Theorem

Fisher's Inequality

Barring trivial ~~expectants~~ exceptions, the inequality $b \geq v$ holds for all connected, equireplicate, variance-balanced design.

Proof» For a connected, variance-balanced design

$$C = \theta\left(I_v - \frac{J_v}{v}\right)$$

The eigen values of C are θ (with multiplicity $v-1$) and

0 (" " 1)

Again, since the design is equireplicate, $D_r - r I_v$ and hence the eigen values of D_r are r with multiplicity v.

Define $P = D_r - C = N D_k^{-1} N'$

Thus, the eigen values of P are $r - \theta$, with multiplicity $v-1$ and r, with multiplicity 1.

$\therefore$ rank $(P) = v$

~~rank~~

P is singular when $r = \theta$, in this case rank$(P) = 1$ and hence rank$(N) = 1$

The columns of P and hence ~~rows~~ those of N are spanned by the vector $\mathbb{1}$ which is the eigen vector corresponding to the zero eigen value of C. Thus it follows that in case of $r = \theta$ the rows of N are identical. If we exclude designs with incidence matrices having identical rows, then we find that for any other equireplicate, variance-balanced design, P is non singular

$rank(P) = v$

$v = rank(P) \leq rank(N) \leq b$

So, $v \leq b$ (Proved)

Recovery of Inter-block Information

While discussing the intra-block analysis of block design, it was stated that the block as well as the trt. effects are fixed. If the block effect is regarded as a random variable the analysis is called inter block analysis or recovery of inter block information.

In the context of incomplete block design Yates noticed that since the allocated trt. to the incomplete blocks is made at random, it is reasonable to assume that the block effect themselves are random variables instead of fixed. If the experimental meterial is farely heterogeneous, treating the block effects as fixed quantities results in the loss of information contained in the block designs.

Assume proper and binary design

$$y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad \begin{pmatrix} i = 1(1)v \\ j = 1(1)b \end{pmatrix}$$

Assumptions ① e_{ij} s are i.i.d. with $E(e_{ij}) = 0$
and $V(e_{ij}) = \sigma^2 \ \forall \ i,j$

② β_j are r.v. with

$$E(\beta_j) = 0, \quad V(\beta_j) = \sigma_b^2 \quad \forall j$$

$$Cov(\beta_j, \beta_{j'}) = 0 \quad \forall j \neq j'$$

③ β_j s are uncorrelated with the error terms e_{ij} s.

We may regard the block totals as obs.

$$B_j = \sum_{i=1}^{v} y_{ij} = k\mu + \sum_{i=1}^{v} n_{ij}\tau_i + \left(k\beta_j + \sum_{i=1}^{v} e_{ij}\right), \quad j = 1(1)b$$

New error terms $d_j = k\beta_j + \sum_{i=1}^{v} e_{ij}, \quad j = 1(1)b$

$$E(d_j) = 0 \quad \forall j = 1(1)b$$

$$Var(d_j) = k^2\sigma_b^2 + k\sigma^2 = \sigma_d^2$$

Inter block estimates are obtained by minimizing the SS due to new errors

$$S = \sum_{j=1}^{b} d_j^2 = \sum_{j=1}^{b}\left(B_j - k\mu - \sum_{i=1}^{v} n_{ij}\tau_i\right)^2$$

$$= \left(\underset{\sim}{B} - k\mu \underset{\sim}{1}_b - N'\underset{\sim}{\tau}\right)'\left(\underset{\sim}{B} - k\mu \underset{\sim}{1}_b - N'\underset{\sim}{\tau}\right)$$

Normal equations

$$\frac{\delta S}{\delta \mu} = 0 \Rightarrow k\sum_{j=1}^{b}\left(B_j - k\mu - \sum_{i=1}^{v} n_{ij}\tau_i\right) = 0$$

$$\text{or, } bk\mu + \sum_{j=1}^{b}\sum_{i=1}^{v} n_{ij}\tau_i = \sum_{j=1}^{b} B_j$$

$$\text{or, } bk\mu + \sum_{j=1}^{b}\sum_{i=1}^{v} n_{ij}\tau_i = G$$

$$\text{or, } bk\mu + \underset{\sim}{1}_b' N'\underset{\sim}{\tau} = G \quad \text{——(1)}$$

$$\frac{\delta S}{\delta \tau_i} = 0, \quad i = 1(1)v$$

19.04.24

$$bk\mu + \underset{\sim}{r}'\underset{\sim}{\tau} = G \quad [\underset{\sim}{1}_b' N' = \underset{\sim}{r}']$$

$$\text{or, } bk\mu + \underset{\sim}{1}_v' D_r \underset{\sim}{\tau} = G \quad \text{——— (1)}$$

$$\frac{\delta S}{\delta \tau_i} = 0 \Rightarrow \sum_{j=1}^{b} n_{ij}\left(B_j - k\mu - \sum_{i=1}^{v} n_{ij}\tau_i\right) = 0, \quad i = 1(1)v$$

$$\text{or, } k\mu \sum_{j=1}^{b} n_{ij} + \sum_{j=1}^{b} n_{ij}\left(\sum_{i=1}^{v} n_{ij}\tau_i\right) = \sum_{j=1}^{b} n_{ij}B_j, \quad i = 1(1)v$$

$$k\mu \begin{pmatrix} \sum_{j=1}^{b} n_{1j} \\ \sum_{j=1}^{b} n_{2j} \\ \vdots \\ \sum_{j=1}^{b} n_{vj} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{b} n_{1j}\left(\sum_{i=1}^{v} n_{ij}\tau_i\right) \\ \sum_{j=1}^{b} n_{2j}\left(\sum_{i=1}^{v} n_{ij}\tau_i\right) \\ \vdots \\ \sum_{j=1}^{b} n_{vj}\left(\sum_{i=1}^{v} n_{ij}\tau_i\right) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{b} n_{1j}B_j \\ \sum_{j=1}^{b} n_{2j}B_j \\ \vdots \\ \sum_{j=1}^{b} n_{vj}B_j \end{pmatrix}$$

$$\text{or, } k\mu \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_v \end{pmatrix} + \begin{pmatrix} \underset{\sim}{n}_1'(N'\underset{\sim}{\tau}) \\ \underset{\sim}{n}_2'(N'\underset{\sim}{\tau}) \\ \vdots \\ n_v'(N'\underset{\sim}{\tau}) \end{pmatrix} = \begin{pmatrix} \underset{\sim}{n}_1'\underset{\sim}{B} \\ \underset{\sim}{n}_2'\underset{\sim}{B} \\ \vdots \\ \underset{\sim}{n}_v' B \end{pmatrix}$$

where $\underset{\sim}{n}_i' = \begin{pmatrix} n_{i1} \\ n_{i2} \\ \vdots \\ n_{ib} \end{pmatrix}'$, $i = 1(1)v$

is the ith row of N

$$\text{or, } k\mu\underset{\sim}{r} + NN'\underset{\sim}{\tau} = N\underset{\sim}{B}$$

$$\text{or, } k\mu D_r \underset{\sim}{1}_v + NN'\underset{\sim}{\tau} = N\underset{\sim}{B} \quad \text{——— (2)}$$

In matrix notation

$$\begin{pmatrix} bk & \underset{\sim}{1}_v' D_r \\ kD_r\underset{\sim}{1}_v & NN' \end{pmatrix} \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix} = \begin{pmatrix} G \\ N\underset{\sim}{B} \end{pmatrix}$$

