UNIVERSITY OF TORONTO
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Financial and Risk Analytics
Lecture 2 – Quantitative Risk Management
Vodafone Big Data Lab

📖 Simulation Modeling

Sums of random variables

- For any random variable x and a constant w

$$\mathbb{E}[w \cdot x] = w \cdot \mathbb{E}[x]$$

- Expectation of the sum of two random variables is equal to the sum of expectations

$$\mathbb{E}[x + y] = \mathbb{E}[x] + \mathbb{E}[y]$$

  and, therefore

$$\mathbb{E}[w_1 \cdot x + w_2 \cdot y] = w_1 \cdot \mathbb{E}[x] + w_2 \cdot \mathbb{E}[y]$$

- Example: expected value of a portfolio

$$\mathbb{E}[0.4 \cdot r_1 + 0.6 \cdot r_2] = 0.4 \cdot \mathbb{E}[r_1] + 0.6 \cdot \mathbb{E}[r_2]$$

- For the variance

$$\mathrm{var}[w \cdot x] = w^2 \cdot \mathrm{var}[x]$$

$$\mathrm{var}[x + y] = \mathrm{var}[x] + \mathrm{var}[y] + 2 \cdot \mathrm{cov}(x, y)$$

$$\mathrm{var}[w_1 \cdot x + w_2 \cdot y] = w_1^2 \cdot \mathrm{var}[x] + w_2^2 \cdot \mathrm{var}[y] + 2 \cdot w_1 \cdot w_2 \cdot \mathrm{cov}(x, y)$$

Sums of random variables

- How to compute the probability distribution of the sum of random variables?

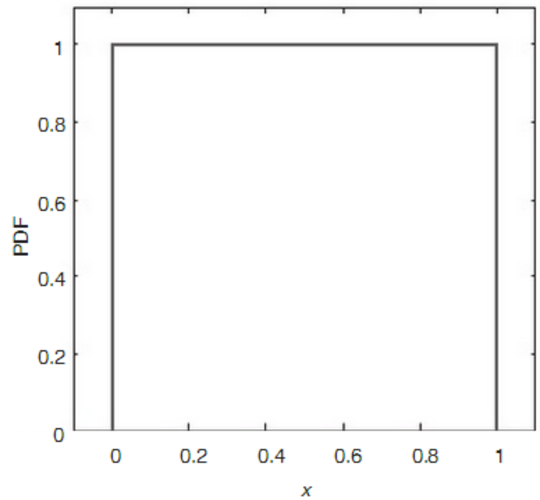
$$z = x + y$$

- We cannot add PDFs or PMFs

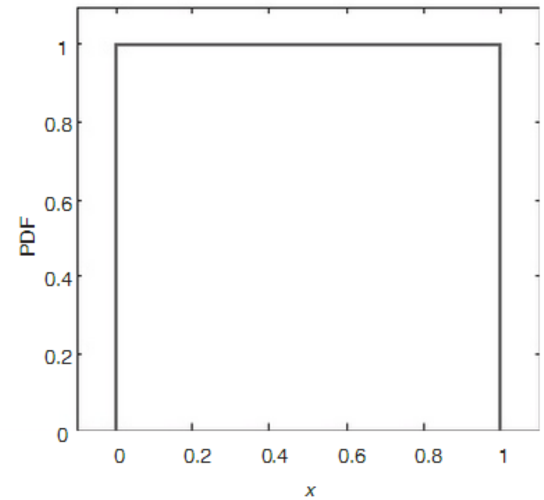
- The formula involves non-trivial integration and is known as convolution:

$$f_z(z) = \int_{-\infty}^{\infty} f_y(z - x)\, f_x(x)\, dx$$

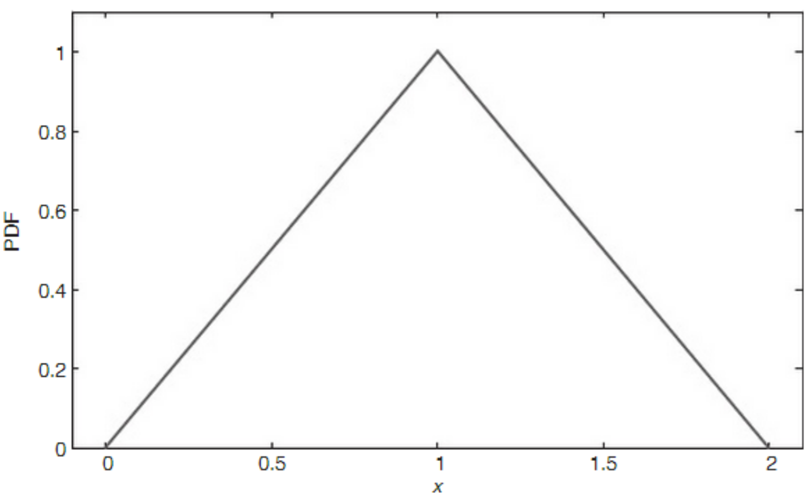
- Use simulation to evaluate such complex integrals



x



y



$z = x + y$

Sums of random variables

$$f_z(z) = \int_{-\infty}^{\infty} f_y(z-x)\, f_x(x)\, dx$$

$f_x(x) = 1$ only in $[0,1]$

$$f_z(z) = \int_0^1 f_y(z-x)\, dx$$

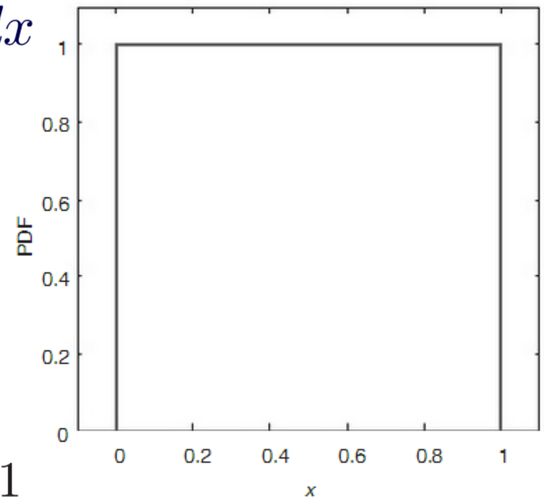
This is zero unless $0 \leq z - x \leq 1$
$(z-1 \leq x \leq z)$

Case 1: $0 \leq z \leq 1$

$$f_z(z) = \int_0^z dx = z$$

Case 2: $1 \leq z \leq 2$

$$f_z(z) = \int_{z-1}^1 dx = 2 - z$$



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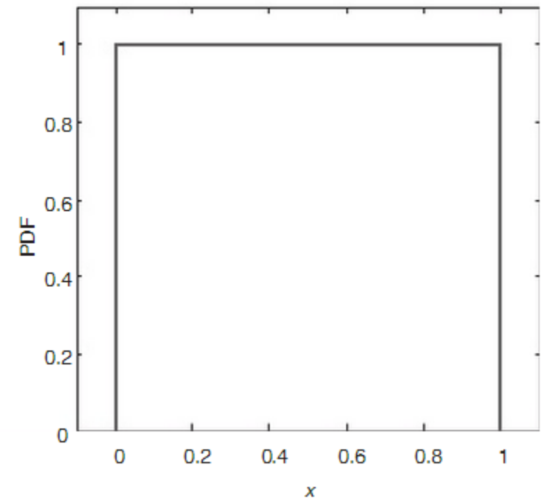

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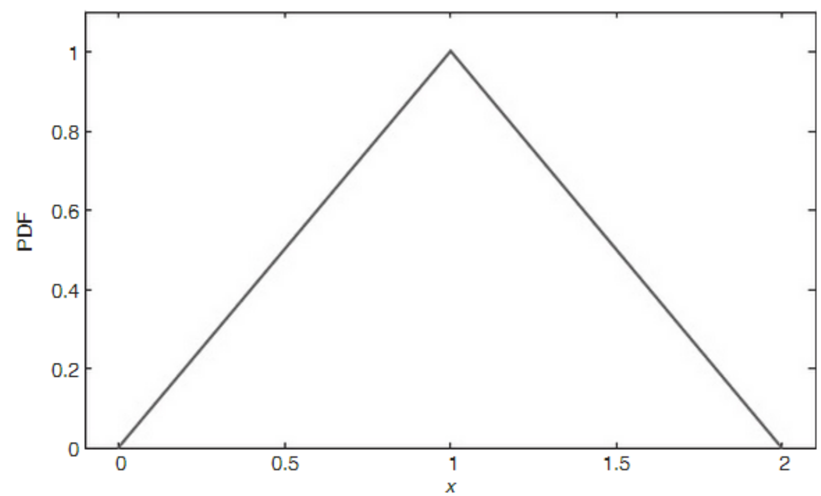
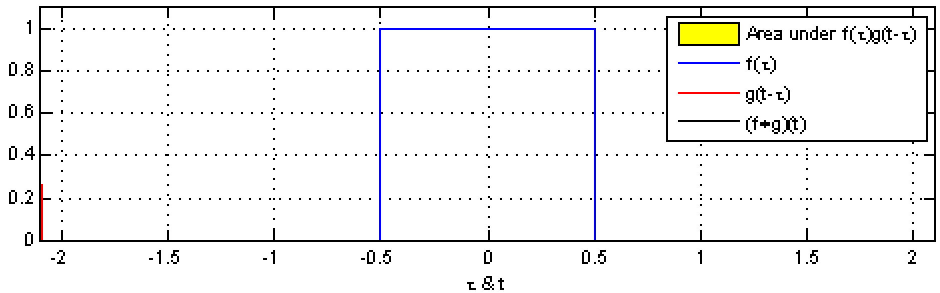


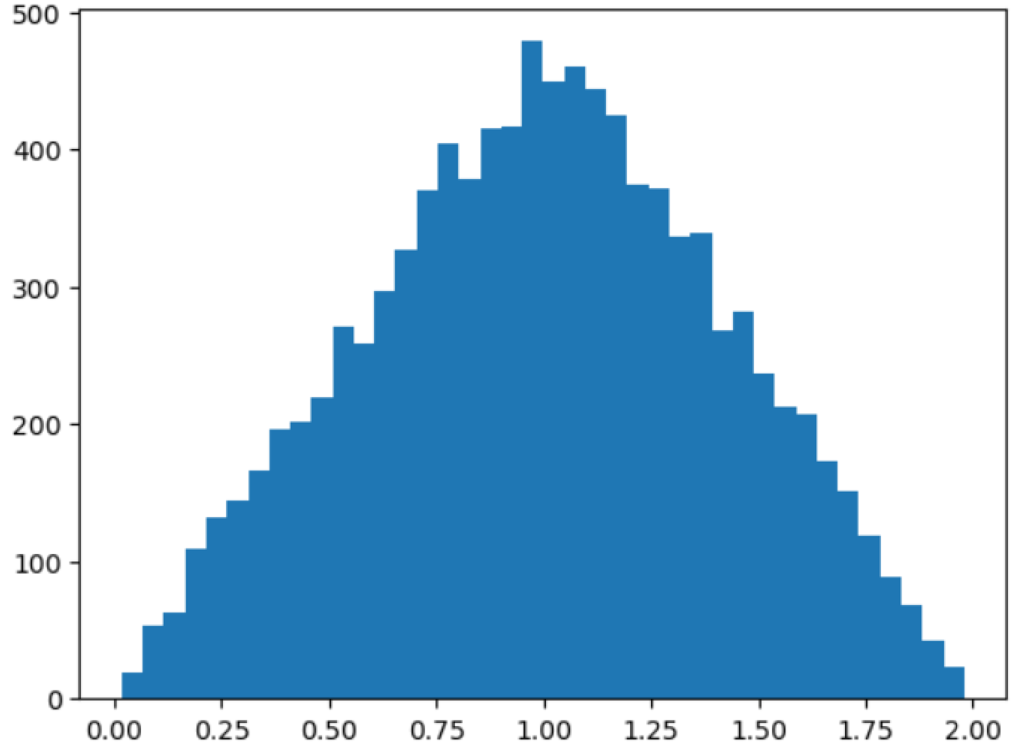
$z = x + y$

Sums of random variables



Sums of random variables via Monte Carlo simulations in Python

```
# Generate random variables x and y from uniform(0,1) distribution
x = np.random.uniform(0,1,size=10000)
y = np.random.uniform(0,1,size=10000)

# Compute random variable z as a linear function z=f(x,y)=x+y
z = x + y

# Plot approximation of distribution of z
plt.hist(z, bins=40);
```



Simulation modeling – example 1

- We want to invest \$1000 in the US stock market for 1 year: $v_0 = 1000$
- Invest into the S&P 500 market index (index fund)
- Value of investment at the end of year 1: v_1
- Market return over the time period [0,1) is $r_{0,1}$

$$v_1 = v_0 + r_{0,1} \cdot v_0 = (1 + r_{0,1}) v_0$$

- Generate scenarios for the market return over the year and compute v_1
  - decide on the number of scenarios and the set of scenarios for $r_{0,1}$
  - generate scenarios
    - ✓ use historic scenarios
    - ✓ draw randomly from historic scenarios (bootstrapping)
    - ✓ draw random numbers from the assumed distribution (Monte Carlo)
  - visualize and analyze the approximate probability distribution of v_1
- In our example we assume that the return of the market over the next year follow Normal distribution

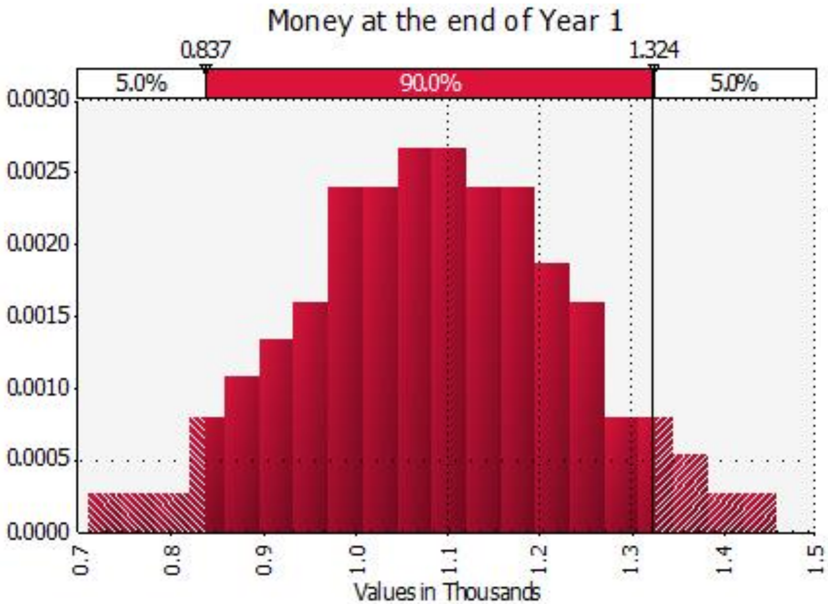
Simulation modeling – example 1

- Between 1977 and 2007, S&P 500 returned 8.79% per year on average with a standard deviation of 14.65%
- Generate 100 scenarios for the market return over the next year (draw 100 random numbers from a Normal distribution with mean 8.79% and standard deviation of 14.65%):

```
r01 = random.normal(0.0879, 0.1465, 100)
 0.099278
-0.004262
 ...
 0.488364
-0.119054
```

- Compute and plot $v_1 = (1 + r_{0,1})v_0$

Number of values	100
Mean	$ 1,087.90
Std Deviation	$ 146.15
Skewness	0.0034442
Kurtosis	2.871695
Mode	$ 1,118.96
5% Perc	$ 837.40
95% Perc	$ 1,324.00
Minimum	$ 708.81
Maximum	$ 1,458.52



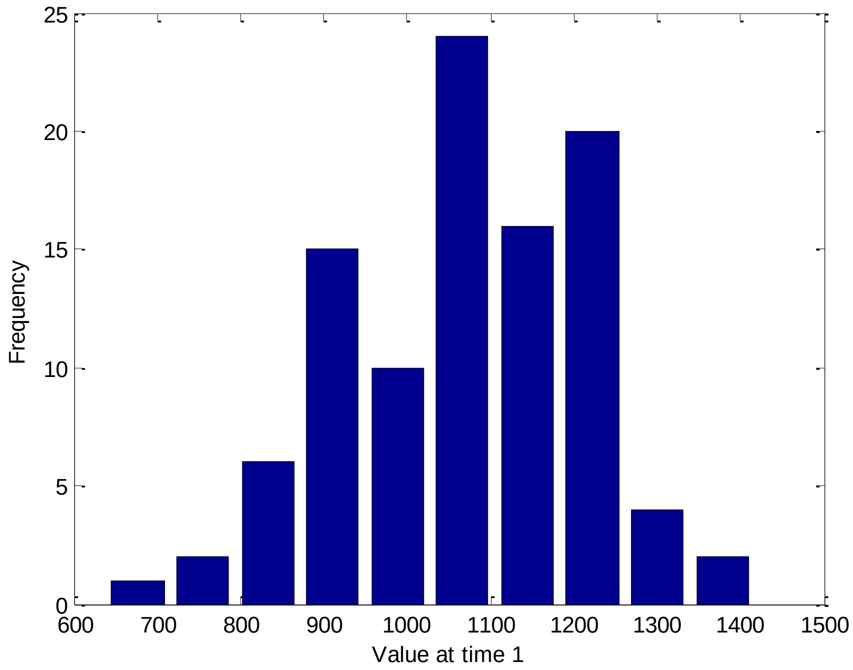
Simulation modeling – example 1 in Python

