ??). We refer the reader to ?? in ?? below, ?? in ?? below, ?? in ?? below, ?? in ?? below, and ?? in ?? below for concrete choices of H in the approximation of various example PDEs.
- (f) For every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$ the optimization step in ?? in ?? above is specified generically in terms of the functions $\psi_m^n: \mathbb{R}^\varrho \rightarrow \mathbb{R}^\vartheta$ and $\Psi_m^n: \mathbb{R}^\varrho \times \mathbb{R}^\vartheta \rightarrow \mathbb{R}^\varrho$ and the random variable $\Xi_m^n: \Omega \rightarrow \mathbb{R}^\varrho$. This generic formulation covers a variety of SGD based optimization algorithms such as Adagrad (**duchi2011adaptive**), RMSprop, or Adam (**Kingma2014**). For example, in order to implement the Adam optimization algorithm, for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$ the random variable Ξ_m^n can be used to hold suitable first and second moment estimates (see ?? and ?? in ?? below for the concrete specification of these functions implementing the Adam optimization algorithm).
- (g) The processes $\mathbb{S}^n: \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^s$, $n \in \{1, 2, \dots, N\}$, and functions $\mathcal{S}^n: \mathbb{R}^s \times \mathbb{R}^\vartheta \times (\mathbb{R}^d)^{\{0,1,\dots,N\} \times \mathbb{N}} \rightarrow \mathbb{R}^s$, $n \in \{1, 2, \dots, N\}$, in ?? above can be used to implement batch normalization; see **ioffe2015batch** for details. Loosely speaking, for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$ the random variable $\mathbb{S}_m^n: \Omega \rightarrow \mathbb{R}^s$ then holds mean and variance estimates of the outputs of each layer of the approximating neural networks related to the minibatches that are used as inputs to the neural networks in computing the loss function at the corresponding gradient descent step.

5.4 Multilevel Picard approximation method for non-local PDEs

In this section we introduce in ?? in ?? below our extension of the full history recursive multilevel Picard approximation method for approximating solutions of non-local nonlinear

ear PDEs with Neumann boundary conditions. The MLP method was first introduced in **Weinan2021**; **Hutzenthaler2020** and later extended in a number of directions; see **E2020** and **Beck2020** for recent surveys. We also refer the reader to **becker2020numerical** and **E2019multilevel** for numerical simulations illustrating the performance of MLP methods across a range of example PDE problems.

In ?? below, we will specify five concrete examples of (non-local) nonlinear PDEs and describe how ?? can be specialized to compute approximate solutions to these example PDEs. These computations will be used in ?? to obtain reference values to compare the deep learning-based approximation method proposed in ?? above against.

5.4.1 Description of the proposed approximation method

Framework 5.4.1 (Multilevel Picard approximation method). *Assume ??, let $c, T \in (0, \infty)$, $\mathfrak{I} = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$, $f \in C([0, T] \times D \times D \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(D, \mathbb{R})$, $u \in C([0, T] \times \mathbb{D}, \mathbb{R})$, assume $u|_{[0, T] \times \mathbb{D}} \in C^{1,2}([0, T] \times \mathbb{D}, \mathbb{R})$, let $\nu_x: \mathcal{B}(\mathbb{D}) \rightarrow [0, 1]$, $x \in \mathbb{D}$, be probability measures, for every $x \in \mathbb{D}$ let $Z_x^i: \Omega \rightarrow \mathbb{D}$, $i \in \mathfrak{I}$, be i.i.d. random variables, assume for every $A \in \mathcal{B}(\mathbb{D})$, $i \in \mathfrak{I}$ that $\mathbb{P}(Z_x^i \in A) = \nu_x(A)$, let $\phi_r: \mathbb{R} \rightarrow \mathbb{R}$, $r \in [0, \infty]$, satisfy for every $r \in [0, \infty]$, $y \in \mathbb{R}$ that*

$$\phi_r(y) = \min\{r, \max\{-r, y\}\}, \quad (5.24)$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{V}^i: \Omega \rightarrow (0, 1)$, $i \in \mathfrak{I}$, be independent $\mathcal{U}_{(0,1)}$ -distributed random variables, let $V^i: [0, T] \times \Omega \rightarrow [0, T]$, $i \in \mathfrak{I}$, satisfy for every $t \in [0, T]$, $i \in \mathfrak{I}$ that

$$V_t^i = t + (T - t)\mathcal{V}^i, \quad (5.25)$$

let $W^i: [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $i \in \mathfrak{I}$, be independent standard Brownian motions, assume that $(\mathcal{V}^i)_{i \in \mathfrak{I}}$ and $(W^i)_{i \in \mathfrak{I}}$ are independent, let $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be globally Lipschitz continuous, for every $x \in \mathbb{R}^d$, $i \in \mathfrak{I}$, $t \in [0, T]$ let $X_t^{x,i} = (X_{t,s}^{x,i})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathbb{R}^d$ be a stochastic process with continuous sample paths, let $(K_{n,l,m})_{n,l,m \in \mathbb{N}_0} \subseteq \mathbb{N}$, for every $i \in \mathfrak{I}$, $n, M \in \mathbb{N}_0$, $r \in [0, \infty]$ let $U_{n,M,r}^i: [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^k$ satisfy for every $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} U_{n,M,r}^i(t, x) = & \left[\sum_{l=0}^{n-1} \frac{(T-t)}{M^{n-l}} \sum_{m=1}^{M^{n-l}} \frac{1}{K_{n,l,m}} \sum_{k=1}^{K_{n,l,m}} \left[f\left(V_t^{(i,l,m)}, X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)}, Z_{X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)}}^{(i,l,m,k)}, \right. \right. \right. \right. \\ & \phi_r\left(U_{l,M,r}^{(i,l,m)}(V_t^{(i,l,m)}, X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)})\right), \phi_r\left(U_{l,M,r}^{(i,l,m)}(V_t^{(i,l,m)}, Z_{X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)}}^{(i,l,m,k)})\right) \\ & - \mathbb{1}_{\mathbb{N}}(l) f\left(V_t^{(i,l,m)}, X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)}, Z_{X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)}}^{(i,l,m,k)}, \phi_r\left(U_{\max\{l-1,0\},M,r}^{(i,l,-m)}(V_t^{(i,l,m)}, X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)})\right)\right), \\ & \left. \phi_r\left(U_{\max\{l-1,0\},M,r}^{(i,l,-m)}(V_t^{(i,l,m)}, Z_{X_{t,V_t^{(i,l,m)}}^{x,(i,l,m)}}^{(i,l,m,k)})\right)\right] \left. \right] + \frac{\mathbb{1}_{\mathbb{N}}(n)}{M^n} \left[\sum_{m=1}^{M^n} g(X_{t,T}^{x,(i,0,-m)}) \right], \end{aligned} \quad (5.26)$$

assume for every $t \in [0, T]$, $x \in \partial_D$ that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$, and assume for every $t \in [0, T]$, $x \in \mathbb{D}$ that $\|u(t, x)\| \leq c(1 + \|x\|^c)$, $u(T, x) = g(x)$, and

$$\begin{aligned} \left(\frac{\partial}{\partial t} u\right)(t, x) + \frac{1}{2} \operatorname{Trace}\left(\sigma(x)[\sigma(x)]^*(\operatorname{Hess}_x u)(t, x)\right) + \langle \mu(x), (\nabla_x u)(t, x) \rangle \\ + \int_D f(t, x, \mathbf{x}, u(t, x), u(t, \mathbf{x})) \nu_x(d\mathbf{x}) = 0. \end{aligned} \quad (5.27)$$

5.4.2 Examples for the approximation method

Example 5.4.2 (Fisher–KPP PDEs with Neumann boundary conditions). *In this example we specialize ?? to the case of certain Fisher–KPP PDEs with Neumann boundary conditions (cf., e.g., (Bian2017; Wang2021)).*

Assume ??, let $\epsilon \in (0, \infty)$ satisfy $\epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $D = [-1/2, 1/2]^d$, and $T \in \{1/5, 1/2, 1\}$, assume for every $n, l, m \in \mathbb{N}$ that $K_{n,l,m} = 1$, assume for every $t \in [0, T]$, $x, \mathbf{x} \in \mathbb{D}$, $y, \mathbf{y} \in \mathbb{R}$, $v \in \mathbb{R}^d$ that $g(x) = \exp(-\frac{1}{4}\|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, and $f(t, x, \mathbf{x}, y, \mathbf{y}) = y(1 - y)$, and assume that for every $x \in \mathbb{R}^d$, $i \in \mathfrak{I}$, $t \in [0, T]$, $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_{t,s}^{x,i} = R\left(x, x + \int_t^s \mu(X_{t,r}^{x,i}) dr + \int_t^s \sigma(X_{t,r}^{x,i}) dW_r^i\right) = R(x, x + \epsilon(W_s^i - W_t^i)). \quad (5.28)$$

The solution $u: [0, T] \times \mathbb{D} \rightarrow \mathbb{R}$ of the PDE in ?? then satisfies that for every $t \in [0, T]$, $x \in \partial_D$ it holds that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$ and that for every $t \in [0, T]$, $x \in \mathbb{D}$ it holds that $u(T, x) = \exp(-\frac{1}{4}\|x\|^2)$ and

$$\left(\frac{\partial}{\partial t} u\right)(t, x) + \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + u(t, x)(1 - u(t, x)) = 0. \quad (5.29)$$

Example 5.4.3 (Non-local competition PDEs). *In this example we specialize ?? to the case of certain non-local competition PDEs (cf., e.g., Doebeli2010; Berestycki2009b; Perthame2007; Genieys2006a).*

Assume ??, let $\mathfrak{s}, \epsilon \in (0, \infty)$ satisfy $\mathfrak{s} = \epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $D = \mathbb{R}^d$, and $T \in \{1/5, 1/2, 1\}$, assume for every $n, l, m \in \mathbb{N}$ that $K_{n,l,m} = 10$, assume for every $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_x(A) = \pi^{-d/2} \mathfrak{s}^{-d} \int_A \exp(-\mathfrak{s}^{-2}\|x - \mathbf{x}\|^2) d\mathbf{x}$, assume for every $t \in [0, T]$, $v, x, \mathbf{x} \in \mathbb{R}^d$, $y, \mathbf{y} \in \mathbb{R}$ that $g(x) = \exp(-\frac{1}{4}\|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, and $f(t, x, \mathbf{x}, y, \mathbf{y}) = y(1 - \mathbf{y}\pi^{d/2}\mathfrak{s}^d)$, and assume that for every $x \in \mathbb{R}^d$, $i \in \mathfrak{I}$, $t \in [0, T]$, $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_{t,s}^{x,i} = x + \int_t^s \mu(X_{t,r}^{x,i}) dr + \int_t^s \sigma(X_{t,r}^{x,i}) dW_r^i = x + \epsilon(W_s^i - W_t^i). \quad (5.30)$$

The solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in ?? then satisfies that for every $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that $u(T, x) = \exp(-\frac{1}{4}\|x\|^2)$ and

$$\left(\frac{\partial}{\partial t} u\right)(t, x) + \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + u(t, x) \left(1 - \int_{\mathbb{R}^d} u(t, \mathbf{x}) \exp\left(-\frac{\|\mathbf{x}-x\|^2}{\mathfrak{s}^2}\right) d\mathbf{x}\right) = 0. \quad (5.31)$$

Example 5.4.4 (Non-local sine-Gordon PDEs). *In this example we specialize ?? to the case of certain non-local sine-Gordon type PDEs (cf., e.g., Hairer2016; Barone1971; Coleman1994).*

Assume ??, let $\mathfrak{s}, \epsilon \in (0, \infty)$ satisfy $\mathfrak{s} = \epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $D = \mathbb{R}^d$, and $T \in \{1/5, 1/2, 1\}$, assume for every $n, l, m \in \mathbb{N}$ that $K_{n,l,m} = 10$, assume for every $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_x(A) = \pi^{-d/2} \mathfrak{s}^{-d} \int_A \exp(-\mathfrak{s}^{-2} \|x - \mathbf{x}\|^2) d\mathbf{x}$, assume for every $t \in [0, T]$, $v, x, \mathbf{x} \in \mathbb{R}^d$, $y, \mathbf{y} \in \mathbb{R}$ that $g(x) = \exp(-\frac{1}{4} \|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, and $f(t, x, \mathbf{x}, y, \mathbf{y}) = \sin(y) - \mathbf{y} \pi^{d/2} \mathfrak{s}^d$, and assume that for every $x \in \mathbb{R}^d$, $i \in \mathfrak{I}$, $t \in [0, T]$, $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_{t,s}^{x,i} = x + \int_t^s \mu(X_{t,r}^{x,i}) dr + \int_t^s \sigma(X_{t,r}^{x,i}) dW_r^i = x + \epsilon(W_s^i - W_t^i). \quad (5.32)$$

The solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in ?? then satisfies that for every $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that $u(T, x) = \exp(-\frac{1}{4} \|x\|^2)$ and

$$\left(\frac{\partial}{\partial t} u \right)(t, x) + \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + \sin(u(t, x)) - \int_{\mathbb{R}^d} u(t, \mathbf{x}) \exp\left(-\frac{\|\mathbf{x}-\mathbf{x}\|^2}{\mathfrak{s}^2}\right) d\mathbf{x} = 0. \quad (5.33)$$

Example 5.4.5 (Replicator-mutator PDEs). In this example we specialize ?? to the case of certain d -dimensional replicator-mutator PDEs (cf., e.g., Hamel2020).