Premultiply both sides by the non singular matrix —

$$\begin{pmatrix} 1 & \underset{\sim}{0}'_v \\ -\frac{1}{b} D_r \underset{\sim}{1}_v & I_v \end{pmatrix}_{(v+1)\times(v+1)}$$

$$\begin{pmatrix} 1 & \underset{\sim}{0}'_v \\ -\frac{1}{b} D_r \underset{\sim}{1}_v & I_v \end{pmatrix} \begin{pmatrix} bK & \underset{\sim}{1}'_v D_r \\ K D_r \underset{\sim}{1}_v & NN' \end{pmatrix} = \begin{pmatrix} 1 & \underset{\sim}{0}'_v \\ -\frac{1}{b} D_r \underset{\sim}{1}_v & I_v \end{pmatrix} \begin{pmatrix} G \\ N\underset{\sim}{B} \end{pmatrix}$$

$$\begin{pmatrix} bK & \underset{\sim}{1}'_v D_r \\ -K D_r \underset{\sim}{1}_v + K D_r \underset{\sim}{1}_v & -\frac{1}{b} D_r \underset{\sim}{1}_v \underset{\sim}{1}'_v D_r + N'N \end{pmatrix} \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix} = \begin{pmatrix} G \\ -\frac{G}{b} D_r \underset{\sim}{1}_v + N\underset{\sim}{B} \end{pmatrix}$$

$$\begin{pmatrix} bK & \underset{\sim}{1}'_v D_r \\ \underset{\sim}{0}_v & NN' - \frac{1}{b} D_r \underset{\sim}{1}_v \underset{\sim}{1}'_v D_r \end{pmatrix} \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix} = \begin{pmatrix} G \\ N\underset{\sim}{B} - \frac{G}{B} D_r \underset{\sim}{1}_v \end{pmatrix}$$

A solu^n of the system of equ^{n}s is obtained by taking the condition $\underset{\sim}{1}'_v D_r \underset{\sim}{\tau} = \underset{\sim}{0}$

i.e $\sum_{i=1}^{v} r_i \tau_i = 0$

and assuming that NN' is non-singular.

we have, $bK\mu + \underset{\sim}{1}'_v D_r \underset{\sim}{\tau} = G$ —— (3)

$NN'\underset{\sim}{\tau} - \frac{1}{b} D_r \underset{\sim}{1}_v \underset{\sim}{1}'_v D_r \underset{\sim}{\tau} = N\underset{\sim}{B} - \frac{G}{b} D_r \underset{\sim}{1}_v$ —— (4)

subject to the above condition

$bK\hat{\mu} = G$

$NN'\hat{\underset{\sim}{\tau}} = N\underset{\sim}{B} - \frac{G}{b} D_r \underset{\sim}{1}_v$

or, $\hat{\mu} = \frac{G}{bk}$

$\hat{\tilde{\tau}} = (NN')^{-1}\left(N\tilde{B} - \frac{G}{b} Dr \tilde{1}_v\right)$

$= (NN')^{-1}\left(N\tilde{B} - \frac{G}{b}\tilde{r}\right)$

$= (NN')^{-1}\left(N\tilde{B} - \frac{G}{b} N\tilde{1}_b\right)$

$= (NN')^{-1}\left(N\tilde{B} - \frac{G}{bk} Nk\tilde{1}_b\right)$ [$\because$ Design is proper $\cdot\ k\tilde{1}_b = \tilde{k}$]

$= (NN')^{-1}\left(N\tilde{B} - \frac{G}{bk} NN'\tilde{1}_v\right)$

$= (NN')^{-1}N\tilde{B} - \frac{G}{bk}\tilde{1}_v$

We have two estimators of $\tilde{\tau}$

(1) Intra-Block estimator $\hat{\tilde{\tau}} = C^{-}Q$

(2) Inter-Block estimator $\hat{\hat{\tilde{\tau}}} = (NN')^{-1}N\tilde{B} - \frac{G}{bk}\tilde{1}_v$

Let $\psi = p'\tilde{\tau}$ be a contrast of trt. effects.

Intra-block estimate of ψ is $\psi_1 = \tilde{p}'C^{-}\tilde{Q}$

with variance $V(\psi_1) = \tilde{p}'C^{-}\,Disp(\tilde{Q})C^{-}\tilde{p}$

$= \tilde{p}'C^{-}(C\sigma^2)C^{-}\tilde{p}$

$= \sigma^2(\tilde{p}'C^{-}\tilde{p})$

Inter-block estimate of ψ is $\psi_2 = \tilde{p}'\left[(NN')^{-1}N\tilde{B} - \frac{G}{bk}\tilde{1}_v\right]$ with variance

$= \tilde{p}'(NN')^{-1}N\tilde{B}$ [$\because \tilde{p}'\tilde{1}_v = 0$]

$var(\psi_2) = \tilde{p}'\,Disp\left((NN')^{-1}N\tilde{B}\right)\tilde{p}$

$= \tilde{p}'(NN')^{-1}N\,Disp(\tilde{B})\,N'(NN')^{-1}\tilde{p}$

$= \tilde{p}'(NN')^{-1}N(\sigma_d^2 I_b)N'(NN')^{-1}\tilde{p}$ [$\because Disp(\tilde{B}) = \sigma_d^2 I_b$]

$= \sigma_d^2\,\tilde{p}'(NN')^{-1}\tilde{p}$

□ ψ_1 and ψ_2 are uncorrelated

$$\psi_1 = \underset{\sim}{p}' C^- \underset{\sim}{Q}$$

$$\psi_2 = \underset{\sim}{p}'(NN')^{-1} N \underset{\sim}{B}$$

We'll show that $\underset{\sim}{Q}$ & $\underset{\sim}{B}$ are uncorrelated

$$\underset{\sim}{Q} = \underset{\sim}{T} - N D_k^{-1} \underset{\sim}{B}$$

$$\text{cov}(\underset{\sim}{Q}, \underset{\sim}{B}) = \text{cov}(\underset{\sim}{T}, B) - N D_k^{-1} \text{Disp}(\underset{\sim}{B})$$

$$\text{cov}(T, B) = \text{cov}(X_\tau' \underset{\sim}{Y}, X_\beta' \underset{\sim}{Y})$$

$$= X_\tau' \text{Disp}(Y) X_\beta$$

$$y_{ij} = \mu + \tau_i + \beta_j = e_{ij}$$

$$v(y_{ij}) = \sigma_b^2 + \sigma_e^2 \ \forall\ i,j$$

$$\text{cov}(y_{ij}, y_{ij'}) = 0$$

$$\text{cov}(y_{ij}, y_{i'j}) = \text{cov}(\beta_j, \beta_j) \ ; \ i \neq i'$$

$$= \sigma_b^2$$

$$T_p = \sum_{j=1}^{b} y_{ij}$$

$$= \sum_{j=1}^{b} (\mu + \tau_i + \beta_j + e_{ij})$$

$$B_j = k\mu + \sum_{i=1}^{v} n_{ij} \tau_i + k\beta_j + \sum_{i=1}^{v} e_{ij}$$

$$\text{cov}(T_i, \beta_j) = \text{cov}\Big(\sum_{j=1}^{b} \beta_j + \sum_{j=1}^{b} e_{ij},\ k\beta_j + \sum_{i=1}^{v} e_{ij}\Big)$$

$$= n_{ij} k \,\text{cov}(\beta_j, \beta_j) + n_{ij} \text{cov}(e_{ij}, e_{ij})$$

$$= n_{ij}[k \text{ var}(\beta_j) + \text{cov}(e_{ij}, e_{ij})]$$

$$= [k\sigma_b^2 + \sigma_e^2] n_{ij}$$

$$= \frac{n_{ij}\sigma_d^2}{k}$$

$$\Big[\because \sigma_d^2 = k^2\sigma_b^2 + k\sigma_e^2\Big]$$

$$\text{cov}(\underset{\sim}{I}, \underset{\sim}{B}) = \frac{N\sigma_d^2}{k}$$

Since the design is proper,

$D_k = kI_b$

$\therefore D_k^{-1} = \frac{1}{k} I_b$

Again, $\text{Disp}(\underset{\sim}{B}) = \sigma_d^2 I$

So, $N D_k^{-1} \text{Disp}(\underset{\sim}{B})$

$$= \frac{N}{k}\sigma_d^2$$

$\therefore \text{cov}(\underset{\sim}{Q}, \underset{\sim}{B}) = \underset{\sim}{0}_{v \times b}$

