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STABLE ∞-CATEGORIES AND THEIR HOMOTOPY CATEGORIES

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ABSTRACT

Stable ∞-Categories and Their Homotopy Categories

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An abstract homotopy theory is a situation in which one has a category with a class of "weak equivalences" that one would like to invert. One recent description of a homotopy theory is as an "∞-category," which is like a category with extra structure that is made to contain homotopical data. Every ∞-category can be flattened into an ordinary category called its "homotopy category," in a way that inverts the weak equivalences.

Among homotopy theories, there are certain ones that are called stable homotopy theories. Accordingly, there is a notion of stable ∞-categories which formalizes them. The homotopy category of a stable ∞-category has a canonical structure of a triangulated category.

A triangulated category is a category which is equipped with some structure that serves as a "weak" or homotopical version of an exact sequence. The theory of stable ∞-categories has some advantages over that of triangulated categories, since ∞-categories retain some homotopical information that is lost in the passage to the homotopy category.

This thesis attempts to do two things: (1) to lay out a self-contained exposition of the theory of ∞-categories, and (2) to describe the relationships between stable ∞-categories and the triangulated structure on their homotopy categories.

1 Introduction

When are two things "the same"? In ordinary category theory we may say that two things are the same if they are isomorphic: that there exist invertible maps between them. It often happens that we are studying a category in which there are objects we want to consider the same but do not have invertible morphisms between them.

- In topology we want to identify spaces that are equivalent up to continuous deformation. This continuous deformation is known as homotopy equivalence. For example, any disk D^n can be continuously deformed to a point. So we want to identify $D^n \sim *$ as being somehow "the same". But they are not homeomorphic (isomorphic in the category of topological spaces); the map $D^n \to *$ does not admit an inverse in the usual sense. The image of any map $* \to D^n$ is a point, so $D^n \to * \to D^n$ sends the disk to a single point in the disk. What is true is that $D^n \to *$ admits an inverse up to homotopy. That is, there is a map $* \to D^n$ and a homotopy h between the map $D^n \to * \to D^n$ and the identity id_{D^n}. We can think of this homotopy as a map between maps, and denote it with a thickened arrow:

$$\begin{tikzcd} & * \arrow[dr] \arrow[d, phantom, "\Downarrow h"] & \\ D^n \arrow[ur] \arrow[rr, "\mathrm{id}"'] & & D^n \end{tikzcd}$$

  We want to construct a new category of spaces in which spaces that are equivalent up to homotopy become categorically isomorphic. We call such a category the homotopy category, and describe it by a functor. In this case:

$$\mathbf{Top} \to ho\mathbf{Top},$$

where $ho\mathbf{Top}$ is the homotopy category of spaces. This functor should be such that $X \sim Y$ (are homotopically equivalent) in **Top** implies that their images $[X] \cong [Y]$ (are isomorphic) in $ho\mathbf{Top}$.

- Analagously, when studying chain complexes $\mathbf{Ch}(R)$ over a commutative ring R, we want to identify chain complexes that have the same homology. So we would want to identify chain complexes that are homotopy equivalent, or those that are quasi-isomorphic.

  This leads one to form a "homotopy category" $K(R)$ by inverting homotopy equivalences of chain complexes, and a "derived category" $D(R)$ by inverting quasi-isomorphisms.

Daniel Quillen abstracted from these examples the idea of an abstract homotopy theory, which he formalized with the notion of a "model category". The study of these abstract homotopy theories he called "Homotopical Algebra" [Qui67]. The name is intentionally reminiscent of homological algebra, which sits within homotopical algebra as the homotopy theory of chain complexes over a commutative ring.

There are other ways to formalize and describe homotopy theories. William Dwyer and Daniel Kan [DK80c; DK80b; DK80a] describe homotopy theories as "simplicial localizations", describing a process of forming out of a category with weak equivalences a new type of category which encodes homotopical information by morphisms of higher dimension. This is done by considering a more elaborate type of category which instead of having hom-sets has hom-simplicial sets.

More recently, homotopy theories have been formalized in terms of ∞-categories. These are like categories with higher morphisms which encode homotopical information. There are many ways to define these, but we will mostly use a model known as quasi-categories. These were first described by Michael Boardman and Rainer Vogt in their book "Homotopy Invariant Algebraic Structures on Topological Spaces" [BV73], in which they called them "weak Kan complexes". This was expanded upon by André Joyal [Joy02; Joy08] who called them quasi-categories, and then Jacob Lurie [Lur08; Lur] who called them ∞-categories.

Among homotopy theories, there are certain ones that are stable. The prototypical example of this is the stable homotopy category of spectra. There is a corresponding notion of a stable ∞-category that model stable homotopy theories.

Stability manifests algebraically (historically in the context of homological algebra and algebraic geometry) in the form of a triangulated category. This is a category with a "weak" type of exact sequences, called distinguished triangles. For example, in the stable homotopy category, taking pieces of fiber or cofiber sequences (which are like homotopical exact sequences) forms distinguished triangles.

In studying a homotopy theory, one wants to form a homotopy category, in which the equivalences are formally inverted. Modelling by an ∞-category, one can construct a homotopy category by "flattening out" the homotopical information.

If the ∞-category is stable, its homotopy category has a canonical triangulated structure. Many of the ideas, constructions, and results from the theory of triangulated categories have analogues in the context of stable ∞-categories.

This thesis is organized as follows:

- Section 2: We lay out the theory of ∞-categories, and see how it generalizes ordinary category theory.
- Section 3: We deal with stability. We describe the theory of stable ∞-categories, and relate them to the theory of triangulated categories.
- There is an appendix for Quillen's theory of model categories, which were one of the first ways to describe homotopy theories, and are useful in describing the homotopy theory of ∞-categories.

NOTES ON NOTATION

- We cite often from the books "Higher Topos Theory" [Lur08] and "Higher Algebra" [Lur17], and the website "Kerodon" [Lur]. For easy reference, we use the shorthands [HTT, $-$], [HA, $-$], and [K, $-$] respectively when citing from these.

  For example [K, 1.3.0.1] refers to Definition 1.3.0.1 in [Lur].
- Internal references will be enclosed in ordinary parentheses ($-$).

  For example, (3.1.10) refers to Definition 3.1.10.
- Some people say 0 is not natural, but it seems pretty natural to this author. To us, natural numbers $\mathbb{N}$ will always include 0.
- We use **Top** to denote the category of continuous maps between "compactly generated weakly Hausdorff spaces" [Mac78, §VII.8].
- The convention for varying squiggly types of "equality":

- (=): equality, eg. when describing a set explicitly.
- ($\cong$): isomorphism in an ordinary category. This will basically only ever be used to denote an isomorphism of sets or groups.
- ($\simeq$): equivalence, which is the homotopical version of isomorphism. Equivalence comes with a direction, so "$X \simeq Y$" should really be understood to be a suppressed version of either "$X \xrightarrow{\sim} Y$" or "$X \xleftarrow{\sim} Y$".
- ($\approx$): homeomorphism (an isomorphism in **Top**).
- ($\sim$): an equivalence relation, usually homotopy.

- Plain capital letters $(C, D, T, M, \ldots)$ will denote ordinary categories, ∞-categories, simplicial sets, objects of a category, or functors. Script letters $(\mathcal{C}, \mathcal{D}, \ldots)$ will essentially always denote ∞-categories.

- The singleton set will be denoted as $\{*\}$. The point in the category of topological spaces will be written as either $*$ or pt.

- Objects of an ordinary category will be written as $\mathrm{ob}(C)$. The collection of all morphisms in a category will be written $\mathrm{Hom}(C) := \bigcup_{x,y \in C} \mathrm{Hom}_C(x, y)$.

- We will use the terms "morphism" and "map" interchangeably.

- Given a category C with objects x and y, we may shorten the hom-set $\mathrm{Hom}_C(x, y)$ to $C(x, y)$.

2 ∞-Categories

> *The real fun begins when one actually starts to* use *∞-category theory, at which point the world becomes a magical place: one's power to make new definitions is limited only by one's imagination, and one's ability to prove new theorems is limited only by the clarity of one's understanding (at least as far as the purely formal aspects are concerned). The many fussy details that arise when one attempts to use point-set techniques to work homotopy-coherently simply melt away: they were in fact irrelevant all along to the true and underlying mathematics, and their disappearance into the ambient machinery brings with it a harmony that is only possible when intuition and language are once again aligned. Thus, paradoxically, by discarding such emotional crutches as underlying sets and strict composition and by embracing the apparent chaos and uncontrol of homotopy-coherence, we acquire a measure of power of which previous generations of mathematicians could barely have dreamed.*
>
> Aaron Mazel-Gee
> "The Zen of ∞-Categories" [Maz]

2.1 Simplicial sets, Kan complexes, quasi-categories

Definition 2.1.1. [the simplex category $\mathbf{\Delta}$]

The $\boxed{\text{simplex category } \mathbf{\Delta}}$ consists of the following:

- (objects): For each $n \in \mathbb{N}$, there is an object $[n] \in \mathbf{\Delta}$ given by the linearly ordered set $[n]\{0, 1, 2, \ldots, n\}$.
- (morphisms): A morphism $f : [n] \to [m]$ is an order-preserving function.

Remark 2.1.2. Intepreting posets as categories with morphisms given by $\leq$ lets us see $\mathbf{\Delta} \subseteq \mathbf{Cat}$ as a full subcategory.

Definition 2.1.3. [simplicial sets]

A $\boxed{\text{simplicial set}}$ is a functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$. In other words, a presheaf on $\mathbf{\Delta}$.

Remark 2.1.4. For $i = 0, 1, \ldots, n$, there are morphisms in $\mathbf{\Delta}$ called coface and codegeneracy maps respectively:

$$\delta_i : [n] \to [n+1], \qquad x \mapsto \begin{cases} x & x < i \\ x+1 & x \geq i. \end{cases}$$

$$\sigma_i : [n] \to [n-1], \qquad x \mapsto \begin{cases} x & x \leq i \\ x-1 & x > i, \end{cases}$$

So δ_i skips i, and σ_i maps to i twice.

$$\delta_i : \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, mapsto] & \ldots \arrow[r] & i-1 \arrow[r] \arrow[d, mapsto] & i \arrow[r] \arrow[dr, mapsto] & \ldots \arrow[r] & n \arrow[dr, mapsto] & \\ 0 \arrow[r] & \ldots \arrow[r] & i-1 \arrow[r] & i \arrow[r] & i+1 \arrow[r] & \ldots \arrow[r] & n+1 \end{tikzcd}$$

$$\sigma_i : \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, mapsto] & \ldots \arrow[r] & i \arrow[r] \arrow[d, mapsto] & i+1 \arrow[r] \arrow[dl, mapsto] & i+2 \arrow[r] \arrow[dl, mapsto] & \ldots \arrow[r] & n \arrow[dl, mapsto] \\ 0 \arrow[r] & \ldots \arrow[r] & i \arrow[r] & i+1 \arrow[r] & \ldots \arrow[r] & n-1 & \end{tikzcd}$$

These satisfy certain identities [Cis19, Prop. 1.2.3], which spell out how these coface and codegeneracies compose with each other. And it turns out that these are the only ones that matter:

Lemma 2.1.5. Any morphism $[n] \to [m]$ in $\mathbf{\Delta}$ has a unique representation as a composite of δ's and σ's. So any simplicial set X is determined by the images

$$d_i := X(\delta_i) : X_n \to X_{n-1}$$
$$s_i := X(\sigma_i) : X_n \to X_{n+1},$$

which we call $\boxed{\text{face and degeneracy maps}}$ respectively. So we can think of a simplicial set as consisting of a collection of sets $\{X_n\}_{n\in\mathbb{N}}$ along with face and degeneracy maps.

For more details, see [GJ09, § 1.1], [Mac78, § 7.5].

Remark 2.1.6. A morphism of simplicial sets, or a simplicial map, is a natural transformation of functors $\mathbf{\Delta}^{op} \to \mathbf{Set}$. The $\boxed{\text{category of simplicial sets}}$ is the functor category $\mathbf{sSet} := \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathbf{Set})$.

Notation 2.1.7.

- We sometimes write "sset" as short for simplicial set.
- Let X be a simplicial set. We will call elements of X_0 "points" or "vertices" of X, and call elements of X_1 "edges" of X. For any $n \in \mathbb{N}$, elements of X_n will be called "n-simplices" of X.

Example 2.1.8. [the standard n-simplex Δ^n]

The $\boxed{\text{standard } n\text{-simplex}}$ is the simplicial set formed by Yoneda embedding

$$\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]) \in \mathbf{sSet}.$$

Remark 2.1.9. Pick an $X \in \mathbf{sSet}$. Yoneda's lemma says that for every $n \in \mathbb{N}$ there is a bijection

$$X_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X).$$

That is an n-simplex $x \in X_n$ corresponds to a simplicial map $x : \Delta^n \to X$. In particular, an i-simplex of Δ^n corresponds to a simplicial map $\Delta^i \to \Delta^n$.

Since any presheaf is a colimit of representables, any $X \in \mathbf{sSet}$ can be written as $X = \operatorname{colim}_{\Delta^n \to X} \Delta^n$. In particular, $\Delta^n = \operatorname{colim}_{\Delta^i \to \Delta^n} \Delta^i$.

Example 2.1.10. [sub-simplices]

Any subset $I = \{i_0, i_1, \ldots, i_k\} \subseteq [n]$ determines a simplicial set

$$\Delta^I := \operatorname{Hom}_{\mathbf{\Delta}}(-, I) \in \mathbf{sSet},$$

which is isomorphic to Δ^k via the isomorphism $I \cong [k]$. The inclusion $I \hookrightarrow [n]$ defines a monomorphism $\Delta^I \hookrightarrow \Delta^n$.

In particular, for $i = 0, \ldots, n$, the i^{th} face of Δ^n is the subsimplex

$$\begin{aligned} d_i(\Delta^n) &= \Delta^{\left\{0,\ldots,\widehat{i},\ldots,n\right\}} \\ &\cong \Delta^{n-1} \\ &\subseteq \Delta^n \end{aligned}$$

(where $\widehat{i}$ means we are omitting i).

The i^{th} degeneracy s_i forms an $(n+1)$-simplex:

$$\begin{aligned} s_i(\Delta^n) &= \Delta^{\{0,\ldots,i,i,\ldots,n\}} \\ &\cong \Delta^{n+1} \end{aligned}$$

with an inclusion $\Delta^n \subseteq \Delta^{n+1}$.

Remark 2.1.11. Let X be a simplicial set.

We call a simplex $x \in X_n$ $\boxed{\text{degenerate}}$ if it can be written as $s_i(y)$ for some

$y \in X_{n-1}$. For any $x \in X_n$, we can find a minimum i such that $x : \Delta^n \to X$ factors as

$$\begin{tikzcd} \Delta^n \arrow[rr, "x"] \arrow[dr] & & X \\ & \Delta^i \arrow[ur] & \end{tikzcd}$$

We call such an i the dimension of the simplex x.

Even such an innocuous simplicial set as Δ^0 has infinitely many degenerate simplices:

$$\Delta^{\{0,0\}} \cong \Delta^1$$
$$\Delta^{\{0,0,0\}} \cong \Delta^2$$
$$\vdots$$

To facilitate working with simplicial sets, we will often focus our attention on the nondegenerate simplices. A simplicial set X with no non-degenerate simplices above dimension n is called n-coskeletal. For example each Δ^n is n-coskeletal.

We will often draw 0-simplices/vertices as points, 1-simplices/edges as arrows, and 2-simplices/faces as thick arrows. We may label a subsimplex by the vertices spanning it, eg.

$$[i_0, \ldots, i_k] : \Delta^{\{i_0,\ldots,i_k\}} \hookrightarrow \Delta^n.$$

For example, nondegenerate simplices of Δ^2 can be labelled:

$$\begin{aligned}
(\Delta^2)_0 &= \left\{\Delta^{\{0\}}, \Delta^{\{1\}}, \Delta^{\{2\}}\right\} \\
&= \{0, 1, 2\} \\
(\Delta^2)_1 &= \left\{\Delta^{\{0,1\}}, \Delta^{\{0,2\}}, \Delta^{\{1,2\}}\right\} \\
&= \{[01], [02], [12]\} \\
(\Delta^2)_2 &= \left\{\Delta^{\{0,1,2\}}\right\} \\
&= \{[012]\}
\end{aligned}$$

and we can draw a picture of Δ^2 as a triangle-shaped diagram:

$$\begin{tikzcd}
& 1 \arrow[dr, "{[12]}"] \arrow[d, phantom, "\Downarrow{[012]}"] & \\
0 \arrow[ur, "{[01]}"] \arrow[rr, "{[02]}"'] & {} & 2
\end{tikzcd}$$

with the assumption that all the degenerate simplices are hidden.

The standard 3-simplex Δ^3 can be drawn as a tetrahedron-shaped diagram with four faces $(\Delta^3)_2 = \{[012], [013], [023], [123]\}$ and a single 3-simplex $[0123]$ we can draw as a thickened arrow $\Rrightarrow$ forming the inside:

$$\begin{tikzcd}
& 1 \arrow[drr] \arrow[ddr] & & \\
& \overset{[0123]}{\Rrightarrow} & & 2 \arrow[dl] \\
0 \arrow[uur] \arrow[urrr] \arrow[rr] & & 3 &
\end{tikzcd}$$

We can isolate a sub-sset $\mathrm{sk}_n(X) \subseteq X$ called the n-skeleton of X, whose nondegenerate simplices are those of X of dimension n and less.

For any n, we can truncate X to an n-coskeletal sset $\mathrm{tr}_{\leq n}(X)$, whose nondegenerate simplices are those of X in degrees $i \leq n$.

Example 2.1.12. [products of simplicial sets]

Products of simplicial sets are defined levelwise. That is, given simplicial sets X and Y, the product $X \times Y$ is defined with n-simplices:

$$(X \times Y)_n := X_n \times Y_n,$$

and face and degeneracy maps formed by face and degeneracies of X and Y in the obvious way:

$$\begin{aligned} s_i &:= (s_i^X, s_i^Y), \\ d_i &:= (d_i^X, d_i^Y). \end{aligned}$$

Using the notation of (2.1.11), we will denote an n-simplex of $X \times Y$ by a pair: $\alpha = (\alpha^X, \alpha^Y) \in (X \times Y)_n$.

Note that an n-simplex in $X \times Y$ may be nondegenerate even when both of its components are degenerate in X and Y.

Example 2.1.13. For example, consider the product $\Delta^1 \times \Delta^1$ [Fri21, Ex. 5.4]. Drawing out all non-degenerate simplices forms a "square":

$$\begin{tikzcd}[column sep=huge, row sep=huge]
([0],[0]) \arrow[r, "{([01],[00])}"] \arrow[d, "{([00],[01])}"'] \arrow[dr, "{([01],[01])}" description] \arrow[dr, phantom, shift left=6ex, "{\swarrow\, ([011],[001])}"] \arrow[dr, phantom, shift right=6ex, "{\nearrow\, ([001],[011])}"'] & ([1],[0]) \arrow[d, "{([11],[01])}"] \\
([0],[1]) \arrow[r, "{([01],[11])}"'] & ([1],[1])
\end{tikzcd}$$

This has a number of non-degenerate simplices arising from degenerate simplices of each Δ^1 factor. Four out of the five edges (namely the horizontal and vertical maps) arise from degenerate simplices of the form [00] and [11]. And there are non-degenerate 2-cells appearing, although Δ^1 has no non-degenerate 2-cells itself.

Example 2.1.14. [constant simplicial sets]

Given a set S, the $\boxed{\text{constant simplicial set}}$ $S_\bullet \in \mathbf{sSet}$ is the simplicial set with $S_n := S$ for all n, and face/degeneracy maps all given by id_S.

Example 2.1.15. [singular complexes]

Let X be a topological space.

The $\boxed{\text{singular complex}}$ $\mathrm{Sing}(X)$ is a simplicial set defined by:

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$$

where $|\Delta^n|$ is the standard topological n-simplex

$$|\Delta^n| = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum_i x_i = 1, x_i \geq 0 \right\},$$

eg. $|\Delta^0|$ is a point, $|\Delta^1|$ is a line segment, $|\Delta^2|$ is a triangular face, $|\Delta^3|$ is a solid tetrahedron, and so on. With appropriate boundary maps (alternating sums of faces) $\mathrm{Sing}(X)$ forms a chain complex, whose homology is the singular homology of the space X.

Face and degeneracy maps of $\mathrm{Sing}(X)$ are induced by topological face and degeneracy maps that are defined as one would expect: face maps of a topological simplex pick out literal geometric faces, and degeneracy maps realize a topological simplex as a face of a higher-dimensional topological simplex. This construction forms a functor $\mathrm{Sing} : \mathbf{sSet} \to \mathbf{Top}$.

For gory details and wonderful pictures see [Fri21].

Remark 2.1.16. The functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ admits a left adjoint

$$|-| : \mathbf{sSet} \to \mathbf{Top},$$

called the $\boxed{\text{geometric realization}}$ functor, which takes $\Delta^n \mapsto |\Delta^n|$. An arbitrary sset X can be written as a colimit

$$X = \operatorname{colim}_{\Delta^n \to X} \Delta^n,$$

so the geometric realization $|X|$ is a space formed by gluing $|\Delta^n|$'s appropriately:

$$|X| = \operatorname{colim}_{\Delta^n \to X} |\Delta^n|.$$

Each topological n-simplex is homeomorphic to an n-disk: $|\Delta^n| \approx D^n$, and the face/degeneracy maps give instructions on gluing these together along their boundaries, so $|X|$ can be thought of as a CW-complex.

Example 2.1.17. [boundaries, horns, spines]

Pick a $\Delta^n \in \mathbf{sSet}$.

- There is a subcomplex $\partial\Delta^n \subseteq \Delta^n$ called the $\boxed{\text{boundary}}$, made up of the faces of Δ^n:

  $$\partial\Delta^n := \bigcup_{i=0,1,\ldots,n} d_i(\Delta^n).$$

  ie. it has all the nondegenerate simplices of Δ^n except for the single one in the highest dimension $[0,1,\ldots,n] \in (\Delta^n)_n$. In terms of geometric realization, this corresponds to throwing away the interior of a topological simplex. That is, this is the same as taking the topological boundary: $\partial_{top}|\Delta^n| = |\partial_{sset}\Delta^n|$.

- For each $i = 0, 1, \ldots, n$, we can throw out the i^{th} face in addition to the interior to form the $\boxed{i^{\text{th}} \text{ horn}}$

$$\Lambda_i^n := \bigcup_{j \neq i} d_j(\Delta^n).$$

  The geometric realization of a horn is homeomorphic to a disk $|\Lambda_i^n| \approx D^{n-1}$, and one can choose a homeomorphism $|\Delta^n| \approx D^n$ so that the inclusion $|\Lambda_i^n| \hookrightarrow |\Delta^n|$ is the inclusion of a hemisphere into an n-disk.

  In particular the horns Λ_0^n and Λ_n^n are called the left and right horns of Δ^n respectively, or collectively outer horns. The horns Λ_i^n for $0 < i < n$ are called inner horns.

- One can throw out everything but the edges of the form $(i \to i+1)$ to form the $\boxed{\text{spine}}$

$$sp(\Delta^n) := \bigcup_{i=0,\ldots,n-1} \{i \to i+1\}.$$

We can illustrate the example of these constructions in the case of Δ^2:

$$\Delta^2 : \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

$$\partial\Delta^2 : \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \Lambda_0^2 : \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad sp(\Delta^2) : \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

Example 2.1.18. [the nerve of a category]

Any category C can be realized as a simplicial set called NC, the $\boxed{\text{nerve}}$ of C, whose n-simplices are

$$NC_n := \mathrm{Fun}([n], C),$$

ie. strings of n composable morphisms in C. An n-simplex $\alpha \in NC_n$ is a string

$$\alpha = (c_1 \xrightarrow{f_1} c_2 \xrightarrow{f_2} \ldots \xrightarrow{f_n} c_n).$$

Images of an n-simplex $\alpha := (c_0 \xrightarrow{f_1} c_1 \to \ldots \xrightarrow{f_n} c_n) \in NC_n$ under the face and degeneracy maps s_i and d_i are given by:

$$s_i(\alpha) = (c_0 \to \ldots c_i \xrightarrow{\mathrm{id}} c_i \to \cdots \to c_n)$$

$$d_i(\alpha) = \begin{cases} (c_0 \to \cdots \to c_{i-1} \xrightarrow{f_{i+1} f_i} c_{i+1} \to \cdots \to c_n) & i \neq 0, n \\ (c_1 \to \cdots \to c_n) & i = 0 \\ (c_0 \to \cdots \to c_{n-1}) & i = n. \end{cases}$$

Sanity check 2.1.19. The nerve of $[m] \in \mathbf{\Delta}$ considered as a poset category looks like:

$$\begin{aligned} N([m])_n &= \mathrm{Fun}([n], [m]) \\ &= (\Delta^m)_n \end{aligned}$$

with face and degeneracy maps given by pulling back along the cosimplicial maps σ_i and δ_i.

This forms an isomorphism of simplicial sets $N([m]) \simeq \Delta^m$.

Since higher simplices in the nerve of a category are precisely the encoding of composable maps, a functor $F : C \to D$ forms a map of simplicial sets $NF : NC \to ND$ given by

$$(c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n) \quad \mapsto \quad (F(c_1) \xrightarrow{F(f_1)} \dots \xrightarrow{F(f_n)} F(c_n)).$$

A composition of functors $C \xrightarrow{F} D \xrightarrow{G} E$ induces a composite of simplicial maps $NC \xrightarrow{NF} ND \xrightarrow{NG} NE$, which agrees with $N(G \circ F)$ since $G(F(f)) = G \circ F(f)$ for any $f \in \mathrm{Hom}(C)$. That is to say the nerve forms a functor $N : \mathbf{Cat} \to \mathbf{sSet}$.

Proposition 2.1.20. [K, 1.2.2.1] The nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful.

2.1.1 Kan complexes

Definition 2.1.21. [Kan complexes]

A $\boxed{\text{Kan complex}}$ $\mathcal{K}$ is a simplicial set with the following “horn-filling property”: for any $i = 0, 1, \dots, n$, a horn $\Lambda^n_i \to \mathcal{K}$ extends to a simplex $\Delta^n \to \mathcal{K}$:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{K} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Kan complexes along with simplicial maps form a subcategory $\mathbf{Kan} \subseteq \mathbf{sSet}$.

Example 2.1.22. Let X be a space. The singular complex $\mathrm{Sing}(X)$ is a Kan complex.

Proof. By the adjunction $(|-| \dashv \mathrm{Sing})$ (2.1.16) a horn-filling problem $\Lambda^n_i \to$

Sing(X) corresponds to a diagram in **Top**:

$$\begin{tikzcd}
|\Lambda^n_i| \approx D^{n-1} \arrow[r, "f"] \arrow[d, hook, "i"'] & X \\
|\Delta^n| \approx D^n \arrow[ur, dotted, "f\circ r"']
\end{tikzcd}$$

where i is the inclusion of a hemisphere into a disk. We can retract a disk to a hemipshere; ie. there is a map $r : D^n \to D^{n-1}$ with $r\circ i = \mathrm{id}_{D^{n-1}}$. The composition $f \circ r : \Delta^n \to X$ gives us a map $\Delta^n \to \mathrm{Sing}(X)$. This makes the diagram commute since $f \circ r \circ i = f$. □

Proposition 2.1.23. [Joy02, Cor. 1.4]

A category X is a groupoid (every morphism is invertible) iff its nerve NX is a Kan complex.

Proof. ⇒: Let X be a groupoid, and say we had a lifting problem:

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & NX \\
\Delta^n \arrow[ur, dotted]
\end{tikzcd}$$

A horn $\Lambda^n_i \to NX$ is by definition a map

$$\Lambda^n_i = \bigcup_{j\neq i} d_j(\Delta^n) \to NX,$$

ie. a collection of maps $d_j(\Delta^n) \to NX$. Each $d_j(\Delta^n)$ is an $(n-1)$-simplex, which we can represent by a string of $(n-1)$ morphisms in X:

$$\begin{tikzcd}
x_0 \arrow[r] & \dots \arrow[r] & x_{j-1} \arrow[r, bend left, "f_{j+1}f_j"] & x_{j+1} \arrow[r] & \dots \arrow[r] & x_n
\end{tikzcd}$$

so Λ^n_i picks out $(n-1)$ strings of morphisms in X, each of the form above, except for the one of the form:

$$\begin{tikzcd} x_0 \arrow[r] & \dots \arrow[r] & x_{i-1} \arrow[r, bend left] & x_{i+1} \arrow[r] & \dots \arrow[r] & x_n \end{tikzcd}$$

We want to build from this an honest simplex $\Delta^n \to NX$; ie. a string $[n] \to X$.

$(i \neq 0,1)$: Take the faces $d_n(\Delta^n)$ and $d_0(\Delta^n)$:

$$d_n(\Delta^n): \quad \begin{tikzcd} x_0 \arrow[r] & x_1 \arrow[r] & \dots \arrow[r] & x_{n-1} \end{tikzcd}$$

$$d_0(\Delta^n): \quad \begin{tikzcd} x_1 \arrow[r] & \dots \arrow[r] & x_{n-1} \arrow[r] & x_n \end{tikzcd}$$

Postcompose $d_n(\Delta^n)$ by the map $x_{n-1} \to x_n$ to get the desired string.

$(i = 0,1)$: When $i = 0$ (the case $i = 1$ is basically the same), consider the faces

$$d_1(\Delta^n): \quad \begin{tikzcd} x_0 \arrow[r, bend left, "f_2 f_1"] & x_2 \arrow[r, "f_3"] & \dots \arrow[r] & x_{n-1} \arrow[r, "f_n"] & x_n \end{tikzcd}$$

$$d_n(\Delta^n): \quad \begin{tikzcd} x_0 \arrow[r, "f_1"] & x_1 \arrow[r, "f_2"] & \dots \arrow[r, "f_{n-1}"] & x_{n-1} \end{tikzcd}$$

We can construct a horn filler $[n] \to X$ by composing the string $d_n(\Delta^n) = (x_0 \to \cdots \to x_{n-1})$ with the map $f_n : x_{n-1} \to x_n$.

$\Leftarrow$: If X is a category whose nerve is a Kan complex, given a morphism $f : x \to y$ in X, we can fill in horns Λ^2_0 and Λ^2_2 in NX of the form:

$$\Lambda^2_0: \quad \begin{tikzcd} & y & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}"'] & & x \end{tikzcd} \qquad \Lambda^2_2: \quad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[rr, "\mathrm{id}"'] & & y \end{tikzcd}$$

which correspond to finding left and right inverses of f. $\square$

Remark 2.1.24. Any topological space is weakly homotopic to a CW-complex (this is called cellular or CW approximation). This can be thought of as a cofibrant replacement in $\mathbf{Top}_{Quillen}$ (4.0.14). Similarly, any simplicial set is weakly equivalent to a Kan complex. Similarly, this can be thought of as a fibrant replacement in $\mathbf{sSet}_{Quillen}$ [K, 3.1.7.2, 3.3.6].

We can make this connection between Kan complexes and CW-complexes tighter by a theorem of Milnor [Mil57], which shows that the homotopy theory of Kan complexes is equivalent to the homotopy theory of CW-complexes [K, 3.5.0.1]. We can see this from the fact that $(|-| \dashv \mathrm{Sing})$ (2.1.16) forms a Quillen equivalence (4.1.5)

$$\mathbf{sSet}_{Quillen} \rightleftarrows \mathbf{Top}_{Quillen}$$

2.1.2 Quasi-categories/∞-categories

In the proof of (2.1.23) we saw a hint that outer horns (Λ^n_0 or Λ^n_n) are related to inverting morphisms. On the other hand, inner horns (Λ^n_i for $0 < i < n$) relate to composing morphisms. For example, $\Lambda^2_1 \to NC$ in the nerve of a category corresponds to a pair of composable morphisms ($\bullet \to \bullet \to \bullet$) in C, and filling in to a 2-simplex $\Delta^2 \to NC$ corresponds to finding a composite map. Composing morphisms is an important part of the structure/definition of a category, but not every category is a groupoid, so it would make sense that a definition of ∞-categories should include inner horn fillers but not necessarily left/right horn fillers.

Indeed, a weakening of the horn-filling condition gives us our weak Kan complexes or quasi-categories.

Definition 2.1.25. [quasi-categories]

A $\boxed{\text{quasi-category}}$ X is a simplicial set satisfying the horn-filling condition for inner horns. That is we can extend horn inclusions $\Lambda^n_i \to X$ for $0 < i < n$ to an n-simplex in X:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Remark 2.1.26. Quasi-categories form a full subcategory $\mathbf{qCat} \subseteq \mathbf{sSet}$. We will use the terms "quasi-category" and "∞-category" interchangeably.

Example 2.1.27. The nerve of an ordinary category is an ∞-category. We showed this as part of the proof of (2.1.23).

Remark 2.1.28. [HTT, 1.1.2.2]

A simplicial set X is equivalent to the nerve of an ordinary category iff horn fillers are unique.

Remark 2.1.29. [Lan21, Prop. 1.2.17, 1.3.23] An ∞-category satisfies a lifting property for spines 2.1.17. That is, given a spine $sp(\Delta^n) \to \mathcal{C}$ in an ∞-category, we can fill it in to a simplex:

$$\begin{tikzcd} sp(\Delta^n) \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

A simplicial set satisfying this "spine lifting property" is called a "composer".

Intuitively this says that given a string $\bullet \to \bullet \to \cdots \to \bullet$ in an ∞-category, we can fill it into an honest simplex. Filling in spines is the higher-categorical way of composing morphisms. So it makes sense that any ∞-category should have this property. But not all composers are ∞-categories.

Definition 2.1.30. [equivalences]

Let $X \in \mathbf{sSet}$. An edge $f \in X_1$ is called an $\boxed{\text{equivalence}}$ (sometimes "isomorphism") if there exists an inverse: an edge $f^{-1} \in X_1$ forming 2-simplices $\Delta^2 \rightrightarrows X$:

$$\begin{tikzcd} & y \arrow[dr, dashed, "f^{-1}"] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, dashed, "f^{-1}"] \arrow[rr, "\mathrm{id}_y"'] & & y \end{tikzcd}$$

Remark 2.1.31. An equivalence is the homotopical/higher-categorical version of an isomorphism in ordinary category theory. One can see that if f is an equivalence, that f^{-1} is not a strict inverse of f, but an inverse up to homotopy.

If an edge $f \in \mathcal{C}$ of a simplicial set is an equivalence, then the corresponding morphism $[f] \in h\mathcal{C}$ in the homotopy category is an isomorphism in the ordinary sense. This follows from the definition (2.1.30), since the 2-simplices appearing exhibit the map $[f^{-1}]$ as an inverse in the ordinary sense of $[f]$ in $h\mathcal{C}$.

For this reason, some authors may refer to equivalences as isomorphisms in an ∞-category.

Remark 2.1.32. It's immediate from the definition that every Kan complex is an ∞-category. In fact, as we saw in the proof of (2.1.23), a Kan complex is an ∞-category in which all edges are invertible; ie. an $(\infty, 0)$-category. An $(\infty, 0)$-category is also called an $\boxed{\infty\text{-groupoid}}$, in analogy with ordinary groupoids.

Alternately, an ∞-groupoid can be defined as a quasi-category X for which any edge $f \in X_1$ is an equivalence. Since any path $f : x \rightsquigarrow y$ in a topological space X can be reversed, this defines an inverse to the corresponding edge $f \in \mathrm{Sing}(X)_1$. That is, the singular complex of a space is an ∞-groupoid.

Grothendieck's homotopy hypothesis [Gro21] says that Kan complexes are precisely the same as ∞-groupoids. See [Lan21, Lem. 1.2.29, Cor. 2.1.12].

Theorem 2.1.33. [Lan21, Rk. 2.1.9][Joy02, Thm. 1.3]

[Joyal's special horn theorem]

Let $\mathcal{C}$ be an ∞-category.

A $\boxed{\text{special horn}}$ is an outer horn $\Lambda^n_0 \to \mathcal{C}$ (2.1.17) in which the first edge φ:

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "\varphi"] & \\ \Lambda^n_0 \arrow[r] & \mathcal{C} \end{tikzcd}$$

is an equivalence (2.1.30). Dually, one can consider an outer horn $\Lambda^n_n \to \mathcal{C}$ in which the final edge $\Delta^{\{n-1,n\}} \to \mathcal{C}$ is an equivalence.

A lifting problem of the form:

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

has a solution iff $\Lambda^n_0 \to \mathcal{C}$ is a special horn. (Dually a horn $\Lambda^n_n \to \mathcal{C}$ can be lifted to a simplex iff it is a special horn.)

Remark 2.1.34. Joyal's theorem makes precise the relationship between invertible maps and outer horns, that we saw a hint of in the discussion about Kan complexes and ∞-groupoids.

Remark 2.1.35. [fundamental ∞-groupoid]

Let X be a topological space. Let $I = [0,1] \in \mathbf{Top}$ denote the unit interval.

The $\boxed{\text{fundamental } \infty\text{-groupoid}}$ $\pi_\infty X$ of X is a simplicial set, with simplices defined:

$$(\pi_\infty X)_0 := \{\text{points } x \in X\} = \mathbf{Top}(*, X)$$
$$(\pi_\infty X)_1 := \{\text{paths in } X\} = \mathbf{Top}(I, X)$$
$$(\pi_\infty X)_2 := \{\text{homotopies of paths in } X\} = \mathbf{Top}(I \times I, X)$$
$$(\pi_\infty X)_3 := \{\text{homotopies of homotopies in } X\} = \mathbf{Top}(I^{\times 3}, X)$$
$$\vdots$$
$$(\pi_\infty X)_n := \mathbf{Top}(I^{\times n}, X)$$

Intuitively this is an ∞-groupoid since for any $n \geq 1$, an n-simplex is a map $I^{\times n} \to X$, which has an inverse formed by homotoping in the reverse direction.

One can exhibit homeomorphisms $I^{\times n} \approx |\Delta^n|$ and use the singular complex $\mathrm{Sing}(X)$ (2.1.15) as a definition of the fundamental ∞-groupoid of X.

Definition 2.1.36. [opposite simplicial set][K, 1.3.2.2]

Let S be a simplicial set.

The $\boxed{\text{opposite simplicial set}}$ $S^{op} \in \mathbf{sSet}$ is the simplicial set defined with:

- (simplices): For any $n \in \mathbb{N}$,

$$(S^{op})_n := S_n.$$

- (face and degeneracy maps): For any $n \in \mathbb{N}$, and any $i = 0, 1, \ldots, n$

$$d_i = d^S_{n-i} : S_n \to S_{n-1},$$
$$s_i = s^S_{n-i} : S_n \to S_{n+1},$$

where $d_\bullet^S$ and $s_\bullet^S$ are the face and degeneracy maps of the sset S.

Proposition 2.1.37. [K, 1.3.2.4, 1.3.2.6]

If $\mathcal{C}$ is an ∞-category, then $\mathcal{C}^{op}$ is an ∞-category.

If C is an ordinary category, then $N(C^{op}) \simeq (NC)^{op}$.

The following diagram of categories lays out what is a nerve of what so far:

$$\begin{tikzcd}
\mathbf{Gpd} \arrow[r, hook] \arrow[d, "N"] & \mathbf{Cat} \arrow[d, "N"] \arrow[dr, bend left, "N"] & \\
\mathbf{Kan} \arrow[r, hook] & \mathbf{qCat} \arrow[r, hook] & \mathbf{sSet}
\end{tikzcd}$$

2.1.3 Connected components

An important construction in topology and homotopy theory is the notion of connected components.

Definition 2.1.38. [connected components of a simplicial set]

There is a functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ called the $\boxed{\text{connected components}}$ functor that sends a simplicial set X to the set

$$\pi_0 X := X_0 / \sim$$

where X_0 is the set of 0-simplices, and $\sim$ is the relation generated by the relation

$$d_1(f) \sim d_0(f)$$

for all edges $f \in X_1$; ie. two points are in the same connected component if they are connected by an edge.

Remark 2.1.39. [K, 1.1.9.10]

When $X \in \mathbf{sSet}$ is a Kan complex, the connected components has a simpler description. Points $x, y \in X_0$ belong to the same connected component iff there is an honest edge $f : x \to y$ in X_1.

This is different than the definition for an arbitrary sset, in which points of a are identified iff there is a <u>finite string</u> of edges connecting them. It may happen that a simplicial set has a string of edges connecting two points, but not an honest edge; eg. the horn Λ^2_1.

Remark 2.1.40. [K, 1.1.7.4]

Take a space $X \in \mathbf{Top}$. We can form its connected components $\pi_0^{top}(X) = \{\text{points } x \in X\}/\ $, with points $x \sim y$ iff there is a path $f : I \to X$ with $f(0) = x, f(1) = y$.

Considering the singular complex $\mathrm{Sing}(X) \in \mathbf{sSet}$ and taking connected components as a simplicial set agrees with the usual connected components. That is, the following commutes:

$$\begin{tikzcd} \mathbf{Top} \arrow[rr, "\pi_0^{top}"] \arrow[dr, "\mathrm{Sing}"'] & & \mathbf{Set} \\ & \mathbf{sSet} \arrow[ur, "\pi_0^{sset}"'] & \end{tikzcd}$$

Proposition 2.1.41. [K, 1.1.6.21]

There's an adjunction:

$$\pi_0 : \mathbf{sSet} \rightleftarrows \mathbf{Set} : (-)_\bullet$$

where $(-)_\bullet$ is the constant simplicial set functor (2.1.14).

Proposition 2.1.42. The functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ preserves coproducts. It also preserves finite products of ssets, and preserves all products of Kan complexes.

Proof. Preserving coproducts follows from π_0 being a left adjoint.

Preserving finite products of arbitrary simplicial sets is [K, 1.1.6.26].

The statement that π_0 preserves all products of Kan complexes can be done component-wise. Say $X = \prod_{i \in I} X_i$ is a product of Kan complexes. Then

$$\pi_0 X = \left\{ (x_0, x_1, \dots) \in \prod_i X_i \right\} / \sim$$

where $\sim$ is the relation $(x_i)_i \sim (y_i)_i$ iff there's an edge $(x_i)_i \to (y_i)_i$ in $\prod_i X_i$. Products of simplicial sets are defined component-wise; in particular, edges are products of edges $(\prod_i X_i)_1 = \prod_i (X_i)_1$. So an edge $f : (x_i)_i \to (y_i)_i$ can be represented by a collection of edges $(x_i \xrightarrow{f_i} y_i)_i \in \prod_i (X_i)_1$.

So $(x_i)_i \sim (y_i)_i$ iff there exist edges $f_i : x_i \to y_i$ for all i – ie. $x_i \sim y_i$ in each X_i respectively. We can define a bijection $\pi_0 \prod_i X_i \to \prod_i \pi_0 X_i$ by:

$$[(x_i)_i] \mapsto ([x_i])_i$$

where square brackets denote equivalence classes. $\square$

Remark 2.1.43. Connected components don't preserve all products of arbitrary simplicial sets. Let X be the simplicial set:

$$X = \left(0 \xrightarrow{f_1} 1 \xrightarrow{f_2} \dots \right)$$

ie. X has points indexed by $\mathbb{N}$, and has a single nondegenerate edge between $n \to (n+1)$ for all $n \in \mathbb{N}$. This sset is contractible, since any two points $i, j \in X_0$ are connected by a finite string $i \to i+1 \to \cdots \to j$.

