(Problems on Neyman-Pearson Fundamental lemma)

① Let $X_1, X_2, \cdots, X_5$ be a random sample of size 5 from $Poi(\lambda)$. Construct an MP test with size $\alpha = 0.05$ for

$$H_0 : \lambda = 1$$
$$vs \quad H_1 : \lambda = 2$$

Also find the power of the test

Ans ⇒ $X_1, X_2, \cdots, X_5 \overset{iid}{\sim} Poi(\lambda)$

$$\therefore \sum_{i=1}^{5} X_i \sim Poi(5\lambda)$$

For Poisson distribution, $Y = \sum_{i=1}^{5} X_i$ is the sufficient statistic for λ

The Most Powerful test for $H_0 : \lambda = 1$ vs $H_1 : \lambda = 2$ is given by,

$$\phi(y) = \begin{cases} 1 & \text{if } Y > c \\ \gamma & \text{if } Y = c \\ 0 & \text{if } Y < c \end{cases}$$

c and γ are to be determined from the size condition.

Given $\alpha = 0.05$

$$\Rightarrow E_{H_0}[\phi(x)] = 0.05$$

$$\Rightarrow P_{H_0}(Y > c) + \gamma P_{H_0}(Y = c) = 0.05$$

$$\Rightarrow P_{H_0}(Y \leq c) - \gamma P_{H_0}(Y = c) = 0.95$$

$$\Rightarrow \gamma = \frac{P_{H_0}(Y \leq c) - 0.95}{P_{H_0}(Y = c)}$$

We have to fix a 'c' such that the randomization probability is positive and also minimized. The table looks like,

c	$P_{H_0}(Y=c)$	$P_{H_0}(Y\leq c)$
0	0.006	0.006
1	0.033	0.039
2	0.084	0.123
3	0.140	0.263
4	0.175	0.438
5	0.175	0.613
6	0.146	0.759
7	0.1044	0.863
8	0.065	0.9284
9	0.036	0.9644

Under H_0, $Y \sim Poi(5)$; $P(Y=y) = \dfrac{e^{-5}5^{y}}{y!}$, $y=0,1,2,\ldots$

Under H_1, $Y \sim Poi(10)$; $P(Y=y) = \dfrac{e^{-10}10^{y}}{y!}$, $y=0,1,2,\ldots$

Here from looking at the table, at $c=9$, $P_{H_0}(Y\leq c)$ becomes larger than 0.95

Hence $\gamma = \dfrac{0.9644-0.95}{0.036} = \boxed{0.4}$

The MP test

$$\phi(y) = \begin{cases} 1 & \text{if } Y>9 \\ 0.4 & \text{if } Y=9 \\ 0 & \text{if } Y<9 \end{cases}$$

power of the test

$E_{H_1}[\phi(x)]$

$= 1 \cdot P(Y>9) + 0.4\, P(Y=9)$

$= 1 - P(Y \le 9) + 0.4 \times (P(Y=9))$

$= 1 - 0.4593 + 0.4 \times 0.1251$

$= \boxed{0.59074}$

Table for Poi(10)

c	$P_{H_1}(Y=c)$	$P_{H_1}(Y \le c)$
0	0.00004	0.00004
1	0.0004	0.00044
2	0.0023	0.00274
3	0.0076	0.0103
4	0.0189	0.0292
5	0.0378	0.0678
6	0.0638	0.1326
7	0.0900	0.2216
8	0.1126	0.3342
9	0.1251	0.4593

(2) Let $X_1, X_2, \dots, X_5$ be i.i.d Ber(p)

Construct a test (MP) for

$H_0: p = 0.3$

against

$H_1: p = 0.5$ with $\alpha = 0.05$

Furthermore find the power for $p = 0.6$, $p = 0.7$, $p = 0.9$

Ans ⇒ $X_1, X_2, \dots, X_5 \overset{iid}{\sim} Ber(p)$

$\therefore \sum_{i=1}^{5} X_i \sim Bin(5, p)$

$Y = \frac{1}{5}\sum_{i=1}^{5} X_i$ is the sufficient statistic for p.

