PROBLEM ON INFERENCE-I

Answer of Q-1:

Using multinomial probability law, we have likelihood function-

$$L = L(\pi) = \frac{n!}{n_1!\,n_2!\,n_3!\,n_4!} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}, \sum_i p_i = 1, \sum_i n_i = n$$

$\Rightarrow \ln(L) = C + n_1 \ln\left(\frac{2+p}{4}\right) + n_2 \ln\left(\frac{1-p}{4}\right) + n_3 \ln\left(\frac{1-p}{4}\right) + n_4 \ln\left(\frac{p}{4}\right)$

$\Rightarrow \quad \ln(L) = C + n_1 \ln(2 + (1-\pi)^2) + (n_2 + n_3) \ln(1 - (1-\pi)^2) + n_4 \ln((1-\pi)^2) - (n_1 + n_2 + n_3 + n_4) \ln 4$

Given that $n_1 = 190$, $n_2 = 36$, $n_3 = 34$ and $n_4 = 27$. We get,

Differentiating with respect to π is given by-

$$\frac{\partial \ln L}{\partial \pi} = -\frac{190 \times 2(1-\pi)}{2 + (1-\pi)^2} + \frac{70 \times 2(1-\pi)}{1 - (1-\pi)^2} - \frac{27 \times 2(1-\pi)}{(1-\pi)^2} = 0$$

$\Rightarrow \frac{190}{2+a} - \frac{70}{1-a} + \frac{27}{a} = 0 \qquad$ [let, $(1-\pi)^2 = a$]

$$\Rightarrow \frac{190a + 54 + 27a}{a(2+a)} = \frac{70}{1-a} \Rightarrow (217a + 54)(1-a) = 70a^2 + 140a$$

$$\Rightarrow 287a^2 - 23a - 54 = 0$$

The roots of the given equation are 0.4757 and -0.39. The value of a is 0.4757 since π cannot be complex.

Therefore,

$$(1-\pi)^2 = 0.4757 \Rightarrow 1 - \pi = 0.6897 \Rightarrow \pi = 0.3103$$

∴ The MLE of π is 0.3103.

Now,

$I(\pi) = nE\left[\frac{\partial \ln L}{\partial \pi}\right]^2$ where total frequency n=287

$= 287 \times \left[\frac{-262.2}{2.4761} + \frac{96.6}{0.5239} - \frac{37.26}{0.4761}\right]^2 = 287 \times [-105.892 + 184.386 - 78.261]^2 = 287 \times [0.233]^2 = 15.5809$

From C-R lower bound,

$$\text{var}(\pi) \geq \frac{1}{I(\pi)} = \frac{1}{15.2809} = 0.06544$$

So, $SE = \sqrt{0.06544} = 0.2558$

∴ Estimate of Standard Error is 0.2558.

Answer of Q-2:

Let, a random sample of size n is drawn from a population with the probability density function $f(x,\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}; x,\theta > 0$.

The likelihood function given by-

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \left(\frac{1}{\theta}\right)^n e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}$$

We know that $\log_e \theta$ is monotonic function of θ. i.e., maximizing $\log_e \theta$ is equivalent to maximizing θ. Therefore, we take $\log_e$ of the likelihood function for getting MLE easily-

$$\ln\big(L(\theta)\big) = -n \ln \theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

Partially differentiate with respect to θ and equate that with zero we get,

$$\frac{\partial \ln(L(\theta))}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0 \Rightarrow \frac{n}{\theta} = \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \Rightarrow \hat{\theta} = \bar{x}$$

Again, partially differentiate with respect to θ and we get-

$$\left|\frac{\partial^2 \ln(L(\theta))}{\partial \theta^2}\right|_{\hat{\theta}=\bar{x}} = \left|\frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i\right|_{\hat{\theta}=\bar{x}} = \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} = -\frac{n}{\bar{x}^2} < 0$$

$\therefore \hat{\theta} = \bar{x}$ is MLE of θ for the above distribution.

Here a random sample of 20 is drawn and the sample mean is 12.6.

So, The MLE of θ is 12.6.

Here, two sample observations are known to exceed 60 only. Then the likelihood function will be $L(\theta) = \prod_{i=1}^{18} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \times (1 - F(60))^2$

where, $F(x)$ be the distribution function of $f(x,\theta)$.

Now,

$$F(x) = \int_0^x f(t,\theta)dt = \int_0^x \frac{1}{\theta} e^{-\frac{t}{\theta}} dt = \frac{\theta}{\theta}\left[-e^{-\frac{t}{\theta}}\right]_0^x = 1 - e^{-\frac{x}{\theta}}$$

Then, the likelihood function be $L(\theta) = \prod_{i=1}^{18} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \times e^{-\frac{120}{\theta}}$

We know that $\log_e \theta$ is monotonic function of θ. i.e., maximizing $\log_e \theta$ is equivalent to maximizing θ. Therefore, we take $\log_e$ of the likelihood function for getting MLE easily-

$$\ln\big(L(\theta)\big) = -18 \ln \theta - \frac{1}{\theta}\sum_{i=1}^{18} x_i - \frac{120}{\theta}$$

Partially differentiate with respect to θ and equate that with zero we get,

$$\frac{\partial \ln(L(\theta))}{\partial\theta} = -\frac{18}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{18} x_i + \frac{120}{\theta} = 0 \Rightarrow \frac{18}{\theta} - \frac{120}{\theta} = \frac{1}{\theta^2}\sum_{i=1}^{18} x_i \Rightarrow \hat{\theta} = \frac{\sum_{i=1}^{18} x_i + 120}{18}$$

$\therefore$ The MLE become, $\hat{\theta} = \frac{\sum_{i=1}^{18} x_i + 120}{18}$

The sample observation exceeding 60 is rejected when sample observations are drawn. So, we need truncated exponential distribution where $0 < x \le 60$.
Let, Y follows the truncated exponential distribution.
The pdf of truncated exponential is,

$$P(Y = x) = \begin{cases} \dfrac{\frac{1}{\theta}e^{-\frac{x}{\theta}}}{1 - e^{-\frac{60}{\theta}}} & ; 0 < x \le 60, 0 < \theta \le 60 \\ 0 & o.w. \end{cases}$$

Since,

$$P(Y = x) = P(X = x | X \le 60) = \frac{P(X = x \cap X \le 60)}{P(X \le 60)} = \frac{P(X = x)}{F(60)}; 0 < x \le 60$$

$$= \frac{\frac{1}{\theta}e^{-\frac{x}{\theta}}}{1 - e^{-\frac{60}{\theta}}}\ ; 0 < x \le 60, 0 < \theta \le 60$$

