Class-15

$GF(p^n)$

$p \to$ prime Number

$x \in GF(p^n)$

$x^{p^n - 1} \equiv 0 \pmod p$

Class-16

* Method of symmetrically repeated difference

→ Pure and Mixed differences :

Consider a module M containing n-elements. To each element of the module, let there correspond m-treatments the treatments corresponding to the element a being denoted by $a_1, a_2, \ldots, a_m$. The treatment a_i is said to belong to the ith class. Thus we have mn treatments n belonging to each of the m-classes. With any ordered pair of distinct treatments a_i and b_j, we associate the difference $a-b$ of the type $[i,j]$. Each difference is an element of the module and is of a certain type. If $i=j$, the difference is said to be a pure difference. Obviously in this case $a \neq b$ as the treatments are distinct. If $i \neq j$, the difference is said to be a mixed difference.

Example

M = residue class mod 5

$M = \{0,1,2,3,4\}$, $n=5$

To each element a of M, let there corresponds two treatments, a_1 and a_2 $(m=2)$

Treatments		Ordered pair	Difference	Difference type
0_1	0_2	$(0_1, 1_1)$	4	[1,1] Pure
1_1	1_2	$(1_1, 0_1)$	1	
2_1	2_2	$(0_1, 2_1)$	3	[1,1] Pure
3_1	3_2	$(2_1, 0_1)$	2	
4_1	4_2	$(3_1, 1_2)$	2	[1,2] Mixed
		$(1_2, 3_1)$	3	[2,1] Mixed
		$(3_2, 4_1)$	4	[2,1] Mixed
		$(3_1, 4_2)$	4	[1,2] Mixed

Now, suppose a block B containing K distinct treatments. From this block we can get $\{K \times (K-1)\}$ ordered pairs of treatments, giving rise to $\{K \times (K-1)\}$ differences. These differences are called differences arising out of the block B:

→ Refer to the previous example :

Suppose $B = (2_1, 4_2, 0_2)$

Differences arising out of the block B

Ordered pair	Difference	Difference type
$(2_1, 4_2)$	3	[1,2] Mixed
$(2_1, 0_2)$	2	[1,2] Mixed
$(4_2, 2_1)$	2	[2,1] Mixed
$(4_2, 0_2)$	4	[2,2] Pure
$(0_2, 2_1)$	3	[2,1] Mix.
$(0_2, 4_2)$	1	[2,2] Pur

→ Since there are m-classes and the no. of elements in the module is n, ~~m~~, there are ~~m-1~~ (n-1) pure differences of the type [i, i], for each $i = 1, 2, .. m$

Total $m(n-1)$ → pure differences. Simillarly there are n mixed differences of each type [i, j] for $i, j = 1(1)m$, $i \neq j$

~~m(~~ Total $nm(m-1)$ → Mixed differences.

→ Now, consider a set of t-blocks, $B_1, B_2, \ldots, B_t$. If among the differences arising from these t-blocks, each possible difference occurs a constant no. of times (say, λ), the differences are said to be symmetrically repeated.

Example

$M = \{0, 1, 2, 3, 4\}$

$m = 3, n = 5, t = 7, \lambda = 1, k = 3$,

To each $a \in M$, there corresponds 3 treatments a_1, a_2, a_3

consider 7 blocks $B_1 = (0_1, 1_1, 0_2)$; $B_2 = (0_2, 1_2, 2_3)$;

$B_3 = (0_3, 1_3, 2_1)$; $B_4 = (0_1, 2_1, 3_2)$; $B_5 = (0_2, 2_2, 0_3)$;

$B_6 = (0_3, 2_3, 0_1)$; $B_7 = (0_1, 2_2, 1_3)$

Blocks	[1,1]	[2,2]	[3,3]	[1,2]	[1,3]	[2,3]	[2,1]	[3,1]	[3,2]
$(0_1, 1_1, 0_2)$	4,1	—	—	0,1	—	—	0,4	—	—
$(0_2, 1_2, 2_3)$	—	4,1	—	—	—	3,4	—	—	1,2
$(0_3, 1_3, 2_1)$	—	—	4,1	—	2,1	—	—	3,4	—
$(0_1, 2_1, 3_2)$	3,2	—	—	2,4	—	—	1,3	—	—
$(0_2, 2_2, 0_3)$	—	3,2	—	—	—	0,2	—	—	0,3
$(0_3, 2_3, 0_1)$	—	—	3,2	—	0,3	—	—	0,2	—
$(0_1, 2_2, 1_3)$	—	—	—	3	4	1	2	1	4

From the table we see that among the non-zero differences arising out of the 7 blocks, each difference is repeated symmetrically. It must be noted that 0 can not occur as a pure difference since the block contents are distinct.