Now let $\overline{X} := \prod_{\mathbb{N}} X = X \times X \times \dots$, an infinite product of X indexed by $\mathbb{N}$. This has vertices and edges given by collections of points and edges in X:

$$\overline{X}_0 = \{x : \mathbb{N} \to X, i \mapsto x_i\}$$
$$\overline{X}_1 = \left\{(x_{i-1} \xrightarrow{f_i} x_i)_{i\in\mathbb{N}}\right\}.$$

Taking connected components,

$$\pi_0\overline{X} = (\overline{X})_0/\sim,$$

where $\sim$ is the equivalence relation generated by the relation $(i \mapsto x_i) \sim (i \mapsto x_{i+1})$ for any point $x = (i \mapsto x_i) \in \overline{X}$. That is, $x = y$ iff there exist finitely many points

$$x \sim z^1 \sim z^2 \sim \cdots \sim z^k \sim y$$

where each $z^j \in \overline{X}_0$ (so each z^j is a function $\mathbb{N} \to X, i \mapsto z_i^j$) is related to z^{j-1} in the following way: given a string of numbers $(z_i)_i$, the next string should consist of adding or subtracting 0 or 1 to each i^{th} entry. The finiteness condition means that under this equivalence relation, two points $x = (i \mapsto x_i)$ and $y = (i \mapsto y_i)$ are identified iff $\{|x_i - y_i|\}_{i\in\mathbb{N}}$ is bounded.

In particular, take the points $x = (i \mapsto 0)$ and $y = (i \mapsto i)$. Since

$$\{|x_i - y_i|\}_{i\in\mathbb{N}} = \mathbb{N}$$

is unbounded, x and y live in separate connected components, making $\pi_0(\overline{X}) \neq \{*\}$, even though $\prod_i \pi_0(X) = \prod_i \{*\} = \{*\}$.

2.2 Travelling between different models of ∞-categories

In this section we will describe some other models of ∞-categories and how they relate to quasi-categories. A good exposition for this is [HTT, Ch.1] as well as [Ber06], which also mentions other models not included here.

Remark 2.2.1. We will take the theory of enriched categories for granted; for details see [Kel82] and [Mac78].

Given a monoidal category V we can consider a category enriched in V – a structure C which is like a category, but instead of hom-sets, hom-objects. So for any $x, y \in \mathrm{ob}(C)$, there exists an object $\mathrm{Hom}_C(x, y) \in \mathrm{ob}(V)$, along with morphisms describing composition, identities, and so on.

Definition 2.2.2. [topologically enriched categories]

A $\boxed{\text{topologically enriched category}}$ $\mathcal{C}$ is a category enriched over **Top** (compactly generated weakly Hausdorff spaces, with monoidal structure given by cartesian product and unit given by the point); ie. $\mathcal{C}$ consists of the following:

- A class of objects $\mathrm{ob}(\mathcal{C})$.
- For each $X, Y \in \mathrm{ob}(\mathcal{C})$, a space $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in$ **Top**.
- For each $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, a continuous function

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

  describing composition.
- For each $X \in \mathrm{ob}(\mathcal{C})$, a point $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$.

Remark 2.2.3. Topologically enriched categories, along with lax-monoidal functors and natural transformations form a 2-category that we call **tCat**.

One way to think of a homotopy theory is as a category enriched over spaces. That is, between any two objects, there is a hom-space, in which maps are encoded as points and homotopies are encoded as paths. That is two maps $x \rightrightarrows y$ (ie. points in the space $\mathcal{C}(x,y)$) are homotopic iff there exists a path $I \to \mathcal{C}(x,y)$ between them.

In this sense, the above gets the job done. One can use topologically enriched categories as a foundation for a theory of ∞-categories, but it has some drawbacks. For example, it's not so easy to define a topologically enriched category of functors between two topologically enriched categories. (On the other hand, it is relatively straightforward to define a quasi-category of functors between quasi-categories (2.8.2).)

We want to translate somehow between topologically enriched categories and quasi-categories. To do this, it turns out that a nice go-between is given by simplicially enriched categories; ie. categories enriched over **sSet**.

Definition 2.2.4. [simplicially enriched categories]

A $\boxed{\text{simplicially enriched category}}$ $\mathcal{C}$ is a category enriched over **sSet** (with tensor product $\times$ and unit Δ^0); ie. $\mathcal{C}$ consists of:

- A class of objects $\mathrm{ob}(\mathcal{C})$.
- For any objects $X, Y \in \mathcal{C}$, a simplicial set called $\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathbf{sSet}$.
- A simplicial map encoding composition for any $X, Y, Z \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z).$$

- A point $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)_0$ for any $X \in \mathcal{C}$.

Remark 2.2.5. We sometimes refer to simplicially enriched categories as simply "simplicial categories". This is an abuse of notation, since there is another notion of a simplicial category which means "a simplicial object in the category **Cat**".

We collect simplicial categories, along with lax-monoidal functors and natural transformations into a 2-category **sCat**.

Proposition 2.2.6. [[Lan21, Lem. 1.2.3.7]

A lax-monoidal functor $f : V \to W$ induces a 2-functor

$$f_* : \mathbf{Cat}_V \to \mathbf{Cat}_W$$

where $\mathbf{Cat}_V$ and $\mathbf{Cat}_W$ are the 2-categories of V-enriched categories and W-enriched categories respectively. An adjunction between monoidal categories induces an adjunction of 2-categories.

Remark 2.2.7. Any ordinary category can be thought of as a simplicially enriched category via the constant functor $(-)_\bullet : \mathbf{Set} \to \mathbf{sSet}$ (2.1.14). The induced functor we call

$$c = ((-)_\bullet)_* : \mathbf{Cat} \to \mathbf{sCat}.$$

This is fully-faithful:

Proof. A functor $F : C_\bullet \to D_\bullet$ between simplicial categories corresponds to an assignment of objects, along with simplicial maps for each $x, y \in C$ between constant simplicial sets:

$$F_{x,y} : \mathrm{Hom}_C(x,y)_\bullet \to \mathrm{Hom}_D(Fx, Fy)_\bullet.$$

By definition this sends identities to identities and is compatible with composition. Since composition on constant hom-ssets are determined by the composition in the ordinary categories, this describes a unique ordinary functor $F : C \to D$. $\square$

This recognizes $\mathbf{Cat} \subseteq \mathbf{sCat}$ as a full subcategory.

The adjunction $(\pi_0 \dashv (-)_\bullet)$ (2.1.41) induces an adjunction

$$\pi = (\pi_0)_* : \mathbf{sCat} \rightleftarrows \mathbf{Cat} : ((-)_\bullet)_* = c.$$

Proposition 2.2.8. [HTT, 1.1.4]

The Quillen equivalence $(|-|, \mathrm{Sing}) : \mathbf{sSet} \rightleftarrows \mathbf{Top}$ (, induces an adjunction (an equivalence) on enriched categories:

$$(|-|_*, \mathrm{Sing}_*) : \mathbf{sCat} \rightleftarrows \mathbf{tCat}.$$

On objects, they are identities, and on morphisms they apply geometric realization and singular simplices on Hom-simplicial sets and Hom-spaces respectively:

$$(\mathcal{C} \in \mathbf{sCat}) \mapsto |\mathcal{C}| := \begin{cases} \mathrm{ob}(|\mathcal{C}|) = \mathrm{ob}(\mathcal{C}) \\ \mathrm{Hom}_{|\mathcal{C}|}(x, y) = |\mathrm{Hom}_{\mathcal{C}}(x, y)| \end{cases}$$

$$(\mathcal{C} \in \mathbf{tCat}) \mapsto \mathrm{Sing}(\mathcal{C}) := \begin{cases} \mathrm{ob}(\mathrm{Sing}(\mathcal{C})) = \mathrm{ob}(\mathcal{C}) \\ \mathrm{Hom}_{\mathrm{Sing}(\mathcal{C})}(x, y) = \mathrm{Sing}(\mathrm{Hom}_{\mathcal{C}}(x, y)) \end{cases}$$

On the other side, we want to translate between simplicially enriched categories and simplicial sets. We will do this by extending the nerve $N : \mathbf{Cat} \to$

sSet to a “simplicial” or “homotopy coherent” nerve $N_\Delta : \mathbf{sCat} \to \mathbf{sSet}$ (sometimes simply also denoted N). This was originally done by Jean-Marc Cordier [Cor82].

Remark 2.2.9. The naive option in defining a simplicial nerve may be to just take strings of vertices $\mathrm{Hom}_{\mathcal{C}}(x,y)_0$ of the hom-sset. But this ignores all higher structure of the hom-sset. We can see this by focusing on the simplest example: a category $[n]$. The ordinary nerve sends $[n] \mapsto N[n] \simeq \Delta^n$. Taking the constant simplicial category $c[n] \in \mathbf{sCat}$ has hom-ssets:

$$\mathrm{Hom}_{c[n]}(i,j)_k = \mathrm{Hom}_{[n]}(i,j)$$

for all k. That is, they are constant simplicial sets. This illuminates our problem: $[n]$, being an ordinary category, lacks any meaningful higher-order structure. The constant simplicial category just encodes a bunch of degenerate simplices.

To construct a thicker nerve, we will first define a thick version of $[n]$ called $\mathfrak{C}[\Delta^n]$ — one that has non-trivial higher-order information (namely homotopies witnessing each string of compositions). For example, the thicker version of $[2]$, called $\mathfrak{C}[\Delta^2]$, includes an explicit homotopy $(0 \to 1 \to 2) \Rightarrow (0 \to 2)$:

$$[2]: \quad \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "="] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \mathfrak{C}[\Delta^2]: \quad \begin{tikzcd} & 1 \arrow[dr] \arrow[d, Rightarrow] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

Higher versions will need to include more and more homotopies (and homotopies between homotopies, and so on).

Definition 2.2.10. [thickened simplicial categories]

For each $n \in \mathbb{N}$, the simplicial category $\boxed{\mathfrak{C}[\Delta^n]}$ (also written $\mathfrak{C}[n]$) has objects $\{0, 1, \dots, n\}$ and hom-simplicial-sets:

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,j) := \begin{cases} N(P_{i,j}) & i \leq j \\ \varnothing & i > j. \end{cases}$$

where $P_{i,j}$ is the partially ordered set $P_{i,j} = \{I \subseteq [n] \text{ of the form } \{i, \dots, j\}\}$ (considered as a category with maps given by $\supseteq$).[1]

Sanity check 2.2.11. In our example, $\mathfrak{C}[\Delta^2]$, when $i = 0, j = 1$, $P_{i,j} = \{\{0,1\}\}$, so the nerve has just one vertex: $\mathrm{Hom}_{\mathfrak{C}[\Delta^2]}(0,1) = NP_{0,1} = \{(0 \to 1)\}$. When $i = 0, j = 2$,

$$P_{0,2} = \{\{0,1,2\} \supseteq \{0,2\}\},$$

so the nerve $NP_{0,2}$ has two vertices (the strings $(0 \to 1 \to 2)$ and $(0 \to 2)$) and an edge between them (the homotopy $(0 \to 1 \to 2) \Rightarrow (0 \to 2)$ that we saw above).

Remark 2.2.12. Given a monotone map $f : [n] \to [m]$, we have a map $\mathfrak{C}(f) : \mathfrak{C}[\Delta^n] \to \mathfrak{C}[\Delta^m]$ sending $i \mapsto f(i)$ and $\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,j) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^m]}(f(i), f(j))$ induced (via the nerve) by the map $P_{i,j} \to P_{f(i),f(j)}, I \mapsto f(I)$. This forms a functor

$$\mathfrak{C} : \mathbf{sSet} \to \mathbf{sCat},$$

which we may call the Cordier map.

Remark 2.2.13. Under the Bergner model structure on **sCat** (4.0.10), these

[1] Sometimes these $P_{i,j}$ are defined with morphisms given by inclusion instead of reverse-inclusion. All this does is "reverse the direction" of homotopies in $\mathfrak{C}[\Delta^n]$. Since all homotopies ought to be invertible in an $(\infty,1)$-category, the direction of these shouldn't matter. Just be careful which is being used.

$\mathfrak{C}[\Delta^n]$ can be thought of as cofibrant replacements (4.0.13) of $[n]$. In fact, for an ordinary category C considered as a simplicial category, the simplicial category $\mathfrak{C}(C)$ is a cofibrant replacement. See [Rie11].

Definition 2.2.14. [homotopy coherent nerve]

Given a simplicial category $\mathcal{C} \in \mathbf{sCat}$, its $\boxed{\text{nerve}}$ is the simplicial set $N_\Delta(\mathcal{C})$ defined:

$$N_\Delta(\mathcal{C})_n := \mathrm{Hom}_{\mathbf{sCat}}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

with face/degeneracy maps induced from the coface and codegeneracies; ie. by precomposing:

$$d_i = - \circ \mathfrak{C}(\delta_i),$$
$$s_i = - \circ \mathfrak{C}(\sigma_i).$$

Remark 2.2.15. Given a functor of simplicial categories $F : \mathcal{C} \to \mathcal{D}$, composing with F forms a simplicial map $N_\Delta F : N_\Delta(\mathcal{C}) \to N_\Delta(\mathcal{D})$, componentwise defined by composing wth F:

$$\underbrace{N_\Delta(\mathcal{C})_n}_{\mathbf{sCat}(\mathfrak{C}[n],\mathcal{C})} \xrightarrow{F\circ -} \underbrace{N_\Delta(\mathcal{D})}_{\mathbf{sCat}(\mathfrak{C}[n],\mathcal{D})} .$$

This forms a functor $N_\Delta : \mathbf{sCat} \to \mathbf{sSet}$, which is right adjoint to $\mathfrak{C}$ (2.2.12).

Sanity check 2.2.16. Sometimes the coherent nerve may be denoted simply N – this has the potential to cause confusion with the ordinary nerve N : $\mathbf{Cat} \to \mathbf{sSet}$, but is justified since one of our motivations was to factor the nerve through simplicial categories. That is, the following diagram of categories should commute:

$$\begin{tikzcd} & \mathbf{sCat} \arrow[dr, "N_\Delta"] \arrow[d, phantom, "?"] & \\ \mathbf{Cat} \arrow[ur, "c"] \arrow[rr, "N"'] & & \mathbf{sSet} \end{tikzcd}$$

Given an ordinary category C, the constant functor interprets it as a simplicial category cC, with all hom- simplicial sets given by constant simplicial sets:

$$\mathrm{Hom}_{cC}(x,y)_n := \mathrm{Hom}_C(x,y)$$

for any $x, y \in C$ and for all n. We can represent these as trivial commuting diagrams in C; eg. an n-simplex in $\mathrm{Hom}_{cC}(x,y)_\bullet$ corresponding to a map $f : x \to y$ can be represented by a commuting diagram of the form:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, equal] \\ x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, equal] \\ \vdots \arrow[d, equal] & \vdots \arrow[d, equal] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

with n strings of identities forming the verticals.

Taking the coherent nerve of this forms a simplicial set $N_\Delta(cC)$ whose simplices are given by:

$$N_\Delta(cC)_n = \mathbf{sCat}(\mathfrak{C}[\Delta^n], cC).$$

A map of simplicial categories $\mathfrak{C}[\Delta^n] \to cC$ will pick out objects $x_0, x_1, \ldots, x_n \in$

C, and for each $i \leq i'$ a map of simplicial sets:

$$\underbrace{\mathfrak{C}[\Delta^n](i,i')}_{NP_{i,i'}} \to \underbrace{cC(x_i, x_{i'})}_{C(x_i, x_{i'})}$$

An n-simplex $\alpha \in NP_{i,i'}$ is a string of subsets of $[n]$ of the form

$$S_0 = \{i, j_1, j_2, \ldots, j_k, i'\} \supseteq S_1 \supseteq \ldots \supseteq S_n.$$

with i and i' included in each subset. For simplicity, we can assume that α is nondegenerate. In this case, that means that all subset inclusions are proper.

Let's label the objects in the image of $S_0 = \{i, j_1, \ldots, j_k, i'\}$ as follows:

$$\begin{aligned} i &\mapsto x \\ j_\ell &\mapsto c_\ell \qquad \text{for } \ell = 1, \ldots, k \\ i' &\mapsto y. \end{aligned}$$

Then the image of S_0 can be interpreted as a string of composable morphisms in C:

$$\begin{array}{cccccc} S_0 = & \{i, & j_1, & \ldots & j_k, & i'\} \\ \\ C: & x \xrightarrow{f_1} & c_1 \longrightarrow & \ldots \longrightarrow & c_k \xrightarrow{f_{k+1}} & y \end{array}$$

Each proper inclusion throws away some elements in between i and i', which we can interpret as composing over the skipped elements. For example, a subset inclusion throwing away a single element

$$\{i, \ldots, j_\ell, \ldots, i'\} \supsetneq \left\{i, \ldots, \widehat{j_\ell}, \ldots, i'\right\}$$

(a non-degenerate 1-simplex in $NP_{i,i'}$), corresponds to the following diagram in C (representing a 1-simplex in $cC(x,y)_\bullet$):

$$\begin{tikzcd}
x \arrow[r, "f_n f_{n-1} \cdots f_1"] \arrow[d, equal] & y \arrow[d, equal] \\
x \arrow[r, "f_n \cdots (f_{\ell+1} f_\ell) \cdots f_1"'] & y
\end{tikzcd}$$

Similarly our nondegenerate n-simplex $S_0 \supsetneq \cdots \supsetneq S_n$ maps to an n-simplex in $cC(x,y)$, where each successive row is a string of morphisms with the skipped elements "composed over".

For clarity, we can look at some small cases. When $n = 2, i = 0, i' = 2$, and the edge $\alpha \in (\mathfrak{C}[\Delta^2](0,2))_1$ given by $\{0,1,2\} \supseteq \{0,2\}$, then the image of α in $cC(x,y)$ can be represented by a commutative diagram in C:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] & c_1 \arrow[r, "g"] & y \arrow[d, equal] \\
x \arrow[rr, "g \circ f"] & & y
\end{tikzcd}$$

which is a triangle witnessing simple composition.

When $n = 3, i = 0, i' = 3$, and $\alpha \in (\mathfrak{C}[\Delta^n](0,3))_2$ the 2-simplex given by:

$$\{0,1,2,3\} \supseteq \{0,1,3\} \supseteq \{0,3\}$$

the corresponding image in $cC(x,y)$ can be represented by a commutative diagram in C:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] & c_1 \arrow[r, "g"] & c_2 \arrow[r, "h"] & y \arrow[d, equal] \\
x \arrow[r, "f"] \arrow[d, equal] & c_1 \arrow[rr, "h\circ g"] & & y \arrow[d, equal] \\
x \arrow[rrr, "(h\circ g)\circ f"] & & & y
\end{tikzcd}$$

which encodes some associativity. Neat!

So a map of simplicial sets $\mathfrak{C}[\Delta^n] \to N_\Delta(cC)$ picks out a string of n composable morphisms in C and equates all the different ways of associating them. This is precisely the ordinary nerve NC.

This sets us up to make precise the relation between simplicial sets and simplicially enriched categories, and more specifically between quasi-categories and Kan-enriched categories.

Proposition 2.2.17. [HTT, 2.2.5.1]

The adjunction

$$\mathfrak{C} : \mathbf{sSet}_{Joyal} \rightleftarrows \mathbf{sCat}_{Bergner} : N_\Delta$$

is a Quillen equivalence (4.1.5).

Remark 2.2.18. Fibrant objects (4.0.13) in $\mathbf{sSet}_{Joyal}$ are precisely the quasi-categories, and the fibrant objects in $\mathbf{sCat}_{Bergner}$ are precisely Kan-enriched categories. This makes us make precise the relationship between quasi-categories and Kan-enriched categories as equivalent models of ∞-categories.

The following lays out the situation between simplicial sets, simplicial categories, and topologically enriched categories, and the adjunctions between them (left adjoints on the left):

$$\begin{tikzcd}
 & \mathbf{sSet} \arrow[d, shift right, "\mathfrak{C}"'] & \\
 & \mathbf{sCat} \arrow[u, shift right, "N_\Delta"'] \arrow[dl, shift right, "|-|_*"'] \arrow[dr, shift left, "\pi"] & \\
\mathbf{tCat} \arrow[ur, shift right, "\mathrm{Sing}_*"'] & & \mathbf{Cat} \arrow[ul, shift left, "c"] \arrow[uul, bend right, "N"']
\end{tikzcd}$$

2.2.1 Other models of ∞-categories

An $(\infty, 1)$-category comes with varying degrees of morphisms, with those above degree 1 all invertible. There are many ways of encoding or approximating such a construction. We talked about three: quasi-categories, simplicially enriched categories, and topologically enriched categories. But there are others (see [Ber06] for a survey), of varying degrees of complexity / equivalence to ∞-categories.

- Rezk spaces (a.k.a. complete Segal spaces)
- Segal categories (a generalization of simplicial categories)
- 1-complicial sets
- Cubical sets

2.3 Morphism spaces

We showed that ∞-categories are the same as Kan-enriched categories (2.2.18). Given points $x, y \in X_0$ in an ∞-category, we can construct a Kan complex whose points correspond to edges of the form $(x \to y) \in X_1$.

Definition 2.3.1. [mapping complex][K, 4.6.1.1]

Let S be a simplicial set, and let $x, y \in S_0$ be points.

The $\boxed{\text{mapping space}}$ of S between x and y is a simplicial set $\mathrm{Map}_S(x,y)$, defined as a fiber product of simplicial sets:

$$\mathrm{Map}_S(x,y) := \{x\}_\bullet \times_{S^{\Delta\{0\}}} S^{\Delta^1} \times_{S^{\Delta\{1\}}} \{y\}_\bullet,$$

where $\{x\}_\bullet$ and $\{y\}_\bullet$ are constant simplicial sets (2.1.14).

Alternatively one can define a mapping space as a pullback of simplicial sets:

$$\begin{tikzcd}
\mathrm{Map}_S(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, S) \arrow[d, "\text{(source,target)}"] \\
\Delta^0 \arrow[r, "{(x,y)}"'] & S \times S
\end{tikzcd}$$

Remark 2.3.2. [K, 4.6.1.1] When S is an ∞-category, $\mathrm{Map}_S(x,y)$ forms a Kan complex, hence the name mapping space.

Sanity check 2.3.3. Limits in simplicial sets are calculated "pointwise" (on the level of sets), so we can draw out some of the low-dimensional simplices of this mapping space by hand and check that they match our intuition of what we think they should be. Namely points in a mapping space $\mathrm{Map}_S(x,y)$ should correspond to edges $(x \to y) \in S_1$, and edges in $\mathrm{Map}_S(x,y)$ should correspond to homotopies.

$$\begin{tikzcd}
 & & \mathrm{Map}_S(x,y)_n \arrow[dll] \arrow[d] \arrow[drr] & & \\
\{x\}_n \arrow[d, hook] & & (S^{\Delta^1})_n \arrow[dll, "i_0^*"] \arrow[drr, "i_1^*"'] & & \{y\}_n \arrow[d, hook] \\
S_n & & & & S_n
\end{tikzcd}$$

$\{x\}_n$ consists of a single degenerate constant n-simplex $(x = x = \cdots = x)$ (similarly $\{y\}_n$). The guy in the middle $(S^{\Delta^1})_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times \Delta^1, S)$.

(0-simplices): We'd expect points of the mapping space between x and y to be maps of the form $(x \to y) \in S_1$.

$$\begin{aligned}\mathrm{Map}_S(x,y)_0 &= \{x\}_0 \times_{S_0} (S^{\Delta^1})_0 \times_{S_0} \{y\}_0 \\ &= \left\{ f : \Delta^1 \to S \ : \ f(0) = x, f(1) = y \right\} \\ &= \{(f : x \to y) \in S_1\}\end{aligned}$$

1-simplices: Edges should be homotopies $f \sim g$ between maps $x \rightrightarrows y$:

$$\mathrm{Map}_S(x,y)_1 = \{x\}_1 \times_{S_1} (S^{\Delta^1})_1 \times_{S_1} \{y\}_1 .$$

Edges of the function complex $(S^{\Delta^1})_1 = \mathrm{Fun}(\Delta^1 \times \Delta^1, S)$ (these look like square-shaped diagrams), and edges of constant simplicial sets are degenerate (eg. $\{x\}_1 = \left\{ x \xrightarrow{\mathrm{id}} x \right\}$). So edges of $\mathrm{Map}_S(x,y)$ are squares $\Delta^1 \times \Delta^1 \to S$ of the form:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr] & y \arrow[d, equal] \arrow[dl, phantom, "\Swarrow"] \\ x \arrow[r, "g"'] \arrow[ur, phantom, "\Nearrow"] & y \end{tikzcd}$$

ie. three maps f, g, h from $x \to y$, and homotopies $\mathrm{id} \circ f = f \sim h$ and $g \circ \mathrm{id} = g \sim h$, which exhibit $f \sim h \sim g$ as homotopic.

Sanity check 2.3.4. What about when our quasi-category is the nerve of a category? Then $\mathrm{Map}_{NC}(x,y)$ is a pullback:

$$\begin{tikzcd} \mathrm{Map}_{NC}(x,y) \arrow[r] \arrow[d] & NC^{\Delta^1} \arrow[d, "(\mathrm{source},\mathrm{target})"] \\ \Delta^0 \arrow[r, "(x,y)"'] & NC \times NC \end{tikzcd}$$

By the pullback, n-simplices of the mapping space $\mathrm{Map}_{NC}(x,y)$ look like diagrams $\Delta^n \times \Delta^1 \to NC$ such that

$$\begin{tikzcd}
\Delta^n \times \{0\} \arrow[d, hook] \arrow[dr, bend left, "x"] & \\
\Delta^n \times \Delta^1 \arrow[r] & NC \\
\Delta^n \times \{1\} \arrow[u, hook] \arrow[ur, bend right, "y"'] &
\end{tikzcd}$$

where $x,y : \Delta^n \rightrightarrows NC$ are degenerate n-simplices constant at x and y respectively.

Simplices of the function complex NC^{Δ^1} look like

$$\begin{aligned}
(NC^{\Delta^1})_n &= \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times \Delta^1, NC) \\
&\cong \mathrm{Hom}_{\mathbf{sSet}}(N([n]\times[1]), NC) \\
&\cong \mathrm{Hom}_{\mathbf{Cat}}([n]\times[1], C),
\end{aligned}$$

ie. diagrams in C of the form

$$\begin{tikzcd}
c_0 \arrow[r] \arrow[d] & c_1 \arrow[r] \arrow[d] & \dots \arrow[r] & c_n \arrow[d] \\
c_0' \arrow[r] & c_1' \arrow[r] & \dots \arrow[r] & c_n'
\end{tikzcd}$$

So n-simplices of $\mathrm{Map}_{NC}(x,y)$ correspond to diagrams in C of the form:

$$\begin{tikzcd}
x \arrow[r, equal] \arrow[d] & x \arrow[r, equal] \arrow[d] & \dots \arrow[r, equal] & x \arrow[d] \\
y \arrow[r, equal] & y \arrow[r, equal] & \dots \arrow[r, equal] & y
\end{tikzcd}$$

which are really just the data of a single morphism $x \to y$ in C. So we have an equivalence

$$\mathrm{Map}_{NC}(x,y) \simeq \mathrm{Hom}_C(x,y)_\bullet$$

where $\mathrm{Hom}_C(x,y)_\bullet$ is the constant simplicial set (2.1.14).

2.4 Homotopy categories

The bare minimum required to present a homotopy theory is an ordinary category with a class of weak equivalences. One would like to study how such a category behaves if these weak equivalences were honest isomorphisms. This construction is known as a homotopy category.

Definition 2.4.1. [homotopy category of a category with weak equivalences]

Let (C, W) consist of an ordinary category C with a collection $W \subseteq \mathrm{Hom}(C)$ of "weak equivalences". The $\boxed{\text{homotopy category}}$ (also called the localization of C at W) is a functor

$$\gamma : C \to hC = C[W^{-1}]$$

that:

- inverts all maps in W; and
- is universal with this property: that any functor $C \to D$ sending all maps in W to isomorphisms in D factors through γ.

Example 2.4.2. The primordial homotopy category is the localization of **Top** at weak homotopy equivalences.

Example 2.4.3. Another example is the homotopy category of chain complexes over a commutative ring R. Let $\mathbf{Ch}(R)$ be the category of chain complexes of R-modules.

The homotopy category is a localization

$$\mathbf{Ch}(R) \to \mathbf{Ch}(R)[(h.e.)^{-1}] = K(R),$$

of chain complexes at homotopy equivalences of chain complexes.

The derived category is a localization

$$\mathbf{Ch}(R) \to \mathbf{Ch}(R)[(q.iso's)^{-1}] = D(R),$$

of chain complexes at quasi-isomorphisms.

2.4.1 Of model categories

Remark 2.4.4. More generally, if M is a model category (4.0.10), there is an explicit description of its homotopy category hM by taking the same objects and taking hom-sets to be $hM(X,Y) := M(X^c, Y^f)$, where X^c and Y^f are cofibrant and fibrant replacements respectively (4.0.13). This exhibits $hM = M[W^{-1}]$ as the localization of M at its weak equivalences [Qui67].

Remark 2.4.5. A Quillen equivalence forms an equivalence (as ordinary categories) of homotopy categories (4.1.5).

- In particular the Quillen equivalence $\mathbf{sSet}_{Quillen} \simeq_Q \mathbf{Top}_{Quillen}$ describes an equivalence of categories $h\mathbf{sSet}_{Quillen} \simeq h\mathbf{Top}_{Quillen}$. Since bifibrant objects in $\mathbf{sSet}_{Quillen}$ are Kan complexes and in $\mathbf{Top}_{Quillen}$ are CW-complexes, we have isomorphisms $h\mathbf{sSet}_{Quillen} \simeq h\mathbf{Kan}$ and $h\mathbf{Top}_{Quillen} \simeq h\mathbf{CW}$, where $\mathbf{Kan} \subseteq \mathbf{sSet}$ and $\mathbf{CW} \subseteq \mathbf{Top}$ are the full subcategories of Kan complexes and CW-complexes. That is, the homotopy theory of Kan complexes is the same as the homotopy theory of CW-complexes.

- Similarly, the Quillen equivalence $\mathbf{sSet}_{Joyal} \simeq_Q \mathbf{sCat}_{Bergner}$ forms an equivalence $h\mathbf{sSet} \simeq h\mathbf{sCat}$. Bifibrant objects in $\mathbf{sSet}_{Joyal}$ are quasi-categories and in $\mathbf{sCat}_{Bergner}$ are Kan-enriched categories, forming an equivalence of homotopy categories $h\mathbf{qCat} \simeq h\mathbf{KCat}$ where $\mathbf{KCat} \subseteq \mathbf{sCat}$ is the full subcategory of Kan-enriched categories.

2.4.2 Of topologically/simplicially enriched categories

We can model a homotopy theory as a topologically or simplicially enriched category. Here homotopies are encoded as paths/edges in hom-spaces/ssets.

Definition 2.4.6. [homotopy category of a topologically/simplicially enriched category]

Let $\mathcal{C}$ be a topologically or simplicially enriched category.

The homotopy category $h\mathcal{C}$ is an ordinary category whose objects are the objects of $\mathcal{C}$, and hom-sets:

$$\mathrm{Hom}_{h\mathcal{C}}(x,y) := \pi_0 \,\mathrm{Hom}_{\mathcal{C}}(x,y),$$

where π_0 is the connected components (2.1.3) of the hom-space or hom-sset. If $f \in \mathcal{C}(x,y)$ is a point in the mapping-space/sset, we call its image in the homotopy category $[f] \in h\mathcal{C}(x,y)$.

Identities and composition come from the enriched category $\mathcal{C}$. That is id_x in $h\mathcal{C}$ is $[\mathrm{id}_x]$, and composition is:

$$\begin{aligned} h\mathcal{C}(x,y) \times h\mathcal{C}(y,z) &\to h\mathcal{C}(x,z) \\ ([f],[g]) &\mapsto [g]\circ[f] := [g\circ f]. \end{aligned}$$

Sanity check 2.4.7. Let $f, f' \in \mathcal{C}(x,y)$ be points in a hom-space/sset. By definition, $[f] = [f']$ in the homotopy category iff they live in the same connected component of the mapping-space/sset $\mathcal{C}(x,y)$. That is, if there is a path $I \to \mathcal{C}(x,y)$ or an edge $\Delta^1 \to \mathcal{C}(x,y)$ connecting them. In other words, $[f] = [f']$ in the homotopy category iff f and f' are homotopic.

If $f : x \to y$ is a weak equivalence in $\mathcal{C}$, by definition there is a map $f^{-1} : y \to x$ such that $f^{-1} \circ f$ is homotopic to id_x (and $f \circ f^{-1}$ is homotopic to id_y). That is, there exists a path/edge $f^{-1}f \to \mathrm{id}_x$ in the space/sset $\mathcal{C}(x,x)$. That is, $f^{-1}f$ and id_x live in the same path-component of $\mathcal{C}(x,x)$, ie.

$$[f^{-1}] \circ [f] = [f^{-1}f] = [\mathrm{id}_x]$$

(similarly $[f] \circ [f^{-1}] = [\mathrm{id}_y]$) in the homotopy category. So indeed a weak equivalence $f \in \mathcal{C}$ is sent to an isomorphism in $h\mathcal{C}$.

Remark 2.4.8. [Rez22, Def. 43.11][HTT, 1.1.3]

There is another construction also called the "homotopy category" of a topologically enriched category. Let $\mathcal{C} \in \mathbf{tCat}$.

Let $\mathcal{H}\mathcal{C}$[2] be the category enriched over $\mathcal{H} = h\mathbf{Kan}$ (ie. enriched over homotopy types) as follows: objects of $\mathcal{H}\mathcal{C}$ are objects of $\mathcal{C}$. Hom-homotopy types are:

$$\mathrm{Hom}_{\mathcal{H}\mathcal{C}}(x,y) := [\mathrm{Sing}(\mathcal{C}(x,y))] \in \mathcal{H},$$

where $\mathcal{C}(x,y) \in \mathbf{Top}$ is the mapping-space, and Sing is the singular complex functor $\mathbf{Top} \to \mathbf{Kan}$ (2.1.22)

[2] In [HTT] this is also denoted $h\mathcal{C}$, which can cause some confusion between the $\mathcal{H}$-enriched homotopy category versus the ordinary homotopy category $h\mathcal{C}$, which is why we have called it $\mathcal{H}\mathcal{C}$.

The two notions of the homotopy category ($\mathcal{HC}$ and $h\mathcal{C}$) of a topologically enriched category $\mathcal{C}$ are related. Taking $X := \mathrm{Hom}_{\mathcal{C}}(c, c') \in \mathbf{Top}$ for objects $c, c' \in \mathcal{C}$, then $[X] = \mathrm{Hom}_{\mathcal{HC}}(c, c') \in \mathcal{H}$ and $\pi_0 X = \mathrm{Hom}_{h\mathcal{C}}(c, c')$ by definition. Connected components are a homotopy invariant, so connected components of the homotopy type $\pi_0[X]$ are the same as $\pi_0 X$.

2.4.3 Of simplicial sets

In the case of simplicial sets, the nerve $N : \mathbf{Cat} \to \mathbf{sSet}$ (2.1.18) admits a left adjoint $h : \mathbf{sSet} \to \mathbf{Cat}$, which sends a simplicial set S to a category hS, called its homotopy category.

Definition 2.4.9. [K, 1.2.5.4]

[homotopy category of a simplicial set]

Let $S \in \mathbf{sSet}$. Its $\boxed{\text{homotopy category}}$ hS is an ordinary category consisting of:

- (objects): Objects are vertices of S, so $\mathrm{ob}(hS) := S_0$;
- (morphisms): Any edge $f \in S_1$ determines a morphism $[f] : d_1(f) \to d_0(f)$ in hS. These generate morphisms of hS under composition, so any string of edges in S:

  $$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n$$

  forms a map $[f_n f_{n-1} \dots f_1] \in hS(x_0, x_n)$.

  Identities and composition are defined by:

  $$[s_0(x)] = \mathrm{id}_x, \quad [d_1(\sigma)] = [d_0(\sigma)] \circ [d_2(\sigma)].$$

for any $x \in S_0$ and any $\sigma \in S_2$. That is, identities are given by degenerate edges, and $[g] \circ [f] = [h]$ iff there is a 2-simplex in S of the form:

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet \end{tikzcd}$$

Proposition 2.4.10. [K, 1.2.5.5] The construction $S \mapsto hS$ described above forms a functor $h : \mathbf{sSet} \to \mathbf{Cat}$, and this functor h is left adjoint to the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$.

Remark 2.4.11 (K, 1.2.5.7)**.** It is not true that every map in hS comes from a single edge in S_1. Generally, a map $x \to y$ in hS can be represented by a sequence of edges $[f_n \dots f_1]$. This makes it not so easy to describe the homotopy category of an arbitrary simplicial set.

When S is an ∞-category, this problem disappears. A sequence of edges in an ∞-category

$$\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \dots \xrightarrow{f_n} \bullet$$

forms the spine (2.1.17) of an $(n+1)$-simplex which we can fill in (2.1.29) and take the edge $\Delta^{\{0,n+1\}} \subseteq \Delta^{n+1}$ as the composite.

2.4.4 Of ∞-categories

Remark 2.4.12. Let $\mathcal{C}$ be a quasi-category, $x, y \in \mathcal{C}_0$, and edges

$$f, g \in \mathcal{C}_1(x, y) := \{\varphi \in \mathcal{C}_1 \ : \ \varphi(0) = x, \varphi(1) = y\}.$$

There are two ways we may describe when f and g are homotopic; call them $\sim_\ell$ and $\sim_r$ for "left" and "right" homotopic.

$$(f \sim_\ell g): \quad \begin{tikzcd} x \arrow[dd, "\mathrm{id}_x"'] \arrow[dr, "f"] & \\ & y \\ x \arrow[ur, "g"'] & \end{tikzcd} \qquad\qquad (f \sim_r g): \quad \begin{tikzcd} & y \arrow[dd, "\mathrm{id}_y"] \\ x \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & y \end{tikzcd}$$

These form two different relations $\sim_\ell$ and $\sim_r$ on $\mathrm{Hom}_{\mathcal{C}}(x,y)$. We can define two hom-ssets

$$\mathrm{Hom}^L_{\mathcal{C}}(x,y) := \mathrm{Hom}_{\mathcal{C}}(x,y)/\sim_\ell$$
$$\mathrm{Hom}^R_{\mathcal{C}}(x,y) := \mathrm{Hom}_{\mathcal{C}}(x,y)/\sim_r .$$

If $\mathcal{C}$ is an ∞-category, the two equivalence relations are the same [Cis19, Lem. 1.6.4], and we can define a mapping space $\mathrm{Hom}_{\mathcal{C}}(x,y)$ that is weakly equivalent to both the left and right hom-ssets [HTT, 4.2.1.8]

$$\mathrm{Hom}^L_{\mathcal{C}}(x,y) \simeq \mathrm{Hom}_{\mathcal{C}}(x,y) \simeq \mathrm{Hom}^R_{\mathcal{C}}(x,y).$$

Definition 2.4.13. [homotopy category of a quasi-category]

Given a quasi-category $\mathcal{C}$, its $\boxed{\text{homotopy category}}$ $h\mathcal{C}$ has objects given by vertices $\mathcal{C}_0$, and morphisms given by $\mathrm{Hom}_{h\mathcal{C}}(x,y) := \mathcal{C}_1(x,y)/\sim$, where

$$\mathcal{C}_1(x,y) := \{f \in \mathcal{C}_1 \ : \ d_1(f) = x, d_0(f) = y\}$$

and $\sim$ is the equivalence relation of homotopy. That is, maps $[f] = [g]$ in $h\mathcal{C}$ iff

$f \sim g$ in $\mathcal{C}$.

Remark 2.4.14. Since the nerve construction $N : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful (2.1.20), for any ordinary category C, the counit $hNC \xrightarrow{\sim} C$ is an equivalence. For an ∞-category $\mathcal{C}$, the unit $\mathcal{C} \to Nh\mathcal{C}$ is an isomorphism iff $\mathcal{C}$ is the nerve of some category [Cis19, Prop. 1.4.11].

Proposition 2.4.15. [homotopy category of a Kan-enriched category][Cis19, Prop. 3.7.2]

An ∞-category $\mathcal{C}$ can be realized as a Kan-enriched category (2.2.18) by the functor $\mathfrak{C} : \mathbf{sSet} \to \mathbf{sCat}$ (2.2.12). We can construct its homotopy category $h^{scat}\mathfrak{C}(\mathcal{C})$ as described for simplicial categories (2.4.6).

Then there is an isomorphism of homotopy categories

$$h^{scat}\mathfrak{C}(\mathcal{C}) \cong h\mathcal{C},$$

where $h\mathcal{C}$ is the homotopy category as a quasi-category (2.4.13).

That is, we can describe the homotopy category $h\mathcal{C}$ as:

$$\mathrm{Hom}_{h\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y),$$

as the connected components (2.1.3) of the mapping space (2.3.1).

Sanity check 2.4.16. Let $\mathcal{C}$ be an ∞-category.

The homotopy category of the opposite ∞-category $\mathcal{C}^{op}$ (2.1.37) is equivalent to the opposite of the homotopy category:

$$h(\mathcal{C}^{op}) \simeq (h\mathcal{C})^{op}.$$

Objects in both are just objects of $\mathcal{C}$. A morphism in $h(\mathcal{C}^{op})$ is an equivalence class of a morphism $f^{op} : x \to y$, which corresponds to an edge $f : y \to x$ in $\mathcal{C}$. The edge f corresponds to an equivalence class $[f] : y \to x$ in $h\mathcal{C}$, which forms a morphism $[f]^{op} : x \to y$ in $(h\mathcal{C})^{op}$. Any morphism in $(h\mathcal{C})^{op}$ can be described in this way.

We've introduced a number of different "homotopy categories":

- (of an ordinary category with weak equivalences): as an ordinary category formed by inverting weak equivalences.
- (of a model category): as a localization inverting weak equivalences, or equivalently described as a category where hom-sets are restricted to cofibrant replacements in the source and fibrant ones in the targets.
- (of a topologically enriched category): as an $\mathcal{H}$-enriched category formed by taking homotopy classes of mapping spaces.
- (of a topologically enriched category): as an ordinary category formed by taking connected components of mapping spaces.
- (of a simplicially enriched category): as an ordinary category formed by taking connected components of mapping simplicial sets.
- (of a simplicial set): as an ordinary category with morphisms generated by edges of the simplicial set (sometimes this is called the fundamental category).
- (of a quasi-category): as an ordinary category formed by taking equivalence classes of edges via homotopy.

2.5 ∞-categories of spaces and ∞-categories

We will construct two ∞-categories: (1) an ∞-category of spaces called $\mathcal{S}$, and (2) an ∞-category of ∞-categories called $\mathbf{Cat}_\infty$.

(1)

Definition 2.5.1. [HTT, 1.2.16.1] The category of Kan complexes is naturally simplicially enriched[3] by taking the mapping complex (2.8.2):

$$\mathrm{Map}_{\mathbf{Kan}}(X,Y)_n = \mathbf{sSet}(X \times \Delta^n, Y).$$

Taking the coherent nerve forms an $\boxed{\text{$\infty$-category of spaces}}$ $\mathcal{S} = N_\Delta(\mathbf{Kan})$.