Then the likelihood function will be $L(\theta) = \prod_{i=1}^{20} \frac{\frac{1}{\theta}e^{-\frac{x_i}{\theta}}}{1 - e^{-\frac{60}{\theta}}}$

We know that $\log_e \theta$ is monotonic function of θ. i.e., maximizing $\log_e \theta$ is equivalent to maximizing θ. Therefore, we take $\log_e$ of the likelihood function for getting MLE easily-

$$\ln\left(L(\theta)\right) = -20\ln\theta - \frac{1}{\theta}\sum_{i=1}^{20} x_i - 20\ln(1 - e^{-\frac{60}{\theta}})$$

Partially differentiate with respect to θ and equate that with zero we get,

$$\frac{\partial \ln\left(L(\theta)\right)}{\partial\theta} = -\frac{20}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{20} x_i + \frac{20e^{-\frac{60}{\theta}} \times \frac{60}{\theta^2}}{1 - e^{-\frac{60}{\theta}}} = 0$$

$$\Rightarrow \frac{20}{\theta} - \frac{20e^{-\frac{60}{\theta}} \times \frac{60}{\theta^2}}{1 - e^{-\frac{60}{\theta}}} = \frac{1}{\theta^2}252 \Rightarrow \frac{252}{\theta^2} - \frac{20}{\theta} + \frac{1200e^{-\frac{60}{\theta}}}{\theta^2\left(1 - e^{-\frac{60}{\theta}}\right)} = 0$$

where, $\sum_{i=1}^{20} x_i = 20 \times \bar{x} = 20 \times 12.6 = 252$

Then using Newton-Raphson method, $\hat{\theta} = 13.25648$

$\therefore$ The MLE become, $\hat{\theta} = 13.25648$

R codes :

```
> library(numDeriv)
> newton_raphson <- function(f, a, b, tol = 1e-5, n = 1000) {
+   require(numDeriv) # Package for computing f'(x)
+
+   x0 <- a # Set start value to supplied lower bound
+   k <- n # Initialize for iteration results
+
+   # Check the upper and lower bounds to see if approximations result i
n 0
+   fa <- f(a)
+   if (fa == 0.0) {
+     return(a)
+   }
+
+   fb <- f(b)
+   if (fb == 0.0) {
+     return(b)
+   }
+
+   for (i in 1:n) {
+     dx <- genD(func = f, x = x0)$D[1] # First-order derivative f'(x0)
+     x1 <- x0 - (f(x0) / dx) # Calculate next value x1
+     k[i] <- x1 # Store x1
+     # Once the difference between x0 and x1 becomes sufficiently small
, output the results.
+     if (abs(x1 - x0) < tol) {
+       root_approx <- tail(k, n=1)
+       res <- list('root approximation' = root_approx, 'iterations' = k
)
+       return(res)
+     }
+     # If Newton-Raphson has not yet reached convergence set x1 as x0 a
nd continue
+     x0 <- x1
+   }
+   print('Too many iterations in method')
+ }
> f2=function(theta){
+   z=(252/theta^2)-(20/theta)+((1200*exp(-60/theta))/(theta^2*(1-exp(-6
0/theta))))
+   return(z)
+ }
> newton_raphson(f2,12.6,15)
$`root approximation`
[1] 13.25648

$iterations
[1] 13.18468 13.25556 13.25648 13.25648
```

Answer of Q-3:

I. Let, a random sample of size n is drawn from a population with the probability density function $f(x,\theta) = \frac{1}{\pi}.\frac{1}{1+(x-\theta)^2}$; $-\infty < x, \theta < \infty$.

The likelihood function given by-

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\pi}.\frac{1}{1+(x_i-\theta)^2} = \left(\frac{1}{\pi}\right)^n.\frac{1}{\prod_{i=1}^{n}[1+(x_i-\theta)^2]}$$

We know that $\log_e \theta$ is monotonic function of θ. i.e., maximizing $\log_e \theta$ is equivalent to maximizing θ. Therefore, we take $\log_e$ of the likelihood function for getting MLE easily-

$$\ln\big(L(\theta)\big) = -n\ln\pi - \sum_{i=1}^{n} \ln[1+(x_i-\theta)^2]$$

Partially differentiate with respect to θ and equate that with zero we get,

$$\frac{\partial \ln(L(\theta))}{\partial\theta} = \sum_{i=1}^{n} \frac{2(x_i-\theta)}{1+(x_i-\theta)^2} = 0$$

It seems very difficult to solve the equation. Therefore, we use Newton-Raphson method to solve equation.

Using R software, the Newton-Raphson process is done.

Then, $\hat{\theta} = 3.934$

$\therefore$ The MLE be, $\hat{\theta} = 3.934$

R codes :

```
> library(numDeriv)
> newton_raphson <- function(f, a, b, tol = 1e-5, n = 1000) {
+   require(numDeriv) # Package for computing f'(x)
+ 
+   x0 <- a # Set start value to supplied lower bound
+   k <- n # Initialize for iteration results
+ 
+   # Check the upper and lower bounds to see if approximations result i
n 0
+   fa <- f(a)
+   if (fa == 0.0) {
+     return(a)
+   }
+ 
+   fb <- f(b)
+   if (fb == 0.0) {
+     return(b)
+   }
+ 
+   for (i in 1:n) {
+     dx <- genD(func = f, x = x0)$D[1] # First-order derivative f'(x0)
+     x1 <- x0 - (f(x0) / dx) # Calculate next value x1
+     k[i] <- x1 # Store x1
+     # Once the difference between x0 and x1 becomes sufficiently small
, output the results.
+     if (abs(x1 - x0) < tol) {
+       root_approx <- tail(k, n=1)
```

```
+       res <- list('root approximation' = root_approx, 'iterations' = k
)
+       return(res)
+     }
+     # If Newton-Raphson has not yet reached convergence set x1 as x0 a
nd continue
+     x0 <- x1
+   }
+   print('Too many iterations in method')
+ }
> x=c(3.7807, 2.9957, 5.2043, 4.8993, 2.6874, 4.9557, 4.9367, 3.4996, 3.
1674)
> f1=function(theta){
+   z=sum((x-theta)/(1+(x-theta)^2))
+   return(z)
+ }
> newton_raphson(f1,median(x),5)
$`root approximation`
[1] 3.934291

$iterations
[1] 3.917908 3.934088 3.934291 3.934291
```