→ First Fundamental Theorem of method of symmetrically repeated difference :-

Let, M be a module containing n-elements, $x^{(0)}, x^{(1)}, \dots, x^{(n-1)}$ and to each element $x^{(u)}$, let there correspond m treatments $x_1^{(u)}, x_2^{(u)}, \dots, x_m^{(u)}$ $(u=0,1,2,\dots,n-1)$. Thus there are mn treatments.

Let, it is possible to find a set of t blocks, $B_1, B_2 \dots B_t$ satisfying the following conditions,

(i) Each block contains k distinct treatments.

(ii) Among the kt treatments occuring in the t-blocks, exactly r- treatments belong to each of the classes.

(iii) The differences arising from the t- blocks are symmetrically repeated, λ times each.

Then the nt blocks obtained by developing the initial blocks $B_1, B_2, \dots, B_t$ provide us with a solution of a BIBD with parameters $v=mn$, $b=nt$, r, k, λ.

→ Given any block B_s containing k- distinct treatment one can obtain n blocks $B_{s,\theta}$ where θ ranges over the elements of M as follows :

corresponding to any treatment $x_i^{(u)}$ of the ith class in B_s, we take the treatment $x_i^{(v)} = x_i^{(u)} + \theta$ of the ith class in $B_{s,\theta}$. The n-blocks $B_{s,\theta}$ are said to be obtained by developing the block B_s ...

Example: Refer to previous example

$n = 5$	$v = 15$
$m = 3$	$b = 35$
$t = 7$	$k = 3$
$\lambda = 1$	$r = 7$
$K = 3$	$\lambda = 1$

$(0_1, 1_1, 0_2)$	$(0_2, 1_2, 2_3)$	$(0_3, 1_3, 2_1)$
$(1_1, 2_1, 1_2)$	$(1_2, 2_2, 3_3)$	$(1_3, 2_3, 3_1)$
$(2_1, 3_1, 2_2)$	$(2_2, 3_2, 4_3)$	$(2_3, 3_3, 4_1)$
$(3_1, 4_1, 3_2)$	$(3_2, 4_2, 0_3)$	$(3_3, 4_3, 0_1)$
$(4_1, 0_1, 4_2)$	$(4_2, 0_2, 1_3)$	$(4_3, 0_3, 1_1)$
$(0_1, 2_1, 3_2)$	$(0_2, 2_2, 0_3)$	$(0_3, 2_3, 0_1)$
$(1_1, 3_1, 4_2)$	$(1_2, 3_2, 1_3)$	$(1_3, 3_3, 1_1)$
$(2_1, 4_1, 0_2)$	$(2_2, 4_2, 2_3)$	$(2_3, 4_3, 2_1)$
$(3_1, 0_1, 1_2)$	$(3_2, 0_2, 3_3)$	$(3_3, 0_3, 3_1)$
$(4_1, 1_1, 2_2)$	$(4_2, 1_2, 4_3)$	$(4_3, 1_3, 4_1)$
$(0_1, 2_2, 1_3)$		
$(1_1, 3_2, 2_3)$		
$(2_1, 4_2, 3_3)$		
$(3_1, 0_2, 4_3)$		
$(4_1, 1_2, 0_3)$		

Example-2

$M = \{0, 1, 2, 3, 4, 5, 6\}$

$n = 7$ $\quad v = b = 7$ $\quad$, $t = 1$ (no. of initial block).