Hence, ψ_1 and ψ_2 are uncorrelated. If we want to combine these two estimators to obtain a estimator with the smallest variance then the combined estimator is obtained by taking a ~~waited~~ weighted avg. of ψ_1 and ψ_2, waigths being the inverse of the variances of these two estimators. Thus the combined estimator is given by

$$\psi^* = \frac{\theta_1\psi_1 + \theta_2\psi_2}{\theta_1 + \theta_2}$$

where, $\theta_1 = (\sigma^2 \underset{\sim}{p}' C^- \underset{\sim}{p})^{-1}$

$\theta_2 = (\sigma_d^2 \underset{\sim}{p}'(NN')^{-1} \underset{\sim}{p})^{-1}$

22.04.24

Mixed effects Model

Let $y_1, y_2 \cdots y_n$ be n obs. such that

$$y_i = a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{ip}\beta_p + b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{iq}x_q + \epsilon_i$$

$i = 1(1)n$

where, $\beta_1, \beta_2 \cdots \beta_p$ are unknown parameters (fixed).

$x_1, x_2 \cdots x_q$ are random variables.

$a_{i1}, a_{i2} \cdots, a_{ip}, b_{i1}, b_{i2} \cdots b_{iq}$ are known constants.

Further let, $E(\epsilon_i) = 0$, $var(\epsilon_i) = \sigma^2 \ \forall i = 1(1)n$

$cov(\epsilon_i, \epsilon_{i'}) = 0 \ \forall i \neq i'$

$E(x_j) = 0$, $Var(x_j) = \sigma_b^2 \ \forall j = 1(1)q$

$cov(x_j, x_{j'}) = 0 \ \forall j \neq j'$

and $cov(\epsilon_i, x_j) = 0$ $\forall i = 1(1)n$ $\forall j = 1(1)q$

A model of this type is called linear mixed effects model.

In matrix notation,

$$\underset{n\times 1}{\underset{\sim}{Y}} = \underset{n\times p}{A}\underset{p\times 1}{\underset{\sim}{\beta}} + \underset{n\times q}{B}\underset{q\times 1}{\underset{\sim}{x}} + \underset{n\times 1}{\underset{\sim}{\epsilon}}$$

where, $\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$; $\underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}$; $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix}$; $\underset{\sim}{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$; $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix}$

$E(\underset{\sim}{\epsilon}) = \underset{\sim}{0}$

$Disp(\underset{\sim}{\epsilon}) = \sigma^2 I_n$

$E(\underset{\sim}{x}) = \underset{\sim}{0}$

$Disp(\underset{\sim}{x}) = \sigma_b^2 I_q$

$cov(\underset{\sim}{x}, \underset{\sim}{\epsilon}) = 0$

$E(\underset{\sim}{Y}) = A\underset{\sim}{\beta}$

$$\begin{aligned} Disp(\underset{\sim}{Y}) &= B\, Disp(\underset{\sim}{x})\, B' + Disp(\underset{\sim}{\epsilon}) \\ &= \sigma_b^2 BB' + \sigma^2 I_n \\ &= \Sigma \text{ (say)} \end{aligned}$$

Normal eq^n $A'\Sigma^{-1}A\underset{\sim}{\beta} = A'\Sigma^{-1}\underset{\sim}{Y}$

Combined inter and intra block analysis

Assumption :- Design is proper.

$$\underset{\sim}{K} = K\underset{\sim}{1}_b$$

$$\& \; D_K = K I_b$$

Model :- $\underset{n\times 1}{\underset{\sim}{Y}} = \mu \underset{n\times 1}{\underset{\sim}{1}} + \underset{n\times v}{X_\tau} \cdot \underset{v\times 1}{\underset{\sim}{\tau}} + \underset{n\times b}{X_\beta} \underset{b\times 1}{\underset{\sim}{\beta}} + \underset{n\times 1}{\underset{\sim}{\epsilon}}$

Treatment effects $\underset{\sim}{\tau}$ are fixed.

Block effects $\underset{\sim}{\beta}$ are randam.

Assumption $E(\underset{\sim}{\epsilon}) = \underset{\sim}{0}$, $Disp(\underset{\sim}{\epsilon}) = \sigma^2 I_n$

$E(\underset{\sim}{\beta}) = \underset{\sim}{0}$ $Disp(\underset{\sim}{\beta}) = \sigma_\beta^2 I_b$

$cov(\underset{\sim}{\beta}, \underset{\sim}{\epsilon}) = 0$

In standard notation,

$$\underset{\sim}{Y} = (\underset{n\times 1}{\underset{\sim}{1}} \;\vdots\; \underset{n\times v}{X_\tau}) \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix} + X_\beta \underset{\sim}{\beta} + \underset{\sim}{\epsilon}$$

$$= A\underset{\sim}{\theta} + B\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$$

where $\underset{n\times(v+1)}{A} = (\underset{\sim}{1} \;\vdots\; X_\tau)$; $B = X_\beta$; $\underset{\sim}{\theta} = \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix}$

The design is proper,

total no of obs. $n = b\cdot K$

Assume that the $n = bk$ obs. are such that the 1st set of k obs. are from block 1, the 2nd set of k obs are from block 2 and finally the last set of k obs are from block b.

Here $X_\beta = ((x_{ij}^\beta))$

where $x_{ij}^\beta = \begin{cases} 1 & \text{if the } i\text{th obs. comes from block } j \\ 0 & \text{o.w.} \end{cases}$

$$X_\beta = \begin{array}{c} \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 1 & 0 & & 0 \\ 1 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & & 0 \\ \hline 0 & 1 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & & 0 \\ \hline \\ \hline 0 & 0 & \cdots & 1 \\ 0 & 0 & & 1 \\ 0 & 0 & & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{array} \right) \end{array} \begin{array}{c} \updownarrow k \\ \\ \updownarrow k \\ \\ \updownarrow k \end{array}$$

$$\text{or, } X_\beta = \begin{pmatrix} \underset{\sim}{1}_k & \underset{\sim}{0}_k & \cdots & \underset{\sim}{0}_k \\ \underset{\sim}{0}_k & \underset{\sim}{1}_k & \cdots & \underset{\sim}{0}_k \\ \vdots & & & \\ \underset{\sim}{0}_k & \underset{\sim}{0}_k & \cdots & \underset{\sim}{1}_k \end{pmatrix}$$

Now, $\Sigma = X_\beta X_\beta' \cdot \sigma_b^2 + I_n \sigma^2 \quad (\mathrm{Disp}(\underset{\sim}{Y}))$

$$X_\beta X_\beta' = \begin{pmatrix} \underset{\sim}{1}_k & \underset{\sim}{0}_k & \cdots & \underset{\sim}{0}_k \\ \underset{\sim}{0}_k & \underset{\sim}{1}_k & \cdots & \underset{\sim}{0}_k \\ \vdots & & & \\ \underset{\sim}{0}_k & \underset{\sim}{0}_k & \cdots & \underset{\sim}{1}_k \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}_k' & \underset{\sim}{0}_k' & \cdots & \underset{\sim}{0}_k' \\ \underset{\sim}{0}_k' & \underset{\sim}{1}_k' & \cdots & \underset{\sim}{0}_k' \\ \vdots & \vdots & & \\ \underset{\sim}{0}_k' & \underset{\sim}{0}_k' & \cdots & \underset{\sim}{1}_k' \end{pmatrix}$$

$$= \begin{pmatrix} J_k & \underset{\sim}{0}_{k\times k} & \underset{\sim}{0}_{k\times k} & \cdots & 0 \\ 0 & J_k & 0 & & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & J_k \end{pmatrix}$$

