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May 27, 2023

# Financial and Risk Analytics

## Lecture 1 – Financial Portfolio Selection and Management

Vodafone Big Data Lab



## About me

### Dr. Oleksandr Romanko

- Associate Director, Financial Risk Quantitative Research at SS&C Algorithmics, formerly with Watson Financial Services, IBM
- Ph.D. in Computer Science from McMaster University
- Author of over 20 papers and reports
- Adjunct professor at University of Toronto, American University Kyiv and Ukrainian Catholic University
- Research areas:
  - business analytics, operational research, optimization, finance
  - portfolio optimization, multi-objective optimization
  - market and credit risk modeling and optimization
  - numerical methods for risk management
  - design of numerical algorithms and their software implementation



# Quick facts about SS&C Algorithmics (formerly IBM Risk Analytics)

Algorithmics offers risk solutions, software and advisory services for

- Banking
- Insurance
- Asset Management
- Hedge Funds
- Pension Funds

Founded in 1989

Acquired by IBM in 2011

Over 800 employees worldwide

- 200 in Research and Development
- 250 in Professional Services
- 110 in Business Lines

Head Office in Toronto

- Primary offices in London and New York
- 23 offices globally in all major financial centers
- Clients in 55 countries

2010 revenue over \$163M USD

Acquired by SS&C from IBM in 2019

## Some of our clients



# Financial concepts and terminology

## ■ Asset classes

- Equities (stocks)
- Fixed-income securities (bonds)
- Derivatives (options)

## ■ Portfolios of assets

## ■ Investment and trading

## ■ Financial risk management

## ■ Valuation (asset pricing)

## ■ Return calculations (see slides by Eric Zivot)

- Compound interest
- Present value and future value
- Calculating rate of return

To familiarize yourself with financial concepts:

<http://www.khanacademy.org/science/core-finance>



# Financial Analytics, Machine Learning and ChatGPT

# Applications of data analytics

Smarter Healthcare



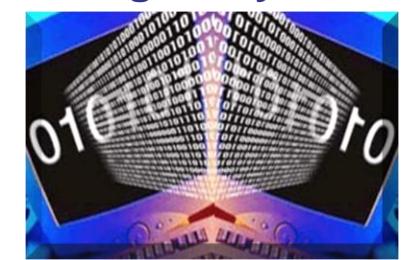
Multi-channel



Finance



Log Analysis



Homeland Security



Traffic Control



Telecom



Search Quality



Manufacturing



Trading Analytics



Fraud and Risk



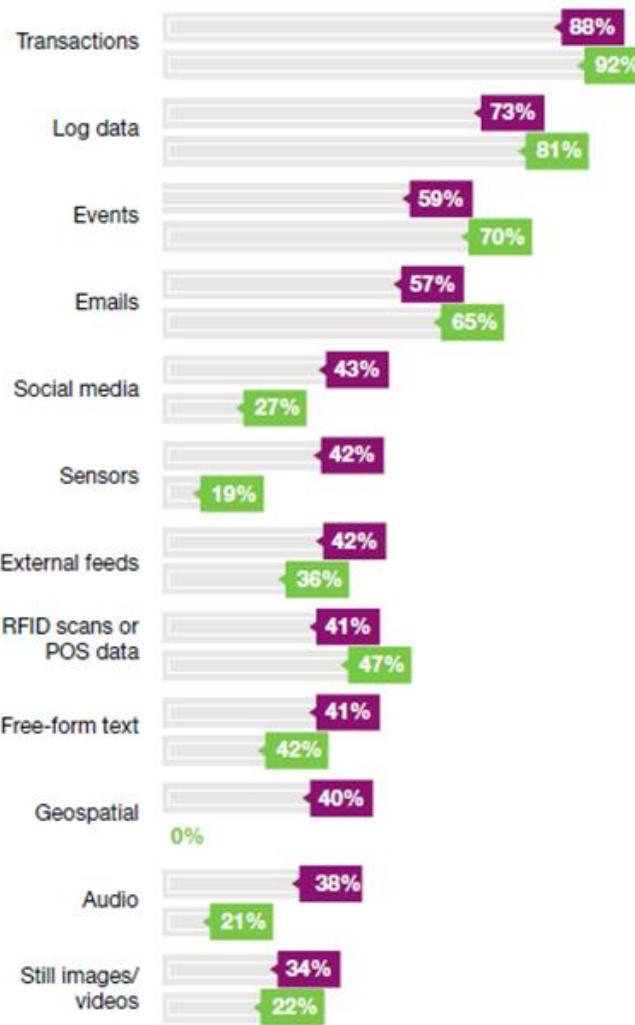
Retail: Churn, NBO





# Use of data and analytics globally and in the financial sector

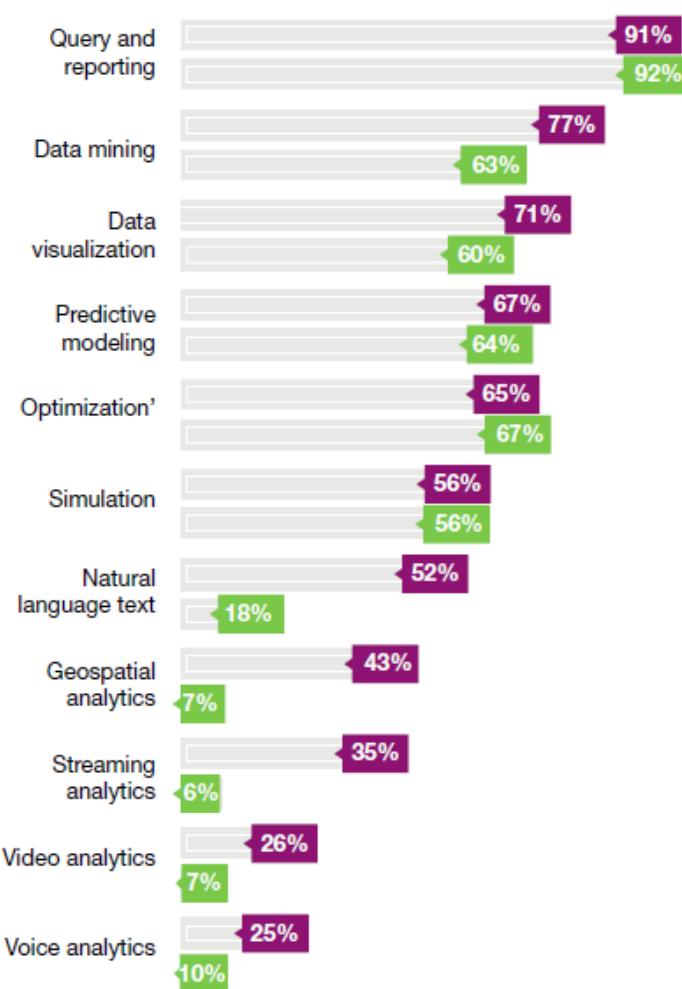
## Big data sources



■ Global

■ Banking and Financial Markets

## Analytics capabilities



■ Global

■ Banking and Financial Markets

Multiple responses accepted

# Models and algorithms used in financial and risk analytics

- **Statistics** – exploratory analysis, distributions and hypothesis testing
  - ✓ Distributions, statistical measures
  - ✓ Timeseries modeling
- **Modeling uncertainty** – what would happen in the future?
  - ✓ Stochastic processes
  - ✓ Monte Carlo simulations
- **Optimizing decisions** – what's best?
  - ✓ Regressions
  - ✓ Optimization
- **Machine learning** – learning from examples
  - ✓ Supervised learning (prediction, classification)
  - ✓ Unsupervised learning (clustering, dimensionality reduction, associations)
  - ✓ Reinforcement learning
- **Finding connections** – is FB related to Cambridge Analytica?
  - ✓ Graph/network models

# Chat-bot for portfolio selection on IBM Cloud

I am investing for 30 years.

Based on the time horizon of your investment, I am applying an aggressive benchmark.

I would like to invest \$30000 in my RRSP this year.

I am adding cash into your optimization formulation.

I do not like having military and tobacco assets in my portfolio.

You don't want this sort of assets? Sure. I will not include them into your portfolio.

# Chat-bot for portfolio selection on IBM Cloud

I want my portfolio to be diversified.

I am adding a constraint on the maximum weight of each asset.

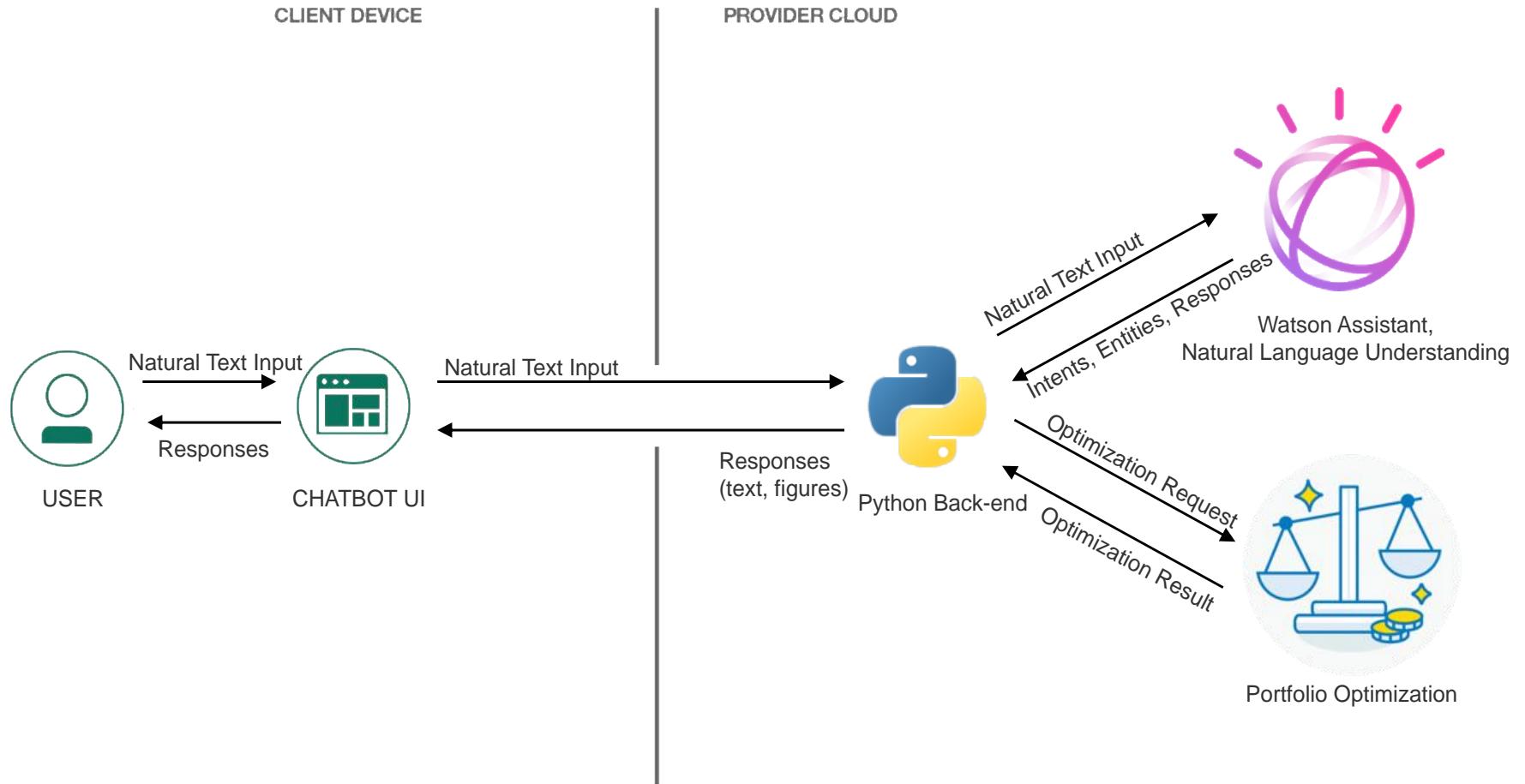
I want my portfolio to have high environmental score.

I see. I will make sure that your portfolio has an overall high score in your selected social responsibility area.

I want my portfolio to have more than 20% IT assets.

Sure I will add constraints on this sector!

# Chat-bot for portfolio selection on IBM Cloud





# Chat-bot for portfolio optimization – demo

★ watson-advisor

Messages About



watson-advisor APP 10:46 AM

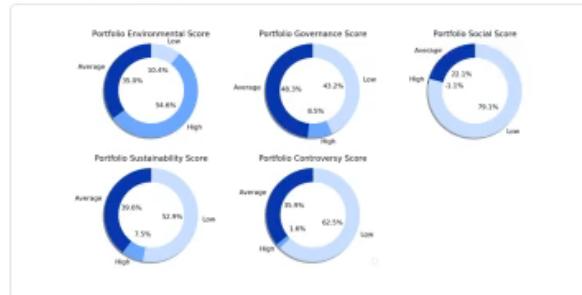
PortfolioBreakdown



Search



Thursday, March 14th



Friday, March 15th



Oleksandr Romanko 1:35 PM

Hi Watson

Today



Oleksandr Romanko 12:32 PM

Hi Watson



watson-advisor APP 12:33 PM

Hi, I'm here to help you build a portfolio of investments with the features you care about. Please tell me your preferences and I'll incorporate them into your portfolio!



Oleksandr Romanko 1:28 PM

lets start from scratch



watson-advisor APP 1:28 PM

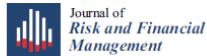
Sure let's start over.



Message watson-advisor



# Chat-bot for portfolio optimization



Article

## Cognitive User Interface for Portfolio Optimization

Yuehuan He <sup>1,\*</sup>, Oleksandr Romanko <sup>2,\*</sup>, Alina Sienkiewicz <sup>1</sup>, Robert Seidman <sup>2</sup> and Roy Kwon <sup>1</sup>

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**Abstract:** This paper describes the development of a chatbot as a cognitive user interface for portfolio optimization. The financial portfolio optimization chatbot is proposed to provide an easy-to-use interface for portfolio optimization, including a wide range of investment objectives and flexibility to include a variety of constraints representing investment preferences when compared to existing online automated portfolio advisory services. Additionally, the use of a chatbot interface allows investors lacking a background in quantitative finance and optimization to utilize optimization services. The chatbot is capable of extracting investment preferences from natural text inputs, handling these inputs with a backend financial optimization solver, analyzing the results, and communicating the characteristics of the optimized portfolio back to the user. The architecture and design of the chatbot are presented, along with an implementation using the IBM Cloud, SS&C Algorithmics Portfolio Optimizer, and Slack as an example of this approach. The design and implementation using cloud applications provides scalability, potential performance improvements, and could inspire future applications for financial optimization services.

**Keywords:** chatbot; portfolio optimization; cognitive user interface



Citation: He, Yuehuan, Oleksandr Romanko, Alina Sienkiewicz, Robert Seidman, and Roy Kwon. 2021.

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### 1. Introduction

In financial decision making, optimization is one of the most critical techniques. Since the development of the mean-variance portfolio optimization method that was introduced by Markowitz (1952), both research and practice in finance have become increasingly technical (Cornuejols and Tütüncü (2006)). While individual investors could potentially benefit from optimizing their portfolios, this would require significant financial literacy (Collins (2012)), the ability to represent investment preferences as an optimization problem, knowledge of the underlying optimization models, coding techniques to solve the optimization problem, and interpretation of the optimal solutions. These requirements represent a significant barrier preventing individual investors from utilizing optimization



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FINANCE

## Morgan Stanley is testing an OpenAI-powered chatbot for its 16,000 financial advisors

PUBLISHED TUE, MAR 14 2023 2:05 PM EDT | UPDATED WED, MAR 15 2023 10:30 AM EDT



Hugh Son  
@HUGH\_SON

WATCH LIVE

### KEY POINTS

- Morgan Stanley is rolling out an advanced chatbot powered by OpenAI's latest technology to help the bank's army of financial advisors, CNBC has learned.
- The bank has been testing the artificial intelligence tool with 300 advisors and plans to roll it out widely in the coming months, according to Jeff McMillan, head of analytics and data at the firm's wealth management division.
- The idea behind the tool, which has been in development for the past year, is to help the bank's 16,000 or so advisors tap the bank's enormous repository of research and data, said McMillan.

# ChatGPT as an interface to financial modeling



I am a Canadian pension fund manager evaluating new ESG policy of divesting from fossil fuel assets. How do I computationally evaluate this strategy for my portfolio?



Evaluating the strategy of divesting from fossil fuel assets as part of your Environmental, Social, and Governance (ESG) policy will require a multi-step approach. You would need to understand the current exposure to fossil fuels, the risks and returns associated with those investments, and the potential impact of divesting. Here's a high-level overview of the steps you might take:

1. **Identify fossil fuel exposure:** Use your portfolio management software to identify all of your holdings that are connected to the fossil fuel industry. This could include not only energy companies, but also financial institutions that heavily invest in fossil fuels, and industrial companies that are heavy users of fossil fuels. You can use industry codes, keywords, and other tools to help identify these holdings.
2. **Quantify current exposure:** Once you've identified the holdings related to fossil fuels, you'll need to quantify your current exposure. This could be as simple as adding up the total value of these holdings, or as complex as calculating the percentage of your total portfolio that these holdings represent.
3. **Analyze historical performance:** Look at the historical performance of your fossil fuel investments. This will give you an idea of the risk and return associated with these investments. Keep in mind that past performance is not a guarantee of future results, but it can provide some insight into potential trends.

4. **Estimate future performance:** Use financial modeling to estimate the future performance of these investments under various scenarios. For example, you could model the impact of increased regulation on fossil fuels, changes in energy prices, or advancements in renewable energy technology. This can help you understand the potential risks and returns associated with these investments.
5. **Identify alternative investments:** If you're considering divesting from fossil fuels, you'll need to identify alternative investments that align with your ESG policy. This could include renewable energy companies, green bonds, or other ESG-focused investments. Use financial modeling to estimate the potential risk and return of these alternative investments.
6. **Model the impact of divestment:** Finally, use your portfolio management software to model the impact of divesting from fossil fuels and investing in your identified alternatives. This should include the potential impact on your portfolio's risk and return, as well as any transaction costs associated with divesting and reinvesting.
7. **Monitor and Adjust:** Once you've implemented your new strategy, it's important to continuously monitor your portfolio's performance and make adjustments as needed. This includes regularly reassessing your portfolio's exposure to fossil fuels and the performance of your alternative investments.

Keep in mind that this is a high-level overview and the actual process could be much more complex, depending on the size and complexity of your portfolio. It's also important to consider the potential social and environmental impact of your investments, in addition to their financial performance.

# ChatGPT as an interface to financial modeling



I do not want to disclose my portfolio holdings. Which hypothetical portfolio of Canadian equities I can use?



If you're looking for a hypothetical portfolio of Canadian equities, the S&P/TSX 60 could be a good starting point. The S&P/TSX 60 Index is a stock market index of 60 large companies listed on the Toronto Stock Exchange (TSX). This index represents a broad cross-section of industries in the Canadian economy, and it's often used as a benchmark for Canadian large-cap equities.

The S&P/TSX 60 Index includes companies from various sectors including financials, energy, materials, industrials, healthcare, and technology, among others. As of my knowledge cutoff in September 2021, some of its largest constituents were Shopify, Royal Bank of Canada, Toronto-Dominion Bank, Enbridge, and Canadian National Railway.

Since the index includes a significant portion of energy companies, which are often involved in fossil fuels, it could serve as a good hypothetical portfolio for your analysis.

Keep in mind that real-world portfolio management would also involve considerations such as transaction costs, tax implications, and liquidity constraints, which wouldn't be present in a hypothetical portfolio. But for the purposes of modeling and analysis, a hypothetical portfolio like the S&P/TSX 60 can be very useful.

Once you have your hypothetical portfolio, you can follow the steps outlined in the previous response to analyze the impact of a fossil fuel divestment strategy.



Can I use S&P/TSX Composite Index instead?

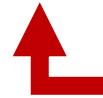


Absolutely, the S&P/TSX Composite Index could also serve as a great hypothetical portfolio for your analysis. The S&P/TSX Composite is broader than the S&P/TSX 60 and includes over 200 of the most significant companies on the Toronto Stock Exchange.

The S&P/TSX Composite Index is often used as a benchmark for the overall Canadian equity market. It covers roughly 95% of the Canadian equities market and represents a variety of sectors.

Using the S&P/TSX Composite Index might provide a more comprehensive view of the market and a more diversified portfolio for your analysis. It would still include a significant portion of energy and related companies, allowing you to evaluate the impact of a fossil fuel divestment strategy.

However, it's important to note that with a larger and more diverse portfolio, the analysis might be more complex and time-consuming. You would need to identify and analyze a larger number of holdings for their exposure to fossil fuels. Nonetheless, the steps you would follow would be similar to those outlined in the previous responses.



get a list of tickers in S&P/TSX Composite Index from Wikipedia and get historical data from Yahoo Finance

# ChatGPT as an interface to financial modeling



Which of the following industries are fossil fuel industries? 'REIT—Office' 'Gold' 'Utilities—Renewable' 'Metal Fabrication'  
 'Asset Management' 'Banks—Regional' 'Banks—Diversified'  
 'Specialty Industrial Machinery' 'Drug Manufacturers—Specialty & Generic'  
 'Telecom Services' 'Utilities—Diversified' 'Advertising Agencies'  
 'Resorts & Casinos' 'Rental & Leasing Services' 'Agricultural Inputs'  
 'Aerospace & Defense' 'Oil & Gas E&P' 'Railroads' 'Electronic Components'  
 'Travel Services' 'Oil & Gas Integrated' 'Security & Protection Services'  
 'Real Estate Services' 'Information Technology Services'  
 'Oil & Gas Midstream' 'Utilities—Regulated Electric' 'REIT—Industrial'  
 'Copper' 'Paper & Paper Products' 'Lodging' 'REIT—Healthcare Facilities'  
 'Other Industrial Metals & Mining' 'Apparel Manufacturing'  
 'Packaged Foods' 'Insurance—Property & Casualty' 'Silver'  
 'Insurance—Life' 'Semiconductors' 'Farm & Heavy Construction Machinery'  
 'Diagnostics & Research' 'Software—Application' 'Beverages—Non-Alcoholic'  
 'Specialty Chemicals' 'Specialty Business Services' 'CATV Systems'  
 'Engineering & Construction' 'Insurance—Diversified'  
 'Household & Personal Products' 'Trucking' 'Steel' 'Specialty Retail'  
 'Waste Management' 'Lumber & Wood Production' 'Auto Parts'



The industries you've listed cover a wide range of sectors. From the list, the industries directly involved in the extraction, production, and distribution of fossil fuels are:

- 'Oil & Gas E&P' (Exploration & Production)
- 'Oil & Gas Integrated'
- 'Oil & Gas Midstream'

These industries are directly related to the fossil fuel industry as they are involved in the exploration, extraction, refining, and transportation of oil and gas.