$m = 1$ $\quad r = k = 4$

$\lambda = 2$

Initial block : (3, 5, 6, 7)

4, 6, 7 1

5 7 1 2

6 1 2 3

7 2 3 4

1 3 4 5

2 4 5 6

Class-17

→ Second fundamental theorem of the method of symmetric difference :

Let, M be a module containing n-elements $x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$, and to each element $x^{(u)}$, let there correspond m-treatments $x_1^{(u)}, x_2^{(u)}, \ldots, x_m^{(u)}$ [$u = 0(1)n-1$]. To these mn treatments, we adjoin a new treatment α (called the invariant treatment) so that we have $v = mn+1$ treatments. consider these (mn+1) treatments. Let, it be possible to find a set of (t+u) block $B_1, B_2, \ldots, B_t, B_1', B_2', \ldots B_u'$ satisfying the following conditions :

(i) Each of the blocks $B_1, B_2, \ldots B_t$ contains k-distinct treatments $x_i^{(u)}$, whereas each of the blocks $B_1', B_2', \ldots B_u'$ contains α and (k-1) distinct treatments $x_i^{(u)}$

(ii) Among the kt treatments occuring in the blocks $B_1, B_2, \ldots, B_t$, exactly $nu - \lambda$ belong to each of the m-classes, whereas among the u(k-1) treatments occuring in the blocks $B_1', B_2', \ldots B_u'$ exactly λ belong to each of the m - classes.

(iii) The differences arising from the $(t+u)$ blocks $B_1, B_2 \ldots B_t, B_1'', B_2'' \ldots B_u''$ are symmetrically repeated λ times, where the blocks B_j'' are obtained from B_j' by deleting the treatment α, for $j = 1(1)u$

Then the $n(t+u)$ blocks, obtained by developing the initial blocks $(B_1, B_2, \ldots, B_t)$ and $B_1', B_2' \ldots B_u'$ provide a solution of a BIBD with parameters $v = mn+1$, $b = n(t+u)$, $r = nu$, k, λ.

[redacted]

→ $\boxed{\text{Steiner's Triplet :- BIBD with } k=3, \lambda=1.}$

A steiner's triplet is an arrangement of v objects in triplet such that every pair of objects appears exactly once in each triplet. obviously if we treat our objects as treatments, steiner's triplet is a BIBD with $k=3$ and $\lambda=1$. From the parametric relations of BIBD, we have : $bk = vr$, $r(k-1) = \lambda(v-1)$

$\therefore$ $3b = vr$

$2r = v-1 \Rightarrow \boxed{v = 2r+1}$

$\Rightarrow 3b = r(2r+1)$

$\Rightarrow b = \dfrac{r(2r+1)}{3}$

$\therefore$ r must be of the form $3t+1$, $3t$

Accordingly we get following two series of Steiner's Triplet :

S_{11} :- $v = 6t+3$
$b = (3t+1)(2t+1)$
$r = 3t+1$
$k = 3$
$\lambda = 1$

S_{12} :- $v = 6t+1$
$b = t(6t+1)$
$r = 3t$
$k = 3$
$\lambda = 1$

Solution of S_{11}

Let, M be the module of residue classes mod $(2t+1)$ and Let it's elements be $0, 1, 2, \ldots, 2t$. To every element $u \in M$, let there correspond 3 treatment u_1, u_2, u_3 so that there are $(6t+3)$ treatments.

consider the pairs :

$$[1, 2t]$$
$$[2, 2t-1]$$
$$\vdots \qquad \vdots$$
$$[i, 2t-i+1]$$
$$\vdots \qquad \vdots$$
$$[t, t+1]$$

The differences arising out from the i^{th} pair are $(2t+1-2i)$ and $2i$. Taking $i = 1(1)t$, we get all the non-zero elements of M, as the differences from the pair.

Now consider the following sets of initial blocks:

$[1_1, (2t)_1, 0_2]$; $[2_1, (2t-1)_1, 0_2]$; $\ldots$ $[t_1, (t+1)_1, 0_2]$

$[1_2, (2t)_2, 0_3]$; $[2_2, (2t-1)_2, 0_3]$; $\ldots$ $[t_2, (t+1)_2, 0_3]$

$[1_3, (2t)_3, 0_1]$; $[2_3, (2t-1)_3, 0_1]$; $\ldots$ $[t_3, (t+1)_3, 0_1]$ —— $(*)$

$[0_1, 0_2, 0_3]$

It is clear that the pure differences of the types $[1,1]$, $[2,2]$ and $[3,3]$ arising from the initial blocks are repeated just once. Again among the pairs, every non-zero element of M occurs once. Hence among the mixed differences of the type arising from the first row, every non-zero element of M occurs just once. The mixed difference 0 of the type $[1,2]$ arises from the last initial block

No mixed difference of the type [1,2] can occur from 2^{nd} and 3rd row. Hence every elements of M occurs exactly once among the mixed differences of type [1,2] arising from the initial blocks.
Simillarly we can proof the same thing for the mixed difference of other types.