Assume ??, let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_d, \mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_d, \mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_d \in \mathbb{R}$ satisfy for every $k \in \{1, 2, \dots, d\}$ that $\mathfrak{m}_k = \frac{1}{10}$, $\mathfrak{s}_k = \frac{1}{20}$, and $\mathfrak{u}_k = 0$, assume that $d \in \{1, 2, 5, 10\}$, $\mathbb{D} = \mathbb{R}^d$, and $T \in \{1/5, 1/2, 1\}$, assume for every $n, l, m \in \mathbb{N}$ that $K_{n,l,m} = 10$, let $a: \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy for every $x \in \mathbb{R}^d$ that $a(x) = -\frac{1}{2} \|x\|^2$, assume for every $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_x(A) = \int_{A \cap [-1/2, 1/2]^d} d\mathbf{x}$, assume for every $t \in [0, T]$, $v = (v_1, \dots, v_d)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\mathbf{x} \in \mathbb{R}^d$, $y, \mathbf{y} \in \mathbb{R}$ that $g(x) = (2\pi)^{-d/2} [\prod_{i=1}^d |\mathfrak{s}_i|^{-1/2}] \exp\left(-\sum_{i=1}^d \frac{(x_i - \mathfrak{u}_i)^2}{2\mathfrak{s}_i}\right)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = (\mathfrak{m}_1 v_1, \dots, \mathfrak{m}_d v_d)$, and

$$f(t, x, \mathbf{x}, y, \mathbf{y}) = y(a(x) - \mathbf{y}a(\mathbf{x})), \quad (5.34)$$

and assume that for every $x \in \mathbb{R}^d$, $i \in \mathfrak{I}$, $t \in [0, T]$, $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_{t,s}^{x,i} = x + \int_t^s \mu(X_{t,r}^{x,i}) dr + \int_t^s \sigma(X_{t,r}^{x,i}) dW_r^i = x + \sigma(0)(W_s^i - W_t^i). \quad (5.35)$$

The solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in ?? then satisfies that for every $t \in [0, T]$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that $u(T, x) = (2\pi)^{-d/2} [\prod_{i=1}^d |\mathfrak{s}_i|^{-1/2}] \exp\left(-\sum_{i=1}^d \frac{(x_i - \mathfrak{u}_i)^2}{2\mathfrak{s}_i}\right)$ and

$$\left(\frac{\partial}{\partial t} u \right)(t, x) + u(t, x) \left(a(x) - \int_{[-1/2, 1/2]^d} u(t, \mathbf{x}) a(\mathbf{x}) d\mathbf{x} \right) + \sum_{i=1}^d \frac{1}{2} |\mathfrak{m}_i|^2 \left(\frac{\partial^2}{\partial x_i^2} u \right)(t, x) = 0. \quad (5.36)$$

Example 5.4.6 (Allen–Cahn PDEs with conservation of mass). In this example we specialize ?? to the case of certain Allen–Cahn PDEs with cubic nonlinearity, conservation of mass, and no-flux boundary conditions (cf., e.g., Rubinstein & Sternberg (RUBINSTEIN1992)).

Assume ??, let $\epsilon \in (0, \infty)$ satisfy $\epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $D = [-1/2, 1/2]^d$, and $T \in \{1/5, 1/2, 1\}$, assume for every $n, l, m \in \mathbb{N}$ that $K_{n,l,m} = 10$, assume for every $x \in D$, $A \in \mathcal{B}(D)$ that $\nu_x(A) = \int_A d\mathbf{x}$, assume for every $t \in [0, T]$, $x, \mathbf{x} \in \mathbb{D}$, $y, \mathbf{y} \in \mathbb{R}$, $v \in \mathbb{R}^d$ that

$g(x) = \exp(-\frac{1}{4}\|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, and $f(t, x, \mathbf{x}, y, \mathbf{y}) = y - y^3 - (\mathbf{y} - \mathbf{y}^3)$, and assume that for every $x \in \mathbb{R}^d$, $i \in \mathcal{I}$, $t \in [0, T]$, $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_{t,s}^{x,i} = R\left(x, x + \int_t^s \mu(X_{t,r}^{x,i}) dr + \int_t^s \sigma(X_{t,r}^{x,i}) dW_r^i\right) = R(x, x + \epsilon(W_s^i - W_t^i)). \quad (5.37)$$

The solution $u: [0, T] \times \mathbb{D} \rightarrow \mathbb{R}$ of the PDE in (??) then satisfies that for every $t \in [0, T]$, $x \in \partial_{\mathbb{D}}$ it holds that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$ and that for every $t \in [0, T]$, $x \in \mathbb{D}$ it holds that $u(T, x) = \exp(-\frac{1}{4}\|x\|^2)$ and

$$\left(\frac{\partial}{\partial t} u\right)(t, x) + \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + u(t, x) - [u(t, x)]^3 - \int_{[-1/2, 1/2]^d} u(t, \mathbf{x}) - [u(t, \mathbf{x})]^3 d\mathbf{x} = 0. \quad (5.38)$$

5.5 Numerical simulations

In this section we illustrate the performance of the machine learning-based approximation method proposed in ?? in ?? above by means of numerical simulations for five concrete (non-local) nonlinear PDEs; see ?????????? below. In each of these numerical simulations we employ the general machine learning-based approximation method proposed in ?? with certain 4-layer neural networks and using the Adam optimizer (cf. (??) and (??) in ?? below and Kingma & Ba (Kingma2014)).

More precisely, in each of the numerical simulations in ?????????? the functions $V_n^{j,s}: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ with $n \in \{1, 2, \dots, N\}$, $j \in \{1, 2, \dots, 8000\}$, $s \in \mathbb{R}^c$ are implemented as N fully-connected feedforward neural networks. These neural networks consist of 4 layers (corresponding to 3 affine linear transformations in the neural networks) where the input layer is d -dimensional (with d neurons on the input layer), where the two hidden layers are both $(d + 50)$ -dimensional (with $d + 50$ neurons on each of the two hidden layers), and where the output layer is 1-dimensional (with 1 neuron on the output layer). We refer to ?? for a graphical illustration of the neural network architecture used in the numerical simulations in ??????????.

As activation functions just in front of the two hidden layers we employ, in ??????? below, multidimensional versions of the hyperbolic tangent function

$$\mathbb{R} \ni x \mapsto (e^x + e^{-x})^{-1}(e^x - e^{-x}) \in \mathbb{R}, \quad (5.39)$$

and we employ, in ?? below, multidimensional versions of the ReLU function

$$\mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R}. \quad (5.40)$$

In addition, in ????? we use the square function $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ as activation function just in front of the output layer and in ??? we use the identity function $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$ as activation function just in front of the output layer. Furthermore, we employ Xavier initialization to initialize all neural network parameters; see glorot2010 for details. We did not employ batch normalization in our simulations.

Each of the numerical experiments presented below was performed with the Julia library HighDimPDE.jl on a NVIDIA TITAN RTX GPU with 1350 MHz core clock and 24 GB GDDR6

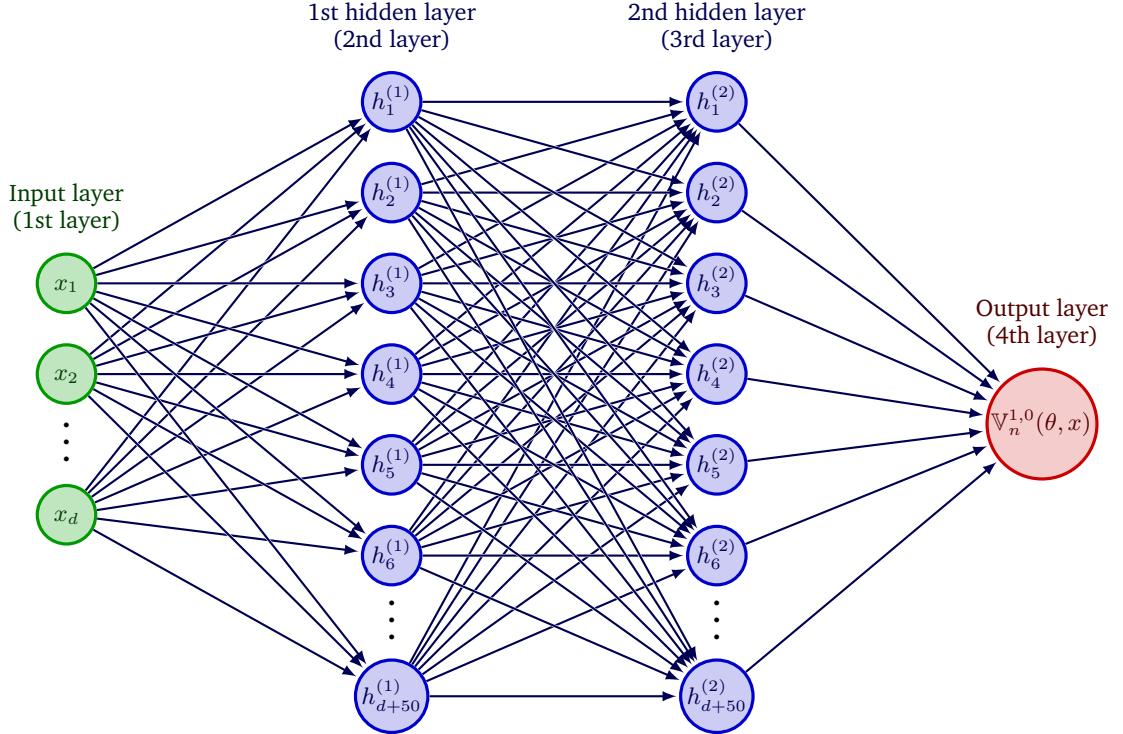


Fig. 5.1: Graphical illustration of the neural network architecture used in the numerical simulations. In ????????? we employ neural networks with 4 layers (corresponding to 3 affine linear transformations in the neural networks) with d neurons on the input layer (corresponding to a d -dimensional input layer), with $d + 50$ neurons on the 1st hidden layer (corresponding to a $(d + 50)$ -dimensional 1st hidden layer), with $d + 50$ neurons on the 2nd hidden layer (corresponding to a $(d + 50)$ -dimensional 2nd hidden layer), and with 1 neuron on the output layer (corresponding to a 1-dimensional output layer) in the numerical simulations.

memory with 7000 MHz clock rate where the underlying system consisted of an AMD EPYC 7742 64-core CPU with 2TB memory running Julia 1.7.2 on Ubuntu 20.04.3.