Please include 'Other Industrial Metals & Mining' industry in addition to the three industries that you suggested and give me an updated list of industries.



Absolutely, I can include 'Other Industrial Metals & Mining' in the list of industries that you might consider divesting from as part of your fossil fuel and related industries divestment strategy. Here's the updated list:

- 'Oil & Gas E&P' (Exploration & Production)
- 'Oil & Gas Integrated'
- 'Oil & Gas Midstream'
- 'Other Industrial Metals & Mining'

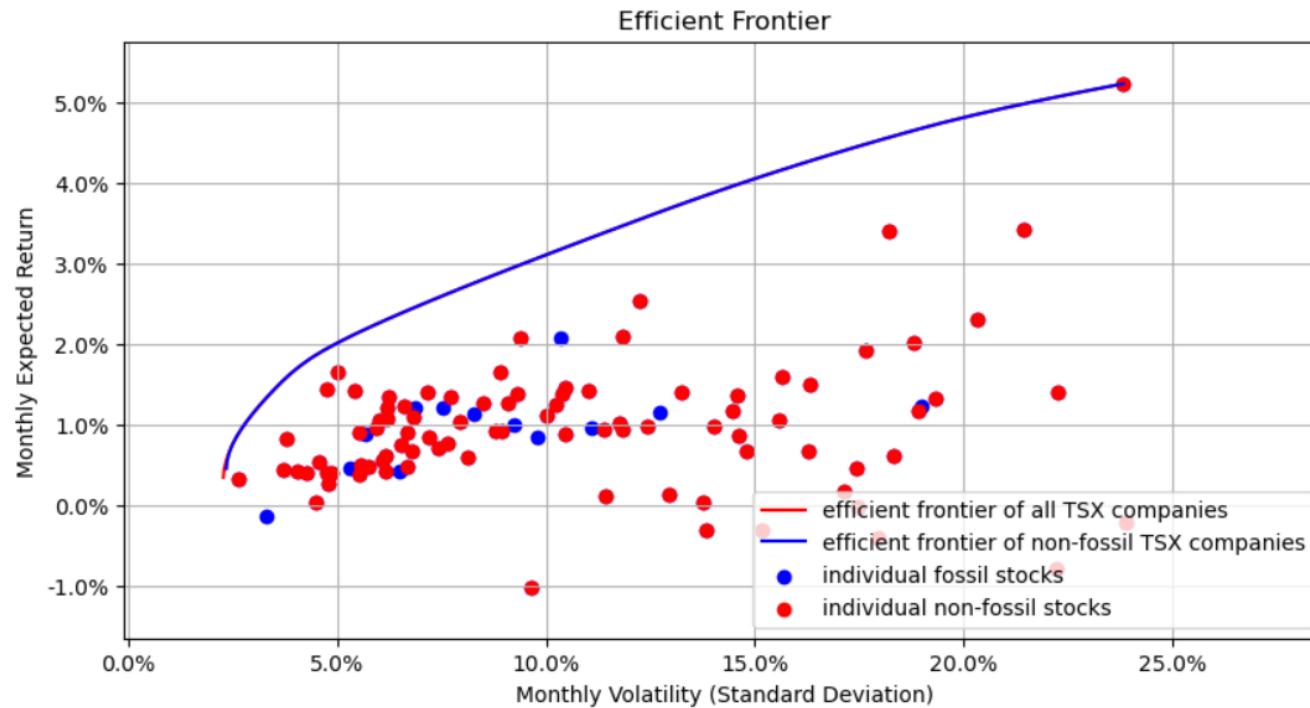
Remember, each industry is diverse, and companies within these industries may have varying levels of exposure to fossil fuels. Therefore, a detailed analysis of individual companies might be necessary to fully understand their specific involvement in the fossil fuel industry or related activities.



compute efficient frontiers with and without fossil fuel stocks, compare efficient frontiers

# ChatGPT as an interface to financial modeling

## Fossil vs. non-fossil stocks (ESG)



# ChatGPT as an interface to financial modeling



While comparing efficient frontiers with and without fossil fuel assets, we found that Spearman correlation coefficient is 99.999999% with p-value of 0.0. Please interpret this results.



ask ChatGPT to compare two efficient frontiers and interpret results



The Spearman correlation coefficient measures the strength and direction of the monotonic relationship between two datasets. In your case, it is measuring the correlation between the efficient frontiers of two portfolios: one with fossil fuel assets and one without.

A Spearman correlation coefficient of 99.999999% indicates an extremely strong positive correlation between the two datasets. This suggests that the efficient frontiers of the two portfolios (with and without fossil fuel assets) move almost perfectly together. In other words, changes in the risk and return trade-off for one portfolio are very closely associated with changes in the risk and return trade-off for the other portfolio.

The p-value in a correlation analysis tests the null hypothesis that the correlation coefficient is not significantly different from zero (no correlation). A p-value of 0.0 suggests that the probability of observing the calculated Spearman correlation coefficient (or a more extreme correlation) by chance alone, assuming the null hypothesis is true, is essentially zero. This means that the correlation is statistically significant and not due to random chance.

In summary, the analysis suggests a very strong, statistically significant correlation between the efficient frontiers of the two portfolios. This means that excluding fossil fuel assets from the portfolio does not significantly change the overall risk-return trade-off in the portfolio, according to the efficient frontier analysis. This is a potentially important finding for your ESG divestment strategy, as it suggests that divesting from fossil fuels may not significantly impact the portfolio's performance. However, it's important to consider other factors, such as the specific risk tolerance of your fund and the broader implications of fossil fuel investments, when making your final decision.



# Mean-Variance Portfolio Modeling

# Terminology

## ■ Random variable

- $y$  is a random variable, takes finite number of values,  $y_j$  for  $j=1,2,\dots,m$
- a probability (associated with each event) represents the relative chance of an occurrence of  $y_j$  such that

$$\sum_{j=1}^m p_j = 1 \text{ and } p_j \geq 0 \text{ for } j = 1, 2, \dots, m \quad \text{relative frequency}$$

## ■ Expected value (mean value or mean)

- average value obtained by regarding probabilities as frequencies

$$\bar{y} = \mathbb{E}(y) = \sum_{j=1}^m p_j y_j \quad \text{finite number of possibilities}$$

## ■ Variance - measure of possible deviation from the mean

$$\begin{aligned} \text{var}(y) &= \mathbb{E} [(y - \bar{y})^2] \\ &= \mathbb{E}(y^2) - 2\mathbb{E}(y) \cdot \bar{y} + \bar{y}^2 \\ &= \mathbb{E}(y^2) - \bar{y}^2 \end{aligned}$$

Expected value of squared variable  
how much  $y$  tends to vary from its mean

# Terminology

## ■ Covariance of two random variables $y_1$ and $y_2$

$$\begin{aligned}\text{cov}(y_1, y_2) &= \sigma_{12} = \mathbb{E} [(y_1 - \bar{y}_1)(y_2 - \bar{y}_2)] \\ &= \mathbb{E}(y_1 y_2) - \bar{y}_1 \bar{y}_2 \\ \sigma_{12} &= \sigma_{21}\end{aligned}$$

## ■ Correlation between two random variables $y_1$ and $y_2$

$$\text{cor}(y_1, y_2) = \frac{\text{cov}(y_1, y_2)}{\sqrt{\text{var}(y_1) \cdot \text{var}(y_2)}} \Rightarrow \rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

$\rho_{12} > 0 \Rightarrow$  positively correlated

sign defines direction of the relationship

$\rho_{12} = 0 \Rightarrow$  uncorrelated (independent)

$\rho_{12} < 0 \Rightarrow$  negatively correlated

$\rho_{12} = \pm 1 \Rightarrow$  perfect +(-) correlation

## Asset returns

- **Asset:** investment instrument that can be bought and sold
  - uncertain asset prices – the return is random uncertainty described in probabilistic terms
- **Asset return:** you purchase an asset today and sell it next year
  - total return on this investment is defined as

$$\text{total return} = \frac{\text{amount received}}{\text{amount invested}} \Leftrightarrow R = \frac{v_1}{v_0}$$

- Rate of return

$$\text{rate of return} = \frac{\text{amount received} - \text{amount invested}}{\text{amount invested}}$$

$$r = \frac{v_1 - v_0}{v_0}$$

- Rate of return acts much like an interest rate

$$R = 1 + r \Leftrightarrow v_1 = (1 + r)v_0$$

# Returns

	Returns	Returns
Time Period	$r_1$	$r_2$
1	6	8
2	4	2
3	7	11
4	3	-1
5	8	12
6	2	-2
7	11	13
8	-1	-3
MEAN	5	5
VARIANCE	3.77964	6.36209



## Portfolio return

Consider  $n$  risky assets to form a portfolio and invest  $V_0$  among  $n$  assets

- Select a position in the  $i$ -th asset (number of units)

$$x_i \text{ for } i = 1, \dots, n \text{ such that } V_0 = \sum_{i=1}^n v_{0i} \cdot x_i = \sum_{i=1}^n x_{0i}$$

- The amounts invested can be expressed as fractions of the total investment

$$x_{0i} = v_{0i} \cdot x_i = w_i \cdot V_0, i = 1, \dots, n$$

where  $w_i$  is fraction or weight of asset  $i$  in portfolio

- Let the total return of asset  $i$  be  $R_i$ . The amount of money gained at the end of the period on asset  $i$  is

$$R_i x_{0i} = R_i w_i V_0, \quad \sum_{i=1}^n w_i = 1$$

- The total return of the portfolio

$$R_p = \frac{\sum_{i=1}^n R_i w_i V_0}{V_0} = \sum_{i=1}^n R_i w_i$$

- The rate of return of portfolio

$$r_p = \sum_{i=1}^n r_i w_i$$

## Portfolio return

- Both the total return and the rate of return of the portfolio of assets are equal to the weighted sum of the corresponding individual asset returns, with the weight of an asset being its relative weight in the portfolio

$$R_p = \sum_{i=1}^n R_i w_i, \quad r_p = \sum_{i=1}^n r_i w_i \Leftrightarrow \sum_{i=1}^n w_i = 1$$

- Expected return of the portfolio** is the weighted sum of the individual expected rates of return

$$r_p = \sum_{i=1}^n r_i w_i \Rightarrow \mathbb{E}(r_p) = \sum_{i=1}^n w_i \cdot \underbrace{\mathbb{E}(r_i)}_{\mu_i}$$

## Example

- Consider a portfolio of a risk-free asset and two risky stocks and three equally likely states

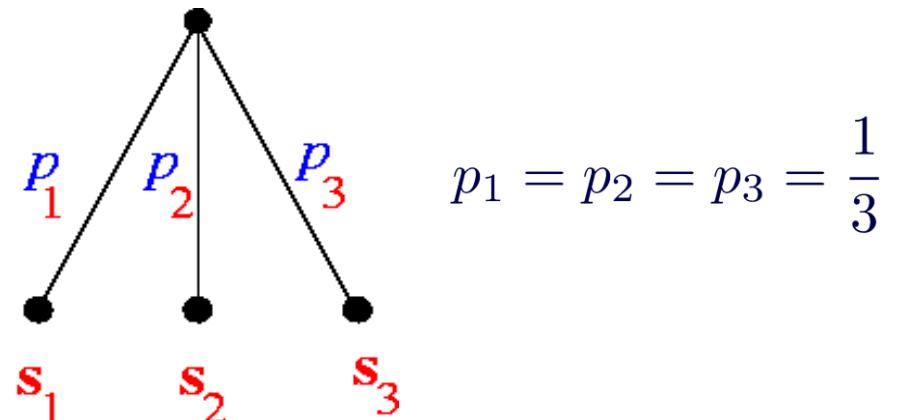
States (s)	T-Bill Return%	Stock A Return%	Stock B Return %
	(i=1)	(i=2)	(i=3)
s <sub>1</sub> "boom"	5	16	3
s <sub>2</sub> "normal"	5	10	9
s <sub>3</sub> "recession"	5	1	15

- Find the expected return of the portfolio formed by 3 assets

- Scenario tree with 3 states

root represents today

future uncertainty is discretized  
by 3 events (states)



Source: P. Parpas. Computational Finance

## Example: expected returns

- The expected return of risk free asset is

$$\mathbb{E}[r_1] = 5\%$$

- For stock A and B, expected return of the portfolio

$$\mathbb{E}[r_2] = \frac{1}{3}[16 + 10 + 1] = 9\%$$

$$\mathbb{E}[r_3] = \frac{1}{3}[3 + 9 + 15] = 9\%$$

- The expected return of the portfolio

$$\mathbb{E}[r_p] = 5 \cdot w_1 + 9 \cdot w_2 + 9 \cdot w_3$$

- For equally weighted portfolio

$$w_1 = w_2 = w_3 = \frac{1}{3} \Rightarrow \mathbb{E}[r_{ew}] = \frac{1}{3}[5 + 2 \cdot 9] = 7.667\%$$

$$\mathbb{E}(\mathbf{w}) = \boldsymbol{\mu}^T \mathbf{w}$$

# Risk

- Risk: a chance that investment's actual return will be different than expected – includes losing some or all of original investment
  - risk averse and risk-seeking (loving)
- Systematic risk: influences a large number of assets, such as political events – it may be impossible to protect yourself against this risk
- Unsystematic risk: (specific risk) affects very small number of assets. For example, news that affects a specific stock such as a sudden strike
  - diversification is the only way to protect yourself
- Others: credit (default) risk, foreign exchange risk, interest rate risk, political risk

# Diversification

- Risk management technique
- Mixes a wide variety of investments within a portfolio
- Objective is to minimize the impact that any one security will have on overall performance of the portfolio
- For the best diversification
  - portfolio should be spread among many different assets; cash, stocks, bonds
  - securities should vary in risk and
  - securities should vary by industry to minimize unsystematic risk to different companies

# Portfolio risk

For  $n$  securities, portfolio risk – variance of the portfolio return

Expected value of squared variable – how much rate of return of portfolio tends to vary from its mean

$$\begin{aligned}
 \sigma_p^2 &= \mathbb{E} [(r_p - \bar{r}_p)^2] = \mathbb{E} \left[ \left( \sum_{i=1}^n r_i w_i - \sum_{i=1}^n \mu_i w_i \right)^2 \right] \\
 &= \mathbb{E} \left[ \left( \sum_{i=1}^n (r_i - \mu_i) w_i \right) \left( \sum_{j=1}^n (r_j - \mu_j) w_j \right) \right] \\
 &= \mathbb{E} \left[ \sum_{i,j=1}^n (r_i - \mu_i)(r_j - \mu_j) w_i w_j \right] \\
 &= \sum_{i,j=1}^n \sigma_{ij} w_i w_j \\
 \sigma_p^2 &= \sum_{i=1}^n \sigma_i^2 w_i^2 + \sum_{i=1}^n \sum_{j=1, i \neq j}^n \sigma_{ij} w_i w_j
 \end{aligned}$$

## Example: variance of portfolio

- The variance and covariance are calculated as

$$\sigma_1^2 = \text{var}[r_1] = 0$$

$$\sigma_2^2 = \text{var}[r_2] = \frac{1}{3} [(16 - 9)^2 + (10 - 9)^2 + (1 - 9)^2] = \frac{114}{3}$$

$$\sigma_3^2 = \text{var}[r_3] = \frac{1}{3} [(3 - 9)^2 + (9 - 9)^2 + (15 - 9)^2] = \frac{72}{3}$$

$$\sigma_{23} = \text{cov}[r_2, r_3] = \frac{1}{3} [7 \cdot (-6) + 1 \cdot 0 + (-8) \cdot 6] = \frac{-90}{3}$$

$$\sigma_{21} = \text{cov}[r_1, r_2] = 0$$

$$\sigma_{31} = \text{cov}[r_1, r_3] = 0$$

## Example: variance of portfolio

- The portfolio risk is

$$\begin{aligned}
 \sigma_p^2 &= \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}^T \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\
 &= \sigma_1^2 w_1^2 + \sigma_2^2 w_2^2 + \sigma_3^2 w_3^2 + \sigma_{12} w_1 w_2 + \sigma_{13} w_1 w_3 \\
 &\quad + \sigma_{21} w_2 w_1 + \sigma_{23} w_2 w_3 + \sigma_{31} w_3 w_1 + \sigma_{32} w_3 w_2
 \end{aligned}$$

$$\begin{aligned}
 \sigma_p^2 &= \sigma_2^2 w_2^2 + \sigma_3^2 w_3^2 + 2\sigma_{23} w_2 w_3 \\
 &= \frac{1}{3}(114 \cdot w_2^2 + 72 \cdot w_3^2 - 180 \cdot w_1 w_2)
 \end{aligned}$$

For equally weighted portfolio  $w_1 = w_2 = w_3 = \frac{1}{3} \Rightarrow \sigma_{\text{ew}}^2 = \frac{6}{27}$

$\text{var}(\mathbf{w}) = \mathbf{w}^T Q \mathbf{w}$

## Short sale

- It is possible to sell an asset that you do not own through the short selling (shorting) the asset
- In order to implement short selling
  - borrow the asset from the owner such as a brokerage firm
  - sell the borrowed asset to someone else, receiving an amount  $v_0$
  - at a later date purchase the asset for  $v_1$ , and return the asset to lender
    - if  $v_1 < v_0$ , then  $profit = v_0 - v_1$
- The short selling is profitable if the asset price declines
- Risky since the potential for loss is unlimited – if asset value increases
  - if  $v_1 > v_0$ , then  $loss = v_1 - v_0$
- Although prohibited within certain financial institutions, not universally forbidden

## Example: short sale

Assume that company X has a poor outlook next month. The stock is now trading at £65, but you see it trading much lower than this price in the future

- You decide to take risk and trade on this stock
- Two things can happen: stock price can go up or down

Action taken	Predicted Stock price £40		Predicted Stock price £85	
	Price(£)	Cost(£)	Price(£)	Cost(£)
Borrowed 100 shares of X & sell	65	6500	65	6500
Bought back 100 shares of X	40	-4000	85	-8500
Profit/Loss		2500		-2000

RISKY: no guarantee that the price of a short stock will drop



# Optimal portfolio

## Portfolio theory

- effects of investor decisions on security prices
- relationship that should exist between the returns and risk
  - possible to have different portfolios varying levels of risk and return
  - decide how much risk you can handle and allocate (or diversify) portfolio according to this decision

## Harry Markowitz - 1990 Nobel Prize winner in Economic Sciences

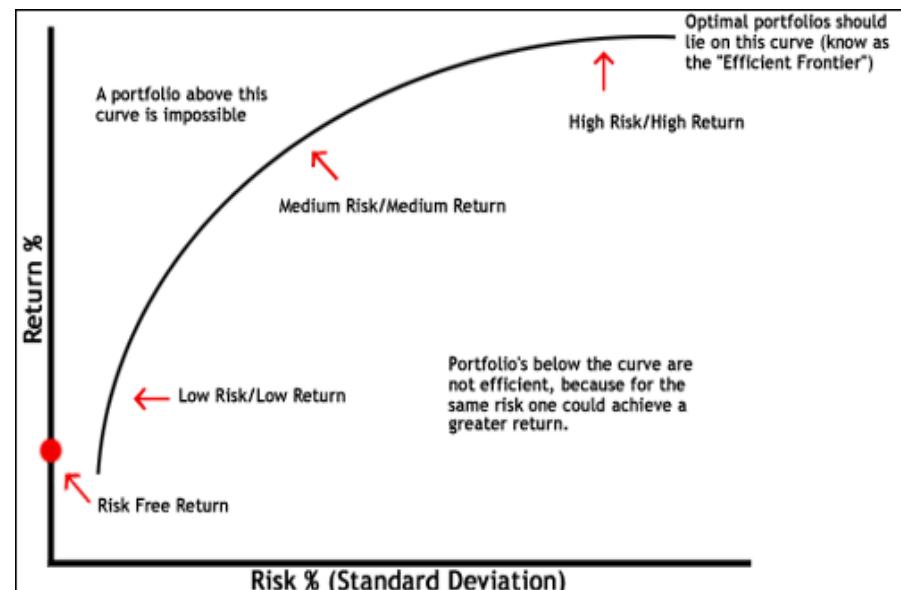
- published in 1952 Journal of Finance titled “Portfolio Selection”
- formalized an integrated theory of diversification, portfolio risks, efficient and inefficient portfolios

**Maximize (expected return of portfolio)**

(weighted sum of individual expected rates of return)

**Minimize (portfolio risk)**

(expected value of squared variable - how much rate of return of portfolio tends to vary from its mean)

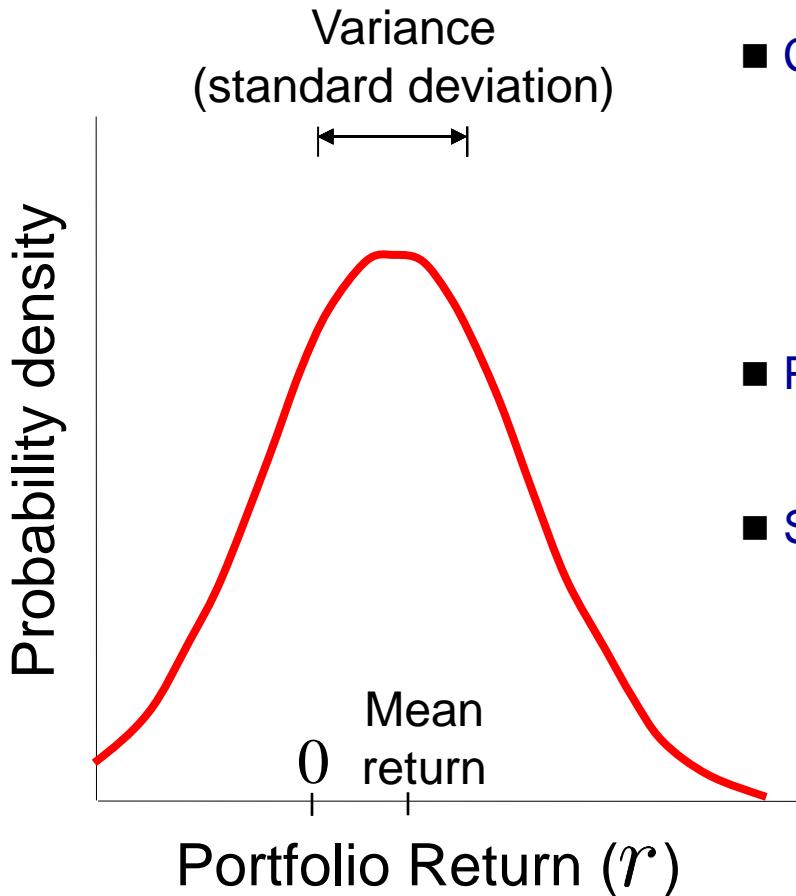




# Mean-Variance Portfolio Selection (part 1)

# Measuring risk and portfolio selection

Portfolio return distribution ( $F_r$ ) is assumed to be Gaussian (Normal)



- Consider  $n$  assets with random returns:
  - $w_i, i = 1, \dots, n$  proportion invested in asset  $i$
  - $\mu_i, \sigma_i$  exp. return and standard dev. of the return of asset  $i$
  - $Q_{ij} = \rho_{ij}\sigma_i\sigma_j$  variance-covariance matrix
- Portfolio expected return and variance:
$$\mathbb{E}(\mathbf{w}) = \boldsymbol{\mu}^T \mathbf{w} \quad \text{var}(\mathbf{w}) = \mathbf{w}^T \mathbf{Q} \mathbf{w}$$
- Set of admissible portfolios:
$$\mathcal{F} = \{\mathbf{w} : \sum_i w_i = 1, \mathbf{w} \geq 0\}$$

## Portfolio selection

- Consider  $n$  assets with random returns:
  - $w_i, i = 1, \dots, n$  proportion of total funds invested in asset  $i$
  - $\mu_i, \sigma_i$  expected return and standard deviation of the return of asset  $i$
  - $\rho_{ij}$  correlation coefficient of  $i$ 's and  $j$ 's returns
  - $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$  vector of expected returns
  - $Q_{ij} = \rho_{ij}\sigma_i\sigma_j$  variance-covariance matrix (PSD)

- Expected return and variance of the resulting portfolio:

$$\mathbb{E}(\mathbf{w}) = \mu_1 w_1 + \dots + \mu_n w_n = \boldsymbol{\mu}^T \mathbf{w}$$

$$\text{var}(\mathbf{w}) = \sum_{i,j} \rho_{ij} \sigma_i \sigma_j w_i w_j = \mathbf{w}^T Q \mathbf{w}$$

- Set of admissible portfolios:

$$\mathcal{F} = \{\mathbf{w} : A\mathbf{w} = \mathbf{b}, \sum_i w_i = 1, \mathbf{w} \geq 0\}$$

# Portfolio selection

A feasible portfolio  $\mathbf{w}$  is efficient if it has:

- maximal expected return among all portfolios with the same variance,
- minimum variance among all portfolios with the same expected return.