Remark 2.5.2. We could have started with **CW**. Two CW complexes X and Y form a CW complex Y^X, whose points are maps $X \to Y$ and which is also a CW-complex. with Kan-enrichment given by singular complexes of mapping spaces, and then taken the simplicial nerve of this to get an alternate version of an ∞-category of spaces. There is an equivalence of ∞-categories $N_\Delta(\mathbf{Kan}) = N_\Delta(\mathbf{CW}) = \mathcal{S}$ (see [Lan21, Obs. 1.3.43]).

Remark 2.5.3. This is a crucial ∞-category, playing the role that the category of sets plays in ordinary category theory. For example, instead of hom- sets we have hom- spaces (2.3.1), which live in $\mathcal{S}$. And presheaves of a ordinary category C are functors $C^{op} \to \mathbf{Set}$, while presheaves of an ∞-category $\mathcal{C}$ are (∞-) functors $\mathcal{C}^{op} \to \mathcal{S}$.

(2)

[3] In fact, Kan-enriched (2.3.2).

Definition 2.5.4. [core/maximal groupoid]

Let C be a category.

The $\boxed{\text{maximal groupoid}}$ is a subcategory $C^{\simeq} \subseteq C$ formed by restricting hom-sets to isomorphisms

$$\mathrm{Hom}_{C^{\simeq}}(X,Y) := \left\{\text{isomorphisms } X \xrightarrow{\sim} Y\right\} \subseteq \mathrm{Hom}_C(X,Y).$$

Using this we can find a maximal ∞-groupoid inside an ∞-category $\mathcal{C}$ by a pullback of simplicial sets:

$$\begin{tikzcd}
\mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
N((h\mathcal{C})^{\simeq}) \arrow[r] & Nh\mathcal{C}
\end{tikzcd}$$

Definition 2.5.5. [the ∞-category of ∞-categories] [HTT, 3.0.0.1]

Let $\mathbf{Cat}_{\infty}^{\Delta}$ be the simplicial category whose objects are quasi-categories, and whose hom simplicial sets are Kan complexes formed by taking the maximal Kan complexes of the usual function complexes:

$$\mathbf{Cat}_{\infty}^{\Delta}(X,Y) = (Y^X)^{\simeq} \subseteq Y^X.$$

Taking the simplicial nerve of this Kan-enriched category forms an ∞-category $\mathbf{Cat}_{\infty} := N_{\Delta}(\mathbf{Cat}_{\infty}^{\Delta})$.

Remark 2.5.6. Note that this process throws away some information. In particular, we lose the information of any non-invertible functors between ∞-categories. In analogy with how **Cat** really wants to form a

2-category, we can see that $\mathbf{Cat}_\infty$ wants to form something like an "$(\infty, 2)$-category". But we will not pursue this.

We want to form an ∞-category, ie. a category enriched in ∞-groupoids (2.1.32). The category of quasi-categories $\mathbf{qCat} \subseteq \mathbf{sSet}$ is simplicially enriched (2.8.2), but a priori the mapping-sset of functors between two ∞-categories need not be an ∞-groupoid. This is the reason for taking the maximal subgroupoid in the construction above.

2.6 Subcategories

Definition 2.6.1. [(∞-) subcategories]

A $\boxed{\text{subcategory}}$ $\mathcal{D}$ of an ∞-category $\mathcal{C}$ is defined via a pullback (of simplicial sets):

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d] \\ N(D) \arrow[r] & N(h\mathcal{C}) \end{tikzcd}$$

where $D \hookrightarrow h\mathcal{C}$ is a subcategory in the ordinary sense.

Remark 2.6.2. Defining subcategories in this way means that many properties of subcategories of an ∞-category $\mathcal{C}$ depend only the level of homotopy categories. For example, a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called full if $h\mathcal{C}' \subseteq h\mathcal{C}$ is full in the usual sense.

2.7 Fibrations

Fibrations show up often in this world, as they allow us a certain level of control rather than working directly with simplicial sets, which often involves keeping track of large amounts of coherence data.

It's easy to lose track of all the various types of fibrations, so we collect these here for reference. A good reference for fibrations in ∞-category theory is [BS16].

Recall that a map p in a category is said to have the right lifting property (RLP) (4.0.6) against a map f if there exists a lift of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

Definition 2.7.1. [fibrations, fibrations, fibrations]

A map $p : X \to Y$ of simplicial sets is called...

- a $\boxed{\text{Kan fibration}}$ if it has the RLP against all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \le i \le n$, for all $n \in \mathbb{N}$.

- a $\boxed{\text{trivial fibration}}$ if it has the RLP against all boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, for all $n \in \mathbb{N}$.

- an $\boxed{\text{isofibration}}$ if it has the RLP aginst the map $\Delta^0 \to J$, where $J = \left\{\bullet \xrightarrow{\sim} \bullet\right\}$ is the walking isomorphism (ie. the preimage of any isomorphism in Y is an isomorphism in X).

- an $\boxed{\text{inner fibration}}$ if it has the RLP against inner-horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i < n$.

- a $\boxed{\text{left fibration}}$ (resp. right fibration) if it has the RLP against horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \le i < n$ (resp. $0 < i \le n$).

- a $\boxed{\text{cocartesian fibration}}$ (resp. cartesian fibration) if it's an inner fibration and has the RLP with respect to the inclusion $\{0\} \hookrightarrow \Delta^1$ (rep. $\{1\} \hookrightarrow \Delta^1$), and the resulting edge $\Delta^1 \to X$ is p-cocartesian (resp. p-cartesian).

These sit in relation to each other as follows:

$$\begin{tikzcd}
 & \{\text{trivial Kan fibrations}\} \arrow[d, hook] & \\
 & \{\text{Kan fibrations}\} \arrow[dl, hook] \arrow[dr, hook] & \\
\{\text{left fibrations}\} \arrow[dr, hook] & & \{\text{right fibrations}\} \arrow[dl, hook] \\
 & \{\text{isofibrations}\} \arrow[d, hook] & \\
 & \{\text{inner fibrations}\} &
\end{tikzcd}$$

2.7.1 Co/cartesian fibrations

Definition 2.7.2. [co/cartesian morphisms in ordinary categories]

Let $p : C \to D$ be a functor of ordinary categories. A map $f : [1] \to C$ is called:

- $\boxed{p\text{-cartesian}}$ if there's a unique solution to the lifting problem:

$$\begin{tikzcd}
(1 \to 2) \arrow[d, hook] \arrow[drr, "f"] & & \\
(1 \to 2 \leftarrow 0) \arrow[rr] \arrow[d, hook] & & C \arrow[d, "p"] \\
{[2]} \arrow[rr] \arrow[urr, dotted, "\exists!"] & & D
\end{tikzcd}$$

- $\boxed{p\text{-cocartesian}}$ if there's a unique solution to the lifting problem:

$$\begin{tikzcd}
(0 \to 1) \arrow[d, hook] \arrow[dr, "f"] & \\
(1 \leftarrow 0 \to 2) \arrow[r] \arrow[d, hook] & C \arrow[d, "p"] \\
{[2]} \arrow[r] \arrow[ur, dotted, "\exists!"'] & D
\end{tikzcd}$$

This formulation is intended to be analagous to the ∞-version, but we can draw out what this looks like in terms of concrete morphisms in C and D. A map $f : c_1 \to c_2$ in C is p-cartesian if for any map $(c_0 \to c_2) \in C$ and a commuting triangle

$$(p(c_0) \to p(c_1) \xrightarrow{p(f)} p(c_2)) \in D,$$

there exists a unique map $(c_0 \to c_1)$ filling in the triangle in C:

$$C: \quad \begin{tikzcd}
& c_1 \arrow[dr, "f"] & \\
c_0 \arrow[ur, dotted, "\exists!"] \arrow[rr] & & c_2
\end{tikzcd}$$

$$D: \quad \begin{tikzcd}
& p(c_1) \arrow[dr, "p(f)"] & \\
p(c_0) \arrow[ur] \arrow[rr] & & p(c_2)
\end{tikzcd}$$

Definition 2.7.3. [HTT, 2.4.1.4]

[co/cartesian morphisms in ∞-categories]

Let $p : X \to Y$ be an inner fibration between simplicial sets.

An edge $f : \Delta^1 \to X$ is called:

- $\boxed{p\text{-cartesian}}$ if for any $n \geq 2$, any lifting problem

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^n_n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution.

- $\boxed{p\text{-cocartesian}}$ if for any $n \geq 2$, any lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution.

Remark 2.7.4. Note that the lifts in the ∞-version are not required to be unique, unlike the ordinary version.

Sanity check 2.7.5. Let $p : C \to D$ be a functor of ordinary categories and $Np : NC \to ND$ the image under the nerve. Then a morphism $f \in \mathrm{Hom}(C)$ is p-cartesian (2.7.2) iff it is Np-cartesian (2.7.3) as an edge in NC.

Proof. $\Rightarrow$: Suppose $f : x \to y$ is a p-cartesian map of C. This defines an edge $f \in NC_1$, and we can ask if this is Np-cartesian in the ∞-sense. Form a lifting problem:

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[d, hook] \arrow[dr, "f"] & \\
\Lambda^n_n \arrow[r] \arrow[d, hook] & NC \arrow[d, "Np"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & ND
\end{tikzcd}$$

An n-simplex of NC is a string $[n] \to C$. In particular, take the face

$$\Delta^{\{0,1,\ldots,\widehat{n-1},n\}} \to \Lambda^n_n.$$

This is a string in C:

$$\begin{tikzcd}
c_0 \arrow[r] & c_1 \arrow[r] & \ldots \arrow[r] & c_{n-2} \arrow[rr, bend left] & & c_n
\end{tikzcd}$$

that skips c_{n-1}. This string is such that it forms an n-simplex in ND, ie. a string in D of the form $p(c_0) \to \cdots \to p(c_{n-1}) \xrightarrow{p(f)} p(c_n)$. That is, we have the following strings in C and D:

$$\begin{tikzcd}
 & & & & c_{n-1} \arrow[dr, "f"] & \\
C: & c_0 \arrow[r] & c_1 \arrow[r] & \ldots \arrow[r] & c_{n-2} \arrow[rr] & & c_n \\
D: & p(c_0) \arrow[r] & p(c_1) \arrow[r] & \ldots \arrow[r] & p(c_{n-2}) \arrow[r] & p(c_{n-1}) \arrow[r, "p(f)"] & p(c_n)
\end{tikzcd}$$

If f is p-cartesian, there exists a map $c_{n-2} \to c_{n-1}$ making the triangle commute and mapping to the edge $p(c_{n-2}) \to p(c_{n-1})$ in D. This forms a simplex $\Delta^n \to NC$ making the diagram commute.

$\Leftarrow$: On the other hand, suppose $f \in NC_1$ is Np-cartesian. Is it p-cartesian?

Since $h(\Delta^n) = h(N[n]) \simeq [n]$, and

$$\begin{aligned} h(\Lambda^n_n) &= h\left(\bigcup_{i \neq n} \Delta^{\{0,\dots,\widehat{i},\dots,n\}}\right) \\ &\simeq \bigcup_i h(\Delta^{n-1}) \\ &\simeq \bigcup_i [n-1], \end{aligned}$$

we can translate the lifting problem via adjunction ($h \dashv N$) (2.4.10) to get a lifting problem in **Cat**:

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^2_2 \arrow[r] \arrow[d] & NC \arrow[d, "Np"] \\ \Delta^2 \arrow[r] \arrow[ur, dotted] & ND \end{tikzcd} \qquad \begin{tikzcd} {[1]} \arrow[d] \arrow[dr, "f"] & \\ (0 \to 2 \leftarrow 1) \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ {[2]} \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

A solution exists since f is Np-cartesian. But it is not necessarily unique. Suppose we had two such solutions g and g':

$$\begin{tikzcd} & c_1 \arrow[dr, "f"] & \\ c_0 \arrow[ur, dashed, shift left, "g"] \arrow[ur, dashed, shift right, "g'"'] \arrow[rr, "\varphi"'] & & c_2 \end{tikzcd}$$

We can organize these into a horn $\Lambda^3_3 \to NC$ formed by the three faces:

$$
\begin{tikzcd}
c_0 \arrow[r, "g"] \arrow[d, equal] \arrow[drr, dashed, "\varphi"] & c_1 \arrow[dr, "f"] & \\
c_0 \arrow[ur, dashed, "g'"] \arrow[rr, "\varphi"'] & & c_2
\end{tikzcd}
\qquad
\begin{tikzcd}
c_0 \arrow[r, "g"] \arrow[drr, "\varphi"'] & c_1 \arrow[dr, "f"] & \\
 & & c_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
c_0 \arrow[d, equal] \arrow[dr, "\varphi"] & \\
c_0 \arrow[r, "\varphi"'] & c_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & c_1 \arrow[dr, "f"] & \\
c_0 \arrow[ur, dashed, "g'"] \arrow[rr, "\varphi"'] & & c_2
\end{tikzcd}
$$

Since f is Np-cartesian the following admits a solution:

$$
\begin{tikzcd}
\Delta^{\{1,2\}} \arrow[d] \arrow[dr, "f"] & \\
\Lambda^3_3 \arrow[r] \arrow[d] & NC \arrow[d, "Np"] \\
\Delta^3 \arrow[r] \arrow[ur, dotted] & ND
\end{tikzcd}
$$

the filled-in face $\Delta^{\{0,1,2\}} \hookrightarrow \Delta^3 \to NC$ is the one witnessing:

$$
\begin{tikzcd}
c_0 \arrow[r, "g"] \arrow[d, equal] & c_1 \\
c_0 \arrow[ur, dashed, "g'"] &
\end{tikzcd}
$$

ie. $[g'] = [g' \circ \mathrm{id}] = [g]$ in the homotopy category $hNC \simeq C$. In other words $g' = g$ in C. $\square$

Definition 2.7.6. [cartesian and cocartesian fibrations]

An inner fibration $p : X \to Y$ between simplicial sets is a:

- $\boxed{\text{cartesian fibration}}$ if any lifting problem

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution which is p-cartesian.

- $\boxed{\text{cocartesian fibration}}$ if any lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution which is p-cocartesian.

2.7.2 The straightening-unstraightening equivalence

The straightening-unstraightening equivalence is the ∞-categorical analogue of the "Grothendieck construction" in ordinary category theory. We will describe the Grothendieck construction and briefly describe the straightening-unstraightening equivalence.

Definition 2.7.7. A $\boxed{\text{Grothendieck fibration}}$ (or categorical fibration) is a functor $p : E \to B$ such that the fibers $E_b := p^{-1}(b)$ depend contravariantly functorially on b for all $b \in B$; ie. a map $b \to b'$ in B gives rise to a functor $E_{b'} \to E_b$ between fibers.

The dual notion where the fibers depend covariantly functorially on b (ie. a map $b \to b'$ forms a map $E_b \to E_{b'}$) is called a $\boxed{\text{Grothendieck opfibration}}$.

Proposition 2.7.8. The Grothendieck construction refers to an equivalence of 2-categories

$$\mathrm{Fun}(B^{op}, \mathbf{Cat}) \simeq \mathrm{Fib}(B)$$

where the left is a 2-category of functors $B^{op} \to \mathbf{Cat}$, and the right is a 2-category of Grothendieck fibrations over B.

(Dually, there is an equivalence $\mathrm{Fun}(B, \mathbf{Cat}) \simeq \mathrm{opFib}(B)$ between covariant functors $B \to \mathbf{Cat}$ and opfibrations over B.)

Remark 2.7.9. Let's unpack this.

The Grothendieck construction starts with a functor $F : B^{op} \to \mathbf{Cat}$ and determines uniquely a fibration $p : E \to B$. A functor $B^{op} \to \mathbf{Cat}$ is a collection of categories indexed by B:

$$\{F(b) \in \mathbf{Cat} \ : \ b \in B\}$$

with functors $F(b) \to F(b')$ for any map $b \leftarrow b'$ in B.

A fibration $p : E \to B$ is a category consisting of subcategories formed by fibers $E_b = p^{-1}(b)$ for each $b \in B$. Each map $b \to b'$ determines a functor between subcategories $E_b \leftarrow E_{b'}$.

Grothendieck's construction defines the fibers as $E_b := F(b)$ for each $b \in B$. One nice thing about this is that we can translate between:

$$\begin{pmatrix} \text{functors between} \\ B\text{-indexed categories} \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} \text{morphisms in a fibered category} \\ E \to B \end{pmatrix}$$

Let's describe how this looks: pick a map $f : b \to b'$. This forms a functor $Ff : F(b') \to F(b)$.

Since p is a cartesian fibration, for any $e' \in E_{b'}$, there is a corresponding element $Ff(e') \in E_b$ and a p-cartesian map $Ff(e') \to e'$ in E:

$$
\begin{tikzcd}
Ff(e') \to e' & E \arrow[d, "p"'] & \\
& B \arrow[r, "F"] & \mathbf{Cat}
\end{tikzcd}
$$

$$
b \xrightarrow{f} b' \qquad \begin{array}{c} F(b) \leftarrow F(b') \\ Ff(e') \leftarrow\!\shortmid e' \end{array}
$$

Let $e \in E_b \simeq F(b)$, and let $e \to e'$ be a morphism in E. Then $e \to e'$ lives over f, and since $Ff(e') \to e'$ is p-cartesian, $e \to e'$ factors uniquely through $Ff(e')$:

$$
\begin{tikzcd}
Ff(e') \arrow[r] & e' \\
e \arrow[u, dotted, "\exists!"] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
E \arrow[d, "p"'] & & & \\
B \arrow[r, "F"] & \mathbf{Cat} & F(b) & F(b') \arrow[l, "Ff"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
b \arrow[r, "f"] & b' \\
b \arrow[u, "\mathrm{id}"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff(e') & e' \arrow[l, mapsto] \\
e \arrow[u, dotted, "\exists!"] &
\end{tikzcd}
$$

Since $E_b \simeq F(b)$, a map $e \to Ff(e')$ in E_b corresponds to a map $e \to Ff(e')$ in $F(b)$. That is,

$$\mathrm{Hom}_E(e, e') \simeq \mathrm{Hom}_{E_b}(e, Ff(e')) \simeq \mathrm{Hom}_{F(b)}(e, Ff(e')).$$

That is, we can study the functor Ff by looking at maps in E.

Remark 2.7.10. The straightening-unstraightening equivalence is an equivalence of ∞-categories

$$\mathrm{Fun}(\mathcal{B}^{op}, \mathbf{Cat}_\infty) \simeq \mathrm{cart}(\mathcal{B})$$

between the ∞-category of functors $\mathcal{B}^{op} \to \mathbf{Cat}_\infty$ and an ∞-category of cartesian fibrations over $\mathcal{B}$. (Dually, an equivalence $\mathrm{Fun}(\mathcal{B}^{op}, \mathbf{Cat}_\infty) \simeq \mathrm{cocart}(\mathcal{B})$.)

Remark 2.7.11. This is the analagous statement to (2.7.9). The straightening-unstraightening equivalence allows us to translate between

$$\begin{pmatrix} \text{functors between} \\ B\text{-indexed } \infty\text{-categories} \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{edges in an } \infty\text{-category} \\ \text{fibered over } B \end{pmatrix}$$

This is useful already in ordinary category theory, and is particularly useful in ∞-category theory, where the large amounts of data involved make it often difficult to make ∞-categorical statements or to work with simplicial sets directly.

For more, see [Maz15], [BS16], and [HTT, §3].

2.8 Functors

Remark 2.8.1. The categories **Top** and **Cat** can be thought of as being "enriched over themselves".

- Given topological spaces X and Y, one can topologize the set of maps $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ by equipping the set with the "compact-open" topology. This forms a <u>space</u> often called Y^X. Points in this space are continuous maps $X \to Y$ and paths in this space are homotopies between maps.
- Given categories C and D, one builds a <u>category</u> $\mathrm{Fun}(C, D)$, whose objects are functors $C \to D$ and whose morphisms are natural transformations.

We can do a similar construction in the category of simplicial sets, in which we can form a simplicial set whose vertices are simplicial maps between two fixed simplicial sets.

Definition 2.8.2. [function complexes]

Let X and Y be simplicial sets.

The $\boxed{\text{function complex}}$ is a simplicial set denoted $\mathrm{Fun}(X,Y)_\bullet$ (sometimes written $\mathrm{Fun}(X,Y)$, $\underline{\mathrm{Hom}}(X,Y)$, or Y^X), which is defined as follows:

- (simplices):

$$\mathrm{Fun}(X,Y)_n := \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times X, Y).$$

- (face and degeneracy maps):

$$s_i : (\Delta^n \times X \xrightarrow{f} Y) \mapsto (\Delta^{n+1} \times X \xrightarrow{s_i \times \mathrm{id}} \Delta^n \times X \xrightarrow{f} Y)$$
$$d_i : (\Delta^n \times X \xrightarrow{f} Y) \mapsto (\Delta^{n-1} \times X \xrightarrow{d_i \times \mathrm{id}} \Delta^n \times X \xrightarrow{f} Y)$$

Proposition 2.8.3. [K, 1.4.3.7] Given a simplicial set X and an ∞-category $\mathcal{C}$, the function complex $\mathrm{Fun}(X,\mathcal{C})_\bullet$ is an ∞-category.

Proposition 2.8.4. [K, 1.4.3.3] There's an equivalence of ∞-categories:

$$N\mathbf{Cat}(C,D) \simeq \mathrm{Fun}(NC,ND).$$

On the left: $\mathbf{Cat}(C,D)$ is the category whose objects are functors between ordinary categories $C \to D$, and whose morphisms are natural transformations. Taking the nerve forms an ∞-category $N\mathbf{Cat}(C,D)$. On the right: since ND is an ∞-category, the function complex $\mathrm{Fun}(NC,ND)_\bullet$ is an ∞-category.

Sanity check 2.8.5. Let X be a simplicial set and Y be an ∞-category.

The homotopy category $h\mathrm{Fun}(X,Y)$ has:

- (objects): simplicial maps $f : X \simeq X \times \Delta^0 \to Y$.
- (morphisms): for two such simplicial maps $f, g : X \rightrightarrows Y$, morphisms $[H] \in \mathrm{Hom}_{h\mathrm{Fun}(X,Y)}(f,g)$ are equivalence classes of simplicial maps $H : X \times \Delta^1 \to Y$ of the form:

$$\begin{tikzcd} X \times \Delta^{\{0\}} \arrow[dr] \arrow[drr, bend left, "f"] & & \\ & X \times \Delta^1 \arrow[r, "H"] & Y \\ X \times \Delta^{\{1\}} \arrow[ur] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

with two such morphisms $[H], [H'] : X \times \Delta^1 \rightrightarrows Y$ being equal in the hom-set $\mathrm{Hom}_{h\mathrm{Fun}(X,Y)}(f,g)$ iff there's a map $\alpha : X \times \Delta^2 \to Y$ of the form:

$$\begin{tikzcd} & g \arrow[dr, equal] & \\ f \arrow[ur, "H"] \arrow[rr, "H'"'] & & g \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & f \arrow[dr, "H"] & \\ f \arrow[ur, equal] \arrow[rr, "H'"'] & & g \end{tikzcd}$$

Proposition 2.8.6. If X and Y are ∞-categories, and $Y = NC$ is the nerve of a category, then there is an equivalence of categories:

$$h\mathrm{Fun}(X,Y) \simeq \mathbf{Cat}(hX, hY),$$

where $h\mathrm{Fun}(X,Y)$ is the homotopy category of the function complex (2.8.2) between X and Y, and $\mathbf{Cat}(hX,hY)$ is the ordinary functor category between the ordinary categories hX and hY.

Proof. We can define a functor $h\mathrm{Fun}(X,Y) \to \mathbf{Cat}(hX,hY)$ as follows:

- (on objects): An object $f \in \mathrm{Fun}(X,Y)$ forms a functor $hf : hX \to hY$ of homotopy categories.

  Since Y is the nerve of some category, $Y \simeq NhY$, so by the adjunction $(h \dashv N)$ (2.4.10), we have a bijection

  $$\mathrm{Hom}_{\mathbf{sSet}}(X,Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X,NhY) \cong \mathrm{Hom}_{\mathbf{Cat}}(hX,hY).$$

  We want to relate the set on the left to the set $\mathrm{Hom}_{\mathbf{sSet}}(X,Y)$; ie. we want to relate NhY to Y. In general the unit gives us a map $NhY \leftarrow Y$, which is an equivalence iff Y is the nerve of an ordinary category.

- (on morphisms): Let f and g be objects in $\mathrm{Fun}(X,Y)$. By definition a morphism $[H] : f \to g$ in $h\mathrm{Fun}(X,Y)$ is represented by a map $H : X \times \Delta^1 \to Y$.

  Given an edge $\varphi : \Delta^1 \to X$ of the form $(x \xrightarrow{\varphi} x')$, we can form a square $\varphi \times \mathrm{id} : \Delta^1 \times \Delta^1 \to X \times \Delta^1$, which we can postcompose with H to get a square $\Delta^1 \times \Delta^1 \to Y$:

  $$\Delta^1 \times \Delta^1 \xrightarrow{\varphi \times \mathrm{id}} X \times \Delta^1 \xrightarrow{H} Y$$

  which we can draw as follows:

  $$\left(\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} x \arrow[r, "\mathrm{id}"] \arrow[d, "\varphi"'] \arrow[dr] & x \arrow[d, "\varphi"] \\ x' \arrow[r, "\mathrm{id}"'] & x' \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} f(x) \arrow[r, "H(\mathrm{id})"] \arrow[d, "f\varphi"'] \arrow[dr] & g(x) \arrow[d, "g\varphi"] \\ f(x') \arrow[r, "H(\mathrm{id})"'] & f(x') \end{tikzcd}\right)$$

The diagram $\Delta^1 \times \Delta^1 \to Y$ corresponds to a diagram in hY

$$h(\Delta^1 \times \Delta^1) \simeq [1] \times [1] \to hY$$

This is a commutative square in hY (square brackets $[-]$ denote equivalence classes):

$$\begin{tikzcd}
f(x) \arrow[r, "H_x = [H(\mathrm{id}_x)]"] \arrow[d, "hf(\varphi) = [f(\varphi)]"'] & g(x) \arrow[d, "hg(\varphi)"] \\
f(x') \arrow[r, "H_{x'}"'] & f(x')
\end{tikzcd}$$

These H_x maps, along with the square above for all $x, x' \in hX$, describes a natural transformation we can call $hH : hf \Rightarrow hg$.

So a morphism $[H] \in \mathrm{Hom}(h\mathrm{Fun}(X,Y))$ corresponds to a morphism $hH \in \mathrm{Hom}(\mathbf{Cat}(hX, hY))$.

Let's check that this is actually a functor:

- (on identities): This sends an identity $[\mathrm{id}_f] \in \mathrm{Hom}_{h\mathrm{Fun}(X,Y)}(f,f)$ to the natural transformation, consisting of component maps in hY:

$$h\,\mathrm{id}_f = \left\{(\mathrm{id}_f)_x = [\mathrm{id}_f(\mathrm{id}_x)] = \mathrm{id}_{f(x)}\right\}_{x \in X},$$

  which is precisely the identity $\mathrm{id}_{hf} \in \mathrm{Hom}_{\mathbf{Cat}(hX,hY)}(hf, hf)$.

- (on composition): When Y is an ∞-category, we can write a composable pair in $h\mathrm{Fun}(X,Y)$ as $f_0 \xrightarrow{[H_1]} f_1 \xrightarrow{[H_2]} f_2$.

  This forms a horn $f_0 \xrightarrow{H_1} f_1 \xrightarrow{H_2} f_2$ in $\mathrm{Fun}(X,Y)$, which fills in with a morphism we call H_2H_1. The corresponding natural transformation $h(H_2H_1)$:

$hf_0 \to hf_2$ is defined

$$\left\{ f_0(x) \xrightarrow{H_1|_x} f_1(x) \xrightarrow{H_2|_x} f_2(x) \right\}$$

which by definition is just $hH_2 \circ hH_1$.

In the other direction, can we form a functor $\mathbf{Cat}(hX, hY) \to h\mathrm{Fun}(X, Y)$? Given a functor $f : hX \to hY$ can we lift this to (an equivalence class of) a map $X \to Y$? If $Y = NC$ is the nerve of a category, then consider a functor $f : hX \to hY$ as an object in the functor category:

$$\begin{aligned} f \in \mathrm{ob}(\mathbf{Cat}(hX, hY)) &= \mathrm{Hom}_{\mathbf{Cat}}(hX, hNC) \\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(X, NhNC) \\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(X, Y). \end{aligned}$$

That is $f \in \mathbf{Cat}(hX, hY)$ corresponds bijectively to a simplicial map $\overline{f} : X \to Y$, forming an object $\overline{f} \in h\mathrm{Fun}(X, Y)$.

Pick a morphism $(\alpha : f \to g)$ in $\mathbf{Cat}(hX, hY)$. This is a natural transformation – a collection of morphisms $(\alpha_x : f(x) \to g(x)) \in hY$ for all $x \in X$, such that for each $(\varphi : x \to x') \in hX$, the following square in hY commutes:

$$\begin{tikzcd} f(x) \arrow[r, "\alpha_x"] \arrow[d, "f(\varphi)"'] & g(x) \arrow[d, "g(\varphi)"] \\ f(x') \arrow[r, "\alpha_{x'}"'] & g(x') \end{tikzcd}$$

We want a morphism in $h\mathrm{Fun}(X, Y)$ between $\overline{f} \to \overline{g}$, ie. (an equivalence class of) a simplicial map $\Delta^1 \times X \to Y$. By cartesian-closedness of $\mathbf{sSet}$, a simplicial map $\Delta^1 \times X \to Y$ corresponds to a simplicial map $X \to Y^{\Delta^1}$. On objects $x \mapsto \alpha_x$.

On morphisms $(\varphi : x \to x')$ maps to a square $\Delta^1 \times \Delta^1 \to Y$ of the form above (witnessing naturality of α). In general, an n-simplex

$$(x_0 \to x_1 \to \cdots \to x_n) \in X_n$$

is mapped to an n-simplex in

$$(Y^{\Delta^1})_n = \mathbf{sSet}(\Delta^1 \times \Delta^n, Y)$$

of the form:

$$\begin{tikzcd}
f(x_0) \arrow[r] \arrow[d, "\alpha_{x_0}"'] & f(x_1) \arrow[r] \arrow[d, "\alpha_{x_1}"] & \cdots \arrow[r] & f(x_n) \arrow[d, "\alpha_{x_n}"] \\
g(x_0) \arrow[r] & g(x_1) \arrow[r] & \cdots \arrow[r] & g(x_n)
\end{tikzcd}$$

This defines a map $\alpha \in \mathrm{Fun}(X, Y)_1$, which forms a morphism $[\alpha] \in h\mathrm{Fun}(X, Y)$. Two morphisms $\alpha, \alpha' \in \mathrm{Fun}(X, Y)_1$ determine the same equivalence class in the homotopy category iff there is a 2-simplex in $\mathrm{Fun}(X, Y)_2 = \mathbf{sSet}(X \times \Delta^2, Y)$ of the form:

$$\begin{tikzcd}
& \overline{g} \arrow[dr, equal] & \\
\overline{f} \arrow[ur, "\alpha"] \arrow[rr, "\alpha'"'] & & \overline{g}
\end{tikzcd}$$

Such a 2-simplex corresponds to a map $X \to Y^{\Delta^2}$, which forms for each object $x \in X_0$, a 2-simplex in Y:

$$\begin{tikzcd}
& g(x) \arrow[dr, equal] & \\
f(x) \arrow[ur, "\alpha_x"] \arrow[rr, "(\alpha')_x"'] & & g(x)
\end{tikzcd}$$

which corresponds precisely to the equality of the morphisms $\alpha = \alpha'$ in $\mathbf{Cat}(hX, hY)$.

So given a morphism $(\alpha : f \to g) \in \mathbf{Cat}(hX, hY)$ we can find a unique equivalence class $[\alpha] : \overline{f} \to \overline{g}$ in $h\mathrm{Fun}(X, Y)$. $\square$

Remark 2.8.7. In general, if Y is not the nerve of some category, then it is <u>not true</u> that $\mathbf{Cat}(hX, hY) \simeq h\mathrm{Fun}(X, Y)$. In this case, we do not have an equivalence $NhY \not\simeq Y$, so functors $hX \to hY$ do not correspond bijectively to simplicial maps $X \to Y$.

2.8.1 Equivalences of ∞-categories

Remark 2.8.8. When are two categories "the same"? One can describe an "isomorphism of categories" as a functor f for which there exists an inverse f^{-1} such that the composites ff^{-1} and $f^{-1}f$ are identity functors. This turns out to be too strict. Instead, the more useful notion is a weaker one that we call an <u>equivalence of categories</u>. Instead of the composites ff^{-1} and $f^{-1}f$ being strictly equal to identity functors, we require a <u>natural isomorphism</u> between them. In practice one often exhibits an equivalence "pointwise": a functor F is an equivalence of categories iff it is (i) essentially surjective and (ii) fully faithful.

Similarly, the right notion of an <u>equivalence of ∞-categories</u> is weaker than an isomorphism of simplicial sets. We will define what an equivalence of ∞-categories means and describe a similar pointwise criterion for checking equivalences of ∞-categories.

Definition 2.8.9. [Rie17, Def. 1.5.4]

[equivalence of ordinary categories]

Let C and D be ordinary categories.

A functor $f : C \to D$ is an $\boxed{\text{equivalence of categories}}$ if there exists a functor

$g : D \to C$, along with natural isomorphisms:

$$\mathrm{id}_C \overset{\sim}{\Rightarrow} gf, \text{ and}$$
$$fg \overset{\sim}{\Rightarrow} \mathrm{id}_D.$$

Remark 2.8.10. This is a form of "homotopy equivalence". Instead of requiring the composites be <u>strictly equal</u> to identities ($gf = \mathrm{id}_C$ and $fg = \mathrm{id}_D$), we require them to be <u>homotopic</u> via natural isomorphisms.

Proposition 2.8.11. [Rie17, Thm. 1.5.9]

A functor $f : C \to D$ is an equivalence of categories iff it is both:

- <u>essentially surjective</u> : For any object $d \in D$ there exists an object $c \in C$ and an isomorphism $f(c) \cong d$.
- <u>fully faithful</u>: For any pair of objects $c, c' \in C$, there is a bijection of hom-sets:

$$\mathrm{Hom}_C(c, c') \cong \mathrm{Hom}_D(f(c), f(c')).$$

Definition 2.8.12. [K, 4.5.1.10]

[equivalence of ∞-categories]

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories.

A functor $f : \mathcal{C} \to \mathcal{D}$ is an $\boxed{\text{equivalence of } \infty\text{-categories}}$ if there exist equivalences in the functor ∞-categories:

$$(\mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} gf) \in \mathrm{Fun}(\mathcal{C}, \mathcal{C})_1$$
$$(fg \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}} \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})_1.$$

Note that this is equivalent to saying that $hF : h\mathcal{C} \to \mathcal{D}$ is an equivalence of homotopy categories in the ordinary sense (2.8.9).

Definition 2.8.13. [(∞-) fully faithful and essentially surjective functors]

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $f : \mathcal{C} \to \mathcal{D}$ be a functor. This induces a map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(f(x), f(y))$$

between mapping spaces. We say that f is:

- $\boxed{\text{fully faithful}}$ if the map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(f(x), f(y))$ is an equivalence in the sense of (2.8.12);
- $\boxed{\text{essentially surjective}}$ if for any $d \in \mathcal{D}_0$ there exists a $c \in \mathcal{C}_0$ and an equivalence $f(c) \simeq d$ in $\mathcal{D}_1$. In other words, if $hf : h\mathcal{C} \to h\mathcal{D}$ is essentially surjective in the ordinary sense.

Proposition 2.8.14. [Rez22, Prop. 44.3, Prop. 44.7]

[pointwise criterion for equivalences of ∞-categories]

A map $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories is an equivalence of ∞-categories (2.8.12) iff it is fully faithful and essentially surjective (2.8.13).

Proposition 2.8.15. [K, 4.5.1.12]

A functor $f : C \to D$ between ordinary categories is an equivalence of categories (2.8.9) iff the induced functor $Nf : NC \to ND$ is an equivalence of ∞-categories (2.8.12).

Proposition 2.8.16. [Rez22, Prop. 22.3]

If $f : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then the induced functor $hf : h\mathcal{C} \to h\mathcal{D}$ is an equivalence of ordinary categories.

Remark 2.8.17. It is <u>not</u> true that an equivalence of ∞-categories can be detected on the homotopy level.

Remark 2.8.18. Recall that a <u>weak homotopy equivalence</u> of simplicial sets (the weak equivalences in $\mathbf{sSet}_{Joyal}$ (4.0.10)) is a map $f : X \to Y$ such that for any ∞-category $\mathcal{C}$, the induced functor $h\mathrm{Fun}(Y, \mathcal{C}) \to h\mathrm{Fun}(X, \mathcal{C})$ is an equivalence of ordinary categories.

There is a finer notion of "categorical equivalence" that lets one work with simplicial sets that may not be ∞-categories.

Definition 2.8.19. [Rez22, Def. 22.5]

[categorical equivalence]

Let X and Y be simplicial sets, and let $f : X \to Y$ be a map.

The map f is a $\boxed{\text{categorical equivalence}}$ if for any ∞-category $\mathcal{C}$, the functor

$$\mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$$

is an equivalence of ∞-categories in the sense of (2.8.12).

Proposition 2.8.20. [Rez22, Prop. 25.13]

[equivalent conditions for a categorical equivalence]

Let $f : X \to Y$ be a map of simplicial sets. The following are equivalent:

(1) The map f is a categorical equivalence in the sense of (2.8.19).

(2) The map f is a weak homotopy equivalence in the sense of (2.8.18).

(3) For any quasi-category $\mathcal{C}$, the map

$$\pi_0(\mathrm{Fun}(Y, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq})$$

is a bijection of sets. (Here $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ is the connected components functor (2.1.3) and $(-)^{\simeq}$ denotes taking the core (2.5.4).)

(4) The functor $\mathfrak{C}f : \mathfrak{C}X \to \mathfrak{C}Y$ (2.2.12) is a Dwyer-Kan equivalence (4.0.10). [HTT 2.2.5.8]

Remark 2.8.21. An equivalence between two ∞-categories (2.8.12) is a categorical equivalence in the sense of (2.8.19).

This weaker notion of categorical equivalence lets us work with simplicial sets that are not ∞-categories, eg. if we want to work in $\mathbf{sSet}_{Joyal}$, which presents the homotopy theory of ∞-categories (4.0.10, 4.1.5).

2.9 Limits and Colimits

In ordinary category theory, there are many ways to formulate definitions of limits and colimits. We will first lay out the one that will look like our eventual definition of ∞-limits.

Definition 2.9.1. [ordinary slice category over an object]

Let C be an ordinary category and $x \in C$ an object.

The $\boxed{\text{slice category of } C \text{ over } x}$ is a category $C_{/x}$ defined as follows:

- (objects): An object in $C_{/x}$ is a morphism $f : c \to x$ whose target is x. We can denote such an object as a pair (c, f).

- (morphisms): A map $(c, f) \to (c', f')$ is given by a map $u : c \to c'$ in C such that $f'u = f$; ie. the following commutes in C:

$$\begin{tikzcd} c \arrow[rr, "u"] \arrow[dr, "f"'] & & c' \arrow[dl, "f'"] \\ & x & \end{tikzcd}$$

Remark 2.9.2. We can "forget" slices to form a functor forget : $C_{/x} \to C$

$$\begin{tikzcd} (c,f) \arrow[d, "u"] & \mapsto & c \arrow[d, "u"] \\ (c',f') & & c' \end{tikzcd}$$

Example 2.9.3. Let $p : I \to C$ be a diagram in an ordinary category C. We can consider p as an object in the functor category $\mathrm{Fun}(I,C)$.

The slice category $\mathrm{Fun}(I,C)_{/p}$ has as objects pairs (F,α), where $F : I \to C$ is a diagram of shape I, and $\alpha : F \Rightarrow p$ is a natural transformation. Morphisms $(F,\alpha) \to (F',\alpha')$ are natural transformations $\varphi : F \Rightarrow F'$ such that the following commutes:

$$\begin{tikzcd} F \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & F' \arrow[dl, "\alpha'"] \\ & p & \end{tikzcd}$$

Definition 2.9.4. [ordinary cone over a diagram]

A $\boxed{\text{cone over the diagram } p}$ is an object in the slice category $\mathrm{Fun}(I,C)_{/p}$ of the form (c_x,α), where $c_x : I \to C$ is the constant diagram at an object $x \in C$, ie.

$$c_x : (i \to j) \mapsto (x \xrightarrow{\mathrm{id}} x)$$

for all $(i \to j) \in I$. Since all the information is contained within α, we may sometimes refer to a cone $\alpha : c_x \to p$ as the cone with apex x.

Definition 2.9.5. [ordinary slice category over a diagram]

Let $p : I \to C$ be a diagram in an ordinary category.

The $\boxed{\text{slice category over } p}$ (or the "over-category") is a category $C_{/p}$ defined by a pullback:

$$\begin{tikzcd} C_{/p} \arrow[r, "\varphi"] \arrow[d] & \mathrm{Fun}(I,C)_{/p} \arrow[d, "\text{forget}"] \\ C \arrow[r, "\text{constant}"'] & \mathrm{Fun}(I,C) \end{tikzcd}$$

In other words, objects of $C_{/p}$ are pairs (x,α) where $x \in \mathrm{ob}(C)$ and $\alpha : c_x \Rightarrow p$ is a cone.

Remark 2.9.6. We can recover the slice over an object $C_{/x}$ by considering x as a constant diagram $x : [0] \to C$.

Remark 2.9.7. There are dual notions of slice categories under a diagram $C_{p/}$ or under an object $C_{x/}$.

Definition 2.9.8. An object $x \in C$ in an ordinary category is called $\boxed{\text{final}}$ if for any other object $c \in C$, there is exactly one map $c \to x$; ie. $\mathrm{Hom}_C(x,c) = \{*\}$.

Dually, x is $\boxed{\text{initial}}$ if $\mathrm{Hom}_C(x,c) = \{*\}$ for any $c \in C$.

Definition 2.9.9. [limits and colimits in ordinary categories]

Given a diagram $p : I \to C$ in an ordinary category, its $\boxed{\text{limit}}$ (if it exists) is a final object $(\lim_I p, \alpha) \in C_{/p}$.

Dually, a $\boxed{\text{colimit}}$ $\mathrm{colim}_I\, p$ is an initial object of the under-category

$$(\mathrm{colim}_I\, p, \alpha) \in C_{p/}.$$

There's a related construction called the join of categories:

Definition 2.9.10. [join of categories]

Given ordinary categories C and D, their $\boxed{\text{join}}$ is the category $C \star D$ with

- (objects): The objects of $C \star D$ are the objects of C along with the objects of D.

- (morphisms): For any $x, y \in C \star D$,

$$\mathrm{Hom}_{C\star D}(x,y) := \begin{cases} \mathrm{Hom}_C(x,y) & x,y \in C \\ \mathrm{Hom}_D(x,y) & x,y \in D \\ \{*\} & x \in C, y \in D \\ \varnothing & x \in D, y \in C. \end{cases}$$

Remark 2.9.11. The join $C\star D$ contains both C and D as full subcategories, and includes a single morphism $c \to d$ for each $c \in C, d \in D$.