II.The pdf of Cauchy distribution is $f(x,\theta) = \frac{1}{\pi}.\frac{1}{1+(x-\theta)^2}$

The cdf of Cauchy distribution is

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi}.\frac{1}{1+(t-\theta)^2}dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{x-\theta} \frac{1}{1+z^2}dz \qquad [Let, t-\theta = z \Rightarrow dt = dz]$$

$$= \frac{1}{\pi}[\tan^{-1} z]_{-\infty}^{x-\theta} = \frac{1}{\pi}\left[\tan^{-1}(x-\theta) + \frac{\pi}{2}\right] = \frac{1}{\pi}\tan^{-1}(x-\theta) + \frac{1}{2}$$

$$F\left(X_p,\theta\right) = \frac{1}{\pi}\tan^{-1}\left(X_p-\theta\right) + \frac{1}{2} = p$$

$$\Rightarrow \tan^{-1}\left(X_p-\theta\right) = \left(p-\frac{1}{2}\right)\pi$$

$$\Rightarrow X_p = \theta + \tan\left(p-\frac{1}{2}\right)\pi$$

$$\Rightarrow \hat{\theta} = X_p + \tan\left(p-\frac{1}{2}\right)\pi$$

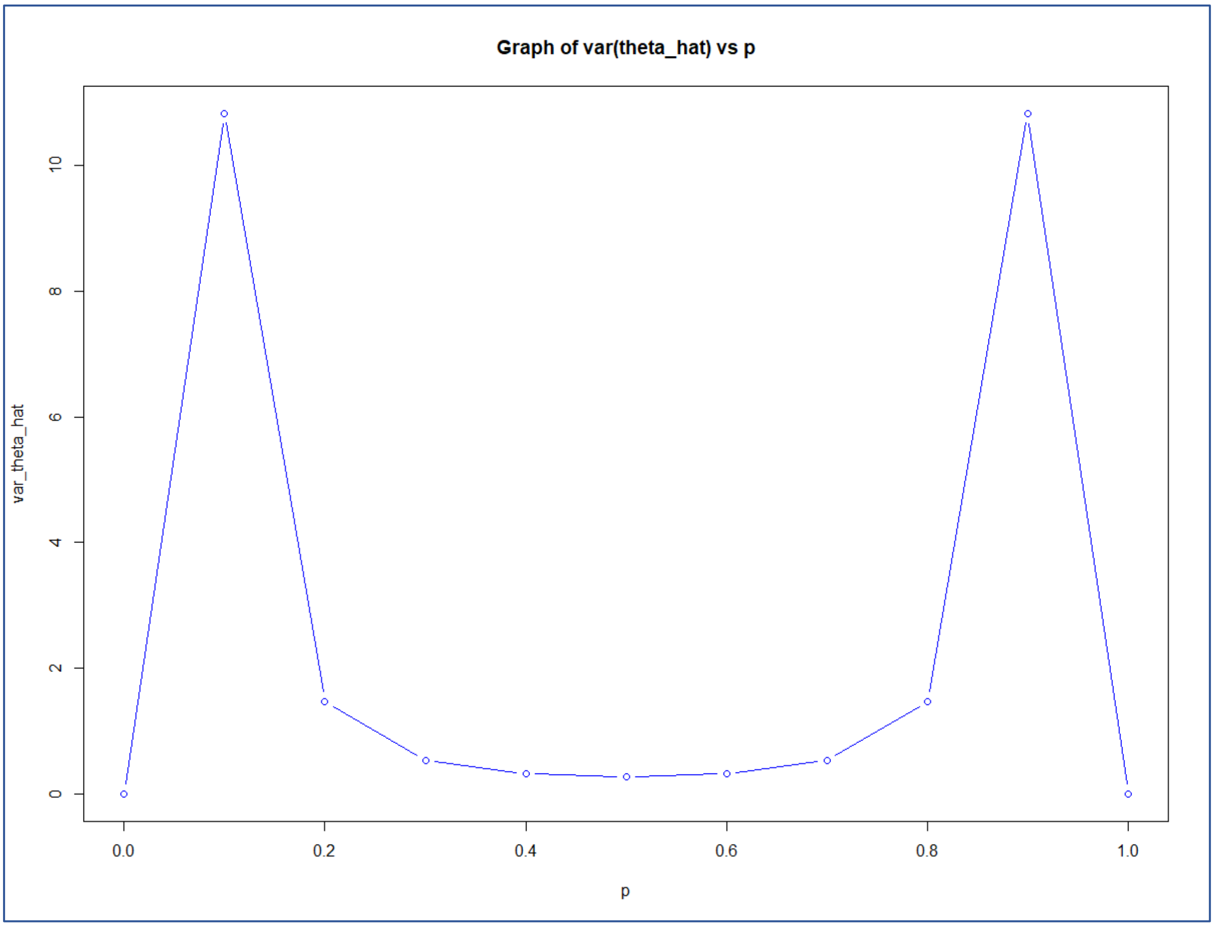
Now, we have to minimize var($\hat{\theta}$) with respect to p.

We know that, asymptotic variance of X_p is,

$$\frac{p(1-p)}{n\left(f(X_p)\right)^2} = \frac{p(1-p)\pi^2\left(1+(X_p-\hat{\theta})^2\right)^2}{9} = \frac{p(1-p)\pi^2\left(1+\left(\tan\left(p-\frac{1}{2}\right)\pi\right)^2\right)^2}{9}$$

$$= \frac{p(1-p)\pi^2\left(\sec\left(p-\frac{1}{2}\right)\pi\right)^4}{9} = \frac{p(1-p)\pi^2}{9\left(\cos\left(p-\frac{1}{2}\right)\pi\right)^4} \quad [\because n = 9]$$

Now, plot var($\hat{\theta}$) vs p to get the value of p for which var($\hat{\theta}$) is minimum.



From the plot, we may observe that for var($\hat{\theta}$) is minimum at p = 0.5.

$\therefore$ Percentile estimate of θ will be $\frac{50}{100} \times (9+1) = 5^{th}$ value of the ordered dataset. i.e., 2.6874, 2.9957, 3.1674, 3.4996, 3.7807, 4.8993, 4.9367, 4.9557, 5.2043.

$\therefore$Percentile estimate of θ is 3.7807.