Mean-variance optimization (Markowitz, 1952):

## Alternative formulations

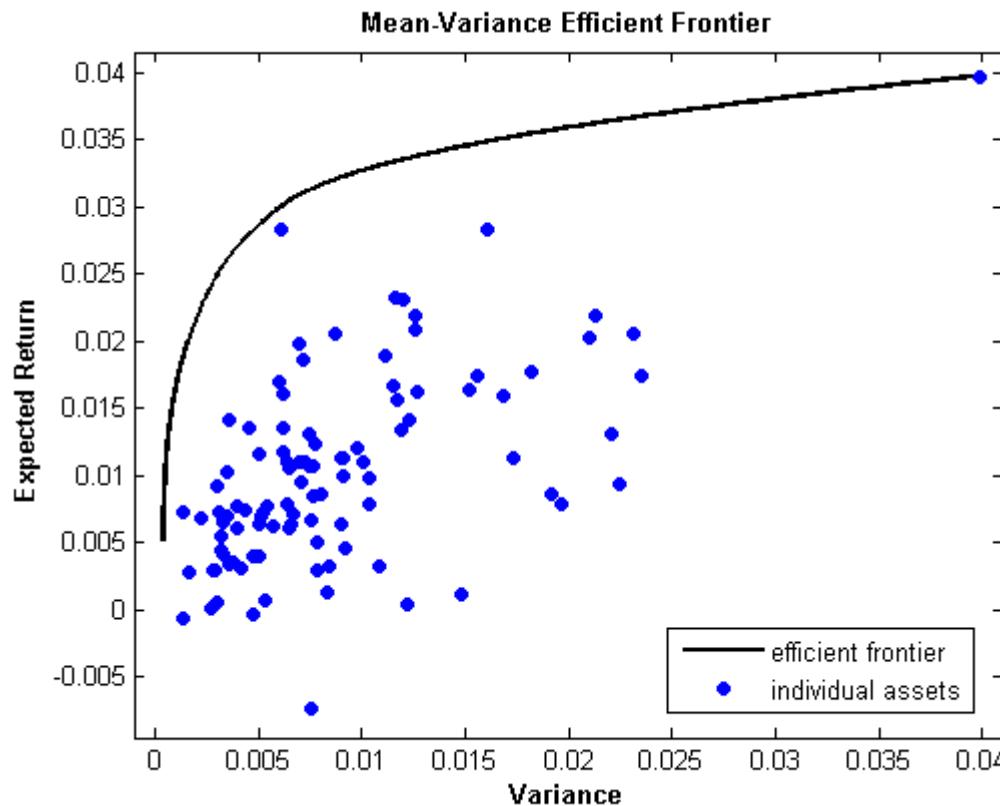
$$\begin{array}{ll}
 \max_{\mathbf{w}} & \boldsymbol{\mu}^T \mathbf{w} \\
 \text{s.t.} & \mathbf{w}^T \mathbf{Q} \mathbf{w} \leq \varepsilon_V \\
 & \mathbf{A} \mathbf{w} = \mathbf{b} \\
 & \sum_i w_i = 1 \\
 & \mathbf{w} \geq 0
 \end{array}
 \quad
 \begin{array}{ll}
 \min_{\mathbf{w}} & \mathbf{w}^T \mathbf{Q} \mathbf{w} \\
 \text{s.t.} & \boldsymbol{\mu}^T \mathbf{w} \geq \varepsilon_R \\
 & \mathbf{A} \mathbf{w} = \mathbf{b} \\
 & \sum_i w_i = 1 \\
 & \mathbf{w} \geq 0
 \end{array}
 \quad
 \xrightarrow{\hspace{1cm}}
 \begin{array}{ll}
 \min_{\mathbf{w}} & -\lambda \boldsymbol{\mu}^T \mathbf{w} + \mathbf{w}^T \mathbf{Q} \mathbf{w} \\
 \text{s.t.} & \mathbf{A} \mathbf{w} = \mathbf{b} \\
 & \sum_i w_i = 1 \\
 & \mathbf{w} \geq 0
 \end{array}$$

Solving for all the values of  $\varepsilon_V$ ,  $\varepsilon_R$ , or  $\lambda$  gives efficient portfolios:

$$\forall \varepsilon_V \in [\varepsilon_{V_{\min}}, \varepsilon_{V_{\max}}] \quad \forall \varepsilon_R \in [\varepsilon_{R_{\min}}, \varepsilon_{R_{\max}}] \quad \forall \lambda \in [0, \lambda_{\max}]$$

# Portfolio selection

Portfolio optimization problem – efficient frontier:





# Overview of Optimization Techniques

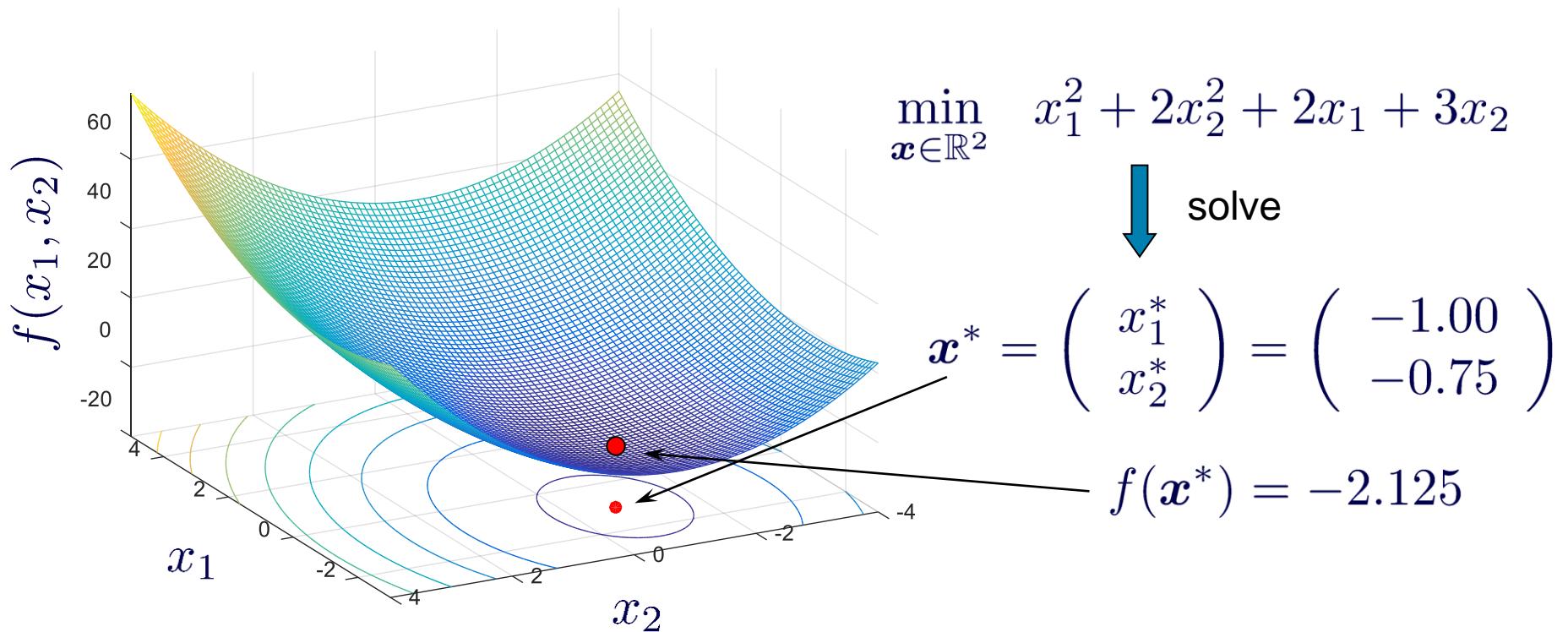
# Optimization

- **Optimization problem**

$$\begin{aligned} & \text{minimize}_{\boldsymbol{x}} \quad f(\boldsymbol{x}) \\ & \text{subject to} \quad \boldsymbol{x} \in \Omega \end{aligned}$$

- **Examples:**

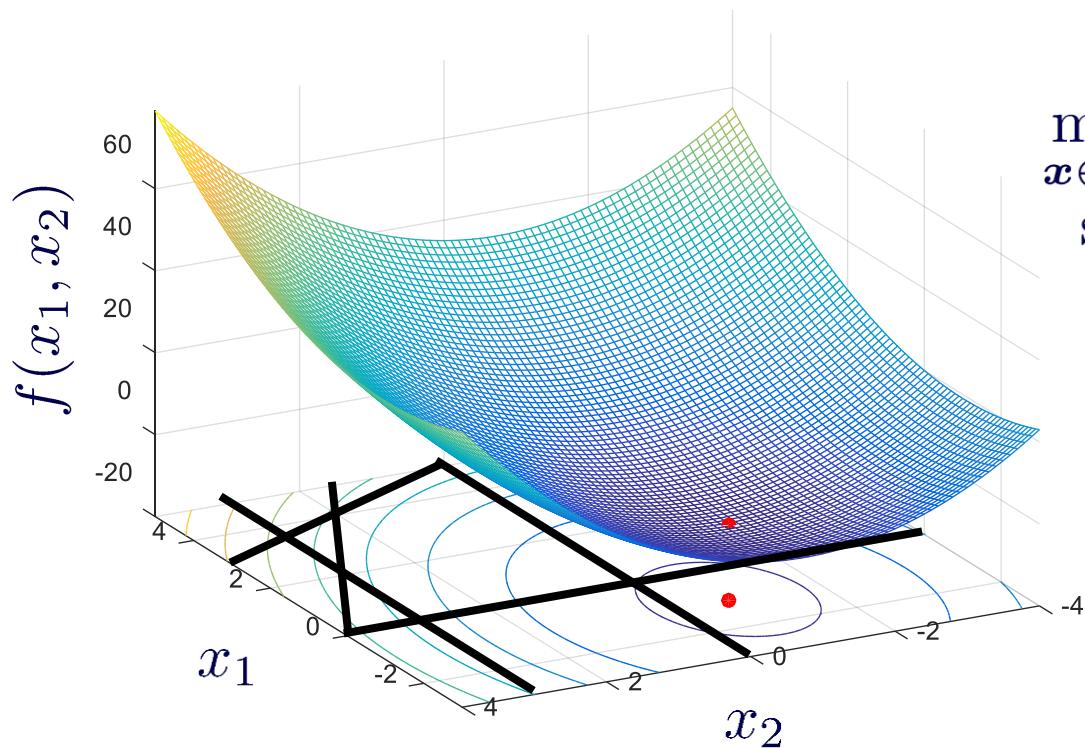
- Minimize **cost**
- Maximize **profit**



# Optimization

- **Optimization problem**

$$\begin{aligned} & \text{minimize}_{\boldsymbol{x}} && f(\boldsymbol{x}) \\ & \text{subject to} && \boldsymbol{x} \in \Omega \end{aligned}$$

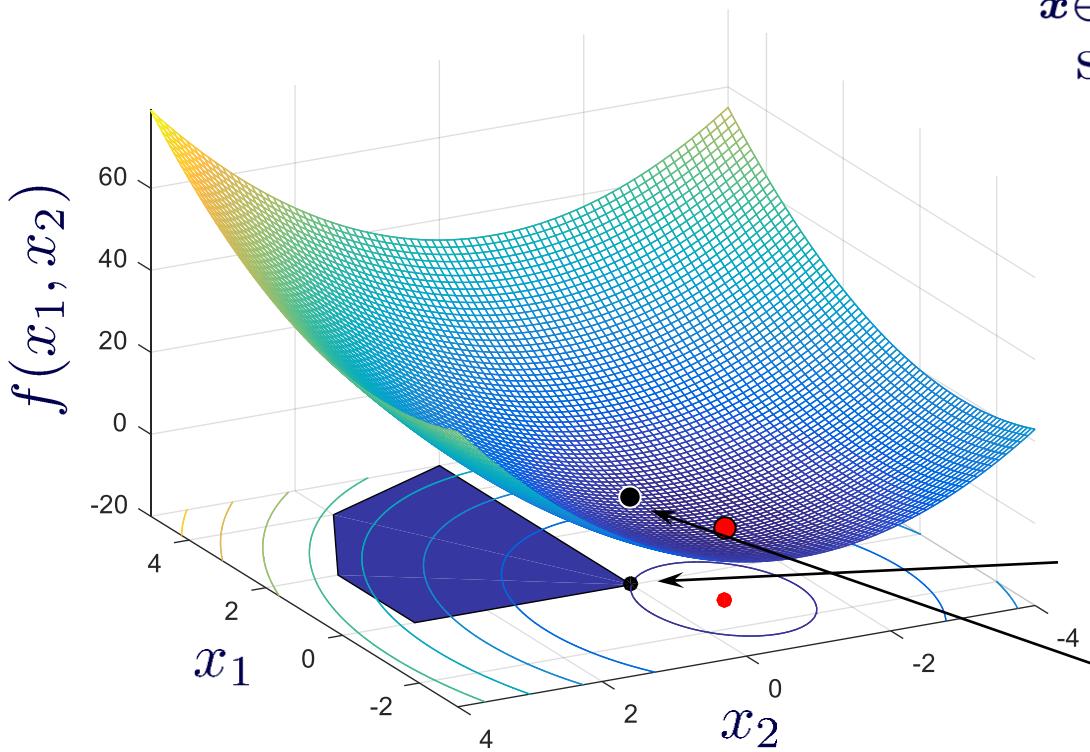


$$\begin{aligned} & \min_{\boldsymbol{x} \in \mathbb{R}^2} && x_1^2 + 2x_2^2 + 2x_1 + 3x_2 \\ & \text{s.t.} && x_1 + 2x_2 \leq 8 \\ & && 2x_1 + x_2 \leq 10 \\ & && x_2 \leq 3 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

# Optimization

- **Optimization problem**

$$\begin{aligned} & \text{minimize}_{\boldsymbol{x}} \quad f(\boldsymbol{x}) \\ & \text{subject to} \quad \boldsymbol{x} \in \Omega \end{aligned}$$



$$\min_{\boldsymbol{x} \in \mathbb{R}^2} \quad x_1^2 + 2x_2^2 + 2x_1 + 3x_2$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \\ & 2x_1 + x_2 \leq 10 \\ & x_2 \leq 3 \end{aligned}$$

$$\begin{aligned} & x_1, x_2 \geq 0 \\ & \downarrow \text{solve} \end{aligned}$$

$$\boldsymbol{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

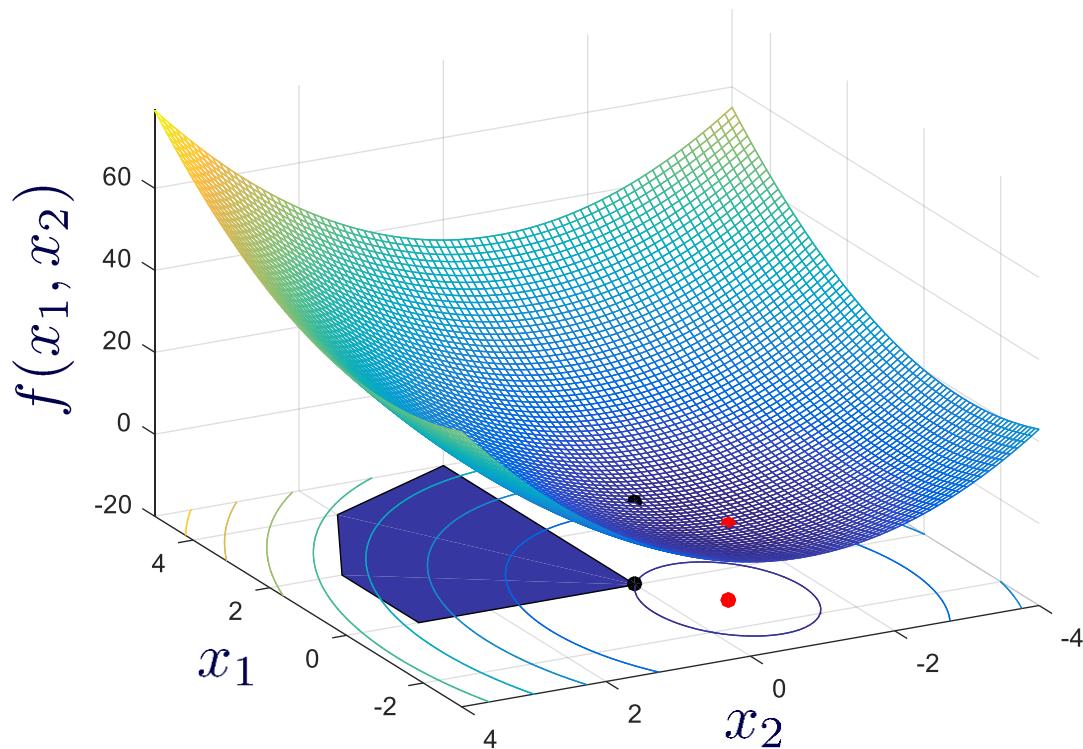
$$f(\boldsymbol{x}^*) = 0$$

# Optimization

- **Optimization problem**

$$\begin{aligned} & \text{minimize}_{\boldsymbol{x}} \quad f(\boldsymbol{x}) \\ & \text{subject to} \quad \boldsymbol{x} \in \Omega \end{aligned}$$

- **Minimizing convex quadratic (QP) objective function over a polyhedron**  
(linear constraints)



$$\begin{aligned} & \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad \mathbf{c}^T \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T Q \boldsymbol{x} \\ & \text{s.t.} \quad \mathbf{l} \leq A \boldsymbol{x} \leq \mathbf{u} \end{aligned}$$

↑ general form

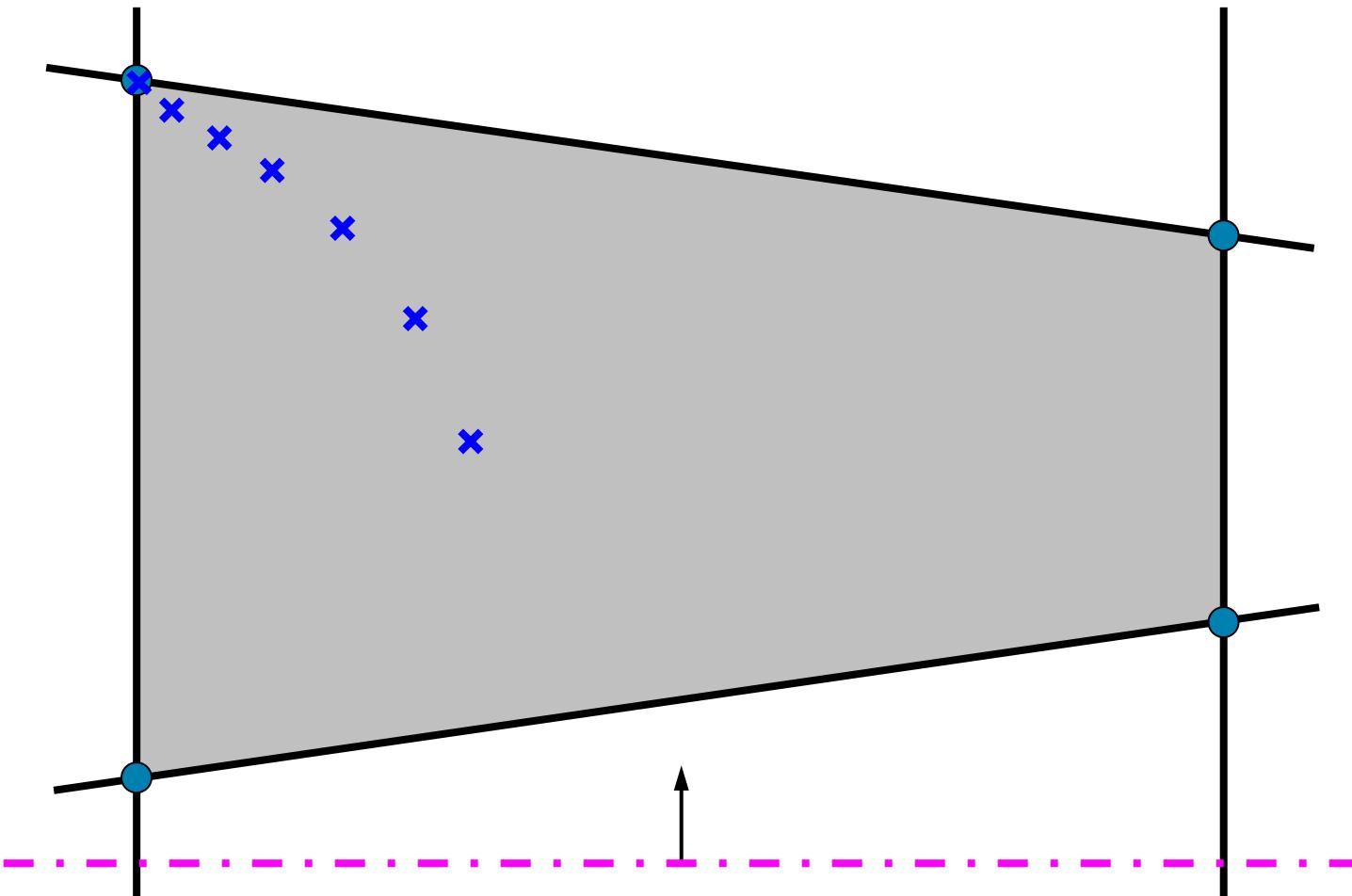
$$\begin{aligned} & \min_{\boldsymbol{x} \in \mathbb{R}^2} \quad x_1^2 + 2x_2^2 + 2x_1 + 3x_2 \\ & \text{s.t.} \quad x_1 + 2x_2 \leq 8 \\ & \quad 2x_1 + x_2 \leq 10 \\ & \quad x_2 \leq 3 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$