We call the inclusions $i_C : C \hookrightarrow C \star D$ and $i_D : D \hookrightarrow C \star D$. We can consider these inclusions as objects $(C\star D, i_C) \in \mathbf{Cat}_{C/}$ and $(C\star D, i_D) \in \mathbf{Cat}_{D/}$ respectively.

Example 2.9.12. Taking either C or D to be the one object category $[0]$, we get cone categories $C^{\triangleright} := C \star [0]$ and $D^{\triangleleft} := [0] \star D$.

A functor $F : D^{\triangleright} \to E$ is a cone $(c_x, \alpha) \in \mathrm{Fun}(D,E)_{/F|_D}$ over $F|_D = F \circ i_D$.

Proposition 2.9.13. [K, 4.3.2.17]

Let C and D be ordinary categories.

There is an adjunction:

$$-\star D : \mathbf{Cat} \rightleftarrows \mathbf{Cat}_{D/} : \mathrm{slice}$$
$$C \mapsto (C \star D, i_D)$$
$$E_{/G} \leftarrow\!\shortmid (E, D \xrightarrow{G} E).$$

(Dually, we have an adjunction $(C \star -, \mathrm{slice}) : \mathbf{Cat} \rightleftarrows \mathbf{Cat}_{C/}$, where the slice this time is defined $(E, C \xrightarrow{G} E) \mapsto E_{G/}$.)

Spelling this out, this means that for any functor $G : D \to E$, there is a bijection

$$\{\text{functors } C \star D \to E \text{ extending G}\} \leftrightarrow \{\text{functors } C \to E_{/G}\},$$

where a functor extending G is a functor $\overline{G}$ such that the following commutes:

$$\begin{tikzcd} D \arrow[r, "G"] \arrow[d, hook] & E \\ C \star D \arrow[ur, "\overline{G}"'] & \end{tikzcd}$$

Remark 2.9.14. Let C and I be ordinary categories and $C^I = \mathbf{Cat}(I, C)$ be the functor category.

Let $c : C \to C^I$ be the formation of constant functors $x \mapsto c_x$ (2.9.4). If C has I-shaped limits/colimits, then the formation of limits/colimits form left/right adjoints to c:

$$\begin{tikzcd} C \arrow[d, "c"] \\ C^I \arrow[u, bend left, dotted, "\lim_I"] \arrow[u, bend right, dotted, "\mathrm{colim}_I"'] \end{tikzcd}$$

2.9.1 (∞-) Limits and Colimits

Definition 2.9.15. [join of simplicial sets]

Let A and B be simplicial sets.

The $\boxed{\text{join}}$ $A \star B$ is the simplicial set with

- (simplices):

$$(A \star B)_n := A_n \cup B_n \cup \Big(\bigcup_{i+j+1=n} A_i \times B_j\Big),$$

- (face and degeneracy maps): for an n-simplex σ, which either looks like an ordinary n-simplex of A_n or B_n, or a pair $\sigma = (\sigma_A, \sigma_B) \in A_i \times B_j$ for some

i, j.

$$
\begin{aligned}
s_k : (A \star B)_n &\to (A \star B)_{n+1} \\
\sigma &\mapsto \begin{cases} s_k^A(\sigma) & \sigma \in A_n \\ s_k^B(\sigma) & \sigma \in B_n \\ (s_k^A(\sigma_A), \sigma_B) & \sigma \notin (A_n \cup B_n), k \leq i \\ (\sigma_A, s_k^B(\sigma_B)) & \sigma \notin (A_n \cup B_n), k > i \end{cases} \\
d_k : (A \star B)_n &\to (A \star B)_{n-1} \\
\sigma &\mapsto \begin{cases} d_k^A(\sigma) & \sigma \in A_n \\ d_k^B(\sigma) & \sigma \in B_n \\ (d_k^A(\sigma_A), \sigma_B) & \sigma \notin (A_n \cup B_n), k \leq i \\ (\sigma_A, d_k^B(\sigma_B)) & \sigma \notin (A_n \cup B_n), k > i, \end{cases}
\end{aligned}
$$

where s_k^A and d_k^A are the face and degeneracy maps of A (and similarly for B).

Sanity check 2.9.16. Intuitively, the join $A \star B$ includes both A and B as subcomplexes, along with a simplex of dimension n joining each pair $(\alpha, \beta) \in A_i \times B_j$. In particular, vertices, edges and faces of $A \star B$ are given by:

- Points of $A \star B$ are the points of both A and B.
- Edges are edges of both A and B, together with a single edge joining each pair of points $(a \in A_0, b \in B_0)$.
- Faces are faces of A and B, along with a single face for each pair $(a \in A_0, f \in B_1)$ and each pair $(f \in A_1, b \in B_0)$.

For example, for each point $a \in A_0$ and edge $(b \xrightarrow{f} b) \in B_1$, we form a face in $A \star B$ of the form:

$$\begin{tikzcd} & b \arrow[dd, "f"] \\ a \arrow[ur] \arrow[dr] & \\ & b' \end{tikzcd}$$

We can think of these faces as joining simplices of different dimensions (namely, a point in A and an edge in B).

In general, for a pair $(\alpha, \beta) \in A_i \times B_j$, if α is spanned by the points $(a_0, \ldots, a_i)$ and β is spanned by the points $(b_0, \ldots, b_j)$, the join $A \star B$ includes an $(i+j+1)$-simplex[4] $\varphi \in (A \star B)_{i+j+1}$ that is spanned by all the points in order

$$(a_0, \ldots, a_i, b_0, \ldots, b_j).$$

Suppose we want to describe a k-subsimplex $\psi = \Delta^{\{i_0, \ldots, i_k\}} =\subseteq \varphi$. Subsimplices come in two flavors:

- All the $i_0, \ldots, i_k$ live in either A or B, in which case ψ is a subsimplex of A or B.
- The collection splits as:

$$\underbrace{\{i_0, \ldots, i_m\}}_{\subseteq A} \cup \underbrace{\{i_{m+1}, \ldots, i_k\}}_{\subseteq B}$$

[4]The +1 appearing here is same +1 that appears in, eg. $\Delta^n \star \Delta^m \simeq \Delta^{n+m+1}$ (2.9.18).

and then the subsimplex

$$\alpha := \Delta^{\{i_0,\dots,i_m\}} \simeq \Delta^m \to A$$

is an m-simplex of A, and the subsimplex

$$\beta : \Delta^{\{i_{m+1},\dots,i_k\}} \simeq \Delta^{k-m-1} \to B$$

is a $(k-m-1)$-simplex of B. The pair $(\alpha, \beta) \in A_m \times B_{k-m-1}$ corresponds to a k-simplex in the join, which is ψ in this case.

This explains why the face and degeneracy maps in the definition appear as they do.

Pick a $\sigma \in (A \star B)_n$. The k^{th} face map picks out an $(n-1)$-simplex. If σ lives entirely in A_n or lives entirely in B_n, this is simply taking the face within A or B as usual. The geometric intuition is that the k^{th} face is formed by deleting the k^{th} vertex, and taking the face living across from it. If the simplex σ lives partly in A_i and B_j (for some i, j such that $i + j = n - 1$), then whether $k \leq i$ or $k > i$ determines whether the vertex lives in the A-side or the B-side of $A \star B$. Then faces are the faces formed by the join, formed by deleting the k^{th} vertex in whichever side it lives in.

For example, here are two 3-simplices in $A \star B$: one formed by joining $\Delta^2_A \star \Delta^0_B$ (a 2-simplex in A and a point in B, and the other formed by joining $\Delta^1_A \star \Delta^1_B$ (an edge each in A and B):

$$\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[drr] & a'' \arrow[dr] & \\
a' \arrow[ur, dashed] \arrow[rr] & & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \\
a' \arrow[ur, dashed] \arrow[r] & b'
\end{tikzcd}$$

The face $s_3(\sigma)$ picks out the face $(a \to a' \to a'') \in A$, and the face $s_0(\sigma)$ picks out the face $a' \to a'' \to b$ joining $(a' \to a'')$ to b.

Proposition 2.9.17. [K, 4.3.3.22]

Let C and D be ordinary categories.

There is an isomorphism of simplicial sets:

$$N(C \star D) \simeq NC \star ND$$

where $C \star D$ is the join as ordinary categories (2.9.10) and $NC \star ND$ is the join of simplicial sets (2.9.15).

Proof. The nerve of the join $N(C \star D)$ has n-simplices

$$N(C \star D)_n = \mathrm{Fun}([n], C \star D).$$

These are strings $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n)$ in $C \star D$. We can find such strings in a few different ways. Either

- all f_i are in C, in which case this is an element of NC_n;
- all f_i are in D, in which case this is an element of ND_n;
- the string breaks apart as

$$\underbrace{(x_0 \to \cdots \to x_i)}_{\in NC_i} \xrightarrow{f_{i+1}} \underbrace{(x_{i+1} \to \cdots \to x_n)}_{\in ND_{n-i-1}},$$

  in which case the map f_{i+1} is the unique map of the join $C \star D$.

Now let's take nerves first and then join them as simplicial sets: $NC \star ND$. Simplices look like:

$$(NC \star ND)_n := (NC_n) \cup (ND_n) \cup (\bigcup_{i+j=n-1} NC_i \times ND_j).$$

We can see that these correspond to breaking apart n-simplices of $N(C \star D)$. We can think of an n-simplex $\sigma \in (NC \star ND)_n$ as either an n-simplex of NC or ND, or a pair

$$(\underbrace{(x_0 \to \cdots \to x_i)}_{\in NC_i}, \underbrace{(x_{i+1} \to \cdots \to x_n)}_{\in ND_{n-i-1}})$$

with $i+j = n-1$ and an edge $f_{i+1} : x_i \to x_{i+1}$ joining them. This gives us levelwise bijections $(NC \star ND)_n \cong N(C \star D)_n$.

We can describe face and degeneracy maps for an n-simplex $\sigma = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n)$ in either $N(C \star D)_n$ or $(NC \star ND)_n$ (here we write $(x_0 \xrightarrow{\cdots} x_n) := (x_0 \to x_1 \to \cdots \to x_n)$ to denote a monotonically increasing sequence).

$$\begin{aligned}
s_k : N(C \star D)_n &\to N(C \star D)_{n+1} \\
\sigma &\mapsto (x_0 \xrightarrow{\cdots} x_k = x_k \xrightarrow{\cdots} x_n) \\
d_k : N(C \star D)_n &\to N(C \star D)_{n-1} \\
\sigma &\mapsto (x_0 \xrightarrow{\cdots} x_{k-1} \xrightarrow{f_{k+1} f_k} x_{k+1} \xrightarrow{\cdots} x_n)
\end{aligned}$$

and

$$
\begin{aligned}
s_k :&(NC\star ND)_n \to (NC\star ND)_{n+1} \\
\sigma &\mapsto \begin{cases} (x_0 \xrightarrow{\cdots} x_k = x_k \xrightarrow{\cdots} x_n) & \sigma \in NC_n \cup ND_n \\ (x_0 \xrightarrow{\cdots} x_k = x_k \xrightarrow{\cdots} x_i) \to (x_{i+1} \xrightarrow{\cdots} x_n) & \sigma \in NC_i \times ND_j, k \leq i \\ (x_0 \xrightarrow{\cdots} x_i) \to (x_{i+1} \xrightarrow{\cdots} x_k = x_k \xrightarrow{\cdots} x_n) & \sigma \in NC_i \times ND_j, k > i \end{cases} \\
d_k :&(NC\star ND)_n \to (NC\star ND)_{n-1} \\
\sigma &\mapsto \begin{cases} (x_0 \xrightarrow{\cdots} x_{k-1} \xrightarrow{f_{k+1} f_k} x_{k+1} \xrightarrow{\cdots} x_n) & \sigma \in NC_n \cup ND_n \\ (x_0 \xrightarrow{\cdots} x_{k-1} \xrightarrow{f_{k+1} f_k} x_{k+1} \xrightarrow{\cdots} x_i) \to (x_{i+1} \xrightarrow{\cdots} x_n) & \sigma \in NC_i \times ND_j, k \leq i \\ (x_0 \xrightarrow{\cdots} x_i) \to (x_{i+1} \xrightarrow{\cdots} x_{k-1} \xrightarrow{f_{k+1} f_k} x_{k+1} \xrightarrow{\cdots} x_n) & \sigma \in NC_i \times ND_j, k > i. \end{cases}
\end{aligned}
$$

Under our bijections, we can see that these are the same, making this an isomorphism $N(C\star D) \cong (NC\star ND)$ of simplicial sets. □

Example 2.9.18. [K, 4.3.3.22] Taking $C := [n], D := [m]$ for some $n, m \in \mathbb{N}$, then

$$
\begin{aligned}
\Delta^n \star \Delta^m &\simeq N[n]\star N[m] \\
&\simeq N([n]\star [m]) \\
&\simeq N[n+m+1] \\
&\simeq \Delta^{n+m+1}.
\end{aligned}
$$

Proposition 2.9.19. Let A and B be ∞-categories.

There is an equivalence of categories:

$$h(A\star B) \simeq hA\star hB$$

where the left is the join of ssets (2.9.15) and the right is the join of homotopy categories (2.9.10).

Proof. Both categories consist of the points of A together with the points of B.

Each edge $f \in (A \star B)_1$ determines a morphism in $[f] \in h(A \star B)$. These come in two flavors:

- The edges living on either side of the join. That is, factoring through A (resp. through B):

$$\begin{tikzcd} & A \arrow[d, hook] \\ \Delta^1 \arrow[ur] \arrow[r, "f"'] & A \star B \end{tikzcd}$$

  Such an edge f corresponds to a homotopy class $[f] : [1] = h\Delta^1 \to h(A \star B)$. Alternately, f corresponds uniquely to an edge $\Delta^1 \to A$ which corresponds to a homotopy class in $hA \subseteq hA \star hB$. Similarly, those factoring through B correspond to homotopy classes in $hB \subseteq hA \star hB$.

We also have for each $a \in A, b \in B$ a single unique edge $a \to b$. These correspond to the unique morphisms in the ordinary join construction. Two such morphisms $(f, f' : a \rightrightarrows b)$ correspond iff there exist 2-simplices in $A \star B$:

$$\begin{tikzcd} a \arrow[dd, equal] \arrow[dr, "f"] & \\ & b \\ a \arrow[ur, "f'"'] & \end{tikzcd}$$

and any face of the forms

$$\begin{tikzcd} a \arrow[dd] \arrow[dr] & \\ & b \\ a' \arrow[ur] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & b \arrow[dd] \\ a \arrow[ur] \arrow[dr] & \\ & b' \end{tikzcd}$$

form commuting triangles in $h(A \star B)$. Such triangles correspond to commuting triangles in the ordinary join $hA \star hB$. □

Remark 2.9.20. In particular, take $A = I^{\triangleleft} = \Delta^0 \star I$, a cone over some diagram sset I, and $B = NC$, the nerve of an ordinary category C. The homotopy category $hA = h(\Delta^0 \star I) = [0] \star hI$. Then the homotopy-nerve adjunction (2.4.10) says there is a bijection

$$\mathbf{Cat}([0] \star hI, C) \cong \mathbf{sSet}(\Delta^0 \star I, NC).$$

That is, ordinary cones $(hI)^{\triangleleft} \to C$ (2.9.12) are the same as a cone of ssets $I^{\triangleleft} \to NC$ (2.9.22) in the nerve.

Proposition 2.9.21. [K, 4.3.3.24] If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, their join $\mathcal{C} \star \mathcal{D}$ is an ∞-category.

Example 2.9.22. In particular, taking $B := \Delta^0$ forms a simplicial set we denote $A^{\triangleright} := A \star \Delta^0$, which looks like

$$(A^{\triangleright})_n = \begin{cases} A_0 \cup \Delta^0 & n = 0 \\ A_n \cup (A_{n-1} \times \Delta^0) & \text{otherwise.} \end{cases}$$

That is, it has one extra vertex given by the Δ^0 (we will denote this point ∞), and an extra n-simplex for each $(n-1)$-simplex of A (these witness joining each $\sigma \in A_{n-1}$ to ∞). All together, we get something like a (simplicially thickened) cone under A (compare with 2.9.12). Similarly, taking $A := \Delta^0$ gives us $B^{\lhd}$, forming a cone over B.

For example, taking $A := (a \to a') \simeq \Delta^1$, the cone $A^{\rhd}$ looks like:

$$\begin{tikzcd} a \arrow[d] \arrow[dr] & \\ a' \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \infty \end{tikzcd}$$

where σ is a 2-simplex filling in the edges in $(A^{\rhd})_1$. That is, all together, $(\Delta^1)^{\rhd} \simeq \Delta^2$.

Definition 2.9.23. [slice simplicial sets]

Given a map of simplicial sets $p : I \to S$, the $\boxed{\text{slice simplicial set}}$ $S_{/p}$ is defined by:

- (simplices):

  $$(S_{/p})_n := \mathrm{Hom}_{\mathbf{sSet}_{I/}}((\Delta^n \star I, i_I), (S, p)),$$

  where i_I is the inclusion $I \hookrightarrow \Delta^n \star I$, so n-simplices are natural transformations $(i_I \to p)$.

  In particular, vertices of $S_{/p}$ are cones $\Delta^0 \star I \to S$ extending p.

- (face and degeneracy maps): for an n-simplex $(f : \Delta^n \star I \to S)$, we can pullback (along the join) via the maps $s_i : \Delta^{n+1} \to \Delta^n$ and $d_i : \Delta^{n-1} \to \Delta^n$

to form composites:

$$s_i : f \mapsto (\Delta^{n+1} \star I \xrightarrow{s_i \star \mathrm{id}} \Delta^n \star I \xrightarrow{f} S)$$
$$d_i : f \mapsto (\Delta^{n-1} \star I \xrightarrow{d_i \star \mathrm{id}} \Delta^n \star I \xrightarrow{f} S).$$

Notation 2.9.24. For simplicial sets I, Y, S, and a diagram $p : I \to S$, we will write:

$$\mathrm{Hom}_p(Y \star I, S) := \mathrm{Hom}_{\mathbf{sSet}_{I/}}((Y \star I, i_I), (S, p)),$$

to denote the set of simplicial maps $Y \star I \to S$ factoring p:

$$\begin{tikzcd} I \arrow[d, hook] \arrow[dr, "p"] & \\ Y \star I \arrow[r] & S \end{tikzcd}$$

Remark 2.9.25. The simplicial set $S_{/p}$ satisfies an analagous universal property to that of the slice construction of ordinary categories (2.9.13):

$$\mathrm{Hom}_{\mathbf{sSet}}(Y, S_{/p}) \cong \mathrm{Hom}_p(Y \star I, S).$$

Proposition 2.9.26. [K, 4.3.6.1] If S is an ∞-category, then $S_{/p}$ is an ∞-category that we call the over-category of S over p (similarly $S_{p/}$ the under-category).

Proposition 2.9.27. Let C be an ordinary category, and $x \in C$ an object. Then:

$$N(C_{/x}) \simeq (NC)_{/x}$$

where the left is the nerve of the ordinary over-category (2.9.1), and the right is the over-category of the nerve (2.9.26).

Proof. The nerve of the over-category $N(C_{/x})$ is an ∞-category, with n-simplices given by $N(C_{/x})_n := \mathrm{Fun}([n], C_{/x})$. Objects in $C_{/x}$ are maps $(c \to x)$ in C and morphisms $(c \to x) \to (c' \to x)$ are given by commuting diagrams of the form:

$$\begin{tikzcd} c \arrow[r] \arrow[d] & c' \arrow[d] \\ x \arrow[r, equal] & x \end{tikzcd}$$

An n-simplex $\sigma \in N(C_{/x})_n$ is a string of composable morphisms in $C_{/x}$, ie. a commutative diagram in C of the form:

$$\sigma : \qquad \begin{tikzcd} c_0 \arrow[r] \arrow[d] & c_1 \arrow[r] \arrow[d] & \dots \arrow[r] & c_n \arrow[d] \\ x \arrow[r, equal] & x \arrow[r, equal] & \dots \arrow[r, equal] & x \end{tikzcd}$$

Face and degeneracy maps look like:

$$d_i(\sigma): \qquad \begin{tikzcd} c_0 \arrow[r] \arrow[d] & \dots \arrow[r] & c_{i-1} \arrow[r] \arrow[d] & c_{i+1} \arrow[r] \arrow[d] & \dots \arrow[r] & c_n \arrow[d] \\ x \arrow[r, equal] & \dots \arrow[r, equal] & x \arrow[r, equal] & x \arrow[r, equal] & \dots \arrow[r, equal] & x \end{tikzcd}$$

$$s_i(\sigma): \qquad \begin{tikzcd} c_0 \arrow[r] \arrow[d] & \dots \arrow[r] & c_i \arrow[r, "\mathrm{id}"] \arrow[d] & c_i \arrow[r] \arrow[d] & \dots \arrow[r] & c_n \arrow[d] \\ x \arrow[r, equal] & \dots \arrow[r, equal] & x \arrow[r, equal] & x \arrow[r, equal] & \dots \arrow[r, equal] & x \end{tikzcd}$$

On the other hand, we can take the nerve first, and then consider the ∞-version of the overcategory: $(NC)_{/x}$. By definition n-simplices are:

$$(NC)_{/x}|_n = \mathrm{Hom}_x(\Delta^n \star \Delta^0, NC).$$

Since $\Delta^n \star \Delta^0 = \Delta^{n+1}$, and the Δ^0 corresponds to the final vertex of Δ^{n+1}, the above consists of $(n+1)$-simplices $\sigma \in (NC)_{n+1}$ whose final vertex is $d_0 \dots d_0(\sigma) = x$. Explicitly these correspond to diagrams in C of the form

$$\begin{tikzcd} c_0 \arrow[r] \arrow[drr] & c_1 \arrow[r] \arrow[dr] & \dots \arrow[r] & c_n \arrow[dl] \\ & & x & \end{tikzcd}$$

which we can see are precisely what we said n-simplices of $N(C_{/x})$ were. Face and degeneracy maps turn out to be the same as well, making the two isomorphic simplicial sets: $N(C_{/x}) \simeq (NC)_{/x}$. □

Proposition 2.9.28. More generally, for a diagram $F : I \to C$ in an ordinary category,

$$N(C_{/F}) \simeq NC_{/NF}.$$

Proof. Let $F : I \to C$ be a diagram in an ordinary category C. We can form the slice category $C_{/F}$, whose objects are pairs $(y \in C, \alpha : c_y \to F)$ of objects in C along with natural transformations from their constant functors. The nerve is an ∞-category $N(C_{/F})$.

Alternately, F lifts to $NF : NI \to NC$, a diagram in the ∞-category NC. We can form the slice simplicial set $NC_{/NF}$:

$$(NC_{/NF})_n = \mathrm{Hom}_{\mathbf{sSet}_{NI/}}((\Delta^n \star NI, i_{NI}), (NC, NF)),$$

ie. maps $f : \Delta^n \star NI \to NC$ extending NF:

$$\begin{tikzcd} NI \arrow[r, "NF"] \arrow[d, hook] & NC \\ \Delta^n \star NI \arrow[ur, "f"'] & \end{tikzcd}$$

Since $\Delta^n \simeq N([n])$ and $N(A \star B) \simeq NA \star NB$,

$$\begin{aligned} \Delta^n \star NI &\simeq N([n]) \star NI \\ &\simeq N([n] \star I) \end{aligned}$$

so $(NC_{/NF})_n$ consists of maps $f : N([n] \star I) \to NC$ extending NF. By fully-faithfulness of the nerve, the diagram of simplicial sets corresponds to a diagram of categories:

$$\begin{tikzcd} NI \arrow[r, "NF"] \arrow[d, hook] & NC \\ N([n] \star I) \arrow[ur, "f"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} I \arrow[r, "F"] \arrow[d, hook] & C \\ {[n] \star I} \arrow[ur, "{[f]}"'] & \end{tikzcd}$$

so our map f extending NF corresponds to a functor $[f] : [n] \star I \to C$ extending F. By our join/slice adjunction,

$$\mathrm{Hom}_{\mathbf{Cat}_{I/}}(([n] \star I, i_I), (C, F)) \cong \mathrm{Fun}([n], C_{/F}),$$

which are n-simplices of $N(C_{/F})$. $\square$

Proposition 2.9.29. Let $F : I \to NC$ be a diagram in the nerve of a category. Then there is an equivalence of categories:

$$C_{/hF} \simeq h(NC_{/F})$$

where $C_{/hF}$ is the ordinary over-category (2.9.5) and $NC_{/F}$ is the ∞-version (2.9.26).

Proof. Objects in the over-category $C_{/hF}$ are cones

$$\overline{x} : [0] \star hI \to C$$

(we use $\overline{x}$ to denote the cone with apex $x \in C$). These correspond bijectively to cones in the nerve (2.9.20):

$$\overline{x} : \Delta^0 \star I \to NC.$$

Morphisms in C_{hF} are natural transformations $\overline{x} \to \overline{y}$. In other words, a single morphism $x \to y$ that commutes with the cones over hF.

A morphism in the slice sset $NC_{/F}$ is a map $H : \Delta^1 \star I \to NC$ as on the left:

$$\begin{tikzcd} \Delta^{\{0\}} \star I \arrow[d, hook] \arrow[dr, bend left, "\overline{x}"] & \\ \Delta^1 \star I \arrow[r, "H"] & NC \\ \Delta^{\{1\}} \star I \arrow[u, hook] \arrow[ur, bend right, "\overline{y}"'] & \end{tikzcd} \qquad \begin{tikzcd} hI \arrow[d, hook] \arrow[dr, bend left, "\overline{x}"] & \\ {[1]} \star hI \arrow[r, "H"] & C \\ hI \arrow[u, hook] \arrow[ur, bend right, "\overline{y}"'] & \end{tikzcd}$$

By adjunction this corresponds to a diagram as on the right. But a map $[1] \star hI \to C$ is precisely a map $x \to y$ commuting with the cones. $\square$

We also want a notion of <u>initial/final objects</u> (2.9.8) in an ∞-category.

Definition 2.9.30. [initial/final objects in a simplicial set]

An object x in a simplicial set $\mathcal{C}$ is $\boxed{\text{initial}}$ if the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is contractible for any object $y \in \mathcal{C}_0$. This means (equivalently):

- The mapping space $\mathrm{Map}_{\mathcal{C}}(x, y) \sim \Delta^0$ in **Kan**, where $\sim$ means weak homotopy equivalent as a simplicial set (4.0.10). That is, taking the geometric realization (2.1.16) gives us a space $|\mathrm{Map}_{\mathcal{C}}(x,y)| \in \mathbf{Top}$ which is contractible in the usual sense.

- The homotopy-type of the mapping-space $[\mathrm{Map}_{\mathcal{C}}(x,y)] \in \mathcal{H} = h\mathbf{Kan}$ is equivalent to the point $* \in \mathcal{H}$.

Dually, x is $\boxed{\text{final}}$ if for all $y \in \mathcal{C}_0$, either of the following equivalent conditions hold:

- The mapping space $\mathrm{Map}_{\mathcal{C}}(y,x) \sim \Delta^0$ in **Kan**.
- The homotopy-type of the mapping-space $[\mathrm{Map}_{\mathcal{C}}(y,x)] \in \mathcal{H} = h\mathbf{Kan}$ is equivalent to the point $* \in \mathcal{H}$.

Remark 2.9.31. Note that the mapping space being contractible is the correct uniqueness statement in higher-categorical terms.

This is the analogue of the statement that there exists a unique map into (resp. out of) a final object (resp. initial object) in an ordinary category.

Proposition 2.9.32. An initial object in a simplicial set is initial when considered as an object in the homotopy category.

Proof. Say $x \in \mathcal{C}_0$ is an initial object. Then $\mathrm{Map}_{\mathcal{C}}(x,y) \simeq *$ for any $y \in \mathcal{C}_0$. Taking connected components, this forms an isomorphism $\pi_0 \mathrm{Map}_{\mathcal{C}}(x,y) \cong \{*\}$, exhibiting x as initial in $h\mathcal{C}$. □

Remark 2.9.33. [K, 4.6.6.19] It is not true that if an object x is initial in the homotopy category of an ∞-category $\mathcal{C}$, then x is initial in the ∞-sense.

Take $\mathcal{C} := \mathrm{Sing}(S^2)$ (2.1.16). This is a Kan complex. An object in a Kan complex is initial iff the Kan complex is contractible [K, 4.6.6.12], so $\mathrm{Sing}(S^2)$ has no initial objects as an ∞-category.

But every point is initial in the homotopy category (this can be shown by the fact that any two paths $I \rightrightarrows S^2$ with fixed endpoints are homotopic.)

Proposition 2.9.34. [K, 4.6.6.15]

[uniqueness of initial/final objects]

Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{C}^{init} \subseteq \mathcal{C}$ be the full subcategory spanned by the initial objects of $\mathcal{C}$ (resp. $\mathcal{C}^{final} \subseteq \mathcal{C}$ spanned by the final objects.)

Then $\mathcal{C}^{init}$ and $\mathcal{C}^{final}$ are both contractible Kan complexes. That is, every object in $\mathcal{C}^{init}$ (resp. $\mathcal{C}^{final}$) is equivalent up to contractible choice of equivalence.

(Note that this is the ∞-categorical way of saying "unique up to unique isomorphism".)

Remark 2.9.35. If $\mathcal{C}$ is an ∞-category, there is an alternate formulation of being initial/final. This was Joyal's definition of initial/final objects in quasicategories [Joy02], which is sometimes called "strongly initial/final" to differentiate from initial/final objects in arbitrary ssets [HTT, 1.2.12.3].

Definition 2.9.36. [strongly initial/final objects in ∞-categories]

An object $x \in \mathcal{C}_0$ of an ∞-category $\mathcal{C}$ is $\boxed{\text{strongly initial}}$ if the forgetful map $\mathcal{C}_{x/} \to \mathcal{C}$ is a trivial fibration (2.7.1); that is, any diagram of the form below admits a lift:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{C} \end{tikzcd}$$

Proposition 2.9.37. [HTT, 1.2.12.5] If $\mathcal{C}$ is an ∞-category, then any object is initial (resp. final) (2.9.30) iff it is strongly initial (resp. strongly final) (2.9.36).

Proposition 2.9.38. Let C be an ordinary category, and $x \in C$ an object. Then

x is initial in C in the ordinary sense (2.9.8) iff it is initial in the nerve in the sense of (2.9.30).

Proof. Suppose $x \in C$ is initial in the ordinary sense. For any $y \in C$ there's a unique morphism $f : x \to y$. Consider the singleton $\{f\} \subseteq \mathrm{Fun}([1], C)$.

Considering $\{f\}_\bullet$ as a constant simplicial set, this sits in a diagram of simplicial sets:

$$\begin{tikzcd} \{f\}_\bullet \arrow[r, hook] \arrow[d] & \mathbf{Cat}([1], C)_\bullet \simeq NC^{\Delta^1} \arrow[d, "\text{(source,target)}"] \\ \Delta^0 \arrow[r, "{(x,y)}"'] & NC \times NC \end{tikzcd}$$

This square is a pullback: take a simplicial set S with a map $F : S \to NC^{\Delta^1}$ and a map $S \to \Delta^0$ making the square commute. So F maps a $\sigma \in S_n$ to some

$$F\sigma \in (NC^{\Delta^1})_n = \mathbf{sSet}(\Delta^1 \times \Delta^n, NC).$$

A map $\Delta^1 \times \Delta^n \to NC$ is a square:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

and commutativity of the square says that $\{0\} \times \Delta^n = (x = x = \cdots = x)$ and $\{1\} \times$

$\Delta^n = (y = y = \cdots = y)$. That is, $F\sigma$ corresponds to a diagram in C of the form:

$$\begin{array}{ccc} x & \longrightarrow & y \\ \| & & \| \\ x & \longrightarrow & y \\ \| & & \| \\ \vdots & & \vdots \\ \| & & \| \\ x & \longrightarrow & y \end{array}$$

Since $\mathrm{Hom}_C(x,y) = \{f\}$, there's only one choice of diagram as above. That is, $F : X \to NC^{\Delta^1} = \mathbf{Cat}([1],C)_\bullet$ has no choice but to factor through $\{f\}_\bullet$, making the square above a pullback.

But this pullback square defines the mapping space. So $\mathrm{Map}_{NC}(x,y) \simeq \{f\}_\bullet \simeq *$, making x initial in NC. The argument can be run in reverse for the backwards direction. $\square$

Definition 2.9.39. [limits and colimits in ∞-categories]

Let $p : I \to \mathcal{C}$ be a diagram in an ∞-category $\mathcal{C}$.

The $\boxed{\text{limit}}$ (if it exists) is a final object of $\mathcal{C}_{/p}$. A colimit is an initial object of $\mathcal{C}_{p/}$.

Example 2.9.40. Let $\mathcal{C}$ be an ∞-category.

An object $x \in \mathcal{C}_0$ is initial (2.9.30) iff it is the limit of the empty diagram $\varnothing \to \mathcal{C}$. (Dually, x is final iff it's the colimit of $\varnothing \to \mathcal{C}$.)

Proof. The object x is initial iff for any $y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(y,x)$ is contractible. There is an equivalence $\mathcal{C}_{/\varnothing} \simeq \mathcal{C}$ formed from the equivalences

$\Delta^n \star \varnothing \simeq \Delta^n$, so:

$$\begin{aligned}\Delta^0 &\simeq \mathrm{Map}_{\mathcal{C}}(y,x)\\ &\simeq \mathrm{Map}_{\mathcal{C}_{/\varnothing}}(y,x),\end{aligned}$$

ie. x is final in the over-category $\mathcal{C}_{\varnothing}$, ie. x is the limit of the diagram $\varnothing \to \mathcal{C}$. $\square$

Example 2.9.41. [K, 7.6.3.1]

An important example for us of limits/colimits are pullback and pushout squares. Let $I = \Delta^1 \cup_{\Delta^{\{1\}}} \Delta^1$:

$$I = \left\{ \begin{tikzcd} & 0' \arrow[d] \\ 0 \arrow[r] & 1 \end{tikzcd} \right\}$$

Then a diagram $p : I \to \mathcal{C}$ in an ∞-category is of the form:

$$\begin{tikzcd} & y \arrow[d] \\ x \arrow[r] & z \end{tikzcd}$$

The $\boxed{\text{pullback}}$ is the limit (2.9.39)

$$x \times_z y = \lim_I p.$$

Explicitly, the pullback is a cone $\Delta^0 \star I \to \mathcal{C}$ (2.9.22). Drawing out the join

$\Delta^0 \star I$ (with the apex labelled $\widehat{0}$):

$$\begin{tikzcd} \widehat{0} \arrow[r] \arrow[d] \arrow[dr] & 0' \arrow[d] \arrow[dl, phantom, "\Swarrow", pos=0.3] \\ 0 \arrow[r] \arrow[ur, phantom, "\Nearrow", pos=0.3] & 1 \end{tikzcd}$$

this is isomorphic as a simplicial set to the square $\Delta^1 \times \Delta^1$ (2.1.13). So alternately, a pullback in $\mathcal{C}$ is a square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ that forms a limit.

Dually, a pushout is a square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ that forms a colimit.

Note that pullbacks and pushouts are <u>not preserved</u> in the passage to the homotopy category [K, 7.6.3.3].

Proposition 2.9.42. Let C be an ordinary category, and $p : I \to C, i \mapsto x_i$ a diagram. Suppose the limit exists and call it $x = \lim_I p \in C$. Then

$$x = \lim_{NI} Np \in NC$$

where the limit is in the sense of (2.9.39).

Proof. The limit satisfies the universal property (a bijection of hom-sets):

$$\lim_{i \in I} \mathrm{Hom}_C(w, x_i) \cong \mathrm{Hom}_C(w, x).$$

Taking nerves, we get a diagram $Np : NI \to NC$. Elements of

$$NI_n = \mathrm{Fun}([n], NI) = \{\text{strings } (i_0 \to i_1 \to \cdots \to i_n) \in I\}.$$

and we expect that a cone $\overline{x} : NI^{\triangleleft} \to NC$ over Np, sending the vertex to x, should define a limit cone in the ∞-sense above.

Let's start with another cone $\overline{w} : NI^{\triangleleft} \to NC$, with vertex $w \in C$. For any string $\sigma = (i_0 \to \cdots \to i_n) \in NI_n$, the composite we have an $(n+1)$-simplex in NC corresponding to a commutative diagram in C of the form:

$$\begin{tikzcd}
 & w \arrow[dl] \arrow[d] \arrow[drrr] & & & \\
x_0 \arrow[r] & x_1 \arrow[r] & \dots \arrow[r] & x_n
\end{tikzcd}$$

That is to say, this w (along with all its maps) forms a cone in the homotopy category $hNC \simeq C$. The universal property of the limit $x = \lim_I x_i$ then induces a unique map $f : w \to x$ making all relevant diagrams involving subdiagrams of I commute. That is, for any string $i_0 \to \cdots \to i_n$ in I, we have a commutative diagram in C:

$$\begin{tikzcd}
 & w \arrow[r, "f"] \arrow[dl] \arrow[d] \arrow[drr] & x \arrow[dll] \arrow[dl] \arrow[dr] & \\
x_0 \arrow[r] & x_1 \arrow[r] & \dots \arrow[r] & x_n
\end{tikzcd}$$

This lifts by definition to an $(n+2)$-simplex in NC.

We want to show that

$$\pi_0 \operatorname{Map}_{NC_{/NI}}(\overline{w}, \overline{x}) \cong \{*\}.$$

This is a set generated by 1-simplices $(NC_{/NI})_1(\overline{w}, \overline{x})$, with an equivalence relation determined by 2-simplices $(NC_{/NI})_2$. A 1-simplex in $NC_{/NI}$ between $\overline{w}$ and $\overline{x}$ is a map $\Delta^1 \star NI \to NC$ restricting to $\overline{w}$ and $\overline{x}$ on $\Delta^{\{0\}} \star NI$ and $\Delta^{\{1\}} \star NI$ respectively. Our unique map $f : w \to x$ (with all the diagrams it makes commute) induces a unique 1-simplex $\overline{f} : \Delta^1 \star NI \to NC$ sending an n-simplex in

$\Delta^1 \star NI$ to an n-simplex in NC of the form created by f:

$$\begin{tikzcd}
& 0 \arrow[r] \arrow[dl] \arrow[d] \arrow[drr] & 1 \arrow[dll] \arrow[dl] \arrow[dr] & & {} \arrow[r, mapsto, "\overline{f}"] & {} & & w \arrow[r, "f"] \arrow[dl] \arrow[d] \arrow[drr] & x \arrow[dll] \arrow[dl] \arrow[dr] & \\
i_0 \arrow[r] & i_1 \arrow[r] & \dots \arrow[r] & i_{n-2} & & & x_0 \arrow[r] & x_1 \arrow[r] & \dots \arrow[r] & x_{n-2}
\end{tikzcd}$$

By uniqueness of the map f, $\overline{f}$ is the only 1-simplex in the mapping space $\mathrm{Map}_{NC_{/NI}}(\overline{w}, \overline{x}) = \left\{\overline{f}\right\} \simeq *$; ie. $\overline{x}$ is final, making $x = \lim_{NI} x_i \in NC$.

On the other hand, suppose we start with a diagram in the nerve $I \to NC$, and say $x = \lim_I x_i$ in the ∞-sense. That is, $\overline{x} \in NC_{/p}$ is a final object; ie. $\mathrm{Hom}_{hNC_{/p}}(\overline{w}, \overline{x}) \simeq *$. Since there is an equivalence of categories $hNC \simeq C$ (2.1.20), the cone $[\overline{x}] \in hNC_{/p}$ is final in the ordinary over-category $C_{/p}$; ie. $x = \lim_I x_i$ in C. □

Proposition 2.9.43. [HTT, 4.2.4.3] Let $F : I \to \mathcal{C}$ be a diagram in an ∞-category $\mathcal{C}$.

Then $x = \lim_I F$ is the limit iff there is a weak equivalence of mapping spaces:

$$\mathrm{Map}_{\mathcal{C}}(w, x) \simeq \mathrm{Map}_{\mathcal{C}^I}(c_w, F),$$

where $c_w : I \to \mathcal{C}$ is the constant diagram at $w \in \mathcal{C}$.

Proposition 2.9.44. [K, 7.1.1.16]

[limits/colimits as adjoint functors]

Let $\mathcal{C}$ be an ∞-category, and let $I \in \mathbf{sSet}$ be a diagram sset.

The constant functor

$$c : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$$

sends an object $X \mapsto c_X$, the constant functor $I \to \mathcal{C}$ at X. Then:

- The ∞-category $\mathcal{C}$ admits all I-indexed limits iff the constant functor c admits a right adjoint:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "c"', shift right] \\ \mathrm{Fun}(I,\mathcal{C}) \arrow[u, dotted, bend right, "\lim_I"'] \end{tikzcd}$$

- The ∞-category $\mathcal{C}$ admits all I-indexed colimits iff the constant functor c admits a left adjoint:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "c", shift left] \\ \mathrm{Fun}(I,\mathcal{C}) \arrow[u, dotted, bend left, "\mathrm{colim}_I"] \end{tikzcd}$$

Remark 2.9.45. The above proposition is reminiscent of the construction of homotopy limit and homotopy colimit functors on a model category (4.2.1).

Remark 2.9.46. Unless an ∞-category is the nerve of a category, translating limits in ∞-land to limits in homotopy-land is in general difficult. That is, a limit of a diagram $I \to \mathcal{C}$ in an arbitrary ∞-category $\mathcal{C}$ does not, in general, produce a limit $\lim_{hI} hF \in h\mathcal{C}$. In general, homotopy categories may not have many limits and colimits.

But we might consider the simpler case of products. A product in an ∞-category is a limit of a diagram I that is discrete; ie. a simplicial set whose nondegenerate simplices are only vertices. This simplifies much of the coherence requirements, and indeed the case of products is as nice as we can hope for.

Proposition 2.9.47. A product $x = \prod_{i \in I} x_i$ (resp. a coproduct) in an ∞-category $\mathcal{C}$ is a product (resp. coproduct) in the homotopy category $h\mathcal{C}$.

Proof. The universal property of a product in an ∞-category (2.12.11) looks like:

$$\mathrm{Map}_{\mathcal{C}}(w,x) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Map}_{\mathcal{C}}(w,x_i).$$

Taking this down to connected components, using that π_0 preserves products of spaces, we have:

$$\begin{aligned}
\mathrm{Hom}_{h\mathcal{C}}(w,x) = \pi_0 \mathrm{Map}_{\mathcal{C}}(w,x) &\xrightarrow{\sim} \pi_0 \prod_I \mathrm{Map}_{\mathcal{C}}(w,x_i) \\
&\cong \prod_I \pi_0 \mathrm{Map}_{\mathcal{C}}(w,x_i) \\
&= \prod_I \mathrm{Hom}_{h\mathcal{C}}(w,x_i),
\end{aligned}$$

satisfying the universal property of the ordinary product.