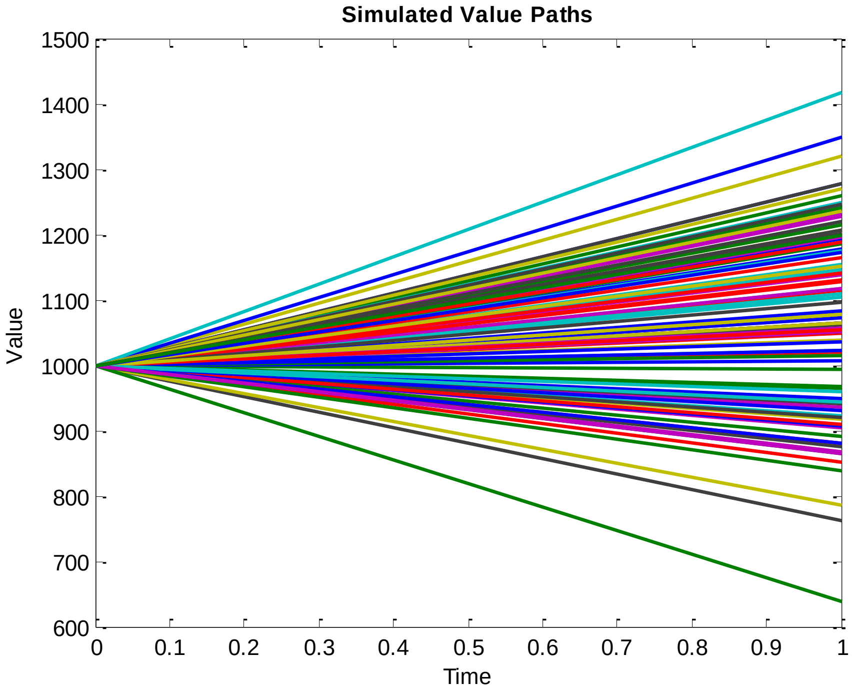
```
v0 = 1000 # initial capital

Ns = 100  # number of scenarios

# Generate Normal random variables
r01 = random.normal(0.0879, 0.1465, Ns)

# Distribution of value at the end of year 1
v1 = (1 + r01) * v0

# Compute statistical measures from the distribution
mean(v1) # mean
std(v1)  # standard deviation
min(v1)  # min
max(v1)  # max

# Compute percentiles/quantiles
percentile(v1, 5)          # 5th percentile
percentile(v1, [5, 50,95]) # 5th percentile, median and 95th percentile
sortedScen = sorted(v1)    # sort scenarios
mean(sortedScen[4:6])      # 5th percentile
mean(sortedScen[94:96])    # 95th percentile
mean(sortedScen[49:51])    # median
# Alternative way to compute percentiles/quantiles
sortedScen[int(Ns-(1-0.05)*Ns)-1] #  5th percentile
sortedScen[int(Ns-(1-0.95)*Ns)-1] # 95th percentile

# Plot a histogram of the distribution of outcomes for v1
hist, bins = histogram(v1)
positions = (bins[:-1] + bins[1:]) / 2
plt.bar(positions, hist, width=60)

# Plot simulated paths over time
for res in v1:
    plt.plot((0,1), (v0, res))
```

```
percentile(v1, 5) =

     1273.3145

sortedScen = sorted(v1)
mean(sortedScen[94:96]) =

     1273.3145

sortedScen[int(Ns-(1-0.95)*Ns)-1] =

     1269.1552
```

Simulation modeling – example 1 in Python

```
v0 = 1000 # initial capital
Ns = 100  # number of scenarios

# Generate Normal random variables
r01 = random.normal(0.0879, 0.1465, Ns)

# Distribution of value at the end of year 1
v1 = (1 + r01) * v0

# Plot a histogram of the distribution of outcomes for v1
hist, bins = histogram(v1)
positions = (bins[:-1] + bins[1:]) / 2
plt.bar(positions, hist, width=60)

# Plot simulated paths over time
for res in v1:
    plt.plot((0,1), (v0, res))
```





Why use simulation?

- Example 1 illustrates very basic Monte Carlo simulation system
- Simulation allows us to evaluate (approximately) a function of a random variable
  - in example 1 the function is simple $v_1 = (1 + r_{0,1})v_0$
  - given distribution of $r_{0,1}$, in some cases we can compute distribution of v_1 in closed form, e.g., if $r_{0,1}$ followed a Normal distribution, then v_1 also follows a Normal distribution with mean $(1 + \mu_{0,1})v_0$ and standard deviation $\sigma_{0,1}v_0$
  - if $r_{0,1}$ was not Normally distributed, or if the output variable v_1 were a more complex function of the input variable $r_{0,1}$, it would be difficult and practically impossible to derive the probability distribution of v_1 from the probability distribution of $r_{0,1}$
- Other advantages of simulation:
  - simulation enables visualizing probability distribution resulting from compounding probability distributions of multiple input variables (example 2)
  - simulation allows incorporating correlations between input variables (example 3)
  - simulation is a low-cost tool for checking the effect of changing a strategy on an output variable of interest (example 4)
- Next, we extend example 1 to illustrate such situations

Simulation modeling – example 2

- You are planning for retirement and decide to invest in the market for the next 30 years (instead of only the next year as in example 1). Your initial capital is still $v_0 = 1000$
- Assume that every year your investment returns from investing into the S&P 500 will follow a Normal distribution with the mean and standard deviation as in example 1.
- Value of investment after 30 years: v_{30}
- The return over 30 years will depend on the realization of 30 random variables

$$r_{0,t} = (1 + r_{0,1})(1 + r_{1,2}) \dots (1 + r_{t-1,t}) - 1$$

$$v_{0,t} = (1 + r_{0,t})v_0$$

$$v_{30} = (1 + r_{0,1})(1 + r_{1,2}) \dots (1 + r_{29,30})v_0$$

- Observations:
  - sum of Normal random variables is Normal
  - here we have multiplication of Normal random variables, is it Normal?

Simulation modeling – example 2

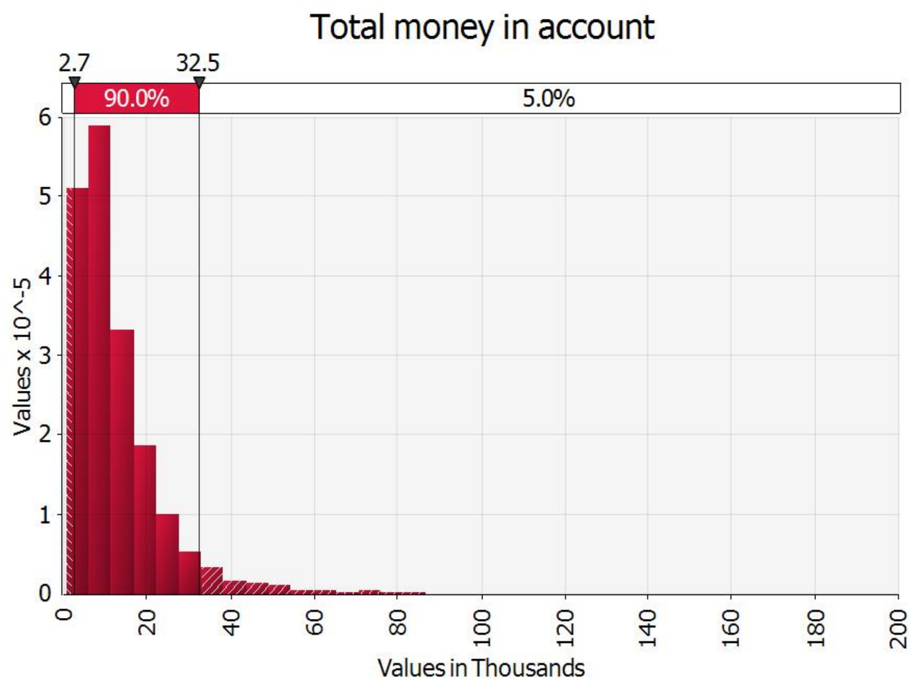
- Between 1977 and 2007, S&P 500 returned 8.79% per year on average with a standard deviation of 14.65%
- Simulate 30 columns of 100 observations each of single period returns:

```
r_speriod30 = random.normal(0.0879, 0.1465, (100,30))
```

$$\begin{array}{cccc} 0.323770 & 0.188574 & \ldots & 0.024316 \\ 0.060499 & 0.142391 & \ldots & 0.093383 \\ \ldots & \ldots & \ldots & \ldots \\ -0.019156 & -0.120207 & \ldots & 0.071931 \\ 0.289694 & 0.038724 & \ldots & 0.356291 \end{array}$$

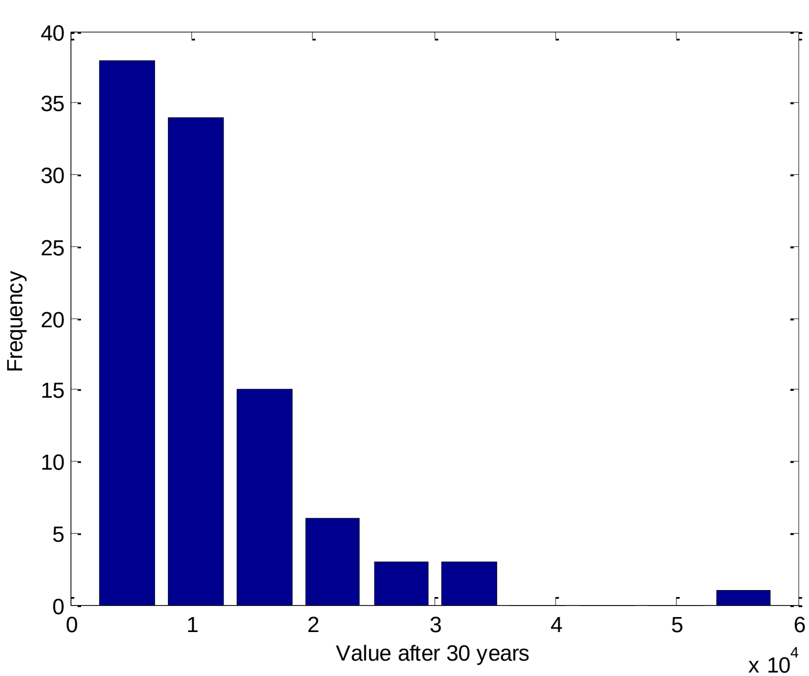
- Compute and plot $v_{30} = (1 + r_{0,30})$

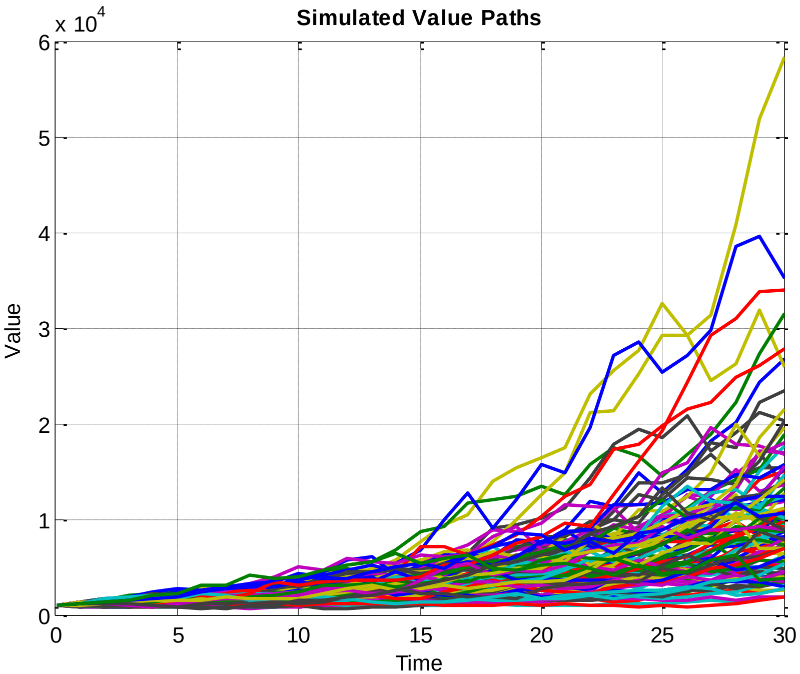
Number of values	5000
Mean	$ 12,587.62
Std Deviation	$ 10,948.39
Skewness	3.349066
Kurtosis	28.24214
Mode	$ 4,458.97
5% Perc	$ 2,655.55
95% Perc	$ 32,481.38
Minimum	$ 609.75
Maximum	$194,355.00



Simulation modeling – example 2 in Python

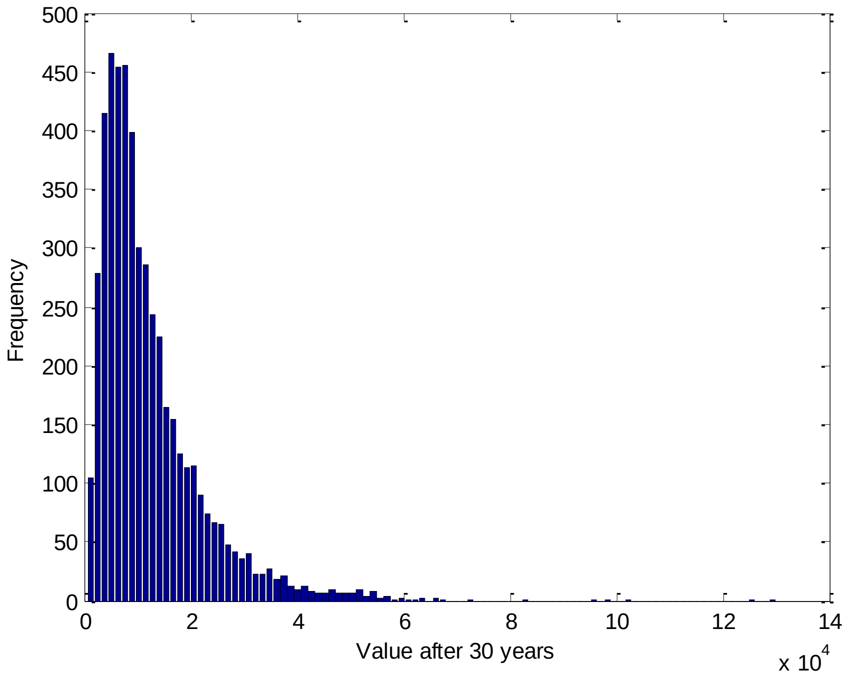
```
v0 = 1000  # initial capital
Ns = 100   # number of scenarios

# Generate Normal random variables
r_speriod30 = random.normal(0.0879, 0.1465, (Ns, 30))

# Distribution of value at the end of year 30
v30 = prod(1 + r_speriod30 , 1) * v0

# Plot a histogram of the distribution of outcomes for v30
hist, bins = histogram(v30); positions = (bins[:-1]+bins[1:])/2; plt.bar(positions, hist)

# Plot simulated paths over time
for scenario in r_speriod30:
    y = [prod(1 + scenario[0:i]) * v0 for i in range(0,31)]
    plt.plot(range(0,31), y)
```