# Solving linear optimization problems

- Maximizing/minimizing linear (LP) function over a polyhedron
- Interior Point Methods vs. Simplex-type Methods

$$\begin{array}{ll}\min_{\boldsymbol{x} \in \mathbb{R}^n} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{l} \leq \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{u}\end{array}$$



# Solving linear optimization problems

## ■ Simplex Method – graphical view

$$\begin{aligned}
 \max_{x \in \mathbb{R}^3} \quad & 3x_1 + 2x_2 + 2x_3 \\
 \text{s.t.} \quad & x_1 + x_3 \leq 8 \\
 & x_1 + x_2 \leq 7 \\
 & x_1 + 2x_2 \leq 12 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$



standard form

$$\begin{aligned}
 \max_{x \in \mathbb{R}^3} \quad & 3x_1 + 2x_2 + 2x_3 \\
 \text{s.t.} \quad & x_1 + x_3 + x_4 = 8 \\
 & x_1 + x_2 + x_5 = 7 \\
 & x_1 + 2x_2 + x_6 = 12 \\
 & x_1, x_2, x_3 \geq 0 \\
 & x_4, x_5, x_6 \geq 0
 \end{aligned}$$

# Solving linear optimization problems

## ■ Simplex Method – graphical view

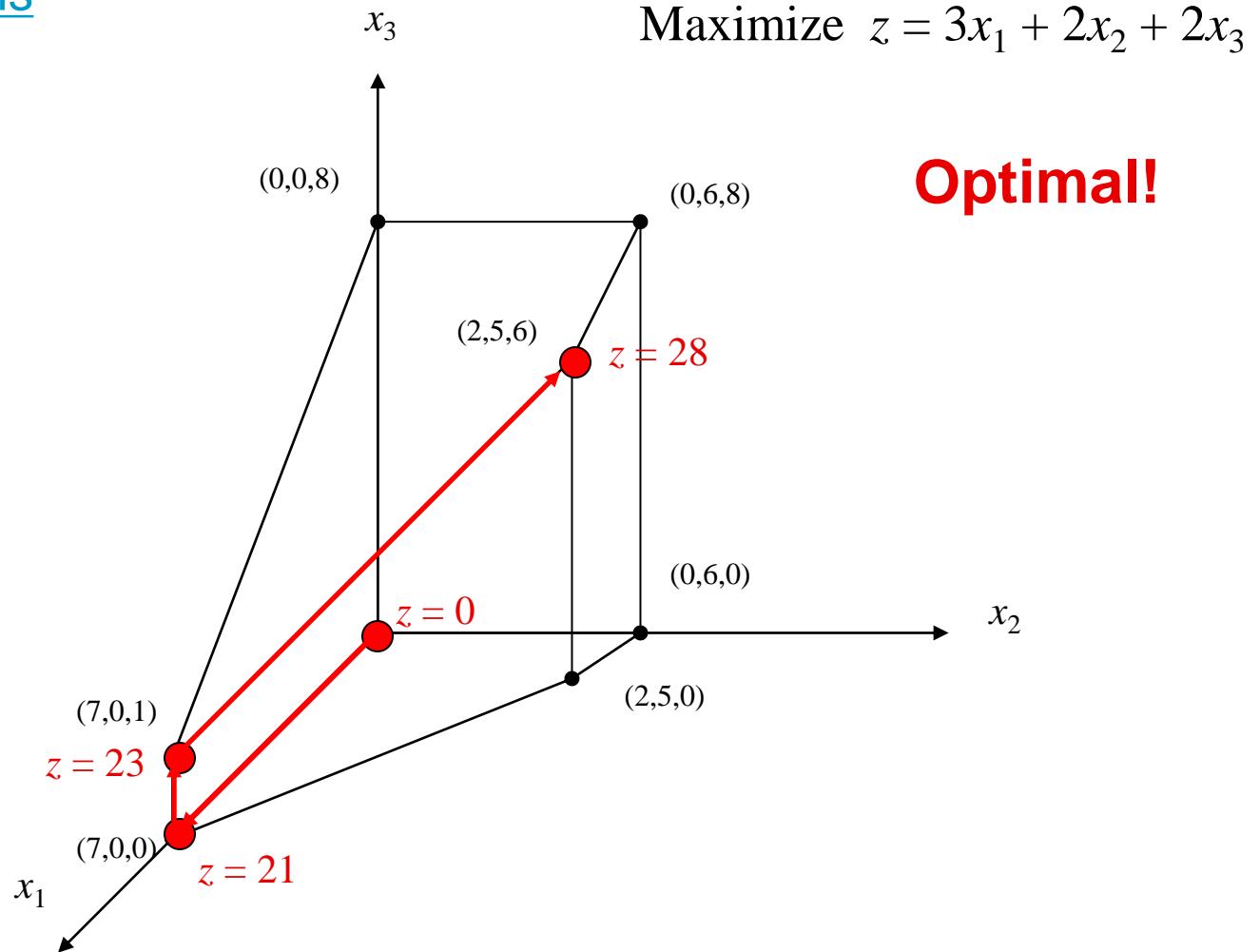
### Current Basis

$(x_4, x_5, x_6)$

$(x_4, x_1, x_6)$

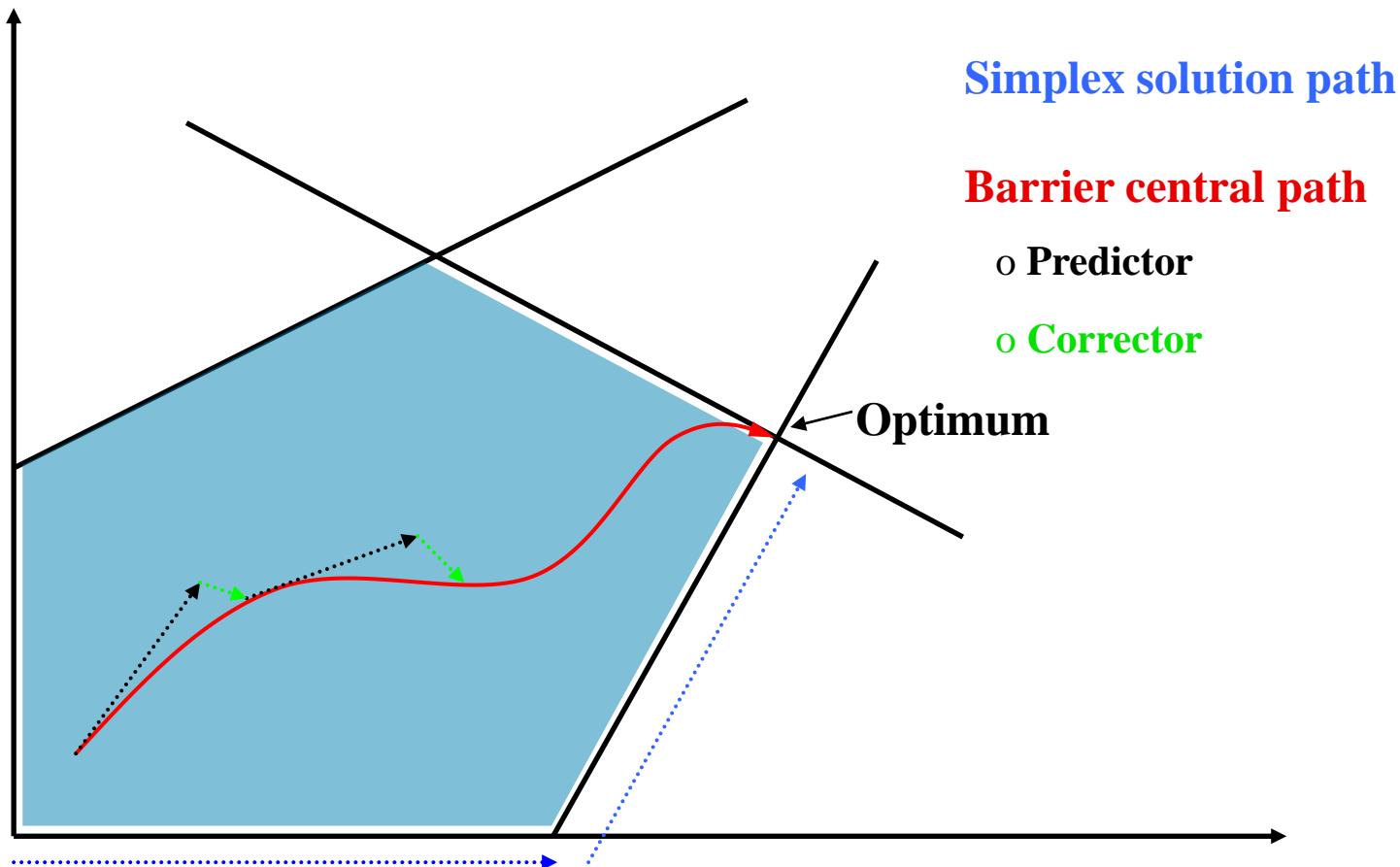
$(x_3, x_1, x_6)$

$(x_3, x_1, x_2)$



# Solving linear optimization problems

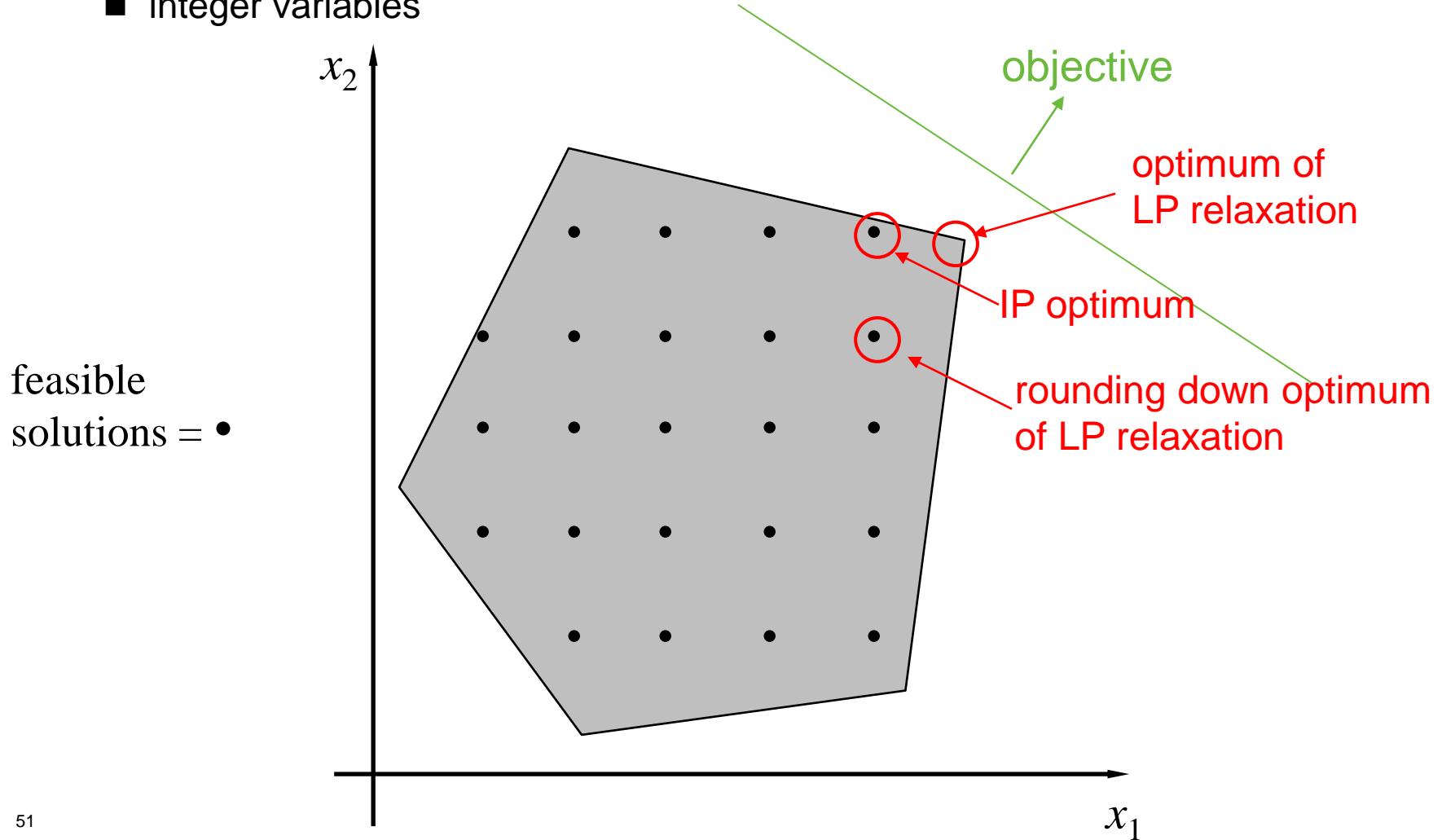
- Interior Point Method (barrier algorithm in CPLEX)



# Solving mixed-integer optimization problems

## ■ Mixed-integer optimization problems (MIP)

- continuous variables
- integer variables



# Quadratic optimization example in Python using CVXPY

```

# Import Python modules
import cvxpy as cp
import numpy as np

# Define linear part of objective
# function and bounds on variables
c = np.array([1., 2., 3.])
ub = np.array([40., np.inf, np.inf])

# Define constraints
A = np.array([[[-1., 1., 1.],
              [1., -3., 1.]]])
b = np.array([20., 30.])

# Define quadratic part of obj. function
Q = np.array([[[-33.0, 6.0, 0.0],
               [6.0, -22.0, 11.5],
               [0., 11.5, -11.0]]])

# Define optimization problem
prob = cp.Problem(cp.Maximize((1/2)*cp.quad_form(x, Q) + c.T@x),
                  [A@x <= b,
                   x >= 0, x <= ub])

# Set CPLEX parameters and optimize the problem
prob.solve(verbose=True)

# Display the solution
print("\nSolution status: ", prob.status)
print("Solution optimal value: ", prob.value)
print("Solution x: ")
print(x.value)

```

$$\begin{aligned}
 \max_x \quad & x_1 + 2x_2 + 3x_3 - 0.5(33x_1^2 + \\
 & 22x_2^2 + 11x_3^2 - 12x_1x_2 - 23x_2x_3) \\
 \text{s.t.} \quad & -x_1 + x_2 + x_3 \leq 20 \\
 & x_1 - 3x_2 + x_3 \leq 30 \\
 & 0 \leq x_1 \leq 40 \\
 & x_2 \geq 0, x_3 \geq 0
 \end{aligned}$$



# Multi-Objective Optimization

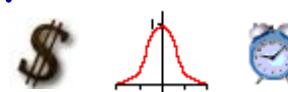
# Multi-objective optimization

- **Multi-objective optimization:** simultaneously optimizing two or more conflicting objectives subject to certain constraints

$$\begin{aligned} \text{minimize} \quad & \{f_1(x), f_2(x), \dots, f_k(x)\} \\ \text{subject to} \quad & x \in \Omega \end{aligned}$$

- Examples:
  - Finance: Minimize **risk** & Maximize **return**
  - Business: Minimize **cost** & Minimize **environmental impact**

- Units of the objectives are typically not the same:  
dollars, probability, units of time, ...



# Multi-objective optimization

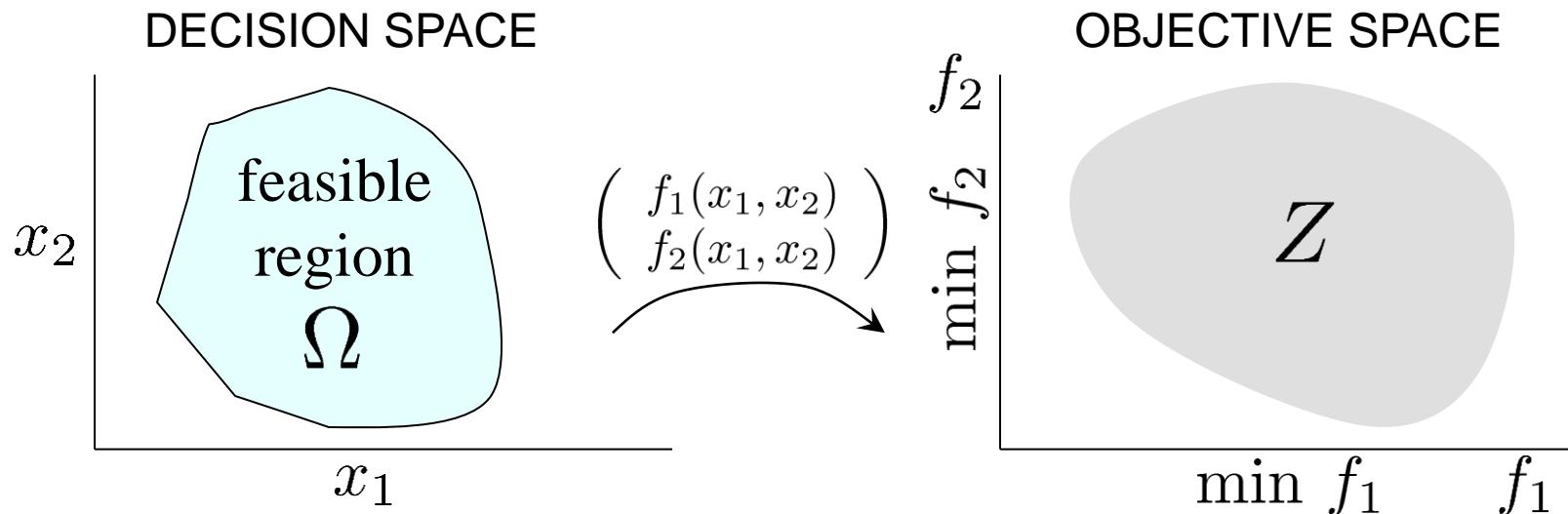
- **Multi-objective optimization:**

$$\begin{aligned} \min \quad & \{f_1(x), f_2(x), \dots, f_k(x)\} \\ \text{s.t} \quad & x \in \Omega \end{aligned}$$

$f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  are (possibly) conflicting objectives and  $\Omega \subseteq \mathbb{R}^n$  is feasible region

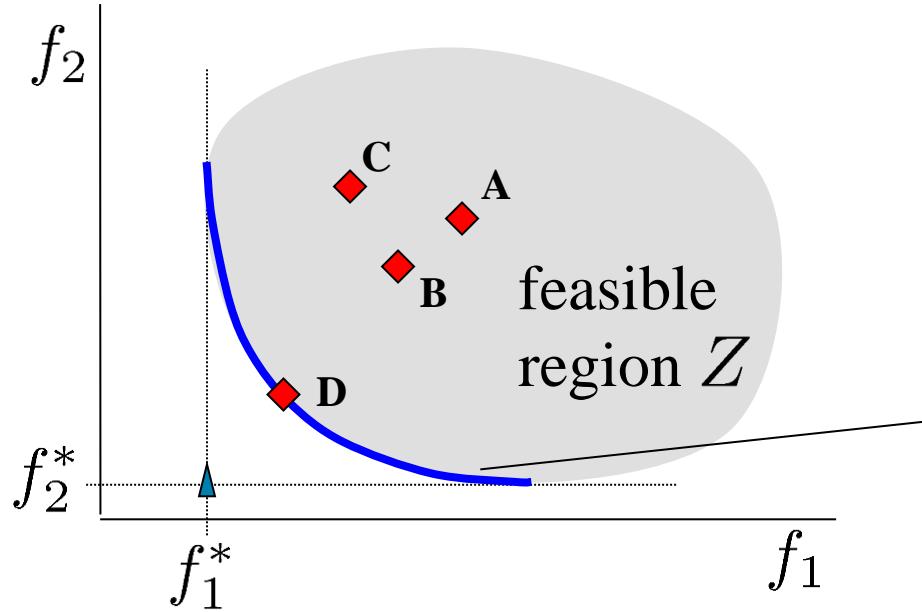
- **Mapping feasible region into the objective space:**

$$Z = \{z \in \mathbb{R}^k : z = ((f_1(x), f_2(x), \dots, f_k(x))^T \ \forall x \in \Omega)\}$$



## Bi-objective example

- $\min f_1 = \text{risk}$ ,  $\min f_2 = \text{loss}$ :