Framework 5.5.1. Assume ??, assume $\mathfrak{d} = (d + 50)(d + 1) + (d + 50)(d + 51) + (d + 51)$, let $\varepsilon, \beta_1, \beta_2 \in \mathbb{R}$, $(\gamma_m)_{m \in \mathbb{N}} \subseteq (0, \infty)$ satisfy $\varepsilon = 10^{-8}$, $\beta_1 = \frac{9}{10}$, and $\beta_2 = \frac{999}{1000}$, let $g: D \rightarrow \mathbb{R}$, $\mu: D \rightarrow \mathbb{R}^d$, and $\sigma: D \rightarrow \mathbb{R}^{d \times d}$ be continuous, let $u = (u(t, x))_{(t, x) \in [0, T] \times \mathbb{D}} \in C^{1,2}([0, T] \times \mathbb{D}, \mathbb{R})$ have at most polynomially growing partial derivatives, assume for every $t \in (0, T]$, $x \in \partial_{\mathbb{D}}$ that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$, assume for every $t \in [0, T]$, $x \in \mathbb{D}$, $j \in \mathbb{N}$, $s \in \mathbb{R}^s$ that $u(0, x) = g(x) = \mathbb{V}_0^{j,s}(\theta, x)$, $\int_{\mathbb{D}} |f(t, x, \mathbf{x}, u(t, x), u(t, \mathbf{x}))| \nu_x(d\mathbf{x}) < \infty$, and

$$\begin{aligned} \left(\frac{\partial}{\partial t} u \right)(t, x) &= \frac{1}{2} \text{Trace}(\sigma(x)[\sigma(x)]^*(\text{Hess}_x u)(t, x)) + \langle \mu(x), (\nabla_x u)(t, x) \rangle \\ &\quad + \int_{\mathbb{D}} f(t, x, \mathbf{x}, u(t, x), u(t, \mathbf{x})) \nu_x(d\mathbf{x}), \end{aligned} \tag{5.41}$$

assume for every $m \in \mathbb{N}$, $i \in \{0, 1, \dots, N\}$ that $J_m = 8000$, $t_i = \frac{iT}{N}$, and $\varrho = 2\delta$, and assume for every $n \in \{1, 2, \dots, N\}$, $m \in \mathbb{N}$, $x = (x_1, \dots, x_\delta)$, $y = (y_1, \dots, y_\delta)$, $\eta = (\eta_1, \dots, \eta_\delta) \in \mathbb{R}^\delta$ that

$$\Xi_0^n(x, y, \eta) = 0, \quad \Psi_m^n(x, y, \eta) = (\beta_1 x + (1 - \beta_1)\eta, \beta_2 y + (1 - \beta_2)((\eta_1)^2, \dots, (\eta_\delta)^2)), \quad (5.42)$$

and

$$\psi_m^n(x, y) = \left(\left[\sqrt{\frac{|y_1|}{1 - (\beta_2)^m}} + \varepsilon \right]^{-1} \frac{\gamma_m x_1}{1 - (\beta_1)^m}, \dots, \left[\sqrt{\frac{|y_\delta|}{1 - (\beta_2)^m}} + \varepsilon \right]^{-1} \frac{\gamma_m x_\delta}{1 - (\beta_1)^m} \right). \quad (5.43)$$

5.5.1 Fisher–KPP PDEs with Neumann boundary conditions

In this subsection we use the machine learning-based approximation method in ?? to approximately calculate the solutions of certain Fisher–KPP PDEs with Neumann boundary conditions (cf., e.g., **Bian2017** and **Wang2021**).

Assume ??, let $\epsilon \in (0, \infty)$ satisfy $\epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $D = [-1/2, 1/2]^d$, $T \in \{1/5, 1/2, 1\}$, $N = 10$, $K_1 = K_2 = \dots = K_N = 1$, and $M_1 = M_2 = \dots = M_N = 500$, assume for every $n, m, j \in \mathbb{N}$, $\omega \in \Omega$ that $\xi^{n,m,j}(\omega) = (0, \dots, 0)$, assume for every $m \in \mathbb{N}$ that $\gamma_m = 10^{-2}$, and assume for every $s, t \in [0, T]$, $x, \mathbf{x} \in \mathbb{D}$, $y, \mathbf{y} \in \mathbb{R}$, $v \in \mathbb{R}^d$ that $g(x) = \exp(-\frac{1}{4}\|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, $f(t, x, \mathbf{x}, y, \mathbf{y}) = y(1 - y)$, and

$$H(t, s, x, v) = R(x, x + \mu(x)(t - s) + \sigma(x)v) = R(x, x + \epsilon v) \quad (5.44)$$

(cf. ?? and ??). The solution $u: [0, T] \times \mathbb{D} \rightarrow \mathbb{R}$ of the PDE in ?? then satisfies that for every $t \in (0, T]$, $x \in \partial_{\mathbb{D}}$ it holds that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$ and that for every $t \in [0, T]$, $x \in \mathbb{D}$ it holds that $u(0, x) = \exp(-\frac{1}{4}\|x\|^2)$ and

$$\left(\frac{\partial}{\partial t} u \right)(t, x) = \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + u(t, x)(1 - u(t, x)). \quad (5.45)$$

In ?? the function $u: [0, T] \times D \rightarrow \mathbb{R}$ models the proportion of a particular type of alleles in a biological population spatially structured over D . For every $t \in [0, T]$, $x \in \mathbb{R}^d$ the number $u(t, x) \in \mathbb{R}$ describes the proportion of individuals with a particular type of alleles located at position $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ at time $t \in [0, T]$. In ?? we use the machine learning-based approximation method in ?? to approximately calculate the mean of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the standard deviation of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the relative L^1 -approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the uncorrected sample standard deviation of the approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, and the average runtime in seconds needed for calculating one realization of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$ based on 5 independent realizations (5 independent runs). The reference value, which is used as an approximation for the unknown value $u(T, (0, \dots, 0))$ of the exact solution of ??, has been calculated via the MLP approximation method for non-local nonlinear PDEs in ?? (cf. ?? and **Beck2017a**).

d	T	N	Mean of the approx. method	Standard deviation of the approx. method	Reference value	Relative L^1 -approx. error	Standard deviation of the error	Average runtime in seconds
1	$\frac{1}{5}$	10	0.9995902	0.0000107	0.9996057	0.0000155	0.0000107	24.887
2	$\frac{1}{5}$	10	0.9991759	0.0000191	0.9991887	0.0000186	0.0000116	26.175
5	$\frac{1}{5}$	10	0.9979572	0.0000388	0.9979693	0.0000303	0.0000235	27.312
10	$\frac{1}{5}$	10	0.9959224	0.0000341	0.9959337	0.0000275	0.0000196	28.972
1	$\frac{1}{2}$	10	0.9992463	0.0000341	0.9992572	0.0000237	0.0000248	26.631
2	$\frac{1}{2}$	10	0.9984982	0.0000287	0.9985442	0.0000460	0.0000287	27.007
5	$\frac{1}{2}$	10	0.9962227	0.0000330	0.9962314	0.0000306	0.0000041	27.632
10	$\frac{1}{2}$	10	0.9925257	0.0001663	0.9921744	0.0003541	0.0001676	28.743
1	1	10	0.9991423	0.0000331	0.9989768	0.0001657	0.0000332	26.601
2	1	10	0.9982349	0.0000782	0.9982498	0.0000605	0.0000430	26.965
5	1	10	0.9956516	0.0000853	0.9957053	0.0000839	0.0000466	27.428
10	1	10	0.9912297	0.0001072	0.9904936	0.0007431	0.0001083	28.521

Tab. 5.1: Numerical simulations for the approximation method in ?? in the case of the Fisher–KPP PDEs with Neumann boundary conditions in (??) in ??.

5.5.2 Non-local competition PDEs

In this subsection we use the machine learning-based approximation method in ?? to approximately calculate the solutions of certain non-local competition PDEs (cf., e.g., **Doebeli2010**; **Berestycki2009b**; **Perthame2007**; **Genieys2006a**).

Assume ??, let $s, \epsilon \in (0, \infty)$ satisfy $s = \epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $\mathbb{D} = \mathbb{R}^d$, $T \in \{1/5, 1/2, 1\}$, $N = 10$, $K_1 = K_2 = \dots = K_N = 5$, and $M_1 = M_2 = \dots = M_N = 500$, assume for every $n, m, j \in \mathbb{N}$, $\omega \in \Omega$ that $\xi^{n,m,j}(\omega) = (0, \dots, 0)$, assume for every $m \in \mathbb{N}$ that $\gamma_m = 10^{-2}$, and assume for every $s, t \in [0, T]$, $v, x, \mathbf{x} \in \mathbb{R}^d$, $y, \mathbf{y} \in \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_x(A) = \pi^{-d/2} s^{-d} \int_A \exp(-s^{-2} \|x - \mathbf{x}\|^2) d\mathbf{x}$, $g(x) = \exp(-\frac{1}{4} \|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, $f(t, x, \mathbf{x}, y, \mathbf{y}) = y(1 - \mathbf{y}s^d \pi^{d/2})$, and

$$H(t, s, x, v) = x + \mu(x)(t - s) + \sigma(x)v = x + \epsilon v \quad (5.46)$$

(cf. (??) and (??)). The solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in (??) then satisfies that for every $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that $u(0, x) = \exp(-\frac{1}{4} \|x\|^2)$ and

$$(\frac{\partial}{\partial t} u)(t, x) = \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + u(t, x) \left(1 - \int_{\mathbb{R}^d} u(t, \mathbf{x}) \exp\left(-\frac{\|\mathbf{x}-x\|^2}{s^2}\right) d\mathbf{x} \right). \quad (5.47)$$

In (??) the function $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ models the evolution of a population characterized by a set of d biological traits under the combined effects of selection, competition and mutation. For every $t \in [0, T]$, $x \in \mathbb{R}^d$ the number $u(t, x) \in \mathbb{R}$ describes the number of individuals with traits $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ at time $t \in [0, T]$. In ?? we use the machine learning-based approximation method in ?? to approximately calculate the mean of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the standard deviation of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the relative L^1 -approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the uncorrected sample standard

d	T	N	Mean of the approx. method	Standard deviation of the approx. method	Reference value	Relative L^1 -approx. error	Standard deviation of the error	Average runtime in seconds
1	$1/5$	5	1.1748404	0.0006512	1.1735975	0.0010591	0.0005549	20.571
2	$1/5$	5	1.2114236	0.0008700	1.2096305	0.0014823	0.0007193	25.042
5	$1/5$	5	1.2186650	0.0007070	1.2159038	0.0022709	0.0005814	54.644
10	$1/5$	5	1.2153864	0.0007789	1.2128666	0.0020776	0.0006422	74.331
1	$1/2$	5	1.4755801	0.0032738	1.4694976	0.0041392	0.0022278	20.182
2	$1/2$	5	1.6112576	0.0110426	1.5948898	0.0103067	0.0068414	25.178
5	$1/2$	5	1.6433913	0.0067468	1.6186897	0.0152602	0.0041681	53.618
10	$1/2$	5	1.6323552	0.0053956	1.6090688	0.0144720	0.0033532	73.648
1	1	5	2.0795628	0.0223341	2.0493301	0.0147525	0.0108982	19.836
2	1	5	2.5651031	0.0513671	2.4683060	0.0392160	0.0208107	24.700
5	1	5	2.6977694	0.0381160	2.5606137	0.0535636	0.0148855	52.343
10	1	5	2.6490054	0.0155291	2.5299994	0.0470380	0.0061380	73.186

Tab. 5.2: Numerical simulations for the approximation method in ?? in the case of the non-local competition PDEs in (??) in ??.

deviation of the approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, and the average runtime in seconds needed for calculating one realization of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$ based on 5 independent realizations (5 independent runs). The reference value, which is used as an approximation for the unknown value $u(T, (0, \dots, 0))$ of the exact solution of (??), has been calculated via the MLP approximation method for non-local nonlinear PDEs in ?? (cf. ?? and Beck2017a).

5.5.3 Non-local sine-Gordon type PDEs

In this subsection we use the machine learning-based approximation method in ?? to approximately calculate the solutions of non-local sine-Gordon type PDEs (cf., e.g., Hairer2016; Barone1971; Coleman1994).