$P(Y = y) = \binom{5}{y} p^y (1-p)^{5-y}, \quad 0<p<1, \; y = 0, 1, \dots, 5$

An MP test is defined by, $\phi(y) = \begin{cases} 1 & \text{if } Y > c \\ \gamma & \text{if } Y = c \\ 0 & \text{if } Y < c \end{cases}$

c and γ are to be determined from the size condition

$E_{H_0}[\phi(x)] = \alpha = 0.05$

$\Rightarrow 1 \cdot P(Y>c) + \gamma P(Y=c) = 0.05$

$\Rightarrow P(Y \le c) - \gamma P(Y=c) = 0.95$

$\Rightarrow \gamma = \dfrac{P_{H_0}(Y \le c) - 0.95}{P_{H_0}(Y=c)}$

We have to fix a 'c' such that the randomization probability is positive and minimized

c	$P_{H_0}(Y=c)$	$P_{H_0}(Y \le c)$
0	0.168	0.168
1	0.360	0.528
2	0.3087	0.8367
3	0.1323	0.969
4	0.02835	0.99735
5	0.002	1

Under Ho,

$P(Y=y) = \binom{5}{y}(0.3)^y(0.7)^{5-y}$

Here, $c=3$ is the point to be considered

Hence, $\gamma = \dfrac{0.969 - 0.95}{0.1323} = \boxed{0.143613}$

$\therefore$ An MP test is,

$$\phi(y) = \begin{cases} 1 & \text{if } Y>3 \\ 0.143613 & \text{if } Y=3 \\ 0 & \text{if } Y<3 \end{cases}$$

Power of the test function

For $p=0.6$

$E_{H_1}[\phi(x)]$

$= P(Y>3) + \gamma P(Y=3)$

$= 0.2592 + 0.078$

$+ 0.049$

$= \boxed{0.3862}$

For $p=0.7$

$E_{H_1}[\phi(x)]$

$= P(Y>3) + \gamma P(Y=3)$

$= 0.360 + 0.168$

$+ 0.044$

$= \boxed{0.572}$

For $p=0.9$

$E_{H_1}[\phi(x)]$

$= P(Y>3) + \gamma P(Y=3)$

$= 0.328 + 0.590$

$+ 0.011$

$= \boxed{0.929}$

③ Let the random variable X has the following distribution

X	0	1	2	3
$P_\theta(X=x)$	θ	2θ	$0.9-2\theta$	$0.1-\theta$

where $0<\theta<1$

for testing $H_0: \theta = 0.05$ vs $H_1: \theta > 0.05$ at $\alpha = 0.05$.
Determine which of the following test is UMP?

i) $\phi(0)=1, \quad \phi(1)=\phi(2)=\phi(3)=0$

ii) $\phi(1)=0.5, \quad \phi(x)=0$ if $x \neq 1$

iii) $\phi(3)=1, \quad \phi(x)=0$ if $x \neq 3$

iv) $\phi(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{if } x=2,3 \end{cases}$

Ans ⇒

X	0	1	2	3
$P(X=x)$	θ	2θ	$0.9-2\theta$	$0.1-\theta$

We test
$H_0: \theta = 0.05$ vs $H_1: \theta > 0.05$

Case-I

$$\phi(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x=1,2,3 \end{cases}$$

Power of the test

$$E_{H_1}[\phi(x)] = 1 \cdot P_{H_1}(X=0) = \theta$$

if $\alpha = 0.05 = P_r(\text{type I error}) =$ size of the test and under H_1, $\theta > 0.05 \Rightarrow$ power > size

⇒ This is an UMP test.

Case II

$$\phi(x) = \begin{cases} 0.5 & \text{if } x=1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

Power of the test

$= 0.5 \times P_{H_1}(X=1) = 0.5 \times 2\theta$
$= \theta$

⇒ This is also an UMP test.