Theorem : The initial blocks ⊛ provide a solution of the series S_{11} of BIBD with parameters $v = 6t+3$, $b = (3t+1)(2t+1)$, $r = 3t+1$, $K = 3$, $\lambda = 1$.

→ Example-1 >> $t = 1$, $[v = 9, b = 12, r = 4, K = 3, \lambda = 1]$

Example-2 >> $t = 2$, $[v = 15, b = 35, r = 7, K = 3, \lambda = 1]$

↓

Initial blocks

$$(1_1, 4_1, 0_2), (2_1, 3_1, 0_2)$$
$$(1_2, 4_2, 0_3), (2_2, 3_2, 0_3)$$
$$(1_3, 4_3, 0_1), (2_3, 3_3, 0_1)$$
$$(0_1, 0_2, 0_3)$$

Solution of S_{12}

$v = 6t+1$

Assume $6t+1 = p^n$ where p is a prime and $n \geq 1$.

Now, $x^{p^n - 1} \equiv 1 \pmod p$

or, $x^{6t} - 1 \equiv 0 \pmod p$.

or, $(x^{3t} - 1)(x^{3t} + 1) \equiv 0 \pmod p$

since x is the primitive element, $x^{3t} - 1 \neq 0$

$\therefore$ $x^{3t} + 1 \equiv 0 \pmod p$

or, $(x^t + 1)(x^{2t} - x^t + 1) = 0$

Now, $x^t + 1 \neq 0$

since, if $x^t = -1$

or. $x^{2t} = 1$

or. $x^{2t} - 1 = 0 \pmod p$

$\Rightarrow\Leftarrow$ as x is primitive element.

$\therefore x^{2t} - x^t + 1 = 0$

$\Rightarrow x^{2t} = x^t - 1$

Consider the set of t-initial blocks

$(x^i, x^{2t+i}, x^{4t+i})$, $i = 0(1)\overline{t-1}$

Differences arising out of the initial blocks

$\pm x^i(x^{2t}-1)$; $x^{2t+i}(x^{2t}-1)$; $\pm x^i(x^{4t}-1)$

Let, $x^{2t} - 1 = x^q$

$\therefore x^i(x^{2t}-1) = x^{i+q}$

$x^i(x^{4t}-1) = x^i(x^{2t}-1)(x^{2t}+1)$

$= x^i \cdot x^q \cdot x^t$ $[\because x^{2t} = x^t - 1]$

$= x^{i+q+t}$

$x^{2t+i}(x^{2t}-1) = x^{q+2t+i}$

$-x^i(x^{2t}-1) = -x^{q+i} = x^{q+3t+i}$ $\left[\begin{array}{l}\because x^{3t}+1=0\\ \quad x^{3t} = -1\end{array}\right]$.

$-x^i(x^{4t}-1) = -x^i \cdot x^{q+t} = x^{q+i+4t}$

$-x^{2t+i}(x^{2t}-1) = -x^{2t+i+q} = x^{5t+i+q}$

Remembering that $x^{6t} = 1$ we see that among the differences arising out of the initial block every non-zero element of $GF(6t+1)$ repeated exactly once.

Theorem

The initial blocks $(x^i, x^{2t+i}, x^{4t+i})$, $i = 0(1)\overline{t-1}$ [x being a primitive element of $GF(6t+1)$] provide a solution of the series S_{12} of BIBD with parameters $v = 6t+1$, $b = t(6t+1)$, $r = 3t$, $k = 3$, $\lambda = 1$ provided v is a prime or prime power.

→ Example

$t = 2$

$v = 13$, $b = 26$, $r = 6$, $K = 3$, $\lambda = 1$.