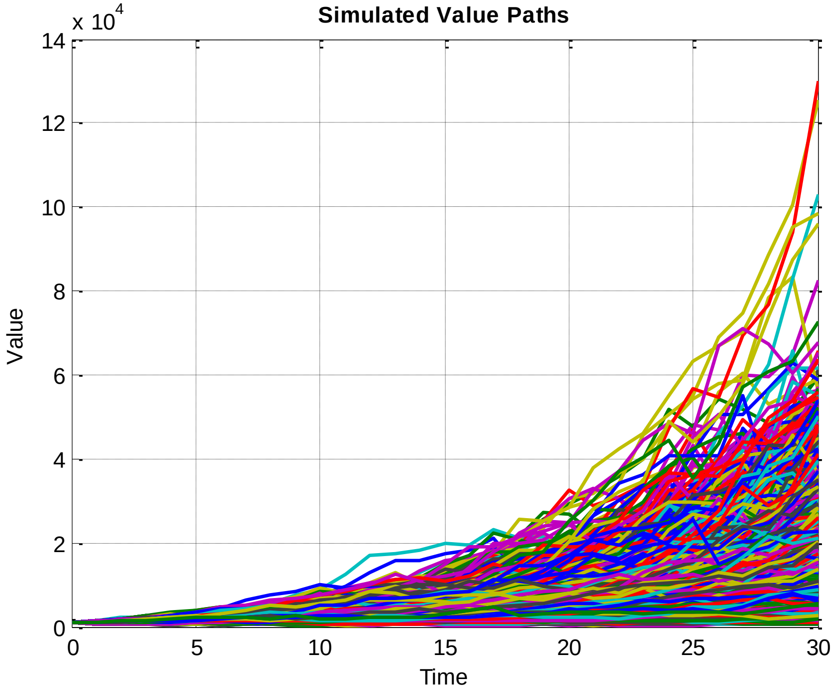




Simulation modeling – example 2 in Python

```
v0 = 1000  # initial capital
Ns = 5000  # number of scenarios

# Generate Normal random variables
r_speriod30 = random.normal(0.0879, 0.1465, (Ns, 30))

# Distribution of value at the end of year 30
v30 = prod(1 + r_speriod30 , 1) * v0

# Plot a histogram of the distribution of outcomes for v30
hist, bins = histogram(v30,bins=100); positions = (bins[:-1]+bins[1:])/2; plt.bar(positions, hist)

# Plot simulated paths over time
for scenario in r_speriod30:
    y = [prod(1 + scenario[0:i]) * v0 for i in range(0,31)]
    plt.plot(range(0,31), y)
```





Simulation modeling – example 3

- You are planning for retirement and decide to invest in the market for the next 30 years. Your initial capital is $v_0 = 1000$
- You have an opportunity to invest in stocks and Treasury bonds:
  - allocate 50% of your capital to the stock market (S&P 500 index fund) today
  - allocate 50% of your capital to bonds today
- Assume that every year your investment returns from investing into the S&P 500 and Treasury bonds will follow a Normal distribution with the mean and standard deviation as in example 2 (for S&P 500), mean 4% and standard deviation 7% for bonds. Assume correlation -0.2 between the stock market and the Treasury bond market.
- Covariance matrix:

$$\begin{pmatrix} 0.1465^2 & -0.2 \cdot 0.1465 \cdot 0.07 \\ -0.2 \cdot 0.1465 \cdot 0.07 & 0.07^2 \end{pmatrix} = \begin{pmatrix} 0.0215 & -0.0021 \\ -0.0021 & 0.0049 \end{pmatrix}$$

- Value of investment after 30 years: v_{30}

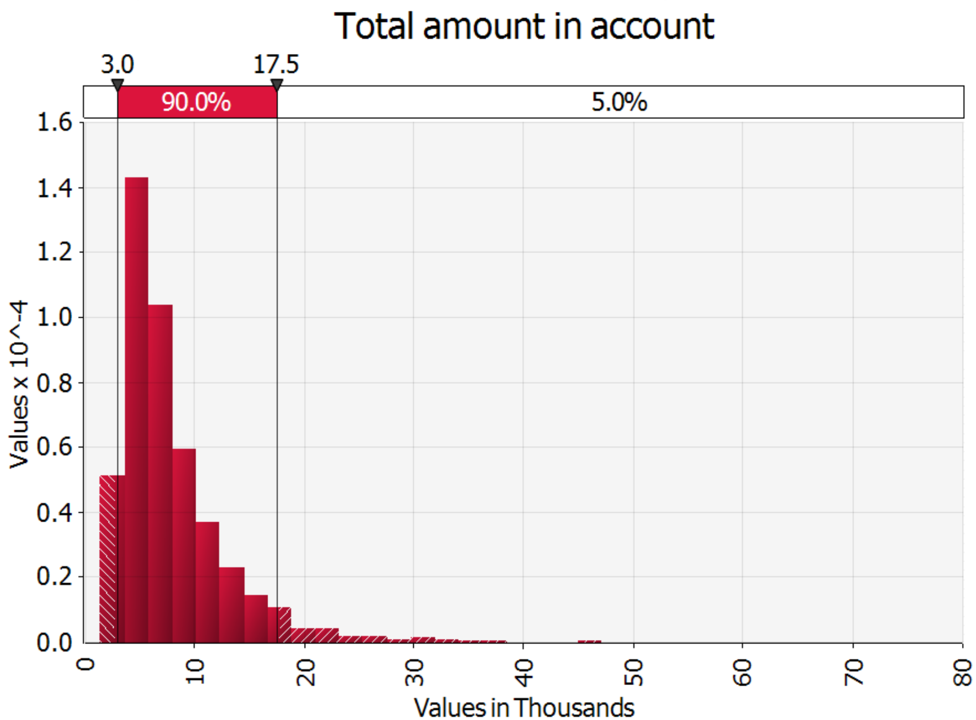
Simulation modeling – example 3

- Simulate 30 years of 100 observations each of single period correlated returns:

```
stockRet = ones(Ns); bondsRet = ones(Ns)

for year in range(1, 31):
    scenarios = random.multivariate_normal(mu, covMat, Ns)
    stockRet *= (1 + scenarios[:,0])
    bondsRet *= (1 + scenarios[:,1])

v30 = 0.5 * v0 * stockRet + 0.5 * v0 * bondsRet
```

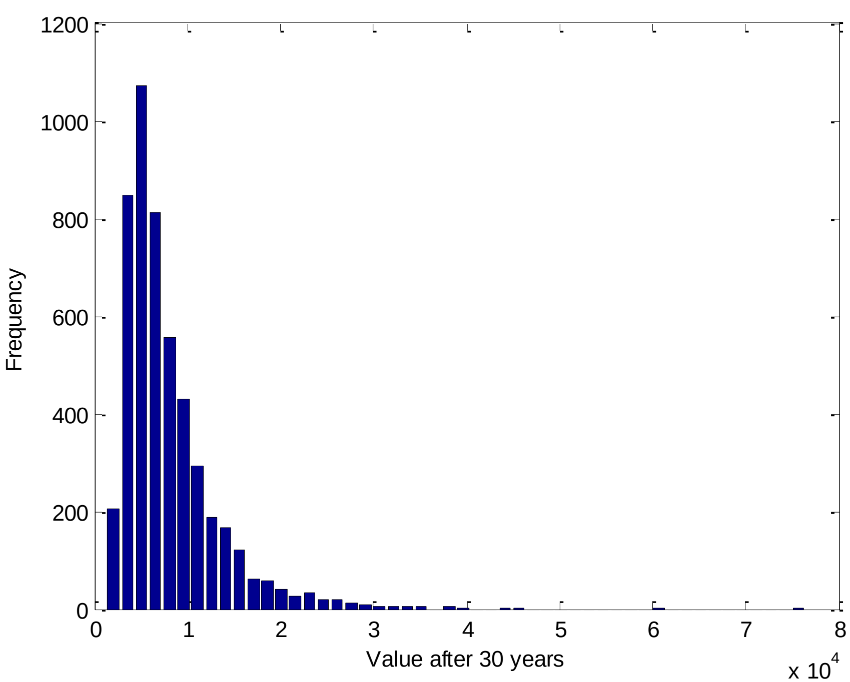
- Compute and plot $v_{30} = 0.5v_0(1 + r^s_{0,30}) + 0.5v_0(1 + r^b_{0,30})$

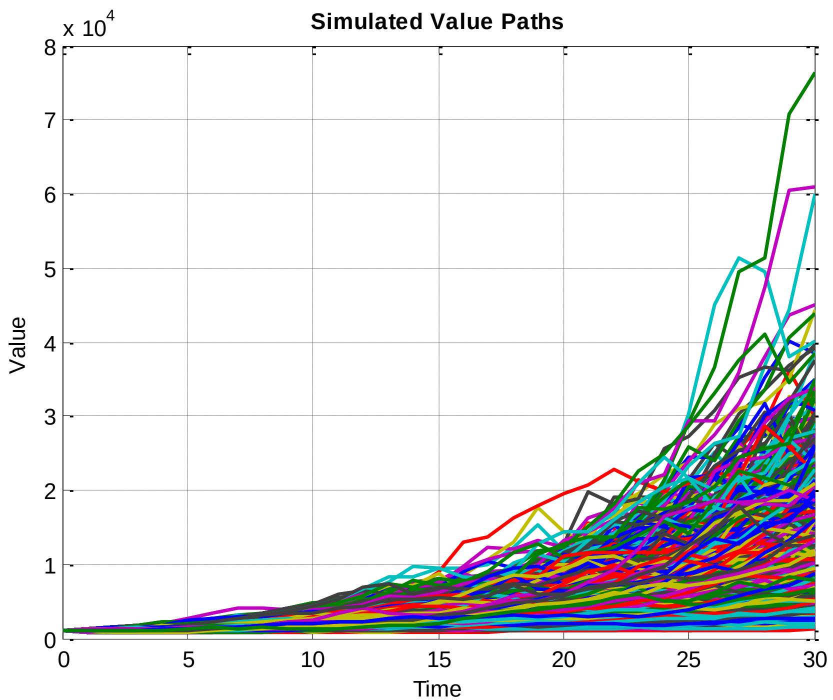
Number of values	5000
Mean	$ 7,892.80
Std Deviation	$ 5,233.10
Skewness	2.921482
Kurtosis	20.48869
Mode	$ 5,050.96
5% Perc	$ 2,951.82
95% Perc	$17,457.43
Minimum	$ 1,408.63
Maximum	$79,729.34



Simulation modeling – example 3 in Python

```
v0 = 1000  # initial capital
Ns = 5000  # number of scenarios

mu = [0.0879, 0.04] # expected return
covMat = [[0.1465**2, -0.0021],[-0.0021, 0.07**2]] # covariance matrix

# Generate correlated Normal random variables
stockRet = ones(Ns)
bondsRet = ones(Ns)
for year in range(1, 31):
    scenarios = random.multivariate_normal(mu, covMat, Ns)
    stockRet *= (1 + scenarios[:,0])
    bondsRet *= (1 + scenarios[:,1])

# Distribution of value at the end of year 30
v30 = 0.5 * v0 * stockRet + 0.5 * v0 * bondsRet
```





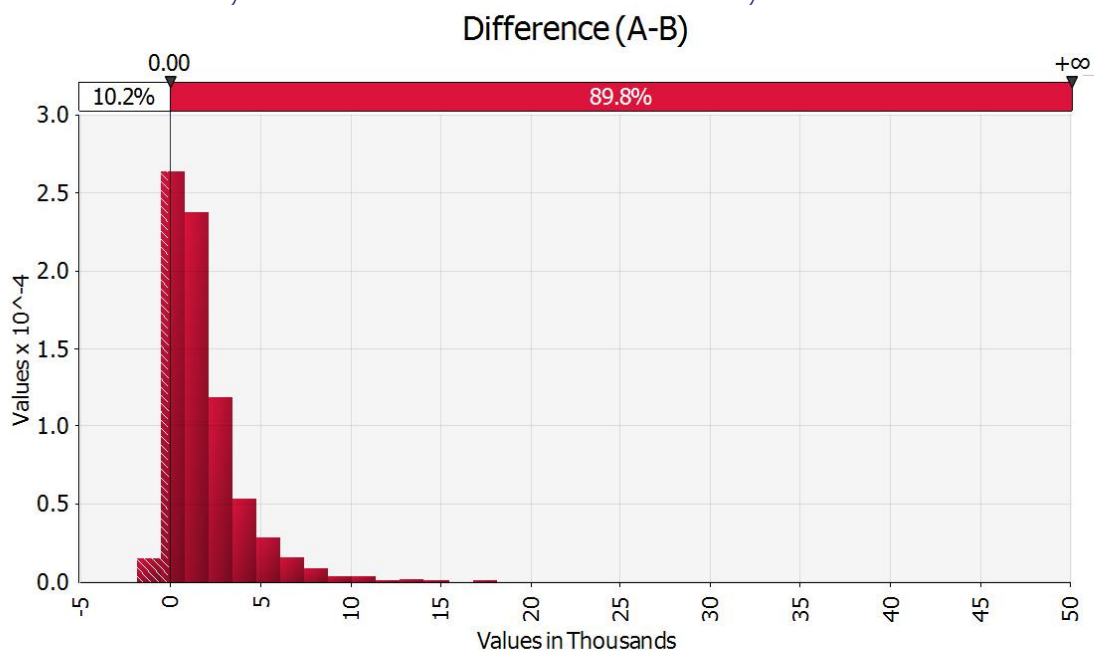
Simulation modeling – example 4

- Using scenario generation procedure from example 3 for decision-making
- Compare portfolios:
  - 50-50 portfolio allocation in stocks and bonds (Strategy A)
  - 30-70 portfolio allocation in stocks and bonds (Strategy B)

```
v30comp = []
for w in arange(0.2, 1.01, 0.2):
    v30comp += [w * v0 * stockRet + (1 - w) * v0 * bondsRet]
```

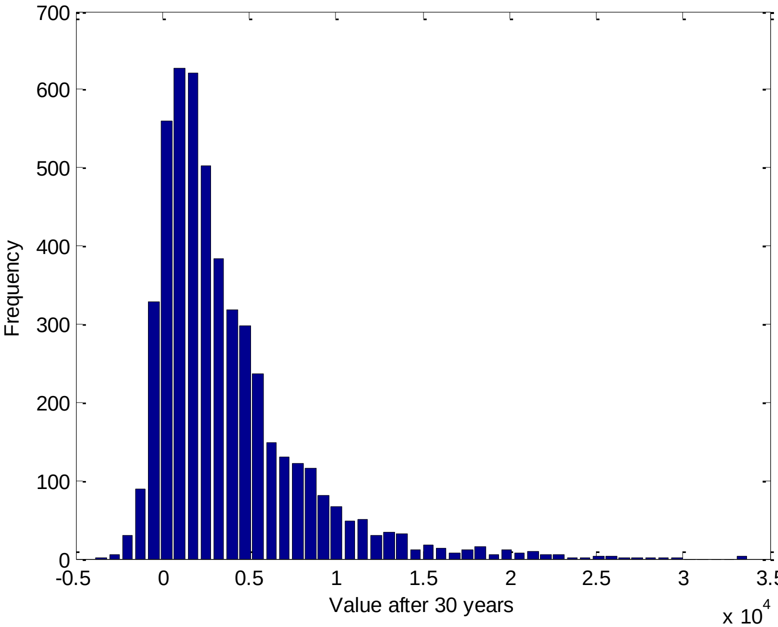
- Compute and plot $v_{30} = w_s v_0(1 + r^s_{0,30}) + w_b v_0(1 + r^b_{0,30})$

Number of values	5000
Mean	$ 1,865.13
Std Deviation	$ 2,214.87
Skewness	3.506451
Kurtosis	40.18968
Mode	$ 687.75
5% Perc	$ -254.41
95% Perc	$ 6,027.23
Minimum	$-1,829.78
Maximum	$45,972.08



Simulation modeling – example 4 in Python

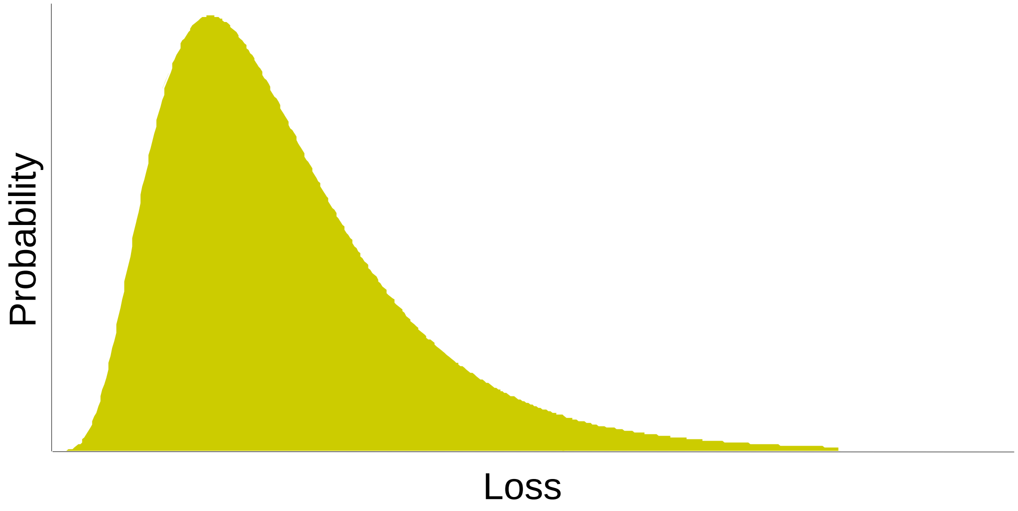
```
v0 = 1000  # initial capital
Ns = 5000  # number of scenarios

mu = [0.0879, 0.04] # expected return
covMat = [[0.1465**2, -0.0021],[-0.0021, 0.07**2]] # covariance matrix

# Generate correlated Normal random variables
stockRet = ones(Ns)
bondsRet = ones(Ns)
for year in range(1, 31):
    scenarios = random.multivariate_normal(mu, covMat, Ns)
    stockRet *= (1 + scenarios[:,0])
    bondsRet *= (1 + scenarios[:,1])

# Compute portfolios by iterating through different combinations of weights
v30comp = []
for w in arange(0.2, 1.01, 0.2):
    v30comp += [w * v0 * stockRet + (1 - w) * v0 * bondsRet]

# Plot a histogram of the distribution of
# differences in outcomes for v30
# (Stratery 4 - Strategy 2)
v30d = v30comp[3] - v30comp[1]

hist, bins = histogram(v30d, bins = 50)
positions = (bins[:-1]+bins[1:])/2
width = (bins[1]-bins[0])*0.9
plt.bar(positions, hist, width=width)
```