Case III

$$\phi(x) = \begin{cases} 1 & \text{if } X=3 \\ 0 & \text{if } X \neq 3 \end{cases}$$

$\therefore$ Power of the test

$E_{H_1}(\phi(x)) = 1 \cdot P_{H_1}(X=3) = 0.1-\theta$

Under H_1, $\theta > 0.05$

$\Rightarrow 0.1-\theta < 0.1-0.05 \Rightarrow$ power $\ngtr$ size

Hence, the test is not unbiased

$\Rightarrow$ the test is not UMP.

Case-IV

$$\phi(x) = \begin{cases} 1 & \text{if } X=1 \\ 0 & \text{o.w} \end{cases}$$

$E_{H_1}\left(\phi(x)\right) = P_{H_1}(X=1) = 2\theta$,

Under $H_1 \Rightarrow$ power $>$ size as $\theta > 0.05$

$\Rightarrow$ This is an UMP test.

④ Let $X_1, X_2, \ldots, X_5 \sim Poi(\lambda)$

$H_0 : \lambda \leq 1$ vs $H_1 : \lambda > 1$

Construct an UMP test with $\alpha = 0.1$

Ans ⇒ For poisson distribution, the sufficient statistics for $\lambda (>0)$ is $\sum_{i=1}^{n} X_i$, where n = sample size

For $n=5$

$$f(x) = \frac{e^{-5\lambda} (\lambda)^{\sum_{i=1}^{5} x_i}}{\prod_{i=1}^{5} x_i!} , \quad \sum_{i=1}^{5} X_i \sim Poi(5\lambda)$$

Now for exponential family, $a(\theta) h(x) e^{c(\theta) T(x)}$ should be matched with $f(x)$.

Hence $f(x)$ is a member of exponential family with $c(\lambda) = \ln \lambda$, UMP test construction can be done on $\sum_{i=1}^{5} X_i$.

The above test in the question may be reformed test in terms of the common limit point of ~~instruction~~ intersection between the H_0 closure set and H_1 closure set.

$$\equiv \quad H_0 : \lambda = 1 \quad vs \quad H_1 : \lambda > 1$$

As per Neyman-Pearson lemma,

UMP will be

$$\phi(t) = \begin{cases} 1 & \text{if } \sum_{i=1}^{5} X_i > c \\ \gamma & \text{if } \sum_{i=1}^{5} X_i = c \\ 0 & \text{if } \sum_{i=1}^{5} X_i < c \end{cases}$$

c and γ are to be determined from size condition.

$E_{H_0}(\phi(t)) = \alpha = 0.1$ (size of the test)

Where $T = \sum_{i=1}^{5} X_i \sim Poi(5\lambda)$

Under $H_0 \Rightarrow T \sim Poi(5)$ $[\because \lambda = 1]$

$\therefore \; 1 \cdot P_{H_0}(T > c) + \gamma P_{H_0}(T = c) = 0.1$

$$\Rightarrow \gamma = \frac{P_{H_0}(T \leq c) - 0.9}{P_{H_0}(T = c)}$$

c	$P_{H_0}(T = c)$	$P_{H_0}(T \leq c)$
0	0.006	0.006
1	0.034	0.040
2	0.084	0.124
3	0.1404	0.264
4	0.175	0.439
5	0.175	0.614
6	0.146	0.760
7	0.104	0.864
8	0.065	0.929

Hence we consider c as $\boxed{8}$ and $\gamma = \frac{P_{H_0}(T \leq 8) - 0.9}{P_{H_0}(T = 8)}$

$$= \frac{0.929 - 0.9}{0.065} = \boxed{0.446}$$

Consequently, the UMP test

$$\phi(t) = \begin{cases} 1 & \text{if } T > 8 \\ 0.446 & \text{if } T = 8 \\ 0 & \text{if } T < 8 \end{cases}, \quad T = \sum_{i=1}^{5} X_i$$