R codes :

```
> p = c(0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1)
> var_theta_hat = (p*(1-p)*pi^2)/(9*(cos(pi*(p-(1/2))))^4)
> plot(p, var_theta_hat, type = "b", main = "Graph of
var(theta_hat) vs p", col = "blue")
```

III. The non-parametric estimate of mean and variance be sample mean ($\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$) and sample variance ($s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$) respectively.

Mean of the given observations is-

$$\bar{x} = \frac{1}{9}\sum_{i=1}^{9} x_i = 4.014089$$

Sample variance of the given observations is-

$$s^2 = \frac{1}{9-1}\sum_{i=1}^{9}(x_i - \bar{x})^2 = 0.9714143$$

∴ Therefore, the non-parametric estimate of mean and variance be 4.014089 and 0.9714143 respectively.

Answer of Q-4:

A random variable X takes values 0, 1, 2 with respective probabilities

$\frac{\theta}{4N}+\frac{1}{2}\left(1-\frac{\theta}{N}\right), \frac{\theta}{2N}+\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right), \frac{\theta}{4N}+\frac{1-\alpha}{2}\left(1-\frac{\theta}{N}\right).$

Now, 1st order raw moment be-

$$\mu_1' = E(X) = 0.\left[\frac{\theta}{4N}+\frac{1}{2}\left(1-\frac{\theta}{N}\right)\right] + 1.\left[\frac{\theta}{2N}+\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)\right] + 2.\left[\frac{\theta}{4N}+\frac{1-\alpha}{2}\left(1-\frac{\theta}{N}\right)\right]$$

$$=\frac{\theta}{2N}+\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)+\frac{\theta}{2N}+\left(1-\frac{\theta}{N}\right)-\alpha\left(1-\frac{\theta}{N}\right)$$

$$=\frac{\theta}{N}-\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)+\left(1-\frac{\theta}{N}\right)=\frac{\theta}{N}+\left(1-\frac{\alpha}{2}\right)\left(1-\frac{\theta}{N}\right)=1-\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)$$

Now, 2nd order raw moment be-

$$\mu_2' = E(X^2) = 0.\left[\frac{\theta}{4N}+\frac{1}{2}\left(1-\frac{\theta}{N}\right)\right] + 1^2.\left[\frac{\theta}{2N}+\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)\right] + 2^2.\left[\frac{\theta}{4N}+\frac{1-\alpha}{2}\left(1-\frac{\theta}{N}\right)\right]$$

$$=\frac{\theta}{2N}+\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)+\frac{\theta}{N}+2\left(1-\frac{\theta}{N}\right)-2\alpha\left(1-\frac{\theta}{N}\right)$$

$$=\frac{\theta}{2N}+\frac{\alpha}{2}-\frac{\theta\alpha}{2N}+\frac{\theta}{N}+2-\frac{2\theta}{n}-2\alpha+\frac{2\alpha\theta}{N}=2-\frac{\theta}{2N}-\frac{3\alpha}{2}\left(1-\frac{\theta}{N}\right)$$

The frequency distribution of the sample be-

X	0	1	2
Frequency (f)	27	38	10

The 1st order raw moment of the sample be-

$$m_1' = \frac{\sum_{i=1}^{3} x_i f_i}{\sum_{i=1}^{3} f_i} = \frac{(0\times 27+1\times 38+2\times 10)}{75} = \frac{58}{75}$$

$$m_2' = \frac{\sum_{i=1}^{3} x_i^2 f_i}{\sum_{i=1}^{3} f_i} = \frac{(0\times 27+1\times 38+4\times 10)}{75} = \frac{78}{75}$$

Equating the sample moments to population moments we get-

$$\mu_1' = m_1' \Rightarrow 1-\frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)=\frac{58}{75} \Rightarrow \frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)=1-\frac{58}{75}=\frac{17}{75} \dots\dots\dots (1)$$

$\mu_2' = m_2' \Rightarrow 2-\frac{\theta}{2N}-\frac{3\alpha}{2}\left(1-\frac{\theta}{N}\right)=\frac{78}{75} \Rightarrow \frac{\theta}{2N}=2-\frac{78}{75}-\frac{51}{75} \Rightarrow \frac{\theta}{2N}=$
$\frac{150-78-51}{75} \Rightarrow \frac{\theta}{2N}=\frac{21}{75} \Rightarrow \hat{\theta}=\frac{42N}{75}$ $\quad\left[\because \frac{\alpha}{2}\left(1-\frac{\theta}{N}\right)=\frac{17}{75}\right]$

Now, put $\hat{\theta}=\frac{42N}{75}$ in (1), we get-

$$\frac{\alpha}{2}\left(1-\frac{42N}{N}\right)=\frac{17}{75}\Longrightarrow\frac{\alpha}{2}\times\frac{33}{75}=\frac{17}{75}\Longrightarrow\hat{\alpha}=\frac{17\times2\times75}{75\times33}=\frac{34}{33}=1.03030303$$

Here, N=25 then, $\hat{\theta}=\frac{42\times25}{75}=14$.

∴ Using method of moments, the estimate of θ and α is 14 and 1.03030303 respectively.

Answer of Q-5:

Given that, $X \sim N(\theta, 1)$, where $\theta \in [-1, 1]$. We have to estimate θ.
On the basis of a sample size of $n(= 10)$, the following estimator has been defined-

$$T = \begin{cases} -1 \ \ if \ \ \bar{X} < -1 \\ \bar{X} \ \ if -1 \leq \bar{X} \leq 1 \\ 1 \ \ if \ \ \bar{X} > 1 \end{cases}$$

Where $\bar{X}$ being sample mean.

We know that $\bar{X} \sim N(\theta, \frac{1}{n})$ [since, if $X \sim N(\theta, \sigma^2)$ $then$ $\bar{X} \sim N(\theta, \frac{\sigma^2}{n})$]

i. We need to plot the risk curve of $\bar{X}$ and T over the range of $\theta \in [-1, 1]$ – assuming squared error loss.

∴The risk function of $\bar{X}$ $\left(R_{\bar{X}}(\theta)\right)$,

$$E(\bar{X} - \theta)^2 = var(\bar{X}) - (E(\bar{X}) - \theta)^2 = \frac{1}{n} - 0 = \frac{1}{n} = \frac{1}{10}$$

∴ So, $R_{\bar{X}}(\theta)$ is independent of θ.