🕮 Simulation and Optimization in Finance and Risk Management

Simulation and optimization in financial risk management

- How much am I likely to lose (gain)?
- Uncertainty pertains to the future values (prices, returns) of financial instruments
- To measure and manage risk, we try to quantify this uncertainty
  - ✓ Identify a set of possible outcomes and their probabilities
- Simulation is central to this process
- Optimization uses the simulation results to construct portfolios

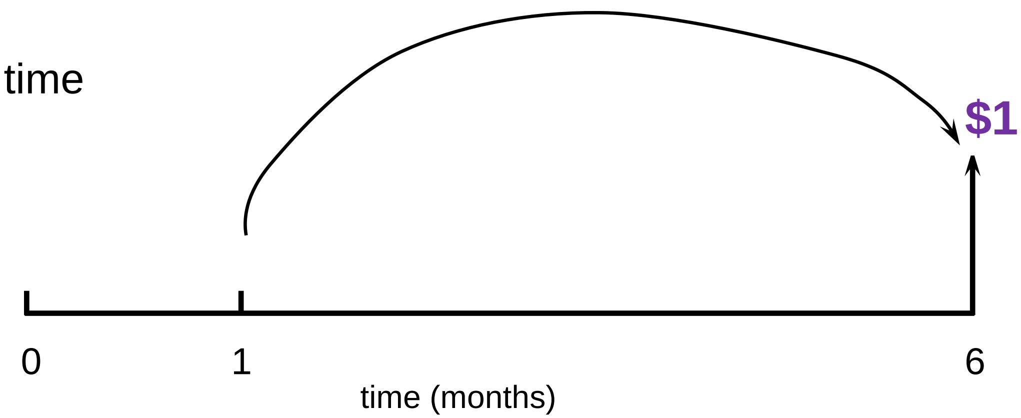


Pricing bond – example

1. Model the risk factor: 5-month interest rate ($r_{[1,5]}$)

t = 0 – current time
τ = 1 – risk horizon
T = 6 – bond maturity time

$r_{[1,5]} \longleftarrow$ random, e.g., log-Normal



2. Compute the zero coupon bond price at time $\tau = 1$

<u>uncertain value (price)</u>

$$v_1 = f(r_{[1,5]}) = e^{-r_{[1,5]} \cdot \frac{5}{12}}$$

<u>sampled value (price)</u>

$$v_{i1} = f(r_{i[1,5]}) = e^{-r_{i[1,5]} \cdot \frac{5}{12}}$$

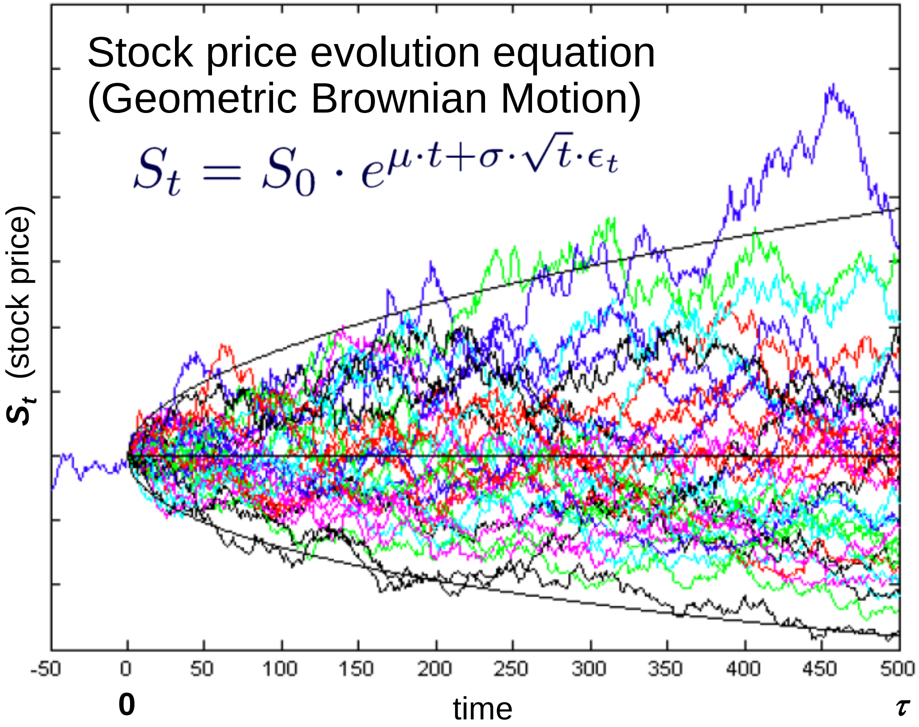
3. Compute the price at $\tau = 1$ if the bond pays a coupon c at time $t = 3$

<u>sampled value (price)</u>

$$v_{i1} = f(r_{i[1,2]}, r_{i[1,5]}) = c \cdot e^{-r_{i[1,2]} \cdot \frac{2}{12}} + e^{-r_{i[1,5]} \cdot \frac{5}{12}}$$

Pricing option – example

1. Model the risk factor: underlying stock price (S_t)



τ is the time at which we compute the option value

T is the option maturity time

2. Model the risk factor: discount rate ($r_{[\tau, T-\tau]}$)

3. Compute option price for path (scenario) $\boldsymbol{i}$ at time $\boldsymbol{\tau}$

$$v_{i\tau} = f(r_{i[\tau, T-\tau]}, S_{i\tau})$$ ⟵ European option, price with Black-Scholes formula

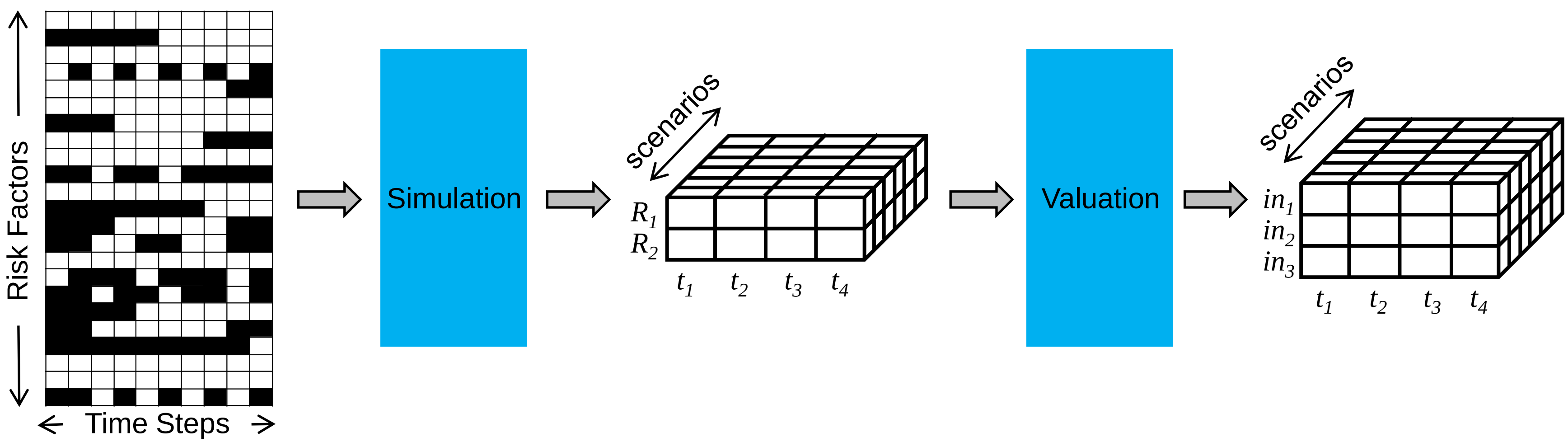
$$v_{i\tau} = f(r_{i[\tau, T-\tau]}, S_{it_1}, S_{it_2}, \ldots, S_{iT})$$ ⟵ Asian option, price is path-dependent

Mark-to-Future framework

Historical Time Series
- Interest rates
- Exchange rates
- Equity indexes
- Commodities
- Individual securities

Risk Factor Scenarios
- Consistent paths of risk factor changes
- Financial instruments may be path-dependent

Instrument Scenarios
- Consistent paths of instrument values
- Mark-to-Future cube



Simulation Methods
- Historical sampling
- Stochastic models
- Calibration
- Codependence
- Transformations (PCA)

Valuation Methods

$$v_{ijt} = f_j(\boldsymbol{R}_{i,t\le\tau})$$

- Linear factor models
- Analytic
- Numerical methods

Portfolio valuation

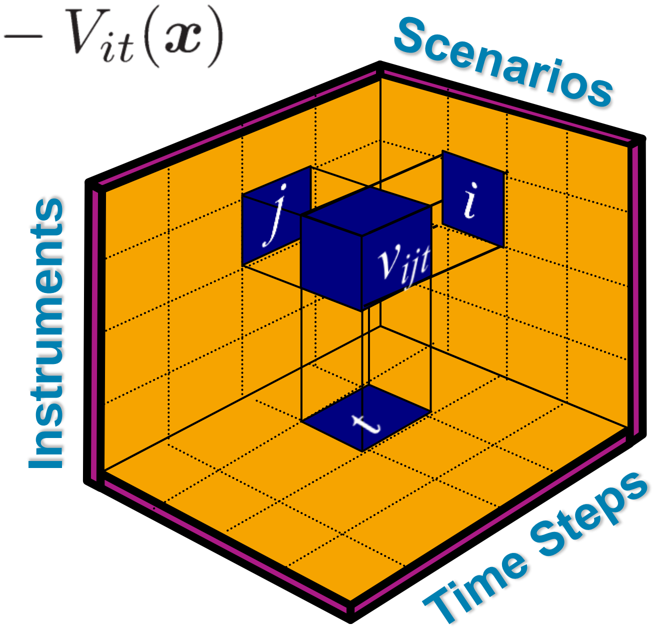
- A portfolio is a set of positions, $\boldsymbol{x}$, where x_j is the number of units of instrument j, $j = 1, \ldots, N$
- From the instrument values in the MtF cube, the portfolio value at time t in scenario i is

$$V_{it}(\boldsymbol{x}) = \sum_{j=1}^{N} v_{ijt} \cdot x_j$$

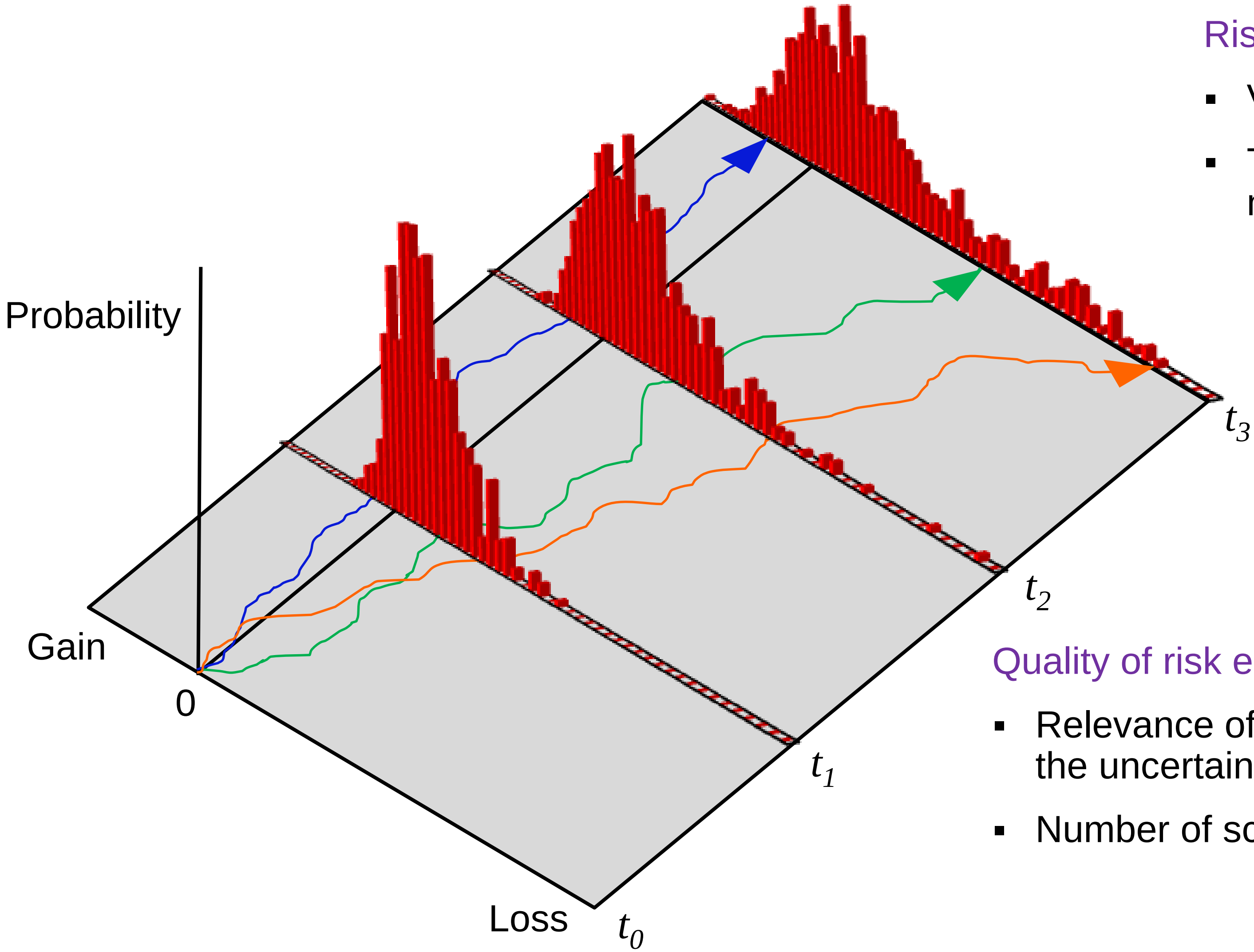
- Given the initial portfolio value, it is straightforward to compute changes in value (profits and losses) and returns in each scenario

$$\text{loss} \longrightarrow \quad \ell_{it}(\boldsymbol{x}) = V_0(\boldsymbol{x}) - V_{it}(\boldsymbol{x})$$

- An MtF cube can be used to value portfolios that hold any subset of its instruments
  - No need to re-simulate when positions change
  - $\boldsymbol{x}$ are decision variables in portfolio optimization