Dually, one can show the corresponding statement for coproducts. $\square$

Proposition 2.9.48. There is another case of a sequential colimit: the colimit of a diagram of the form:

$$X_0 \to X_1 \to \dots$$

This can be constructed by taking a coequalizer :

$$\begin{tikzcd} \coprod_I c_i \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\coprod_I f_i"'] & \coprod_I c_i \arrow[r] & \mathrm{hocolim}\, c_i \end{tikzcd}$$

The universal property of the coequalizer:

$$\begin{tikzcd} C(\mathrm{hocolim}_I c_i, y) \arrow[r] \arrow[d] & C(\coprod_I c_i, y) \arrow[d, "-\circ \mathrm{id}"] \\ C(\coprod_I c_i, y) \arrow[r, "-\circ(\coprod_I f_i)"'] & C(\coprod_I c_i, y) \end{tikzcd}$$

ie. a map $(\mathrm{hocolim}_I c_i \to y)$ corresponds to a pair $\Phi, \Phi' : \coprod_I c_i \rightrightarrows y$ such that

$\Phi \circ \mathrm{id} = \Phi' \circ \mathrm{shift}$. Both Φ and Φ' are just collections $(\Phi_i : c_i \to y)_i$ and $(\Phi'_i : c_i \to y)_i$, so $\Phi = \Phi' \circ \mathrm{shift}$ is the condition that for all i, $\Phi_i = \Phi'_{i+1} \circ f_{i+1}$.

We can interpret this as a colimit (in the ∞-sense) of the nerve. Let $I \in \mathbf{sSet}$ be the simplicial set:

$$I := \Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \cdots = (\bullet \to \bullet \to \bullet \to \dots)$$

Let $F : I \to NC$ be the resulting diagram, which forms a sequential diagram in C:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \dots$$

Then the sequential colimit $\mathrm{colim}_{hI}\, c_i$ calculated as a coequalizer is the same as the colimit $\mathrm{colim}_I F$ as an ∞-category.

Proof. The colimit $\mathrm{colim}_I F \in NC$ is defined by an equivalence of spaces:

$$\mathrm{Map}_{NC}(\mathrm{colim}_I F, y) \simeq \mathrm{Map}_{NC^I}(F, c_y).$$

Since $I = (\Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \dots)$, the function complex

$$\begin{aligned} NC^I &= \mathrm{Fun}(I, C) \\ &\simeq NC^{\Delta^1} \times_{NC} NC^{\Delta^1} \times_{NC} \cdots \end{aligned}$$

so a map $(\mathrm{colim}_I F \to y)$ is the same as a natural transformation $(F \to c_y)$, ie. an edge in NC^I. By the above, edges of NC^I are

$$(NC^I)_1 = (NC^{\Delta^1})_1 \times_{NC_1} (NC^{\Delta^1})_1 \times_{NC_1} \cdots$$

so an edge $F \to c_y$ is a collection of squares ($\Delta^1 \times \Delta^1 \to NC$) of the form:

$$\begin{tikzcd}
c_0 \arrow[r, "f_1"] \arrow[d, "\varphi_0"'] \arrow[dr] & c_1 \arrow[d, "\varphi_1"] & c_1 \arrow[r, "f_2"] \arrow[d, "\varphi_1"'] \arrow[dr] & c_2 \arrow[d, "\varphi_2"] & \cdots \\
y \arrow[r, equal] & y & y \arrow[r, equal] & y &
\end{tikzcd}$$

ie. a collection of maps $(\varphi_i : c_i \to y)_{i \in I}$ along with homotopies $\varphi_{i+1} f_{i+1} \sim \varphi_i$ for all i. Down in the homotopy category $hNC \simeq C$, these are commuting triangles witnessing $\varphi_{i+1} f_{i+1} = \varphi_i$. This is precisely the universal property of the coequalizer above. $\square$

2.10 Adjunctions

We know that ordinary category is actually just a specific case of a 2-category; that a category of categories, **Cat**, wants to include natural transformations as 2-morphisms. One idea in ordinary categories which is 2-categorical in nature is the notion of an adjunction between categories.

Accordingly, we can formulate adjunctions more generally using the language of 2-categories.

Definition 2.10.1. [adjunctions in a 2-category] [K, 6.1.1.1]

Let C be a 2-category.

An $\boxed{\text{adjunction}}$ in C consists of a pair of objects

$$x, y \in C,$$

a pair of 1-morphisms

$$f : x \rightleftarrows y : g,$$

and a pair of 2-morphisms

$$(\eta : \mathrm{id}_x \to gf) \in \mathrm{Hom}(C(x,x))$$
$$(\varepsilon : fg \to \mathrm{id}_y) \in \mathrm{Hom}(C(y,y)),$$

that we call the unit and counit respectively. These must form commutative triangles in hom-categories:

$$\begin{tikzcd} & fgf \arrow[dr, "\varepsilon\circ \mathrm{id}_f"] & \\ f = f\circ \mathrm{id}_x \arrow[ur, "\mathrm{id}_f\circ\eta"] \arrow[rr, "\mathrm{id}_f"'] & & \mathrm{id}_y\circ f = f \end{tikzcd} \qquad \text{in } C(x,y)$$

$$\begin{tikzcd} & gfg \arrow[dr, "\mathrm{id}_g\circ\varepsilon"] & \\ g = \mathrm{id}_x\circ g \arrow[ur, "\eta\circ \mathrm{id}_g"] \arrow[rr, "\mathrm{id}_g"'] & & g\circ \mathrm{id}_y = g \end{tikzcd} \qquad \text{in } C(y,x)$$

Remark 2.10.2. [Rie17, Prop. 4.2.6] An ordinary adjunction between (1-) categories is an adjunction in the above sense in the 2-category **Cat**.

Given an adjunction $F : C \rightleftarrows D : G$ between ordinary categories, the conditions on the unit and counit making the triangles commute is equivalent to exhibiting bijections

$$\mathrm{Hom}_C(c, Gd) \cong \mathrm{Hom}_D(Fc, d)$$

that are natural in both variables c and d.

Definition 2.10.3. [adjunctions of ∞-categories] [K, 6.2.1.2]

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let F and G be functors between them:

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G.$$

An $\boxed{\text{adjunction}}$ between F and G is a pair of edges in functor categories (natural transformations)

$$(\eta : \mathrm{id}_{\mathcal{C}} \to GF) \in \mathrm{Fun}(\mathcal{C}, \mathcal{C})_1,$$
$$(\varepsilon : FG \to \mathrm{id}_{\mathcal{D}}) \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})_1$$

that are compatible up to homotopy [K, 6.2.1.1] in the sense that there exist 2-simplices $\alpha : \Delta^2 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\beta : \Delta^2 \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ of the form:

$$\alpha : \quad \begin{tikzcd} & F \circ G \circ F \arrow[dr, "\varepsilon \circ \mathrm{id}_F"] & \\ F = F \circ \mathrm{id}_{\mathcal{C}} \arrow[ur, "\mathrm{id}_F \circ \eta"] \arrow[rr, "\mathrm{id}_F"'] & & \mathrm{id}_{\mathcal{D}} \circ F = F \end{tikzcd} \quad \text{in } \mathrm{Fun}(\mathcal{C}, \mathcal{D})_2$$

$$\beta : \quad \begin{tikzcd} & G \circ F \circ G \arrow[dr, "\mathrm{id}_G \circ \varepsilon"] & \\ G = \mathrm{id}_{\mathcal{C}} \circ G \arrow[ur, "\eta \circ \mathrm{id}_G"] \arrow[rr, "\mathrm{id}_G"'] & & G \circ \mathrm{id}_{\mathcal{D}} = G \end{tikzcd} \quad \text{in } \mathrm{Fun}(\mathcal{D}, \mathcal{C})_2$$

In this case, we again call η the counit and ε the unit of an adjunction.

Remark 2.10.4. We can describe adjunctions in terms of fibered categories. Recall Grothendieck's construction (2.7.8), equivalences of 2-categories:

$$\mathrm{Fib}(B) \simeq \mathrm{Fun}(B^{op}, \mathbf{Cat})$$
$$\mathrm{opFib}(B) \simeq \mathrm{Fun}(B, \mathbf{Cat})$$

identifying fibrations (resp. opfibrations) over B and contravariant functors (resp. covariant functors) $B \to \mathbf{Cat}$.

Then an adjunction in the sense of (2.10.1) is a functor of 2-categories $p : E \to [1]$ (considering $[1] = \{0 \to 1\}$ as a 2-category) that is both a fibration and an opfibration (2.7.7).

Proof. The functor p determines two functors $[1] \rightrightarrows$ **Cat**, one covariant and one contravariant. That is, it determines a pair of categories C and D and two functors between them in opposite directions

$$F : C \rightleftarrows D : G,$$

along with equivalences of categories with the fibers: $C \simeq E_0$ and $D \simeq E_1$.

We want to exhibit a unit η and a counit ε.

We want a natural transformation $\eta : \mathrm{id}_C \to GF$ forming a commutative triangle in the functor category $\mathrm{Fun}(C, C)$:

$$\begin{tikzcd} & FGF \arrow[dr, "\varepsilon \circ \mathrm{id}_F"] & \\ F = F \circ \mathrm{id}_C \arrow[ur, "\mathrm{id}_F \circ \eta"] \arrow[rr, "\mathrm{id}_F"'] & & \mathrm{id}_D \circ F = F \end{tikzcd}$$

By our discussion in (2.7.9), we have bijections of hom-sets for any $c \in C \simeq E_0$ and $d \in D \simeq E_1$:

$$\mathrm{Hom}_C(c, Gd) \cong \mathrm{Hom}_E(c, d) \cong \mathrm{Hom}_D(Fc, d)$$

Fixing $c \in C$, the bijection above gives us:

$$\mathrm{Hom}_C(c, GFc) \cong \mathrm{Hom}_E(c, Fc) \cong \mathrm{Hom}_D(Fc, Fc).$$

Let $\eta_c : c \to GFc$ be the image under bijection to the map $\mathrm{id}_{Fc} \in \mathrm{Hom}_D(Fc, Fc)$. By the bijection, the naturality square on the left corresponds to a square in D

on the right:

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "\varphi"'] & GFc \arrow[d, "GF\varphi"] \\ c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[r, "\mathrm{id}_{Fc}"] \arrow[d, "F\varphi"'] & Fc \arrow[d, "F\varphi"] \\ Fc \arrow[r, "\mathrm{id}_{Fc}"'] & Fc \end{tikzcd}$$

in C in D

The square on the right commutes so the naturality square we need commutes.

We form a counit $\varepsilon : FG \to \mathrm{id}_D$ in an analagous way – fixing a $d \in D$ and using the bijections on hom-sets to take the image of id_{Gd} to form $\varepsilon_d : FGd \to d$. Showing this is natural proceeds in the same way. □

Definition 2.10.5. An $\boxed{\text{adjunction}}$ is a map $\mathcal{E} \to \Delta^1$ that is both a cartesian and a cocartesian fibration (2.7.1).

Remark 2.10.6. The process of constructing an ∞-categorical adjunction between ∞-categories $\mathcal{C} \rightleftarrows \mathcal{D}$ proceeds analagously to (2.10.4).

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is left adjoint (equivalently, admits a right adjoint) if there is a bicartesian fibration $\mathcal{E} \to \Delta^1$ along with equivalences of ∞-categories $\mathcal{C} \simeq \mathcal{E}_0$ and $\mathcal{D} \simeq \mathcal{E}_1$.

Given a bicartesian fibration $\mathcal{E} \to \Delta^1$, we can show that both fibers $\mathcal{E}_0 = p^{-1}(0)$ and $\mathcal{E}_1 = p^{-1}(1)$ are ∞-categories by describing them as pullbacks of simplicial sets:

$$\begin{tikzcd} \mathcal{E}_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \mathcal{E} \arrow[d, "p"] \\ \{i\} \arrow[r, hook] & \Delta^1 \end{tikzcd}$$

Since bicartesian fibrations are in particular inner fibrations, one can show inner-horn lifting for both $\mathcal{E}_0$ and $\mathcal{E}_1$. So these fibers form ∞-categories.

By straightening-unstraightening (2.7.10), a bicartesian fibration p corresponds to a pair of (∞-) functors

$$\mathcal{E}_0 \rightleftarrows \mathcal{E}_1$$

which we can compose with the equivalences $\mathcal{C} \simeq \mathcal{E}_0$ and $\mathcal{D} \simeq \mathcal{E}_1$ to get an adjunction of ∞-categories:

$$\begin{tikzcd} \mathcal{C} \arrow[rrrr, bend left, "F"] & \simeq & \mathcal{E}_0 \arrow[r, shift left] & \mathcal{E}_1 \arrow[l, shift left] & \simeq & \mathcal{D} \arrow[lllll, bend left, "G"] \end{tikzcd}$$

Proposition 2.10.7. Let C and D be ordinary categories.

Then a pair of functors

$$F : C \rightleftarrows D : G$$

forms an adjoint pair in the sense of (2.10.1) iff the pair of functors between their nerves

$$NF : NC \rightleftarrows ND : NG$$

forms an (∞-) adjunction in the sense of (2.10.3).

Proof. Suppose we have an ordinary adjunction $F : C \rightleftarrows D : G$. This comes with a pair of natural transformations $\eta : \mathrm{id}_C \to G \circ F$ and $\varepsilon : F \circ G \to \mathrm{id}_D$.

Taking nerves, we get edges

$$(N\eta : N\,\mathrm{id}_C \to N(GF)) \in N\,\mathrm{Fun}(C, C)_1$$
$$(N\varepsilon : N(FG) \to N\,\mathrm{id}_D) \in N\,\mathrm{Fun}(D, D)_1.$$

Since $N\,\mathrm{Fun}(D,D) \simeq \mathrm{Fun}(ND,ND)$ (2.8.4) these correspond to edges

$$(\overline{\eta} : \mathrm{id}_{NC} \to NG \circ NF) \in \mathrm{Fun}(NC,NC)_1$$
$$(\overline{\varepsilon} : NF \circ NG) \to \mathrm{id}_{ND}) \in \mathrm{Fun}(ND,ND)_1.$$

By definition, these form a commutative triangle in $\mathrm{Fun}(C,D)$ corresponding to a 2-simplex in $N\,\mathrm{Fun}(C,D) \simeq \mathrm{Fun}(NC,ND)$:

$$\begin{tikzcd}
 & NF \circ NG \circ NF \arrow[dr, "\overline{\varepsilon} \circ \mathrm{id}_{NF}"] & \\
NF \simeq NF \circ \mathrm{id}_{NC} \arrow[ur, "\mathrm{id}_{NF} \circ \overline{\eta}"] \arrow[rr, "\mathrm{id}_{NF}"'] & & \mathrm{id}_{ND} \circ NF \simeq NF
\end{tikzcd}$$

A similar argument gets the other relevant 2-simplex in $\mathrm{Fun}(ND,NC)$. These define precisely the 2-simplices needed to exhibit $\overline{\eta}$ and $\overline{\varepsilon}$ as the unit and counit of an adjunction of ∞-categories $NF : NC \rightleftarrows ND : NG$. $\square$

The equivalence $N\,\mathrm{Fun}(C,D) \simeq \mathrm{Fun}(NC,ND)$ lets us run this argument in the backwards direction.

Proposition 2.10.8. [K, 6.2.1.14]

Say $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint of ∞-categories. Given a functor $G : \mathcal{D} \to \mathcal{C}$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$, then the following are equivalent:

(i) The map η is the unit of an adjunction between ∞-categories, exhibiting G as right adjoint to F.

(ii) The induced map $\eta' : \mathrm{id}_{h\mathcal{C}} \to hG \circ hF$ is the unit of an ordinary adjunction $(hF, hG) : h\mathcal{C} \rightleftarrows h\mathcal{D}$.

Remark 2.10.9. Beware that the assumption above that F admits an adjoint in the ∞-sense is necessary for the implication (ii) $\Rightarrow$ (i). It is not true in general that adjoints are detectable on the homotopy level. That is if F is a functor of ∞-categories and hF admits an adjoint, it is not true in general that F admits an adjoint.

Nonetheless, there are some things we can say in this direction. (1) While we can't test adjoints on the level of ordinary homotopy categories, we can on the level of <u>enriched</u> homotopy categories (2.4.8). (2) If $\mathcal{D}$ and G satisfy some completeness conditions, we can detect adjoints on the ordinary category.

Definition 2.10.10. [adjunction between enriched categories]

Let V be a monoidal category and let C and D be categories enriched over V (2.2.1).

An $\boxed{\text{enriched adjunction}}$ between V-enriched categories C and D is a pair of V-enriched functors

$$F : C \rightleftarrows D : G$$

along with natural isomorphisms between the hom-functors $D \times C \to V$:

$$C(F(-), -) \simeq D(-, G(-)).$$

Proposition 2.10.11. [HTT, 5.2.2.9, 5.2.2.12]

Let $\mathcal{C}$ be an ∞-category. Recall the enriched homotopy category $\mathcal{HC}$, which is enriched over $\mathcal{H} = h\mathbf{Kan}$ (2.4.8).

Then a (∞-) functor $F : \mathcal{C} \to \mathcal{D}$ admits a (∞-) right adjoint iff the induced functor $\mathcal{H}F : \mathcal{HC} \to \mathcal{HD}$ admits an enriched adjoint (2.10.10).

Theorem 2.10.12. [NRS20, Thm. 3.3.1]

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. Suppose $\mathcal{D}$ admits finite limits and G preserves them.

Then G admits a left adjoint iff $hG : h\mathcal{D} \to h\mathcal{C}$ does.

2.11 ∞-categories underlying model categories

The fundamental construction associated to an ∞-category is its homotopy category. As we know, the passage from an ∞-category to its homotopy category represents a massive loss of higher-order information. For example, we lose a lot of information needed to describe homotopy limits/colimits, which are not often easy to get a hold of in the homotopy category, but are described easily in the ∞-categorical level.

Before ∞-categories, we could describe a homotopy theory via a model structure (4.0.10). More generally, realizing that the weak equivalences are the important part of a homotopy theory, we can describe a more general idea of a category with weak equivalences only. These are sometimes called categories with weak equivalences, relative categories, or homotopical categories.[5] (For more on the relation between ∞-categories and relative/homotopical categories, see [BK11] and [Rie20].)

Given either a model category or a relative category M, there is a way to construct an ∞-category M^∞ that describes the same homotopy theory. This ∞-category we call the underlying ∞-category of M, and "having the same homotopy theory" means that there is an isomorphism of homotopy categories:

$$ho(M) \simeq h(M^\infty),$$

where $ho(M)$ is the homotopy category of the model category (4.0.2) and $h(M^\infty)$

[5]There may be different conditions about closure properties of the weak equivalences, so beware.

is the homotopy category of the ∞-category (2.4.13).

This section will be devoted to constructing the underlying M^∞, as well as a related construction called Dwyer-Kan localization.

2.11.1 Dwyer-Kan localization

Historically, this type of construction was done first in the language of simplicial model categories[6] – model categories that are simplicially enriched (2.2.4) in a way that's compatible with the model structure.

In a series of papers [DK80c; DK80a; DK80b] William Dwyer and Daniel Kan described a process of simplicial localization (a.k.a. Dwyer-Kan localization or DK localization), which takes a simplicial model category and builds a simplicial category that records the higher-order homotopical information. This construction goes as follows:

(i) Starting with a model/relative category M (4.0.3);

(ii) realize it as a simplicial model category via simplicial (a.k.a. Dwyer-Kan) localization, to form a simplicial category $L^H M \in \mathbf{sCat}$ (2.2.4);

(iii) taking the homotopy category (2.4.6) recovers the homotopy category of the model category (4.0.2), ie. there are isomorphisms:

$$\mathrm{Hom}_{ho(M)}(x,y) \cong \pi_0(L^H M(x,y)_\bullet)$$

for all $x, y \in M$.

[6]Many model categories we care about are already naturally simplicial model categories; eg. $\mathbf{sSet}_{Quillen}$ and $\mathbf{Top}_{Quillen}$. Although there are many model categories that aren't naturally simplicially enriched, Dugger [Dug01] showed that for a certain class of particularly nice model categories, there is a way to construct a Quillen equivalence from each to a simplicial model category.

Definition 2.11.1. [Dwyer-Kan localization]

Let M be a model category (4.0.3) or, more generally, a relative category. The $\boxed{\text{Dwyer-Kan localization}}$ (or hammock localization) functor

$$L^H : \mathbf{RelCat} \to \mathbf{sCat}$$

takes a model/relative category to a simplicially enriched category $L^H M$, which is defined as follows:

- (objects): Objects are zig-zags (4.0.2) in M: a zig-zag of length i is a string in M of the form:

  $$x \to m_1 \xleftarrow{\sim} m_2 \to \cdots m_{i-1} \to y$$

  where each leftwards map is a weak equivalence. Note that, although we have drawn them as above, there is no reason that the first and last maps need be as above. These are precisely the same objects as in the homotopy category $ho(M)$ (4.0.2).

- (hom-ssets): Given objects $x, y \in M$, the hom-sset $L^H M(x,y)_\bullet \in \mathbf{sSet}$ is defined with n-simplices given by $\underline{\text{reduced hammocks of height } n}$ – diagrams of the form

$$\begin{tikzcd}
 & m_{0,1} \arrow[r, dash] \arrow[d, "\sim"] & \cdots \arrow[r, dash] & m_{0,i-1} \arrow[d, "\sim"'] \arrow[dr, dash] & \\
x \arrow[ur, dash] \arrow[r, dash] \arrow[ddr, dash] & m_{1,1} \arrow[r, dash] \arrow[d, "\sim"] & \cdots \arrow[r, dash] & m_{1,i-1} \arrow[r, dash] \arrow[d, "\sim"'] & y \arrow[ddl, dash] \\
 & \vdots \arrow[d, "\sim"] & & \vdots \arrow[d, "\sim"'] & \\
 & m_{n,1} \arrow[r, dash] & \cdots \arrow[r, dash] & m_{n,i-1} &
\end{tikzcd}$$

such that:

- The horizontal strings are zig-zags between x and y of any length i. Backwards (left-facing) maps in a zig-zag are by definition required to be weak equivalences. The zig-zags are "parallel" in the sense that maps in the same column face the same direction.
- All vertical maps are weak equivalences (marked with $\sim$).
- No column consists of only identity maps.

Face maps are given by deleting and composing over a row, and degeneracies are given by repeating a row. For example, given a 2-simplex $\alpha \in L^H M(x,y)_2$ of the form below, its 1^{st} face and 0^{th} degeneracy look like:

$$\alpha : \quad \begin{tikzcd} & a \arrow[d] & a' \arrow[l, "\sim"'] \arrow[d] \arrow[dr] & \\ x \arrow[ur] \arrow[r] \arrow[dr] & b \arrow[d] & b' \arrow[l, "\sim"'] \arrow[d] \arrow[r] & y \\ & c & c' \arrow[l, "\sim"'] \arrow[ur] & \end{tikzcd}$$

$$d_1\alpha : \quad \begin{tikzcd} & a \arrow[dd, bend left] & a' \arrow[l, "\sim"'] \arrow[dd, bend left] \arrow[dr] & \\ x \arrow[ur] \arrow[dr] & & & y \\ & c & c' \arrow[l, "\sim"'] \arrow[ur] & \end{tikzcd} \qquad s_0\alpha : \quad \begin{tikzcd} & a \arrow[d, equal] & a' \arrow[l, "\sim"'] \arrow[d, equal] \arrow[ddr] & \\ & a \arrow[d] & a' \arrow[l, "\sim"'] \arrow[d] \arrow[dr] & \\ x \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & b \arrow[d] & b' \arrow[l, "\sim"'] \arrow[d] \arrow[r] & y \\ & c & c' \arrow[l, "\sim"'] \arrow[ur] & \end{tikzcd}$$

Remark 2.11.2. Dwyer and Kan actually construct a different version first in [DK80c], and then construct the hammock localization in [DK80a] where they show that the two constructions form equivalent simplicial categories [DK80a, Prop. 2.2]

Proposition 2.11.3. [DK80a, Prop. 3.1]

There is an equivalence of categories:

$$h(L^H M) \simeq ho(M),$$

where $h(L^H M)$ is the homotopy category of the simplicial category (2.4.6) and $ho(M)$ is the homotopy category of the model category (4.0.2).

Remark 2.11.4. Since simplicial categories are a model of ∞-categories, we could take Dwyer-Kan localization as one description of an underlying ∞-category of M.

If we want to describe the underlying as an quasi-category, there are two constructions. Both will be shown to be equivalent.

Definition 2.11.5. [underlying ∞-category of a model category (def. 1)]

Let M be a model category.

The $\boxed{\text{underlying } \infty\text{-category of } M}$ (we may also call it simply "the underlying of M") is a quasi-category we call M^∞ that is constructed as follows:

$$(-)^\infty : \mathbf{RelCat} \xrightarrow{L^H} \mathbf{sCat} \xrightarrow{(-)^f} \mathbf{sCat} \xrightarrow{N_\Delta} \mathbf{sSet}$$

To break this down:

- L^H is Dwyer-Kan localization (2.11.1).
- $(-)^f$ is fibrant replacement in $\mathbf{sCat}_{Bergner}$ (4.0.10). Recall that fibrant objects (4.0.13) in $\mathbf{sCat}_{Bergner}$ are Kan-enriched categories (4.0.20) (ie. ∞-categories (4.1.6)).
- Taking the coherent nerve (2.2.14) of a Kan-enriched category forms a fibrant object in $\mathbf{sSet}_{Joyal}$, ie. quasi-category (4.0.20) $M^\infty = N_\Delta(L^H M)^f$.

Remark 2.11.6. [HTT, A.2]

If one is starting with a simplicial model category, there is a construction of the underlying ∞-category as follows :

(i) Restrict to the subcategory of bifibrant objects $M^{cf} \subseteq M$.

(ii) Take the coherent nerve (2.2.14) $N_\Delta(M^{cf})$ to get a quasi-category.

Remark 2.11.7. [HA, 1.3.4.15]

We can describe the underlying ∞-category via universal property as follows: let M be a model category and $M^c \subseteq M$ the subcategory of cofibrant objects (4.0.18).

A functor of ∞-categories

$$N(M^c) \to \mathcal{C}$$

is said to exhibit $\mathcal{C}$ as the $\boxed{\text{underlying } \infty\text{-category}}$ of M if it induces an equivalence

$$N(M^c)[W^{-1}] \simeq \mathcal{C},$$

where $W \subseteq N(M^c)$ is the subcategory spanned by weak equivalences, and the term on the left represents localization of $N(M^c)$ at W (2.14.1).

Proposition 2.11.8. [HA, 1.3.4.20]

Let M be a simplicial model category (2.2.4, 4.0.3). Then the ∞-category $N_\Delta(M^{cf})$ (2.11.6) is the underlying ∞-category of M in the sense of (2.11.7).

Proposition 2.11.9. Taking the homotopy category exhibits $hM^\infty = ho(M) = M[W^{-1}]$, as the usual model-categorical localization.

Proposition 2.11.10. [Hin15, Prop. 1.5.1]

A Quillen adjunction (4.1.2)

$$f : M \rightleftarrows N : g$$

between model categories induces an adjunction (2.10.3)

$$\mathbb{L}f : M^\infty \rightleftarrows N^\infty : \mathbb{R}g$$

between their underlying ∞-categories.

2.12 Yoneda embedding

Definition 2.12.1. [presheaves on ordinary categories]

Let C be an ordinary category.

A $\boxed{\text{presheaf}}$ on a category C is a functor $C^{op} \to \mathbf{Set}$.

The category of presheaves on C is the functor category

$$\mathcal{P}(C) := \mathrm{Fun}(C^{op}, \mathbf{Set}).$$

Remark 2.12.2. We can recognize a category C as a full subcategory of its presheaves by the Yoneda embedding

$$\begin{aligned} j : C &\to \mathcal{P}(C), \\ X &\mapsto \mathrm{Hom}_C(-, X). \end{aligned}$$

The Yoneda lemma says that the functor j is fully faithful.

Remark 2.12.3. The category of presheaves $\mathcal{P}(C)$ is complete and cocomplete — just take limits/colimits of sets, eg. for a collection $\{F_i\}_I$ of presheaves on C, the colimit presheaf is defined by

$$\operatorname{colim}_I F_i : X \mapsto \operatorname{colim}_I F_i(X).$$

This defines a universal property on presheaves: any colimit-preserving functor $F : C \to D$, from a category C to a cocomplete category D will factor through the Yoneda emedding and a colimit-preserving map $\overline{F}$:

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "j"'] & & D \\ & \mathcal{P}(C) \arrow[ur, dotted, "\overline{F}"'] & \end{tikzcd}$$

So forming a presheaf category can be thought of as formally sticking in all limits and colimits to a category.[7]

Definition 2.12.4. [presheaves on simplicial sets]

Let X be a simplicial set. Then the simplicial set of $\boxed{\text{presheaves on } X}$ is the function complex (2.8.2):

$$\mathrm{Fun}(X^{op}, \mathcal{S}).$$

When X is an ∞-category, then $\mathcal{P}(X)$ is an ∞-category. (2.8.3).

Definition 2.12.5. [(∞-) Yoneda embedding] [HTT, 5.1.3]

Let $\mathcal{C}$ be an ∞-category.

The adjoint to the coherent nerve (2.2.12) forms a Kan-enriched category

[7]This sometimes destroys honest limits/colimits one might have had in the original category. To rectify this, one can shift to sheaves, imposing "relations" to recover colimits from the original category as colimits in the presheaf category. See [Dug].

$\mathfrak{C}(\mathcal{C}) \in \mathbf{sCat}$ (2.2.18). There is a simplicial functor

$$\mathfrak{C}(\mathcal{C})^{op} \times \mathfrak{C}(\mathcal{C}) \to \mathbf{Kan}$$
$$(X, Y) \mapsto \mathrm{Hom}_{\mathfrak{C}(\mathcal{C})}(X, Y),$$

where **Kan** is the simplicially enriched category of Kan complexes. Composing with the natural map $\mathfrak{C}(\mathcal{C}^{op} \times \mathcal{C}) \to \mathfrak{C}(\mathcal{C})^{op} \times \mathfrak{C}(\mathcal{C})$ gives a simplicial functor:

$$\mathfrak{C}(\mathcal{C}^{op} \times \mathcal{C}) \to \mathbf{Kan},$$

which corresponds by adjunction $(\mathfrak{C} \dashv N_\Delta)$ (2.2.17) to a map of ∞-categories:

$$\mathcal{C}^{op} \times \mathcal{C} \to N_\Delta(\mathbf{Kan}) = \mathcal{S}.$$

Since **sSet** is cartesian-closed, we can identify this with a map

$$j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) = \mathcal{P}(\mathcal{C}),$$

which we call the $\boxed{\infty\text{-categorical Yoneda embedding}}$.

Proposition 2.12.6. [(∞-) Yoneda lemma] [HTT, 5.1.3.1]

Let $\mathcal{C}$ be an ∞-category. The Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ (2.12.5) is fully-faithful (2.8.13).

Proposition 2.12.7. [HTT, 5.1.3.2]

The Yoneda embedding $j : \mathcal{C}^{op} \to \mathcal{P}(\mathcal{C})$ preserves all small limits.

Remark 2.12.8. [HTT, 5.1.2.4]

Let $\mathcal{C}$ be an ∞-category. Then the ∞-category of presheaves $\mathcal{P}(\mathcal{C})$ is complete and co-complete (it has all small limits and colimits).

Remark 2.12.9. Let X be an object in an ∞-category $\mathcal{C}$.

There is an evaluation map $\mathrm{ev}_X : \mathcal{P}(\mathcal{C}) \to \mathcal{S}$ that sends a presheaf $F \mapsto F(X)$. Composing with the Yoneda embedding j (2.12.5) forms a functor

$$j_X := \mathrm{ev}_X \circ j : \mathcal{C} \to \mathcal{S},$$

which we call the functor corepresented by X. Explicitly, on objects, this sends $Y \in \mathcal{C}_0$ to the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$.

Dually, there is a functor:

$$j^X := \mathrm{ev}_X \circ j : \mathcal{C}^{op} \to \mathcal{S}$$

called the functor represented by X. Explicitly, on objects this sends $Y \in \mathcal{C}_0$ to the mapping space $\mathrm{Map}_{\mathcal{C}}(Y, X)$.

Proposition 2.12.10. [K, 7.4.5.12]

Let X be an object in an ∞-category $\mathcal{C}$.

The functors corepresented and represented by X:

$$h_X : \mathcal{C} \to \mathcal{S}$$
$$h^X : \mathcal{C}^{op} \to \mathcal{S}$$

both preserve I-indexed limits, for any simplicial set I.

Remark 2.12.11. In particular, suppose $F : I \to \mathcal{C}$ was a diagram with a limit $\lim_I F$. Then the proposition (2.12.10) says that there is an equivalence of spaces:

$$h_X(\lim_I F) = \mathrm{Map}_{\mathcal{C}}(X, \lim_I F) \simeq \lim_I \mathrm{Map}_{\mathcal{C}}(X, F(i)).$$

Dually, there is an equivalence of spaces:

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_I F, X) \simeq \lim_I \mathrm{Map}_{\mathcal{C}}(F(i), X).$$

2.13 Presentability, Accessibility

Size issues arise in ordinary category theory (eg. when do certain constructions form proper <u>sets</u>?), leading to the notion of a <u>locally presentable category</u> [GU71]. A standard reference for these in English is Adámek and Rosický's book [AR94]. There is an analagous class of ∞-categories called presentable[8] ∞-categories. For a reference, see [HTT, A.1.1, Ch.5].

Definition 2.13.1. [regular cardinals]

A $\boxed{\text{regular cardinal}}$ is a cardinal α such that given a collection of cardinals $\{\beta_i\}_{i\in I}$ with each $\beta_i < \alpha$ and the cardinality of the indexing set $|I| < \alpha$, then the sum $\sum_{i\in I} \beta_i < \alpha$.

Example 2.13.2. The simplest example of a regular cardinal is the smallest infinite cardinal $\omega = \aleph_0 = |\mathbb{N}|$.[9] In this case the statement above says that a finite sum of finite cardinals is always less than $\aleph_0$.

Definition 2.13.3. [small ordinary categories]

Let α be a regular cardinal.

[8] This is the terminology used in [HTT], which we will stick to. To avoid confusion, some people may still say "locally presentable ∞-categories".

[9] Sometimes people will make a distinction between $\omega = \{0, 1, 2, \ldots\}$ as an <u>ordinal</u> and $\aleph_0 = |\omega|$ as the corresponding <u>cardinal</u>. To us $\omega = \aleph_0$, and we'll use both interchangeably.

A category C is called $\boxed{\alpha\text{-small}}$ if C has fewer than α morphisms; ie. the cardinality $|\mathrm{Hom}(C)| < \alpha$.

(Note that this implies that C has $< \alpha$ objects.)

Definition 2.13.4. [filtered ordinary categories]

A category I is called $\boxed{\alpha\text{-filtered}}$ if for any subset $I_{<\alpha} \subseteq I$ with cardinality $|I_{<\alpha}| < \alpha$, there exists a co-cone of I_α – ie. an element $i \in I$ with morphisms $j \to i$ for all $j \in I_{<\alpha}$ that commute with the morphisms in $I_{<\alpha}$.

An $\boxed{\alpha\text{-filtered diagram}}$ is a diagram $I \to C$ such that the indexing category I is α-filtered.

Definition 2.13.5. [filtered ordinary colimits]

Let C be an ordinary category.

A $\boxed{\alpha\text{-filtered colimit}}$ is the colimit of a diagram $I \to C$, where the diagram category I is α-filtered (2.13.4).

Definition 2.13.6. [compact objects in an ordinary category]

Let C be an ordinary category with small colimits and $x \in C$ be an object. Let α be a regular cardinal.

We say that x is $\boxed{\alpha\text{-compact}}$ if for any α-filtered diagram $I \to C, i \mapsto y_i$ (2.13.4) with colimit $\mathrm{colim}_{i \in I} y_i$, the induced map

$$\mathrm{colim}_I \mathrm{Hom}_C(x, y_i) \xrightarrow{\sim} \mathrm{Hom}_C(x, \mathrm{colim}_I y_i)$$

is a bijection.

We say that x is $\boxed{\text{small}}$ if it is α-compact for some regular cardinal α.

We call x $\boxed{\text{compact}}$ if it's ω-compact.

Definition 2.13.7. [generation under colimits]

Let C be an ordinary category. Let α be a regular cardinal (2.13.1).

A set of objects $G \subseteq \mathrm{ob}(C)$ is said to $\boxed{\text{generate } C \text{ under } \alpha\text{-filtered colimits}}$ if any $x \in C$ can be written as a α-filtered colimit (2.13.5)

$$x = \mathrm{colim}_{i \in I}\, g_i$$

where I is α-filtered (2.13.4) and $g_i \in G$ for all $i \in I$.

This leads to a related, weaker notion of inductive limits:

Definition 2.13.8. [Ind-objects of a category]

We can form a category called $\boxed{\mathrm{Ind}(C)}$, defined as:

- (objects): are Ind-objects, ie. diagrams $X : I \to C$. We may call these such a diagram $[\varinjlim_i X_i]$. This is meant to suggest that Ind-objects are kind of formal colimits.
- (morphisms):

$$\mathrm{Hom}_{\mathrm{Ind}(C)}\left([\varinjlim_i X_i], [\varinjlim_j Y_j] \right) := \varprojlim_i \varinjlim_j \mathrm{Hom}_C(X_i, Y_j).$$

  where the colimit and limit on the right are taken in **Set**.

For an arbitrary regular cardinal α, the category $\mathrm{Ind}_\alpha(C)$ is defined as above, with Ind_α-objects being $\underline{\alpha\text{-filtered diagrams}}$ $I \to C$ (2.13.4).

Remark 2.13.9. Taking the indexing category $I := [0]$ gives us the objects of C as the Ind-objects, and morphisms given by the morphisms in C. That is C can be thought of as a full subcategory $C \hookrightarrow \mathrm{Ind}(C)$. This is kind of like a weaker Yoneda embedding (compare 2.12.2).

Remark 2.13.10. We think of $\mathrm{Ind}(C)$ as a formal cocompletion (along filtered or α-filtered diagrams). The category $\mathrm{Ind}_\alpha(C)$ satisfies a similar universal property as the category of presheaves (2.12.3): given a category D with α-filtered colimits (2.13.5), any α-filtered-colimit preserving functor $C \to D$ factors as

$$\begin{tikzcd} C \arrow[rr] \arrow[dr, hook] & & D \\ & \mathrm{Ind}_\alpha(C) \arrow[ur, dotted] & \end{tikzcd}$$

Now we can talk about a more general notion of accessibility. We broaden the generating set to include α-compact objects, and allow them to generate via α-filtered colimits.

Definition 2.13.11. [accessible 1-categories/functors]

A category C is $\boxed{\alpha\text{-accessible}}$ if it's generated by a set of α-compact (2.13.6) objects and has all α-filtered colimits (2.13.5). That is, $C = \mathrm{Ind}_\alpha(S)$ (2.13.8) for some small category S.

We'll say C is $\boxed{\text{accessible}}$ if it's α-accessible for some α.

If C and D are α-accessible categories, then a functor $F : C \to D$ is called an $\boxed{\alpha\text{-accessible functor}}$ if it preserves all α-filtered colimits.

If our category contains not just α-filtered colimits, but all small colimits (ie. it is cocomplete), we have a stronger (and much nicer) notion of local presentability:

Definition 2.13.12. [locally presentable 1-categories]

A category C is called $\boxed{\text{locally presentable}}$ if it's accessible and cocomplete. In other words:

(i) C is locally small.

(ii) C is cocomplete.

(iii) There's a regular cardinal α and a small set of objects $S \subseteq \mathrm{ob}(C)$ that are α-small and that generate C under α-filtered colimits.

Remark 2.13.13. In the case of $\alpha = \omega$, the above says that C is ω-accessible (ie. finitely accessible) if it's generated by a small set of compact objects and closed under finite colimits.

Remark 2.13.14. Presentability is a stronger version of accessibility: we've strengthened the filtered colimit condition to include all small colimits. Presentable categories are better behaved in a number of ways. A simple example: an α-presentable category is α'-presentable for all $\alpha' > \alpha$, but the analagous statement in terms of accessibility is not true.

A meatier example are the Adjoint Functor Theorem(s). First the "ordinary" adjoint functor theorems:

Proposition 2.13.15. [(General) Adjoint Functor Theorems]

A functor $F : C \to D$ out of a small and complete category C is a

(i) left adjoint iff

- F preserves all colimits.
- F satisfies the <u>solution set condition</u>: for any $d \in D$, there's a *set* I and a family of maps $\{f_i : F(c_i) \to d\}_{i \in I}$ in D such that any map $\varphi : F(c) \to d$ factors as:

$$\begin{tikzcd} F(c) \arrow[rr, "\varphi"] \arrow[dr, "F(g_i)"'] & & d \\ & F(c_i) \arrow[ur, "f_i"'] & \end{tikzcd}$$

  for some $i \in I$ and a $g_i \in \mathrm{Hom}_C(c, c_i)$.

(ii) right adjoint iff

- F preserves all limits.
- F satisfies the solution set condition: for any $d \in D$, there's a *set* I and a family of maps $\{f_i : d \to F(c_i)\}_{i \in I}$ in D such that any map $\varphi : d \to F(c)$ factors as:

$$\begin{tikzcd} d \arrow[rr, "\varphi"] \arrow[dr, "f_i"'] & & F(c) \\ & F(c_i) \arrow[ur, "F(g_i)"'] & \end{tikzcd}$$

for some $i \in I$ and a $g_i \in \mathrm{Hom}_C(c_i, c)$.

The solution set conditions can be interpreted as some kind of smallness condition. The fact that the family $\{f_i\}_{i \in I}$ is indexed by a set I is important here.

The punchline here is that when C and D are both locally presentable, the above reduces to:

Proposition 2.13.16. [(Presentable) Adjoint Functor Theorem]

A functor $F : C \to D$ between locally presentable categories (2.13.12) is a

(i) left adjoint iff it preserves all small colimits;

(ii) right adjoint iff it preserves all small limits and is accessible (ie. preserves α-filtered colimits for some α).

There is an ∞-version of this theorem that is worded exactly the same (HTT 5.5.2.9), but we need to unpack it a little bit.

Definition 2.13.17. [small simplicial sets]

Let α be a regular cardinal (2.13.1).

A simplicial set $X \in \mathbf{sSet}$ is called $\boxed{\alpha\text{-small}}$ if it has $< \alpha$ nondegenerate simplices.

A diagram in an ∞-category $K \to \mathcal{C}$ is called $\underline{\alpha\text{-small}}$ if K is α-small. For example, a colimit of a diagram $K \to \mathcal{C}$ is called an α-small colimit if K is an α-small diagram.

Definition 2.13.18. [filtered ∞-categories & ∞-colimits]

Let $\mathcal{C}$ be an ∞-category.

We say that $\mathcal{C}$ is $\boxed{\alpha\text{-filtered}}$ if for any α-small diagram $p : K \to \mathcal{C}$ (2.13.17), the slice category $\mathcal{C}_{p/}$ (2.9.26) is nonempty. [HTT, 5.3.1.9]

Equivalently, $\mathcal{C}$ is α-filtered if for any α-small simplicial set K, a map $K \to \mathcal{C}$ extends to a map $K^{\triangleright} \to \mathcal{C}$ (2.9.22). [HTT, 5.3.1.7]

We'll call $\mathcal{C}$ filtered if its ω-filtered.

We can generalize inductive limits/colimits (2.13.8) to ∞-categories:

Definition 2.13.19. [(∞-) inductive limits]

An $\boxed{\text{Ind-object}}$ of an ∞-category $\mathcal{C}$ is a small filtered diagram $X : I \to \mathcal{C}$.