∴The risk function of T $\left(R_T(\theta)\right)$

$$E(T - \theta)^2$$

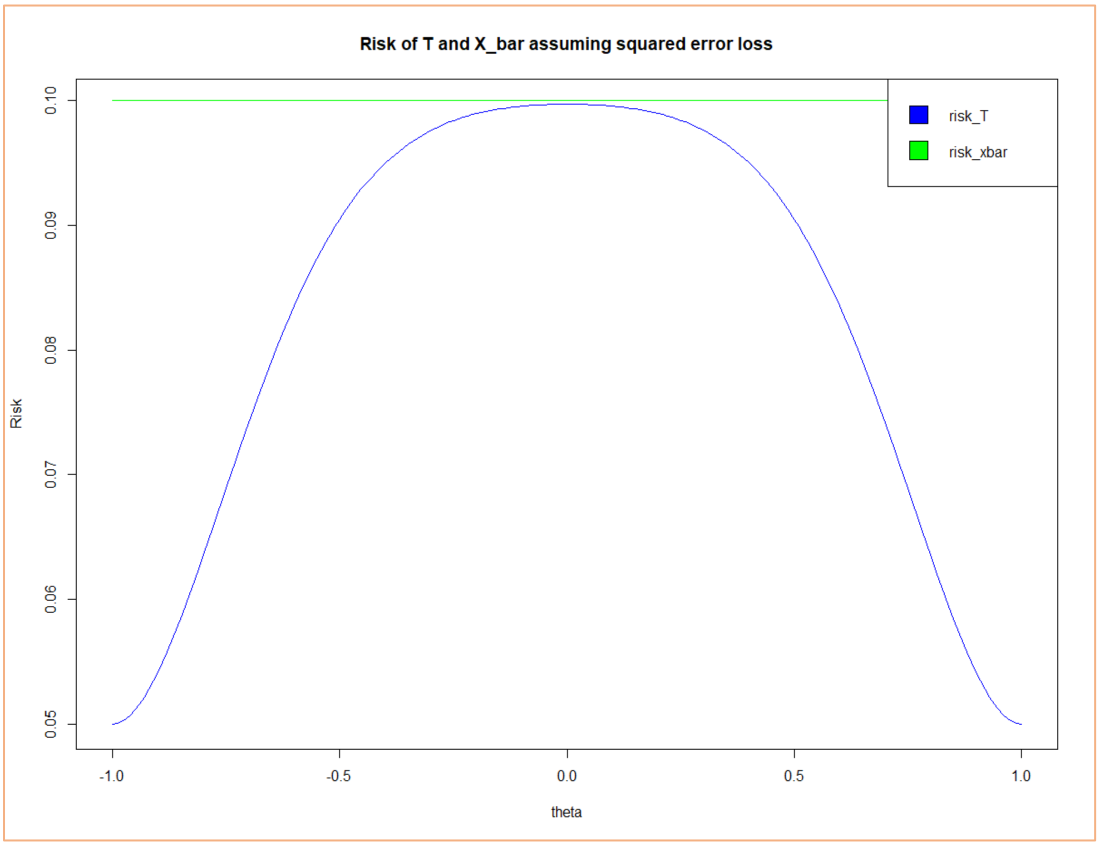
$$= \int_{-\infty}^{-1} (-1-\theta)^2 \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(\bar{X}-\theta)^2} d\bar{x} + \int_{-1}^{1} (\bar{X}-\theta)^2 \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(\bar{X}-\theta)^2} d\bar{x} + \int_{1}^{\infty} (1-\theta)^2 \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(\bar{X}-\theta)^2} d\bar{x}$$

Let, $\sqrt{n}(\bar{X} - \theta) = z \Longrightarrow \sqrt{n}\, d\bar{x} = dz$

$$= \int_{-\infty}^{-\sqrt{n}(1+\theta)} (1+\theta)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \int_{-\sqrt{n}(1+\theta)}^{\sqrt{n}(1-\theta)} \frac{1}{n\sqrt{2\pi}} z^2 e^{-\frac{z^2}{2}} dz + \int_{\sqrt{n}(1-\theta)}^{\infty} (1-\theta)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= (1+\theta)^2 \Phi\left(-\sqrt{n}(1+\theta)\right) + \int_{-\sqrt{n}(1+\theta)}^{\sqrt{n}(1-\theta)} \frac{1}{n\sqrt{2\pi}} z^2 e^{-\frac{z^2}{2}} dz + (1-\theta)^2 \Phi\left(-\sqrt{n}(1-\theta)\right)$$

Therefore, plotting the risk curve of $\bar{X}$ and T over the range of $\theta \in [-1, 1]$ in R software we get-



∴ From the diagram, it may be observed that risk based on estimator T is less than risk based on estimator $\bar{X}$. So, T is better estimator than $\bar{X}$.

R codes :

```
> f1 <- function(z){0.1 * (sqrt(2 * pi))^(-1) * z^2 * exp(-(z^2)/2)
}
> sum1 <- function(theta){
+   int1 = (1 + theta)^2 * pnorm(((-1) * sqrt(10) * (1 + theta)), m
ean = 0, sd = 1)
+   int3 = (1 - theta)^2 * pnorm(((-1) * sqrt(10) * (1 - theta)), m
ean = 0, sd = 1)
+   return(int1+int3)
+ }
> sum2 <- function(theta){
+   low = -sqrt(10) * (1+theta)
+   upp = sqrt(10) * (1-theta)
+   int2 = integrate(f1,low,upp)
+   return(int2$value)
+ }
> theta = seq(-1,1,0.01)
> risk_T = vector()
> risk_xbar = rep(0.1,201)
> for (x in theta){
+   risk_T = append(risk_T, sum1(x) + sum2(x))
+ }
> plot(theta, risk_T, type = "line", col = "blue", main = "Risk of
T and     X_bar assuming squared error loss", ylab ="Risk")
> lines(theta, risk_xbar, col = "green")
> legend("topright", c("risk_T","risk_xbar"), fill = c("blue",
"green"))
```

ii. We need to plot the risk curve of $\bar{X}$ and T over the range of $\theta \in [-1, 1]$ – assuming absolute error loss.