$J_k = \underset{\sim}{1}_k \underset{\sim}{1}_k'$

$= (k \times k)$ matrix with all entries 1

$$= \text{Diag}(J_k \; J_k \cdots J_k)$$

$$\therefore \Sigma = \sigma^2 I_n + \sigma_b^2 \, \text{Diag}(\underbrace{J_k, J_k \cdots J_k}_{b \text{ times}})$$

$$= \sigma^2 \, \text{Diag}(\underbrace{I_k, I_k, \cdots I_k}_{b\text{ times}}) + \sigma_b^2 \, \text{Diag}(\underbrace{J_k, J_k \cdots J_k}_{b \text{ times}})$$

$$= \text{Diag}(\underbrace{\sigma^2 I_k + \sigma_b^2 J_k, \sigma^2 I_k + \sigma_b^2 J_k, \cdots, \sigma^2 I_k + \sigma_b^2 J_k}_{b \text{ times}})$$

$$= \text{Diag}(L, L, \cdots L), \text{ where } L = \sigma^2 I_k + \sigma_b^2 J_k$$

$$\therefore \Sigma^{-1} = \text{Diag}(L^{-1}, L^{-1} \cdots L^{-1}) \; ; \; L^{-1} = (\sigma^2 I_k + \sigma_b^2 J_k)^{-1}$$

$$= \frac{1}{\sigma^2}\left(I_k + \frac{\sigma_b^2}{\sigma^2} \underset{\sim}{1}_k \underset{\sim}{1}_k'\right)$$

$$(A + \underset{\sim}{u}\underset{\sim}{v}')^{-1} = A^{-1} - \frac{(A^{-1}\underset{\sim}{u})(\underset{\sim}{v}'A^{-1})}{1 + \underset{\sim}{v}'A^{-1}\underset{\sim}{u}}$$

The matrix $I_k + \frac{\sigma_b^2}{\sigma^2} \underset{\sim}{1}_k \underset{\sim}{1}_k'$ is of the form $A + \underset{\sim}{u}\underset{\sim}{v}'$ with

$A = I_k$

$\underset{\sim}{u} = \frac{\sigma_b^2}{\sigma^2} J_k$

$\underset{\sim}{v} = \underset{\sim}{1}_k$

$$\therefore \left(I_k + \frac{\sigma_b^2}{\sigma^2} \underset{\sim}{1}_k \underset{\sim}{1}_k'\right)^{-1} = I_k - \frac{\left(\frac{\sigma_b^2}{\sigma^2} \underset{\sim}{1}_k\right)\left(\underset{\sim}{1}_k'\right)}{1 + \underset{\sim}{1}_k' \underset{\sim}{1}_k \left(\frac{\sigma_b^2}{\sigma^2}\right)}$$

$$= I_k - \left(\frac{\sigma_b^2}{\sigma^2 + k\sigma_b^2}\right) J_k$$

$$\therefore L^{-1} = \frac{1}{\sigma^2} I_k - \frac{\sigma_b^2}{\sigma^2(\sigma^2 + k\sigma_b^2)} J_k$$

Define $w_1 = \frac{1}{\sigma^2}$ (reciprocal of intra-block variance)

$w_2 = \frac{1}{\sigma^2 + k\sigma_b^2}$ (reciprocal of inter-block variance)

$$\therefore w_1 - w_2 = \frac{k\sigma_b^2}{\sigma^2(\sigma^2 + k\sigma_b^2)}$$

$$\therefore L^{-1} = w_1 I_k - \left(\frac{w_1 - w_2}{k}\right) J_k$$

$$\therefore \Sigma^{-1} = \text{Diag}\left(L^{-1}, L^{-1} \cdots L^{-1}\right)$$

$$= \text{Diag}\Big(\underbrace{w_1 I_k - \left(\frac{w_1 - w_2}{k}\right) J_k, \cdots, w_1 I_k - \left(\frac{w_1 - w_2}{k}\right) J_k}_{b \text{ times.}}\Big)$$

$$= w_1 I_n - \left(\frac{w_1 - w_2}{k}\right) \text{Diag}\left(J_k, J_k \cdots J_k\right)$$

$$= w_1 I_n - \left(\frac{w_1 - w_2}{k}\right) x_\beta x_\beta'$$

Normal equations.

$$A' \Sigma^{-1} A \underline{\theta} = A' \Sigma^{-1} \underset{\sim}{Y}$$

$$A = [1 : x_\tau] \; ; \; \underset{\sim}{\theta} = \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix}$$

$$A^{-1} \Sigma^{-1} A = \begin{bmatrix} \underset{\sim}{1}' \\ x_\tau' \end{bmatrix} \Sigma^{-1} [\underset{\sim}{1}_n : x_\tau] = \begin{pmatrix} \underset{\sim}{1}_n' \Sigma^{-1} \underset{\sim}{1}_n & \underset{\sim}{1}_n' \Sigma^{-1} x_\tau \\ x_\tau' \Sigma^{-1} \underset{\sim}{1}_n & x_\tau' \Sigma^{-1} x_\tau \end{pmatrix}$$

$$\therefore \begin{pmatrix} \underset{\sim}{1}_n' \Sigma^{-1} \underset{\sim}{1}_n & \underset{\sim}{1}_n' \Sigma^{-1} x_\tau \\ x_\tau' \Sigma^{-1} \underset{\sim}{1}_n & x_\tau' \Sigma^{-1} x_\tau \end{pmatrix} \underset{\sim}{\theta} = \begin{pmatrix} \underset{\sim}{1}' \Sigma^{-1} \underset{\sim}{Y} \\ x_\tau' \Sigma^{-1} \underset{\sim}{Y} \end{pmatrix}$$

We have

$$\underset{\sim n}{1'}\Sigma^{-1}\underset{\sim n}{1} = \underset{\sim}{1'}\left[w_1 I_n - \frac{w_1-w_2}{k} x_\beta x_\beta'\right]\underset{\sim n}{1}$$

$$= \cancel{w_1 \underset{\sim n}{1'} - \frac{(w_1-w_2)}{k}(\underset{\sim n}{1}\underset{\sim n}{1'})}$$

$$= w_1 \underset{\sim n}{1'}\underset{\sim n}{1} - \frac{(w_1-w_2)}{k}(\underset{\sim n}{1'}x_\beta)(x_\beta'\underset{\sim n}{1})$$

$$= w_1 n - \frac{(w_1-w_2)}{k}(\underset{\sim}{k'})(\underset{\sim}{k})$$

$$= w_1 n - \frac{(w_1-w_2)}{k}(k\underset{\sim b}{1'})(k\underset{\sim b}{1}) \quad [\because \text{proper design}]$$

$$= w_1 n - k(w_1-w_2)(\underset{\sim b}{1'}\underset{\sim b}{1})$$

$$= w_1 n - bk(w_1-w_2)$$

$$= w_1 n - n(w_1-w_2)$$

$$= nw_2$$

$$\underset{\sim n}{1'}\Sigma^{-1}X_\tau = w_2\underset{\sim}{r'}$$

$$X_\tau'\Sigma^{-1}X_\tau = w_1 D_r - \frac{(w_1-w_2)}{k} NN'$$

$$\underset{\sim n}{1'}\Sigma^{-1}X_\tau = \underset{\sim n}{1'}\left[w_1 I_n - \frac{w_1-w_2}{k} x_\beta x_\beta'\right]X_\tau$$

$$= w_1\underset{\sim n}{1'}X_\tau - \frac{w_1-w_2}{k}(\underset{\sim}{1'}x_\beta)(x_\beta' x_\tau)$$

$$= w_1\underset{\sim}{r'} - \frac{w_1-w_2}{k}(\underset{\sim}{k'})(N')$$

$$= w_1\underset{\sim}{r'} - \frac{w_1-w_2}{k}(k\underset{\sim b}{1'})(N\underset{\sim b}{1})'$$