Empirical portfolio loss distributions over time



Risk

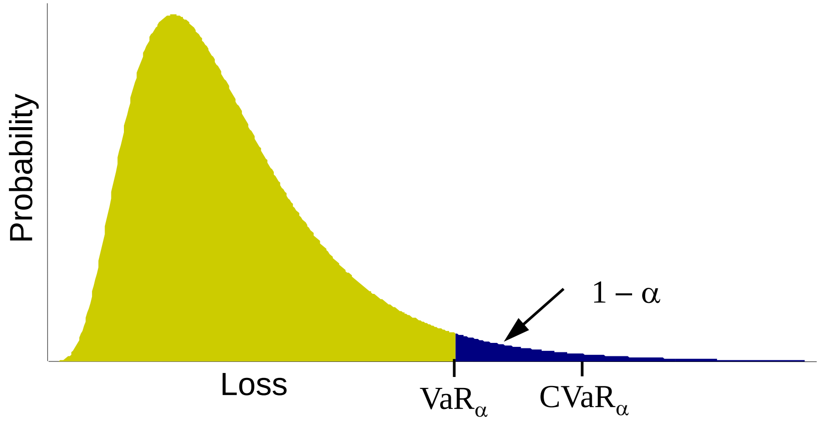
- Variance
- Tail-based risk measures

Quality of risk estimates depends on

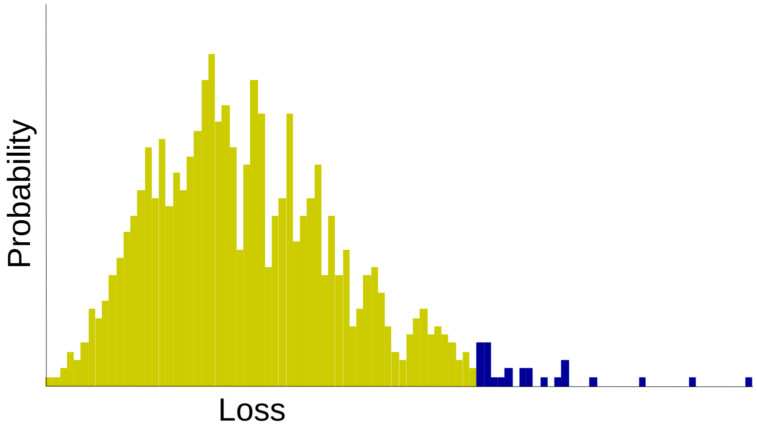
- Relevance of scenarios (capture the uncertainty)
- Number of scenarios

Tail-based risk measures

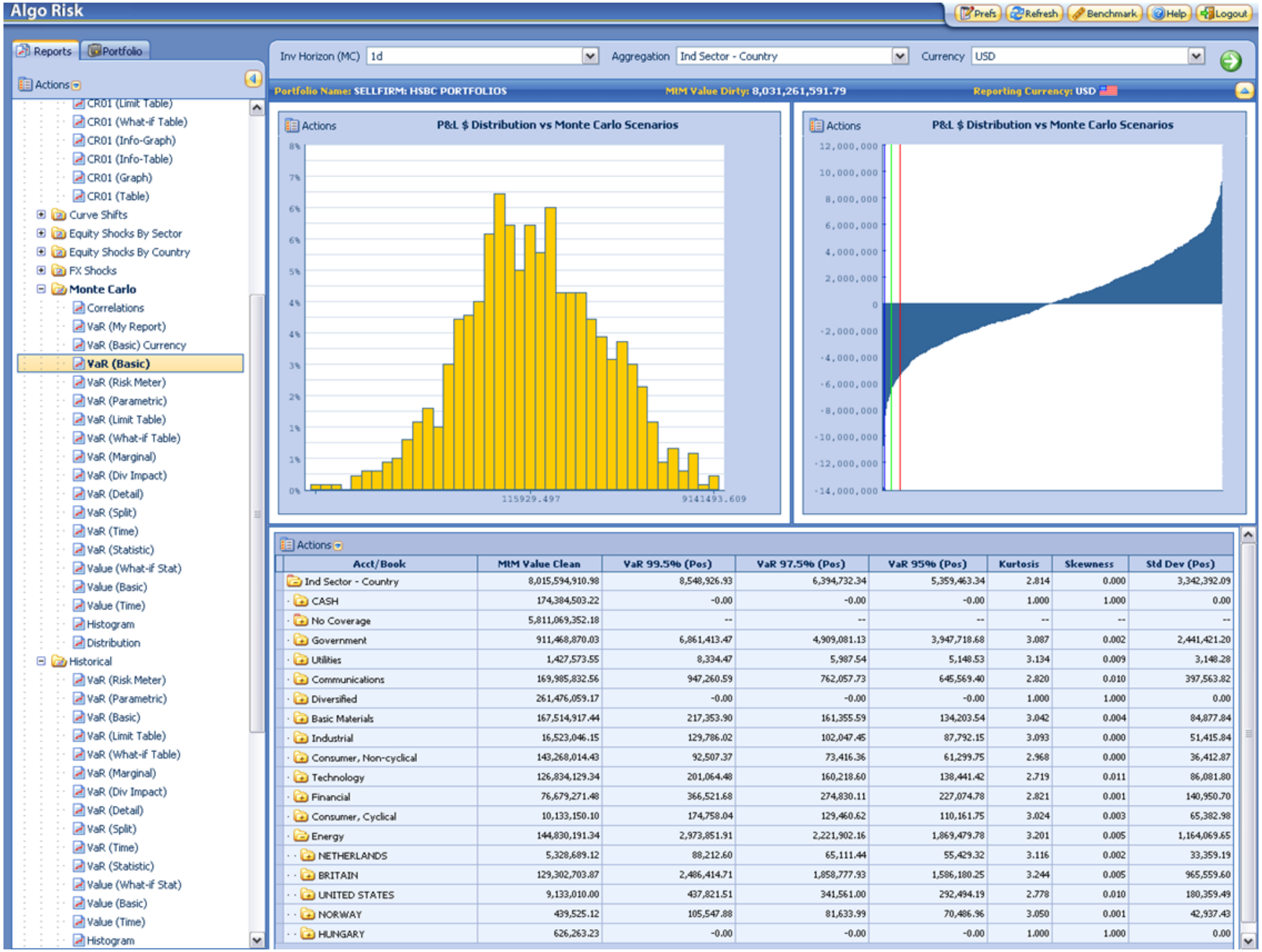
- "Value-at-Risk": VaR_α is the loss that is likely to be exceeded with probability $(1 - \alpha)$
- "Conditional Value-at-Risk" or "Expected Shortfall": CVaR_α is the average loss beyond VaR_α



- Given a random sample of M losses, ordered from smallest to largest, with $M\alpha$ an integer
  - VaR_α is estimated by loss $M\alpha$
  - CVaR_α is estimated by the average of losses $M\alpha+1$ through M
  - E.g., if $M = 100$ and $\alpha = 95\%$ then VaR_α is estimated by the 6[th] largest loss and CVaR_α is estimated by the average of the 5 largest losses

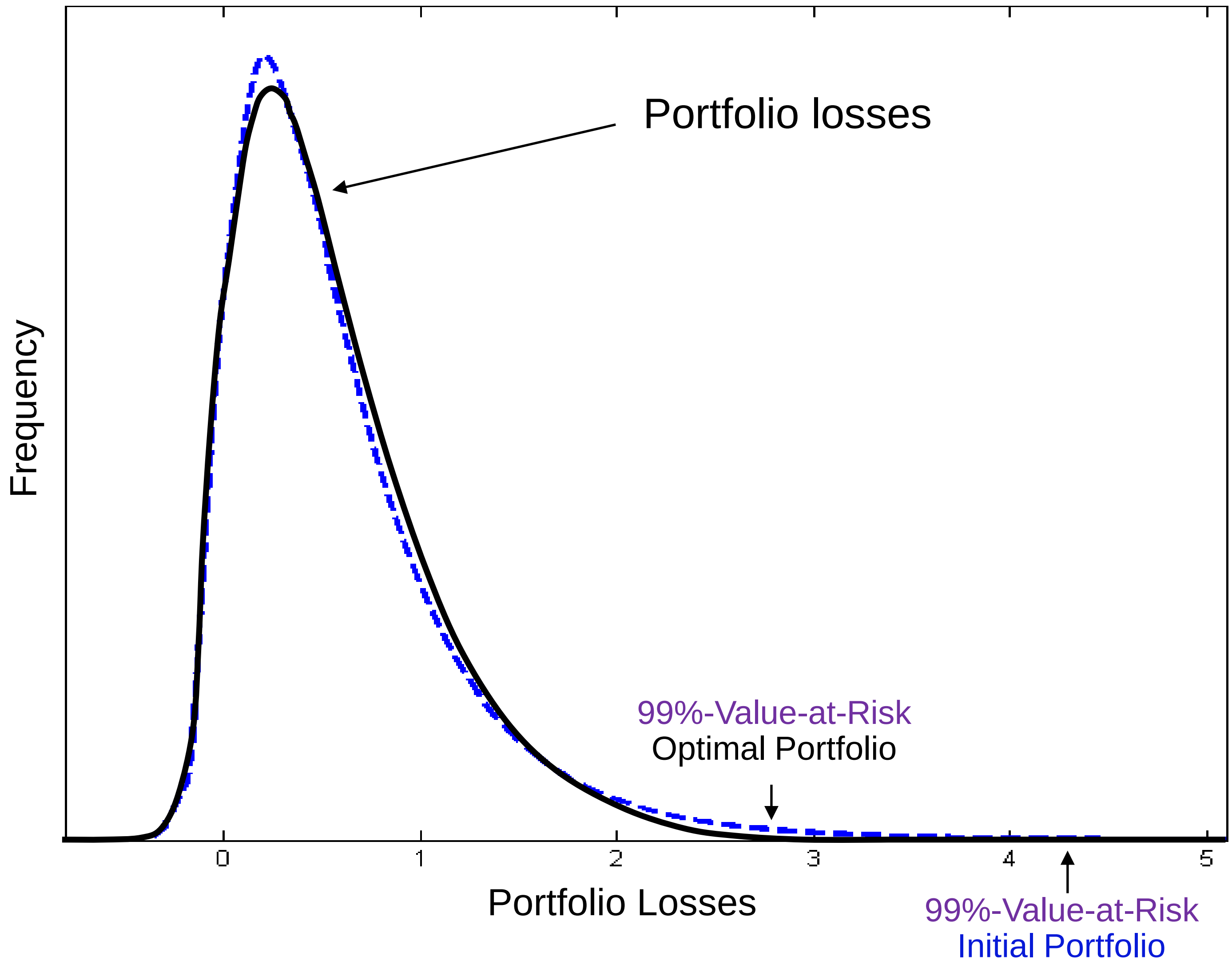


Risk reporting – VaR and P&L



Acct/Book	MtM Value Clean	VaR 99.5% (Pos)	VaR 97.5% (Pos)	VaR 95% (Pos)	Kurtosis	Skewness	Std Dev (Pos)
Ind Sector - Country	8,015,594,910.98	8,548,926.93	6,394,732.34	5,359,463.34	2.814	0.000	3,342,392.09
CASH	174,384,503.22	-0.00	-0.00	-0.00	1.000	1.000	0.00
No Coverage	5,811,069,352.18	--	--	--	--	--	--
Government	911,468,870.03	6,861,413.47	4,909,081.13	3,947,718.68	3.087	0.002	2,441,421.20
Utilities	1,427,573.55	8,334.47	5,987.54	5,148.53	3.134	0.009	3,148.28
Communications	169,985,832.56	947,260.59	762,057.73	645,569.40	2.820	0.010	397,563.82
Diversified	261,476,059.17	-0.00	-0.00	-0.00	1.000	1.000	0.00
Basic Materials	167,514,917.44	217,353.90	161,355.59	134,203.54	3.042	0.004	84,877.84
Industrial	16,523,046.15	129,786.02	102,047.45	87,792.15	3.093	0.000	51,415.84
Consumer, Non-cyclical	143,268,014.43	92,507.37	73,416.36	61,299.75	2.968	0.000	36,412.87
Technology	126,834,129.34	201,064.48	160,218.60	138,441.42	2.719	0.011	86,081.80
Financial	76,679,271.48	366,521.68	274,830.11	227,074.78	2.821	0.001	140,950.70
Consumer, Cyclical	10,133,150.10	174,758.04	129,460.62	110,161.75	3.024	0.003	65,382.98
Energy	144,830,191.34	2,973,851.91	2,221,902.16	1,869,479.78	3.201	0.005	1,164,069.65
NETHERLANDS	5,328,689.12	88,212.60	65,111.44	55,429.32	3.116	0.002	33,359.19
BRITAIN	129,302,703.87	2,486,414.71	1,858,777.93	1,586,180.25	3.244	0.005	965,559.60
UNITED STATES	9,133,010.00	437,821.51	341,561.00	292,494.19	2.778	0.010	180,359.49
NORWAY	439,525.12	105,547.88	81,633.99	70,486.96	3.050	0.001	42,937.43
HUNGARY	626,263.23	-0.00	-0.00	-0.00	1.000	1.000	0.00

Portfolio tail risk optimization



🕮 Financial Risk Management

Which risks are worth taking?

Financial risk management

- Risk management is a systematic approach for minimizing exposure to risk
- Risks should be measured and managed across diverse financial instruments, geographies and risk types
- Financial risks are classified to:
  - market risk – movement of an entire market
  - credit risk – risk that an obligor may default
  - operational risk – impact of operational events
  - liquidity risk – difficulty of selling an asset
  - currency risk – risk from international exposure
  - company-specific risks, etc.

Total Risk

Non-Business Risk (regulatory capital):

Market risk
Credit risk
Operational risk

Business Risk (no regulatory capital):

Risk from strategic decisions
Reputation risk

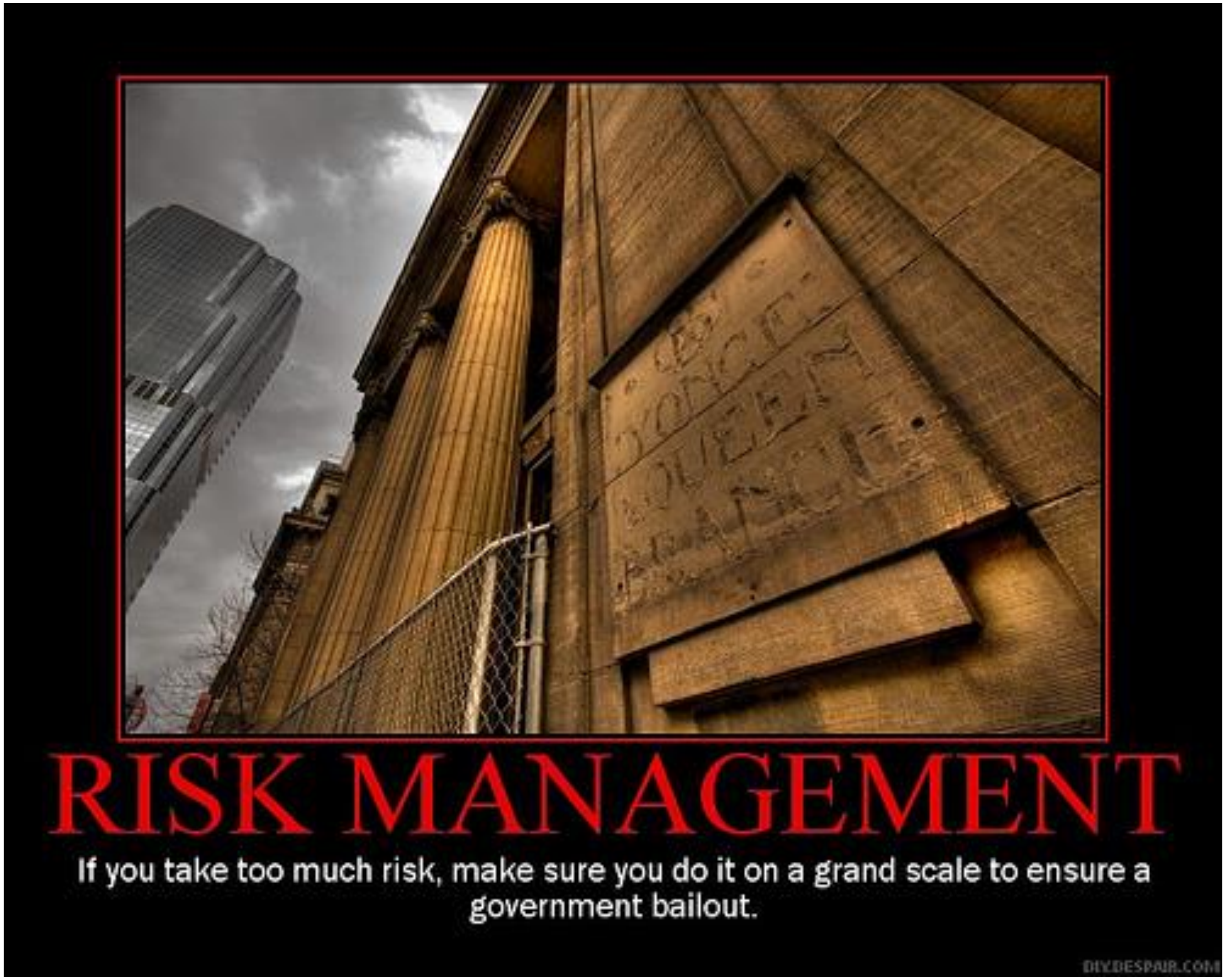
Source: J. Hull. Risk Management and Financial Institutions, 2012