Assume ??, let $\mathfrak{s}, \epsilon \in (0, \infty)$ satisfy $\mathfrak{s} = \epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $\mathbb{D} = \mathbb{R}^d$, $T \in \{1/5, 1/2, 1\}$, $N = 10$, $K_1 = K_2 = \dots = K_N = 5$, and $M_1 = M_2 = \dots = M_N = 500$, assume for every $n, m, j \in \mathbb{N}$, $\omega \in \Omega$ that $\xi^{n,m,j}(\omega) = (0, \dots, 0)$, assume for every $m \in \mathbb{N}$ that $\gamma_m = 10^{-3}$, and assume for every $s, t \in [0, T]$, $v, x, \mathbf{x} \in \mathbb{R}^d$, $y, \mathbf{y} \in \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_x(A) = \pi^{-d/2} \mathfrak{s}^{-d} \int_A \exp(-\mathfrak{s}^{-2} \|x - \mathbf{x}\|^2) d\mathbf{x}$, $g(x) = \exp(-\frac{1}{4} \|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, $f(t, x, \mathbf{x}, y, \mathbf{y}) = \sin(y) - \mathbf{y} \pi^{d/2} \mathfrak{s}^d$, and

$$H(t, s, x, v) = x + \mu(x)(t - s) + \sigma(x)v = x + \epsilon v \quad (5.48)$$

(cf. (??) and (??)). The solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in (??) then satisfies that for every $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that $u(0, x) = \exp(-\frac{1}{4} \|x\|^2)$ and

$$(\frac{\partial}{\partial t} u)(t, x) = \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + \sin(u(t, x)) - \int_{\mathbb{R}^d} u(t, \mathbf{x}) \exp\left(-\frac{\|x - \mathbf{x}\|^2}{\mathfrak{s}^2}\right) d\mathbf{x}. \quad (5.49)$$

d	T	N	Mean of the approx. method	Standard deviation of the approx. method	Reference value	Relative L^1 -approx. error	Standard deviation of the error	Average runtime in seconds
1	$1/5$	10	1.1363013	0.0000101	1.1366512	0.0003079	0.0000089	23.635
2	$1/5$	10	1.1678476	0.0000118	1.1685004	0.0005586	0.0000101	24.788
5	$1/5$	10	1.1731812	0.0000087	1.1740671	0.0007546	0.0000074	24.233
10	$1/5$	10	1.1704700	0.0000063	1.1715686	0.0009377	0.0000054	24.767
1	$1/2$	10	1.3514235	0.0000152	1.3529022	0.0010930	0.0000112	22.622
2	$1/2$	10	1.4393708	0.0000245	1.4423641	0.0020753	0.0000170	23.419
5	$1/2$	10	1.4546282	0.0000816	1.4598476	0.0035754	0.0000559	23.739
10	$1/2$	10	1.4473282	0.0000739	1.4503958	0.0021150	0.0000510	24.222
1	1	10	1.7114614	0.0000309	1.7136091	0.0012533	0.0000180	22.067
2	1	10	1.9019763	0.0000288	1.9062322	0.0022326	0.0000151	22.707
5	1	10	1.9364921	0.0000602	1.9411610	0.0024052	0.0000310	22.899
10	1	10	1.9223347	0.0001494	1.9272222	0.0025360	0.0000775	23.719

Tab. 5.3: Numerical simulations for the approximation method in ?? in the case of the non-local sine-Gordon PDEs in (??) in ??.

In ?? we use the machine learning-based approximation method in ?? to approximately calculate the mean of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the standard deviation of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the relative L^1 -approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the uncorrected sample standard deviation of the approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, and the average runtime in seconds needed for calculating one realization of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$ based on 5 independent realizations (5 independent runs). The reference value, which is used as an approximation for the unknown value $u(T, (0, \dots, 0))$ of the exact solution of (??), has been calculated via the MLP approximation method for non-local nonlinear PDEs in ?? (cf. ?? and Beck2017a).

5.5.4 Replicator-mutator PDEs

In this subsection we use the machine learning-based approximation method in ?? to approximately calculate the solutions of certain replicator-mutator PDEs describing the dynamics of a phenotype distribution under the combined effects of selection and mutation (cf., e.g., Hamel2020).

Assume ??, let $\mathcal{D} \subseteq \mathbb{R}^d$, $m_1, m_2, \dots, m_d, s_1, s_2, \dots, s_d, u_1, u_2, \dots, u_d, t \in \mathbb{R}$ satisfy for every $k \in \{1, 2, \dots, d\}$ that $m_k = \frac{1}{10}$, $s_k = \frac{1}{20}$, $u_k = 0$, and $t = \frac{1}{50}$, assume that $d \in \{1, 2, 5, 10\}$, $\mathbb{D} = \mathbb{R}^d$, $T \in \{1/10, 1/5, 1/2\}$, $N = 10$, $K_1 = K_2 = \dots = K_N = 5$, let $a \in C(\mathbb{R}^d, \mathbb{R})$, $\delta \in C(\mathbb{R}^d, (0, \infty))$ satisfy for every $x \in \mathbb{R}^d$ that $a(x) = -\frac{1}{2}\|x\|^2$, and assume for every $s, t \in [0, T]$, $v = (v_1, \dots, v_d)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\mathbf{x} \in \mathbb{R}^d$, $y, \mathbf{y} \in \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_x(A) = \int_{A \cap \mathcal{D}} \delta(\mathbf{x}) d\mathbf{x}$, $g(x) = (2\pi)^{-d/2} [\prod_{i=1}^d |s_i|^{-1/2}] \exp(-\sum_{i=1}^d \frac{(x_i - u_i)^2}{2s_i})$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = (m_1 v_1, \dots, m_d v_d)$, $f(t, x, \mathbf{x}, y, \mathbf{y}) = y(a(x) - \mathbf{y}a(\mathbf{x})[\delta(\mathbf{x})]^{-1})$, and

$$H(t, s, x, v) = x + \mu(x)(t - s) + \sigma(x)v = x + (m_1 v_1, \dots, m_d v_d) \quad (5.50)$$

(cf. (??) and (??)). The solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in (??) then satisfies that for every $t \in [0, T]$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$u(0, x) = (2\pi)^{-d/2} \left[\prod_{i=1}^d |\mathfrak{s}_i|^{-1/2} \right] \exp \left(- \sum_{i=1}^d \frac{(x_i - \mathfrak{u}_i)^2}{2\mathfrak{s}_i} \right) \quad (5.51)$$

and

$$\left(\frac{\partial}{\partial t} u \right)(t, x) = u(t, x) \left(a(x) - \int_{\mathcal{D}} u(t, \mathbf{x}) a(\mathbf{x}) d\mathbf{x} \right) + \sum_{i=1}^d \frac{1}{2} |\mathfrak{m}_i|^2 \left(\frac{\partial^2}{\partial x_i^2} u \right)(t, x). \quad (5.52)$$

In (??) the function $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ models the evolution of the phenotype distribution of a population composed of a set of d biological traits under the combined effects of selection and mutation. For every $t \in [0, T]$, $x \in \mathbb{R}^d$ the number $u(t, x) \in \mathbb{R}$ describes the number of individuals with traits $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ at time $t \in [0, T]$. The function a models a quadratic Malthusian fitness function.

In ?? we use the machine learning-based method in ?? to approximately solve the PDE in ?? above in the case $\mathcal{D} = \mathbb{R}^d$. More precisely, we assume for every $n, m, j \in \mathbb{N}$ that $\xi^{n,m,j} = 0$, $\gamma_m = 1/100$, $M_n = 1000$ and we assume for every $\mathbf{x} \in \mathbb{R}^d$ that $\delta(\mathbf{x}) = (2\pi)^{-d/2} t^{-d} \exp(-\frac{\|\mathbf{x}\|^2}{2t^2})$ to approximately calculate the mean of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the standard deviation of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the relative L^1 -approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the uncorrected sample standard deviation of the approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, and the average runtime in seconds needed for calculating one realization of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$ based on 5 independent realizations (5 independent runs). The value $u(T, (0, \dots, 0))$ of the exact solution of (??) has been calculated by means of ?? below.

In ?? we use the machine learning-based method in ?? to approximate the solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in ?? above with $d = 5$, $T = 1/2$, and $\mathcal{D} = \mathbb{R}^d$. The right-hand side of ?? shows a plot of $[-1/4, 1/4] \ni x \mapsto u(t, (x, 0, \dots, 0)) \in \mathbb{R}$ for $t \in \{0, 0.05, 0.1, 0.15\}$ where u is the exact solution of the PDE in ?? with $d = 5$, $T = 1/2$, and $\mathcal{D} = \mathbb{R}^d$ computed via ?? in ?? below. The left-hand side of ?? shows a plot of $[-1/4, 1/4] \ni x \mapsto \mathbb{V}_n^{1,0}(\Theta_{M_n}^n(\omega), (x, 0, \dots, 0)) \in \mathbb{R}$ for $n \in \{0, 1, 2, 3\}$ and one realization $\omega \in \Omega$ where the functions $\mathbb{R}^d \ni x \mapsto \mathbb{V}_n^{1,0}(\Theta_{M_n}^n(\omega), x) \in \mathbb{R}$ for $n \in \{0, 1, 2, 3\}$, $\omega \in \Omega$ were computed via ?? as an approximation of the solution of the PDE in ?? with $d = 5$, $T = 1/2$, and $\mathcal{D} = [-1/2, 1/2]^d$. For the approximation, we take $M_1 = M_2 = \dots = M_N = 2000$, $\gamma_1 = \gamma_2 = \dots = \gamma_{2000} = 1/200$, and $\delta = \mathbb{1}_{\mathbb{R}^d}$ and we take $\xi^{n,m,j}: \Omega \rightarrow \mathbb{R}^d$, $n, m, j \in \mathbb{N}$, to be independent $\mathcal{U}_{[-1/2, 1/2]^d}$ -distributed random variables. Note that the solution of the PDE in ?? in the case $\mathcal{D} = [-R, R]^d$ with $R \in (0, \infty)$ sufficiently large is a good approximation of the solution $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the PDE in ?? in the case $\mathcal{D} = \mathbb{R}^d$ since we have that for all $t \in [0, T]$ the value $u(t, x)$ of the solution u of the PDE in ?? in the case $\mathcal{D} = \mathbb{R}^d$ quickly tends to 0 as $\|x\|$ tends to ∞ .

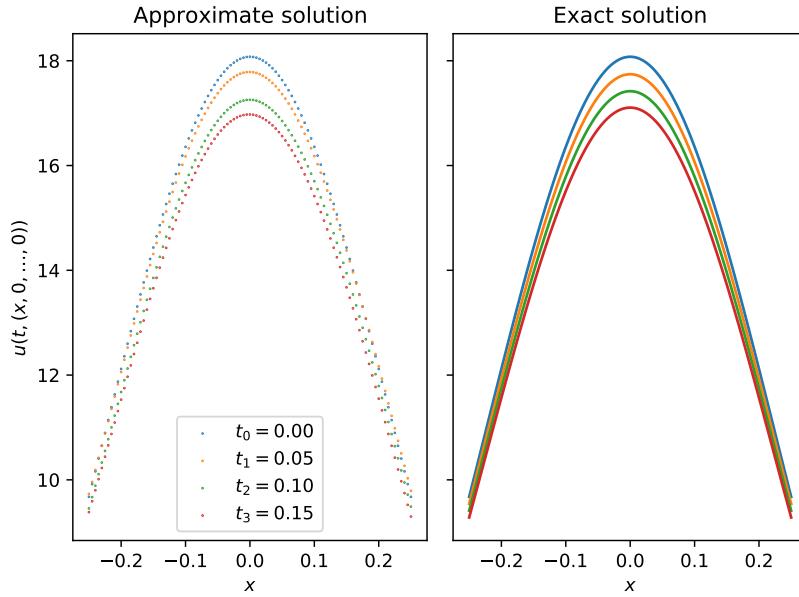


Fig. 5.2: Plot of a machine learning-based approximation of the solution of the replicator-mutator PDE in ?? in the case $d = 5$, $T = 1/2$, and $\mathcal{D} = \mathbb{R}^d$. The left-hand side shows a plot of $[-1/4, 1/4] \ni x \mapsto \mathbb{V}_n^{1,0}(\Theta_{M_n}^n(\omega), (x, 0, \dots, 0)) \in \mathbb{R}$ for $n \in \{0, 1, 2, 3\}$ and one realization $\omega \in \Omega$ where the functions $\mathbb{R}^d \ni x \mapsto \mathbb{V}_n^{1,0}(\Theta_{M_n}^n(\omega), x) \in \mathbb{R}$ for $n \in \{0, 1, 2, 3\}$, $\omega \in \Omega$ were computed via ?? as an approximation of the solution of the PDE in ?? with $d = 5$, $T = 1/2$, and $\mathcal{D} = [-1/2, 1/2]^d$ where we take $M_1 = M_2 = \dots = M_N = 2000$, $\gamma_1 = \gamma_2 = \dots = \gamma_{2000} = 1/200$, and $\delta = \mathbb{1}_{\mathbb{R}^d}$ and where we take $\xi^{n,m,j}: \Omega \rightarrow \mathbb{R}^d$, $n, m, j \in \mathbb{N}$, to be independent $\mathcal{U}_{[-1/2, 1/2]^d}$ -distributed random variables. The right-hand side of ?? shows a plot of $[-1/4, 1/4] \ni x \mapsto u(t, (x, 0, \dots, 0)) \in \mathbb{R}$ for $t \in \{0, 0.05, 0.1, 0.15\}$ where u is the exact solution of the PDE in ?? with $d = 5$, $T = 1/2$, and $\mathcal{D} = \mathbb{R}^d$.