$$= w_1\underset{\sim}{r'} - (w_1-w_2)(N\underset{\sim}{1})$$

$$= w_1\underset{\sim}{r'} - \frac{(w_1-w_2)}{k}(k\,\underset{\sim}{r'}) \qquad k$$

$$= w_1\underset{\sim}{r'} - w_1\underset{\sim}{r'} + w_2\underset{\sim}{r'}$$

$$= w_2\underset{\sim}{r'}$$

$$X_\tau'\Sigma^{-1}X_\tau = X_\tau'\left[w_1 I_n - \frac{w_1-w_2}{k} x_\beta x_\beta'\right]X_\tau$$

$$= w_1 X_\tau' X_\tau - \frac{w_1-w_2}{k}(X_\tau' x_\beta)(x_\beta' x_\tau)$$

$$= w_1 D_r - \frac{w_1-w_2}{k} NN'$$

$$\underset{\sim}{1}_n' \Sigma^{-1} \underset{\sim}{Y} = \underset{\sim}{1}_n' \left[w_1 I_n - \frac{(w_1 - w_2)}{k} x_\beta x_\beta' \right] \underset{\sim}{Y}$$

$$= w_1 \underset{\sim}{1}_n' \underset{\sim}{Y} - \frac{(w_1 - w_2)}{k} (\underset{\sim}{1}_n' x_\beta)(x_\beta' \underset{\sim}{Y})$$

$$= w_1 \underset{\sim}{1}_n' \underset{\sim}{Y} - \frac{(w_1 - w_2)}{k} (k \underset{\sim}{1}_b')(\underset{\sim}{B})$$

$$= w_1 G - (w_1 - w_2) G \quad \left[\because \underset{\sim}{1}_n' \underset{\sim}{Y} = \underset{\sim}{1}_b' \underset{\sim}{B} = G \right]$$

$$x_\tau' \Sigma^{-1} \underset{\sim}{Y} = x_\tau' \left[w_1 I_n - \frac{w_1 - w_2}{k} x_\beta x_\beta' \right] \underset{\sim}{Y}$$

$$= w_1 x_\tau' \underset{\sim}{Y} - \frac{w_1 - w_2}{k} (x_\tau' x_\beta)(x_\beta' \underset{\sim}{Y})$$

$$= w_1 \underset{\sim}{T} - \frac{w_1 - w_2}{k} N \underset{\sim}{B}$$

$$= w_1 T - w_1 \frac{N\underset{\sim}{B}}{k} + w_2 \frac{N\underset{\sim}{B}}{k}$$

$$= w_1 \left(T - \frac{N\underset{\sim}{B}}{k} \right) + w_2 \left(\frac{N\underset{\sim}{B}}{k} \right)$$

$$= w_1 Q + w_2 (T - Q)$$

$$Q = T - N D_k^{-1} \underset{\sim}{B}$$

$$= T - \frac{NB}{k}$$

for proper design

$$\therefore \begin{pmatrix} w_2 n & w_2 \underset{\sim}{r}' \\ w_2 \underset{\sim}{r} & w_1 D_r - \frac{(w_1 - w_2)}{k} N N' \end{pmatrix} \begin{pmatrix} \mu \\ \tau \end{pmatrix} = \begin{pmatrix} w_2 G \\ w_1 \underset{\sim}{Q} + w_2 (T - \underset{\sim}{Q}) \end{pmatrix}$$

$$n w_2 \hat{\mu} + w_2 \underset{\sim}{r}' \hat{\underset{\sim}{\tau}} = w_2 G \quad \text{--- (1)}$$

$$w_2 \underset{\sim}{r}' \hat{\mu} + w_1 D_r \hat{\underset{\sim}{\tau}} - \frac{(w_1 - w_2)}{k} N N' \hat{\underset{\sim}{\tau}} = w_1 \underset{\sim}{Q} + w_2 (\underset{\sim}{T} - \underset{\sim}{Q}) \quad \text{--- (2)}$$

Premultiply eqⁿ (1) by $\underset{\sim}{r}$

$$\cancel{n w_2}\; w_2 \underset{\sim}{r} \hat{\mu} + w_2 \frac{\underset{\sim}{r} \underset{\sim}{r}'}{n} \hat{\underset{\sim}{\tau}} = \frac{w_2}{n} \underset{\sim}{r} G \quad \text{--- (3)}$$

$\cancel{w_2 \underset{\sim}{r}}$ (2)-(3) given

$$\left(w_1 D_r - \frac{(w_1 - w_2)}{k} N N' - w_2 \frac{\underset{\sim}{r} \underset{\sim}{r}'}{n} \right) \hat{\underset{\sim}{\tau}} = w_1 \underset{\sim}{Q} + w_2 (\underset{\sim}{T} - \underset{\sim}{Q}) - \frac{w_2}{n} \underset{\sim}{r} G$$

$$\text{or, } \left[w_1 \left(D_r - \frac{NN'}{k} \right) + w_2 \left(\frac{NN'}{k} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{n} \right) \right] \hat{\underset{\sim}{\tau}} = w_1 \underset{\sim}{Q} + w_2 \left(T - Q - \frac{\underset{\sim}{r} G}{n} \right)$$

$$\text{or, } \left(w_1 C + w_2 C^* \right) \hat{\underset{\sim}{\tau}} = w_1 \underset{\sim}{Q} + w_2 \underset{\sim}{Q}^*$$

where

$$C^* = \frac{NN'}{k} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{n}$$

$$Q^* = \underset{\sim}{T} - Q - \frac{\underset{\sim}{r} G}{n}$$

Eq^n (4) is called adjusted inter and intra block normal eq^n.

23. 04. 2024

Result 1 $E(\underset{\sim}{Q}) = C\tau$

Proof>> $\underset{\sim}{Q} = \underset{\sim}{T} - \frac{N\underset{\sim}{B}}{k}$

$$\text{or, } \underset{\sim}{Q} = X_\tau' \underset{\sim}{Y} - \frac{N X_\beta' \underset{\sim}{Y}}{k} = \left(X_\tau' - \frac{N X_\beta'}{k} \right) \underset{\sim}{Y}$$

$$E(\underset{\sim}{Q}) = \left(X_\tau' - \frac{N X_\beta'}{k} \right) E(\underset{\sim}{Y})$$

$$= \left(X_\tau' - \frac{N X_\beta'}{k} \right) \left(\mu \underset{\sim}{1}_n + X_\tau \cdot \underset{\sim}{\tau} \right)$$

$$= \mu X_\tau' \underset{\sim}{1}_n - \frac{\mu}{k} N X_\beta' \underset{\sim}{1}_n + X_\tau' X_\tau \cdot \underset{\sim}{\tau} - \frac{N X_\beta' X_\tau \underset{\sim}{\tau}}{k}$$

$$= \mu \underset{\sim}{r} - \frac{\mu}{k} N \underset{\sim}{k} + D_r \underset{\sim}{\tau} - \frac{NN' \underset{\sim}{\tau}}{k}$$

$$= \mu \underset{\sim}{r} + \frac{\mu}{k} N (k \underset{\sim}{1}_b) + \left(D_r - \frac{NN'}{k} \right) \underset{\sim}{\tau}$$

$$= \mu \underset{\sim}{r} - \mu (N \underset{\sim}{1}_b) + C\tau \quad \left[\because \underset{\sim}{k} = k \underset{\sim}{1}_b,\ C = D_r - \frac{NN'}{k} \right]$$

$$= \mu \underset{\sim}{r} - \mu \underset{\sim}{r} + C \underset{\sim}{\tau}$$

$$= C \underset{\sim}{\tau}$$

Result 2 $E(\underset{\sim}{Q}^*) = C^* \underset{\sim}{\tau}$

Proof>> $\underset{\sim}{Q}^* = \underset{\sim}{T} - \underset{\sim}{Q} - \frac{\underset{\sim}{r} G}{n} = \frac{N\underset{\sim}{B}}{k} - \frac{\underset{\sim}{r} G}{n}$