Primitive element of $GF(13)$ is 2

$i = 0, 1$

Initial Blocks : $(2^0, 2^4, 2^8) \equiv (1, 3, 9)$

$(2^1, 2^5, 2^9) \equiv (2, 6, 5)$

$\boxed{\text{Class-18}}$

→ $\boxed{\text{BIBD with } K = 4, \lambda = 1}$

$bK = vr$

$r(K-1) = \lambda(v-1)$

$\therefore$ $4b = vr$ $\qquad b = \frac{r(3r+1)}{4}$

$3r = v - 1$

$\Rightarrow r = 4t$ or $r = 4t+1$

S_{31} :- $v = 12t+1$, $b = t(12t+1)$, $r = 4t$, $K = 4$, $\lambda = 1$

S_{32} :- $v = 12t+4$, $b = (3t+1)(4t+1)$, $r = 4t+1$, $K = 4$, $\lambda = 1$

S_{31} :- $12t + 1 = p^n$ (p is a prime number and $n > 1$)

$x^{12t} - 1 = 0 \pmod{p} \Rightarrow x^{6t} + 1 = 0$ [$\because x^{6t} - 1 \neq 0$ as x is the primitive element]

Initial blocks :- $(0, x^{2i}, x^{4t+2i}, x^{8t+2i})$ $i = 0(1)\overline{t-1}$

Let, $x^{4t} - 1 = x^q$

e.g : $t = 2$

$v = 25,\ b = 50,\ r = 8,\ k = 4,\ \lambda = 1$

$v = 25 = 5^2$

Minimal polynomial of $GF(5^2)$ is $x^2 + 2x + 3$

Initial blocks $(0, x^0, x^8, x^{16})$

$(0, x^2, x^{10}, x^{18})$

$$x^2 = -2x - 3 = 3x + 2 \pmod 5 \longrightarrow [ax + b,\ a, b \in GF(5)]$$

$$x^3 = 3x^2 + 2x = 3(-2x - 3) + 2x = -6x - 9 + 2x = -4x - 9$$
$$= x + 1 \pmod 5$$

$$x^4 = x^2 + x = -2x - 3 + x = -x - 3 = 4x + 2 \pmod 5$$

$$x^8 = 16x^2 + 16x + 4 = x^2 + x + 4 \pmod 5 = -2x - 3 + x + 4$$
$$= -x + 1$$
$$= 4x + 1 \pmod 5$$

$$x^{10} = (4x + 1)(3x + 2) = 12x^2 + 11x + 2 = 2x^2 + x + 2$$
$$= 6x + 4 + x + 2$$
$$= 7x + 6$$
$$= 2x + 1 \pmod 5$$

$$x^{16} = (4x + 1)(4x + 1) = 16x^2 + 8x + 1$$
$$= 48x + 32 + 8x + 1$$
$$= 56x + 33$$
$$= x + 3 \pmod 5$$

$$x^{18} = (x + 3)x^2 = (x + 3)(3x + 2)$$
$$= 3x^2 + 2x + 9x + 6$$
$$= 3x^2 + 11x + 6$$
$$= 9x + 6 + 11x + 6$$
$$= 20x + 12$$
$$= 2 \pmod 5$$

Initial block : $0, 3x + 2$.

Theorem

A solution of the series S_{32} with parameters

$v = 12t+4$, $b = (3t+1)(4t+1)$

$r = (4t+1)$, $K = 4$, $\lambda = 1$ is provided by the initial blocks

$$(x_1^{2i}, x_1^{2t+2i}, x_2^{\beta+2i}, x_2^{\beta+2t+2i}) ; i = 0(1)\overline{t-1}$$

$$(x_2^{2i}, x_2^{2t+2i}, x_3^{\beta+2i}, x_3^{\beta+2t+2i}), \quad i = 0(1)\overline{t-1}$$

$$(x_3^{2i}, x_3^{2t+2i}, x_1^{\beta+2i}, x_1^{\beta+2t+2i}), \quad i = 0(1)\overline{t-1}$$

$$(\alpha, 0, 0_2, 0_3)$$

Provided that,

1. $(4t+1)$ is a prime or prime power.
2. To every element of $GF(4t+1)$ there correspond 3 treatments and we adjoin the invariant treatment α to these treatments.
3. x be a primitive element of $GF(4t+1)$
4. It is possible to find an odd integer β such that:

$$\frac{x^\beta+1}{x^\beta-1} = x^q, \text{ where } q \text{ is an odd integer.}$$

e.g : $t=1$

$v = 16$, $b = 20$, $r = 5$, $K = 4$, $\lambda = 1$

$4t+1 = 5 \rightarrow$ Prime or prime power

$GF(5) \longrightarrow$ Primitive element 2

$$\beta = 1 \Rightarrow \frac{2^1+1}{2^1-1} = 3 \equiv 8 \pmod 5$$
$$= 2^3 \pmod 5.$$

$\exists\ q = 3$

$\beta = 1$

Initial blocks:

$(2_1^{0}, 2_1^{2+0}, 2_2^{1+0}, 2_2^{1+2+0})$

$(2_2^{0}, 2_2^{2+0}, 2_3^{1+0}, 2_3^{1+2+0})$

$(2_3^{0}, 2_3^{2+0}, 2_1^{1+0}, 2_1^{1+2+0})$

$(\alpha, 0_1, 0_2, 0_3)$

$\Rightarrow$ Initial blocks: $(1_1, 4_1, 2_2, 3_2)$

$(1_2, 4_2, 2_3, 3_3)$

$(1_3, 4_3, 2_1, 3_1)$.