d	T	N	Mean of the approx. method	Standard deviation of the approx. method	Reference value	Relative L^1 -approx. error	Standard deviation of the error	Average runtime in seconds
1	$1/10$	10	1.7650547	0.0048907	1.7709574	0.0033330	0.0027616	43.949
2	$1/10$	10	3.1210874	0.0015513	3.1362901	0.0048474	0.0004946	45.002
5	$1/10$	10	17.1948978	0.0160821	17.4196954	0.0129048	0.0009232	45.934
10	$1/10$	10	295.8776489	0.0572639	303.4457874	0.0249407	0.0001887	47.750
1	$1/5$	10	1.7499938	0.0005580	1.7582066	0.0046711	0.0003174	43.129
2	$1/5$	10	3.0621917	0.0027811	3.0912904	0.0094131	0.0008996	44.443
5	$1/5$	10	16.3846066	0.0139748	16.8015567	0.0248162	0.0008318	45.019
10	$1/5$	10	268.2944397	0.0623432	282.2923073	0.0495864	0.0002208	45.612
1	$1/2$	10	1.7018557	0.0060157	1.7222757	0.0118564	0.0034929	42.092
2	$1/2$	10	2.8911286	0.0027431	2.9662336	0.0253200	0.0009248	42.657
5	$1/2$	10	14.2520916	0.1356645	15.1535149	0.0594861	0.0089527	43.338
10	$1/2$	10	201.6446228	0.3009756	229.6290127	0.1218678	0.0013107	44.190

Tab. 5.4: Numerical simulations for the approximation method in ?? in the case of the replicator-mutator PDEs in (??) in ?? where we assume for every $n, m, j \in \mathbb{N}$ that $\mathcal{D} = \mathbb{R}^d$, $\xi^{n,m,j} = 0$, $\gamma_m = 1/100$, and $M_n = 1000$ and where we assume for every $\mathbf{x} \in \mathbb{R}^d$ that $\delta(\mathbf{x}) = (2\pi)^{-d/2} t^{-d} \exp(-\frac{\|\mathbf{x}\|^2}{2t^2})$.

Lemma 5.5.2. Let $d \in \mathbb{N}$, $u_1, u_2, \dots, u_d \in \mathbb{R}$, $m_1, m_2, \dots, m_d, s_1, s_2, \dots, s_d \in (0, \infty)$, let $a: \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy for every $x \in \mathbb{R}^d$ that $a(x) = -\frac{1}{2}\|x\|^2$, for every $i \in \{1, 2, \dots, d\}$ let $\mathfrak{S}_i: [0, \infty) \rightarrow (0, \infty)$ and $\mathfrak{U}_i: [0, \infty) \rightarrow \mathbb{R}$ satisfy for every $t \in [0, \infty)$ that

$$\mathfrak{S}_i(t) = m_i \left[\frac{m_i \sinh(m_i t) + s_i \cosh(m_i t)}{m_i \cosh(m_i t) + s_i \sinh(m_i t)} \right] \quad \text{and} \quad \mathfrak{U}_i(t) = \frac{m_i u_i}{m_i \cosh(m_i t) + s_i \sinh(m_i t)}, \quad (5.53)$$

and let $u: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ that

$$u(t, x) = (2\pi)^{-d/2} \left[\prod_{i=1}^d |\mathfrak{S}_i(t)|^{-1/2} \right] \exp \left(- \sum_{i=1}^d \frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)} \right). \quad (5.54)$$

Then

- (i) it holds that $u \in C^{1,2}([0, \infty) \times \mathbb{R}^d, \mathbb{R})$,
- (ii) it holds for every $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ that

$$u(0, x) = (2\pi)^{-d/2} \left[\prod_{i=1}^d |\mathfrak{s}_i|^{-1/2} \right] \exp \left(- \sum_{i=1}^d \frac{(x_i - u_i)^2}{2s_i} \right), \quad (5.55)$$

and

- (iii) it holds for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ that

$$(\frac{\partial}{\partial t} u)(t, x) = u(t, x) \left(a(x) - \int_{\mathbb{R}^d} u(t, \mathbf{x}) a(\mathbf{x}) d\mathbf{x} \right) + \sum_{i=1}^d \frac{1}{2} |m_i|^2 (\frac{\partial^2}{\partial x_i^2} u)(t, x). \quad (5.56)$$

Proof of ??. First, note that the fact that for every $i \in \{1, 2, \dots, d\}$ it holds that $\mathfrak{S}_i \in C^\infty([0, \infty), (0, \infty))$, the fact that for every $i \in \{1, 2, \dots, d\}$ it holds that $\mathfrak{U}_i \in C^\infty([0, \infty), \mathbb{R})$,

and (??) establish ???. Moreover, observe that the fact that for every $i \in \{1, 2, \dots, d\}$ it holds that $\mathfrak{S}_i(0) = \mathfrak{s}_i$, the fact that for every $i \in \{1, 2, \dots, d\}$ it holds that $\mathfrak{U}_i(0) = \mathfrak{u}_i$, and (??) prove ???. Next note that (??) ensures that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$u(t, x) = \prod_{i=1}^d \left[(2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right]. \quad (5.57)$$

The product rule hence implies that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} & (\frac{\partial}{\partial t} u)(t, x) \\ &= \frac{\partial}{\partial t} \left(\prod_{i=1}^d \left[(2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right] \right) \\ &= \sum_{i=1}^d \left[\left[\prod_{j \in \{1, \dots, d\} \setminus \{i\}} \left((2\pi \mathfrak{S}_j(t))^{-1/2} \exp\left(-\frac{(x_j - \mathfrak{U}_j(t))^2}{2\mathfrak{S}_j(t)}\right) \right) \right] \right. \\ &\quad \cdot \left. \left[\frac{\partial}{\partial t} \left((2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right) \right] \right]. \end{aligned} \quad (5.58)$$

The chain rule, the product rule, and (??) therefore show that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} & (\frac{\partial}{\partial t} u)(t, x) \\ &= \sum_{i=1}^d \left[\left[\prod_{j \in \{1, \dots, d\} \setminus \{i\}} \left((2\pi \mathfrak{S}_j(t))^{-1/2} \exp\left(-\frac{(x_j - \mathfrak{U}_j(t))^2}{2\mathfrak{S}_j(t)}\right) \right) \right] \right. \\ &\quad \cdot \left[\left(\frac{\partial}{\partial t} \left((2\pi \mathfrak{S}_i(t))^{-1/2} \right) \right) \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right. \\ &\quad + (2\pi \mathfrak{S}_i(t))^{-1/2} \left(\frac{\partial}{\partial t} \left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)} \right) \right) \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \left. \right] \\ &= \sum_{i=1}^d \left[\left[\prod_{j \in \{1, \dots, d\} \setminus \{i\}} \left((2\pi \mathfrak{S}_j(t))^{-1/2} \exp\left(-\frac{(x_j - \mathfrak{U}_j(t))^2}{2\mathfrak{S}_j(t)}\right) \right) \right] \right. \\ &\quad \cdot \left[-(2\pi \mathfrak{S}_i(t))^{-1/2} \left[\frac{(\frac{\partial}{\partial t} \mathfrak{S}_i)(t)}{2\mathfrak{S}_i(t)} \right] \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right. \\ &\quad + (2\pi \mathfrak{S}_i(t))^{-1/2} \left(\frac{2(\frac{\partial}{\partial t} \mathfrak{U}_i)(t)(x_i - \mathfrak{U}_i(t))}{2\mathfrak{S}_i(t)} \right. \\ &\quad \left. \left. + \frac{(x_i - \mathfrak{U}_i(t))^2(\frac{\partial}{\partial t} \mathfrak{S}_i)(t)}{2|\mathfrak{S}_i(t)|^2} \right) \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right] \left. \right] \\ &= u(t, x) \left[\sum_{i=1}^d \left(\frac{-(\frac{\partial}{\partial t} \mathfrak{S}_i)(t)}{2\mathfrak{S}_i(t)} + \frac{2\mathfrak{S}_i(t)(\frac{\partial}{\partial t} \mathfrak{U}_i)(t)(x_i - \mathfrak{U}_i(t)) + (x_i - \mathfrak{U}_i(t))^2(\frac{\partial}{\partial t} \mathfrak{S}_i)(t)}{2|\mathfrak{S}_i(t)|^2} \right) \right]. \end{aligned} \quad (5.59)$$

Moreover, observe that (??), the chain rule, and the product rule ensure that for every $i \in \{1, \dots, d\}$, $t \in [0, \infty)$ it holds that

$$\begin{aligned} \left(\frac{\partial}{\partial t} \mathfrak{U}_i\right)(t) &= \frac{\partial}{\partial t} \left(\frac{\mathfrak{m}_i \mathfrak{u}_i}{\mathfrak{m}_i \cosh(\mathfrak{m}_i t) + \mathfrak{s}_i \sinh(\mathfrak{m}_i t)} \right) \\ &= -|\mathfrak{m}_i|^2 \mathfrak{u}_i \left[\frac{\mathfrak{m}_i \sinh(\mathfrak{m}_i t) + \mathfrak{s}_i \cosh(\mathfrak{m}_i t)}{[\mathfrak{m}_i \cosh(\mathfrak{m}_i t) + \mathfrak{s}_i \sinh(\mathfrak{m}_i t)]^2} \right] \\ &= -\mathfrak{S}_i(t) \mathfrak{U}_i(t) \end{aligned} \quad (5.60)$$

and

$$\begin{aligned} &\left(\frac{\partial}{\partial t} \mathfrak{S}_i\right)(t) \\ &= \frac{\partial}{\partial t} \left(\mathfrak{m}_i \left[\frac{\mathfrak{m}_i \sinh(\mathfrak{m}_i t) + \mathfrak{s}_i \cosh(\mathfrak{m}_i t)}{\mathfrak{m}_i \cosh(\mathfrak{m}_i t) + \mathfrak{s}_i \sinh(\mathfrak{m}_i t)} \right] \right) \\ &= |\mathfrak{m}_i|^2 \left[\frac{\mathfrak{m}_i \cosh(\mathfrak{m}_i t) + \mathfrak{s}_i \sinh(\mathfrak{m}_i t)}{\mathfrak{m}_i \cosh(\mathfrak{m}_i t) + \mathfrak{s}_i \sinh(\mathfrak{m}_i t)} \right] - |\mathfrak{m}_i|^2 \left[\frac{\mathfrak{m}_i \sinh(\mathfrak{m}_i t) + \mathfrak{s}_i \cosh(\mathfrak{m}_i t)}{\mathfrak{m}_i \cosh(\mathfrak{m}_i t) + \mathfrak{s}_i \sinh(\mathfrak{m}_i t)} \right]^2 \\ &= |\mathfrak{m}_i|^2 - |\mathfrak{S}_i(t)|^2. \end{aligned} \quad (5.61)$$