$$= \frac{N X_\beta' \underset{\sim}{Y}}{k} - \frac{\underset{\sim}{r} \underset{\sim}{1}_n' \underset{\sim}{Y}}{n}$$

$$= \left(\frac{NX\beta'}{K} - \frac{\underset{\sim}{r}\underset{\sim}{1}_n'}{n}\right)\underset{\sim}{Y}$$

$$E(\underset{\sim}{Q}^*) = \left(\frac{NX\beta'}{K} - \frac{\underset{\sim}{r}\underset{\sim}{1}_n'}{n}\right) E(\underset{\sim}{Y})$$

$$= \left(\frac{NX\beta'}{K} - \frac{\underset{\sim}{r}\underset{\sim}{1}_n'}{n}\right)\left(\mu \underset{\sim}{1}_n + X_\tau \underset{\sim}{\tau}\right)$$

$$= \mu \frac{NX\beta'}{K}\underset{\sim}{1}_n - \frac{\mu \underset{\sim}{1}_n \underset{\sim}{1}_n' \underset{\sim}{r}}{n} + \frac{NX\beta'}{K} X_\tau \underset{\sim}{\tau} - \frac{\underset{\sim}{r}\underset{\sim}{1}_n'}{n} X_\tau \underset{\sim}{\tau}$$

$$= \frac{\mu N k \cdot 1}{k} - \frac{\mu\, n \underset{\sim}{r}}{n} + \frac{NN'\underset{\sim}{\tau}}{k} - \frac{\underset{\sim}{r}\underset{\sim}{r}'\underset{\sim}{\tau}}{n}$$

$$= \mu \underset{\sim}{r} - \mu \underset{\sim}{r} + \left(\frac{NN'}{k} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{n}\right)\underset{\sim}{\tau}$$

$$= C^* \underset{\sim}{\tau}$$

Result 3

$$\text{Disp}\begin{pmatrix} \underset{\sim}{Q} \\ \underset{\sim}{Q}^* \\ G \end{pmatrix} = \begin{pmatrix} C\sigma^2_{v\times v} & 0_{v\times v} & \underset{\sim}{0}_{v\times 1} \\ 0_{v\times v} & C^*/W_2 & \underset{\sim}{0}_{v\times 1} \\ \underset{\sim}{0}'_{1\times v} & \underset{\sim}{0}'_{1\times v} & \left(\frac{n}{W_2}\right)_{1\times 1} \end{pmatrix}$$

Proof » To be shown $\text{Disp}(\underset{\sim}{Q}) = C\sigma^2$

$$\text{Disp}(\underset{\sim}{Q}^*) = C^*/W_2$$

$$\text{Var}(G) = \frac{n}{W_2}$$

$$\text{Cov}(\underset{\sim}{Q}, \underset{\sim}{Q}^*) = 0_{v\times v}$$

$$\text{Cov}(\underset{\sim}{Q}, G) = \underset{\sim}{0}_{v\times 1}$$

$$\text{Cov}(\underset{\sim}{Q}^*, G) = \underset{\sim}{0}_{v\times 1}$$

$$\underset{\sim}{Q} = \left(X_\tau' - \frac{N X_\beta'}{k} \right) \underset{\sim}{Y}$$

$$\text{Disp}(\underset{\sim}{Q}) = \left(X_\tau' - \frac{N X_\beta'}{k} \right) \text{Disp}(\underset{\sim}{Y}) \left(X_\tau - \frac{X_\beta N'}{k} \right)$$

$$= \left(X_\tau' - \frac{N X_\beta'}{k} \right) \Sigma \left(X_\tau - \frac{X_\beta N'}{k} \right)$$

$$= \cancel{\left(X_\tau' \Sigma - \frac{N X_\beta' \Sigma X_\tau}{k} \right.}$$

$$= X_\tau' \Sigma X_\tau - \frac{N X_\beta' \Sigma X_\tau}{k} - \frac{X_\tau' \Sigma X_\beta N'}{k} + \frac{N X_\beta' \Sigma X_\beta N'}{k^2}$$

$$X_\tau' \Sigma X_\tau = X_\tau' \left[\sigma^2 I_n + \sigma_b^2 X_\beta \cdot X_\beta' \right] X_\tau$$

$$= X_\tau' X_\tau + \sigma_b^2 (X_\tau' X_\beta)(X_\beta' X_\tau)$$

$$= \sigma^2 D_r + \sigma_b^2 N N'$$

$$X_\tau' \Sigma X_\beta = X_\tau' \left[\sigma^2 I_n + \sigma_b^2 X_\beta \cdot X_\beta' \right] X_\beta$$

$$= \sigma^2 X_\tau' X_\beta + \sigma_b^2 (X_\tau' X_\beta)(X_\beta' X_\beta)$$

$$= \sigma^2 N + k \sigma_b^2 N \quad [\because X_\beta' X_\beta = D_k = k I_b]$$

$$= N(\sigma^2 + k\sigma_b^2) = \frac{N}{w_2} \qquad \left[\because w_2 = \frac{1}{\sigma^2 + k\sigma_b^2}\right]$$

$$X_\beta' \Sigma X_\beta = X_\beta' (\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta') X_\beta$$

$$= \sigma^2 X_\beta' X_\beta + \sigma_b^2 (X_\beta' X_\beta)(X_\beta' X_\beta)$$

$$= (k\sigma^2 + k^2 \sigma_b^2) I_b$$

$$= k(\sigma^2 + k\sigma_b^2) I_b$$

$$= \frac{k I_b}{w_2}$$

$$= \sigma^2 D_r + \sigma_b^2 N N' - \frac{(\sigma^2 + k\sigma_b^2)}{k} N N' - \frac{(\sigma^2 + k\sigma_b^2)}{k} N N' + k\frac{(\sigma^2 + k\sigma_b^2)}{k^2} N N'$$

$\underset{\sim}{Y}'$

$$= \sigma^2 D_r + \sigma_b^2 NN' - \frac{(\sigma^2 + k\sigma_b^2)}{k} NN'$$

$$= \sigma^2 \left(D_r - \frac{NN'}{k} \right)$$

$$= C\sigma^2$$

$$\beta^* = \left(\frac{N X_\beta'}{k} - \frac{\underset{\sim}{r}\, \underset{\sim}{1}_n'}{n} \right) \underset{\sim}{Y}$$

$$\text{Disp}(\underset{\sim}{\beta}^*) = \left(\frac{N X_\beta'}{k} - \frac{\underset{\sim}{r}\, \underset{\sim}{1}_n'}{n} \right) \text{Disp}(\underset{\sim}{Y}) \left(\frac{X_\beta N'}{k} - \frac{\underset{\sim}{1}_n \underset{\sim}{r}'}{n} \right)$$

$$= \frac{N X_\beta' \Sigma X_\beta N'}{k^2} - \frac{\underset{\sim}{r}\, \underset{\sim}{1}_n' \Sigma X_\beta N'}{nk} - \frac{N X_\beta' \Sigma \underset{\sim}{1}_n \underset{\sim}{r}'}{nk} + \frac{\underset{\sim}{r}\, \underset{\sim}{1}_n' \Sigma \underset{\sim}{1}_n \underset{\sim}{r}'}{n^2}$$

$$\underset{\sim}{1}_n' \Sigma \underset{\sim}{1}_n = \sigma^2 \underset{\sim}{1}_n' \underset{\sim}{1}_n + \sigma_b^2 (\underset{\sim}{1}_n' X_\beta)(X_\beta' \underset{\sim}{1}_n)$$