$(\alpha, 0_1, 0_2, 0_3)$

$\rightarrow$ BIBD with parameters $b = v = 4t-1$, $r = k = 2t-1$, $\lambda = t-1$

<u>Case-I</u> >> when $4t-1 = p^n$ (Prime or Prime power)

Then the initial block is $(x^0, x^2, x^4, \ldots, x^{4t-4})$

where x is a primitive element of $GF(p^n)$

<u>Case-II</u> >> when $4t-1$ is not a prime or prime power

<u>Defn</u>: A square matrix H of order n with entries -1 and $+1$ is said to be <u>Hadamard matrix</u> if $HH' = nI$. It can be seen that $H'H = HH' = nI$.

Also, it follows that if any row or column of a Hadamard matrix is multiplied by (-1), the matrix remains a Hadamard matrix. A necessary condition for the existance of a Hadamard matrix of order n is that $n \equiv 0 \pmod 4$, $n=2$ is the trivial case.

since, A Hadamard matrix remains Hadamard when any of it's rows of columns is multiplied by (-1), it is always possible to write a Hadama matrix with its first row and first column with +1 only. This form is called the normal form.

Let, H be a Hadamard matrix of order $n = 4t$ in it's normal form and B be a matrix obtained from H by deleting it's first row and first column. Obviously B is square matrix of order $(4t-1)$.

Let, $N = \dfrac{(B + \underset{\sim}{1}\underset{\sim}{1}')}{2}$

where $\underset{\sim}{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(4t-1)\times 1}$

Then it can be easily proved that N is the incidence matrix of a BIBD with the above mentioned parameters. Conversely if N is the incidence matrix of a BIBD with above parameters, then changing the zero's by -1 in N and bordering the resulting matrix by a row and a column of all $+1$ we get the Hadamard matrix of order $4t$.

Result

A Hadamard matrix of order $4t$ coexists with a BIBD with parameters $b = v = 4t-1$, $r = k = 2t-1$, $\lambda = t-1$.

Illustration $t = 4$

$v = 15$ is not a prime or prime power so, method of difference will not work.

we construct H_{16}.

$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$; $\otimes \rightarrow$ Kronecker Product

$H_4 = H_2 \otimes H_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

$$H_{16} = H_4 \otimes H_4$$

$$= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{pmatrix}$$

$$N = \begin{matrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{matrix}$$

BIBD

2	4	6	8	10	12	14
1	4	5	8	9	12	13
3	4	7	8	11	12	15
1	2	3	8	9	10	11
2	5	7	8	10	13	15
1	6	7	8	9	14	15
3	5	6	8	11	13	14
1	2	3	4	5	6	7
2	4	6	9	11	13	15
1	4	5	10	11	14	15
3	4	7	9	10	13	14
1	2	3	12	13	14	15
2	5	7	9	11	12	14
1	6	7	10	11	12	13
3	5	6	9	10	12	15

Class-19

* 3^n Factorial Experiment :-

Start with 3^2

Two factors A and B, each at 3 levels 0 (Low), 1 (intermediate), 2 (High)

Total $3^2 = 9$ treatment combinations

$$\begin{array}{|ccc|} a_0b_0 & a_0b_1 & a_0b_2 \\ a_1b_0 & a_1b_1 & a_1b_2 \\ a_2b_0 & a_2b_1 & a_2b_2 \end{array} \quad \text{or} \quad \begin{array}{|ccc|} 00 & 01 & 02 \\ 10 & 11 & 12 \\ 20 & 21 & 22 \end{array} \quad \text{or} \quad \begin{array}{|ccc|} (1) & (b) & (b^2) \\ (a) & (ab) & (ab^2) \\ (a^2) & (a^2b) & (a^2b^2) \end{array}$$