(5) Let $X_1, X_2, \dots, X_{10}$ be iid $N(\mu, 1)$. Propose an UMP test for $H_0: \mu \le 2$ or $\mu \ge 3$ Vs $H_1: 2 < \mu < 3$

Ans $\Rightarrow$ Normal population is a member of exponential family. As per NP lemma UMP test will be of the form of the sufficient statistic $T = \sum_i X_i$ as follows.

$$\phi(t) = \begin{cases} 1 & \text{if } c_1 < \sum_{i=1}^{n} X_i < c_2 \\ \gamma_j & \text{if } \sum_{i=1}^{n} X_i = c_j \\ 0 & \text{o.w} \end{cases}$$

Since normal is a continuous distribution, non-randomised test exists.

$$\phi(t) = \begin{cases} 1 & \text{if } c_1 < \sum_{i=1}^{n} X_i < c_2 \\ 0 & \text{o.w} \end{cases}$$

where c_1, c_2 are determined from size condition.

$$E_{\mu=\mu_0}(\phi(t)) = \alpha \Rightarrow P_{H_0}\left(c_1 < \sum_{i=1}^{n} X_i < c_2\right) = \alpha$$

$$\Rightarrow P_{H_0}\left(\frac{c_1 - n\mu_0}{\sqrt{n}} < \frac{\sum_{i=1}^{n} X_i - n\mu_0}{\sqrt{n}} < \frac{c_2 - n\mu_0}{\sqrt{n}}\right) = \alpha$$

And $E_{\mu=\mu_1}(\phi(t)) = \alpha$

$$\Rightarrow P_{H_0}\left(\frac{c_1 - n\mu_1}{\sqrt{n}} < \frac{\sum_{i=1}^{n} X_i - n\mu_1}{\sqrt{n}} < \frac{c_2 - n\mu_1}{\sqrt{n}}\right) = \alpha$$

i.e $\alpha = \Phi\left(\frac{c_2 - n\mu_1}{\sqrt{n}}\right) - \Phi\left(\frac{c_1 - n\mu_1}{\sqrt{n}}\right)$

$\boxed{\sum_{i=1}^{n} X_i \sim N(n\mu, \sqrt{n})}$

and

$$\alpha = \Phi\left(\frac{c_2 - n\mu_0}{\sqrt{n}}\right) - \Phi\left(\frac{c_1 - n\mu_0}{\sqrt{n}}\right)$$

For $\alpha = 0.05$, $n = \sqrt{10}$, $\mu_0 = 2$, $\mu_1 = 3$, we have,

(P.T.O)

$$\alpha = 0.05 = \Phi\left(\frac{c_2 - 30}{\sqrt{10}}\right) - \Phi\left(\frac{c_1 - 30}{\sqrt{10}}\right)$$

Also

$$\alpha = \Phi\left(\frac{c_2 - 20}{\sqrt{10}}\right) - \Phi\left(\frac{c_1 - 20}{\sqrt{10}}\right) \quad \left[\Phi(\cdot) = \text{c.d.f of standard normal}\right]$$

left limit can be considered as 0.8

right limit can be considered as 1 from Z score table.

Thus, $\frac{c_2 - 20}{\sqrt{10}} = 1 \Rightarrow c_2 = \sqrt{10} + 20 \doteq 23.162$

$\frac{c_1 - 20}{\sqrt{10}} = 0.8 \Rightarrow c_1 = 22.53$

One choice of such (c_1, c_2) is $(22.53, 23.162)$

$\therefore$ The UMP test

$$\phi(t) = \begin{cases} 1 & \text{if } 22.53 < T < 23.162 \\ 0 & \text{o.w} \end{cases}$$

⑥ Let X and Y be independent poisson(λ) and poisson(μ)

We want to test

$H_0 : \mu \leq \lambda$

ag $H_1 : \mu > \lambda$

(i) Construct a UMP test.