We can collect these into an ∞-category $\boxed{\mathrm{Ind}(\mathcal{C})}$, where morphism spaces are defined analagously as before; for $(X : I \to \mathcal{C}), (Y : J \to \mathcal{C})$ in $\mathrm{Ind}(\mathcal{C})$,

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(X, Y) := \varprojlim_I \varinjlim_J \mathrm{Map}_{\mathcal{C}}(X_i, Y_j),$$

where the limit and colimit on the RHS are co/limits in the ∞-category of spaces $\mathcal{S}$.

This gives us a notion of accessibility (compare to 2.13.11):

Definition 2.13.20. [accessible ∞-categories/functors]

An ∞-category $\mathcal{C}$ is $\boxed{\alpha\text{-accessible}}$ if there exists a small ∞-category $\mathcal{C}^0$ and an equivalence

$$\mathrm{Ind}_\alpha(\mathcal{C}^0) \to \mathcal{C}$$

We say that $\mathcal{C}$ is accessible if it's α-accessible for some α.

When $\alpha = \omega$, we say that $\mathcal{C}$ is finitely accessible if it is ω-accessible.

A theorem of Carlos Simpson gives a list of equivalent conditions for an ∞-category to be presentable, which we can take as a definition.

Definition 2.13.21. [presentable ∞-category] [HTT, 5.5.1.1, Simpson]

The following are equivalent for an ∞-category $\mathcal{C}$:

(1) $\mathcal{C}$ is $\boxed{\text{presentable}}$.

(2) $\mathcal{C}$ is accessible, and for every regular cardinal α, the full subcategory $\mathcal{C}^\alpha \subseteq \mathcal{C}$ (of α-compact objects) admits α-small colimits.

(3) There exists some regular cardinal α such that $\mathcal{C}$ is α-accessible and $\mathcal{C}^\alpha$ admits α-small colimits.

(4) There's an equivalence $\mathcal{C} \simeq \mathrm{Ind}_\alpha(\mathcal{C}_0)$, where $\mathcal{C}_0$ is a small ∞-category admitting α-small colimits.

(5) $\mathcal{C}$ is the accessible localization of a presheaf category $\mathcal{P}(\mathcal{D})$ for some small ∞-category $\mathcal{D}$.

(6) $\mathcal{C}$ is locally small, admits small colimits, and there's a regular cardinal α and a small set of α-compact objects $S \subseteq \mathcal{C}^\alpha$ such that any object $x \in \mathcal{C}$ can be written as a colimit of a small diagram landing in the full subcategory $\overline{S} \subseteq \mathcal{C}$ spanned by S.

Definition 2.13.22. [compactly generated ∞-categories]

Let $\mathcal{C}$ be an ∞-category and let α be a regular cardinal (2.13.1).

We say that $\mathcal{C}$ is $\boxed{\alpha\text{-compactly generated}}$ if it is:

- presentable (2.13.21); and
- α-accessible (2.13.20).

Equivalently, $\mathcal{C}$ is α-compactly generated iff there is a small ∞-category $\mathcal{C}^0$ which admits α-small colimits (2.13.17), and there is an equivalence

$$\mathcal{C} \simeq \mathrm{Ind}_\alpha(\mathcal{C}^0)$$

(2.13.19, 2.13.20).

When $\alpha = \omega$, we say that $\mathcal{C}$ is $\boxed{\text{compactly generated}}$. In other words, $\mathcal{C}$ is compactly generated if it is:

- presentable; and
- finitely accessible.

Equivalently,$\mathcal{C}$ is compactly generated iff there is a small ∞-category $\mathcal{C}^0$ which admits finite colimits, and there is an equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^0)$.

Proposition 2.13.23. [(Presentable) Adjoint Functor Theorem] [HTT, 5.5.2.9]

A functor $F : \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories is a

(i) left adjoint iff it preserves all small colimits;

(ii) right adjoint iff it preserves all small limits and is accessible (2.13.20) (ie. preserves α-filtered colimits for some α).

Another nice quality of presentable ∞-categories is that it is straightforward to tell if a presheaf on a presentable category is representable:

Proposition 2.13.24. [HTT, 5.5.2.2]

If $\mathcal{C}$ is a presentable ∞-category, then a presheaf $F : \mathcal{C}^{op} \to \mathcal{S}$ is representable iff it preserves small limits.

Finally, an important characterization of presentable ∞-categories as essentially equivalent to combinatorial model categories:

Proposition 2.13.25. [Cis19, Thm. 7.11.16, Rk. 7.11.17], [HTT, A.3.7.6]

An ∞-category $\mathcal{C}$ is presentable (2.13.21) iff it underlies a combinatorial (4.4.1) model category M; ie. there's an equivalence $\mathcal{C} \simeq M^{\infty}$, where M^{∞} is the underlying ∞-category (2.11.5) of M.

Remark 2.13.26. See also [Pav21] for the stronger statement that an ∞-category of presentable ∞-categories is (∞-) equivalent to an ∞-category of combinatorial model categories.

2.14 Localization

Localization is crucial in studying homotopy theories. We can localize a category with weak equivalences by formally turning them into isomorphisms (4.0.1). Given a homotopy theory, we can also localize further, with respect to some morphisms which are not already weak equivalences. There exists an analagous processes in ∞-category theory. In this section we lay these out and compare them.

Definition 2.14.1. [localization via universal property] [HA, 1.3.4.1]

Let $\mathcal{C}$ be an ∞-category and $W \subseteq \mathcal{C}_1$ some collection of morphisms.

There exists an ∞-category $\mathcal{C}[W^{-1}]$ along with a functor $f : \mathcal{C} \to \mathcal{C}[W^{-1}]$ that exhibits $\boxed{\mathcal{C}[W^{-1}]}$ as the ∞-category obtained from $\mathcal{C}$ by inverting morphisms in W. This means that precomposing with f induces a fully faithful embedding (2.8.13)

$$- \circ f : \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

for any ∞-category $\mathcal{E}$, and the essential image of $- \circ f$ is functors $\mathcal{C} \to \mathcal{E}$ sending all maps in W to equivalences in $\mathcal{E}$.

Remark 2.14.2 (HA, 1.3.4.2)**.** One can construct $\mathcal{C}[W^{-1}]$ by considering $(\mathcal{C}, W)$ as a "marked simplicial set". A marked simplicial set is a simplicial set with some edges identified. These form a category that Lurie denotes $\mathbf{sSet}^+$. One can give this a model structure and realize such a localization as a fibrant replacement.

Theorem 2.14.3. [Cis19, Thm. 7.9.8]

Let C be a model category and let $\mathcal{C} = NC$ be its nerve. The collection of weak equivalences in C forms a collection of edges $W \subseteq \mathcal{C}_1$. Let I be a diagram category.

There is an equivalence of ∞-categories:

$$\mathrm{Fun}(NI, \mathcal{C})[\overline{W}^{-1}] \xrightarrow{\sim} \mathrm{Fun}(NI, \mathcal{C}[W^{-1}])$$

where $\overline{W} \subseteq \mathrm{Fun}(NI, \mathcal{C})_1$ is the collection of "fiberwise weak equivalences"[10]:

$$\begin{aligned} \overline{W} := &\left\{ f : NI \times \Delta^1 \to \mathcal{C} \ : \ f(i \times \Delta^1) \in W \text{ for all } i \in I \right\} \\ &\subseteq \mathrm{Fun}(NI, \mathcal{C})_1 \end{aligned}$$

[10]This is analagous to how for a model category M, the category of functors M^I has an induced model structure.

That is, localizing the functor category

$$\mathrm{Fun}(NI, \mathcal{C}) \to \mathrm{Fun}(NI, \mathcal{C})[(\overline{W})^{-1}]$$

with respect to the fiberwise weak equivalences is the same as considering functors from NI into the localization $\mathcal{C}[W^{-1}]$.

2.14.1 Bousfield localization

Categories obtained as localizations in the sense of (2.14.1) are not always so easy to get a handle on. For example, it may happen that $\mathcal{C}$ is locally small but $\mathcal{C}[W^{-1}]$ is not. In certain cases, we can get a nicer description of $\mathcal{C}[W^{-1}]$ as a full subcategory sitting inside of $\mathcal{C}$. This is called reflective localization or Bousfield localization.

Definition 2.14.4. [reflective/Bousfield localization of ∞-categories]

A functor $l : \mathcal{C} \to \mathcal{D}$ between ∞-categories is a $\boxed{\text{reflective localization}}$ (or Bousfield localization) if it admits a fully-faithful right adjoint r.

Remark 2.14.5. Given a reflective localization $(l, r) : \mathcal{C} \rightleftarrows \mathcal{D}$ as above, we consider the localization as the subcategory formed by the essential image $r(\mathcal{D}) \subseteq \mathcal{C}$.

Remark 2.14.6. We will usually call $L := r \circ l : \mathcal{C} \to \mathcal{C}$ a localization functor. For an ordinary category C we may call $L(C) := L(NC)$ the localization of the nerve.

Definition 2.14.7. [accessible localization] [HTT, 5.5.1.2]

A localization

$$L : \mathcal{C} \to \mathcal{C}$$

is called $\boxed{\text{accessible}}$ if either of the following (equivalent) conditions hold:

(1) L is an accessible functor (preserves κ-filtered colimits for some κ).

(2) The essential image $L(\mathcal{C}) \subseteq \mathcal{C}$ is an accessible subcategory.

(3) There's an equivalence $L \simeq r \circ l$, where (l, r) form an adjoint pair:

$$l : \mathcal{C} \rightleftarrows \mathcal{D} : r,$$

and $\mathcal{D}$ is accessible.

Remark 2.14.8. If $(l, r) : \mathcal{C} \rightleftarrows \mathcal{D}$ is a reflective localization, then $l : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as a localization $\mathcal{D} = \mathcal{C}[S^{-1}]$ of $\mathcal{C}$ by the class

$$S = \{f \in \mathcal{C}_1 \text{ sent to equivalences under } l\}.$$

In [HTT], this type of localization is called simply a localization. While reflective/Bousfield localization is a localization in the sense of inverting some morphisms, the converse is not true. That is, not every localization can be described as a reflective localization; ie. not every $\mathcal{C} \to \mathcal{C}[S^{-1}]$ admits a fully-faithful right adjoint.

There are partial things in the opposite direction, assuming conditions on $\mathcal{C}$ and on the collection $S \subseteq \mathcal{C}_1$.

Definition 2.14.9. [S-local objects]

Let $\mathcal{C}$ be an ∞-category, and $S \subseteq \mathcal{C}_1$ a collection of morphisms.

An object $Z \in \mathcal{C}$ is called $\boxed{S\text{-local}}$ if for any map $s : X \to Y$ in S, precomposing induces an equivalences of spaces:

$$- \circ s : \mathrm{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(X, Z).$$

We can collect all S-local objects into a full subcategory we call $\boxed{S^{-1}\mathcal{C}}$.

A map $f : X \to Y$ is called an $\boxed{S\text{-equivalence}}$ if precomposing induces an equivalence $\mathrm{Map}_{\mathcal{C}}(Y,Z) \to \mathrm{Map}_{\mathcal{C}}(X,Z)$ as above, for any $Z \in S^{-1}\mathcal{C}$.

Definition 2.14.10. [strong saturation]

Let $\mathcal{C}$ be an ∞-category with small colimits, and $S \subseteq \mathcal{C}_1$ a collection of morphisms.

The collection $\mathcal{S}$ is $\boxed{\text{strongly saturated}}$ if it satisfies the following:

(i) S is closed under pushouts.

(ii) The full subcategory $S' \subseteq \mathcal{C}^{\Delta^1}$ spanned by S is closed under small colimits.

(iii) S satisfies a 2-out-of-3 property.

Remark 2.14.11. Given a collection S in an ∞-category with small colimits, one can always find a smallest saturated collection containing S – we will usually call such a set the saturated closure, denoted $\overline{S}$.

If S is a small collection (a set), then we say that $\overline{S}$ is $\boxed{\text{of small generation}}$.

Proposition 2.14.12. [HTT, 5.5.4.15]

Let $\mathcal{C}$ be a presentable ∞-category, $S \subseteq \mathcal{C}_1$ a small collection of morphisms, and $\overline{S}$ the saturated set generated by S.

(i) The full subcategory $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ of S-local objects is a Bousfield localization $\mathcal{C} \to L\mathcal{C}$.

(ii) $S^{-1}\mathcal{C}$ is presentable.

(iii) There is an equivalence:

$$S^{-1}\mathcal{C} \simeq \mathcal{C}[(\overline{S})^{-1}],$$

where the left is Bousfield localization with respect to S, and the right is ordinary localization with the saturated closure $\overline{S}$, which is precisely $\overline{S} = \{f : Lf \text{ is an equivalence}\}$.

3 Stable ∞-categories and triangulated categories

> *We worked on this book with the disquieting feeling that the development of homological algebra is currently in a state of flux, and that the basic definitions and constructions of the theory of triangulated categories, despite their widespread use, are of only preliminary nature (this applies even more to homotopic algebra).*
>
> Sergei I. Gelfand, Yuri I. Manin,
> "Methods of Homological Algebra" [GM11]

3.1 Stable ∞-categories

There are special homotopy theories called stable homotopy theories. The prototypical example of a stable homotopy theory is the stable homotopy category of spectra **SHC**. As ∞-categories present homotopy theories, there is a notion of a stable ∞-category which presents a stable homotopy theory. This section will be devoted to the theory of stable ∞-categories. The primary source for this is Chapter 1 of [Lur17].

Definition 3.1.1. [zero objects, pointed ∞-categories]

An object in an ∞-category is called a $\boxed{\text{zero object}}$ if it is both initial and terminal (2.9.30).

An ∞-category is called $\boxed{\text{pointed}}$ if it has a zero object.

Definition 3.1.2. [HTT, 7.2.2.1]

Let $\mathcal{C}$ be an ∞-category with a final object $* \in \mathcal{C}_0$.

The $\boxed{\text{category of pointed objects}}$ of $\mathcal{C}$ is the subcategory

$$\mathcal{C}_* \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$$

spanned by edges $\Delta^1 \to \mathcal{C}$ of the form $(* \to X)$.

Note that $\mathcal{C}_*$ is a pointed category (3.1.1), with zero object given by the identity $(* \xrightarrow{\mathrm{id}} *)$.

Definition 3.1.3. [pointed functors]

A functor between pointed ∞-categories is called a pointed functor if it preserves zero objects. That is $f : \mathcal{C} \to \mathcal{D}$ is pointed iff $f(0_{\mathcal{C}})$ is a zero object in $\mathcal{D}$.

Remark 3.1.4. An initial object (resp. a final object) in an ∞-category can be described as a colimit (resp. a limit) of an empty diagram (2.9.30). As limits and colimits are unique up to equivalence, zero objects are unique up to equivalence.

Remark 3.1.5. Let $\mathcal{C}$ be an ∞-category. A zero object $0 \in \mathcal{C}$ in the sense of (3.1.1) is a zero object (in the ordinary sense) in the homotopy category $h\mathcal{C}$.

Proof. This follows immediately from (2.9.32). □

Remark 3.1.6. Note that the converse is <u>not true</u>, since initial/final objects in the homotopy category may not be initial/final in the ∞-category (2.9.33).

Definition 3.1.7. [co/fiber sequences in ∞-categories]

Let $\mathcal{C}$ be an ∞-category.

A $\boxed{\text{triangle}}$ in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
0 \arrow[r] & \bullet
\end{tikzcd}$$

where 0 is a zero object of $\mathcal{C}$.

A $\boxed{\text{fiber sequence}}$ in $\mathcal{C}$ is a triangle that forms a pullback square in $\mathcal{C}$. A $\boxed{\text{cofiber sequence}}$ in $\mathcal{C}$ is a triangle that forms a pushout square. Note these are pullbacks and pushouts in the sense of (2.9.41).

Remark 3.1.8. We draw a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ as a square as in the definition above, but it should be noted that we are hiding some information. A more accurate picture of a triangle that includes all non-degenerate simplices might look like:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d] \arrow[dr, "h"] & \bullet \arrow[d, "g"] \arrow[dl, phantom, "\Leftarrow"] \\
0 \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & \bullet
\end{tikzcd}$$

That is, a triangle consists of:

- A pair of maps, along with a 2-simplex witnessing composition:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dr, "h"'] & \bullet \arrow[d, "g"] \arrow[dl, phantom, "\Leftarrow"] \\
 & \bullet
\end{tikzcd}$$

- A 2-simplex witnessing a "null-homotopy" of the map h:

$$\begin{tikzcd}
\bullet \arrow[d] \arrow[dr, "h"] & \\
0 \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & \bullet
\end{tikzcd}$$

Such a diagram corresponds to an ordinary commutative square $[1] \times [1] \to h\mathcal{C}$ in the homotopy category.

A pushout (resp. pullback) square in an ∞-category corresponds to a "homotopy pushout" (resp. homotopy pullback) (4.2.1) in the homotopy category. Pushing out along a zero map means that this corresponds to a "homotopy cokernel" (resp. homotopy kernel), ie. a "weak cokernel" (resp. weak kernel).

Definition 3.1.9. Let $\mathcal{C}$ be an ∞-category with a morphism $f \in \mathcal{C}_1$.

A $\boxed{\text{fiber}}$ of f is a fiber sequence in $\mathcal{C}$ (3.1.7) of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, "f"] \\ 0 \arrow[r] & \bullet \end{tikzcd}$$

Recall a fiber sequence is a pullback square. We may use the term "fiber of f" when referring to the pullback object in the square above, and denote the object $\mathrm{fib}(f)$.

A $\boxed{\text{cofiber}}$ of f is a fiber sequence in $\mathcal{C}$ of the form:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d] & \bullet \arrow[d] \\ 0 \arrow[r] & \bullet \end{tikzcd}$$

Similarly to the fiber, we may refer to the pushout object in the square above as the "cofiber of f", denoted $\mathrm{cofib}(f)$.

Definition 3.1.10. [HA, 1.1.1.9]

[stable ∞-categories]

An ∞-category $\mathcal{C}$ is called $\boxed{\text{stable}}$ if

(i) it's pointed (3.1.1), ie. has a zero object $0 \in \mathcal{C}_0$;

(ii) every map $f \in \mathcal{C}_1$ admits a fiber and cofiber (3.1.9);

(iii) fiber sequences are the same as cofiber sequences (3.1.7).

Remark 3.1.11. The final condition that fiber and cofiber sequences are the same is precisely what makes the stable homotopy category **SHC** stable. The definition of a stable ∞-category axiomatizes this stable behavior.

Example 3.1.12. [examples of stable ∞-categories]

(a) Starting from an abelian category A, [HA, 1.3] constructs a construction of a stable ∞-category called $\mathcal{D}(A)$, whose homotopy category is the usual derived category of A. [HA, 1.3]

(b) In [HA, 1.4] there is constructed a stable ∞-category of spectra, **Sp**, whose homotopy category is the stable homotopy category **SHC** of stable homotopy theory.

3.1.1 Functoriality of cofibers, loops and suspensions

Let $\mathcal{C}$ be a stable ∞-category (3.1.1), and let $f \in \mathcal{C}_1$ be a morphism. We defined a cofiber of f (3.1.9) as a (∞-) pushout:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d] & \bullet \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \end{tikzcd}$$

We can make this construction functorial. That is, we construct a functor of ∞-categories

$$\mathrm{cofib} : \mathcal{C}^{\Delta^1} \to \mathcal{C},$$

which takes an edge $f \in \mathcal{C}_1$ to the cofiber $\mathrm{cofib}(f)$.

First we will the following statement, [HTT, 4.3.2.15], which is used regularly and unceremoniously throughout the rest of [HTT] and [HA]. The statement in [HTT] is written more generally than we need, so we present a simplified version.

Proposition 3.1.13. [HTT 4.3.2.15] (simplified)

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and $\mathcal{C}^0 \subseteq \mathcal{C}$ a full subcategory. Let $\mathcal{K} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory

$$\mathcal{K} := \{\text{functors } F : \mathcal{C} \to \mathcal{D} \text{ that are left Kan extensions of functors } F|_{\mathcal{C}^0}\},$$

(respectively one can take right Kan extensions) and $\mathcal{K}^0 \subseteq \mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$ the full subcategory of functors $F^0 : \mathcal{C}^0 \to \mathcal{D}$ such that for any $x \in \mathcal{C}$, the induced diagram

$$\mathcal{C}^0_{/x} \to \mathcal{D}$$

has a colimit (resp. $\mathcal{C}^0_{x/} \to \mathcal{D}$ has a limit). (Here $\mathcal{C}^0_{/x}$ is the slice category restricted to the subcategory $\mathcal{C}^0$ – if you like, the pullback $\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}^0$.)

Then the restriction $\mathcal{K} \to \mathcal{K}^0$ is a trivial fibration (2.7.1) of simplicial sets. In other words, its fibers are contractible Kan complexes. That is, we can extend such a functor from a subcategory $\mathcal{C}^0$ to a functor on the whole category $\mathcal{C} \to \mathcal{D}$, and such an extension is unique up to homotopy (which is the right notion of uniqueness in ∞-categories).

The dual version

Remark 3.1.14. [HA, 1.1.1.7], [Gar12]

Let $\mathcal{D}$ be a stable ∞-category.

We construct a functor $\mathrm{cofib} : \mathcal{D}^{\Delta^1} \to \mathcal{D}$ that sends a map

$$(f : X \to Y) \in (\mathcal{D}^{\Delta^1})_0 = \mathcal{D}_1$$

to its cofiber $\mathrm{cofib}(f)$ (3.1.9).

We'll do this using (3.1.13) twice:

(1) Take $\mathcal{C}^0 := \Delta^1$ and $\mathcal{C} := \Delta^1 \cup_{\Delta^{\{0\}}} \Delta^1$. Then we will Kan extend edges $\Delta^1 \to \mathcal{D}$ to diagrams $\Delta^1 \cup_{\Delta^{\{0\}}} \Delta^1 \to \mathcal{D}$ of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ 0_{\mathcal{D}} & \end{tikzcd}$$

where $0_{\mathcal{D}} \in \mathcal{D}$ is a zero object.

In this case,

$$\begin{aligned} \mathcal{K} &= \left\{ \text{diagrams } \Delta^1 \cup_{\Delta^{\{0\}}} \Delta^1 \to \mathcal{D} \text{ as above that are Kan extensions} \right\} \\ \mathcal{K}^0 &= \left\{ \text{edges in } \mathcal{D} \text{ such that } (\Delta^1)_{x/} \to \mathcal{D} \text{ has a limit for any } x \in \Delta^1 \cup_{\Delta^{\{0\}}} \Delta^1 \right\} \end{aligned}$$

In this case $\mathcal{C}$ only has three objects (the two objects of $\mathcal{C}^0$ and the object mapping to $0_{\mathcal{D}}$), and diagrams $\mathcal{C}^0_{x/} = \Delta^1_{x/} \to \mathcal{D}$ are 2-simplices of $\mathcal{D}$. Since $\mathcal{D}$ is stable, it has all finite limits, so $\mathcal{K}^0 = \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) = \mathcal{D}_1$ is simply the collection of all edges of $\mathcal{D}$.

The theorem then says that the restriction $\mathcal{K} \to \mathcal{K}^0$ is a trivial fibration. In other words, we can extend an edge $(\bullet \to \bullet) \in \mathcal{D}$ to a diagram $(0_{\mathcal{D}} \leftarrow \bullet \to \bullet) \in \mathcal{D}$, and this extension is unique (up to homotopy).

(2) Kan extending diagrams $\mathcal{K}^0 := \left\{\Delta^1 \cup_{\Delta^{\{0\}}} \Delta^1 \to \mathcal{D}\right\}$ of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ 0_{\mathcal{D}} & \end{tikzcd}$$

to diagrams $\mathcal{K} = \left\{\Delta^1 \times \Delta^1 \to \mathcal{D}\right\}$ of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ 0_{\mathcal{D}} \arrow[r] & \bullet \end{tikzcd}$$

Left Kan extensions of this type are colimits, ie. pushout squares. In other words, $\mathcal{K}^0$ consists of diagrams that want to be completed to cofiber sequences (3.1.7), and $\mathcal{K}$ consists of those cofiber sequences. Cofibers always exist in a stable ∞-category, so the restriction $\mathcal{K} \to \mathcal{K}^0$ is also a trivial fibration. That is, these homotopy cofibers are unique up to homotopy.

These trivial fibrations compose to a trivial fibration

$$\left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ 0_{\mathcal{D}} \arrow[r] & \bullet \end{tikzcd}\right\} \to \left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ 0_{\mathcal{D}} & \end{tikzcd}\right\} \to \{\bullet \to \bullet\} = \mathcal{D}^{\Delta^1}$$

In other words, we can extend an edge in $\mathcal{D}$ to an exact triangle, which is unique up to homotopy. Let $s : \mathcal{D}^{\Delta^1} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{D})$ be a section representing such an extension. And let i be the inclusion:

$$i : \Delta^{\{1\}} \times \Delta^{\{1\}} \hookrightarrow \Delta^1 \times \Delta^1$$

Pulling back by i forms a functor

$$i^* : \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{D}) \to \mathrm{Fun}(\Delta^0, \mathcal{D}) \simeq \mathcal{D}$$

that "evaluates" a square in $\mathcal{D}$ by sending it to the bottom-right object. In particular, it sends an exact triangle diagram to the cofiber object.

Composing these gives us our functor

$$\mathrm{cofib} = \mathrm{ev} \circ s : \mathcal{D}^{\Delta^1} \to \mathcal{D}.$$

Remark 3.1.15. Using the dual version of (3.1.13), one can construct a similar functor

$$\mathrm{fib} : \mathcal{D}^{\Delta^1} \to \mathcal{D}$$

that forms fibers (3.1.9).

Remark 3.1.16. [HA, 1.1.2.6]

[suspension and loops functors on a pointed ∞-category]

Let $\mathcal{C}$ be a pointed ∞-category (3.1.10) with fibers and cofibers (3.1.9).

Let $M^\Sigma \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be the subcategory spanned by pushout squares of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0' \arrow[r] & \bullet \end{tikzcd}$$

with 0 and $0'$ both zero-objects in $\mathcal{C}$. In other words, M^Ω consists of cofibers of 0-maps. The same argument as in (3.1.14) shows $M^\Sigma \to \mathcal{C}$ to be a trivial fibration. Let $s : \mathcal{C} \to M^\Sigma$ be a section and $e : M^\Sigma \to \mathcal{C}$ be evaluation at the final vertex (the object Y in the diagram above). We call the composite functor the

$\boxed{\text{suspension functor}}$ $\Sigma : \mathcal{C} \to \mathcal{C}$.

Similarly, taking $M^\Omega \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ to be the subcategory spanned by fibers of 0-maps, ie. pullback squares of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & \bullet \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

lets us define a $\boxed{\text{loops functor}}$ $\Omega : \mathcal{C} \to \mathcal{C}$.

For example, the suspension and loops of an object $X \in \mathcal{C}_0$ are described by pushout and pullback squares respectively of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 0 \arrow[d] \\ 0' \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & X \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

3.1.2 Stabilization

Given an ∞-category $\mathcal{C}$, there is a way to construct a stable ∞-category, called the stabilization of $\mathcal{C}$, and denoted $\mathbf{Sp}(\mathcal{C})$.

This section will lay out the process of constructing the stabilization of an ∞-category. The ∞-category of spectra $\mathbf{Sp}$ then can be constructed as the stabilization of the ∞-category $\mathcal{S}$ of spaces (2.5.1). A reference for this section is [HA, 1.4].

Definition 3.1.17. [excisive functors]

Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.

Then f is called $\boxed{\text{excisive}}$ if it takes pushouts to pullbacks. That is, if $\alpha : \Delta^1 \times \Delta^1 \to \mathcal{C}$ is a pushout square (2.9.41), the diagram $f \circ \alpha : \Delta^1 \times \Delta^1 \to \mathcal{D}$ is a pullback square.

Remark 3.1.18. Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories. Define the following subsets of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$:

$$\begin{aligned}\mathrm{Fun}_*(\mathcal{C},\mathcal{D}) &= \{\text{pointed functors } \mathcal{C}\to\mathcal{D}\}\\ \mathrm{Exc}(\mathcal{C},\mathcal{D}) &= \{\text{excisive functors } \mathcal{C}\to\mathcal{D}\}\\ \mathrm{Exc}_*(\mathcal{C},\mathcal{D}) &= \{\text{pointed, excisive functors } \mathcal{C}\to\mathcal{D}\}\end{aligned}$$

Definition 3.1.19. [HA, 1.4.2.5]

Recall the ∞-category of spaces $\mathcal{S}$ (2.5.1).

There is a final object in $\mathcal{S}$ given by the point Δ^0. Let $\mathcal{S}_*$ be the ∞-category of pointed objects of $\mathcal{S}$ (3.1.2). Explicitly, $\mathcal{S}_*$ is the full subcategory of $\mathrm{Fun}(\Delta^1,\mathcal{S})$ spanned by morphisms of the form $(\Delta^0\to X)$.

Let $\mathcal{S}^{fin}\subseteq\mathcal{S}$ be the smallest full subcategory that contains the final object $*$ and is closed under finite colimits. We will call $\mathcal{S}^{fin}$ the ∞-category of finite spaces.

Let $\mathcal{S}_*^{fin}\subseteq\mathcal{S}_*$ be the ∞-category of pointed objects of $\mathcal{S}^{fin}$ (that is, the subcategory of $\mathrm{Fun}(\Delta^1,\mathcal{S})$ spanned by morphisms of the form $(*\to X)$, where X is a finite space.

For each $n\in\mathbb{N}$, we call $S^n\in\mathcal{S}_*$ a representative for the pointed n-sphere. For example, using the description $\mathcal{S}=N_\Delta(\mathbf{Kan})$ (2.5.1), we can take the n-sphere to be the Kan complex formed by the singular complex (2.1.15) of the ordinary n-sphere in **Top**.

Definition 3.1.20. [spectrum objects]

A $\boxed{\text{spectrum object}}$ of an ∞-category $\mathcal{C}$ with finite limits is a pointed, excisive functor $\mathcal{S}_*^{fin}\to\mathcal{C}$.

The collection of spectrum objects in $\mathcal{C}$ forms an ∞-category

$$\mathbf{Sp}(\mathcal{C}) := \mathrm{Exc}_*(\mathcal{S}_*^{fin}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{fin}, \mathcal{C}).$$

Remark 3.1.21. One way of describing a spectrum object that is useful is as a functor $X : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$ such that:

- $X(i,j) = 0$ whenever $i \neq j$;
- For each i, the square

  $$\begin{tikzcd} X(i,i) =: X_i \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X_{i+1} \end{tikzcd}$$

  is a pullback in $\mathcal{C}$ (ie. $X_i \simeq \Omega X_{i+1}$).

Proposition 3.1.22. [HA, 1.4.2.16]

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, with $\mathcal{C}$ admitting finite limits and $\mathcal{D}$ admitting finite colimits.

Then $\mathrm{Exc}_*(\mathcal{D}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is a <u>stable ∞-category</u> (3.1.10).

In particular, taking $\mathcal{D} = \mathcal{S}_*^{fin}$ (3.1.19), then $\mathrm{Exc}_*(\mathcal{S}_*^{fin}, \mathcal{C}) = \mathbf{Sp}(\mathcal{C})$ is stable.

Remark 3.1.23. [HA, 1.4.2.18]

Let $\mathcal{C}$ be an ∞-category with finite limits, and let $\mathcal{C}_*$ be its category of pointed objects (3.1.2). The forgetful functor $\mathcal{C}_* \to \mathcal{C}$ induces an equivalence of ∞-categories $\mathbf{Sp}(\mathcal{C}_*) \to \mathbf{Sp}(\mathcal{C})$.

Definition 3.1.24. [Ω^∞ functor]

Let $\mathcal{C}$ be a pointed ∞-category, and $\mathbf{Sp}(\mathcal{C})$ be the category of spectrum objects.

We define a functor

$$\Omega^\infty : \mathbf{Sp}(\mathcal{C}) \to \mathcal{C}$$

sending an object in $\mathbf{Sp}(\mathcal{C})$ (a functor $f : \mathcal{S}_*^{fin} \to \mathcal{C}$) to the functor evaluated at the 0-sphere $f(S^0) \in \mathcal{C}$.

Proposition 3.1.25. [HA, 1.4.2.21]

Let $\mathcal{C}$ be an ∞-category admitting finite limits.

Then $\mathcal{C}$ is stable iff the functor $\Omega^\infty : \mathbf{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories.

Proposition 3.1.26. [HA, 1.4.2.23]

Let $\mathcal{C}$ be an ∞-category with finite limits. We can describe $\mathbf{Sp}(\mathcal{C})$ with the following universal property.

Let $\mathcal{D}$ be a stable ∞-category (3.1.10), and define the subcategories:

$$\mathrm{Fun}'(\mathcal{D}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{D}, \mathcal{C})$$
$$\mathrm{Fun}'(\mathcal{D}, \mathbf{Sp}(\mathcal{C})) \subseteq \mathrm{Fun}(\mathcal{D}, \mathbf{Sp}(\mathcal{C}))$$

to be those spanned by functors which preserve finite limits. Then composition with Ω^∞ forms an equivalence of ∞-categories:

$$\Omega^\infty \circ - : \mathrm{Fun}'(\mathcal{D}, \mathbf{Sp}(\mathcal{C})) \xrightarrow{\sim} \mathrm{Fun}'(\mathcal{D}, \mathcal{C}).$$

That is, given a stable ∞-category $\mathcal{D}$ and a fiber-preserving (3.1.9) map $\mathcal{D} \to \mathcal{C}$, it factors uniquely through $\mathbf{Sp}(\mathcal{C})$:

$$\begin{tikzcd} & \mathbf{Sp}(\mathcal{C}) \arrow[d, "\Omega^\infty"] \\ \mathcal{D} \arrow[ur, dotted, "\exists!"] \arrow[r] & \mathcal{C} \end{tikzcd}$$

Informally, $\mathbf{Sp}(\mathcal{C})$ is the closest stable ∞-category to $\mathcal{C}$.

Proposition 3.1.27. [HA 1.4.2.24]

Let $\mathcal{C}$ be a pointed ∞-category admitting finite limits. Then $\Omega^\infty : \mathbf{Sp}(\mathcal{C}) \to \mathcal{C}$ can be lifted to an equivalence of ∞-categories:

$$\mathbf{Sp}(\mathcal{C}) \simeq \lim(\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C}).$$

Example 3.1.28. [the ∞-category of spectra $\mathbf{Sp}$]

The $\boxed{\infty\text{-category of spectra}}$ is the stabilization (3.1.20) of the ∞-category of spaces $\mathcal{S}$. We denote it

$$\mathbf{Sp} := \mathbf{Sp}(\mathcal{S}).$$

An object in $\mathbf{Sp}$ is a pointed, excisive functor $X : \mathcal{S}_*^{fin} \to \mathcal{S}$. Denote $X_i := X(S^i)$ for each $i \in \mathbb{N}$. By excision (3.1.17) X sends the pushout square on the left to a pullback square on the right:

$$\begin{tikzcd} S^i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d] \\ * \arrow[r] & \Sigma S^i \simeq S^{i+1} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} X_i \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X_{i+1} \end{tikzcd}$$

The square on the right being a pullback square forms an equivalence $X_i \simeq \Omega X_{i+1}$. This is precisely the data of an Ω-spectrum.

The functor $\Omega^\infty : \mathbf{Sp} \to \mathcal{S}$ (3.1.24) sends a spectrum X to $X(S^0) = X_0$.

3.1.3 As spectrally enriched

One perspective of an ∞-category is as a category weakly enriched in spaces (2.2.18). Spectra are the stable analogue of spaces (3.1.28), and accordingly there is an analagous perspective on stable ∞-categories, as categories weakly enriched in spectra.

Definition 3.1.29. [BGT13, Def. 2.15]

Let $\mathcal{C}$ be a stable ∞-category.

The $\boxed{\text{spectral Yoneda embedding}}$ is the composite:

$$\mathcal{C} \simeq \mathbf{Sp}(\mathcal{C}_*) \to \mathbf{Sp}(\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})_*) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Sp}).$$

Here the equivalence

$$\mathcal{C} \simeq \mathbf{Sp}(\mathcal{C}) \simeq \mathbf{Sp}(\mathcal{C}_*)$$

follows from (3.1.25, 3.1.23), the functor in the middle

$$\mathbf{Sp}(\mathcal{C}_*) \to \mathbf{Sp}(\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})_*)$$

is induced from the ∞-categorical Yoneda embedding (2.12.5), and the last equivalence

$$\mathbf{Sp}(\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})_*) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Sp})$$

follows from the fact that limits in a functor category are calculated pointwise.

Remark 3.1.30. There is a mapping spectrum functor

$$\mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Sp}$$

that sends a pair of objects $(X,Y) \mapsto \mathrm{Map}_{\mathcal{C}}(X,Y)_\bullet \in \mathbf{Sp}$ to the mapping spectrum. Informally, we can describe the mapping spectrum by:

$$\mathrm{Map}_{\mathcal{C}}(X,Y)_n \simeq \mathrm{Map}_{\mathcal{C}}(X,\Sigma^n Y)$$

3.1.4 Some useful facts

Here we collect some useful assorted facts about stable ∞-categories.

Proposition 3.1.31. [HA, 1.4.2.21]

There are many equivalent conditions for an ∞-category to be stable.

Let $\mathcal{C}$ be a pointed ∞-category. Then the following are equivalent:

(i) $\mathcal{C}$ is stable.

(ii) Every map in $\mathcal{C}$ admits a fiber (3.1.9) and $\Sigma : \mathcal{C} \to \mathcal{C}$ (3.1.16) is an equivalence (2.8.12).

(iii) Every map in $\mathcal{C}$ admits a cofiber and $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence.

(iv) $\mathcal{C}$ has finite co/limits, and a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ is a pullback iff it's a pushout (2.9.41) [HA, 1.1.3.4].

(v) Every map in $\mathcal{C}$ admits a fiber and cofiber (3.1.9), and a triangle (3.1.7) is a fiber sequence iff it's a cofiber sequence (3.1.7).

(vi) The functor $\Omega^\infty : \mathbf{Sp}(\mathcal{C}) \to \mathcal{C}$ (3.1.24) is an equivalence of ∞-categories. (3.1.25)

Proposition 3.1.32. [HA, 1.4.4.1]

(1) A stable ∞-category has small colimits iff it has small coproducts.

(2) A functor between stable ∞-categories preserves small colimits iff it preserves small coproducts.

(3) An object X in a stable ∞-category $\mathcal{C}$ is compact (3.2.37) iff for any small coproduct $\coprod_I Y_i \in \mathcal{C}$, any map $X \to \coprod_I Y_i$ factors through a finite subcoproduct – that is, there exists a finite subcomplex $I^0 \subseteq I$ and a 2-simplex $\Delta^2 \to \mathcal{C}$ of the form:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \coprod_I Y_i \\ & \coprod_{I^0} Y_i \arrow[ur, hook] & \end{tikzcd}$$

Proposition 3.1.33. [HA, 1.4.4.4]

Let $\mathcal{C}$ be a presentable ∞-category (2.13.21).

- The ∞-category $\mathbf{Sp}(\mathcal{C})$ (3.1.20) is stable.
- The functor $\Omega^\infty : \mathbf{Sp}(\mathcal{C}) \to \mathcal{C}$ (3.1.24) admits a left adjoint $\Sigma^\infty_+ : \mathcal{C} \to \mathbf{Sp}(\mathcal{C})$.

Proposition 3.1.34. [HA, 1.1.3.1]

Let $\mathcal{C}$ be a stable ∞-category and K a simplicial set. Then the functor category $\mathrm{Fun}(K, \mathcal{C})$ is a stable ∞-category.

Proposition 3.1.35. [BGT13, Prop. 5.10, 5.11]

A functor $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories is fully faithful (2.8.13) iff the induced functor $hF : h\mathcal{C} \to h\mathcal{D}$ is fully faithful in the ordinary sense.

The functor F is an equivalence of ∞-categories (2.8.12) iff hF is an equivalence of ordinary categories.

3.2 Triangulated categories

3.2.1 Basics of triangulated categories

Definition 3.2.1. [triangulated categories] (Verdier)

A $\boxed{\text{triangulated category}}$ is a triple (T, Σ, Δ), where:

- T is an additive category;
- $\Sigma : T \to T$ is a self equivalence called suspension (sometimes $(-)[1]$ is also used to denote suspension);
- $\Delta \subseteq T^{[3]}$ is a collection of diagrams in T of the form:

$$X \to Y \to Z \to \Sigma X$$

called <u>distinguished triangles</u> (or exact triangles). These exact triangles satisfying the following:

(TR0) – The collection of distinguished triangles is stable under isomorphism. That is, given a distinguished triangle $(X \to Y \to Z \to \Sigma X)$, along with isomorphisms:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\sim"'] & Y \arrow[r] \arrow[d, "\sim"'] & Z \arrow[r] \arrow[d, "\sim"'] & \Sigma X \arrow[d, "\sim"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma(X') \end{tikzcd}$$

then the triangle $(X' \to Y' \to Z' \to \Sigma(X'))$ is a distinguished triangle.

– For any $X \in T$, the string

$$X \xrightarrow{\text{id}} X \to 0 \to \Sigma X$$

is a distinguished triangle.

(TR1) Every morphism $X \xrightarrow{f} Y$ in T can be extended to a distinguished triangle:

$$X \xrightarrow{f} Y \to Z \to \Sigma X.$$

(TR2) Distinguished triangles can be "shifted." That is, a string

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a distinguished triangle iff the "shift"

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

(TR3) Given a commuting diagram between two distinguished triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma Z'
\end{tikzcd}$$

then there exists a map $Z \to Z'$ making the diagram commute.

(TR4) Given three distinguished triangles

$$X \xrightarrow{f} Y \to Y/X \to \Sigma X$$
$$Y \xrightarrow{g} Z \to Z/Y \to \Sigma Y$$
$$X \xrightarrow{g\circ f} Z \to Z/X \to \Sigma X$$

Then there exists a distinguished triangle

$$Y/X \to Z/X \to Z/Y \to \Sigma(Y/X)$$

making the following commute:

$$\begin{tikzcd}
X \arrow[rr, bend left, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left] \arrow[dr] & & Z/Y \arrow[rr, bend left, dashed] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr] & & Z/X \arrow[ur, dashed] \arrow[dr] & & \Sigma Y \arrow[ur] & \\
& & Y/X \arrow[ur, dashed] \arrow[rr, bend right] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

Remark 3.2.2. Exact triangles are a generalization of exact sequences, and also of fiber and cofiber sequences in homotopy theory. They can be thought of as “weak exact sequences” or ”homotopy exact sequences.”

In an exact triangle ($X \xrightarrow{f} Y \to Z \to \Sigma X$), the morphism $Y \to Z$ is sometimes called the cofiber of the morphism f, and the object Z is called the cone of f. One can think of this cone as a “weak cokernel” or “homotopy cofiber” of f (this suggests the notation, eg. Y/X, used in (TR4)).

Remark 3.2.3. Our labelling and formulation of the axioms differ from some sources. We follow mostly the numbering in [HA, 1.1.2.5].