Risk limits

- Risk limits
  - Risk must be quantified and risk limits set
  - Exceeding risk limits not acceptable even when profits result
  - Do not assume that you can outguess the market
  - Be diversified
  - Scenario analysis and stress testing is important
    - Scenario analysis (simulation modeling) involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities
    - Stress testing is a simulation technique used for financial instruments to determine their reactions to different financial situations (e.g., instruments may be tested under the following stresses: 50% drop in equity prices; 200% rise in oil prices; unemployment at 13 percent; 21 percent decline in housing prices)
- Big losses
  - Barings ($1 billion)
  - Enron's counterparties ($ billions in lawsuits)
  - LTCM ($4 billion)
  - Orange County ($2 billion)
  - Soc Gen ($7 billion)
  - Subprime mortgage losses ($ tens of billions)
  - UBS (2.3 billion)

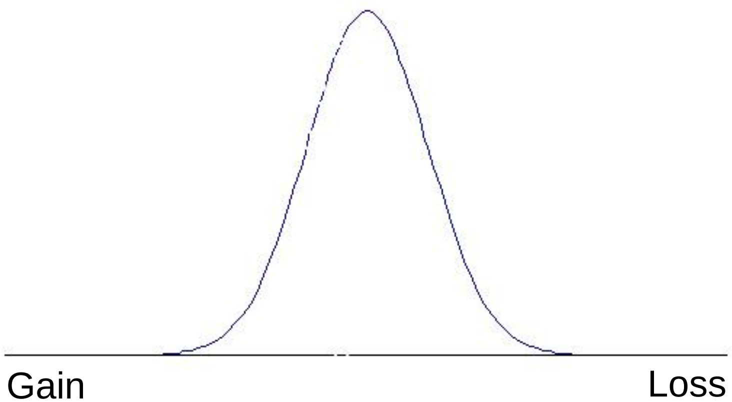
Source: J. Hull. Risk Management and Financial Institutions, 2012

Big losses

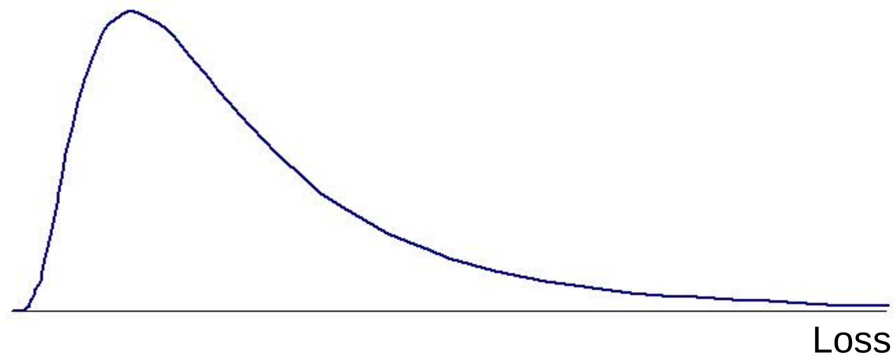

RISK MANAGEMENT
If you take too much risk, make sure you do it on a grand scale to ensure a government bailout.
DIY.DESPAIR.COM

Loss distributions

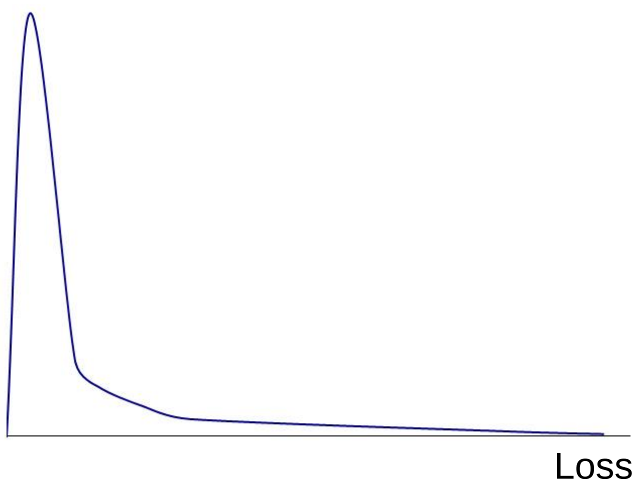
- One-year market risk gain/loss distribution



- One-year credit risk loss distribution



- One-year operational risk loss distribution



Source: J. Hull. Risk Management and Financial Institutions, 2012

Characteristics of distributions

	Second Moment (Variance)	**Third Moment (Skewness)**	**Fourth Moment (Kurtosis)**
Market Risk	High	Zero	Low
Credit Risk	Moderate	Moderate	Moderate
Operational Risk	Low	High	High

Source: J. Hull. Risk Management and Financial Institutions, 2012

Importance of risks

Type of Business	**Most Important Risk**
Commercial Banking	Credit Risk
Investment Banking & Trading	Market Risk and Credit Risk
Asset Management	Operational Risk

Source: J. Hull. Risk Management and Financial Institutions, 2012

Operational risk

- "Operational risk is the risk of loss resulting from inadequate or failed internal processes, people, and systems, or from external events" (Basel Committee, 2001), it includes people risks, technology and processing risks, physical risks, legal risks
  - Internal fraud
  - External fraud
  - Employment practices and workplace safety
  - Clients, products and business practices
  - Damage to physical assets
  - Business disruption and system failures
  - Execution, delivery and process management
- Regulatory Capital – in Basel II there is a capital charge for Operational Risk
  - Basic Indicator (15% of annual gross income)
  - Standardized (different percentage for each business line, e.g, for trading and sales, retail banking, commercial banking, asset management)
  - Advanced Measurement Approach (AMA)
- Example of operational risk in asset management:
  - No more than 10% of European Growth Trust (EGT) could be invested in unlisted (OTC) securities
  - Peter Young, the fund manager, violated this rule
  - The cost to Deutsche Bank was about $200 million

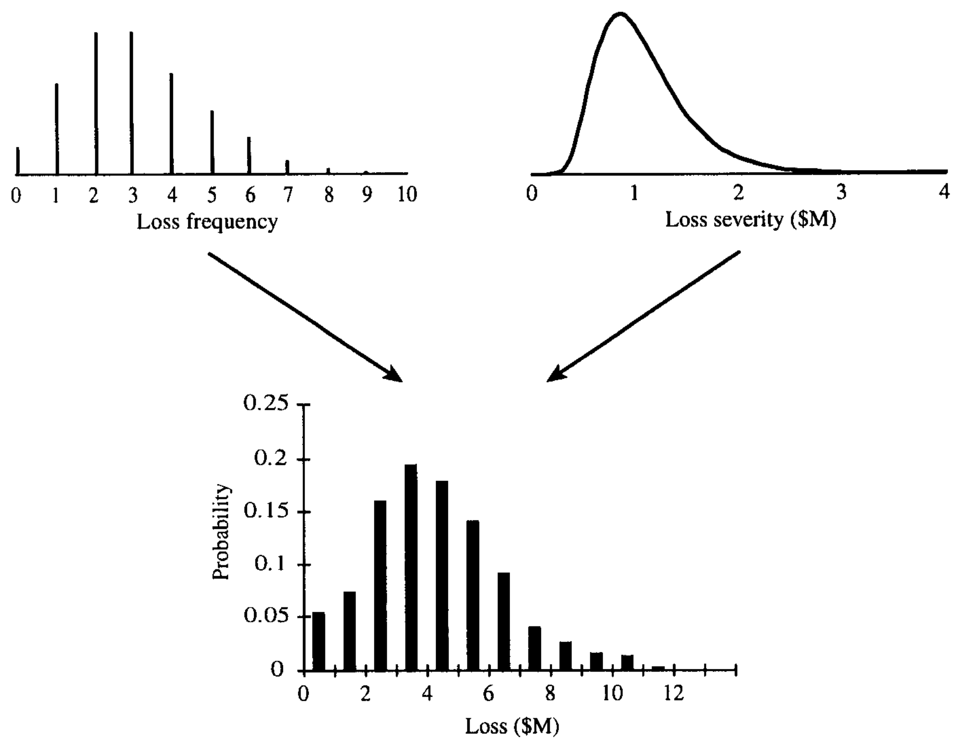
Source: J. Hull. Risk Management and Financial Institutions, 2012

Operational risk modeling

- Using Monte Carlo simulations to model operational risk
  - Loss frequency should be estimated from the banks own data as far as possible. One possibility is to assume a Poisson distribution so that we need only estimate an average loss frequency. Probability of n events in time T is then

$$e^{-\lambda T}\,\frac{(\lambda T)^n}{n!}$$

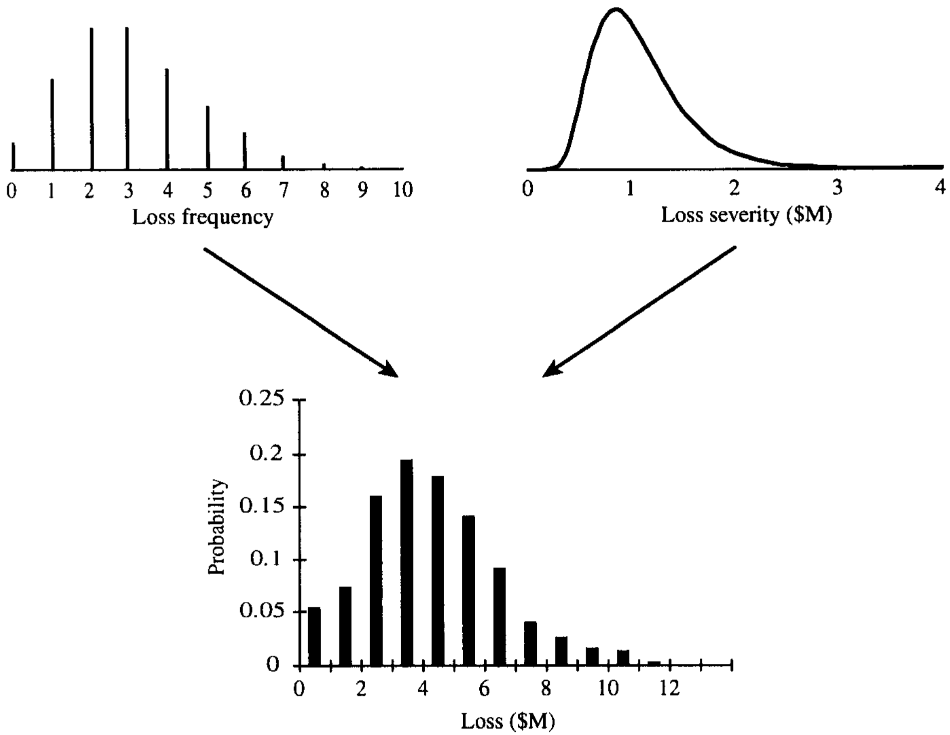
  - Loss severity can be based on internal and external historical data (one possibility is to assume a lognormal distribution, so that we need only estimate the mean and standard deviation of losses)



Source: J. Hull. Risk Management and Financial Institutions, 2012

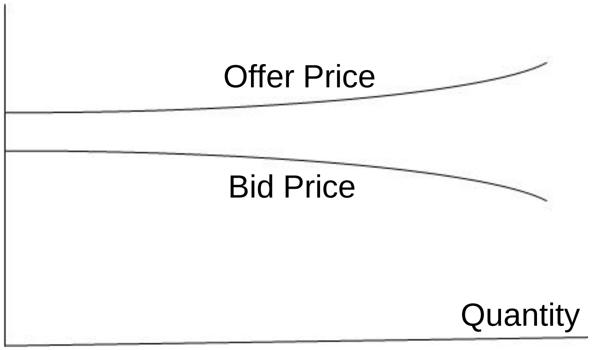
Operational risk modeling

- Using Monte Carlo simulations to combine the distributions
  - Sample from frequency distribution to determine the number of loss events (=*n*)
  - Sample *n* times from the loss severity distribution to determine the loss severity for each loss event
  - Sum loss severities to determine total loss



Source: J. Hull. Risk Management and Financial Institutions, 2012

Liquidity risk

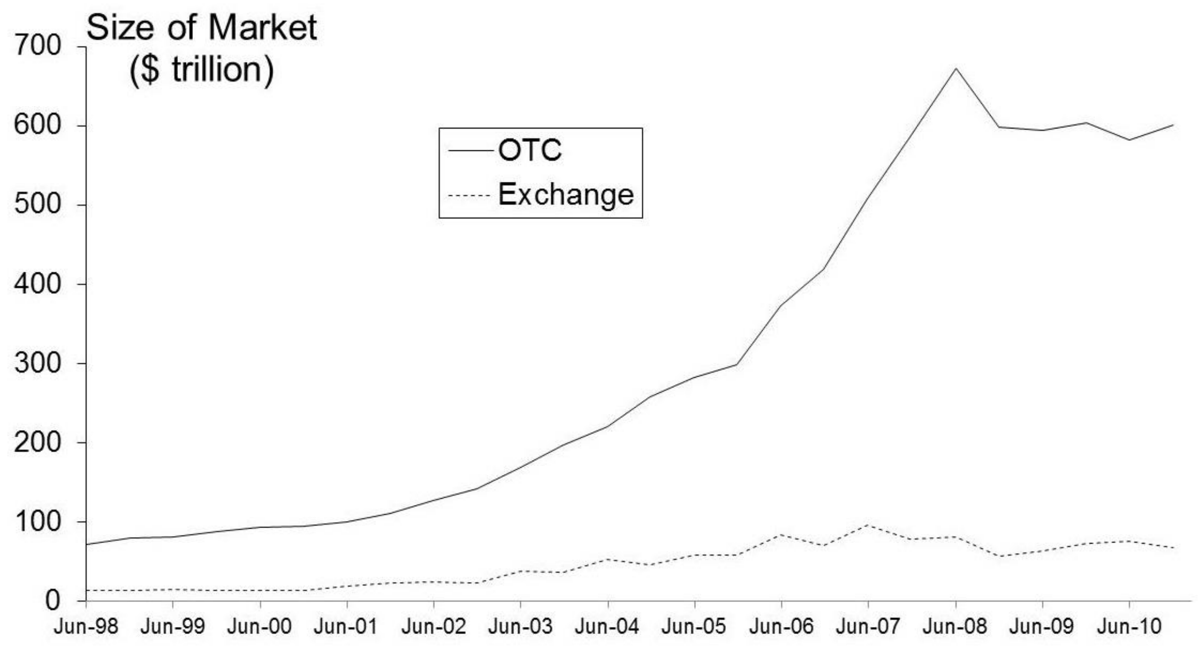


- Types of liquidity risk
  - Liquidity Trading Risk – an asset cannot be sold due to lack of liquidity in the market (sub-set of market risk), price received for an asset depends on
    - mid-market price
    - how much is to be sold, how quickly it is to be sold
    - economic environment
  - Liquidity Funding Risk – risk that liabilities cannot be met when they fall due or can only be met at an uneconomic price, sources of liquidity
    - liquid assets, ability to liquidate trading positions
    - wholesale and retail deposits, lines of credit
    - central bank borrowing
- Liquidity black holes – liquidity black hole occurs when most market participants want to take one side of the market and liquidity dries up
  - Crash of 1987
  - LTCM
- Credit crisis of 2007 has emphasized the importance of liquidity risk
- Is liquidity improving?
  - Spreads are narrowing
  - Risks of liquidity black holes are now (arguably) greater than they used to be
  - Need more diversity in financial markets where different groups of investors are acting independently of each other

Source: J. Hull. Risk Management and Financial Institutions, 2012

Financial markets

- Exchange traded
  - Traditionally exchanges have used the open-outcry system, but electronic trading has now become the norm
  - Contracts are standard; there is virtually no credit risk
- Over-the-counter (OTC)
  - A computer- and telephone-linked network of dealers at financial institutions, corporations, and fund managers
  - Contracts can be non-standard; there is some credit risk
- Derivatives
  - Forwards
  - Futures
  - Swaps
  - Options
  - Exotics



Source: J. Hull. Risk Management and Financial Institutions, 2012

🕮 Risk Measures

Portfolio risk measures

- Variance (standard deviation) in Markowitz model:

$$\sigma_P^2 = \sum_{j=1}^{J} \sigma_j^2 w_j^2 + \sum_{j=1}^{J} \sum_{k=1, j \neq k}^{J} \sigma_{jk} w_j w_k = \boldsymbol{w}^T \boldsymbol{Q} \boldsymbol{w}$$

- Beta of the portfolio in CAPM:

$$\beta_P = \sum_{j=1}^{J} \beta_j w_j = \boldsymbol{\beta}^T \boldsymbol{w}$$

- Value-at-Risk (VaR) in market and credit risk models
- Conditional value-at-risk (CVaR), also known as expected shortfall (ES) or expected tail loss (ETL), in market and credit risk models
- While standard deviation is usually computed for portfolio return, VaR and CVaR are computed for portfolio value or portfolio return