(ii) Also find the power of the test given

$\lambda = 1$, $\mu = 0.2$

, given $\alpha = 0.1$

$\lambda = 10$, $\mu = 20$

Ans ⟹ The test will be constructed on $U = Y$ where the critical point will be found from conditional distribution $Y | X+Y = t$

(i) $$Y | X+Y \sim Bin\left(t, \frac{\mu}{\mu+\lambda}\right)$$

The probability table goes as follows

Y	$X+Y=t$	$P(Y \mid X+Y=t)$
0	1	$P(Y=0 \mid X+Y=1) = 1/2$
1		$P(Y=1 \mid X+Y=1) = 1/2$
0	2	$P(Y=0 \mid X+Y=2) = 1/4$
1		$P(Y=1 \mid X+Y=2) = 1/2$
2		$P(Y=2 \mid X+Y=2) = 1/4$
0	3	$P(Y=0 \mid X+Y=3) = 1/8$
1		$P(Y=1 \mid X+Y=3) = 3/8$
2		$P(Y=2 \mid X+Y=3) = 3/8$
3		$P(Y=3 \mid X+Y=3) = 1/8$
0	4	$P(Y=0 \mid X+Y=4) = 1/16$
1		$P(Y=1 \mid X+Y=4) = 4/16$
2		$P(Y=2 \mid X+Y=4) = 6/16$
3		$P(Y=3 \mid X+Y=4) = 4/16$
4		$P(Y=4 \mid X+Y=4) = 1/16$

The modified hypothesis based on the limit point of Θ_0 and Θ_1 is

$$H_0^* : \mu = \lambda$$

ag.

$$H_1^* : \mu > \lambda$$

Thus,

$$E_{H_0}(U) = \alpha = 0.1$$

$$\Rightarrow P_{H_0}(U > c(t)) + \gamma P_{H_0}(U = c(t)) = 0.1$$

$$\Rightarrow P_{H_0}(U \leq c(t)) - \gamma P_{H_0}(U = c(t)) = 0.9$$

$$\Rightarrow P_{H_0}(U < c(t)) + (1-\gamma) P_{H_0}(U = c(t)) = 0.9$$

$$\Rightarrow (1-\gamma) = \frac{0.9 - P_{H_0}(U < c(t))}{P_{H_0}(U = c(t))}$$

When $c(t) = 1 \Rightarrow t = 1$ then the UMP test will be

$$\phi(u) = \begin{cases} 1 & \text{if } u > 1 \\ 0.2 & \text{if } u = 1 \\ 0 & \text{o.w} \end{cases}$$

$$\left[(1-\gamma) = \frac{0.9 - P_{H_0}(U<1)}{P_{H_0}(U=1)} = \frac{0.9-0.5}{0.5} = 4/5 \quad \therefore \gamma = 1 - 4/5 = 0.2 \right]$$

But practically this not feasible since the rejection region contains no points.

For $t = 4$, $c(t) = 1$

$$1-\gamma = \frac{0.9 - P_{H_0}(U<1)}{P_{H_0}(U=1)} \Rightarrow 1-\gamma = \frac{0.9 - (1/16)}{4/16} \Rightarrow \text{not feasible}$$

Therefore, For $t = 4$, $c(t) = 3$

$$1-\gamma = \frac{0.9 - P_{H_0}(U<3)}{P_{H_0}(U=3)} = \frac{0.9 - 0.6875}{0.25} = 0.85$$

$\therefore \boxed{\gamma = 0.15}$ $\Rightarrow$ The UMP test looks like,

$$\phi(u) = \begin{cases} 1 & \text{if } U = 3 \\ 0.15 & \text{if } U = 3 \\ 0 & \text{o.w} \end{cases} \quad \text{for } X+Y = 4$$