*Pareto frontier or efficient frontier  
(all non-dominated solutions)*

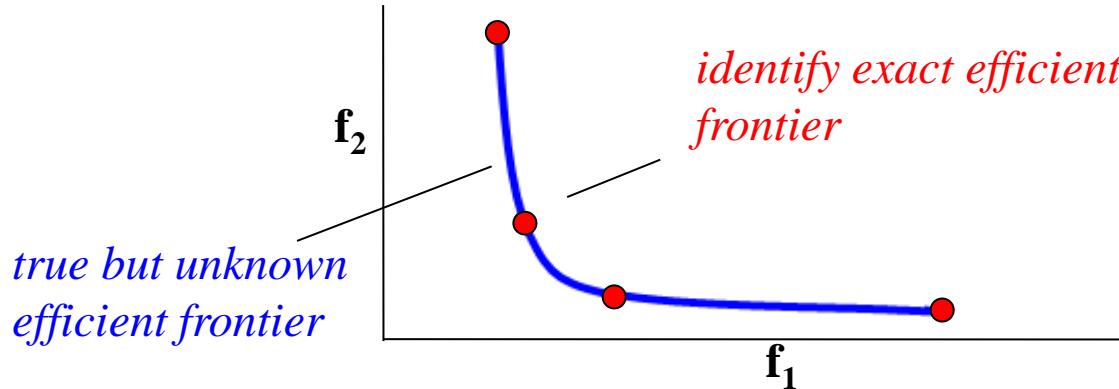
- **Pareto efficiency**: solutions with characteristics like **D**, are called tradeoff, Pareto optimal or non-dominated
- **Multi-objective optimization goal**: find solution(s) on the efficient frontier according to the decision maker preferences

## Computing efficient frontiers - possibilities

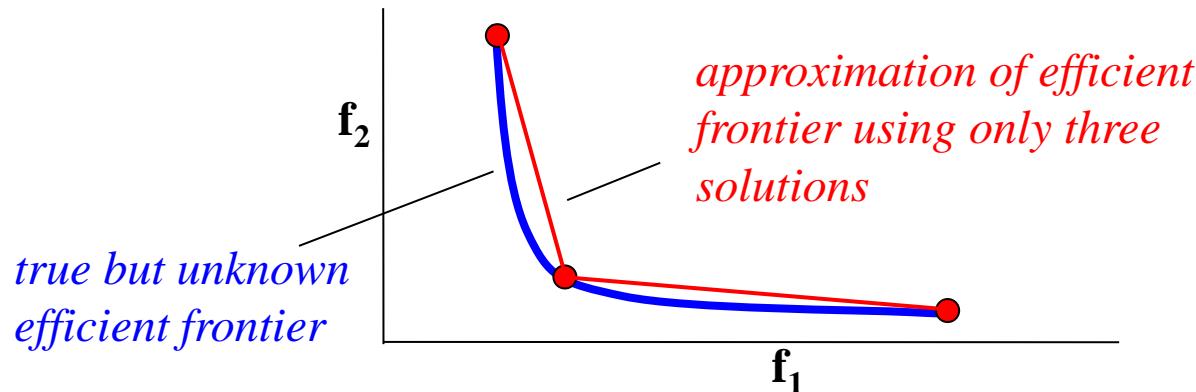
Multi-objective analysis involves **computing the efficient frontier**, evaluating it (if possible, out-of-sample) and selecting the final solution based on the decision maker preferences

### Computing efficient frontiers:

- Ideal (often unrealistic) goal: compute exact frontier



- Typical (more realistic) goal: approximate the frontier



# Solving multi-objective optimization problems

- **Convert multi-objective optimization problem to a series of single-objective optimization problems**
  
- Methods:
  - Weighting Method
  - $\varepsilon$ -Constraint (Hierarchical) Method

## Weighting method

- Assign weights to each objective
- Optimize the weighted sum of the objectives
- Multi-objective optimization with weighting method:

$$\begin{aligned} \min \quad & \omega_1 \cdot f_1(x) + \omega_2 \cdot f_2(x) \\ \text{s.t.} \quad & x \in \Omega \end{aligned}$$

$f_i$  is convex function (linear, convex quadratic)

$\Omega \subseteq \mathbb{R}^n$  (convex)

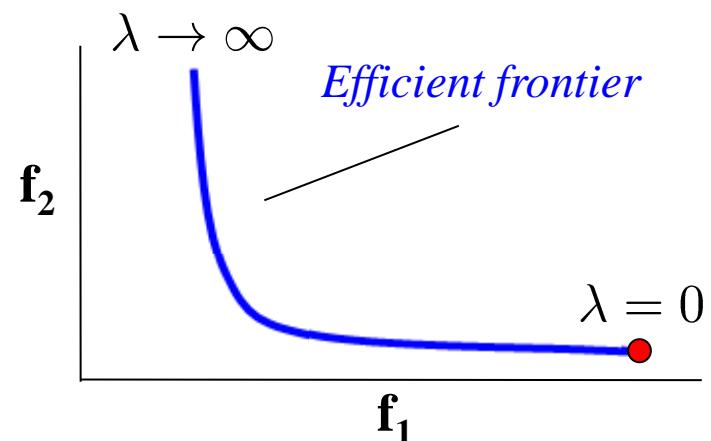
$\omega_i \in \mathbb{R}$  is the weight of the  $i$ -th objective

$\omega_i \geq 0$ ,  $i = 1, 2$  and  $\omega_1 + \omega_2 = 1$

- Easier formulation:

$$\downarrow \quad \lambda = \frac{\omega_1}{\omega_2}$$

$$\begin{aligned} \min \quad & \lambda f_1(x) + f_2(x) \\ \text{s.t.} \quad & x \in \Omega \end{aligned}$$



## $\varepsilon$ - constrained method

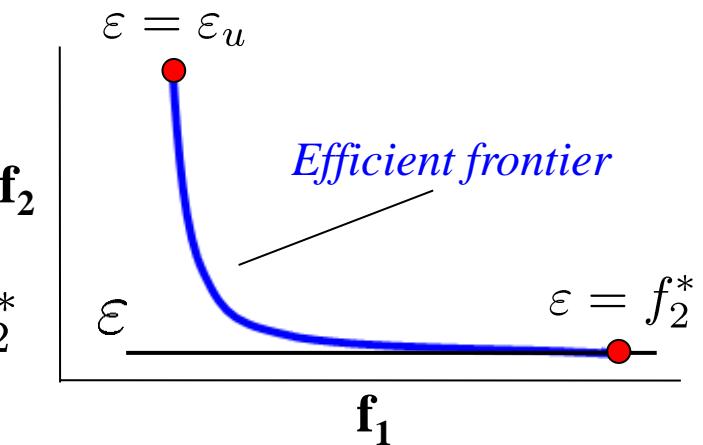
- Optimize one objective
- Convert other objectives into constraints
- Multi-objective optimization with  $\varepsilon$ -constrained method:

First step

$$\begin{array}{ll} \min & f_2(x) \\ \text{s.t.} & x \in \Omega \\ & \downarrow \\ & f_2^* \end{array}$$

Second step

$$\begin{array}{ll} \min & f_1(x) \\ \text{s.t.} & x \in \Omega \\ & f_2(x) \leq (1 + \epsilon)f_2^* \end{array}$$



## Multi-objective optimization examples

- Multi-objective optimization in finance and risk management:

$$\min w_1 \cdot (\text{performance measure}) + w_2 \cdot (\text{risk measure})$$

$$\min w_1 \cdot (\text{risk measure } 1) + w_2 \cdot (\text{risk measure } 2)$$

$$\min \text{risk measure}$$

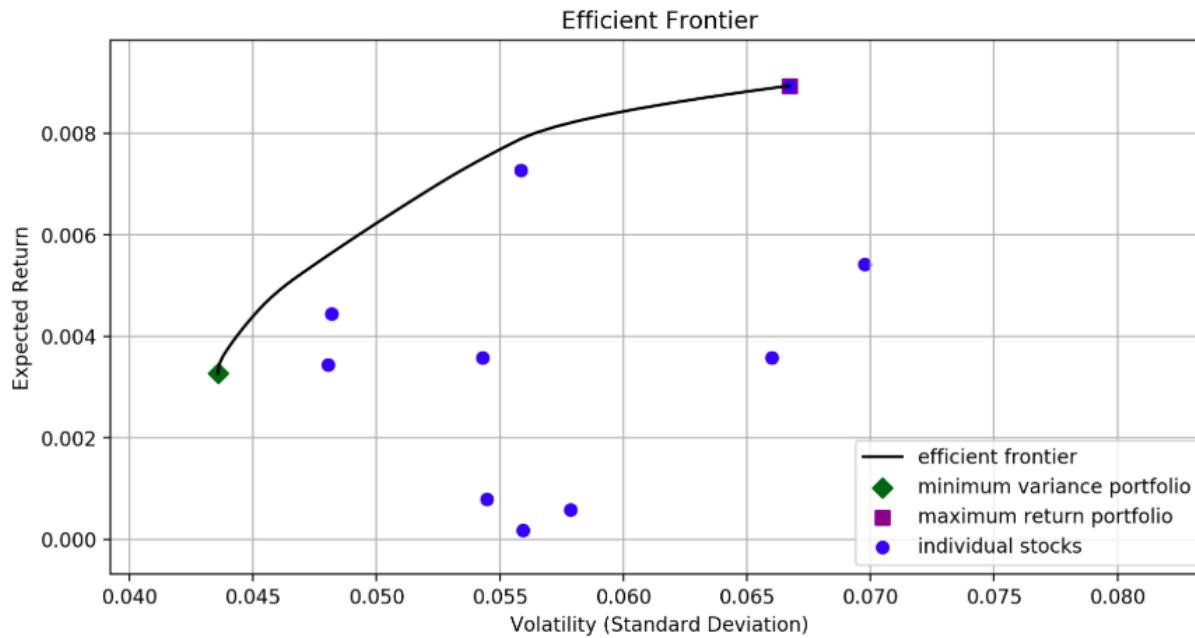
s.t. performance measure  $\geq \varepsilon + w_2 \cdot (\text{perform. measure 2})$

- Firm's performance measures: profits, sales, stock price, growth, market share, P/E ratio, ...
- Bank's performance measures: return, loss, liquidity, ...
- Risk measures:
  - variance
  - value-at-risk (VaR)
  - conditional value-at-risk (CVaR)
  - tracking error



# Mean-Variance Portfolio Selection (part 2)

# Mean-variance portfolio selection in Python



Step 1 – compute minimum variance portfolio

$$\begin{aligned} \min_w \quad & w^T Q w \\ \text{s.t.} \quad & \sum_i w_i = 1 \\ & w \geq 0 \end{aligned}$$

Step 2 – compute maximum return portfolio

$$\begin{aligned} \max_w \quad & \mu^T w \\ \text{s.t.} \quad & \sum_i w_i = 1 \\ & w \geq 0 \end{aligned}$$

Step 3 – compute efficient frontier (mean-variance)

$$\begin{aligned} \min_w \quad & w^T Q w \\ \text{s.t.} \quad & \mu^T w \geq \varepsilon \\ & \sum_i w_i = 1 \\ & w \geq 0 \end{aligned}$$

$$\varepsilon \in [\varepsilon_{R_{\min \text{Var}}}, \varepsilon_{R_{\max \text{Ret}}}]$$

# Mean-variance portfolio selection in Python

```

# Random data for 10 stocks
n = 10
Q = np.random.random((n,n))
Q = np.dot(Q,Q.T)/1000 # covariance matrix
mu = np.random.rand(n) / 100 # expected return

# Compute minimum variance portfolio
w1 = cp.Variable(n)
prob1 = cp.Problem(cp.Minimize(cp.quad_form(w1, Q)),
                   [sum(w1) == 1,
                    w1 >= 0])
prob1.solve(verbose=True)

# Display minimum variance portfolio
w_minVar = w1.value
var_minVar = np.dot(w_minVar, np.dot(Q, w_minVar))
ret_minVar = np.dot(mu, w_minVar)
print("Minimum variance portfolio:\n")
print("    Solution status =", prob1.status)
print("    Solution value =", prob1.value)
print("    Variance =", var_minVar)
print("    Expected return =", ret_minVar)
print("Standard deviation =", np.sqrt(var_minVar))

```

Minimum variance portfolio:

```

Solution status = optimal
Solution value = 0.001899092402601308
Variance = 0.001899092402601308
Expected return = 0.003267477841642341
Standard deviation = 0.043578577335673864

```

**Step 1 – compute minimum variance portfolio**

$$\begin{aligned}
 \min_w \quad & w^T Q w \\
 \text{s.t.} \quad & \sum_i w_i = 1 \\
 & w \geq 0
 \end{aligned}$$

# Mean-variance portfolio selection in Python

```

# Random data for 10 stocks
n = 10
Q = np.random.random((n,n))
Q = np.dot(Q,Q.T)/1000      # covariance matrix
mu = np.random.rand(n) / 100 # expected return

# Compute maximum return portfolio
w2 = cp.Variable(n)
prob2 = cp.Problem(cp.Maximize(mu.T@w2),
                   [sum(w2) == 1,
                    w2 >= 0])
prob2.solve(verbose=True)

# Display maximum return portfolio
w_maxRet = w2.value
var_maxRet = np.dot(w_maxRet, np.dot(Q, w_maxRet))
ret_maxRet = np.dot(mu, w_maxRet)
print("Maximum return portfolio:\n")
print("    Solution status =", prob2.status)
print("    Solution value =", prob2.value)
print("    Expected return =", ret_maxRet)
print("    Variance =", var_maxRet)
print("Standard deviation =", np.sqrt(var_maxRet))

```

Maximum return portfolio:

```

Solution status = optimal
Solution value = 0.00893492393857598
Expected return = 0.00893492393857598
Variance = 0.00445041529455067
Standard deviation = 0.0667114330122706

```

**Step 2 – compute maximum return portfolio**

$$\begin{aligned} \max_w \quad & \mu^T w \\ \text{s.t.} \quad & \sum_i w_i = 1 \\ & w \geq 0 \end{aligned}$$

# Mean-variance portfolio selection in Python

```

# Target returns
targetRet = np.linspace(ret_minVar, ret_maxRet, 50)

# Compute efficient frontier
w = cp.Variable(n)
eps = cp.Parameter()
eps.value = targetRet[0]
prob3 = cp.Problem(cp.Minimize(cp.quad_form(w, Q)),
                    [mu.T@w >= eps,
                     sum(w) == 1,
                     w >= 0])

w_front = []
var_front = []
ret_front = []
for epsilon in targetRet:
    eps.value = epsilon
    prob3.solve(verbos=False)
    w_front.append(w.value)
    var_front.append(np.dot(w.value, np.dot(Q, w.value)))
    ret_front.append(np.dot(mu, w.value))

# Plot efficient frontier
ax = plt.gca()
ax.scatter(x=np.sqrt(var_minVar), y=ret_minVar, marker='D', s=50, color='DarkGreen',
            label='minimum variance portfolio')
ax.scatter(x=np.sqrt(var_maxRet), y=ret_maxRet, marker='s', s=50, color='DarkMagenta',
            label='maximum return portfolio')
ax.plot(np.sqrt(var_front), ret_front, 'k-', label='efficient frontier')
ax.scatter(x=np.sqrt(np.diag(Q)), y=mu, color='Blue', label='individual stocks')
ax.set_xlabel('Volatility (Standard Deviation)')
ax.set_ylabel('Expected Return')
plt.title('Efficient Frontier')
ax.legend(loc="lower right")
ax.grid()
plt.show()

```

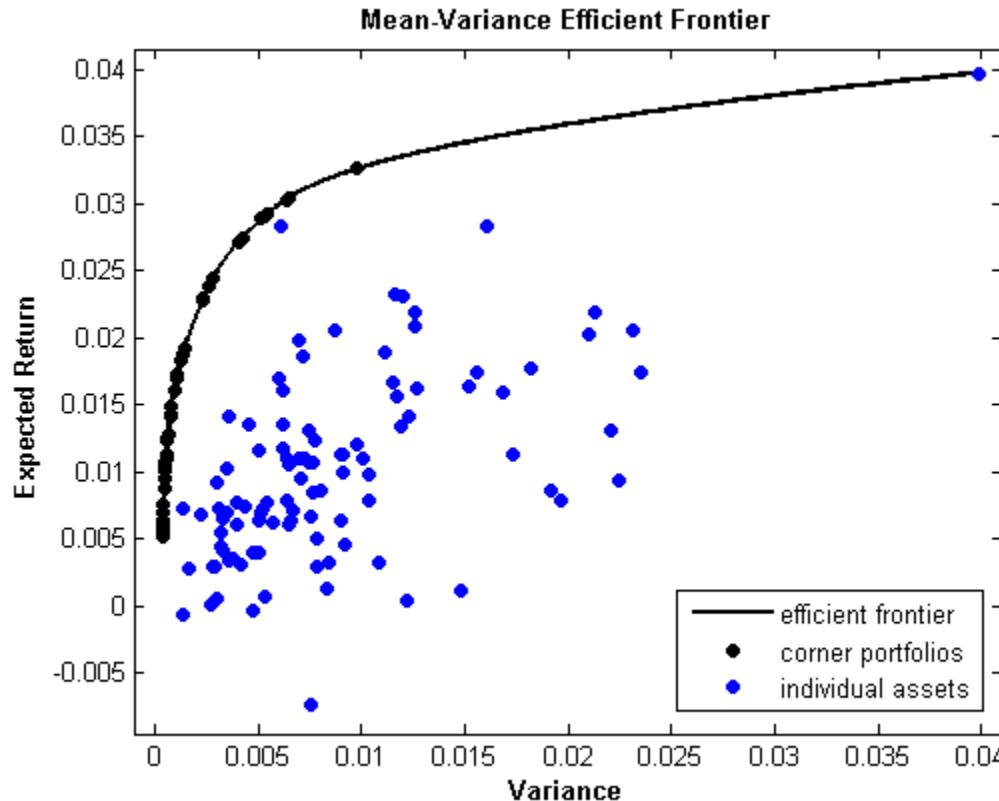
**Step 3 – compute efficient frontier (mean-variance)**

$$\begin{aligned}
 \min_w \quad & w^T Q w \\
 \text{s.t.} \quad & \mu^T w \geq \varepsilon \\
 & \sum_i w_i = 1 \\
 & w \geq 0 \\
 \varepsilon \in & [\varepsilon_{R_{\minVar}}, \varepsilon_{R_{\maxRet}}]
 \end{aligned}$$

## Portfolio selection

- **Mean-variance portfolio optimization problem** – two objectives:

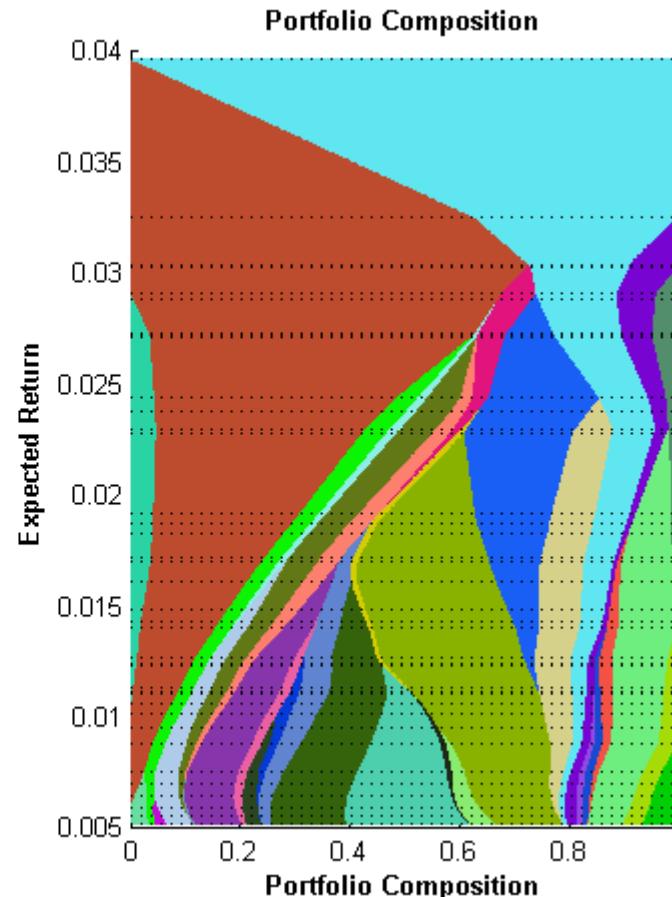
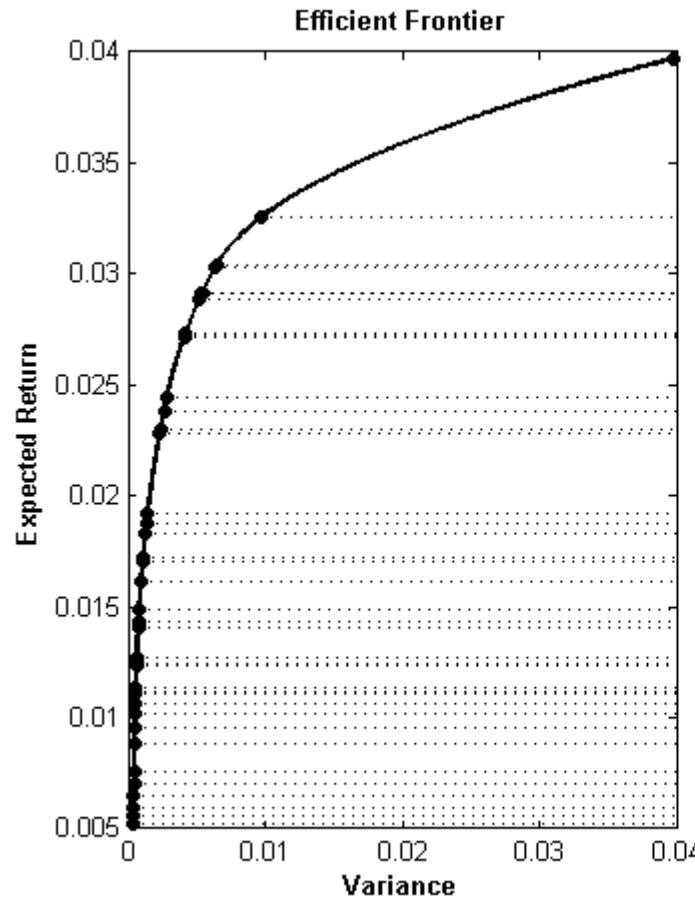
$$\begin{aligned} \min_w \quad & w^T Q w \\ \text{s.t.} \quad & \mu^T w \geq \varepsilon \\ & \sum_i w_i = 1 \\ & w \geq 0 \end{aligned}$$