Combining this with (??) implies that for every $i \in \{1, 2, \dots, d\}$, $t \in [0, \infty)$ it holds that

$$\begin{aligned} \left(\frac{\partial}{\partial t} u\right)(t, x) &= \frac{u(t, x)}{2} \sum_{i=1}^d \left[\frac{-[|\mathfrak{m}_i|^2 - |\mathfrak{S}_i(t)|^2]}{\mathfrak{S}_i(t)} \right. \\ &\quad \left. + \frac{2|\mathfrak{S}_i(t)|^2 \mathfrak{U}_i(t) (\mathfrak{U}_i(t) - x_i) + (x_i - \mathfrak{U}_i(t))^2 (|\mathfrak{m}_i|^2 - |\mathfrak{S}_i(t)|^2)}{|\mathfrak{S}_i(t)|^2} \right] \\ &= \frac{u(t, x)}{2} \sum_{i=1}^d \left[|\mathfrak{m}_i|^2 \left(\left(\frac{x_i - \mathfrak{U}_i(t)}{\mathfrak{S}_i(t)} \right)^2 - \frac{1}{\mathfrak{S}_i(t)} \right) \right. \\ &\quad \left. + \mathfrak{S}_i(t) + 2(|\mathfrak{U}_i(t)|^2 - \mathfrak{U}_i(t) x_i) - (|x_i|^2 - 2\mathfrak{U}_i(t) x_i + |\mathfrak{U}_i(t)|^2) \right] \\ &= \frac{u(t, x)}{2} \sum_{i=1}^d \left[|\mathfrak{m}_i|^2 \left(\left(\frac{x_i - \mathfrak{U}_i(t)}{\mathfrak{S}_i(t)} \right)^2 - \frac{1}{\mathfrak{S}_i(t)} \right) + \mathfrak{S}_i(t) + |\mathfrak{U}_i(t)|^2 - |x_i|^2 \right]. \end{aligned} \quad (5.62)$$

Furthermore, note that (??) and the product rule show that for every $i \in \{1, 2, \dots, d\}$, $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \left(\frac{\partial}{\partial x_i} u\right)(t, x) &= \frac{\partial}{\partial x_i} \left[\prod_{j=1}^d \left[(2\pi \mathfrak{S}_j(t))^{-1/2} \exp\left(-\frac{(x_j - \mathfrak{U}_j(t))^2}{2\mathfrak{S}_j(t)}\right) \right] \right] \\ &= \left[\frac{\partial}{\partial x_i} \left[(2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(x_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right] \right] \\ &\quad \cdot \prod_{j \in \{1, 2, \dots, d\} \setminus \{i\}} \left[(2\pi \mathfrak{S}_j(t))^{-1/2} \exp\left(-\frac{(x_j - \mathfrak{U}_j(t))^2}{2\mathfrak{S}_j(t)}\right) \right] \\ &= -u(t, x) \left(\frac{x_i - \mathfrak{U}_i(t)}{\mathfrak{S}_i(t)} \right) = u(t, x) \left(\frac{\mathfrak{U}_i(t) - x_i}{\mathfrak{S}_i(t)} \right). \end{aligned} \quad (5.63)$$

The product rule therefore assures that for every $i \in \{1, 2, \dots, d\}$, $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \left(\frac{\partial^2}{\partial x_i^2} u\right)(t, x) &= \frac{\partial}{\partial x_i} \left(u(t, x) \left(\frac{\mathfrak{U}_i(t) - x_i}{\mathfrak{S}_i(t)} \right) \right) \\ &= \left(\frac{\partial}{\partial x_i} u \right)(t, x) \left(\frac{\mathfrak{U}_i(t) - x_i}{\mathfrak{S}_i(t)} \right) - \frac{u(t, x)}{\mathfrak{S}_i(t)} = u(t, x) \left[\left(\frac{x_i - \mathfrak{U}_i(t)}{\mathfrak{S}_i(t)} \right)^2 - \frac{1}{\mathfrak{S}_i(t)} \right]. \end{aligned} \quad (5.64)$$

Hence, we obtain that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\sum_{i=1}^d \left[\frac{1}{2} |\mathfrak{m}_i|^2 \left(\frac{\partial^2}{\partial x_i^2} u \right)(t, x) \right] = \frac{u(t, x)}{2} \sum_{i=1}^d \left[|\mathfrak{m}_i|^2 \left(\left(\frac{x_i - \mathfrak{U}_i(t)}{\mathfrak{S}_i(t)} \right)^2 - \frac{1}{\mathfrak{S}_i(t)} \right) \right]. \quad (5.65)$$

Next observe that (??) and Fubini's theorem ensure that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &u(t, x) \left(a(x) - \int_{\mathbb{R}^d} u(t, \mathbf{x}) a(\mathbf{x}) d\mathbf{x} \right) \\ &= u(t, x) \left(-\frac{1}{2} \left[\sum_{i=1}^d |x_i|^2 \right] - \int_{\mathbb{R}^d} -\frac{1}{2} \left[\sum_{i=1}^d |\mathbf{x}_i|^2 \right] u(t, \mathbf{x}) d\mathbf{x} \right) \\ &= \frac{u(t, x)}{2} \left(- \left[\sum_{i=1}^d |x_i|^2 \right] \right. \\ &\quad \left. + \sum_{i=1}^d \left[\int_{\mathbb{R}} |\mathbf{x}_i|^2 (2\pi\mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(\mathbf{x}_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) d\mathbf{x}_i \right. \right. \\ &\quad \left. \cdot \left(\prod_{j \in \{1, 2, \dots, d\} \setminus \{i\}} \int_{\mathbb{R}} (2\pi\mathfrak{S}_j(t))^{-1/2} \exp\left(-\frac{(\mathbf{x}_j - \mathfrak{U}_j(t))^2}{2\mathfrak{S}_j(t)}\right) d\mathbf{x}_j \right) \right]. \end{aligned} \quad (5.66)$$

This and the fact that for every $i \in \{1, 2, \dots, d\}$, $t \in [0, \infty)$ it holds that

$$\int_{\mathbb{R}} (2\pi\mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(x - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) dx = 1 \quad (5.67)$$

imply that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &u(t, x) \left(a(x) - \int_{\mathbb{R}^d} u(t, \mathbf{x}) a(\mathbf{x}) d\mathbf{x} \right) \\ &= \frac{u(t, x)}{2} \sum_{i=1}^d \left[-|x_i|^2 + \int_{\mathbb{R}} |\mathbf{x}_i|^2 (2\pi\mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(\mathbf{x}_i - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) d\mathbf{x}_i \right]. \end{aligned} \quad (5.68)$$

Next observe that the integral transformation theorem demonstrates that for every $i \in \{1, 2, \dots, d\}$, $t \in [0, \infty)$ it holds that

$$\begin{aligned}
& \int_{\mathbb{R}} x^2 \left[(2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{(x - \mathfrak{U}_i(t))^2}{2\mathfrak{S}_i(t)}\right) \right] dx \\
&= \int_{\mathbb{R}} (x + \mathfrak{U}_i(t))^2 \left[(2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{x^2}{2\mathfrak{S}_i(t)}\right) \right] dx \\
&= \int_{\mathbb{R}} x^2 \left[(2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{x^2}{2\mathfrak{S}_i(t)}\right) \right] dx \\
&\quad + \int_{\mathbb{R}} |\mathfrak{U}_i(t)|^2 \left[(2\pi \mathfrak{S}_i(t))^{-1/2} \exp\left(-\frac{x^2}{2\mathfrak{S}_i(t)}\right) \right] dx \\
&= \mathfrak{S}_i(t) + |\mathfrak{U}_i(t)|^2.
\end{aligned} \tag{5.69}$$

Combining this with (??) ensures that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$u(t, x) \left(a(x) - \int_{\mathbb{R}^d} u(t, \mathbf{x}) a(\mathbf{x}) d\mathbf{x} \right) = \frac{u(t, x)}{2} \sum_{i=1}^d (\mathfrak{S}_i(t) + |\mathfrak{U}_i(t)|^2 - |x_i|^2). \tag{5.70}$$

This and (??) demonstrate that for every $t \in [0, \infty)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
& u(t, x) \left(a(x) - \int_D u(t, \mathbf{x}) a(\mathbf{x}) d\mathbf{x} \right) + \sum_{i=1}^d \frac{1}{2} |\mathfrak{m}_i|^2 \left(\frac{\partial^2}{\partial x_i^2} u \right)(t, x) \\
&= \frac{u(t, x)}{2} \sum_{i=1}^d \left[|\mathfrak{m}_i|^2 \left(\left(\frac{x_i - \mathfrak{U}_i(t)}{\mathfrak{S}_i(t)} \right)^2 - \frac{1}{\mathfrak{S}_i(t)} \right) + \mathfrak{S}_i(t) + |\mathfrak{U}_i(t)|^2 - |x_i|^2 \right].
\end{aligned} \tag{5.71}$$

Combining this with (??) proves ???. The proof of ?? is thus complete. \square

5.5.5 Allen–Cahn PDEs with conservation of mass

In this subsection we use the machine learning-based approximation method in ?? to approximately calculate the solutions of certain Allen–Cahn PDEs with cubic nonlinearity, conservation of mass and no-flux boundary conditions (cf., e.g., Rubinstein & Sternberg (RUBINSTEIN1992)).

Assume ??, let $\epsilon \in (0, \infty)$ satisfy $\epsilon = \frac{1}{10}$, assume that $d \in \{1, 2, 5, 10\}$, $\mathbb{D} = [-1/2, 1/2]^d$, $T \in \{1/5, 1/2, 1\}$, $N = 10$, $K_1 = K_2 = \dots = K_N = 5$, and $M_1 = M_2 = \dots = M_N = 500$, assume that $\xi^{n,m,j}$, $n, m, j \in \mathbb{N}$, are independent \mathcal{U}_D -distributed random variables, assume for every $m \in \mathbb{N}$ that $\gamma_m = 10^{-2}$, and assume for every $s, t \in [0, T]$, $x, \mathbf{x} \in \mathbb{D}$, $y, \mathbf{y} \in \mathbb{R}$, $v \in \mathbb{R}^d$, $A \in \mathcal{B}(D)$ that $\nu_x(A) = \int_A d\mathbf{x}$, $g(x) = \exp(-\frac{1}{4}\|x\|^2)$, $\mu(x) = (0, \dots, 0)$, $\sigma(x)v = \epsilon v$, $f(t, x, \mathbf{x}, y, \mathbf{y}) = y - y^3 - (\mathbf{y} - \mathbf{y}^3)$, and

$$H(t, s, x, v) = R(x, x + \mu(x)(t - s) + \sigma(x)v) = R(x, x + \epsilon v) \tag{5.72}$$

d	T	N	Mean of the approx. method	Standard deviation of the approx. method	Reference value	Relative L^1 -approx. error	Standard deviation of the error	Average runtime in seconds
1	$1/5$	10	0.9947184	0.0021832	0.9932255	0.0015709	0.0021380	31.417
2	$1/5$	10	0.9908873	0.0027061	0.9868883	0.0040521	0.0027421	35.069
5	$1/5$	10	0.9942151	0.0052064	0.9710707	0.0238340	0.0053615	38.363
10	$1/5$	10	0.9792556	0.0203935	0.9514115	0.0292661	0.0214350	42.782
1	$1/2$	10	0.9870476	0.0014673	0.9880013	0.0014996	0.0007477	30.297
2	$1/2$	10	0.9763564	0.0030895	0.9750274	0.0024841	0.0021561	34.922
5	$1/2$	10	0.9518845	0.0051304	0.9431354	0.0092766	0.0054398	37.963
10	$1/2$	10	0.9249420	0.0052786	0.9063239	0.0205424	0.0058242	43.139
1	1	10	0.9823494	0.0003647	0.9780817	0.0043633	0.0003729	29.250
2	1	10	0.9659823	0.0004128	0.9658025	0.0003195	0.0003137	34.485
5	1	10	0.9209547	0.0019223	0.9158821	0.0055385	0.0020988	39.318
10	1	10	0.8693402	0.0029947	0.8683143	0.0030165	0.0015052	44.258

Tab. 5.5: Numerical simulations for the approximation method in ?? in the case of the Allen–Cahn PDEs with conservation of mass in (??) in ??.