(ii) Power of the test for $\lambda = 0.1$, $\mu = 0.2$

$$Y | X+Y \sim Bin\left(t, \frac{0.2}{0.2+0.1}\right)$$

$\therefore E_{H_1}[\phi(U) | T=t]$, for $t=4$

$$= 1 \cdot P_{H_1}(U=4) + 0.15 \cdot P_{H_1}(U=3)$$

$$= {}^4C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^0 + 0.15 \times {}^4C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)$$

$$= 16/81 + 4.8/81 = 0.257$$

Power of the test for $\lambda = 10$, $\mu = 20$

$$Y | X+Y \sim Bin\left(t, \frac{20}{20+10}\right)$$

$\therefore E_{H_1}[\phi(U) | T=t]$ for $t=4$

$$= 1 \cdot P_{H_1}(U > 3) + 0.15 \cdot P_{H_1}(U=3)$$

$= 0.257$ (same as above)

(7) Construct a test $H_0: P_1 = P_2$ vs $H_1: P_1 > P_2$
where $X \sim Bin(m, P_1)$, $Y \sim Bin(n, P_2)$
Construct an UMP test for $\alpha = 0.1$, $m = 3$, $n = 4$.

Ans $\Rightarrow$ The joint pmf,

$$f(x,y) = \binom{m}{x}\binom{n}{y} P_1^x P_2^y (1-P_1)^{m-x} (1-P_2)^{n-y}$$

$$= \binom{m}{x}\binom{n}{y} (1-P_1)^m (1-P_2)^n e^{x \log\left(\frac{P_1(1-P_2)}{P_2(1-P_1)}\right) + (x+y)\log \frac{P_2}{1-P_2}}$$

$$\theta = \log \frac{P_1(1-P_2)}{(1-P_1)P_2}, \quad U = X$$

$$\eta = \log\left(\frac{P_2}{1-P_2}\right), \quad T = X+Y$$

Then a UMP can be constructed as,

$$\phi(u) = \begin{cases} 1 & \text{if } U > c_0(t) \ \cancel{\text{or } U < c_1(t)} \\ \gamma_0 & \text{if } U = c_0(t) \\ 0 & \cancel{\text{if } \ldots} \ \text{o.w.} \end{cases}$$

$U \mid T = t \Rightarrow X \mid X + Y = t \sim$ Hypergeometric distribution

$$P(X = x \mid T = t) = \frac{\binom{m}{x}\binom{n}{t-x}}{\binom{m+n}{t}} = \frac{\binom{3}{x}\binom{4}{t-x}}{\binom{7}{t}}$$

$c(t)$ and γ are determined from the size condition

$$E_{H_0}\left[\phi(U) \mid T = t\right] = \alpha = 0.1$$

$$\Rightarrow 1 \cdot P_{H_0}(U > c(t)) + \gamma P_{H_0}(U = c(t)) = 0.1$$

$$\Rightarrow 1 - \gamma = \cancel{0.9 =} \frac{0.9 - P_{H_0}(U < c(t))}{P_{H_0}(U = c(t))}$$

t	$X = x$	$P(X = x \mid T = t)$
1	0	$0.5714 > 0.1$
	1	$0.4286 > 0.1$
2	0	$0.2857 > 0.1$
	1	$0.5714 > 0.1$
	2	$0.1429 > 0.1$
3	0	$0.1143 > 0.1$
	1	$0.5143 > 0.1$
	2	$0.3429 > 0.1$
	3	$0.0286 < 0.1$

Hence for $t = 3$, $X = 3$ is in the critical region

$P_{H_0}(X = 2) + P_{H_0}(X = 3) = 0.3715 > 0.1 \Rightarrow X = 2$ is a randomized point.

$E_{H_0}[\phi(u) \mid T=3] = 0.1$

$$\Rightarrow \cancel{(1-\gamma)\}} \quad \gamma = \frac{0.1 - P_{H_0}(U>2)}{P_{H_0}(U=2)} \qquad \left[\because c(t)=2 \text{ is the randomization point}\right]$$

$$= \frac{0.1 - 0.0286}{0.3429} = 0.21$$

So the UMP test for $H_0: p_1 = p_2$ ag. $H_1: p_1 > p_2$ is

$$\phi(u) = \begin{cases} 1 & \text{if } x > 2 \\ 0.21 & \text{if } x = 2 \\ 0 & \text{o.w} \end{cases} \qquad \text{for } t = 3$$