Extensions of mean-variance model: introduce transaction costs

# Portfolio selection

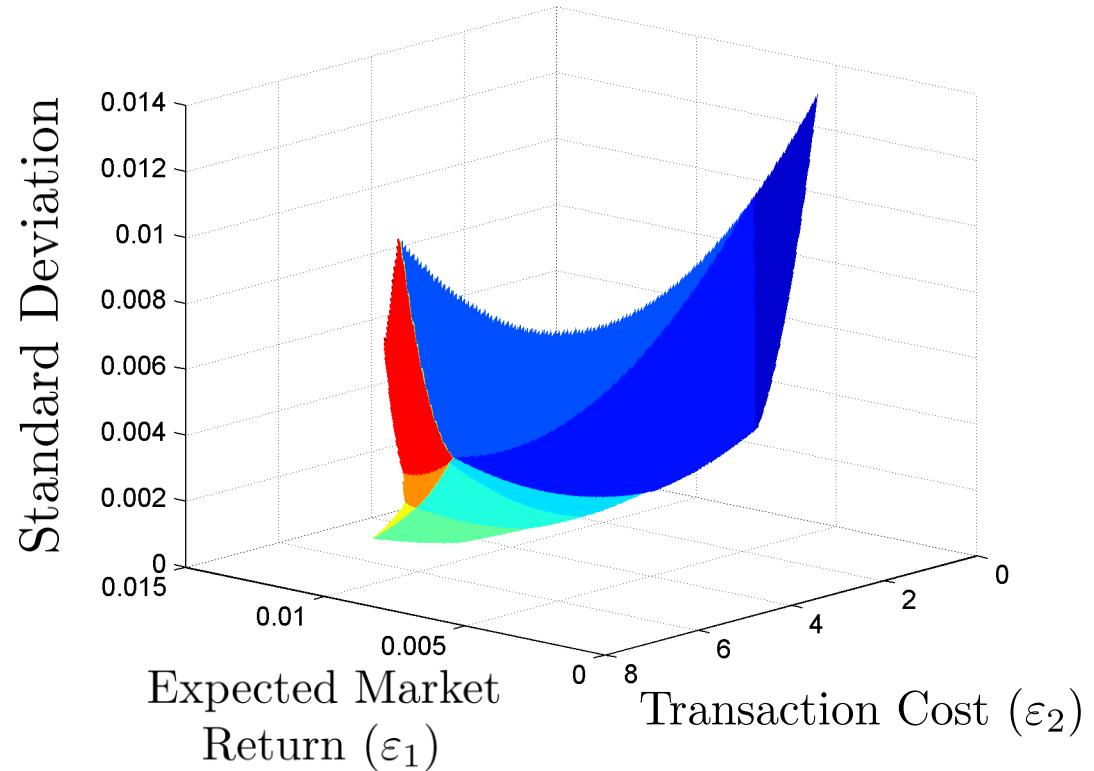
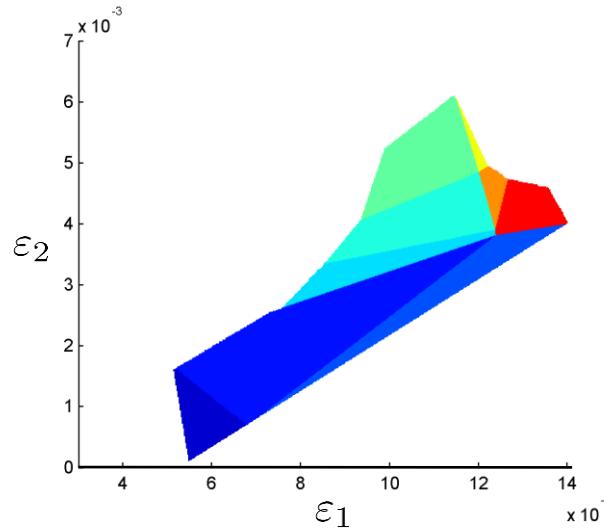
- **Mean-variance portfolio optimization problem** – efficient frontier and portfolio composition:



# Portfolio selection

- **Mean-variance portfolio optimization problem with transaction cost** – three objectives:

$$\begin{aligned} \min_w \quad & w^T Q w \\ \text{s.t.} \quad & \mu^T w \geq \varepsilon_1 \\ & c^T w \leq \varepsilon_2 \\ & \sum_i w_i = 1 \\ & w \geq 0 \end{aligned}$$



# Illustration of mean-variance analysis

- Investor has a portfolio entirely in the U.S. capital markets, but is considering going onto non-U.S. markets

- Asset classes:

- U.S. stocks
- Non-U.S. stocks
- U.S. bonds
- Non-U.S. bonds

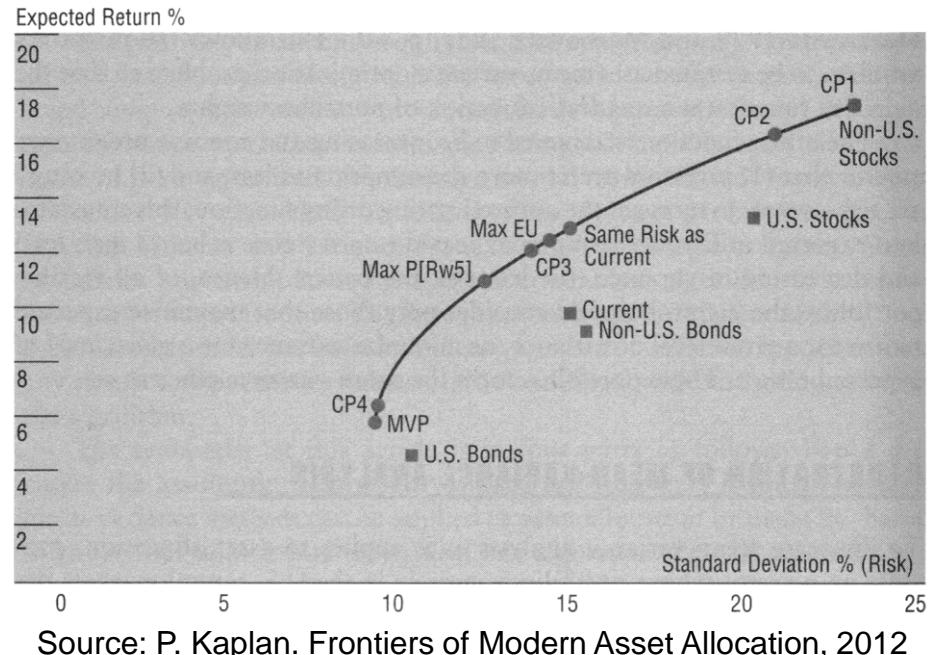
Asset Class	Expected Return	Standard Deviation	U.S. Stocks	Non-U.S. Stocks	U.S. Bonds	Non-U.S. Bonds
U.S. Stocks	15.00	20.00	1.00			
Non-U.S. Stocks	18.00	23.00	0.58	1.00		
U.S. Bonds	8.50	9.80	0.54	0.21	1.00	
Non-U.S. Bonds	12.00	15.00	0.03	0.52	0.15	1.00

- Portfolios:

- CP1, ..., CP4 – corner portfolios
- MVP – minimum variance portfolio
- Max EU – portfolio for a risk tolerance  $\lambda$
- Max P – Roy's safety first portfolio (maximize probability that realized return is greater than the threshold)

$$\max_p \mathbb{P}(r_p > r_{\text{th}}) = \max_p SF Ratio_p$$

$$SF Ratio_p = \frac{\mathbb{E}(w) - r_{\text{th}}}{\sqrt{\text{var}(w)}}$$



Source: P. Kaplan. Frontiers of Modern Asset Allocation, 2012



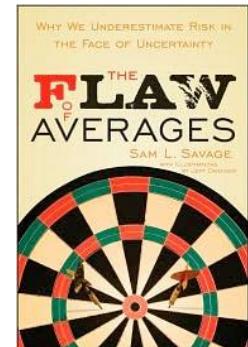
# Sharpe Ratio

## Sharpe ratio

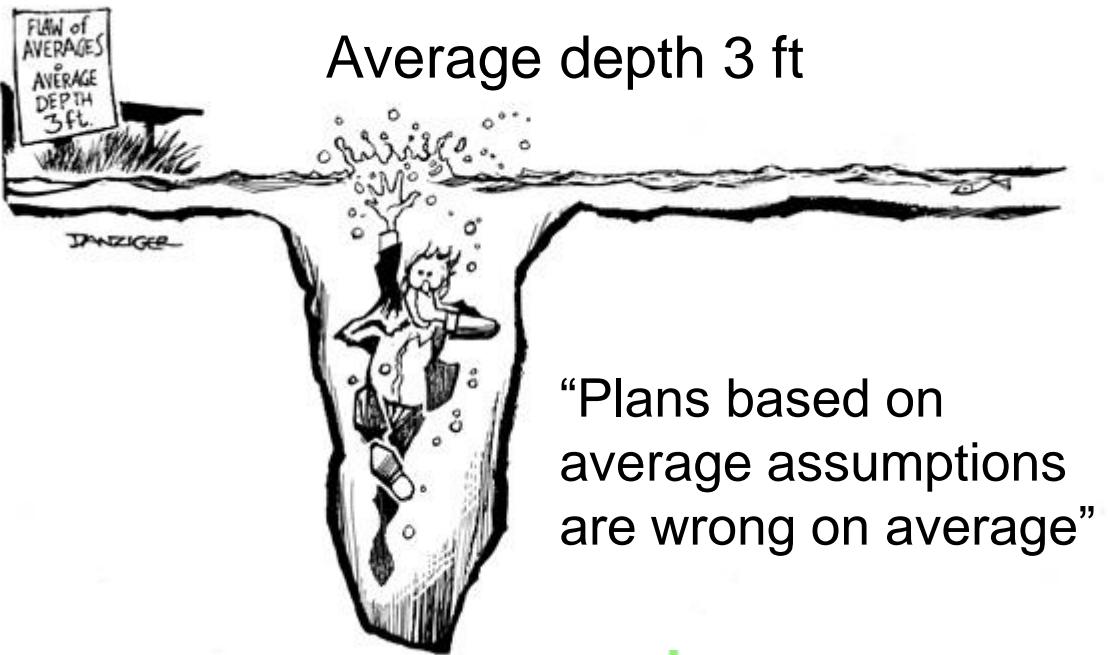
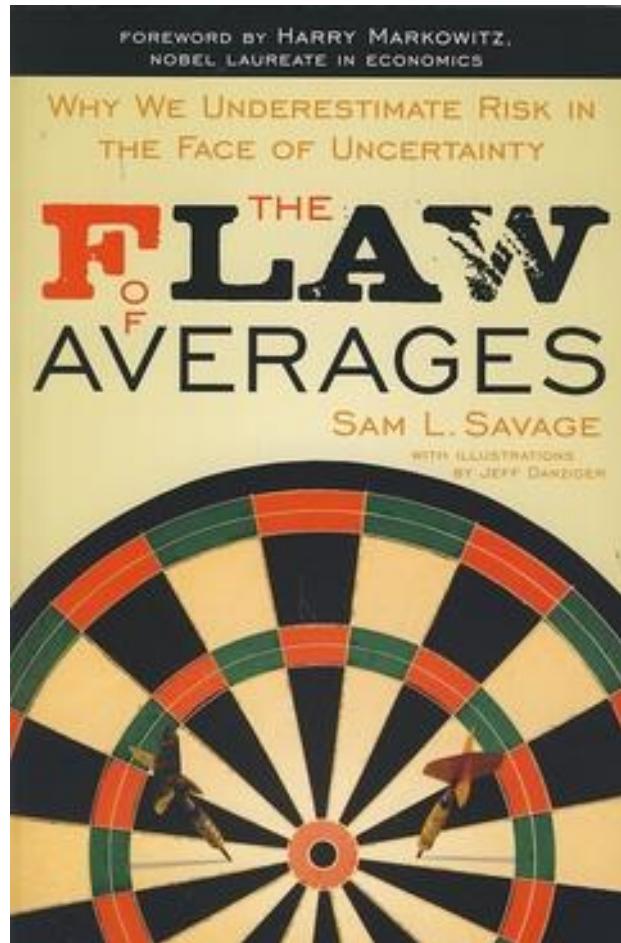
- **Sharpe ratio (reward-to-variability ratio)** measures the **excess return** (risk premium) per unit of **risk** (deviation) in an investment asset or a portfolio
- **Idea:** how much **additional return** you are receiving for the **additional volatility** of holding the risky asset or portfolio over a **risk-free asset** – the higher the better:

$$SR_i = \frac{r_i - r_f}{\sigma_i} \quad SR_p = \frac{r_p - r_f}{\sigma_p} = \frac{\mathbb{E}(w) - r_f}{\sqrt{\text{var}(w)}}$$

- **Sharpe ratio** is a **risk-adjusted measure of return** that can be used to evaluate performance of a portfolio (ratio helps to make portfolios' performance **comparable** by making an adjustment for risk)
- **Example:**
  - Manager A generates a return of 15%, while manager B generates a return of 12%. **Is manager A performing better than manager B?**
  - Risk free-rate is 5%, manager A's portfolio has a standard deviation of 8%, while manager B's portfolio has a standard deviation of 5%.
  - Sharpe ratio for **manager A** is **1.25**, while **manager B**'s ratio is **1.4**, which is better than manager A.
  - Based on these calculations, **manager B** was able to generate a higher return on a risk-adjusted basis.
- **Flaw of averages**



# Flaw of averages



"Plans based on average assumptions are wrong on average"

# Maximizing Sharpe ratio

- **Sharpe ratio** for a portfolio = excess expected return per unit of risk:

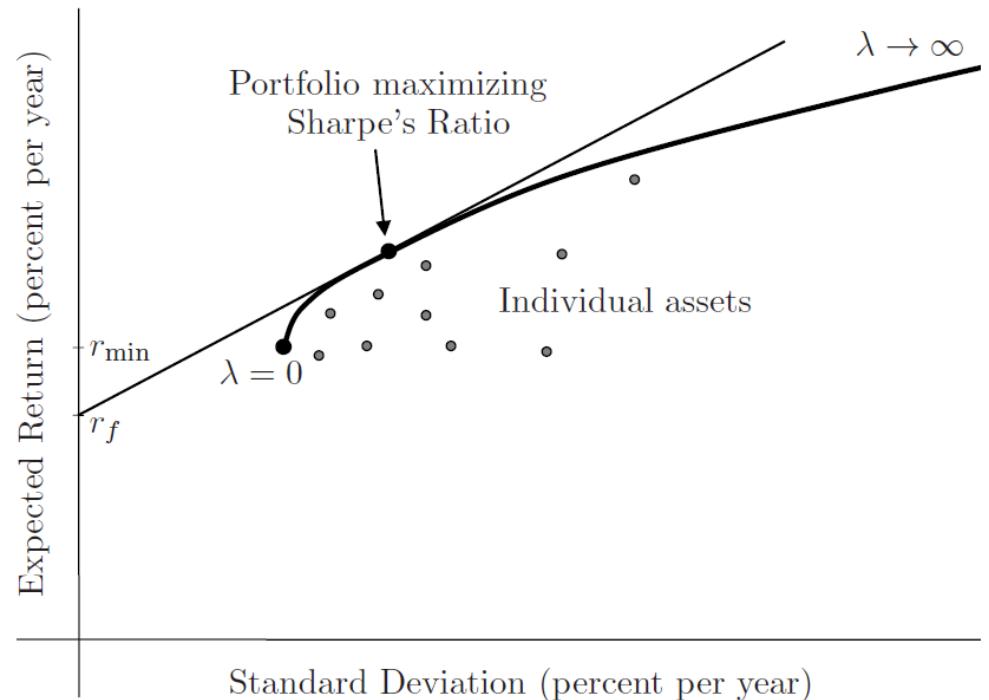
$$SR_p = \frac{r_p - r_f}{\sigma_p} = \frac{\mathbb{E}(w) - r_f}{\sqrt{\text{var}(w)}} = \frac{\mu^T w - r_f}{\sqrt{w^T Q w}}$$

- Optimization problem that corresponds to **maximizing Sharpe's ratio**:

$$\begin{aligned} \max_w \quad & \frac{\mu^T w - r_f}{\sqrt{w^T Q w}} \\ \text{s.t.} \quad & w \in \mathcal{F} \end{aligned}$$

- **Capital Market Line (CML)**:

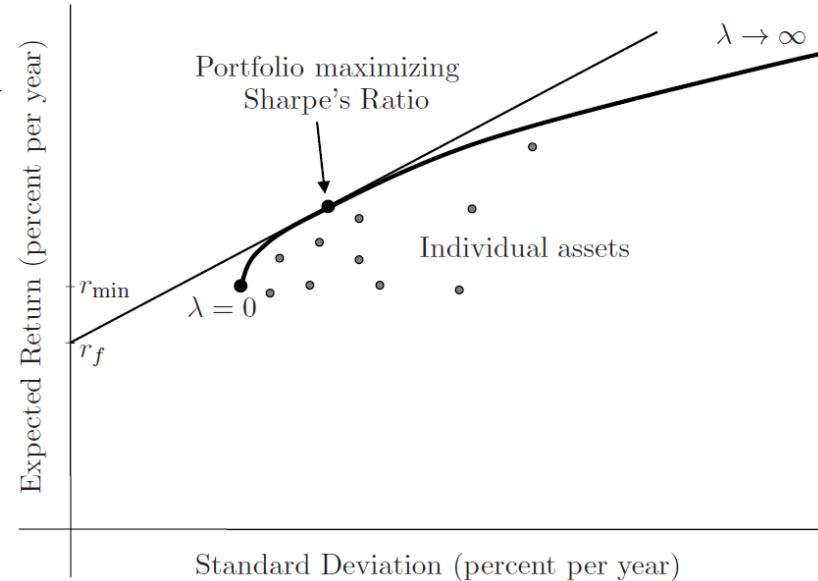
- Capital Allocation Line (CML) is a ray in  $(r_p, \sigma_P)$  coordinates that starts at the point  $(r_f, 0)$
- A portfolio that maximizes Sharpe ratio corresponds to the point where CML is tangential to the Markowitz efficient frontier
- In the presence of a risk-free asset, all efficient frontier portfolios lie on the CML and it gives the trade-off between portfolio risk and return



# Maximizing Sharpe ratio

- Riskless asset  $r_f$  such that  $r_f \geq 0$ ,  $r_f < r_{\min}$
- Optimization problem that corresponds to **maximizing Sharpe's ratio**:

$$\begin{array}{ll} \max_w & \frac{\mu^T w - r_f}{\sqrt{w^T Q w}} \\ \text{s.t.} & w \in \mathcal{F} \end{array} \quad \begin{array}{l} \text{non-linear} \\ \text{possibly non-convex} \\ \text{difficult to solve} \end{array}$$



- A portfolio  $w^*$  with the maximum Sharpe ratio can be found by solving:

$$\begin{array}{ll} \min_{y \in \mathbb{R}^n, \kappa \in \mathbb{R}} & y^T Q y \\ \text{s.t.} & \sum_i (\mu_i - r_f) y_i = 1 \\ & (\kappa, y) \in \left\{ \kappa > 0, \frac{y}{\kappa} \in \mathcal{F} \right\} \cup (0, 0) \end{array}$$

and computing  $w^* = \frac{y}{\kappa}$

# Maximizing Sharpe ratio

- Optimization problem that corresponds to **maximizing Sharpe's ratio**:

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \frac{\mu^T w - r_f}{\sqrt{w^T Q w}} \\ \text{s.t.} \quad & w \in \mathcal{F} \end{aligned}$$

$\leftarrow \mathcal{F} = \{\sum_i w_i = 1, l \leq Aw \leq u\}$

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \frac{\mu^T w - r_f}{\sqrt{w^T Q w}} \\ \text{s.t.} \quad & \sum_i w_i = 1 \\ & l \leq Aw \leq u \end{aligned}$$



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and computing  $w^* = \frac{y}{\kappa}$

$$\begin{aligned} \min_{y \in \mathbb{R}^n, \kappa \in \mathbb{R}} \quad & y^T Q y \\ \text{s.t.} \quad & \sum_i (\mu_i - r_f) y_i = 1 \\ & \sum_i y_i = \kappa \\ & l \cdot \kappa \leq A y \leq u \cdot \kappa \\ & \kappa \geq 0 \end{aligned}$$



## Maximizing Sharpe ratio (proof idea)

- Full proof can be found in the book **Optimization Methods in Finance by Gerard Cornuejols and Reha Tutuncu, 2006**

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \frac{\mu^T w - r_f}{\sqrt{w^T Q w}} \\ \text{s.t.} \quad & w \in \mathcal{F} \end{aligned}$$

Assume that there exists a feasible portfolio  $\hat{w}$  with  $\mu^T \hat{w} > r_f$   
 $\Rightarrow (\mu - r_f e)^T \hat{w} > 0$ . Let us make the change of variables:

$$\kappa = \frac{1}{(\mu - r_f e)^T \hat{w}}, \quad y = \kappa w$$

Now, objective function becomes:

$$\frac{\mu^T w - r_f}{\sqrt{w^T Q w}} = \frac{(\mu - r_f e)^T w}{\frac{1}{\kappa} \sqrt{y^T Q y}} = \frac{\frac{1}{\kappa}}{\frac{1}{\kappa} \sqrt{y^T Q y}} = \frac{1}{\sqrt{y^T Q y}}$$

Maximizing  $\frac{1}{\sqrt{y^T Q y}}$  is equivalent to minimizing  $y^T Q y$ . Also

$$(\mu - r_f e)^T w > 0, \quad w \in \mathcal{F} \Leftrightarrow \kappa > 0, \quad \frac{y}{\kappa} \in \mathcal{F} \quad \text{and}$$

$$\kappa = \frac{1}{(\mu - r_f e)^T \hat{w}} \Leftrightarrow (\mu - r_f e)^T y = 1 \quad \text{given } y/\kappa = w$$

# Comparing Sharpe ratios

- For comparison purposes, **Sharpe ratio** is usually annualized

$$\frac{\mu^T w - r_f}{\sqrt{w^T Q w}} = \frac{\mathbb{E}[r^T w - r_f]}{\sqrt{\text{var}[r^T w - r_f]}}$$

- E.g., if  $r$  is the vector of daily returns (i.i.d.) and the number of trading days in a year is equal to 252, then annualized Sharpe ratio is:

$$\begin{aligned} \frac{\mathbb{E}\left[\sum_{t=1}^{252} r_t^T w - 252 \cdot r_f\right]}{\sqrt{\text{var}\left[\sum_{t=1}^{252} r_t^T w - 252 \cdot r_f\right]}} &= \frac{252 \cdot (w^T \mathbb{E}[r] - r_f)}{\sqrt{252 \cdot \text{var}[r^T w]}} \\ &= \sqrt{252} \cdot \frac{\mu^T w - r_f}{\sqrt{w^T Q w}} \end{aligned}$$