Remark 3.2.4. The axiom (TR4) is called the "octahedral axiom"[11], so called because it can be described by an octahedron-shaped diagram. One picture that is sometimes drawn is:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[rrrr, bend left, "gf"] & & Y \arrow[dl] \arrow[rr, "g"] & & Z \arrow[dl] \arrow[ddll, bend left] \\
& Y/X \arrow[ul, "+1"] \arrow[dr, dotted] & & Z/Y \arrow[ul, "+1"'] \arrow[ll, dotted, "+1"] & \\
& & Z/X \arrow[uull, bend left, "+1"] \arrow[ur, dotted] & &
\end{tikzcd}$$

(in which a map marked $A \xrightarrow{+1} B$ denotes a map $A \to \Sigma B$). There are three exact triangles (the two small ones, and the larger one on the outside), and the axiom forms a fourth (dotted). We can form an diagram in the shape of an octahedron by rearranging terms as follows (by "picking up Y"):

$$\begin{tikzcd}
& Y \arrow[ddl] \arrow[drr, "g"] & & \\
& X \arrow[u, "f"] \arrow[rr, "gf"] & & Z \arrow[dl] \arrow[ddl] \\
Y/X \arrow[ur, "+1"] \arrow[drr, dotted] & & Z/Y \arrow[uul, "+1"'] \arrow[ll, dotted, "+1"] & \\
& & Z/X \arrow[uul, "+1"] \arrow[u, dotted] &
\end{tikzcd}$$

The idea is not as complicated as it looks. Given composable maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ with composite $X \xrightarrow{g \circ f} Z$, one can extend each into an exact triangle

[11] Also "Verdier's axiom".

by (TR1). These triangles should fit together in a coherent way, which is what the octahedral axiom is describing. Distinguished triangles are like weak exact sequences, which motivates the notation (3.2.2) $Y/X, Z/Y, Z/X$, which we can think of as "weak cokernels". The octahedral axiom forms another distinguished triangle involving the cones

$$Y/X \to Z/X \to Z/Y \to \Sigma(Y/X),$$

which exhibits $Z/Y = \mathrm{cone}(Y/X \to Z/X)$, which we can think of as a "weak cokernel" $(Z/X)/(Y/X)$. This is reminiscent of an isomorphism theorem.

Proposition 3.2.5. [HA, 1.1.2.14]

Let $\mathcal{C}$ be a stable ∞-category (3.1.10).

Then its homotopy category $h\mathcal{C}$ (2.4.13) has a canonical triangulated structure described as follows:

- The homotopy category is additive by [HA, 1.1.2.9].
- Let $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ denote the suspension functor (3.1.16) on $\mathcal{C}$. The suspension functor on $h\mathcal{C}$ is the functor formed by taking the image of $\Sigma_{\mathcal{C}}$ under the homotopy functor $h : \mathbf{qCat} \to \mathbf{Cat}$ (2.4.10):

  $$\Sigma_{h\mathcal{C}} = h(\Sigma_{\mathcal{C}}) : h\mathcal{C} \to h\mathcal{C}.$$

  The functor $h(\Sigma_{\mathcal{C}})$ is an equivalence of categories since $\Sigma_{\mathcal{C}}$ is an equivalence of ∞-categories (3.1.31).
- A string in $h\mathcal{C}$

  $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is an exact triangle iff there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ of the form:

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\overline{g}"'] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r, "\overline{h}"] & X' \end{tikzcd}$$

where:

- The objects 0 and $0'$ are zero objects (3.1.1) in $\mathcal{C}$;
- The maps $f = [\overline{f}]$ and $g = [\overline{g}]$ (that is, the maps f and g in the homotopy category are represented by $\overline{f}$ and $\overline{g}$ respectively);
- Both squares are pushout squares (2.9.41) in $\mathcal{C}$.
- Since both inner squares are pushouts, the outer square is also a pushout; this forms an equivalence $\varphi : X' \simeq \Sigma X$. The map $h : Z \to \Sigma X$ is given by:

$$h = [\varphi] \circ [\overline{h}].$$

Example 3.2.6. (a) Let A be an abelian category. Then its derived ∞-category $\mathcal{D}(A)$ [HA, 1.3] is a stable ∞-category, and its homotopy category is the ordinary derived category: $h\mathcal{D}(A) = D(A)$. Since $\mathcal{D}(A)$ is stable, the usual derived category $D(A)$ has triangulated structure.

(b) The ∞-category of spectra **Sp** (3.1.28) is a stable ∞-category, and its homotopy category is the stable homotopy category: **SHC** $= h$**Sp**. This gives the stable homotopy category a triangulated structure.

Remark 3.2.7. While every stable ∞-category forms a canonical triangulated structure on its homotopy category, it is not true that all triangulated categories arise in this way. It's often said that essentially all triangulated categories arising "in nature" arise in this way, but as far as this author knows there is not a precise characterization.

There are advantages and disadvantages of working on the level of the "underlying" stable ∞-category rather than that of on the triangulated homotopy level. One of the advantage is that being stable is a property whereas being triangulated is a structure. That is, an ∞-category is either stable or it's not, and one knows how to check this (3.1.31). But given an additive category C, it doesn't make sense to ask "Is C triangulated?" without specifying some additional data – namely, what suspension is, what exact triangles look like, etc.

Let us look at some inconveniences with triangulated categories that one finds mentioned:

- non-functoriality of mapping cones (3.2.8);
- non-existence of homotopy limits/colimits;
- given triangulated categories T and T', the functor category $\mathrm{Fun}(T, T')$ doesn't form a triangulated category;
- hard to doing gluing/descent arguments.

The ∞-category holds more information – in particular homotopical information. This makes problems like non-functoriality of the cone and non-existence of homotopy limits/colimits go away: it is relatively easy to describe a cone functor on the ∞-level, and homotopy limits/colimits are the only type of limits/colimits one can talk about on the ∞-level (2.9.39) and a stable ∞-category comes with all finite limits/colimits (3.1.31).

Remark 3.2.8. The most often cited inconvenience with triangulated categories is the fact that "cones are not functorial". That is, given a triangulated category T, there is no functor

$$T^{[1]} \to T$$

that takes a map f to its cone cone(f) (3.2.2).

However, if $T = h\mathcal{C}$ is the homotopy category of a stable ∞-category $\mathcal{C}$, we constructed the cofiber as a functor between ∞-categories (3.1.14)

$$\mathrm{cofib} : \mathcal{C}^{\Delta^1} \to \mathcal{C}.$$

A distinguished triangle $(X \xrightarrow{f} Y \to Z \to \Sigma X)$ in $h\mathcal{C}$ lifts to a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r] & X' \end{tikzcd}$$

where $f = [\overline{f}]$, and both squares are pushout squares. In particular, the square on the left being a pushout is the definition of a cofiber sequence (3.1.9). This means $Z \simeq \mathrm{cofib}(X \xrightarrow{f} Y)$, the image of f in $\mathcal{C}^{\Delta^1}$ under the cofiber functor.

3.2.2 Higher triangulation

The octahedral axiom [TR4] (3.2.1, 3.2.4) is formulated to relate the cones of a pair of composable morphisms to the cone of their composite. But what if we wanted to make a similar statement about a string of n composable morphisms for $n > 2$?

We will describe what these "higher octahedra" or n-triangles, look like, leading to the notion of an n-triangulated category (one that satisfies octahedral axioms up to level n) and show that the homotopy category of a stable ∞-category is n-triangulated for all n.

For this section we follow [Bal11].

Definition 3.2.9. [suspended categories]

A $\boxed{\text{suspended category}}$ is an additive category C with an auto-equivalence $\Sigma : C \xrightarrow{\sim} C$ that we call the suspension.

Example 3.2.10. The homotopy category of a stable ∞-category is a suspended category, with suspension induced from suspension on the ∞-category (3.1.16).

Definition 3.2.11. [n-triangles]

Let C be a suspended category (3.2.9) with a suspension Σ.

For any $n \geq 1$, an $\boxed{n\text{-triangle}}$ α in C is a commutative diagram $\alpha : [n] \times [n] \to C$ of the form:

$$\begin{tikzcd}
a_{0,1} \arrow[r, "f"] \arrow[d] & a_{0,2} \arrow[r] \arrow[d, "g"] & a_{0,3} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,n} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & a_{1,2} \arrow[r] \arrow[d] & a_{1,3} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r, "h"] \arrow[d] & \Sigma(a_{0,1}) \arrow[d, "\Sigma f"] \\
 & 0 \arrow[r] & a_{2,3} \arrow[r] \arrow[d] & \ldots \arrow[r] & a_{2,n} \arrow[d] \arrow[r] & \Sigma(a_{0,2}) \arrow[d] \\
 & & 0 & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & \ddots & 0 \arrow[r] & a_{n-1,n} \arrow[r] \arrow[d] & \Sigma(a_{0,n-1}) \arrow[d] \\
0 & & & & 0 \arrow[r] & \Sigma(a_{0,n})
\end{tikzcd}$$

Here we've omitted indexing on the morphisms for legibility, but the ones to keep track of are f (horizontal), g (vertical), and h (into the suspensions). Notice that all objects in the bottom left half are zero objects (we will often omit all the zeros entirely), and that the row of $(n-1)$ morphisms along the top

$$a_{0,1} \xrightarrow{f} a_{0,2} \to \cdots \to a_{0,n}$$

corresponds to the vertical string of suspensions on the very right

$$\Sigma a_{0,1} \xrightarrow{\Sigma f} \Sigma a_{0,2} \to \cdots \to \Sigma a_{0,n}.$$

We call the row $(a_{0,1} \xrightarrow{f} a_{0,2} \to \cdots \to a_{0,n})$ along the top the $\boxed{\text{base}}$ of the n-triangle.

To make this very precise, we may draw a picture of the category $[n] \times [n]$ with objects (i,j) for $0 \le i, j \le n$, and maps H and V (horizontal and vertical) organized into a commutative diagram as follows:

$$\begin{tikzcd}
{} \arrow[r, squiggly, "j"] \arrow[d, squiggly, "i"'] & {} & & & \\
{} & (0,0) \arrow[r, "H_{0,0}"] \arrow[d, "V_{0,0}"'] & (0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,n) \arrow[d] \\
& (1,0) \arrow[r] \arrow[d] & (1,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (1,n) \arrow[d] \\
& \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\
& (n,0) \arrow[r] & (n,1) \arrow[r] & \cdots \arrow[r] & (n,n)
\end{tikzcd}$$

Then we have the following conditions on the diagram $\alpha : [n] \times [n] \to C$:

- (on objects):

$$\sigma(i,j) = \begin{cases} 0 & i > j \\ a_{i,j+1} & i < j, j \neq n \\ 0 & i = 0, j = n \\ \Sigma a_{0,i} & i \neq 0, j = n \end{cases}$$

- (on maps):

$$\sigma(V_{i,j}) = \begin{cases} 0 & i \geq j \\ g_{i,j+1} & i < j, j < n \\ 0 & i = 0, j = n \\ \Sigma f_{0,i} & i \neq 0, j = n \end{cases}, \qquad \sigma(H_{i,j}) = \begin{cases} 0 & i > j \\ f_{i,j+1} & i \leq j, j < n-1 \\ 0 & i = 0, j = n-1. \end{cases}$$

Remark 3.2.12. Given an n-triangle $\alpha : [n] \times [n] \to C$ as in (3.2.11), we can "extend" the diagram α down and to the right indefinitely, forming a commutative diagram in C of the form:

$$\begin{tikzcd}
a_{0,1} \arrow[r, "f"] \arrow[d] & a_{0,2} \arrow[r] \arrow[d, "g"] & \cdots \arrow[r] & a_{0,n} \arrow[r] \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & a_{1,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r, "h"] \arrow[d] & \Sigma(a_{0,1}) \arrow[r, dashed] \arrow[d, "\Sigma f"] & 0 \arrow[d, dashed] & \\
& 0 \arrow[r] & \cdots \arrow[r] & a_{2,n} \arrow[r] \arrow[d] & \Sigma(a_{0,2}) \arrow[r, dashed, "\Sigma g"] \arrow[d] & \Sigma(a_{1,2}) \arrow[r, dashed] \arrow[d, dashed] & \cdots \\
& & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \\
& & & a_{n-1,n} \arrow[r] \arrow[d] & \Sigma(a_{0,n-1}) \arrow[r, dashed] \arrow[d] & \Sigma(a_{1,n-1}) \arrow[r, dashed] \arrow[d, dashed] & \cdots \\
& & & 0 \arrow[r] & \Sigma(a_{0,n}) \arrow[r, dashed] & \Sigma(a_{1,n}) \arrow[r, dashed] \arrow[d, dashed] & \cdots \\
& & & & & \vdots &
\end{tikzcd}$$

This is like extending a cofiber sequence in stable homotopy theory. In [Bal11], n-triangles are defined in this way, as diagrams $\mathbb{Z} \times \mathbb{Z} \to C$. The two notions are equivalent, but we've defined it this way since such a construction is determined by the finite diagram α.

Sanity check 3.2.13. For example, when $n = 1$:

$$\begin{tikzcd}
a_{0,1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma a_{0,0}
\end{tikzcd}$$

When $n = 2$:

$$\begin{tikzcd}
a_{0,1} \arrow[r, "f"] \arrow[d] & a_{0,2} \arrow[r] \arrow[d, "g"'] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & a_{1,2} \arrow[r, "h"] \arrow[d] & \Sigma a_{0,1} \arrow[d, "\Sigma f"] \\
0 \arrow[r] & 0 \arrow[r] & \Sigma a_{0,2}
\end{tikzcd}$$

in which we can see something that looks like a distinguished triangle (3.2.1) of a triangulated category: $(a_{0,1} \to a_{0,2} \to a_{1,2} \to \Sigma a_{0,1})$. In particular, the two pushout squares along the top look like the type of diagram in a stable ∞-category that forms distinguished triangles in the homotopy category (3.2.5).

When $n = 3$:

$$\begin{tikzcd}
a_{0,1} \arrow[r, "f"] \arrow[d] & a_{0,2} \arrow[r] \arrow[d, "g"'] & a_{0,3} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & a_{1,2} \arrow[r] \arrow[d] & a_{1,3} \arrow[r, "h"] \arrow[d] & \Sigma a_{0,1} \arrow[d, "\Sigma f"] \\
 & 0 \arrow[r] & a_{2,3} \arrow[r] \arrow[d] & \Sigma a_{0,2} \arrow[d] \\
 & & 0 \arrow[r] & \Sigma a_{0,3}
\end{tikzcd}$$

Here we have omitted the 0's in the bottom left corner. If we squint, we may be able to see the octahedral axiom in this. Rename the objects to match the notation of (TR4) (3.2.4):

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "f'"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Y/X \arrow[r] \arrow[d] & Z/X \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"] \\
& 0 \arrow[r] & Z/Y \arrow[r] \arrow[d] & \Sigma Y \arrow[d] \\
& & 0 \arrow[r] & \Sigma Z
\end{tikzcd}$$

The base of our triangle is the string $X \xrightarrow{f} Y \xrightarrow{f'} Z$. We can pick out the three triangles that the axiom requires, starting with f, f', and $f' \circ f$ respectively:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[d] & \\
& Y/X \arrow[r, bend left] & \Sigma X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "f'"] & Z \arrow[d, bend left] & \\
& Z/Y \arrow[r] & \Sigma Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, bend left, "f'\circ f"] & Z \arrow[d] & \\
& Z/X \arrow[r] & \Sigma X
\end{tikzcd}$$

along with the triangle that the axiom produces:

$$\begin{tikzcd}
Y/X \arrow[r, dotted] & Z/X \arrow[d, dotted] & \\
& Z/Y \arrow[r, dotted, bend left] & \Sigma(Y/X)
\end{tikzcd}$$

Note that this required extending the diagram as in (3.2.12) to get a map $\Sigma Y \to \Sigma(Y/X)$.

Remark 3.2.14. [Bal11, Rk. 5.3]

There is a simplicial structure on these n-triangles. Given an n-triangle α in a category C, the face map d_k forms an $(n-1)$-triangle by deleting all objects $a_{i,j}$ with i or j equal to k, and composing over the gaps. The degeneracy map s_k forms an $(n+1)$-triangle by sticking in a row and a column of identities in the k^{th} place. For example, given an n-triangle as in (3.2.11), its k^{th} face is a diagram $[n-1]\times[n-1]\to C$:

$$
\begin{tikzcd}
a_{0,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,k-1} \arrow[r, bend left] \arrow[d] & a_{0,k+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,n} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \cdots \arrow[r] & a_{1,k-1} \arrow[r, bend left] \arrow[d] & a_{1,k+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r] \arrow[d] & (a_{0,1})' \arrow[d] \\
 & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & a_{k-1,k-1} \arrow[r, bend left] \arrow[d, bend left] & a_{k-1,k+1} \arrow[r] \arrow[d, bend left] & \cdots \arrow[r] & a_{i-1,n-1} \arrow[r] \arrow[d, bend left] & (a_{0,k-1})' \arrow[d, bend left] \\
 & & 0 \arrow[r, bend left] & a_{k+1,k+1} \arrow[r] & \cdots \arrow[r] & a_{k+1,n-1} \arrow[r] \arrow[d] & (a_{0,k+1})' \arrow[d] \\
 & & & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & & & 0 \arrow[r] & (a_{0,n})'
\end{tikzcd}
$$

and its k^{th} degeneracy is a diagram $[n+1]\times[n+1]\to C$:

$$
\begin{tikzcd}
a_{0,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,k} \arrow[r, equal] \arrow[d] & a_{0,k} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,n} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \cdots \arrow[r] & a_{1,k} \arrow[r, equal] \arrow[d] & a_{1,k} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r] \arrow[d] & \Sigma(a_{0,1}) \arrow[d] \\
 & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & a_{k-1,k} \arrow[r, equal] \arrow[d] & a_{k-1,k} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{k-1,n-1} \arrow[r] \arrow[d] & \Sigma(a_{0,k-1}) \arrow[d] \\
 & & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & a_{k,n-1} \arrow[r] \arrow[d, equal] & \Sigma(a_{0,k}) \arrow[d, equal] \\
 & & 0 \arrow[r, equal] & 0 \arrow[r] & \cdots \arrow[r] & a_{k,n-1} \arrow[r] \arrow[d] & \Sigma(a_{0,k}) \arrow[d] \\
 & & & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & & & 0 \arrow[r] & \Sigma(a_{0,n})
\end{tikzcd}
$$

Definition 3.2.15. [symmetric and translate of an n-triangle]

Let (C,Σ) be a suspended category (3.2.9) and let α be an n-triangle (3.2.11) in C.

The $\boxed{\text{symmetric}}$ of α is an n-triangle denoted $\sigma(\alpha)$ formed by applying the suspension Σ to every object and changing the sign of every horizontal morphism in the last column:

$$
\begin{tikzcd}
\Sigma(a_{0,1}) \arrow[r, "\Sigma f"] & \Sigma(a_{0,2}) \arrow[r] \arrow[d, "\Sigma g"] & \cdots \arrow[r] & \Sigma(a_{0,n}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
& \Sigma(a_{1,2}) \arrow[r] & \cdots \arrow[r] & \Sigma(a_{1,n}) \arrow[r, "-\Sigma h"] \arrow[d] & \Sigma^2(a_{0,1}) \arrow[d] \\
& & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
& & & a_{n-1,n} \arrow[r] & \Sigma^2(a_{0,n-1}) \arrow[d] \\
& & & & \Sigma^2(a_{0,n})
\end{tikzcd}
$$

The $\boxed{\text{translate}}$ is an n-triangle $\tau(\alpha)$ formed by shifting the indices of everything up by 1. In other words, extending the triangle (3.2.12) by one row and one column, and then truncating the first row and column:

$$
\begin{tikzcd}
a_{1,2} \arrow[r] & a_{1,3} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r, "h"] \arrow[d] & \Sigma(a_{0,1}) \arrow[r] \arrow[d, "\Sigma f"] & 0 \arrow[d] \\
& a_{2,3} \arrow[r] & \cdots \arrow[r] & a_{2,n} \arrow[r] \arrow[d] & \Sigma(a_{0,2}) \arrow[r, "\Sigma g"] \arrow[d] & \Sigma(a_{1,2}) \arrow[d] \\
& & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
& & & a_{n-1,n} \arrow[r] & \Sigma(a_{0,n-1}) \arrow[r] \arrow[d] & \Sigma(a_{1,n-1}) \arrow[d] \\
& & & & \Sigma(a_{0,n}) \arrow[r] & \Sigma(a_{1,n}) \arrow[d] \\
& & & & & \Sigma^2(a_{0,1})
\end{tikzcd}
$$

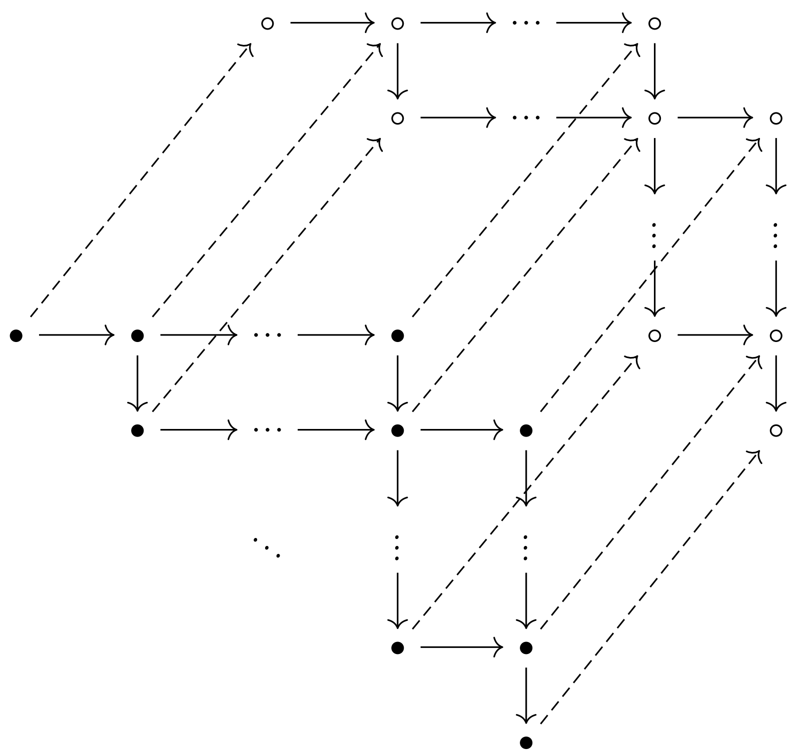
Definition 3.2.16. [N-triangulated categories]

Let $N \geq 2$ and let C be a suspended category (3.2.9).

We say that C is $\boxed{N\text{-triangulated}}$ (or N-angulated) if it is equipped with a collection of distinguished n-triangles for all $2 \leq n \leq N$ satisfying the axioms:

(TC 3.1) (Bookkeeping:)

(TC 3.1.a) Any n-triangle isomorphic to a distinguished n-triangle is itself distinguished. That is, given two n-triangles $\alpha, \beta : [n] \times [n] \rightrightarrows C$ along with a natural isomorphism $\alpha \simeq \beta$, then α is distinguished iff β is distinguished. A natural transformation $\alpha \to \beta$ can be described by a diagram $[n] \times [n] \times [1] \to C$:



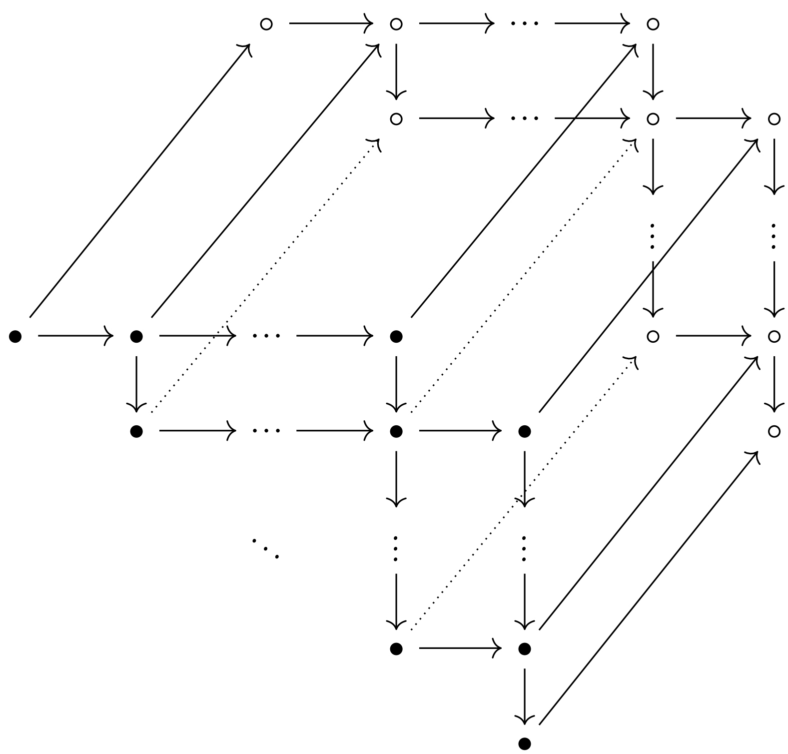
where the $\bullet$'s are the diagram α and the $\circ$'s are the diagram β, and the dashed lines are the maps forming a natural transformation $\alpha \to \beta$.

(TC 3.1.b) Distinguished triangles are preserved by the simplicial structure à la Waldhausen (3.2.14): degeneracies of distinguished $(n-1)$-triangles are distinguished n-triangles and faces of distinguished n-triangles are distinguished $(n-1)$-triangles.

(TC 3.1.c) A n-triangle α is distinguished iff its symmetric $\sigma(\alpha)$ and translate $\tau(\alpha)$ (3.2.15) are both distinguished n-triangles.

(TC 3.2) Every $(n-1)$-tuple of composable morphisms is the base of a distinguished n-triangle.

(TC 3.3) Given two distinguished n-triangles, every morphism between bases extends to a morphism of n-triangles. That is, given two triangles with a map of bases (drawn with solid lines below):



extends to a map of triangles (the dotted lines).

We say that $\mathcal{C}$ is $\boxed{\infty\text{-triangulated}}$ if it's N-triangulated for all $N \geq 2$.[12]

Remark 3.2.17. [n-angulated structure on the homotopy category of a stable ∞-category]

Let $\mathcal{C}$ be a stable ∞-category, and $h\mathcal{C}$ its homotopy category. We'll call a diagram $\alpha : [n] \times [n] \to h\mathcal{C}$ of the form:

$$\begin{tikzcd}
a_{0,1} \arrow[r, "{[f]}"] & a_{0,2} \arrow[r] \arrow[d, "{[g]}"] & \cdots \arrow[r] & a_{0,n} \arrow[d] & \\
& a_{1,2} \arrow[r] & \cdots \arrow[r] & a_{1,n} \arrow[r, "{[h]}"] \arrow[d] & \Sigma a_{0,1} \arrow[d, "{\Sigma[f]}"] \\
& & \ddots & a_{2,n} \arrow[r] \arrow[d] & \Sigma a_{0,2} \arrow[d] \\
& & & \vdots \arrow[d] & \vdots \arrow[d] \\
& & & a_{n-1,n} \arrow[r] & \Sigma a_{0,n-1} \arrow[d] \\
& & & & \Sigma a_{0,n}
\end{tikzcd}$$

a distinguished n-triangle if it arises as the image under the homotopy functor

[12]Note that the ∞ in "∞-triangulated" has nothing to do with the ∞ in "∞-category".

h of a diagram $\widetilde{\alpha} : \Delta^n \times \Delta^n \to \mathcal{C}$ of the form:

$$\begin{tikzcd}
a_{0,1} \arrow[r, "f"] \arrow[d] & a_{0,2} \arrow[r] \arrow[d, "g"] & \cdots \arrow[r] & a_{0,n} \arrow[rr] \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & a_{1,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[rr, "h'"] \arrow[d] & & (a_{0,1})' \arrow[d, "f'"] \\
& 0 \arrow[r] & \ldots \arrow[r] & a_{2,n} \arrow[rr] \arrow[d] & & (a_{0,2})' \arrow[d] \\
& & \ddots & \vdots \arrow[d] & & \vdots \arrow[d] \\
& & & a_{n-1,n} \arrow[rr] \arrow[d] & & (a_{0,n-1})' \arrow[d] \\
& & & 0 \arrow[rr] & & (a_{0,n})'
\end{tikzcd}$$

satisfying the conditions:

- Each square making up $\widetilde{\alpha}$ is a (∞-) pushout square (2.9.41). Note that this forms equivalences for each $i = 1, \ldots, n$:

  $$(a_{0,i})' \simeq \Sigma(a_{0,i}),$$

  which induce isomorphisms in $h\mathcal{C}$.

- For each $i = 1, \ldots, n$, each map $f' : (a_{0,i})' \to (a_{0,i+1})'$ in $\mathcal{C}$ forms the following commutative diagram $h\mathcal{C}$:

  $$\begin{tikzcd}
(a_{0,i})' \arrow[r, "\sim"] \arrow[d, "{[f']}"'] & \Sigma a_{0,j} \arrow[d, "{\Sigma_{h\mathcal{C}}[f]=[\Sigma_{\mathcal{C}} f]}"] \\
(a_{0,i+1})' \arrow[r, "\sim"] & \Sigma a_{0,j}
\end{tikzcd}$$

- For each $i = 1, \ldots, n$, for each map $[h] : a_{i,n} \to \Sigma(a_{0,i})$ in $h\mathcal{C}$, there is a 2-simplex $\Delta^2 \to \mathcal{C}$ of the form

$$\begin{tikzcd} & a'_{0,i} \arrow[dr, "\sim"] & \\ a_{i,n} \arrow[ur, "h'"] \arrow[rr, "h"'] & & \Sigma a_{0,i} \end{tikzcd}$$

  where $(a_{0,j})' \xrightarrow{\sim} a_{0,j}[1]$ is the induced equivalence as above. That is, in the homotopy category, the map $[h']$ composes with the equivalence to give the map $[h]$ of the n-triangle.

Example 3.2.18. [A "3-triangulated" category]

Let $\mathcal{C}$ be a stable ∞-category.

Given a composite, $X \to Y \to Z$ in the homotopy category $h\mathcal{C}$, a 3-triangle in $h\mathcal{C}$ looks like:

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r] \arrow[d] & Z \arrow[d] & \\ & Y/X \arrow[r] & Z/X \arrow[r] \arrow[d] & X[1] \arrow[d] \\ & & Z/Y \arrow[r] & Y[1] \arrow[d] \\ & & & Z[1] \end{tikzcd}$$

where $X[1], Y[1], Z[1]$ are the suspensions of X, Y, Z.

This 3-triangle is distinguished if we have a diagram $\Delta^3 \times \Delta^3 \to \mathcal{C}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Y/X \arrow[r] \arrow[d] & Z/X \arrow[r] \arrow[d] & X' \arrow[d] \\
 & 0 \arrow[r] & Z/Y \arrow[r] \arrow[d] & Y' \arrow[d] \\
 & & 0 \arrow[r] & Z'
\end{tikzcd}$$

where each square is a pushout square in $\mathcal{C}$, each morphism is a representative of the corresponding morphisms in the triangle, and the (homotopy classes of) maps $Z/X \to X', Z/Y \to Y'$ should compose with the isomorphisms $X' \cong X[1]$ and $Y' \cong Y[1]$ in $h\mathcal{C}$ to give us the corresponding maps in our triangle.

It may be helpful to have pictures for the axioms:

(TC 3.1.a) Any 3-triangle isomorphic to a distinguished 3-triangle is itself distinguished:

(TC 3.1.b) Degeneracies of distinguished 2-triangles (regular old triangles) are distinguished 3-triangles and faces of distinguished 3-triangles are distinguished 2-triangles:

Given a (2-)triangle $X \to Y \to Z \to X[1]$, its degeneracies are:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y \arrow[d, equal] & \\
 & X \arrow[r] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & & Z \arrow[r] & X[1] \arrow[d] \\
 & & & Y[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, equal] & X \arrow[r] \arrow[d] & Y \arrow[d] & \\
 & 0 \arrow[r] & Z \arrow[r] \arrow[d, equal] & X[1] \arrow[d, equal] \\
 & & Z \arrow[r] & X[1] \arrow[d] \\
 & & & Y[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & \\
 & Z \arrow[r, equal] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
 & & 0 \arrow[r] & Y[1] \arrow[d, equal] \\
 & & & Y[1]
\end{tikzcd}$$

and given a 3-triangle as above, its faces are:

$$X \to Y \to Y/X \to X[1]$$

$$Y \to Z \to Z/Y \to Y[1]$$

$$X \to Z \to Z/X \to X[1]$$

$$Y/X \to Z/X \to Z/Y \to (Y/X)[1]$$

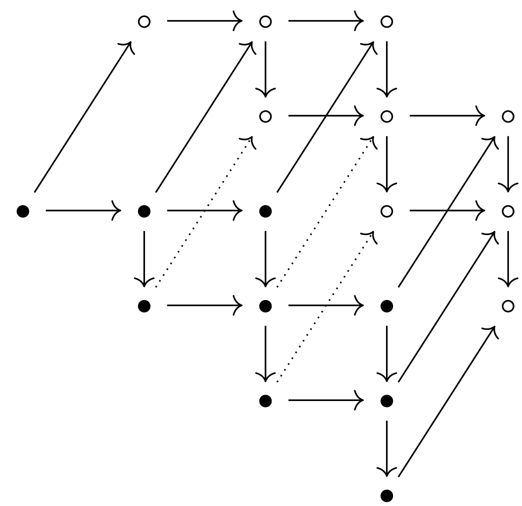
(TC 3.1.c) A 3-triangle α is distinguished iff its symmetric $\sigma(\alpha)$ and translate $\tau(\alpha)$ are both distinguished:

$$\begin{tikzcd}
X[1] \arrow[r] & Y[1] \arrow[r] \arrow[d] & Z[1] \arrow[d] & \\
& Y/X[1] \arrow[r] & Z/X[1] \arrow[r] \arrow[d] & X[2] \arrow[d] \\
& & Z/Y[1] \arrow[r] & Y[2] \arrow[d] \\
& & & Z[2]
\end{tikzcd}
,
\begin{tikzcd}
Y/X \arrow[r] & Z/X \arrow[r] \arrow[d] & X[1] \arrow[d] & \\
& Z/Y \arrow[r] & Y[1] \arrow[r] \arrow[d] & Y/X[1] \arrow[d] \\
& & Z[1] \arrow[r] & Z/X[1] \arrow[d] \\
& & & X[2]
\end{tikzcd}$$

(TC 3.2) Every pair of composable morphisms $X \to Y \to Z$ is the base of a distinguished 3-triangle.

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r] \arrow[d, dotted] & Z \arrow[d, dotted] & \\
& Y/X \arrow[r, dotted] & Z/X \arrow[r, dotted] \arrow[d, dotted] & X[1] \arrow[d] \\
& & Z/Y \arrow[r, dotted] & Y[1] \arrow[d] \\
& & & Z[1]
\end{tikzcd}$$

(TC 3.3) Given two distinguished 3-triangles, every morphism between bases extends to a morphism of 3-triangles.



Proposition 3.2.19. Given a stable ∞-category $\mathcal{C}$, its homotopy category $h\mathcal{C}$ is ∞-triangulated, with distinguished n-triangles as in.

Proof. Bookkeeping:

(TC 3.n.a) Isomorphisms in $h\mathcal{C}$ lift to equivalences in $\mathcal{C}$. Then we can translate the corresponding diagram $\Delta^n \times \Delta^n \to \mathcal{C}$ to the one we want via these equivalences.

(TC 3.n.b) Given an n-triangle induced by a diagram of pushouts $\mathcal{C}$ as in Def 3.2, extending by pushout squares extends the diagram by one row and one column of identities:

$$
\begin{tikzcd}
a_{0,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,i} \arrow[r, equal] \arrow[d] & a_{0,i} \arrow[r] \arrow[d] & a_{0,i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,n} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \cdots \arrow[r] & a_{1,i} \arrow[r, equal] \arrow[d] & a_{1,i} \arrow[r] \arrow[d] & a_{1,i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r] \arrow[d] & (a_{0,1})' \arrow[d] \\
& \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
& & a_{i-1,i} \arrow[r, equal] \arrow[d] & a_{i-1,i} \arrow[r] \arrow[d] & a_{i-1,i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{i-1,n-1} \arrow[r] \arrow[d] & (a_{0,i-1})' \arrow[d] \\
& & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & a_{i,i+1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & a_{i,n-1} \arrow[r] \arrow[d, equal] & (a_{0,i})' \arrow[d, equal] \\
& & 0 \arrow[r, equal] & 0 \arrow[r] & a_{i,i+1} \arrow[r] & \cdots \arrow[r] & a_{i,n-1} \arrow[r] \arrow[d] & (a_{0,i})' \arrow[d] \\
& & & & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
& & & & & & 0 \arrow[r] & (a_{0,n})'
\end{tikzcd}
$$

making the i^{th} degeneracy a distinguished $(n+1)$-triangle.

On the other hand, a face of the n-triangle deletes one row and one column, by composing over the gap. Say we take the i^{th} face; ie. delete the i^{th} row and column.

When $i = 0$, this is just deleting the top row. Extending what's left by pushouts as follows:

$$
\begin{tikzcd}
0 \arrow[r] & a_{1,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r] \arrow[d] & (a_{0,1})' \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 \arrow[r] & \dots \arrow[r] & a_{2,n} \arrow[r] \arrow[d] & (a_{0,2})' \arrow[r] \arrow[d] & (a_{1,2})' \arrow[d] \\
 & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & a_{n-1,n} \arrow[r] \arrow[d] & (a_{0,n-1})' \arrow[r] \arrow[d] & (a_{1,n-1})' \arrow[d] \\
 & & & 0 \arrow[r] & (a_{0,n})' \arrow[r] \arrow[d] & (a_{1,n})' \arrow[d] \\
 & & & & 0 \arrow[r] & (a_{0,1})''
\end{tikzcd}
$$

we get equivalences $(a_{1,j})' \simeq \Sigma a_{1,j}$ for each $j = 2, \dots, n$, and $(a_{0,n})'' \simeq \Sigma^2 a_{0,n}$, making the induced triangle distinguished in $h\mathcal{C}$.

This will affect the following subdiagram of the larger diagram in $\mathcal{C}$:

$$\begin{tikzcd}
a_{0,i-1} \arrow[r] \arrow[d] & a_{0,i} \arrow[r] \arrow[d] & a_{0,i+1} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
a_{j-1,i-1} \arrow[r] \arrow[d] & a_{j-1,i} \arrow[r] \arrow[d] & a_{j-1,i+1} \arrow[d] \\
a_{j,i-1} \arrow[r] \arrow[d] & a_{j,i} \arrow[r] \arrow[d] & a_{j,i+1} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & a_{i-1,i} \arrow[r] \arrow[d] & a_{i-1,i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{i-1,k} \arrow[r] \arrow[d] & a_{i-1,k+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & (a_{0,i-1})' \arrow[d] \\
& 0 \arrow[r] & a_{i,i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{i,k} \arrow[r] \arrow[d] & a_{i,k+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & (a_{0,i})' \arrow[d] \\
& & 0 \arrow[r] & \cdots \arrow[r] & a_{i+1,k} \arrow[r] & a_{i+1,k+1} \arrow[r] & \cdots \arrow[r] & (a_{0,i+1})'
\end{tikzcd}$$

Each square is already a pushout, so the "pushout pairs" make pushout rectangles

$$\begin{tikzcd}
a_{j-1,i-1} \arrow[r] \arrow[d] & a_{j-1,i} \arrow[r] \arrow[d] & a_{j-1,i+1} \arrow[d] \\
a_{j,i-1} \arrow[r] & a_{j,i} \arrow[r] & a_{j,i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
a_{i-1,k} \arrow[r] \arrow[d] & a_{i-1,k+1} \arrow[d] \\
a_{i,k} \arrow[r] \arrow[d] & a_{i,k+1} \arrow[d] \\
a_{i+1,k} \arrow[r] & a_{i+1,k+1}
\end{tikzcd}$$

for $0 \leq j < i < k \leq n$.

But these outer rectangles are precisely what is left after applying the face maps:

$$
\begin{tikzcd}
a_{0,1} \arrow[r] \arrow[d] & a_{0,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,i-1} \arrow[r] \arrow[d] & a_{0,i+1} \arrow[r] \arrow[d] & a_{0,i+2} \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & a_{1,2} \arrow[r] & \cdots \arrow[r] & a_{1,i-1} \arrow[r] \arrow[d] & a_{1,i+1} \arrow[r] \arrow[d] & a_{1,i+2} \arrow[r] & \cdots \arrow[r] & (a_{0,1})' \\
 & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & & \vdots \arrow[d] \\
 & & & a_{i-2,i-1} \arrow[r] \arrow[d] & a_{i-2,i+1} \arrow[d] & & \ddots & (a_{0,i-2})' \arrow[d] \\
 & & & 0 \arrow[r] & a_{i-1,i+1} \arrow[r] \arrow[d] & a_{i-1,i+2} \arrow[r] \arrow[d] & \cdots \arrow[r] & (a_{0,i-1})' \arrow[d] \\
 & & & & 0 \arrow[r] & a_{i+1,i+2} \arrow[r] & \cdots \arrow[r] & (a_{0,i+1})' \arrow[d] \\
 & & & & & & \ddots & \vdots \arrow[d] \\
 & & & & & & & (a_{0,n})'
\end{tikzcd}
$$

making the induced $(n-1)$-triangle in $h\mathcal{C}$ distinguished.

(TC 3.n.c) $\Rightarrow$: Given a distinguished triangle induced by the same diagram, we can extend by pushouts (the dashed arrows):

$$
\begin{tikzcd}
a_{0,1} \arrow[r] \arrow[d] & a_{0,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,n} \arrow[r] \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & a_{1,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r] \arrow[d] & (a_{0,1})' \arrow[r, dashed] \arrow[d] & 0 \arrow[d, dashed] & & \\
& 0 \arrow[r] & \ldots \arrow[r] & a_{2,n} \arrow[r] \arrow[d] & (a_{0,2})' \arrow[r, dashed] \arrow[d] & (a_{1,2})' \arrow[r, dashed] \arrow[d, dashed] & 0 & \\
& & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \ddots & \\
& & & a_{n-1,n} \arrow[r] \arrow[d] & (a_{0,n-1})' \arrow[r, dashed] \arrow[d] & (a_{1,n-1})' \arrow[d, dashed] & & \\
& & & 0 \arrow[r] & (a_{0,n})' \arrow[r, dashed] \arrow[d, dashed] & (a_{1,n})' \arrow[r, dashed] \arrow[d, dashed] & \cdots \arrow[r, dashed] & 0 \arrow[d, dashed] \\
& & & & 0 \arrow[r, dashed] & (a_{0,1})'' \arrow[r, dashed] & \cdots \arrow[r, dashed] & (a_{0,n})''
\end{tikzcd}
$$

with equivalences $(a_{i,j})' \simeq \Sigma a_{i,j}$, and $(a_{0,i})'' \simeq \Sigma^2 a_{0,i}$. We can see the symmetric hiding in the above – the chain of suspensions $(a_{0,1})' \to \cdots \to (a_{0,n})'$ forms the base of an n-triangle "turned on its side". The fact that it's turned on its side is what will give us the negative $-h[1]$ maps that appear in the symmetric via [HA, 1.1.2.10].