(cf. (??) and (??)). The solution $u: [0, T] \times \mathbb{D} \rightarrow \mathbb{R}$ of the PDE in (??) then satisfies that for every $t \in (0, T]$, $x \in \partial\mathbb{D}$ it holds that $\langle \mathbf{n}(x), (\nabla_x u)(t, x) \rangle = 0$ and that for every $t \in [0, T]$, $x \in \mathbb{D}$ it holds that $u(0, x) = \exp(-\frac{1}{4}\|x\|^2)$ and

$$(\frac{\partial}{\partial t} u)(t, x) = \frac{\epsilon^2}{2} (\Delta_x u)(t, x) + u(t, x) - [u(t, x)]^3 - \int_{[-1/2, 1/2]^d} u(t, \mathbf{x}) - [u(t, \mathbf{x})]^3 \, d\mathbf{x}. \quad (5.73)$$

In ?? we use the machine learning-based approximation method in ?? to approximately calculate the mean of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the standard deviation of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the relative L^1 -approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, the uncorrected sample standard deviation of the approximation error associated to $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$, and the average runtime in seconds needed for calculating one realization of $\mathbb{V}_N^{1,0}(\Theta_{M_N}^N, (0, \dots, 0))$ based on 5 independent realizations (5 independent runs). The reference value, which is used as an approximation for the unknown value $u(T, (0, \dots, 0))$ of the exact solution of (??), has been calculated via the MLP approximation method for non-local nonlinear PDEs in ?? (cf. ?? and Beck2017a).

5.6 Code availability

The machine learning-based method and the MLP method are implemented the Julia package **HighDimPDE.jl** available at <https://github.com/SciML/HighDimPDE.jl>, and the simulation code is available at https://github.com/vboussange/HighDimPDE_examples.

5.7 Acknowledgements

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Discussion

“ To be but one with all living things, to return, by a radiant self-forgetfulness, to the All of Nature.

— Friedrich Hölderlin (1770-1843)

6.1 Contributions

Understanding biological and economic systems involves the understanding of the necessary and sufficient elemental processes, and their couplings, that result in mechanisms constraining their dynamics (Levin2002). Along this line, my work aimed at advancing our quantitative understanding of how ecological and evolutionary processes, and their interplay, shape the dynamics of biological and economic systems. In particular, this thesis contributed to

- (i) a fundamental understanding of the role of eco-evolutionary processes in shaping the dynamics of biological populations structured in complex landscapes (chap1),
- (ii) the quantification of the effect of mechanisms associated with eco-evolutionary processes in economic systems ??,
- (iii) methodological advances in the forward and inverse modelling of eco-evolutionary dynamics ??????.

In the following, I discuss the chapters of my thesis collectively within the broader context of our current understanding of the dynamics of biological and economic systems, and the accompanying modelling paradigm.

6.1.1 Advances in the fundamental understanding of biological and economic systems

Linking eco-evolutionary processes to patterns of differentiation

Spatial patterns of biodiversity result from microscopic processes acting upon individual organisms (Champagnat2006), which include reproduction, competition, mutation and dispersal. Mutations result in genetic drift, which promotes stochastic variations in the allelic proportions and phenotypes of biological populations (XXX). In spatially structured populations, this results in turn to "neutral differentiation", where spatially isolated populations are characterised by differentiated alleles and traits (XXX). Dispersal tends to reduce neutral differentiation, and this effect is modulated by landscape connectivity (Wright1943; McRae2006; McRae2007) through the mechanism of "isolation by limited

dispersal" (**Orsini2013**). By increasing the dispersal ability of organisms, landscape connectivity decreases neutral differentiation. When landscapes present heterogeneous habitats, natural selection can supplement the effect of genetic drift and increase the sole effect of stochasticity on differentiation. Under this scenario, local environment conditions select individuals with traits that provide them higher fitness (XXX). At the population level, this results in populations adapting to their local environment, a mechanism coined "local adaptation" (**Kawecki2004**) and resulting in patterns of "adaptive differentiation". Adaptive differentiation is hindered by dispersal, which prevents local adaptation by bringing maladapted individuals, that destabilise the evolution of traits towards the optimal. While adaptive differentiation concerns traits under selection, it indirectly affects the differentiation of neutral traits, that are co-evolving with traits under selection through linkages (XXX). This results in turn to the mechanism of "isolation by adaptation", where habitat heterogeneity, rather than landscape connectivity, increases neutral differentiation (**nosil2008**). Simple mechanisms resulting in neutral and adaptive differentiation are identified, but how they are modulated by eco-evolutionary feedbacks and landscape complexity is unclear.

In ??, I demonstrate a novel mechanism, involving the ecological process of competition, that considerably affects neutral differentiation. Through the creation of unbalanced migration fluxes which increases the intensity of competition in highly connected populations, heterogeneity in connectivity reduces gene flow and reinforces neutral differentiation. I also investigate the mechanism of local adaptation and how it results in adaptive differentiation in complex landscapes, where habitats are arranged in a realistic fashion. I show that the complexity of habitat spatial distribution can be reduced to a measure of habitat spatial auto-correlation, coined the "habitat assortativity". Landscapes characterised by a high habitat assortativity systematically support populations that are locally better adapted than in landscape with low assortativity, resulting in higher adaptive differentiation. Specifically, I provide an analytical condition for local adaptation that sheds light on how it relates to dispersal intensity, selection strength, habitat heterogeneity, and habitat assortativity.

Because habitat assortativity affects local adaptation, it must also affect neutral differentiation through the mechanism of isolation by adaptation. Closing the loop, I demonstrate that habitat assortativity affects population differentiation through two antagonistic effects. By favoring local adaptation, it promotes isolation by adaptation, therefore increasing neutral differentiation. In parallel, it also favors gene flow within clusters of similar environmental conditions, decreasing isolation by limited dispersal. This results in habitat assortativity decreasing neutral differentiation for low dispersal intensity, and increasing neutral differentiation for high dispersal intensity. This complex feedback is essential to understand population differentiation in complex landscapes.

I provide a summary of the processes and resulting mechanisms shaping neutral and adaptive differentiation in ???. Overall, ?? links processes to patterns, establishing a map of the causal pathways involved in the phenotypic differentiation of populations.

Linking economic patterns to eco-evolutionary processes

?? provides a novel understanding of the role of eco-evolutionary processes in economic systems. Neoclassical economics and evolutionary economics seek to explain economic change with formal modelling, focusing on the relationships between economic variables

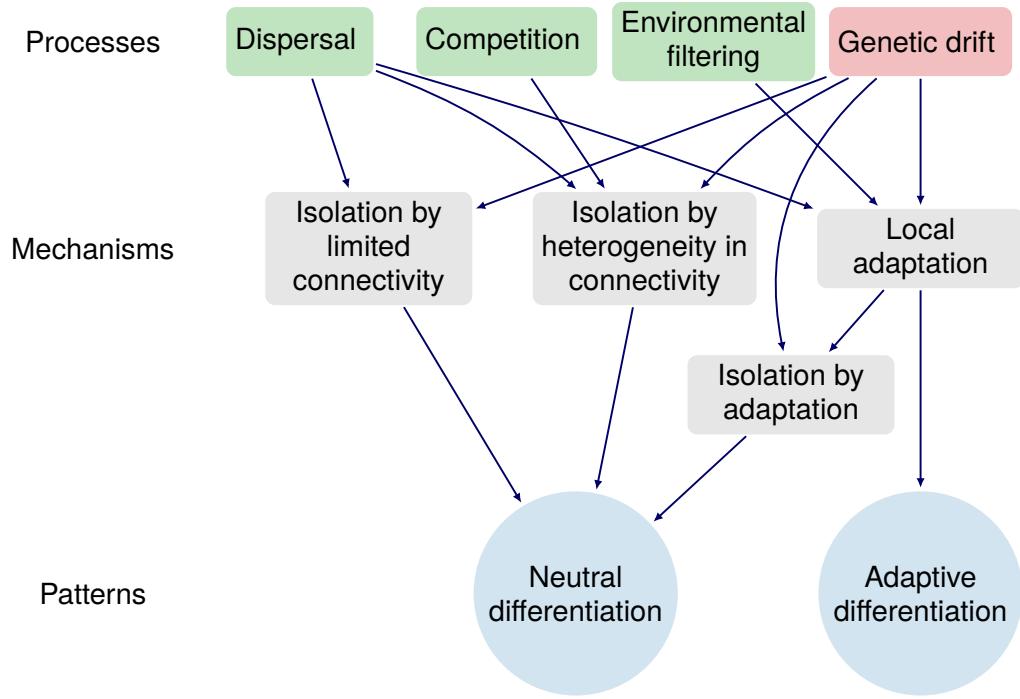


Fig. 6.1: Summary of the causal pathways involved in neutral and adaptive differentiation, disentangled in ???. Ecological processes are displayed in green boxes, evolutionary processes are displayed in red boxes.

such as output, employment and productivity (Boschma2005a). In particular, evolutionary economics is concerned with explaining economic change by endogenous forces, such as interactions between firms and economic activities, and evolutionary processes acting upon them (Hodgson2019; Metcalfe2006). In opposition to this approach, complexity economics (Hidalgo2021) seeks to predict variations in national income by using fine-grained data of economic activity outputs and dimensionality reduction techniques (Mitchell), without assumptions on the underlying processes (Hidalgo). While a current concern in evolutionary economics is to obtain an agreement between the mechanisms proposed and empirical observations (XXX), complexity economics seeks to understand the causal processes underlying the success of the dimensionality reduction technique (XXX).

By being agnostic to economic variables yet providing a causal link with the most probable processes underlying economic development, ??bridges evolutionary economics and economic complexity. Our approach relies on the simple observation that the dynamics of economic activities contains signatures from the many complex processes that underpin them. These signatures consist in peculiar temporal variations and couplings that can be conveniently modelled by biologically inspired population dynamic models, which use are additionally justified by deep analogies between economic activities and biological populations. Analogously to biological populations that are characterised by genes, economic activities are characterised by organizational routines (NelsonWinter), which experience evolutionary processes and define how they engage in ecological processes (NelsonWinter). As a result, and similarly to the size of biological populations, the capital growth of economic

activities is determined by the ensemble of organizational routines that characterise them (**Boschma2005a**). Similarly to approaches used in ecology and evolution (**Skeels**), inverse modelling approaches can then be used to disentangle, from their signatures (**Skeels**) left on the dynamics of economic activities, the role of the eco-evolutionary processes acting upon them.

Specifically, ??explores the effect of eco-evolutionary processes, proposed in the evolutionary economics and geography economics literature, on the dynamics of economic activities at the national scale. In particular, ??seeks to test whether the dynamics of economic activities can be explained by different types of interdependencies, including positive (XXX) and negative interactions (XXX), spatial transfers (XXX), and transformations (XXX). Using population dynamic models capturing the different interdependencies, ??provides empirical evidence that economic activities engage in positive interactions, and benefit from spatial transfers of knowledge and routines. Positive interactions may arise from a variety of processes proposed in the evolutionary economic literature, such as supply chains (**Ozman2009; Saavedra2009a**) and knowledge spillovers (**Menon2015**). Its support implies that diversity promotes economic development (**Hidalgo2018**), as economic activities promote each other. The support for spatial transfers implies that, besides international patent laws (XXX), transfers of knowledge and organisational routines have a considerable impact on economic activities. Nevertheless, discrepancies in the strength of evidence obtained for spatial transfers across countries highlight that some countries are overall more akin to spatial transfers than others, which may be explained by differences in cognitive, organizational, social, institutional or geographic proximities across countries (XXX). I provide a summary of the mechanisms evidenced in economic systems in ???. Overall, ??bridges approaches in economics and biology to understand the mechanisms shaping the endogenous dynamics of economic systems.