$\therefore$The risk function of $\bar{X}$ $\left(R_{\bar{X}}(\theta)\right)$,

$$E(|\bar{X}-\theta|) = \frac{\sqrt{n}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\bar{X}-\theta|\, e^{-\frac{n}{2}(\bar{X}-\theta)^2}d\bar{x}$$

Let, $\sqrt{n}(\bar{X}-\theta) = z \Longrightarrow \sqrt{n}\, d\bar{x} = dz$

$$= \frac{1}{\sqrt{2n\pi}}\int_{-\infty}^{\infty}|z|\, e^{-\frac{z^2}{2}}dz = \frac{\sqrt{2}}{\sqrt{n\pi}}\int_{0}^{\infty} z\, e^{-\frac{z^2}{2}}dz$$

$$= \frac{\sqrt{2}}{\sqrt{n\pi}} \times \frac{\Gamma(2)}{2\left(\frac{1}{2}\right)^{\frac{2}{2}}} = \frac{\sqrt{2}}{\sqrt{n\pi}}$$

$\therefore$ So, $R_{\bar{X}}(\theta)$ is independent of θ.

$\therefore$The risk function of T $\left(R_T(\theta)\right)$

$$E(|T-\theta|)$$

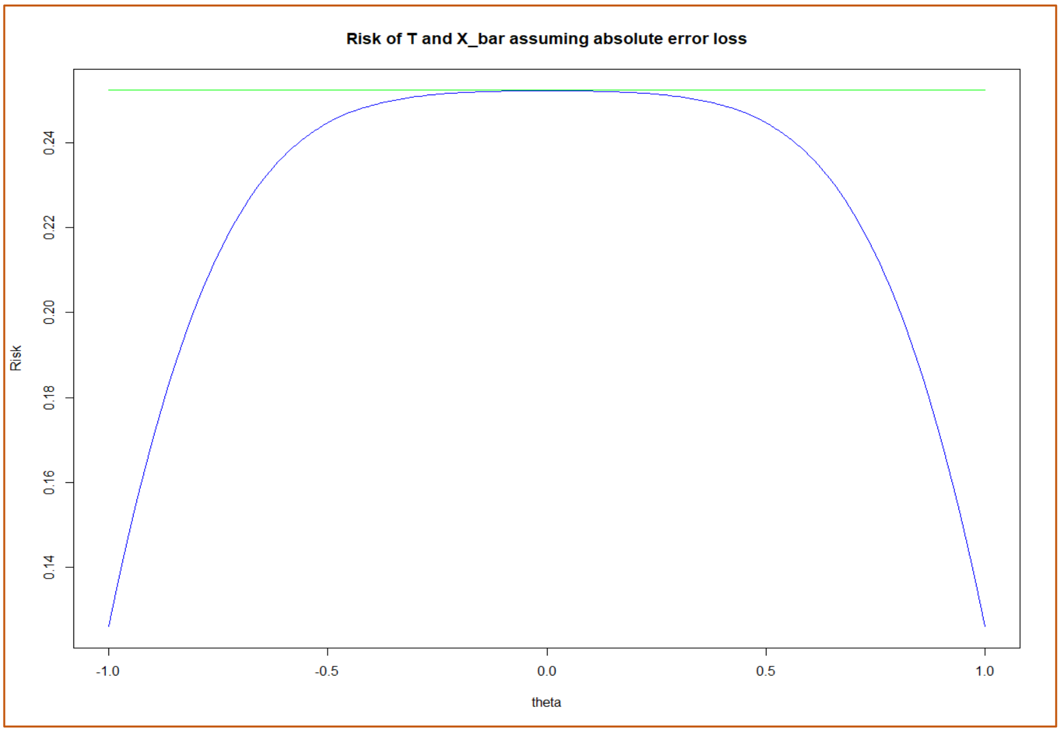
$$= \int_{-\infty}^{-1}|-1-\theta|\frac{\sqrt{n}}{\sqrt{2\pi}}e^{-\frac{n}{2}(\bar{X}-\theta)^2}d\bar{x} + \int_{-1}^{1}|\bar{X}-\theta|\frac{\sqrt{n}}{\sqrt{2\pi}}e^{-\frac{n}{2}(\bar{X}-\theta)^2}d\bar{x}$$
$$+ \int_{1}^{\infty}|1-\theta|\frac{\sqrt{n}}{\sqrt{2\pi}}e^{-\frac{n}{2}(\bar{X}-\theta)^2}d\bar{x}$$

Let, $\sqrt{n}(\bar{X}-\theta) = z \Longrightarrow \sqrt{n}\, d\bar{x} = dz$

$$= \int_{-\infty}^{-\sqrt{n}(1+\theta)}(1+\theta)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \int_{-\sqrt{n}(1+\theta)}^{\sqrt{n}(1-\theta)}\frac{1}{\sqrt{2n\pi}}|z|e^{-\frac{z^2}{2}}dz +$$
$$\int_{\sqrt{n}(1-\theta)}^{\infty}|1-\theta|\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$= (1+\theta)\Phi\left(-\sqrt{n}(1+\theta)\right) + \int_{-\sqrt{n}(1+\theta)}^{\sqrt{n}(1-\theta)}\frac{1}{\sqrt{2n\pi}}|z|e^{-\frac{z^2}{2}}dz +$$
$$(1-\theta)\Phi\left(-\sqrt{n}(1-\theta)\right)$$

Therefore, plotting the risk curve of $\bar{X}$ and T over the range of $\theta \in [-1, 1]$ in R software we get-



∴ From the diagram, it may be observed that risk based on estimator T is less than risk based on estimator $\bar{X}$. So, T is better estimator than $\bar{X}$.

R codes :

```
> f1 <- function(z){(sqrt(2 * pi * 10))^(-1) * abs(z) * exp(-(z^2)/
2)}
> sum1 <- function(theta){
+   int1 = (1 + theta) * pnorm(((-1) * sqrt(10) * (1 + theta)), mea
n = 0, sd = 1)
+   int3 = (1 - theta) * pnorm(((-1) * sqrt(10) * (1 - theta)), mea
n = 0, sd = 1)
+   return(int1+int3)
+ }
> sum2 <- function(theta){
+   low = -sqrt(10) * (1+theta)
+   upp = sqrt(10) * (1-theta)
+   int2 = integrate(f1,low,upp)
+   return(int2$value)
+ }
> theta = seq(-1,1,0.01)
> risk_T = vector()
> risk_xbar = rep(0.2523,201)
> for (x in theta){
+   risk_T = append(risk_T, sum1(x) + sum2(x))
+ }
> plot(theta, risk_T, type = "line", col = "blue", main = "Risk of
T and     X_bar assuming absolute error loss", ylab ="Risk")
> lines(theta, risk_xbar, col = "green")
> legend("topright", c("risk_T","risk_xbar"), fill = c("blue",
"green"))
```

Answer of Q-6:

The total amount of claims for each year from a portfolio of five insurance policies over t years were found to be $X_1, X_2, \dots \dots X_t$.