# Equity Portfolio Selection in Practice

# Investment process

- **Setting investment objectives** (the target return and risk level can be defined as absolute or relative to a benchmark):
  - **Banks**: earn a return on invested funds that is higher than the cost of acquiring those funds
  - **Pension plans**: Generate sufficient cash flow from the investment portfolio to satisfy the plan's pension obligations
  - **Life insurance companies**: earn a higher return on the premium it invests than the interest rate it has guaranteed policyholders
  - **Mutual funds, hedge funds**: maximize expected returns at a certain level of risk
- **Developing and implementing a portfolio strategy**
  - Selecting the type of investment strategy
    - ◆ **Active strategies** (manager's intuition or quantitative risk models)
    - ◆ **Passive strategies** (indexing)
  - Formulating the inputs for portfolio construction
  - Constructing the portfolio (**optimization**)
- **Monitoring the portfolio** (performance measurement over a specified time interval)
- **Adjusting the portfolio (portfolio rebalancing)**

# Portfolio constraints

- Single-period view of investing

- Stock weights

- at the beginning of the period  $w_0$
  - at the end of the period  $w$

- Long-only (no short-selling) constraints

$$w \geq 0$$

- Holding constraints

$$l \leq w \leq u$$

- consider a portfolio of 10 stocks
  - issuers of assets 1, 3 and 5 are in the same industry
  - limit portfolio exposure to that industry to be at least 20%, but at most 40%

$$0.20 \leq w_1 + w_3 + w_5 \leq 0.40$$

- Turnover constraints

$$|w_i - w_{0i}| \leq u_i$$

## Cardinality constraints

- Portfolio **cardinality** is the number of non-zero components in  $w$ :

$$\text{card}(w) = \sum_i w_i^0 = \|w\|_0 \text{ with } 0^0 = 0$$

- Restrict **number of assets** in the portfolio to be less than  $K$ :

$$\begin{aligned} \min_w \quad & w^T Q w \\ \text{s.t.} \quad & \sum_{i=1}^n w_i = 1 \\ & l \leq w \leq u \\ & \text{card}(w) \leq K \end{aligned}$$



$$\begin{aligned} \min_{w,z} \quad & w^T Q w \\ \text{s.t.} \quad & \sum_{i=1}^n w_i = 1 \\ & \sum_{i=1}^n z_i = K \\ & l \cdot z \leq w \leq u \cdot z \\ & z = \{0, 1\} \end{aligned}$$

- **Sparse and stable Markowitz portfolios:**

- Sparse portfolios are stable as there is less estimation error, those are also easier to manage
- Portfolio optimization problem is more difficult as it contains binary variables  $z_i$  – need mixed-integer optimization

# Cardinality constraints in Python using CVXPY

```

# Random data for 10 stocks
n = 10
Q = np.random.random((n,n))
Q = np.dot(Q,Q.T)/1000           # covariance matrix
mu = np.random.rand(n) / 100     # expected return

# Define cardinality-constrained optimization problem
w = cp.Variable(n)
z = cp.Variable(n, boolean=True)
K = cp.Parameter()
K.value = n
prob = cp.Problem(cp.Minimize(cp.quad_form(w, Q)),
                  [sum(w) == 1,
                   sum(z) == K,
                   w - z <= 0,
                   w >= 0])

# Compute minimum variance portfolio
prob.solve(verbose=True)

# Compute cardinality-constrained minimum variance portfolios
w_cardP = []
var_cardP = []
p_cardP = []
for card in range(1,n+1):
    K.value = card
    p_cardP.append(card)
    prob.solve(solver=cp.CPLEX, verbose=False, cplex_params={"threads": 4})
    w_cardP.append(w.value)
    var_cardP.append(np.dot(w.value, np.dot(Q, w.value)))

# Plot minimum variance portfolios for cardinality 1, 2, ..., 10
ax = plt.gca()
ax.plot(p_cardP, np.sqrt(var_cardP), 'bo')
ax.set_xlim([0,n+1])
ax.set_xlabel('Portfolio cardinality')
ax.set_ylabel('Portfolio standard deviation')
plt.title('Minimum Variance Portfolios with Cardinality Constraint')
ax.grid()
plt.show()

```

$$\begin{aligned}
 & \min_{w,z} && w^T Q w \\
 & \text{s.t.} && \sum_{i=1}^n w_i = 1 \\
 & && \sum_{i=1}^n z_i = K \\
 & && l \cdot z \leq w \leq u \cdot z \\
 & && z = \{0, 1\}
 \end{aligned}$$

# Cardinality constraints in Python

## Mathematical formulation

$$\min_{w,z} w^T Q w$$

$$\text{s.t. } \sum_{i=1}^n w_i = 1$$

$$\sum_{i=1}^n z_i = K$$

$$l \cdot z \leq w \leq u \cdot z$$

$$z = \{0, 1\}$$

$$\rightarrow x = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

## Solver formulation

$$\min_x \tilde{c}^T x + \frac{1}{2} x^T \tilde{Q} x$$

$$\text{s.t. } \tilde{l} \leq \tilde{A}x \leq \tilde{u}$$

$x_1, x_2, x_3$  continuous vars

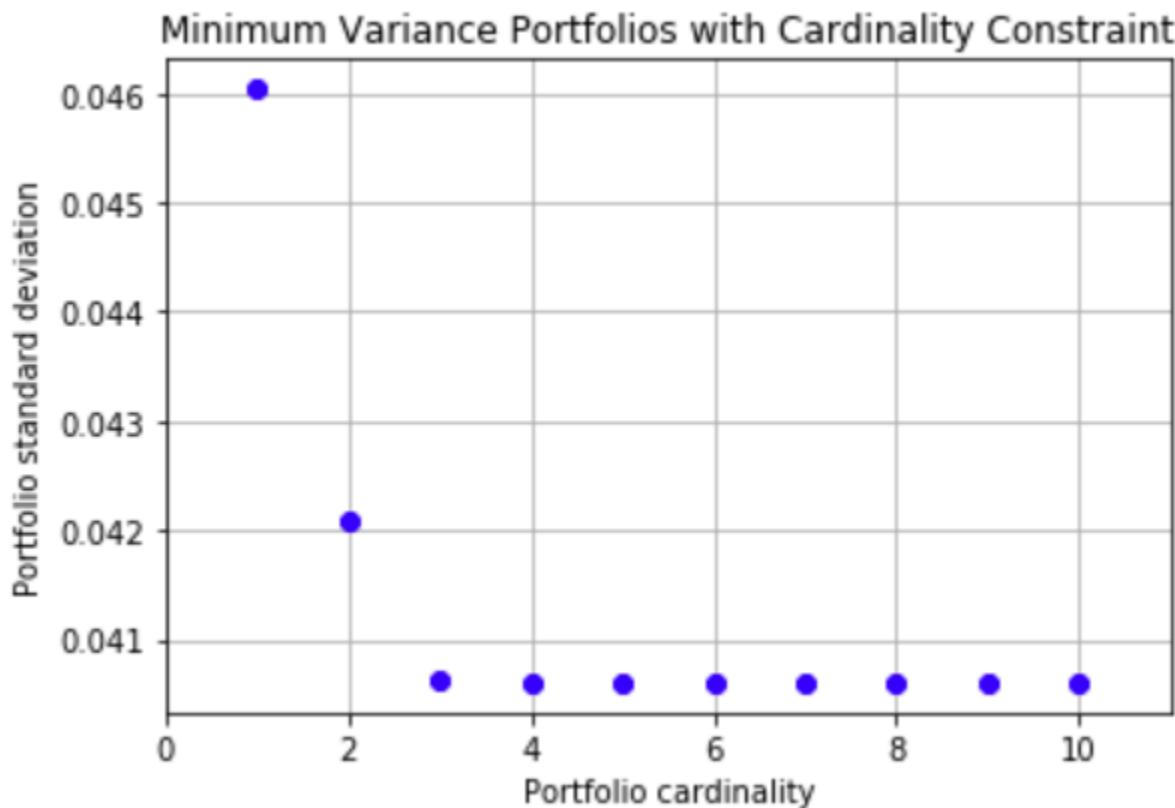
$x_4, x_5, x_6$  binary vars

$$\tilde{c} = (0 \ 0 \ 0 \ 0 \ 0 \ 0)^T$$

$$\tilde{Q} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & 0 & 0 & 0 \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & 0 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}}_{\tilde{A}} \cdot x \leq \underbrace{\begin{pmatrix} 1 \\ K \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\tilde{u}}$$

# Cardinality constraints in Python



$$\begin{aligned}
 & \min_{w,z} && w^T Q w \\
 & \text{s.t.} && \sum_{i=1}^n w_i = 1 \\
 & && \sum_{i=1}^n z_i = K \\
 & && l \cdot z \leq w \leq u \cdot z \\
 & && z = \{0, 1\}
 \end{aligned}$$

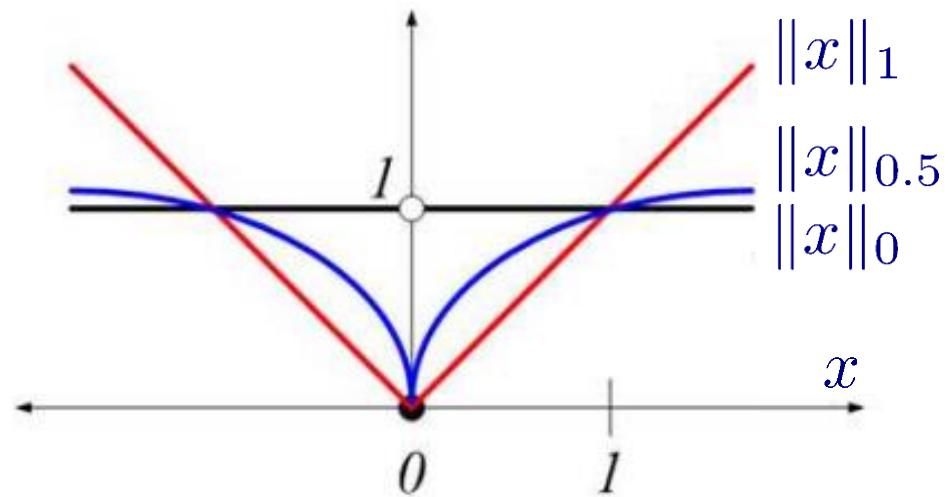
## Trading budget constraint

- **Regularized optimization** – replace cardinality constraint with norm constraint

$$\begin{array}{ll} \min_w & w^T Q w \\ \text{s.t.} & \begin{aligned} w \in \mathcal{F} \\ \|w\|_1 \leq \varepsilon \end{aligned} \end{array} \quad \begin{aligned} \text{card}(w) = \sum_i w_i^0 &= \|w\|_0 \quad \text{with } 0^0 = 0 \\ \|w\|_0 \leq K &\rightarrow \|w\|_1 \leq \varepsilon \end{aligned}$$

- $\ell_1$ -norm constraint:

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$



## Trading budget constraint

- **Regularized optimization** – replace cardinality constraint with norm constraint

$$\begin{array}{ll} \min_w & w^T Q w \\ \text{s.t.} & \begin{array}{l} \text{card}(w) = \sum_i w_i^0 = \|w\|_0 \text{ with } 0^0 = 0 \\ \|Dw\|_1 \leq \varepsilon \end{array} \end{array}$$

$$\|w\|_0 \leq K \rightarrow \|w\|_1 \leq \varepsilon$$

- The total percentage traded to construct a portfolio cannot exceed budget  $\varepsilon$ :

unit-based trading costs  $\sum_{i=1}^n |w_i| \leq \varepsilon \rightarrow \|w\|_1 \leq \varepsilon$

$d$ -based trading costs  $\sum_{i=1}^n d_i |w_i| \leq \varepsilon \rightarrow \|Dw\|_1 \leq \varepsilon, D = \text{diag}(d_j)$

$\uparrow D = I$

# Trading budget constraint

## ■ Regularized optimization

$$\begin{array}{ll} \min_w & w^T Q w \\ \text{s.t.} & \text{card}(w) = \sum_i w_i^0 = \|w\|_0 \quad \text{with } 0^0 = 0 \\ & \|w\|_1 \leq \varepsilon \\ & \|w\|_0 \leq K \rightarrow \|w\|_1 \leq \varepsilon \end{array}$$

- The total percentage traded to construct or rebalance a portfolio cannot exceed budget  $\varepsilon$ :

Constructing portfolio

$$\sum_{i=1}^n |w_i| \leq \varepsilon$$

$w_i^+ + w_i^-$

$$\begin{cases} |w_i| = w_i^+ + w_i^- \\ w_i = w_i^+ - w_i^- \\ w_i^+ \geq 0, w_i^- \geq 0 \end{cases}$$

Rebalancing portfolio

$$\sum_{i=1}^n |w_i - w_{0i}| \leq \varepsilon$$

$w_i^+ + w_i^-$

Optimization ensures that in an optimum  $w_i^+ = 0$  or  $w_i^- = 0$

# Transaction costs

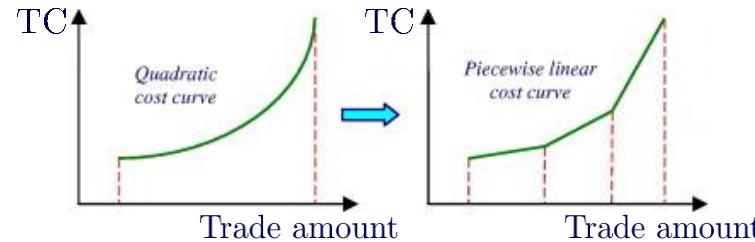
## ■ Transaction costs objective function

$$\begin{array}{ll} \min_{w \in \mathbb{R}^n} & -\mu^T w + \lambda \cdot w^T Q w + \lambda_{TC} \cdot TC \\ \text{s.t.} & w \in \mathcal{F} \end{array} \Leftrightarrow \begin{array}{ll} \min_{w \in \mathbb{R}^n} & -\mu^T w + \lambda w^T Q w \\ \text{s.t.} & TC \leq \varepsilon_{TB} \\ & w \in \mathcal{F} \end{array}$$

## ■ Linear transaction costs

$$TC = \sum_{i=1}^n c_i \cdot |w_i - w_{0i}|$$

## ■ Piecewise-linear transaction costs



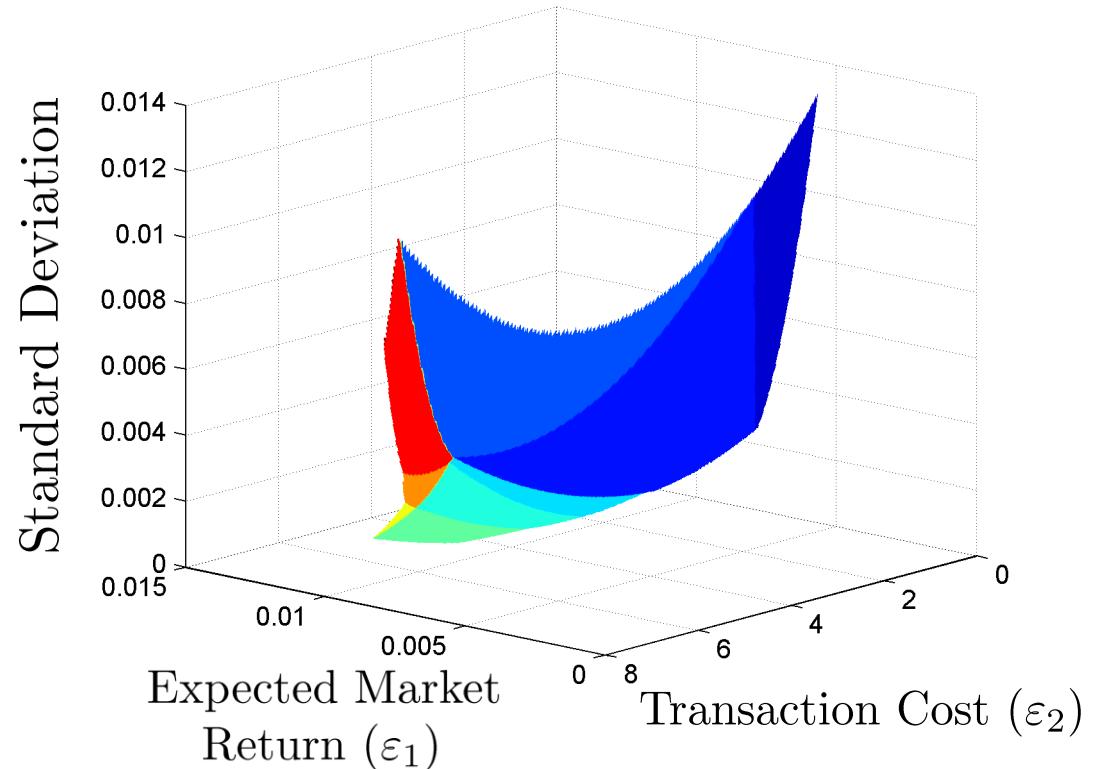
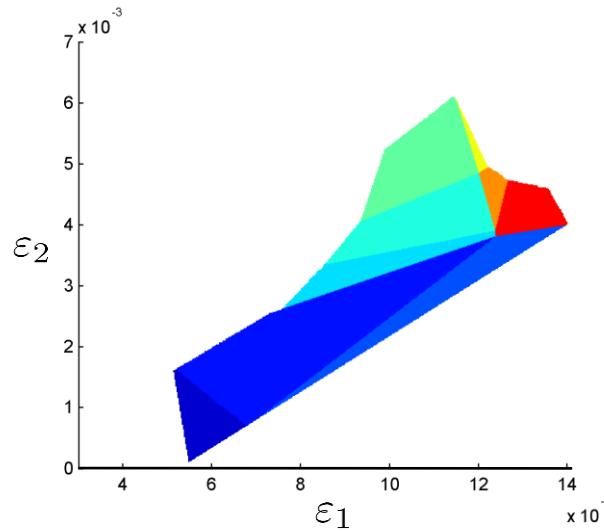
## ■ Fixed transaction costs

$$TC_i = a_i \cdot y_i, \quad y_i \in \{0, 1\}$$

## Portfolio selection – linear transaction cost

- **Mean-variance portfolio optimization problem with transaction cost** – three objectives:

$$\begin{aligned} \min_w \quad & w^T Q w \\ \text{s.t.} \quad & \mu^T w \geq \varepsilon_1 \\ & c^T w \leq \varepsilon_2 \\ & \sum_i w_i = 1 \\ & w \geq 0 \end{aligned}$$



# Benchmark tracking

- **Passive investment strategy**
  - in the absence of any superior forecasting ability, investors might assume that they cannot beat the market
  - hold a broadly diversified portfolio
  - manage portfolio relative to a benchmark or stock universe, such as the S&P 500
  - active investment strategies, e.g., mean-variance framework, identify attractive investments without relying on benchmarks
- **Keep tracking error relative to the **benchmark** small**
- **Indexing** – instead of minimizing variance (as in Markowitz model), **minimize tracking error** with respect to a **benchmark** that represents the **market** (S&P 500)
- **Tracking error** – the **variance** (or standard deviation) of the difference between the **portfolio return**  $r^T \mathbf{w}$  and the **return of the benchmark**  $r^T \mathbf{w}_b$

# Benchmark tracking

## ■ Asset weights

- in the benchmark (e.g., S&P 500):  $\mathbf{w}_b$
- in a benchmark tracking portfolio:  $\mathbf{w}$

■ Tracking error squared

$$\begin{aligned} \text{TE}^2(\mathbf{w}) &= \text{var}(\mathbf{r}^T \mathbf{w} - \mathbf{r}^T \mathbf{w}_b) \\ &= \text{var}(\mathbf{r}^T (\mathbf{w} - \mathbf{w}_b)) \\ &= (\mathbf{w} - \mathbf{w}_b)^T \cdot \text{var}(\mathbf{r}) \cdot (\mathbf{w} - \mathbf{w}_b) \\ &= (\mathbf{w} - \mathbf{w}_b)^T \mathbf{Q} (\mathbf{w} - \mathbf{w}_b) \end{aligned}$$

## ■ Tracking error minimization (indexing)

$$\begin{array}{ll} \min_{\mathbf{w} \in \mathbb{R}^n} & \text{TE}^2(\mathbf{w}) \quad \rightarrow \\ \text{s.t.} & \mathbf{w} \in \mathcal{F} \end{array} \qquad \begin{array}{ll} \min_{\mathbf{w} \in \mathbb{R}^n} & (\mathbf{w} - \mathbf{w}_b)^T \mathbf{Q} (\mathbf{w} - \mathbf{w}_b) \\ \text{s.t.} & \mathbf{w} \in \mathcal{F} \end{array}$$

## ■ Tracking error constraint

$$(\mathbf{w} - \mathbf{w}_b)^T \mathbf{Q} (\mathbf{w} - \mathbf{w}_b) \leq \sigma_{\text{TE}}^2 \quad \leftarrow \text{limit on the tracking error}$$



# Factor Models

# Factor models

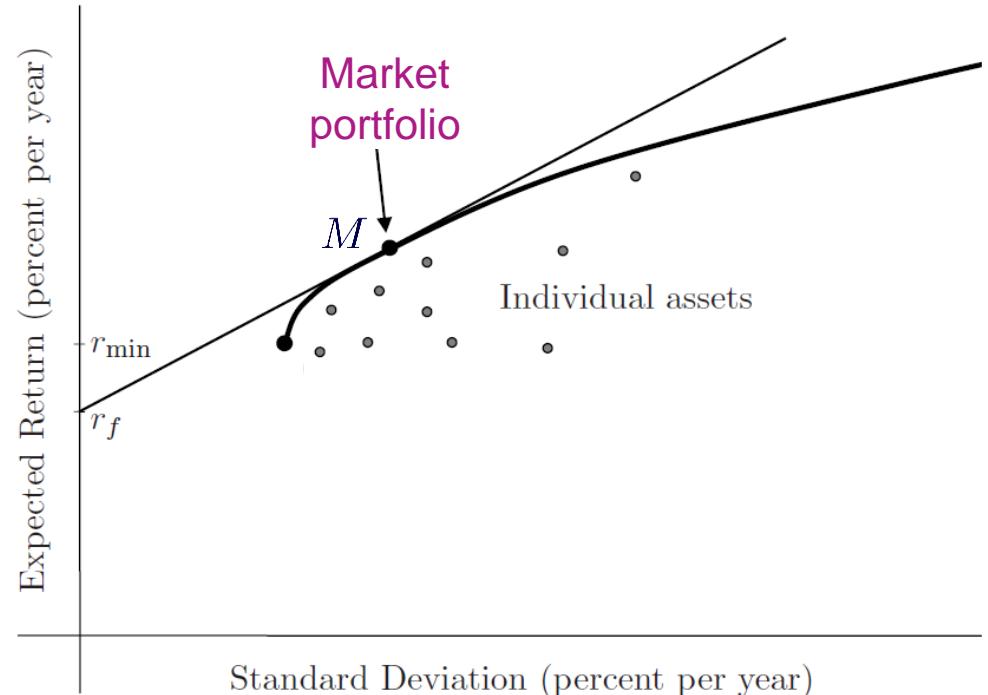
- Factor analysis attempts to simplify the joint behaviour of assets by looking at key drivers:
  - Statistical factor analysis: identify principal components via principal-component analysis (PCA), but those components are not interpretable
  - Fundamental factor analysis: uses real drivers (indexes, interest rates, exchange rates, commodities), but may fail to include key drivers and is sensitive to data