Portfolio risk measures

- Variance (standard deviation) in Markowitz model
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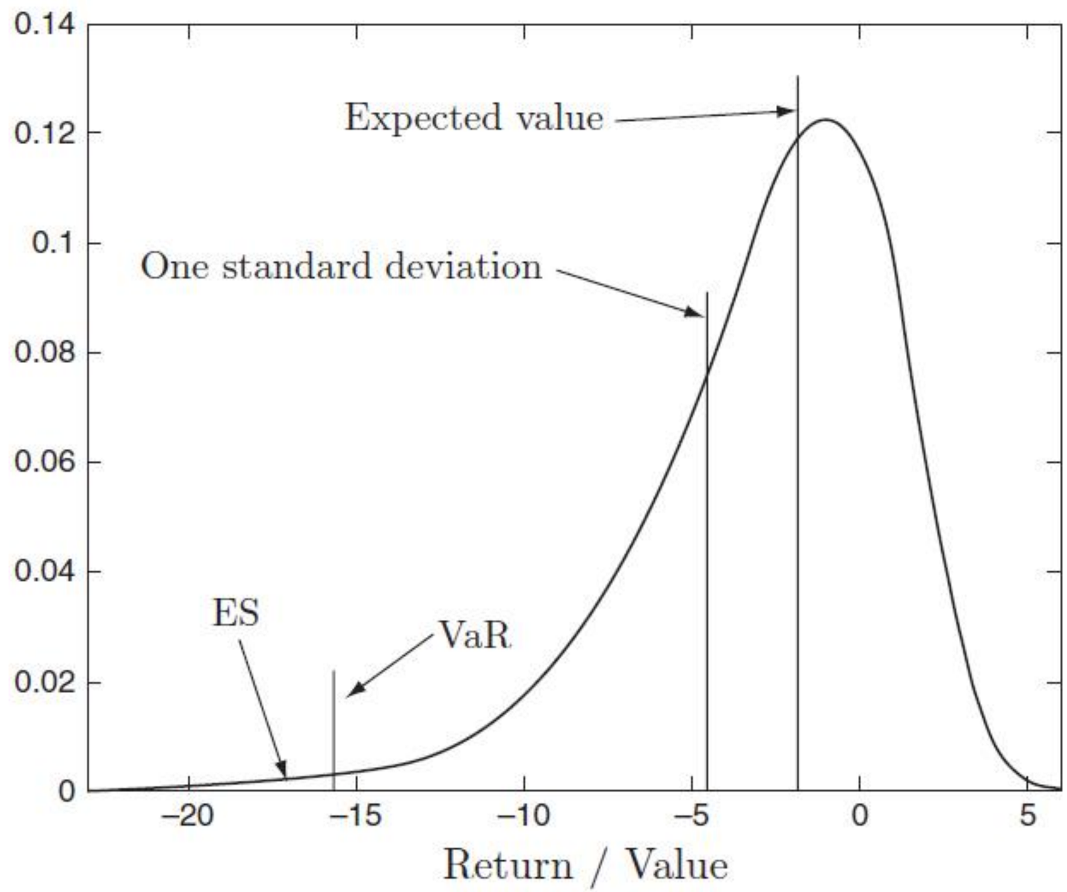


Portfolio return and portfolio value

- Consider *J* risky assets to form a portfolio and invest v_0 among *J* assets
  - assets can be represented by positions $x_j, \;\; j = 1, \ldots, J$ (measured in units)
  - assets can be represented by weights $w_j, \;\; j = 1, \ldots, J$ (measured in percentages)

- Asset value (price) and asset return
  - v_j is the value of asset j at time 1
  - μ_j is the expected market return of asset j

- Total portfolio value $$v_P = \sum_{j=1}^{J} v_j x_j$$

- Asset weight $$w_j = \frac{v_j x_j}{v_P}$$

$$\sum_{j=1}^{J} w_j = 1$$

- Portfolio return $$r_P = \sum_{j=1}^{J} \mu^T w_j$$

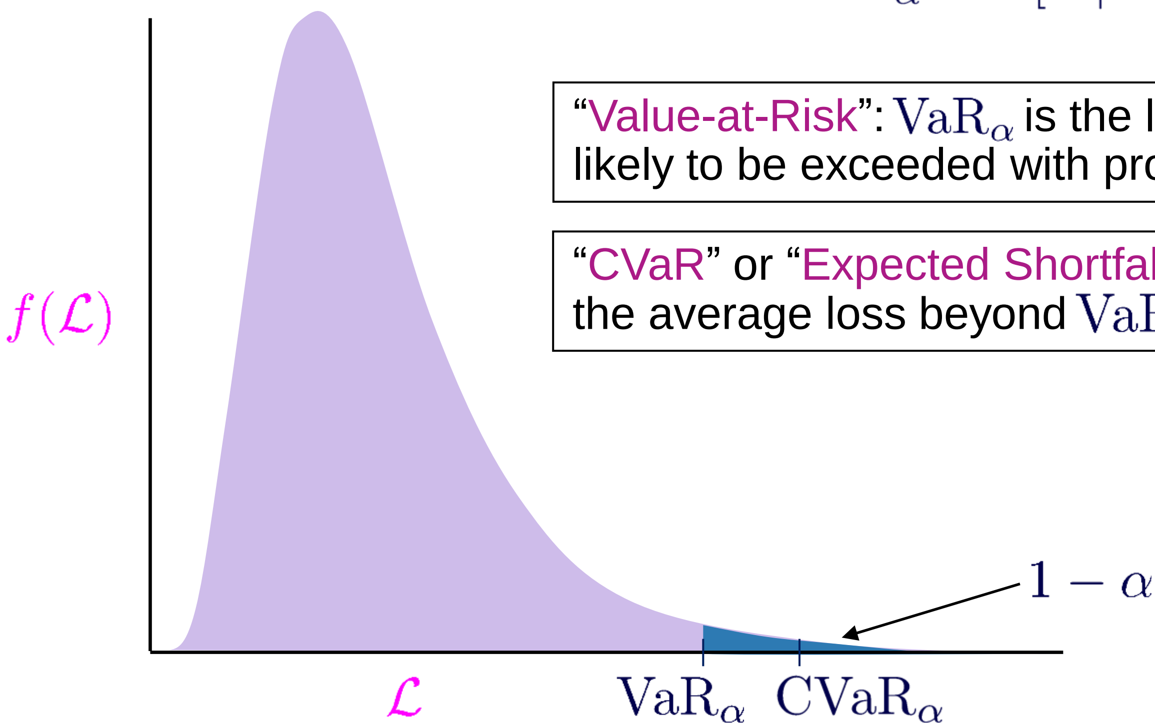
Portfolio return and portfolio value

- Total portfolio value $$v_P = \sum_{j=1}^{J} v_j x_j$$
- Asset weight $$w_j = \frac{v_j x_j}{v_P}$$
- Portfolio return $$r_P = \sum_{j=1}^{J} \mu^T w_j$$

- Portfolio loss in scenario *i* (total *N* scenarios)
  - Loss (return): $\ell_i(\boldsymbol{w}) = \sum_{j=1}^{J} -r_{ij} w_j$
  - Loss (value): $\ell_i(\boldsymbol{x}) = \sum_{j=1}^{J} -(v_{ij} - v_{0j}) x_j$

Tail-based risk measures

- Notation
  - $\boldsymbol{w} \in \Omega \subseteq \mathbb{R}^J$ is a portfolio, where w_j is the weight of asset j
  - F is the multivariate distribution of asset returns
  - $\mathrm{VaR}_\alpha(\boldsymbol{w})$ is the *actual risk* (out-of-sample VaR) of the portfolio $\boldsymbol{w}$
  - $\mathrm{VaR}_{\alpha,N}(\boldsymbol{w})$ is an *estimate* of $\mathrm{VaR}_\alpha(\boldsymbol{w})$ based on a sample of size N from F
- Consider a continuous random variable $\mathcal{L}$ with distribution F
  - the value-at-risk (VaR) at level α: $\mathrm{VaR}_\alpha = \min\{\ell : \mathbb{P}\{\mathcal{L} \leq \ell\} \geq \alpha\} = F^{-1}(\alpha)$
  - the conditional value-at-risk (CVaR) at level α: $\mathrm{CVaR}_\alpha = \mathbb{E}[\mathcal{L} \mid \mathcal{L} > \mathrm{VaR}_\alpha]$



"Value-at-Risk": VaR_α is the loss that is likely to be exceeded with probability $(1-\alpha)$

"CVaR" or "Expected Shortfall": CVaR_α is the average loss beyond VaR_α

Estimators

- Given a random sample of size N, let $\ell_{(k)}$ be the k^{th} order statistic, i.e., $\ell_{(1)} \leq \ell_{(2)} \leq \ldots \leq \ell_{(N)}$
  - An estimate of VaR_α is $\mathrm{VaR}_{\alpha,N} = \ell_{(\lceil N\alpha \rceil)}$ ← `math.ceil`
  - An estimate of CVaR_α is

$$\mathrm{CVaR}_{\alpha,N} = \frac{1}{N(1-\alpha)} \left[\left(\lceil N\alpha \rceil - N\alpha \right) \ell_{(\lceil N\alpha \rceil)} + \sum_{k=\lceil N\alpha \rceil + 1}^{N} \ell_{(k)} \right]$$

$\cdots$	$\ell_{(98)}$	$\ell_{(99)}$	$\ell_{(100)}$
$\cdots$	$\boxed{0.42}$	0.44	0.50

$\mathrm{VaR}_{0.98,100} = 0.42$ $\quad$ $\mathrm{CVaR}_{0.98,100} = 0.47$

$\mathrm{VaR}_{0.975,100} = 0.42$ $\quad$ $\mathrm{CVaR}_{0.975,100} = 0.46$

More observations in the tail → less noise → more robust estimates

Computing VaR and CVaR from simulation modeling – modified example 2

- You are planning for retirement and decide to invest in the market for the next 3 years. Your initial capital is $v_0 = 1000$
- Assume that every year your investment returns from investing into the S&P 500 will follow a Normal distribution with the mean 8.79% per year and standard deviation of 14.65%.
- Value of investment after 3 years: v_3
- Loss in portfolio value after 3 years: $-(v_3 - v_0)$
- The return over 3 years will depend on the realization of 3 random variables

$$r_{0,t} = (1 + r_{0,1})(1 + r_{1,2}) \dots (1 + r_{t-1,t}) - 1$$

$$v_{0,t} = (1 + r_{0,t})v_0$$

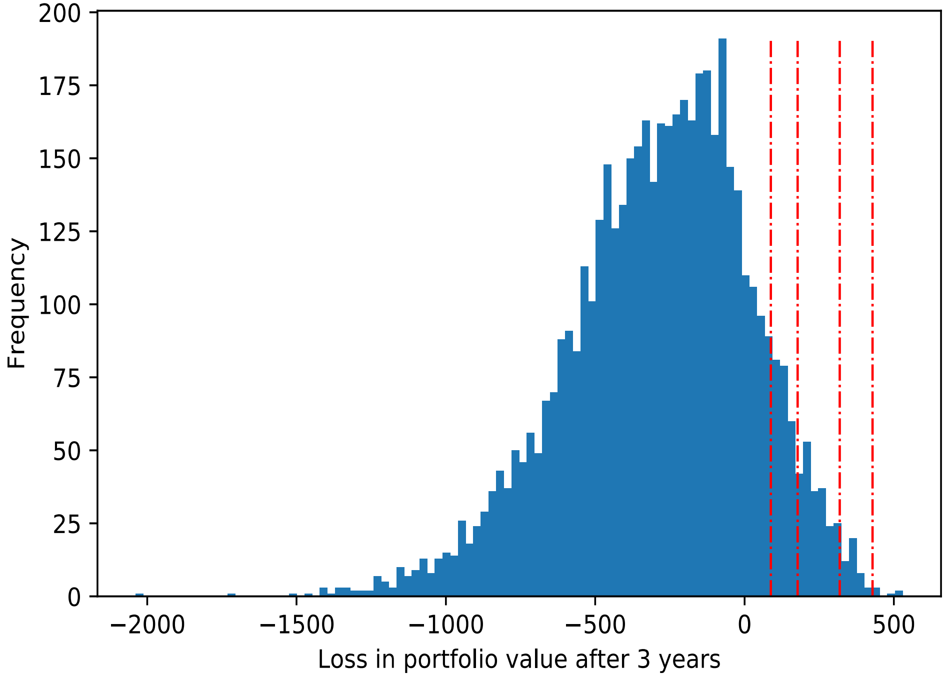
- Compute VaR and CVaR at quantile levels 90%, 95%, 99% and 99.9% for:
  - portfolio value after 3 years
  - portfolio return over 3 years

Computing VaR and CVaR from simulation modeling – example 2 in Python

```
# Initial capital
v0 = 1000

# Number of scenarios
N = 5000

# Generate Normal random variables
r_speriod3 = np.random.normal(0.0879, 0.1465, (N, 3))

# Distribution of value at the end of year 3
v3 = v0 * np.prod(1 + r_speriod3, 1)
# Distribution of return over 3 years
r3 = np.prod(1 + r_speriod3, 1) - 1

# Losses (value and return)
loss_v3 = np.sort(-(v3 - v0))
loss_r3 = np.sort(-r3)

# Quantile levels (90%, 95%, 99%, 99.9%)
alphas = [0.90, 0.95, 0.99, 0.999]

# Compute VaR and CVaR
VaRv = []; VaRr = []; CVaRv = []; CVaRr = []
print ('Loss in value after 3 years:')
for q in range(len(alphas)):
   alf = alphas[q]
   VaRv.append(loss_v3[int(math.ceil(N * alf)) - 1])
   VaRr.append(loss_r3[int(math.ceil(N * alf)) - 1])
   CVaRv.append((1/(N*(1-alf)))*((math.ceil(N*alf)-N*alf)*VaRv[q]+sum(loss_v3[int(math.ceil(N*alf)):])))
   CVaRr.append((1/(N*(1-alf)))*((math.ceil(N*alf)-N*alf)*VaRr[q]+sum(loss_r3[int(math.ceil(N*alf)):])))
   print ('VaR %4.1f%% = $%6.2f, CVaR %4.1f%% = $%6.2f' % (100 * alf, VaRv[q], 100 * alf, CVaRv[q]))
print ('\nLoss return over 3 years:')
for q in range(len(alphas)):
   print ('VaR %4.1f%% = %6.2f%%, CVaR %4.1f%% = %6.2f%%' % (100*alphas[q], 100*VaRr[q], 100*alphas[q],
          100*CVaRr[q]))

# Plot a histogram of the distribution of losses in value after 3 years
frequencyCounts, binLocations, patches = plt.hist(loss_v3, 100)
for q in range(len(alphas)):
   plt.plot([VaRv[q], VaRv[q]], [0, max(frequencyCounts)], color='r', linewidth=1, linestyle='-.')
```

```
Loss in value after 3 years:
VaR 90.0% = $ 87.88, CVaR 90.0% = $198.11
VaR 95.0% = $177.81, CVaR 95.0% = $267.44
VaR 99.0% = $318.50, CVaR 99.0% = $374.27
VaR 99.9% = $428.27, CVaR 99.9% = $480.35

Loss return over 3 years:
VaR 90.0% =   8.79%, CVaR 90.0% =  19.81%
VaR 95.0% =  17.78%, CVaR 95.0% =  26.74%
VaR 99.0% =  31.85%, CVaR 99.0% =  37.43%
VaR 99.9% =  42.83%, CVaR 99.9% =  48.04%
```

Computing VaR and CVaR from simulation modeling – example 2 in Python

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# Distribution of return over 3 years
r3 = np.prod(1 + r_speriod3, 1) - 1