The insurer believes that the annual claims (X_i) have a normal distribution with mean μ and variance σ^2.

$$X_i \sim N(\mu, \sigma^2), \text{for } \forall i = i(1)t \text{ where, } \sigma^2 \text{ is known and } \mu \text{ is unknown.}$$

We have to estimate μ.

The prior distribution of μ is assumed to be normal with mean γ and variance η^2. i.e., $\mu \sim N(\gamma, \eta^2)$.

We know that $\bar{X}_t$ is sufficient for Θ.

i. Now, $\bar{X} \sim N(\mu, \frac{\sigma^2}{t})$ where $\theta \in \mathbb{R} = \Theta$.

$$P(\mu|\bar{X}) \propto P(\bar{X}|\mu)P(\mu)$$

$$= \text{constant}. e^{-\frac{t}{2\sigma^2}(\bar{X}-\mu)^2} . e^{-\frac{1}{2\eta^2}(\mu-\gamma)^2}$$

$$= \text{constant}. e^{-\frac{1}{2}\left[\frac{t}{\sigma^2}(\bar{X}^2+\mu^2-2\mu\bar{X})+\frac{1}{\eta^2}(\mu^2+\gamma^2-2\mu\gamma)\right]}$$

$$= \text{constant}. e^{-\frac{1}{2}\left[\left(\frac{t}{\sigma^2}+\frac{1}{\eta^2}\right)\mu^2-2\mu\left(\frac{t\bar{X}}{\sigma^2}+\frac{\gamma}{\eta^2}\right)+\left(\frac{t\bar{X}^2}{\sigma^2}+\frac{\gamma^2}{\eta^2}\right)\right]} \dots \dots \dots . . (1)$$

We know that $(\mu|\bar{X})$ follows normal distribution but we have to calculate the mean and the variance of the distribution.

For getting a normal distribution we have to get a format like-

$$\text{constant}. e^{-\frac{1}{2b^2}(\mu^2+a^2-2\mu a)} \dots \dots \dots . (2) \quad \text{where, mean } a \text{ and variance } b^2.$$

Equating (1) and (2), we get-

So, $-\frac{\mu^2}{2b^2} = -\frac{\mu^2}{2}\left(\frac{t}{\sigma^2}+\frac{1}{\eta^2}\right) \Longrightarrow \frac{1}{b^2} = \left(\frac{1}{\frac{\sigma^2}{t}}+\frac{1}{\eta^2}\right) \Longrightarrow b^2 = \frac{\eta^2\frac{\sigma^2}{t}}{\eta^2+\frac{\sigma^2}{t}} \quad \dots \dots \dots . (3)$

Again, $\frac{\mu a}{b^2} = \mu\left(\frac{t\bar{X}}{\sigma^2}+\frac{\gamma}{\eta^2}\right) \Longrightarrow \frac{\mu a}{b^2} = \mu\left(\frac{\bar{X}}{\frac{\sigma^2}{t}}+\frac{\gamma}{\eta^2}\right) \Longrightarrow \frac{a}{b^2} = \frac{\bar{X}}{\frac{\sigma^2}{t}}+\frac{\gamma}{\eta^2}$

$$\Longrightarrow \frac{a}{b^2} = \frac{\bar{X}\eta^2+\frac{\sigma^2}{t}\gamma}{\eta^2\frac{\sigma^2}{t}} \Longrightarrow a = \frac{\bar{X}\eta^2+\frac{\sigma^2}{t}\gamma}{\eta^2\frac{\sigma^2}{t}} \times \frac{\eta^2\frac{\sigma^2}{t}}{\eta^2+\frac{\sigma^2}{t}} \Longrightarrow a = \frac{\bar{X}\eta^2+\frac{\sigma^2}{t}\gamma}{\eta^2+\frac{\sigma^2}{t}}$$

$$\Longrightarrow a = \frac{\bar{X}\eta^2-\eta^2\gamma+\eta^2\gamma+\frac{\sigma^2}{t}\gamma}{\eta^2+\frac{\sigma^2}{t}} = \frac{\left(\eta^2+\frac{\sigma^2}{t}\right)\gamma}{\eta^2+\frac{\sigma^2}{t}} + \frac{\eta^2(\bar{X}-\gamma)}{\eta^2+\frac{\sigma^2}{t}} = \gamma + \frac{\eta^2}{\eta^2+\frac{\sigma^2}{t}}(\bar{X}-\gamma)$$

$$\therefore (\mu|\bar{X}) \sim N\left(\gamma + \frac{\eta^2}{\eta^2+\frac{\sigma^2}{t}}(\bar{X}-\gamma),\ \frac{\eta^2\frac{\sigma^2}{t}}{\eta^2+\frac{\sigma^2}{t}}\right)$$

$\therefore$The posterior distribution of μ is $N\left(\gamma + \frac{\eta^2}{\eta^2+\frac{\sigma^2}{t}}(\bar{X}-\gamma),\ \frac{\eta^2\frac{\sigma^2}{t}}{\eta^2+\frac{\sigma^2}{t}}\right)$.

ii. We know that,

$$d_0(\underline{x}) = d_0(\bar{x}) = E(\mu|\bar{X} = \bar{x}) = \gamma + \frac{\eta^2}{\eta^2+\frac{\sigma^2}{t}}(\bar{X} - \gamma) = \frac{\bar{X}\eta^2+\frac{\sigma^2}{t}\gamma}{\eta^2+\frac{\sigma^2}{t}} = \frac{\frac{t}{\sigma^2}\bar{X}+\frac{\gamma}{\eta^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}}$$

$\therefore$ Bayesian point estimate of μ under the quadratic (squared error) loss function $\frac{\frac{t}{\sigma^2}\bar{X}+\frac{\gamma}{\eta^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}}$.

iii. So, $\frac{\frac{t}{\sigma^2}\bar{X}+\frac{\gamma}{\eta^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}} = \frac{\frac{t}{\sigma^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}}\bar{X} + \left(1 - \frac{\frac{t}{\sigma^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}}\right)\gamma = z\bar{X} + (1-z)\gamma$

where, $z = \frac{\frac{t}{\sigma^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}}$ and z be the credibility factor.

iv. The all or nothing loss function is

$$L(\theta, d) = \begin{cases} 1 \; if \; d \neq \theta \\ 0 \; if \; d = \theta \end{cases}$$

We know for the above loss function the Bayesian estimate of population mean is the mode of posterior distribution. For Normal distribution is symmetric i.e., mean, median, mode be same.