# Capital Market Line (CML)

## ■ Capital Market Line (CML):

- Capital Allocation Line (CML) is a ray in  $(r_P, \sigma_P)$  coordinates that starts at the point  $(r_f, 0)$
- A portfolio that maximizes Sharpe ratio corresponds to the point where CML is tangent to the Markowitz efficient frontier
- In the presence of a risk-free asset, all efficient frontier portfolios lie on the CML and it gives the trade-off between portfolio risk and return



- CML describes all possible mean-variance efficient portfolios that are a combination of the **risk free asset** and **market portfolio**:
  - buy risk free asset (between  $M$  and  $r_f$ ) or
  - sell risk free asset (beyond point  $M$ ) and
  - hold the same portfolio  $M$  of risky assets
- Every portfolio on CML is an efficient fund of risky assets (**market portfolio**) and a risk free asset (**a bond that matures at the end of investment horizon**)

# Capital Market Line (CML)

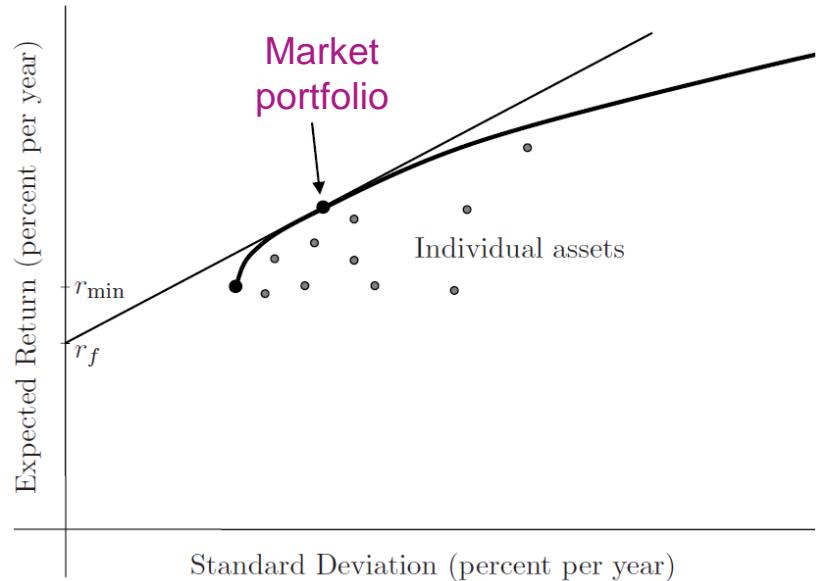
- Equation describes all portfolios on CML:

$$\bar{r}_P = r_f + \frac{\bar{r}_M - r_f}{\sigma_M} \cdot \sigma_P$$

$\mathbb{E}[r_M] = \bar{r}_M$  expected market rate of return

$\sigma_M$  standard deviation of market rate of return

- CML relates the expected rate of return of an efficient portfolio to its standard deviation
- The slope the CML is called the price of **risk**



# Capital Asset Pricing Model (CAPM)

- How does the expected **rate of return** of an individual asset relate to its individual **risk**? (*model developed by Sharpe, Lintner and Mossin in the 60's*)
- CAPM relates the random return on the  $i$ -th investment  $r_i$  to the random return on the market  $r_M$  by

$$r_i - r_f = \beta_i(r_M - r_f) + \epsilon_i$$

$\epsilon_i$  is a random variable with zero mean and  $\sigma_{\epsilon_i}$  standard deviation  
 $\epsilon_i$  is uncorrelated with the market return  $r_M$  and  $\text{cov}(\epsilon_i, \epsilon_j) = 0$

- Taking the expectation we get the **CAMP equation**:

$$\mathbb{E}[r_i] - r_f = \beta_i(\mathbb{E}[r_M] - r_f)$$

$$\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)$$

$\sigma_M$  is the standard deviation of market return,  $\mathbb{E}[\epsilon_i] = 0$   
 $\text{cov}(\epsilon_i, r_M) = \text{cov}(r_i, r_M) - \beta_i \cdot \text{var}(r_M) = \sigma_{iM} - \beta_i \sigma_M^2 = 0$

- **Beta of an asset** (risk premium):

$$\beta_i = \frac{\text{cov}(r_i, r_M)}{\text{var}(r_M)} = \frac{\sigma_{iM}}{\sigma_M^2}$$

# Capital Asset Pricing Model (CAPM)

- CAPM describes relationship between risk and expected return of asset
- Expected excess rate of return of an asset is proportional to the expected excess rate of return of the market portfolio – proportional factor is the beta of the asset

$$\bar{r}_i - r_f = \beta_i (\bar{r}_M - r_f)$$

- Beta of an asset:
  - beta of an asset measures the risk of the asset with respect to the market portfolio  $M$
  - high beta assets earn higher average return in equilibrium because of  $\beta_i(\bar{r}_M - r_f)$
  - beta of market portfolio – average risk of all assets

$$\beta_M = \frac{\text{cov}(r_M, r_M)}{\text{var}(r_M)} = \frac{\sigma_M^2}{\sigma_M^2} = 1$$

- If the betas of the individual assets are known, then the beta of the portfolio is

$$\beta_P = \sum_{i=1}^n \beta_i w_i$$

Show it by using  $r_P = \sum_{i=1}^n r_i w_i$  and  $\text{cov}(r_P, r_M) = \sum_{i=1}^n w_i \cdot \text{cov}(r_i, r_M)$

# Systematic and specific risk

- CAPM divides total risk of holding risky assets into two parts:
  - systematic risk (risk of holding the market portfolio)
  - specific risk
- CAPM relates the random return on the  $i$ -th investment  $r_i$  to the random return on the market  $r_M$  by

$$r_i - r_f = \beta_i(r_M - r_f) + \epsilon_i$$

- Total risk of holding risky asset  $i$  is

$$\underbrace{\sigma_i^2}_{\text{total risk}} = \text{var}[r_i] = \underbrace{\beta_i^2 \cdot \sigma_M^2}_{\text{systematic risk}} + \underbrace{\sigma_{\epsilon_i}^2}_{\text{specific risk}} = \beta_i \cdot \sigma_{iM} + \sigma_{\epsilon_i}^2$$

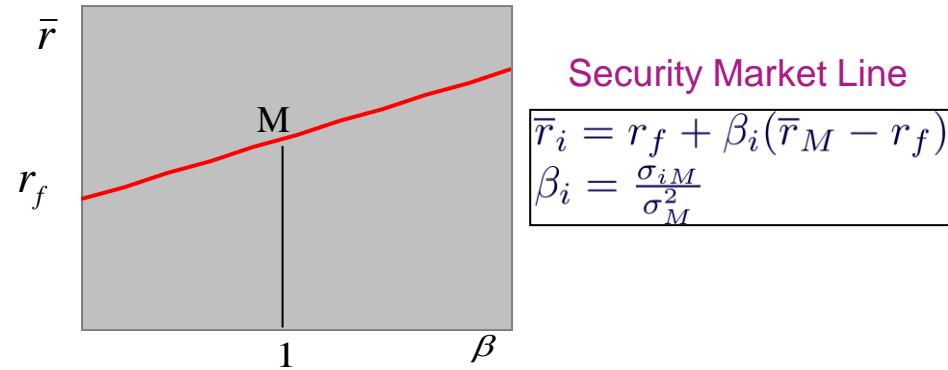
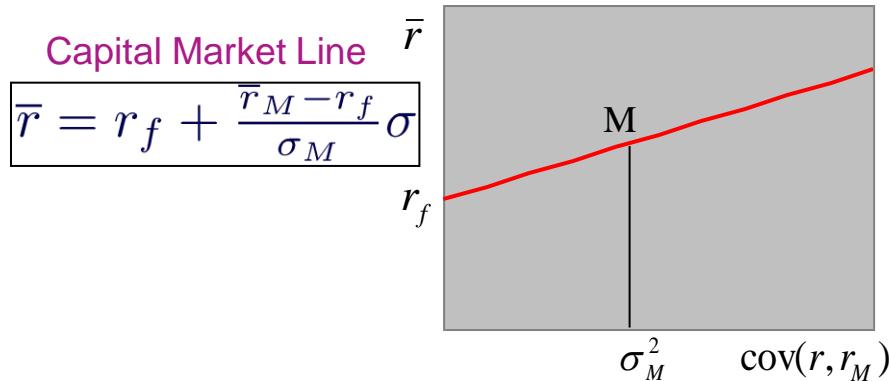
- For a portfolio:

$$\bar{r}_P = r_f + \beta_P(\bar{r}_M - r_f)$$

$$\sigma_P = |\beta_P| \sigma_M \quad \beta_P = \sum_{i=1}^n \beta_i w_i$$

## Beta of the market

- Average risk of all assets is 1 (beta of the market portfolio)
- Beta of market portfolio is used as a reference point to measure risk of other assets



- Assets or portfolios with betas greater than 1 are above average risk: tend to move more than market. **Example:**
  - If risk free rate is 5% per year and market rises by 10%, then assets with a beta of 2 will tend to increase by 15%
  - If market falls by 10%, then assets with a beta of 2 will tend to fall by 25% on average
- Assets or portfolios with betas less than 1 are of below average risk: tend to move less than market

# Single-Factor Models and Multi-Factor Models

- Consider  $n$  assets with rates of return  $r_i$  for  $i=1,2,\dots,n$  and one factor  $f$  which is a random quantity such as **inflation**, **interest rate**
- Assume that the rates of return and single factor are linearly related

$$r_i = \underbrace{\alpha_i}_{\text{constant}} + \underbrace{\beta_i \cdot f}_{\text{constant}} + \underbrace{\epsilon_i}_{\text{random}}$$

↑                   ↑                   ↑  
**Intercept**      **Factor loadings**      **Error**

- Errors:
  - have zero mean
  - are uncorrelated with the factor
  - are uncorrelated with each other
- Single-factor model can be extended to have more than one factor (**multi-factor**):

$$r_i = \alpha_i + \sum_{k=1}^m \beta_{ik} \cdot f_k + \epsilon_i$$

# Multi-Factor Models and Mean-Variance Analysis

- Multi-factor model:  $r_i = \alpha_i + \sum_{k=1}^m \beta_{ik} \cdot f_k + \epsilon_i$

- Multi-factor model in vector form:

$$\mathbf{r} = \boldsymbol{\alpha} + \mathbf{B} \cdot \mathbf{f} + \boldsymbol{\epsilon}$$

$\mathbf{r}$   $n$ -dimensional vector of returns  
 $\boldsymbol{\alpha}$   $n$ -dimensional vector of mean returns  
 $\mathbf{f}$   $m$ -dimensional vector of factors  
 $\mathbf{B}$   $m \times n$  matrix of factor loadings  
 $\boldsymbol{\epsilon}$   $n$ -dimensional vector of residual errors

- Expected portfolio return:

Expected excess portfolio return

$$\boldsymbol{\alpha}^T \cdot \mathbf{w}$$

Expected portfolio return

$$\boldsymbol{\alpha}^T \cdot \mathbf{w} + \mathbf{B} \cdot \mathbf{f} \cdot \mathbf{w}$$

- Variance of the portfolio return:

$$\mathbf{w}^T \cdot (\mathbf{B}^T \cdot \mathbf{Q}_f \cdot \mathbf{B} + \mathbf{D}) \cdot \mathbf{w}$$

$\mathbf{Q}_f$  factor covariance matrix  
 $\mathbf{D}$  variance of error terms (diagonal matrix)

# How to select factors?

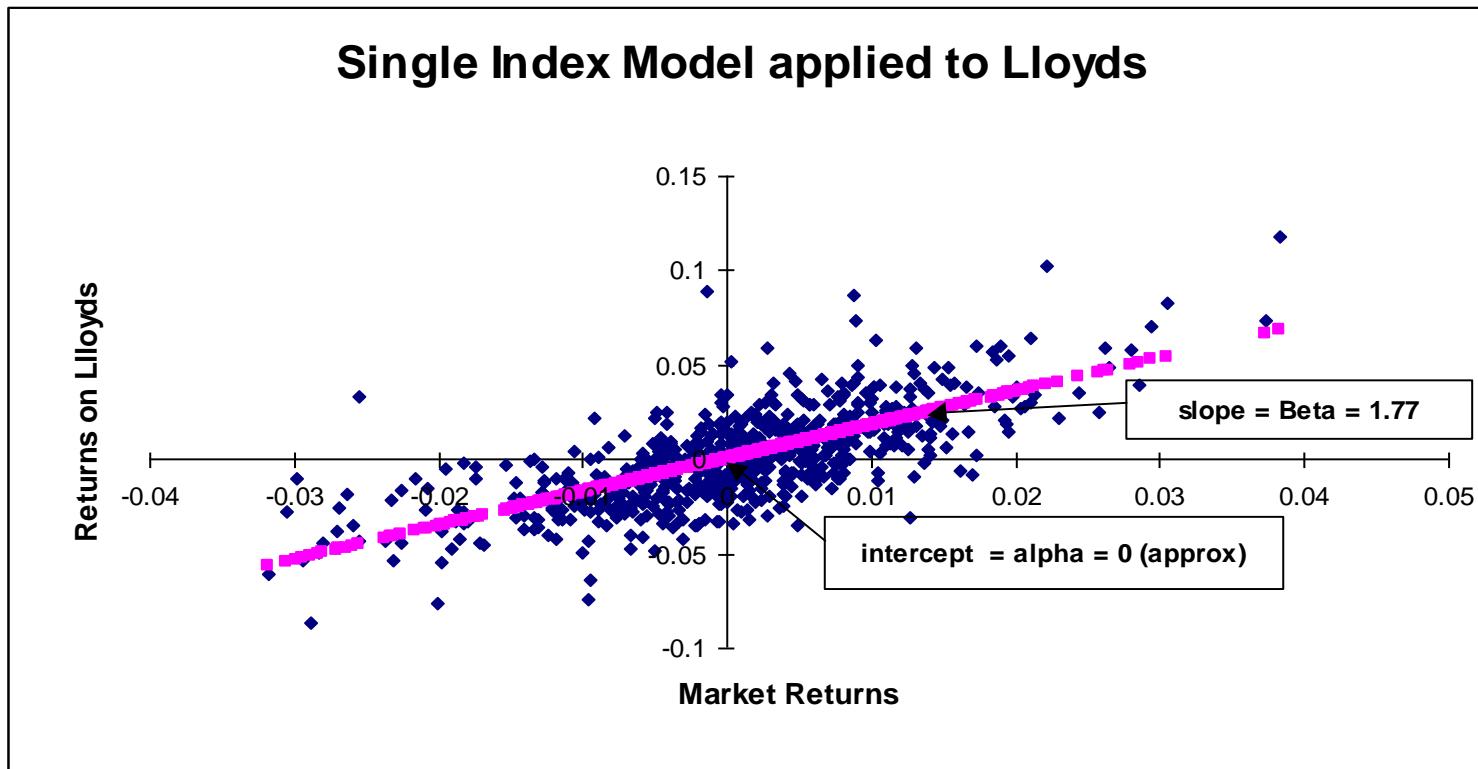
- Factors are **external** to securities, e.g., economic factors:
  - gross domestic product (GDP)
  - consumer price index (CPI)
  - unemployment rate
  - credit spreads on bonds
- Factors are **extracted** from known information about security returns:
  - rate of return on the market portfolio (**market risk**)
  - average of the return of stocks in a particular country and industry (utilities, transportation, aerospace, financial, materials, manufacturing, etc.)
- Firm characteristics:
  - price earning ratio
  - dividend payout ratio
  - earnings growth forecast
- **How to select factors: It is part science and part art**
  - Statistical approach – principal component analysis
  - Fundamental approach – uses real factors, i.e., beta, inflation rate, interest rate, industrial production, etc.



# CAPM as a Factor Model – example

## ■ Single factor model equation defines a linear fit to data

- Imagine several independent observations of the rate of return and factor
- Straight line, defined by single factor model equation, is fitted through these points such that average value of errors is zero
- Error is measured by the vertical distance from a point to the line



# Estimating betas in CAPM model

- Security's **beta** can be estimated from a set of observed returns for the security and the market return using **linear regression**:

$$r_{it} - r_{ft} = \alpha_i + \beta_i \cdot (r_{Mt} - r_{ft}) + e_{it}$$

$r_{it}$  observed return on security  $i$  for time  $t$

$r_{ft}$  observed return on the risk-free asset for time  $t$

$r_{Mt}$  observed return on the market portfolio for time  $t$

$e_{it}$  error term for time  $t$

Intercept term  $\alpha_i$  should be statistically 0  
for this equation to be consistent with CAPM

- Sample data:

Date	S&P return	Risk-free rate	S&P - risk-free rate	Oracle return	Oracle excess return
01/12/2000	0.03464	0.00473	0.02990	0.00206	-0.00267
01/01/2001	-0.09229	0.00413	-0.09642	-0.34753	-0.35165
01/02/2001	-0.06420	0.00393	-0.06813	-0.21158	-0.21550
01/03/2001	0.07681	0.00357	0.07325	0.07877	0.07521
01/04/2001	0.00509	0.00321	0.00188	-0.05322	-0.05643

# Estimating betas in CAPM model in Python

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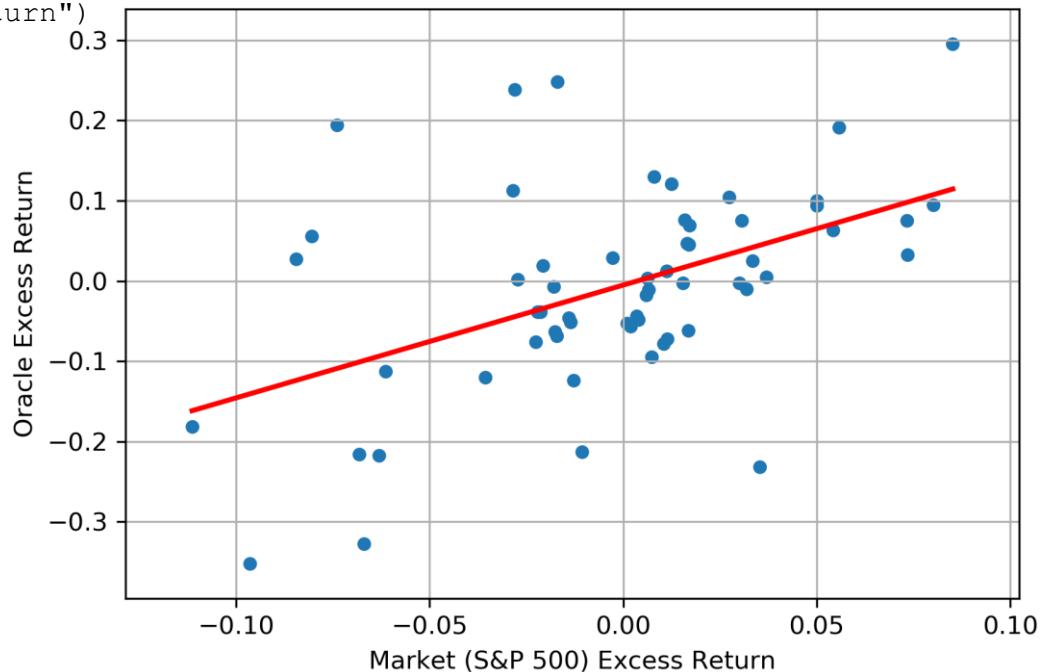
data = pd.read_excel('Beta.xls', sheet_name='Data', index_col=0, skiprows=2)
SP500ExcessRet = data["S&P - Risk-free rate"].values
OracleExcessRet = data["Oracle excess return"].values

# Run linear regression
model = sf.ols('OracleExcessRet ~ SP500ExcessRet', data)
fitted_model = model.fit()
beta = fitted_model.params
pvals = fitted_model.pvalues
betaint = fitted_model.conf_int()

New_SnPexcess = pd.DataFrame({'SP500ExcessRet': [SP500ExcessRet.min(), SP500ExcessRet.max()]})
Oracle_predictions = fitted_model.predict(New_SnPexcess)
ax = data.plot(kind='scatter', x='S&P - Risk-free rate', y='Oracle excess return', grid=True)
plt.plot(New_SnPexcess, Oracle_predictions, c='red', linewidth=2)
ax.set_xlabel("Market (S&P 500) Excess Return")
ax.set_ylabel("Oracle Excess Return")

```

<b>beta</b>	
Intercept	-0.005063
SP500ExcessRet	1.405566
<b>pvals</b>	
Intercept	0.729056
SP500ExcessRet	0.000113
<b>betaint</b>	
-0.034179	0.024054
0.726236	2.084897



# Arbitrage Pricing Theory (APT)

- CAPM is criticised for two assumptions:
  - investors are mean-variance optimizers
  - model is single-period
- Stephen Ross ("The Arbitrage Pricing Theory of Capital Asset Pricing", Journal of Economic Theory, 1976) developed an alternative model based on arbitrage arguments
- APT vs. CAPM:
  - APT is a more general approach to asset pricing than CAPM
  - CAPM considers variances and covariance's as possible measures of risk while APT allows for a number of risk factors
  - APT postulates that a security's expected return is influenced by a variety of factors, as opposed to just the single market index of CAPM
  - APT does not specify what factors are, but assumes that the relationship between security returns and factors is linear

$$r_i = \mathbb{E}[r_i] + \beta_{i1} \cdot f_1 + \dots + \beta_{iK} \cdot f_K$$