# Losses (value and return)
loss_v3 = np.sort(-(v3 - v0))
loss_r3 = np.sort(-r3)

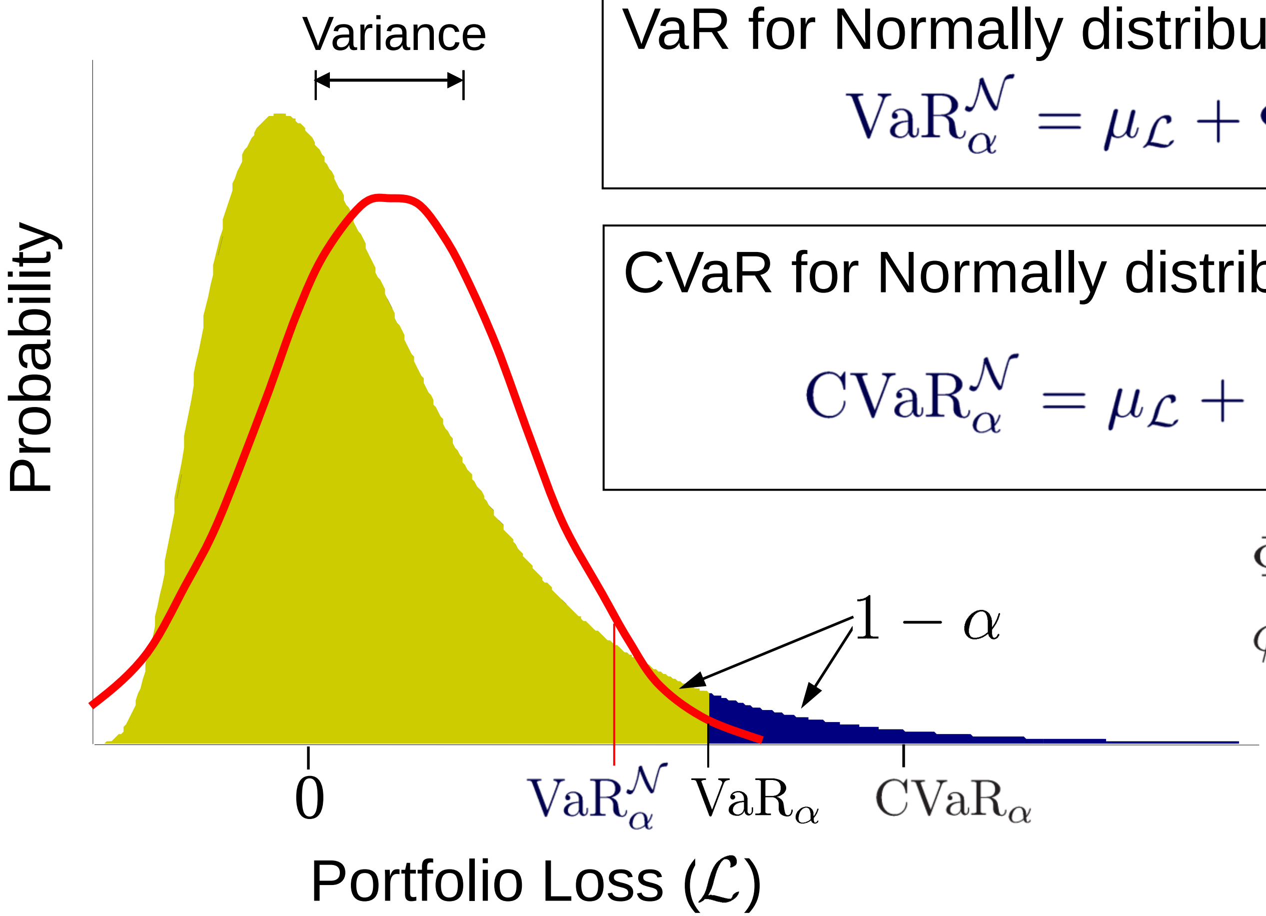
# Quantile levels (90%, 95%, 99%, 99.9%)
alphas = [0.90, 0.95, 0.99, 0.999]

# Compute VaR and CVaR
VaRv = []; VaRr = []; CVaRv = []; CVaRr = []
print ('Loss in value after 3 years:')
for q in range(len(alphas)):
   alf = alphas[q]
   VaRv.append(loss_v3[int(math.ceil(N * alf)) - 1])
   VaRr.append(loss_r3[int(math.ceil(N * alf)) - 1])
   CVaRv.append((1/(N*(1-alf)))*((math.ceil(N*alf)-N*alf)*VaRv[q]+sum(loss_v3[int(math.ceil(N*alf)):])))
   CVaRr.append((1/(N*(1-alf)))*((math.ceil(N*alf)-N*alf)*VaRr[q]+sum(loss_r3[int(math.ceil(N*alf)):])))
   print ('VaR %4.1f%% = $%6.2f, CVaR %4.1f%% = $%6.2f' % (100 * alf, VaRv[q], 100 * alf, CVaRv[q]))
print ('\nLoss return over 3 years:')
for q in range(len(alphas)):
   print ('VaR %4.1f%% = %6.2f%%, CVaR %4.1f%% = %6.2f%%' % (100*alphas[q], 100*VaRr[q], 100*alphas[q],
          100*CVaRr[q]))

# Plot a histogram of the distribution of losses in value after 3 years
frequencyCounts, binLocations, patches = plt.hist(loss_v3, 100)
for q in range(len(alphas)):
   plt.plot([VaRv[q], VaRv[q]], [0, max(frequencyCounts)], color='r', linewidth=1, linestyle='-.')
```

VaR and CVaR for Normal distributions

For Normal distributions, VaR and CVaR is the mean loss + a constant multiplied by the standard deviation of losses



VaR for Normally distributed losses:

$$\mathrm{VaR}_\alpha^{\mathcal{N}} = \mu_{\mathcal{L}} + \Phi^{-1}(\alpha) \cdot \sigma_{\mathcal{L}}$$

CVaR for Normally distributed losses:

$$\mathrm{CVaR}_\alpha^{\mathcal{N}} = \mu_{\mathcal{L}} + \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} \cdot \sigma_{\mathcal{L}}$$

Φ is the cdf of $\mathcal{N}(0,1)$
ϕ is the pdf of $\mathcal{N}(0,1)$

Computing VaR and CVaR from historical scenarios – example

- Historical distribution of losses was obtained from the daily stock prices of a major financial institution (Citigroup)
- In total 1007 historical scenarios (from beginning of 2005 till the end of 2008)
- Portfolio is a long position in 1 unit of stock
- Portfolio 1-day loss: $\ell_t = -(v_t - v_{t-1})$

- Compute VaR and CVaR at quantile level 99% from:
  - historical scenarios
  - if we assume Normal distribution of losses (mean and standard deviation are computed from the historical scenarios)

Computing VaR and CVaR from historical scenarios – example in Python

```
# Specify quantile level for VaR/CVaR
alf = 0.99

# Read Profit/Loss (P/L) data
xlsfile = pd.ExcelFile('Hist_prices_Citigroup.xls')
PLData = xlsfile.parse('VaR').iloc[1:,3].to_numpy().real

# Number of historical scenarios
N = len(PLData)

# Sort loss data in increasing order
loss_1d = np.sort(-PLData)

# Compute Historical 1-day VaR from the data
VaR = loss_1d[int(math.ceil(N * alf)) - 1]
# Compute Historical 1-day CVaR from the data
CVaR = (1 / (N*(1-alf))) * ((math.ceil(N*alf) - N*alf) * VaR + sum(loss_1d[int(math.ceil(N*alf)):]))

# Compute Normal 1-day VaR from the data
VaRn = np.mean(loss_1d) + scs.norm.ppf(alf) * np.std(loss_1d)
# Compute Normal 1-day CVaR from the data
CVaRn = np.mean(loss_1d) + (scs.norm.pdf(scs.norm.ppf(alf)) / (1 - alf)) * np.std(loss_1d)

print ('Historical 1-day VaR %4.1f%% = $%6.2f, Historical 1-day CVaR %4.1f%% = $%6.2f' % (100 * alf,
VaR, 100 * alf, CVaR))
print (' Normal 1-day VaR %4.1f%% = $%6.2f, Normal 1-day CVaR %4.1f%% = $%6.2f\n' % (100 * alf, VaRn,
100 * alf, CVaRn))
```

```
Historical 1-day VaR 99.0% = $  2.08,   Historical 1-day CVaR 99.0% = $  2.64
    Normal 1-day VaR 99.0% = $  1.67,       Normal 1-day CVaR 99.0% = $  1.90
```

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# Plot a histogram of the distribution of losses in portfolio value
frequencyCounts, binLocations, patches = plt.hist(loss_1d, 100)
normf = (1 / (np.std(loss_1d) * math.sqrt(2 * math.pi))) * np.exp(-0.5 * ((binLocations -
         np.mean(loss_1d)) / np.std(loss_1d)) ** 2)
normf = normf * sum(frequencyCounts) / sum(normf)
plt.plot(binLocations, normf, color='r', linewidth=3.0)
plt.plot([VaRn, VaRn], [0, max(frequencyCounts) / 2], color='r', linewidth=1, linestyle='-.')
plt.plot([VaR, VaR], [0, max(frequencyCounts)/2], color='r', linewidth=1, linestyle='-.')
plt.text(0.98 * VaR, max(frequencyCounts) / 1.9, 'VaR')
plt.text(0.7 * VaRn, max(frequencyCounts) / 1.9, 'VaRn')
plt.xlabel('1-day loss in $ value on 1 unit on stock')
plt.ylabel('Frequency')
```

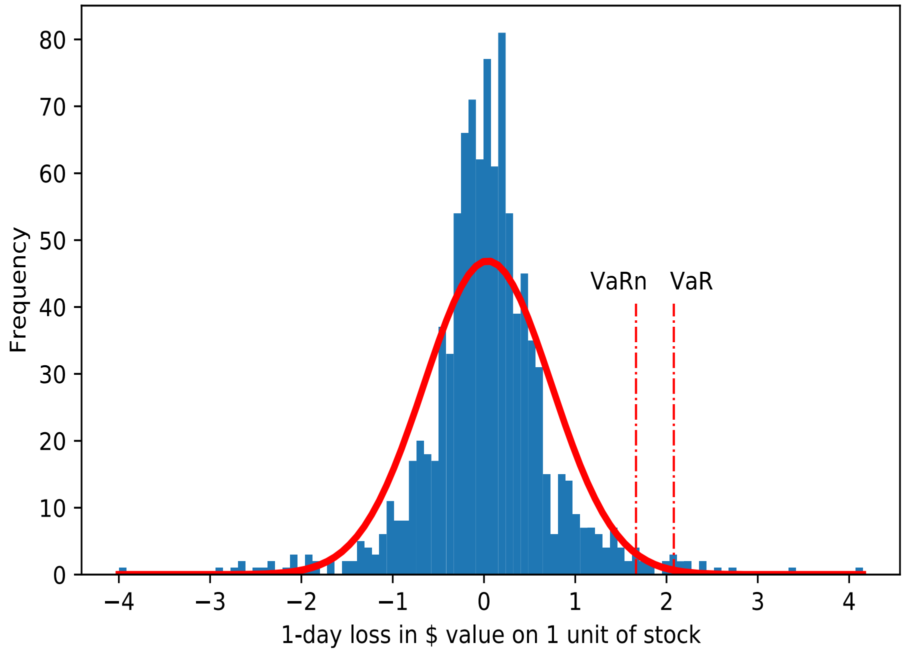

Value-at-risk (VaR)

- The question being asked in VaR:
  - "What loss level is such that we are (100α)% confident it will not be exceeded in *M* business days?"
- VaR and Regulatory Capital
  - Regulators base the capital they require banks to keep on VaR
  - The market-risk capital is *k* times the 10-day VaR 99% where *k* is at least 3.0
  - Under Basel II, capital for credit risk and operational risk is based on a one-year VaR 99.9%
- Advantages of VaR:
  - It captures an important aspect of risk in a single number
  - It is easy to understand
  - It asks the simple question: "How bad can things get?"
- Disadvantages of VaR:
  - Unlike CVaR, VaR is not a coherent risk measure as it does not satisfy sub-additivity property
    - "risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged"
  - Two portfolios with the same VaR can have very different CVaRs

Source: J. Hull. Risk Management and Financial Institutions, 2012

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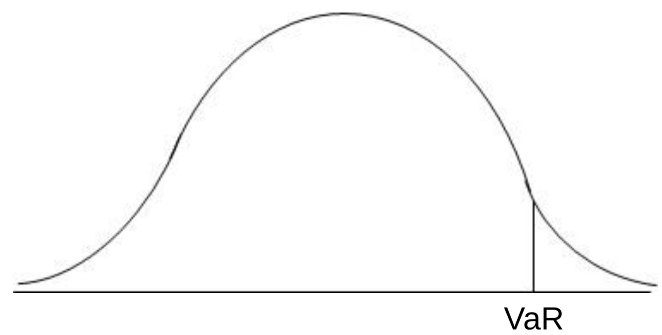
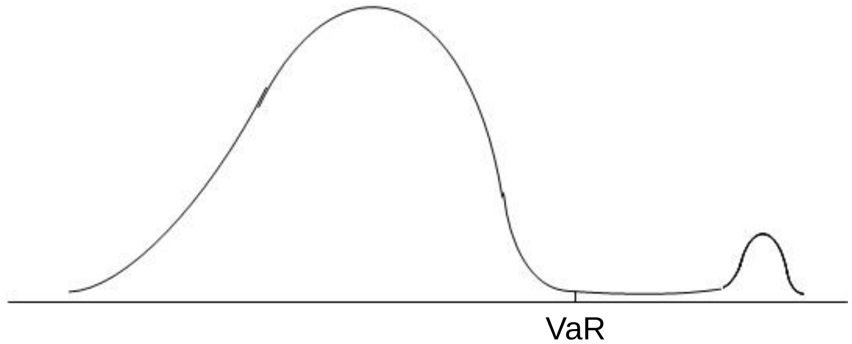




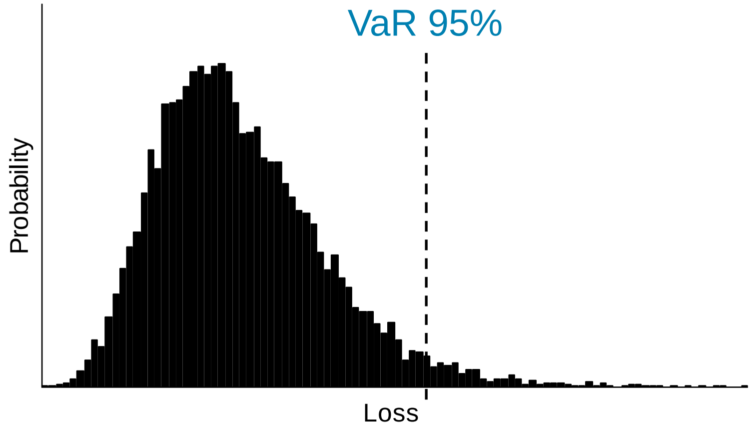
Source: J. Hull. Risk Management and Financial Institutions, 2012

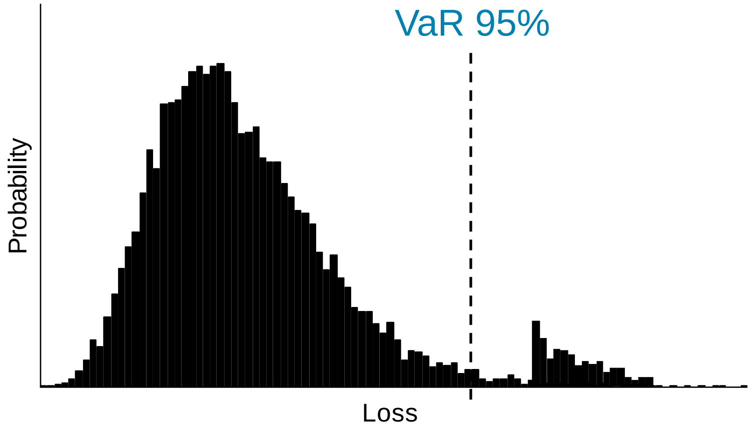
Decision Making and Financial Crisis

Risk Management and Financial Crisis

Did risk management fail?

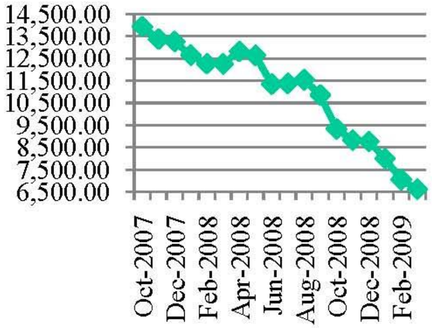
- Risk analytics was done wrong (Basel II)
- Blame everything on VaR
- Risk analytics was not taken into account during decision making
- Decisions were made from the wrong perspective





Decision making

- **Lesson**: know who makes the decisions and what her/his objective is
- **Did risk management fail in these crises?**



Financial Crisis

BP Oil Spill

Volcanic Eruption

- Business analytics shares some blame, but
  - problems were created by viewing decisions from the wrong perspective
  - we can improve the value of models by recognizing decision-makers' self interested behavior and designing policies with consistent incentives

BP oil spill

- Water depth and experience at depth may have increased chances for failures
- Blowout preventer (blind shear ram) installed to shear and seal pipe in event of leak
- One blowout preventer installed but not two – a mistake in assessment?

- It is best for BP to install 2 BOPs
- Did BP just get the probabilities wrong?
- What decisions are in the interest of the managing executives?
- Executives losses are limited on the downside
- Early completion as well as revenues leads to early share improvement and bonus compensation
- **Even with best knowledge of probabilities, decisions might have been the same**

BP oil spill