$\therefore$ Bayesian point estimate of μ under the all or nothing loss function $\frac{\frac{t}{\sigma^2}\bar{X}+\frac{\gamma}{\eta^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}}$.

Since it is same as squared error loss function. So, it can be written as a function of credibility factor.

v. Given that, $(X_1, X_2, X_3, X_4, X_5) = (1050, 1175, 1100, 1200, 1150)$
Here, t = 5

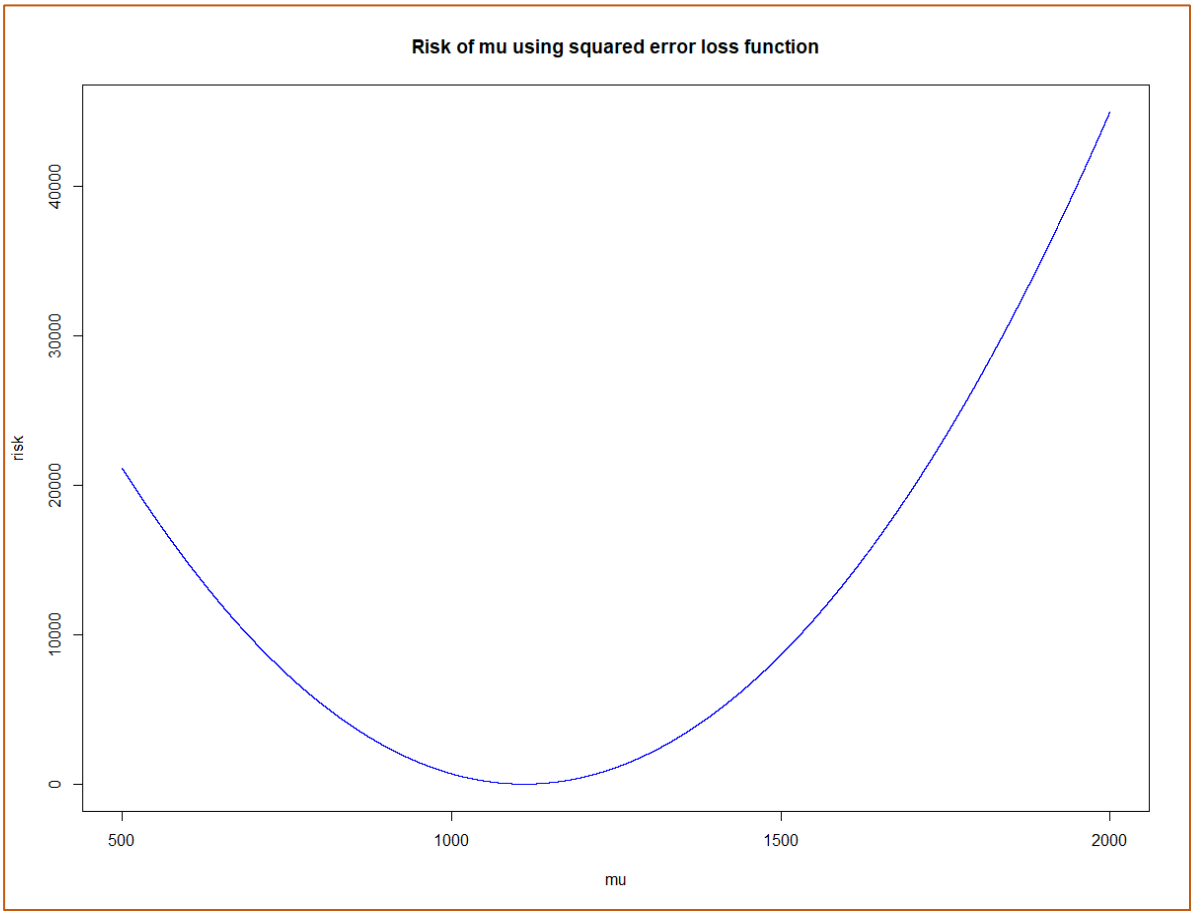
$\therefore$ It is calculated that Bayesian point estimate of μ under the quadratic (squared error) loss function $\frac{\frac{t}{\sigma^2}\bar{X}+\frac{\gamma}{\eta^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}} = \frac{\frac{5\times1135}{400}+\frac{1110}{256}}{\frac{1}{256}+\frac{5}{400}} = \frac{14.1875+4.3359}{0.0125+0.0039} = 1129.4756$

$\therefore$ the credibility factor is $= \frac{\frac{t}{\sigma^2}}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}} = \frac{\frac{5}{400}}{\frac{1}{256}+\frac{5}{400}} = \frac{0.0125}{0.0164} = 0.7622$

The risk of Bayes estimate is $(R_{d_0}(\mu) = E_{\bar{x}|\mu}\{d_0(\bar{x}) - \theta\}^2$

$$= \frac{\frac{t}{\sigma^2}}{\left(\frac{1}{\eta^2}+\frac{t}{\sigma^2}\right)^2} + \left[\frac{\frac{1}{\eta^2}(\gamma - \mu)}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}}\right]^2$$

The plot of risk function using R programming is given below-



R codes :

```
> f1 = function(mu){46.4684+(0.0567*(1110-mu)^2)}
> mu = seq(500, 2000, 0.01)
> risk_mu = vector()
> for (x in mu){risk_mu = append(risk_mu, f1(x))}
> plot(mu, risk_mu, type = "l", col = "blue")
> plot(mu, risk_mu, type = "l", col = "blue", ylab = "risk")
> plot(mu, risk_mu, type = "l", col = "blue", ylab = "risk", main =
"Risk of mu using squared error loss function")
```

$\therefore$ Bayes' risk of the estimate $= r_{d_0} = E_\theta\left[R_{d_0}(\theta)\right] = \dfrac{\left(\frac{1}{\eta^2}+\frac{t}{\sigma^2}\right)}{\left(\frac{1}{\eta^2}+\frac{t}{\sigma^2}\right)^2} = \dfrac{1}{\frac{1}{\eta^2}+\frac{t}{\sigma^2}} =$ 60.9756.