6.1.2 Methodological advances in forward and inverse modelling

Advances in the modelling of realistic spatial and phenotypic structures

???? provide new tools to better understand mechanisms resulting from ecological and evolutionary processes, and their interplay, in the context of realistic population structures and phenotypic distributions. Evolutionary dynamics have been traditionally studied in the context of homogeneous or spatially extended populations (**LiebermanHauert2005**). For instance, (**Slatkin1973; Slatkin1978; Kirkpatrick1997; Polechova2015; Polechova2018; AndradeRestrepo2019; Doebeli2003; Meszena1997; Yeaman2011; Debarre2013; Mirrahimi2020**) consider regular spatial structures, missing the effect of the complexity of spatial structures on population differentiation. Nevertheless, biological habitats differ in their connectivity (XXX), and economic entities are structured through complex networks (XXX). The work of (**LiebermanHauert2005**) and subsequent studies of "evolutionary dynamics on graphs" (XXX) show that this complexity affects the interplay between selection and drift. However, evolutionary dynamics on graphs does not consider eco-evolutionary feedbacks (**Govaert2019a**). Thus far, models that include frequency dependence together with realistic, complex population structures were missing.

While a vast majority of the work on eco-evolutionary feedbacks has focused on the evolution of scalar phenotypes (Doebeli2011), in most organisms many phenotypic properties combine in complicated ways to determine ecological interactions, and hence frequency-dependent selection (Doebeli2014) (Doebeli2011) shows that the consideration of multiple traits with complex interactions relaxes the unrealistic conditions of strong frequency dependence required to generate diversity in one dimensional phenotype spaces, calling for a better understanding of evolutionary dynamics in high dimensional spaces. ?? demonstrates that the co-evolution of traits proved to have genuine consequences on differentiation, pointing towards the inclusion of multiple traits to understand the dynamics of ecological interactions. Other works on eco-evolutionary dynamics, including cancer cell evolution (XXX) and plankton dynamics (XXX), highlight the importance of considering multiple traits. The simulation of high dimensional models of trait leads to complications, since the numerical cost of traditional methods grows exponentially in the number of dimensions. Novel numerical methods have been proposed to simulate high dimensional models, but they could not handle non-local terms, which capture non-local interactions between microscopic agents.

?? develops a generic modelling framework to capture the effect of eco-evolutionary processes on biological populations structured in complex landscapes, and ?? provides tools to efficiently simulate the resulting high-dimensional models. The IBM presented in ?? involves the combination of graphs and highly dimensional phenotypic spaces, together with eco-evolutionary feedbacks, to model population structures. The modelling framework presented can readily be generalised, and the code associated to the numerical experiments in ?? include a Julia library, **Evoid.jl**, that implements a more general version of the model. As such, the modelling framework presented in ?? may be used to investigate other questions involving complex population structures and the co-evolution of characteristics.

Reproducing the discrete and stochastic nature of ecological and evolutionary processes (Champagnat2006), simulations of the IBM may not provide a general understanding of system investigated (XXX), and cannot be scaled to simulate large systems involving millions of individuals (XXX). Nevertheless, the IBM proposed in ?? is mathematically tractable under simplifying assumptions, and can be efficiently simulated with a deterministic PDE approximation. Tractability allows to obtain analytical insights on how structural properties affect macroscopic population under simplifying assumptions (??). The PDE approximation, combined with the numerical methods presented in ??, further allow efficient simulations. I have implemented the numerical methods for simulating high dimensional models in the Julia library **HighDimPDE.jl** (HighDimPDE), a registered Julia package belonging to the SciML organisation (XXX). The user interface respects standards from the SciML organisation, meaning that Julia users can easily adopt it. The package aims at hosting any solver algorithms that break down the curse of dimensionality, and is currently receiving contributions to implement the DeepBSDE scheme (Han2018).

The combination of analytical insights and numerical simulations can help to elucidate the links between microscopic processes and macroscopic properties (Levin). ???? provide novel analytical and numerical tools to link the interplay between eco-evolutionary processes and complex spatio-evolutionary structures to the resulting mechanisms and patterns.

Advances in inference methods for the investigation of eco-evolutionary processes

???? develop and test a novel inverse modelling method that allows to infer highly nonlinear dynamical processes from observation data. The method opens up new venues to further our understanding of the dynamics of biological and economic systems, and to provide predictions. The most developed inference methods in current use for inverse modelling are Bayesian inference methods with Markov Chain Monte Carlo (XXX) and variational methods (XXX). Bayesian inference methods require a large number of forward model integrations (**Schneider2017**), and are highly affected by the number of model parameters (**Csillery2010**). Variational methods require the model sensitivity to its parameters (XXX) and are prone to converge to local minima, especially with complex models (XXX). Those central issues likely explain the very poor number of studies that have used inverse modelling to further our knowledge on eco-evolutionary dynamics (XXX).

?? presents a novel inference framework that can handle with high computational efficiency dynamic eco-evolutionary models. The framework is based on a simple variational method, but relies on recent development in ML, including AD (**Rackauckas2020a**) and state-of-the-art optimizers (**Kingma2014**), and implements a learning strategy based on a mini-batch method, in order to circumvent the drawbacks of variational methods and adapt to the specificities of eco-evolutionary models. The use of AD simply eliminates the effort required to obtain the model sensitivity to its parameters, and the state-of-the-art optimizers, together with the mini-batch method, ensure the efficiency and reliability of the method in handling highly nonlinear models. By contrasting competing hypotheses embedded in alternative models, ??? provide concrete examples, both with synthetic and empirical data, that the ML framework can successfully elucidate eco-evolutionary mechanistic pathways. This work is part of an ongoing effort to blend ML and traditional models to gain scientific understanding and extrapolability (XXX). Importantly, in contrast to physical systems such as ocean and atmospheric systems (XXX), invariant laws are yet to be formulated (if they exist). While ML is mostly used to improve model forecast skill in weather and climate modelling (XXX), the fields of Biology and Economic can gain scientific knowledge through model selection. Nevertheless, the proposed method is also relevant for improving the forecast of predictive eco-evolutionary model (**Urban2016**), and integrates the practical constraints of datasets including short time series with partial, noisy, shallow and independent observations (**Dornelas2018**).

The ML framework is implemented in the multi-purpose Julia package **MiniBatchInference.jl** (**MiniBatchInference**), readily available to the scientific community. **MiniBatchInference.jl** is built around the celebrated differential equation solver **DifferentialEquations.jl** and the deep learning library **Flux.jl**. As such, the use of **MiniBatchInference.jl** requires very limited efforts to any user familiar with those libraries.

Overall, the method proposed in ?? successfully blends ML methods with mechanistic ecosystem models to improve our gain scientific knowledge from observation data. Concret case examples in ??? show that it enables the testing of eco-evolutionary theories against data, advancing our understanding and the improvement of current mechanistic models, with the potential to provide better forecasts of ecosystems states (**Urban2016**).

CV

Personal Information

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Personal skills

Languages	English (fluent) French (native) Spanish (B2) German (B1)
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Sports	Ski mountaineering Alpinism Rock climbing Enduro mountainbiking Surfing
Alpine CV	[vboussange.github.io/pages/alpine_cv/]

Education

- 10.2022 **Ph.D in Environmental Sciences**, Swiss Federal Institute for Forest, Snow and Landscape (WSL | Swiss Federal Institute of Technology Zurich, ETH), Switzerland
 09.2018 *Forward and inverse modelling of eco-evolutionary processes in biological and economic systems.* Under the guidance of Prof. Dr. Loïc Pellissier.

- 06.2017 **Full year academic exchange**, University of New South Wales (UNSW Sydney),
 09.2016 Australia
- 06.2017 **Master thesis in theoretical geomechanics**, UNSW Sydney | CSIRO, Australia
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Numerical continuation and bifurcation analysis for unconventional geomechanics.
 Under the guidance of Dr. Thomas Poulet.
- 08.2018 **M.S. in Energy and Environmental Engineering**, Institut National Des Sciences Appliquées de Lyon (INSA Lyon), France
 09.2013 Three-year undergraduate engineering course in Energy and Environmental Systems, focused on Advanced Energy Systems and Efficiency.
- 08.2018 **B.S. in Mathematics and Physics**, Institut National Des Sciences Appliquées de Lyon (INSA Lyon), France
 09.2013 Ranking : 21/650 students.

Professional appointments

- 08.2018 **R&D intern**, Compagnie National du Rhône (CNR), France
 03.2018 Development of an Energy Management System based on various optimisation techniques for optimal production of renewable resources. Applications to EU sponsored projects: **Jupiter1000** (power-to-gas), **Move in pure** (vehicle-to-grid), **Marie-Galante island** (micro-grid)

Publications

Peer-reviewed

1. **Boussange, V.** & Pellissier, L., *Eco-evolutionary model on spatial graphs reveals how habitat structure affects phenotypic differentiation.* **Commun Biol** 5, 668 (2022). [[bioRxiv](#)]

Preprints

1. **Boussange, V.**, Vilimelis-Aceituno, P., Pellissier, L., *Mini-batching ecological data to improve ecosystem models with machine learning* [[bioRxiv](#)] (2022), 46 pages. In review.
2. **Boussange, V.**, Becker, S., Jentzen, A., Kuckuck, B., Pellissier, L., *Deep learning approximations for non-local nonlinear PDEs with Neumann boundary conditions*. [[arXiv](#)]

(2022), 59 pages. Revision requested from Partial Differential Equations and Applications.

Proceedings

1. Poulet, T., Alevizos, S., Veveakis, M., **Boussange, V.**, Regenauer-Lieb, K., *Episodic mineralising fluid injection through chemical shear zones*, ASEG Extended Abstracts (2018), 5 pages.

In preparation

1. **Boussange, V.**, Sornette, D., Lischke, H., Pellissier, L., *Analogous forces to ecological interactions, dispersal and mutations shape the dynamics of economic activities*.

Talks

- 07.2022 **Speaker**, HIGHDIMPDE.JL: A Julia package for solving high-dimensional PDEs, JuliaCon2022, online. youtube.com/watch?v=4sXqGhhknT4
- 06.2022 **Speaker**, Interpretable machine learning for forecasting dynamical processes in ecosystems, World Biodiversity Forum, Davos, Switzerland.
- 06.2022 **Invited speaker**, Investigating empirical patterns of biodiversity with mechanistic eco-evolutionary models, Seminar at the Theoretical Ecology and Evolution group, Universität Bern.
- 11.2021 **Invited speaker**, Numerical approximations of solutions of highly dimensional, non-local nonlinear PDEs, StAMBio seminar, St Andrews, UK.
- 10.2021 **Speaker**, Graph topology and habitat assortativity drive phenotypic differentiation in an eco-evolutionary model, Conference on Complex Systems, Lyon, France.
- 10.2021 **Speaker**, Using graph-based metrics to assess the effect of landscape topography on diversification, ECBC, Amsterdam, Netherlands.
- 09.2021 **Speaker**, Solving non-local nonlinear Partial Differential Equations in high dimensions with HighDimPDE.jl, International Conference on Computational Methods in Systems Biology, Bordeaux, France.
- 04.2021 **Speaker**, Responses of neutral and adaptive diversity to complex geographic population structure, Mathematical Population Dynamics, Ecology and Evolution, CIRM Marseille, France.

Softwares

- 2022 **MiniBatchInference.jl** Julia
github.com/vboussange/MiniBatchInference.jl
A Julia package for maximum likelihood estimation and model selection of strongly nonlinear dynamical models.

2021	HighDimPDE.jl github.com/vboussange/HighDimPDE.jl	Julia
A Julia package that breaks down the curse of dimensionality in solving non local, non linear PDEs.		
2021	EvoId.jl	Julia
2019	github.com/vboussange/EvoId.jl	
	Evolutionary individual based modelling, mathematically grounded.	
2018	OptiVPP confidential	Python, GAMS
	Energy Management System for Virtual Power Plants.	

Open source software contributions

SciML
 DiffEqFlux.jl
 CUDA.jl
 Flux.jl
 LightGraphs.jl

Teaching and supervision

12.2020	701-3001-00L Environmental Systems Data Science , ETH Zürich, D-USYS,
09.2020	Switzerland
06.2020	262-0100-00L Lab rotation , ETH Zürich, D-BSSE, Switzerland
04.2020	
12.2020	Taste of research internship , Polytech Nice-Sophia, France
09.2020	

Reviews

2022	Journal of Open Source Software
2019	Journal of Theoretical Biology

List of Figures