$$= \sigma^2 n + \sigma_b^2 (\underline{k}')(\underline{k})$$

$$= \sigma^2 n + k^2 \sigma_b^2 (\underset{\sim}{1}_b' \underset{\sim}{1}_b)$$

$$= \sigma^2 n + b k^2 \sigma_b^2$$

$$= n(\sigma^2 + k\sigma_b^2) \qquad [bk = n]$$

$$= \frac{n}{w_2}$$

$$X_\beta' \Sigma \underset{\sim}{1}_n = \sigma^2 X_\beta' \underset{\sim}{1}_n + \sigma_b^2 (X_\beta' X_\beta)(X_\beta' \underset{\sim}{1}_n)$$

$$= \sigma^2 (k \underset{\sim}{1}_b) + \sigma_b^2 (k I_b)(k \underset{\sim}{1}_b)$$

$$= k(\sigma^2 + k\sigma_b^2) \underset{\sim}{1}_b$$

$$= \frac{k}{w_2} \underset{\sim}{1}_b$$

$$= \frac{k\, NN'}{k^2 w_2} - \frac{k}{w_2} \cdot \frac{r\, \underset{\sim}{1}_b' N'}{nk} - \frac{k}{w_2} \frac{N \underset{\sim}{1}_b \underset{\sim}{r}'}{nk} + \frac{n}{w_2} \frac{\underset{\sim}{r} \underset{\sim}{r}'}{n^2}$$

$$= \frac{NN'}{kW_2} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{nW_2} + \cancel{\frac{\underset{\sim}{r}\underset{\sim}{r}'}{nW_2}} + \cancel{\frac{\underset{\sim}{r}\underset{\sim}{r}'}{W_2 n}}$$

$\Big| N\underset{\sim}{1}_b = \underset{\sim}{r}$

$$= \frac{1}{W_2}\left(\frac{NN'}{k} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{n}\right)$$

$$= \frac{C^*}{W_2}$$

$$\text{var}(G) = \text{var}(\underset{\sim}{1}_n'\underset{\sim}{Y})$$

$$= \underset{\sim}{1}_n'\Sigma\underset{\sim}{1}_n$$

$$= \frac{n}{W_2} \quad [\text{already found}]$$

$$\text{cov}(\underset{\sim}{Q}, \underset{\sim}{Q}^*) = \text{cov}\left(\left(X_\tau' - \frac{NX_\beta'}{k}\right)\underset{\sim}{Y}, \left(\frac{NX_\beta}{k} - \frac{\underset{\sim}{r}\underset{\sim}{1}_n'}{n}\right)\underset{\sim}{Y}\right)$$

$$= \left(X_\tau' - \frac{NX_\beta'}{k}\right)\Sigma\left(\frac{X_\beta N'}{k} - \frac{\underset{\sim}{1}_n\underset{\sim}{r}'}{n}\right)$$

$$= \frac{X_\tau'\Sigma X_\beta N'}{k} - \frac{NX_\beta'\Sigma X_\beta N'}{k^2} - \frac{X_\tau'\Sigma\underset{\sim}{1}_n\underset{\sim}{r}'}{n} + \frac{NX_\beta'\Sigma\underset{\sim}{1}_n\underset{\sim}{r}'}{nk}$$

$$X_\tau'\Sigma\underset{\sim}{1}_n = \sigma^2 X_\tau'\underset{\sim}{1}_n + \sigma_b^2(X_\tau'X_\beta)(X_\beta'\underset{\sim}{1}_n)$$

$$= \sigma^2\underset{\sim}{r} + \sigma_b^2 N(k\underset{\sim}{1}_b)$$

$$= \sigma^2\underset{\sim}{r} + k\sigma_b^2\underset{\sim}{r}$$

$$= \frac{\underset{\sim}{r}}{W_2}$$

$$= \frac{NN'}{W_2 k} - \frac{NkI_b N'}{W_2 k^2} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{W_2 n} + \frac{Nk\underset{\sim}{1}_b\underset{\sim}{r}'}{W_2 nk}$$

$$= \cancel{\frac{NN'}{W_2 k}} - \cancel{\frac{NN'}{W_2 k}} - \cancel{\frac{\underset{\sim}{r}\underset{\sim}{r}'}{W_2 n}} + \cancel{\frac{\underset{\sim}{r}\underset{\sim}{r}'k}{W_2 nk}}$$

$$= 0$$

$$\operatorname{cov}(\underset{\sim}{Q}, G) = \operatorname{cov}\left(\left(X\tau' - \frac{NX\beta'}{k}\right)\underset{\sim}{Y}, \underset{\sim}{1}_n'\underset{\sim}{Y}\right)$$

$$= \left(X\tau' - \frac{NX\beta'}{k}\right)\Sigma\, \underset{\sim}{1}_n$$

$$= X\tau'\Sigma\, \underset{\sim}{1}_n - \frac{NX\beta'\Sigma\, \underset{\sim}{1}_n}{k}$$

$$= \frac{\underset{\sim}{r}}{W_2} - \frac{k}{W_2 k} N\, \underset{\sim}{1}_b$$

$$= \frac{\underset{\sim}{r}}{W_2} - \frac{\underset{\sim}{r}}{W_2}$$

$$= \underset{\sim}{0}_{v\times 1}$$

$$\operatorname{cov}(\underset{\sim}{Q}^*, G) = \operatorname{cov}\left(\left(\frac{NX\beta'}{k} - \frac{r\, 1_n'}{n}\right)\underline{Y}, 1_n'\underline{Y}\right)$$

$$= \left(\frac{NX\beta'}{k} - \frac{r\, 1_n'}{n}\right)\Sigma\, 1_n$$

$$= \frac{NX\beta'\Sigma\, 1_n}{k} - \frac{r\, 1_n'\Sigma\, 1_n}{n}$$

$$= \frac{N\cancel{K}\, 1_r}{\cancel{K}W_2} - \frac{r\cancel{n}}{\cancel{n}W_2}$$

$$= \frac{r}{W_2} - \frac{r}{W_2}$$

$$= \underset{\sim}{0}$$

Estimation of Variance Component

25.4.24

Normal eq^n $(w_1 C + w_2 C^*)\tilde{\underset{\sim}{\tau}} = w_1 \underset{\sim}{Q} + w_2 \underset{\sim}{Q}^*$

$$w_1 = \frac{1}{\sigma^2}, \quad w_2 = \frac{1}{\sigma^2 + k\sigma_b^2}$$

In reality, we don't know the value of σ^2 and σ_b^2, so we need to estimate them from the data. We'll follow the method by R.C. Bose (Assuming proper and binary design)

In intra-Block analysis

(1) Total SS $= S^2 = \sum_i \sum_j y_{ij}^2 - \frac{G^2}{n}$

(2) Unadjusted block SS $= S_b'^2 = \sum_j \frac{B_j^2}{k} - \frac{G^2}{n}$

(3) Unadjusted treatment SS $= S_t'^2 = \sum_i \frac{T_i^2}{r_i} - \frac{G^2}{n}$

(4) Adjusted trt. SS $= S_t^2 = \hat{\tau}' \underset{\sim}{Q} =$

(5) Residual SS $= R_0^2$; $S^2 = S_b'^2 + S_t^2 + R_0^2 = \underset{\substack{\downarrow \\ \text{Adjusted block SS} \\ \hat{\beta}' \underset{\sim}{P}}}{S_b^2} + S_t'^2 + R_0^2$

We find the expectations of these SS under mixed effect model.

$$\begin{aligned}
E(G) &= E(\underset{\sim}{1}_n' \underset{\sim}{Y}) \\
&= \underset{\sim}{1}_n' E(\underset{\sim}{Y}) \\
&= \underset{\sim}{1}_n' (\underset{\sim}{1}_n : X_\tau) \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix} \\
&= \underset{\sim}{1}_n' \underset{\sim}{1}_n \mu + \underset{\sim}{1}_n' X_\tau \cdot \underset{\sim}{\tau} \\
&= n\mu + \underset{\sim}{r}' \underset{\sim}{\tau} \\
&= n\mu + \sum_{i=1}^{v} r_i \tau_i \\
&= n(\mu + \bar{\tau})
\end{aligned}$$

$$\left[\bar{\tau} = \sum_{i=1}^{v} r_i \tau_i \Big/ \sum_{i=1}^{v} r_i\right]$$

$$E\left(\frac{G^2}{n}\right) = \frac{1}{n}\left[V(G) + (E(G))^2\right]$$

$$= \frac{1}{n}\left[n(\sigma^2 + k\sigma_b^2) + n^2(\mu + \bar{\tau})^2\right] \longrightarrow (1)$$

$$E\left(\frac{T_i^2}{r_i}\right) = \frac{1}{r_i}\left[Var(T_i) + (E(T_i))^2\right]$$

$$\underset{\sim}{T} = X'_\tau \underset{\sim}{Y}$$

$$E(\underset{\sim}{T}) = X'_\tau E(\underset{\sim}{Y})$$

$$= X'_\tau(\underset{\sim}{1}_n : X_\tau)\begin{pmatrix}\mu \\ \underset{\sim}{\tau}\end{pmatrix}$$

$$= X'_\tau \underset{\sim}{1}_n \mu + X'_\tau X_\tau \cdot \underset{\sim}{\tau}$$

$$= \underset{\sim}{r}\mu + D_r \underset{\sim}{\tau}$$

$$E(T_i) = r_i\mu + r_i\tau_i$$

$$= r_i(\mu + \tau_i)$$

$$Disp(\underset{\sim}{T}) = X'_\tau \, Disp(\underset{\sim}{Y}) \, X_\tau$$

$$= X'_\tau \Sigma X_\tau$$

$$= X'_\tau(\sigma^2 I_n + \sigma_b^2 X_\beta X'_\beta)X_\tau$$

$$= \sigma^2 D_r + \sigma_b^2 NN'$$

$$V(\tau_i) = \sigma^2 r_i + \sigma_b^2 \sum_{j=1}^{b} n_{ij}^2 \quad ; \quad i = 1(1)v$$

$$= \sigma^2 r_i + \sigma_b^2 \sum_{j=1}^{b} n_{ij} \quad [\because \text{The design is binary}]$$

$$= \sigma^2 r_i + \sigma_b^2 r_i$$

$$= r_i(\sigma^2 + \sigma_b^2)$$

$$E\left(\frac{T_i^2}{r_i}\right) = \frac{1}{r_i}\left[r_i(\sigma^2 + \sigma_b^2) + r_i^2(\mu + \tau_i)^2\right] \longleftarrow (2)$$

$$E(S_t'^2) = E\left(\sum_i \frac{T_i^2}{r_i}\right) - E\left(\frac{G^2}{n}\right)$$

$$= \sum_i E\left(\frac{T_i^2}{r_i}\right) - E\left(\frac{G^2}{n}\right)$$

$$= \sum_{i=1}^{v}\left[(\sigma^2+\sigma_b^2) + r_i(\mu+\tau_i)^2\right] - (\sigma^2+k\sigma_b^2) - n(\mu+\bar{\tau})^2$$

$$= (v-1)\sigma^2 + (v-k)\sigma_b^2 + \sum_{i=1}^{v} r_i(\mu+\tau_i)^2 - n(\mu+\bar{\tau})^2$$

$$= (v-1)\sigma^2 + (v-k)\sigma_b^2 + \sum_{i=1}^{v} r_i(\tau_i-\bar{\tau})^2 \quad \text{———} (3)$$

$$E(\text{Total SS}) = \sum_{u=1}^{n} E(y_u^2) - E\left(\frac{G^2}{n}\right)$$

$$\text{Total ss} = \sum_i^b \sum_j^v y_{ij}^2 - \frac{G^2}{n} = \sum_{u=1}^{n} y_u^2 - \frac{G^2}{n}$$

$$= \sum_{u=1}^{n}\left[var(y_u) - \{E(y_u)\}^2\right] - E\left(\frac{G^2}{n}\right)$$

$$= \sum_{u=1}^{n}\left[(\sigma^2+\sigma_b^2) + \sum_{i=1}^{b}\sum_{j=1}^{v} n_{ij}(\mu+\tau_i)^2\right] - (\sigma^2+k\sigma_b^2) - n(\mu+\bar{\tau})^2$$

$$= (n-1)\sigma^2 + (n-k)\sigma_b^2 + \sum_{i=1}^{v} r_i(\tau_i-\bar{\tau})^2 \quad \text{———} (4)$$

$$\sum_{i=1}^{v}\sum_j^{b} n_{ij}(\mu+\tau_i)^2 - n(\mu+\bar{\tau})^2$$

$$= \sum\sum n_{ij}(\mu^2+2\mu\tau_i+\tau_i^2) - n(\mu^2+2\mu\bar{\tau}+\bar{\tau}^2)$$

$$= n\mu^2 + 2\mu\sum\sum n_{ij}\tau_i + \sum\sum n_{ij}\tau_i^2 - n\mu^2 - 2n\mu\bar{\tau} + n\bar{\tau}^2$$

$$= 2\mu\left(\sum_i r_i\right)\tau_i + \sum_i r_i\tau_i^2 - n\bar{\tau}^2 - 2n\mu\bar{\tau}$$

$$= 2n\mu\bar{\tau} + \sum r_i\tau_i^2 - n\bar{\tau}^2 - 2n\mu\bar{\tau}$$

$$E(R_0^2) = (n-b-v+1)\sigma^2 \ (\text{connected design})$$

$$E(S_b^2) = E(\text{Adjusted block SS})$$

$$= E(\text{Total SS}) - E(S_t'^2) - E(R_0^2)$$

$$= (n-1)\sigma^2 + (n-k)\sigma_b^2 + \sum_{i=1}^{v} r_i(\tau_i-\bar{\tau})^2 - (v-1)\sigma^2 - (v-k)\sigma_b^2 - \sum_{i=1}^{v} r_i(\tau_i-\bar{\tau})^2 - (n-b-v+1)\sigma^2$$

$$= (b-1)\sigma^2 + (n-v)\sigma_b^2 \quad \text{———} (5)$$

$$\hat{\sigma}^2 = \frac{R_0}{n-b-v+1}$$

$$E\left[\frac{S_b^2}{n-v} - \left(\frac{b-1}{n-v}\right)\frac{R_0^2}{(n-b-v+1)}\right] = \sigma_b^2$$

$$\hat{\sigma}_b^2 = \left(\frac{b-1}{n-v}\right)\left(\frac{S_b^2}{b-1} - \frac{R_0^2}{n-b-v+1}\right)$$

III Balanced Incomplete Block Design (BIBD)

30.4.24

(ii) Each trt. appears in r blocks.

iii) Every pair of trt. appear together in λ blocks.

The parameters v, b, r, k, λ are called the parameters of the BIBD. They are related by the following identities.

i) $b \cdot k = v \cdot r$

ii) $r(k-1) = \lambda(v-1)$

BIBD is the binary design as in this case the incidence matrix has only two elements 0, 1

Example

1	2	4
2	3	5
3	4	6
4	5	7
5	6	1
6	7	2
7	1	3

$v = 7$
$b = 7$
$r = 3$
$k = 3$
$\lambda = 1$

Result >>

For a BIBD, $r(k-1) = \lambda(v-1)$

Proof >> As a BIBD is proper and equireplicate.

$N \underline{1}_b = \underline{r} = r \underline{1}_v$

$\underline{1}'_v N = \underline{k}' = k \underline{1}'_b$

The $(i, j)^{th}$ element of NN' is

$$\sum_{l=1}^{b} n_{il} n_{jl}$$