Collapsing relevant pushout squares gives us these pushouts hiding in the above:

$$
\begin{tikzcd}
a_{i,n} \arrow[r] \arrow[d] & (a_{0,i})' \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & (a_{0,n})' \arrow[r] \arrow[d] & (a_{i,n})' \arrow[d] \\
& 0 \arrow[r] & (a_{0,i})''
\end{tikzcd}
$$

The morphisms above induce a map between pushout squares (the rectangle on top, and the rectangle on the right), meaning the map $(a_{i,n})' \to$

$(a_{0,n})''$ classifies the map $-h[1] : a_{i,n}[1] \to a_{0,n}[2]$ in $h\mathcal{C}$. All other maps are suspensions of the relevant maps from the original triangle. The resulting diagram then makes the symmetric $\sigma(\Theta)$ a distinguished triangle.

The translate is easier. Extend by pushouts:

$$
\begin{tikzcd}
a_{0,1} \arrow[r] \arrow[d] & a_{0,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,n} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & a_{1,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n} \arrow[r] \arrow[d] & (a_{0,1})' \arrow[r, dashed] \arrow[d] & 0 \arrow[d, dashed] \\
& 0 \arrow[r] & \ldots \arrow[r] & a_{2,n} \arrow[r] \arrow[d] & (a_{0,2})' \arrow[r, dashed] \arrow[d] & (a_{1,2})' \arrow[d, dashed] \\
& & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, dashed] \\
& & & a_{n-1,n} \arrow[r] \arrow[d] & (a_{0,n-1})' \arrow[r, dashed] \arrow[d] & (a_{1,n-1})' \arrow[d, dashed] \\
& & & 0 \arrow[r] & (a_{0,n})' \arrow[r, dashed] \arrow[d, dashed] & (a_{1,n})' \arrow[d, dashed] \\
& & & & 0 \arrow[r, dashed] & (a_{0,1})''
\end{tikzcd}
$$

and the diagrams will induce equivalences $(a_{1,i})' \simeq a_{1,i}[1]$ for $i = 1, \ldots, n$, and $(a_{0,1})'' \simeq a_{0,1}[2]$ playing well with the morphisms in the original triangle. The diagram induces the translate $\tau(\Theta)$ as a distinguished triangle.

$\Leftarrow$: If the symmetric $\sigma(\Theta)$ is distinguished, we can translate by Σ^{-2} to get a distinguished triangle $\Sigma^{-2}\sigma(\Theta)$:

$$\begin{tikzcd}
a_{0,1}[-1] \arrow[r, "{f[-1]}"] \arrow[d] & a_{0,2}[-1] \arrow[r] \arrow[d, "{g[-1]}"] & \cdots \arrow[r] & a_{0,n}[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & a_{1,2}[-1] \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{1,n}[-1] \arrow[r, "{-h[-1]}"] \arrow[d] & (a_{0,1})'[-1] \arrow[d, "{f'[-1]}"] \\
 & 0 \arrow[r] & \ldots \arrow[r] & a_{2,n}[-1] \arrow[r] \arrow[d] & (a_{0,2})'[-1] \arrow[d] \\
 & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & a_{n-1,n}[-1] \arrow[r] \arrow[d] & (a_{0,n-1})'[-1] \arrow[d] \\
 & & & 0 \arrow[r] & (a_{0,n})'[-1]
\end{tikzcd}$$

and then see that the symmetric of the above is just what we want:

$$\sigma\Sigma^{-2}\sigma(\Theta) = \Theta.$$

(TC 3.2) Let $\mathcal{E} \subseteq \mathrm{Fun}(\Delta^n \times \Delta^n, \mathcal{C})$ consist of diagrams in $\mathcal{C}$ of the form:

$$\mathcal{E} = \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ & 0 \arrow[r] & \ldots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\ & & & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ & & & 0 \arrow[r] & \bullet \end{tikzcd} \right\}$$

with n nonzero objects forming the base at the top, and all squares being pushouts.

Let evaluation $e : \mathcal{E} \to \mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$ pick out the base of a diagram in $\mathcal{E}$. We can play the same game as in (3.1.14), using (3.1.13) to decompose e as a bunch of trivial fibrations and full inclusions, showing e to be a trivial fibration itself. In particular, any element in $\mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$ lifts to a diagram in $\mathcal{E}$; ie. any string of $(n-1)$ composable morphisms in $h\mathcal{C}$ extends to a distinguished n-triangle.

(TC 3.3) Let $\alpha, \beta \in \mathcal{E}$ denote diagrams of n-triangles. Let $a_{i,j}, b_{i,j}$ denote the objects in α and β respectively.

A morphism between bases is a bunch of maps $a_{0,i} \to b_{0,i}$ $(1 \leq i \leq n)$

forming a commutative rectangle in $h\mathcal{C}$:

$$\begin{tikzcd} a_{0,1} \arrow[r] \arrow[d] & a_{0,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{0,n} \arrow[d] \\ b_{0,1} \arrow[r] & b_{0,2} \arrow[r] & \cdots \arrow[r] & b_{0,n} \end{tikzcd}$$

This lifts to a morphism $\varphi : e(\alpha) \to e(\beta)$ in $\mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$. Since e is a trivial fibration, this lifts to a morphism of triangles $\alpha \to \beta$ in $\mathcal{E}$.

Since n was arbitrary, this makes $h\mathcal{C}$ in fact ∞-triangulated.

□

3.2.3 Exact functors

Definition 3.2.20. [exact functors between triangulated categories]

Let T and T' be triangulated categories with suspensions Σ_T and $\Sigma_{T'}$.

An $\boxed{\text{exact functor}}$ is an additive functor $F : T \to T'$ along with natural isomorphisms $F \circ \Sigma_T \simeq \Sigma_{T'} \circ F$, that preserves triangles. That is, given a distinguished triangle in T:

$$X \to Y \to Z \to \Sigma_T X,$$

the induced string in T':

$$F(X) \to F(Y) \to F(Z) \to F(\Sigma_T X) \simeq \Sigma_{T'}(F(X))$$

is a distinguished triangle in T'.

The corresponding ∞-version is also called an exact functor:

Definition 3.2.21. [exact functors between stable ∞-categories]

An $\boxed{\text{exact functor}}$ $F : \mathcal{C} \to \mathcal{C}'$ between stable ∞-categories is a functor that:

- is pointed;
- sends fiber sequences to fiber sequences (3.1.7): that is, sends a (∞-) pullback square in $\mathcal{C}$ of the form on the left to a pullback square in $\mathcal{C}'$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} FX \arrow[r] \arrow[d] & FY \arrow[d] \\ F0 \arrow[r] & FZ \end{tikzcd}$$

Remark 3.2.22. Notice that we do not require natural isomorphisms $F \circ \Sigma_{\mathcal{C}} \simeq \Sigma_{\mathcal{C}'} \circ F$. This is absorbed into the preservation of fiber sequences.

Remark 3.2.23. [HA, 1.1.4.1]

Stable ∞-categories have all finite limits and colimits (3.1.31). An exact functor $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories preserves all finite limits and colimits.

Sanity check 3.2.24. An exact functor of stable ∞-categories $F : \mathcal{C} \to \mathcal{D}$ induces an exact functor of triangulated categories $hF : h\mathcal{C} \to h\mathcal{D}$.

Proof.
- (additivity): The exact functor F preserves zero objects by definition, and zero objects of an ∞-category are zero objects in the homotopy category (2.9.32), so hF preserves zero objects on the homotopy-level.

  We need to check that it preserves direct sums. We can interpret direct sum as a product/coproduct $X \oplus Y = X \sqcup Y \xrightarrow{\sim} X \times Y$. Since products/coproducts are preserved under formation of the homotopy category (2.9.47), this is the same as the direct sum in the homotopy category.

  Exact functors preserve finite limits and colimits (3.2.23), so there are isomorphisms $F(X \oplus Y) \simeq\simeq FX \oplus FY$ in $h\mathcal{C}$ for all $X, Y \in \mathcal{C}_0$.

- (respects suspensions): We want a natural isomorphism $\Sigma_{h\mathcal{D}} hF \simeq hf\Sigma_{h\mathcal{C}}$.

  Let X be an object in $\mathcal{C}$. Suspension in a stable ∞-category is a fiber sequence $\Delta^1 \times \Delta^1 \to \mathcal{C}$ (3.1.7) of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

  The functor F sends this to a fiber sequence in $\mathcal{D}$, forming an equivalence $F(\Sigma_{\mathcal{C}} X) \simeq \Sigma_{\mathcal{D}}(FX)$. Since suspension on the homotopy-level is simply induced from the ∞-category (3.2.5), this is an isomorphism in $h\mathcal{C}$:

$$F(\Sigma_{h\mathcal{C}} X) \simeq \Sigma_{h\mathcal{D}}(FX).$$

  To check naturality, let $[f] : X \to Y$ be a map in $h\mathcal{C}$. This forms a map in $h\mathcal{C}$:

$$\Sigma_{\mathcal{C}} X \xrightarrow{\Sigma_{h\mathcal{C}}[f]=[\Sigma_{\mathcal{C}} f]} \Sigma_{\mathcal{C}} Y,$$

  which under hF is sent to a map in $h\mathcal{D}$:

$$F(\Sigma_{\mathcal{C}} X) \xrightarrow{hF[\Sigma_{\mathcal{C}} f]=[F\Sigma_{\mathcal{C}} f]} F(\Sigma_{\mathcal{C}} Y)$$

  On the other hand, $[f]$ is sent under hF to a map in $h\mathcal{D}$:

$$FX \xrightarrow{F[f]=[Ff]} FY,$$

  which suspends to:

$$\Sigma_{\mathcal{D}} FX \xrightarrow{\Sigma_{h\mathcal{D}}[Ff]=[\Sigma_{\mathcal{D}} Ff]} \Sigma_{\mathcal{D}} FY.$$

All together, the square we want to commute for naturality is a square in $h\mathcal{D}$:

$$\begin{tikzcd} F(\Sigma_{\mathcal{C}}X) \arrow[r, "{[F\Sigma_{\mathcal{C}}f]}"] \arrow[d, "\sim"] & F(\Sigma_{\mathcal{C}}Y) \arrow[d, "\sim"] \\ \Sigma_{\mathcal{D}}(FX) \arrow[r, "{[\Sigma_{\mathcal{D}}Ff]}"] & \Sigma_{\mathcal{D}}(FY) \end{tikzcd}$$

which corresponds to a square $\Delta^1 \times \Delta^1 \to \mathcal{D}$:

$$\begin{tikzcd} F(\Sigma_{\mathcal{C}}X) \arrow[r, "F\Sigma_{\mathcal{C}}f"] \arrow[d, "\sim"] & F(\Sigma_{\mathcal{C}}Y) \arrow[d, "\sim"] \\ \Sigma_{\mathcal{D}}(FX) \arrow[r, "\Sigma_{\mathcal{D}}Ff"] & \Sigma_{\mathcal{D}}(FY) \end{tikzcd}$$

ie., an edge $\Sigma_{\mathcal{D}}Ff \to F\Sigma_{\mathcal{C}}f$ in $\mathrm{Fun}(\Delta^1, \mathcal{D})$.

Since $\Sigma_{\mathcal{D}}$ is a self-equivalence, the induced edge $\Sigma_{\mathcal{D}}\circ - : Ff \to \Sigma_{\mathcal{D}}\circ Ff$ is an equivalence in the functor category $\mathrm{Fun}(\Delta^1, \mathcal{D})$. This forms the following horn $\Lambda^2_1 \to \mathrm{Fun}(\Delta^1, \mathcal{D})$:

$$\begin{tikzcd} & \Sigma_{\mathcal{D}}Ff \arrow[dr, dotted] & \\ Ff \arrow[ur, "\sim"] \arrow[rr] & & F\Sigma_{\mathcal{C}}f \end{tikzcd}$$

which fills in to a 2-simplex by Joyal's theorem (2.1.33), giving us the edge $\Sigma_{\mathcal{D}}Ff \to F\Sigma_{\mathcal{C}}f$ that we want.

- (preserves triangles): Since F preserves zero objects and fiber sequences, it maps a diagram of interated fiber sequences in $\mathcal{C}$ to a similar diagram in $\mathcal{D}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0_{\mathcal{C}} \arrow[d] \\ 0_{\mathcal{C}} \arrow[r] & Z \arrow[r] & X' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} FX \arrow[r] \arrow[d] & FY \arrow[r] \arrow[d] & 0_{\mathcal{D}} \arrow[d] \\ 0_{\mathcal{D}} \arrow[r] & FZ \arrow[r] & F(X') \end{tikzcd}$$

By stability of $\mathcal{D}$, the outer rectangle of the diagram on the right is a pushout, giving an equivalence $F(X') \simeq \Sigma_{\mathcal{D}}(FX)$, so the diagram represents a distinguished triangle $FX \to FY \to FZ \to \Sigma_{\mathcal{D}}(FX)$ in $h\mathcal{D}$.

□

Remark 3.2.25. More generally, for $n \geq 2$, we can consider functors on n-angulated categories (3.2.16).

Let T and T' be n-angulated categories for some $n \geq 2$.

An $\boxed{n\text{-angulated functor}}$ is an additive functor $f : T \to T'$ along with a natural isomorphism $f \circ \Sigma_T \simeq \Sigma_{T'} \circ f$ that preserves $\underline{i\text{-triangles}}$ for all $i = 1, 2, \ldots, n$. That is, given a distinguished i-triangle $\alpha : [i] \times [i] \to T$, the induced triangle $f \circ \alpha$ is a distinguished i-triangle in T'.

If $T = h\mathcal{C}$ and $T' = h\mathcal{D}$ for stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, and $F : \mathcal{C} \to \mathcal{D}$ is an exact functor, then the induced functor $hF : h\mathcal{C} \to h\mathcal{D}$ is n-angulated.

Proof. We proved additivity and suspension isomorphisms above. We only need to show that it preserves i-triangles for all $i = 1, 2, \ldots, n$.

Pick $i \in \{1, 2, \ldots, n\}$. An i-triangle in $h\mathcal{C}$ lifts to a diagram $\Delta^i \times \Delta^i \to \mathcal{C}$, in which every square is a pullback square (3.2.17). An exact functor $F : \mathcal{C} \to \mathcal{D}$ preserves pullbacks, forming a diagram $\Delta^i \times \Delta^i \to \mathcal{D}$ which forms an i-triangle in $h\mathcal{D}$. □

3.2.4 Subcategories

Recall that full subcategories of an ∞-category $\mathcal{C}$ are defined via (ordinary) full subcategories of its homotopy category $h\mathcal{C}$ (2.6.1).

Definition 3.2.26. [Nee01, Def. 1.5.1]

[triangulated subcategories]

Let T be a triangulated category and $T^0 \subseteq T$ a full, replete (closed under isomorphism) subcategory.

We call $T^0 \subseteq T$ a $\boxed{\text{triangulated subcategory}}$ if it satisfies the following:

- it's closed under suspensions and desuspensions; and
- given an exact triangle $X \to Y \to Z \to X[1]$ in T, if two out of three of $\{X, Y, Z\}$ are in T^0, then so is the third. (Note that this implies that $T^0 \subseteq T$ is an additive subcategory.)

Alternately, T^0 is a triangulated subcategory of T if the inclusion $T^0 \hookrightarrow T$ is an exact functor.

The corresponding notion for stable ∞-categories is that of a stable subcategory:

Definition 3.2.27 (HA, 1.1.3.2)**.** [stable subcategories]

Let $\mathcal{C}$ be a stable ∞-category, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory.

We call $\mathcal{C}^0$ a $\boxed{\text{stable subcategory}}$ if it contains a zero object (3.1.1) and is closed under fibers and cofibers (3.1.9).

Remark 3.2.28. There are also weaker conditions one can consider to check that a subcategory is stable. Equivalently, a stable subcategory is a full subcategory that's either

- closed under suspensions and desuspensions and cofibers [HA, 1.1.3.3]; or
- closed under finite limits and colimits (this implies closed under suspensions and desuspensions, as well as under cofibers).

Remark 3.2.29. Let $\mathcal{C}$ be a stable ∞-category. Then there is a correspondence:

$$\{\text{triangulated subcategories of } h\mathcal{C}\} \quad \leftrightsquigarrow \quad \{\text{stable subcategories of } \mathcal{C}\}$$

In other words, given a subcategory $C^0 \subseteq h\mathcal{C}$, and letting $\mathcal{C}^0 \subseteq \mathcal{C}$ be the corresponding (∞-) subcategory (a pullback of simplicial sets):

$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \mathcal{C} \arrow[d] \\
N(C^0) \arrow[r, hook] & Nh\mathcal{C}
\end{tikzcd}$$

Then C^0 is a triangulated subcategory of $h\mathcal{C}$ iff $\mathcal{C}^0$ is a stable subcategory of $\mathcal{C}$.

Proof. $\Rightarrow$: Suppose $C^0 \subseteq h\mathcal{C}$ is a triangulated subcategory. Then since suspension on the ∞-level and the homotopy-level are the same (3.2.5), $\mathcal{C}^0$ is closed under suspension and desuspension iff C^0 is.

Let $f : X \to Y$ be an edge in $\mathcal{C}^0$. This forms a cofiber sequence in $\mathcal{C}$ (3.1.7), ie. a pushout square:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & \mathrm{cofib}(f)
\end{tikzcd}$$

which we can extend to a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{cofib}(f) \arrow[r] & X'
\end{tikzcd}$$

in which both squares are pushout squares, and there is an equivalence $X' \simeq \Sigma X$. This forms a distinguished triangle in $h\mathcal{C}$

$$X \xrightarrow{[f]} Y \to \mathrm{cofib}(f) \to \Sigma X.$$

Since f was taking from $\mathcal{C}^0$, the map $[f]$ is in C^0, and since C^0 is a triangulated subcategory, it contains the cofiber $\mathrm{cofib}(f)$, meaning $\mathcal{C}^0$ contains $\mathrm{cofib}(f)$.

$\Leftarrow$: Suppose $\mathcal{C}^0$ is a stable subcategory of $\mathcal{C}$.

- (closed under suspension/desuspension): Since suspension and desuspension were defined as fibers and cofibers respectively (3.1.16) of a zero map over a zero-object, and a stable subcategory $\mathcal{C}_0$ is closed under fibers and cofibers, then it contains its suspensions and desuspensions (and so, so does C^0).

- (two-out-of-three): Say $(X \to Y \to Z \to \Sigma X)$ was a distinguished triangle in $h\mathcal{C}$. That is, we have a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r] & X'
\end{tikzcd}$$

  with and both inner squares pushouts (and so the outer square also a pushout, giving an equivalence $\Sigma X \simeq X'$). Suppose that two out of three of $\{X, Y, Z\}$ are in $h\mathcal{C}_0$ – that is, two out of three are vertices in $\mathcal{C}_0$.

If X and Y are the two objects, then since $\mathcal{C}_0$ is closed under cofibers, the pushout Z is also in $\mathcal{C}_0$, ie. is an object in the homotopy category. Since pushout squares are pullback squares in a stable quasi-category, the other cases are proven similarly, using closedness of fibers/cofibers appropriately.

□

Definition 3.2.30. [thick subcategories]

(1-): A triangulated subcategory $T^0 \subseteq T$ is called $\boxed{\text{thick}}$ if it's closed under retracts. In other words, if $Y \in T^0$ and there are maps in T:

$$X \xrightarrow{i} Y \xrightarrow{p} X$$

such that $p \circ i = \mathrm{id}_X$, then X is in T^0. Equivalently, a thick subcategory is a full subcategory that's closed under direct summands.

(∞-): A stable (∞-) subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ is called $\boxed{\text{thick}}$ if it's closed under retracts. In other words, given a 2-simplex in $\mathcal{C}$ of the form:

$$\begin{tikzcd} & Y \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "\mathrm{id}"'] & & X \end{tikzcd}$$

if $Y \in \mathcal{C}^0$, then $X \in \mathcal{C}^0$.

Remark 3.2.31. Let $\mathcal{C}$ be a stable ∞-category. Then there is a correspondence:

$$\{\text{thick subcategories of } h\mathcal{C}\} \quad \leftrightsquigarrow \quad \{\text{thick subcategories of } \mathcal{C}\}.$$

Proof. A retract diagram in $h\mathcal{C}$, ie. a commutative diagram:

$$\begin{tikzcd} & Y \arrow[dr, "{[p]}"] & \\ X \arrow[ur, "{[i]}"] \arrow[rr, "\mathrm{id}=[\mathrm{id}_X]"'] & & X \end{tikzcd}$$

corresponds precisely to a 2-simplex $\Delta^2 \to \mathcal{C}$:

$$\begin{tikzcd} & Y \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "\mathrm{id}"'] & & X \end{tikzcd}$$

□

Definition 3.2.32. [localizing subcategories]

(1-): A $\boxed{\text{localizing subcategory}}$ in a triangulated category is a subcategory that is:

- a thick subcategory (as a triangulated subcategory) (3.2.30); and
- is closed under small coproducts (direct sums).

(∞-): A $\boxed{\text{localizing subcategory}}$ in a stable ∞-category is a subcategory that is:

- a thick subcategory (as a stable subcategory); and
- is closed under small colimits.

More generally, for an infinite cardinal α, a subcategory is $\boxed{\alpha\text{-localizing}}$ if it's thick and is closed under $\underline{\alpha\text{-small}}$ coproducts. So a subcategory is localizing if it's α-localizing for all α.

Remark 3.2.33. Let $\mathcal{C}$ be a stable ∞-category. There is a correspondence:

$$\{\text{localizing subcategories of } h\mathcal{C}\} \quad \leftrightsquigarrow \quad \{\text{localizing subcategories of } \mathcal{C}\}.$$

Proof. We showed that thick subcategories in the ∞-category vs. the homotopy category correspond (3.2.31).

If a subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ is closed under small colimits, it's in particular closed under small coproducts, and so $h\mathcal{C}^0 \subseteq h\mathcal{C}$ is closed under small coproducts (2.9.47). In the reverse direction, $h\mathcal{C}^0$ is closed under small coproducts iff $\mathcal{C}^0$ is closed under small coproducts. A stable ∞-category $\mathcal{C}^0$ has small coproducts iff it has small colimits [HA, 1.4.4.1]. □

By definition localizing (3.2.32) $\implies$ thick (3.2.30) $\implies$ triangulated/stable (3.2.26, 3.2.27). So we have the following correspondence for a stable ∞-category $\mathcal{C}$:

$$\begin{tikzcd}
\{\text{triangulated subcategories of } h\mathcal{C}\} \arrow[r, leftrightsquigarrow] & \{\text{stable subcategories of } \mathcal{C}\} \\
\{\text{thick subcategories of } h\mathcal{C}\} \arrow[r, leftrightsquigarrow] \arrow[u, hook] & \{\text{thick subcategories of } \mathcal{C}\} \arrow[u, hook] \\
\{\text{localizing subcategories of } h\mathcal{C}\} \arrow[r, leftrightsquigarrow] \arrow[u, hook] & \{\text{localizing subcategories of } \mathcal{C}\} \arrow[u, hook]
\end{tikzcd}$$

3.2.5 Compactly-generated categories

There is a nice class of triangulated categories that are called compactly generated. These have some nice properties – for example, Brown representability holds in them [Nee01].

The original "Brown representability theorem" [Bro62] was in the case of the stable homotopy category of spectra **SHC**. In his paper, he showed that

SHC is compactly generated by the sphere spectrum, and then used this to show that it satisfies representability – namely that any cohomology theory can be represented by a spectrum.

In his book [Nee01], Neeman introduces a generalization of compactly generated categories which he calls well-generated triangulated categories. He shows that Brown representability holds for well-generated categories, and shows that they are nice in some ways. For example, Bousfield localization (3.2.76) of a well-generated triangulated category is well-generated, but the corresponding statement in terms of compactly-generated triangulated categories is not true.

For us, starting with a stable ∞-category, we can ask of the relation between the homotopy category being compactly-generated or well-generated, and the underlying ∞-category. The main theorems of this section are:

- (3.2.49), in which we show that a stable ∞-category is compactly generated as an ∞-category iff its homotopy category is compactly-generated as a triangulated category.
- (3.2.55), in which we show that the homotopy category of a presentable (2.13.21) stable ∞-category is well-generated.

Definition 3.2.34. [compactness in ordinary categories]

Let C be a locally small category with (α-) filtered colimits (2.13.5).

An object $X \in C$ is called $\boxed{\text{($\alpha$-) compact}}$ if the functor

$$C(X,-) : C \to \mathbf{Set}$$

preserves (α-)filtered colimits, ie. given a map $F : I \to C$ from a filtered category I, the induced map

$$\operatorname{colim}_I C(X, F(i)) \xrightarrow{\sim} C(X, \operatorname{colim}_I F)$$

is an isomorphism.

Definition 3.2.35. [compactness in triangulated categories]

Let T be a triangulated category (3.2.1), and let α be a regular cardinal (2.13.1).

The full subcategory $\boxed{T^\alpha \subseteq T}$ is the maximal subcategory satisfying the conditions:

- (α-smallness): The objects in T^α are $\underline{\alpha\text{-small}}$: that is, for $X \in T^\alpha$, a map $X \to \coprod_{i \in I} Y_i$ into an abitrary coproduct factors through an α-small subcoproduct:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \coprod_{i \in I} Y_i \\ & \coprod_{i \in I_{<\alpha}} Y_i \arrow[ur, hook] & \end{tikzcd}$$

  where $I_{<\alpha} \subseteq I$ with cardinality $|I_{<\alpha}| < \alpha$.

- (α-perfectness): The objects in T^α are $\underline{\alpha\text{-perfect}}$: that is, for $X \in T^\alpha$, any map $X \to \coprod_{i \in I} Y_i$ with $|I| < \alpha$ factors as:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \coprod_i Y_i \\ & \coprod_i X_i \arrow[ur, "\coprod_i \varphi_i"'] & \end{tikzcd}$$

  where each $X_i \in T^\alpha$ and each $\varphi_i \in T(X_i, Y_i)$.

Neeman shows that there's a unique maximal subcategory $T^\alpha \subseteq T$ satisfying these conditions, and we say that the objects in T^α are $\boxed{\alpha\text{-compact}}$. If $T = T^\alpha$, we say that T is $\boxed{\alpha\text{-compact}}$.

Remark 3.2.36. [(ω-) compactness in triangulated categories]

Let T be a triangulated category, and pick an object $X \in \mathrm{ob}(T)$ which is ω-compact (3.2.35).

The first condition (ω-smallness) says that any map of the form $X \to \coprod_i Y_i$ factors through a coproduct of finitely many of the Y_i's.

The second condition (ω-perfectness) turns out to be trivial: a finite coproduct in a triangulated category is a direct sum $\bigoplus_i Y_i$, and a map $X \to \bigoplus_i Y_i$ can be written as a collection $(f_i)_{i \in I}$ with each $f_i : X \to Y_i$. This factors as:

$$X \xrightarrow{\Delta} \bigoplus_i X^{(i)} \xrightarrow{\begin{bmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{bmatrix}} \bigoplus_i Y_i$$

where each $X^{(i)} = X$ so $\bigoplus_i X^{(i)} = X \oplus \cdots \oplus X$, a direct-sum of n copies of X, and $\Delta : x \mapsto (x, x, \dots)$ is the diagonal map.

We will often refer to ω-compact objects as simply $\boxed{\text{compact}}$. Equivalently, an object X in a triangulated category T is compact if for any coproduct $\coprod_I Y_i \in T$, the induced map

$$\coprod_I T(X, Y_i) \xrightarrow{\sim} T(X, \coprod_I Y_i)$$

is an isomorphism of abelian groups. This recovers the original notion of compactness (2.13.6).

Definition 3.2.37. [compactness in ∞-categories] [HTT, 5.3.4.5]

An object X in an ∞-category $\mathcal{C}$ admitting (α-)filtered colimits is called $\boxed{(\alpha\text{-})\text{compact}}$ if the functor corepresented by X (2.12.9)

$$j_X : \mathcal{C} \to \widehat{\mathcal{S}}$$

preserves (α-)filtered colimits. That is, given an (α-) filtered sset I, for any functor $F : I \to \mathcal{C}$, the induced map

$$\operatorname{colim}_I \operatorname{Map}_{\mathcal{C}}(X, F(i)) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_I F)$$

is an equivalence of spaces.

Remark 3.2.38. [(ω-) compactness in stable ∞-categories]

Let $\mathcal{C}$ be a stable ∞-category which is (ω-) compact (which we simply call "compact").

In a stable ∞-category $\mathcal{C}$, we can simplify compactness similarly to (3.2.35), as a statement about coproducts.

An object X in a stable ∞-category $\mathcal{C}$ is compact if and only if any map $X \to \coprod_{i \in I} Y_i$ into an arbitrary coproduct factors (up to homotopy) through a finite sub-coproduct – that is, there exists a 2-simplex in $\mathcal{C}$ of the form:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \coprod_I Y_i \\ & \coprod_{I_0} Y_i \arrow[ur, hook] & \end{tikzcd}$$

where $I_0 \subseteq I$ is a finite subset. (3.1.32)

Remark 3.2.39. Comparing (ω-) compactness in stable ∞-categories (3.2.38) versus in triangulated categories (3.2.36), we can see that they are essentially the same. That is, an object X in a stable ∞-category $\mathcal{C}$ is compact in the ∞-sense in $\mathcal{C}$ iff it is compact in the triangulated-sense in the homotopy category $h\mathcal{C}$.

However, the story for larger cardinals $\alpha > \omega$ is unclear. It is less clear how to relate being α-compact in an ∞-category versus being α-compact in a triangulated category. While α-smallness may look the same, it is not clear how α-perfectness should translate.

There are some varying notions of "generation" in the literature. In [Nee01] Neeman uses two notions of generation. We will only use the following.

Definition 3.2.40. [generation in triangulated categories]

Let T be a triangulated category

A collection of objects $G \subseteq \mathrm{ob}(T)$ is said to $\boxed{\text{generate}}$ T if it's closed under suspensions and desuspensions, and for all $t \in T$:

$$T(g,t) = 0 \text{ for all } g \in G \qquad \Longrightarrow \qquad t = 0.$$

Intuitively, this says that if t is "orthogonal" to every $g \in G$, then $t = 0$.

Remark 3.2.41. [other notions of generation in triangulated categories]

There are other forms of generation that are sometimes conidered, stronger than the above. We will not use these, but mention them briefly here for completeness.

Let T be a triangulated category and let $G \subseteq \mathrm{ob}(T)$. For a cardinal α, we denote a subcategory

$$\mathrm{Loc}_\alpha(G) \subseteq T$$

which is defined to be the smallest α-localizing subcategory (3.2.32) containing

G – ie. the smallest thick subcategory (3.2.30) containing G, and that is closed under taking coproducts of less than α objects.

We call

$$\mathrm{Loc}(G) := \bigcup_{\alpha} \mathrm{Loc}_{\alpha}(G)$$

the smallest localizing subcategory (3.2.32) containing G – ie. the smallest thick subcategory containing G that is closed under taking arbitrary small coproducts.

Then we say that G:

- "classically generates" T if $T = \mathrm{Loc}(G)$.
- "strongly generates" T if $T = \mathrm{Loc}_n(G)$ for some finite n.

Definition 3.2.42. [generation in ∞-categories]

Let $\mathcal{C}$ be a stable ∞-category, and $G \subseteq \mathcal{C}_0$ a collection of objects.

We say that G $\boxed{\text{generates}}$ $\mathcal{C}$ if it is closed under suspension and desuspensions (3.1.16), and for all $Y \in \mathcal{C}$:

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(g, Y) \simeq \{*\} \text{ for all } g \in G \qquad \Longrightarrow \qquad Y \simeq 0_{\mathcal{C}}.$$

where $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is the mapping space (2.3.1), $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ is the connected components functor (2.1.3), $\{*\}$ is the singleton set, and $0_{\mathcal{C}} \in \mathcal{C}$ is a zero object (3.1.1).

An object $X \in \mathcal{C}_0$ is said to generate $\mathcal{C}$ if for all $Y \in \mathcal{C}_0$, a similar statement holds:

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) \cong \{*\} \qquad \Longrightarrow \qquad Y \simeq 0_{\mathcal{C}}.$$

Remark 3.2.43. A set G generates $\mathcal{C}$ iff the object

$$X := \coprod_{i\in\mathbb{Z}, g\in G} \Sigma^i g$$

formed by taking all suspensions and desuspensions of objects of G and taking their coproduct generates $\mathcal{C}$ as an object.

Proof. Pick an object $Y \in \mathcal{C}_0$. The mapping space:

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(X,Y) &= \mathrm{Map}_{\mathcal{C}}(\coprod_{i\in\mathbb{Z}, g\in G} \Sigma^i g, Y) \\ &\simeq \prod_{i\in\mathbb{Z}, g\in G} \mathrm{Map}_{\mathcal{C}}(\Sigma^i g, Y) \end{aligned}$$

and since connected components preserve products (2.1.3),

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \prod_{i\in\mathbb{Z}, g\in G} \pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^i g, Y).$$

Then $\pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \{*\}$ iff $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^i g, Y) \simeq \{*\}$ for all $i \in \mathbb{Z}, g \in G$. $\square$

Remark 3.2.44. A collection of objects $G \subseteq \mathcal{C}_0$ in a stable ∞-category $\mathcal{C}$ generates in the sense of (3.2.42) iff it generates $h\mathcal{C}$ in the sense of (3.2.40).

Since for any objects $X, Y \in \mathcal{C}$, the hom-set of the homotopy category can be calculated as

$$h\mathcal{C}(X,Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y),$$

(2.4.13), the statements in the stable ∞-setting (3.2.42) agree precisely with the statement in the triangulated-setting (3.2.40).

Now we can define what "compactly generated" means, both in the world of triangulated categories and of ∞-categories.

Definition 3.2.45. [compactly generated triangulated categories]

Let T be a triangulated category which has small coproducts.

We say that T is $\boxed{\text{compactly generated}}$ if there exists a set $G \subseteq \mathrm{ob}(T)$ of compact generators:

- (compactness): Objects $X \in G$ are compact (3.2.36). ($T(X,-) : T \to \mathbf{Set}$ preserves coproducts.)
- (generation): The set G generates T (3.2.40). (G is closed under Σ, Σ^{-1}, and $T(X,Y) = 0$ for all Y means $Y = 0$.)

Definition 3.2.46. [compactly generated stable ∞-categories]

Recall that an ∞-category $\mathcal{C}$ is compactly generated (2.13.22 if it's accessible (2.13.20) and presentable (2.13.21). In other words, there is a set of compact objects that generate $\mathcal{C}$ under finite colimits.

In the case that $\mathcal{C}$ is stable, we can re-state this as follows:

A stable ∞-category $\mathcal{C}$ is $\boxed{\text{compactly generated}}$ if there is a set $G \subseteq \mathcal{C}_0$ of objects such that:

- (compactness): For any $X \in G$, any map $X \to \coprod_I Y_i$ factors through a finite sub-coproduct. (3.1.32)
- (generation): Any $X \in G$ satisfies the property that for any $Y \in \mathcal{C}$,

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y) \cong \{*\} \qquad \Longrightarrow \qquad Y \simeq 0_{\mathcal{C}}.$$

Proposition 3.2.47. [HA, 1.4.4.2]

Let $\mathcal{C}$ be a stable ∞-category.

Then $\mathcal{C}$ is presentable (2.13.21) iff the following are satisfied:

- The ∞-category $\mathcal{C}$ admits all small coproducts.
- The homotopy category $h\mathcal{C}$ is locally small.
- There exists a regular cardinal α (2.13.1) and an α-compact object $X \in \mathcal{C}_0$ that generates $\mathcal{C}$ as an object (3.2.42).

Remark 3.2.48. In the proof of [HA, 1.4.4.2], starting from a presentable stable ∞- category $\mathcal{C}$, he constructs the α-compact generator X as follows. A presentable ∞-category $\mathcal{C}$ can be written as the localization of a presheaf ∞-category (2.13.21) $\mathcal{P}(\mathcal{C}^0)$ where $\mathcal{C}^0$ is a small ∞-category. Consider the composition:

$$\mathcal{C}^0 \xrightarrow{j} \mathcal{P}(\mathcal{C}^0) \xrightarrow{l} \mathcal{C}$$

where j is the Yoneda embedding (2.12.5) and l is the localization functor (2.14.4). Each object $c \in \mathcal{C}^0$ forms an object $l \circ j(c) \in \mathcal{C}$. Then the α-compact generator X is constructed as:

$$X = \coprod_{i \in \mathbb{Z}, c \in \mathcal{C}^0} \Sigma^i l \circ j(c)$$

as a coproduct of all suspensions and desuspensions of objects in $\mathcal{C}^0$. In other words, we can rephrase the above to say that $\mathcal{C}$ is generated by the collection of objects (3.2.42)

$$G := \left\{ l \circ j(c) \ : \ c \in \mathcal{C}^0 \right\}.$$

In particular, if $\mathcal{C}$ is <u>compactly generated</u> (2.13.22), it is both presentable and finitely accessible.

Then the proposition above (3.2.47) says that there exists an (ω-) compact object $X \in \mathcal{C}_0$ (3.2.38) that generates $\mathcal{C}$, ie. a collection of objects $G \subseteq \mathcal{C}_0$ that are compact and generate $\mathcal{C}$.

Proposition 3.2.49. [HA, 1.4.4.3]

A stable ∞-category $\mathcal{C}$ is compactly generated in the ∞-sense (3.2.46) if and only if its homotopy category $h\mathcal{C}$ is compactly generated as a triangulated category (3.2.45).

Proof. Let $G \subseteq \mathcal{C}_0$ be a set of objects, and pick an object $X \in G$. We want to show that G consists of compact generators in the ∞-sense iff it is compact and generates in the ordinary sense.

(i) (compactness):

$\Rightarrow$: Coproducts are the same in an ∞-category and its homotopy category. So a map in the homotopy category $[f] : X \to \coprod_I Y_i$ lifts to an edge $f : X \to \coprod_I Y_i$ in $\mathcal{C}$. Since X is compact in the ∞-sense, we can fill the edge f to a 2-simplex in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & \coprod_I Y_i \\ & \coprod_{I_0} Y_i \arrow[ur, hook] & \end{tikzcd}$$

where $I_0 \subseteq I$ is a finite subset forming a finite subcoproduct. Since coproducts in an ∞-category and its homotopy category agree, the 2-simplex above forms a commutative triangle in $h\mathcal{C}$, factoring $[f]$ through a finite subcoproduct $\coprod_{I_0} Y_i \hookrightarrow \coprod_I Y_i$.

$\Leftarrow$: The same argument can be run backwards. Suppose G consists of compact objects in $h\mathcal{C}$. Then an edge $(f : X \to \coprod_I Y_i) \in \mathcal{C}_1$ corresponds to a morphism $[f] : X \to \coprod_I Y_i$ in $h\mathcal{C}$. By compactness we can factor $[f]$ through a finite subcoproduct $\coprod_{I_0} Y_i \hookrightarrow \coprod_I Y_i$. This forms a commutative triangle, which lifts to a 2-simplex in $\mathcal{C}$.

(ii) (generation): If $\mathcal{C}$ is compactly generated, then there exists a set G of compact objects that generates $\mathcal{C}$ (3.2.42). We showed in (3.2.44) that generation on the stable ∞-level and the triangulated-level agree.

$\square$

Remark 3.2.50. In [Nee09, Rk. 0.4], Neeman describes that satisfying Brown representability is a condition that's preserved under localization of triangulated categories, but being compactly-generated is not. So it was clear that Brown representability in fact holds for a larger class of triangulated categories. It was the search to understand this that led Neeman [Nee01] to describe the more general notion of well-generated triangulated categories:

Definition 3.2.51. [Nee01, Rk. 8.17]

[well-generated triangulated categories (Neeman)]

Let T be a triangulated category that has small coproducts.

The triangulated category T is called $\boxed{\text{well-generated}}$ if there is a regular cardinal α (2.13.1) and an α-generating set $G \subseteq \mathrm{ob}(\mathcal{T})$ (3.2.40). That is:

- (G is weakly generating): G is closed under suspensions and desuspensions, and ($\mathcal{T}(G, Y) = 0$) iff ($Y = 0$), where $\mathcal{T}(G, Y) = 0$ means that $\mathcal{T}(X, Y) = 0$ for all $X \in G$.

- (objects in G are α-small): All $X \in G$ are α-small: ie. any map $X \to \coprod_I Y$ factors as $X \to \coprod_{I_0 \subseteq I} Y_i \hookrightarrow \coprod_I Y_i$, where $|I_0| < \alpha$.

- (objects in G are α-perfect): For any $X \in G$, any map $X \to \coprod_{I_0} Y_i$ with $|I_0| < \alpha$ factors as

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \coprod_{I_0} Y_i \\ & \coprod_{I_0} X_i \arrow[ur] & \end{tikzcd}$$

with all the $X_i \in G$.

A simpler version shown to be equivalent by Krause [Kra01]:

Definition 3.2.52. [well-generated triangulated categories (à la Krause)]

Let T be a triangulated category that has small coproducts.

Then T is $\boxed{\text{well-generated}}$ if

- There's a weakly generating set $G \subseteq \mathrm{ob}(T)$ (3.2.40).
- All objects of G are α-small (3.2.35) for some regular cardinal α.
- Given a collection of maps $(f_i : Y_i \to Z_i)_{i \in I}$ (where the index I is a set), then $(f_i)_* : T(X, Y_i) \to T(X, Z_i)$ surjective for all $X \in G$ implies that the induced map $(\coprod_i f_i)_* : T(X, \coprod_i Y_i) \to T(X, \coprod_i Z_i)$ is surjective.

Remark 3.2.53. Let $\mathcal{C}$ be a stable ∞-category. We saw that being compactly generated (3.2.46) depends only on the homotopy category (3.2.49). In light of this, we might ask if we can we get a similar result for well-generated categories. In other words, when is the homotopy category of a stable ∞-category well-generated, or conversely, given a homotopy category that is well-generated, can we say anything about the underlying ∞-category?

The answer is not so straightforward, due to the fact that it is unclear how to translate the condition of α-perfectness (3.2.35) to the ∞-categorical level (see 3.2.39).

There are, however some partial results in this direction.

Theorem 3.2.54. [Ros09, Thm. 4.9]

If M is a model category (4.0.3) that is stable (4.3.1) and combinatorial (4.4.1), then its homotopy category $ho(M)$ (4.0.2) is a well-generated triangulated category (3.2.51).

Combinatorial model categories are "the same as" presentable ∞-categories (2.13.25). And stable model categories model stable ∞-categories (4.3.3).

Proposition 3.2.55. Let $\mathcal{C}$ be a presentable stable ∞-category (2.13.21, 3.1.10).

Then its homotopy category $h\mathcal{C}$ is well-generated (3.2.51).

Proof. Since any presentable ∞-category arises as the underlying ∞-category of a combinatorial model categories (2.13.25), and stable ∞-categories underly stable model categories (4.3.3), if $\mathcal{C}$ is presentable and stable, we can write it as $\mathcal{C} \simeq M^\infty$, where M^∞ is the underlying ∞-category (2.11.5) of a combinatorial stable model category M (4.4.1, 4.3.1). Then by the proposition above (3.2.54), the homotopy category

$$ho(M) \cong h(M^\infty) \simeq h\mathcal{C}$$

is well-generated. □

3.2.6 Brown representability and adjoint functor theorems

Definition 3.2.56. [representable functors in ordinary categories]

Let C be an ordinary category.

A functor $f : C^{op} \to \mathbf{Set}$ is $\boxed{\text{representable}}$ if there's an object $x \in C$ and a natural isomorphism $f \simeq C(-, x)$.

Definition 3.2.57. [Nee01, Def. 8.2.1]

[Brown representability in triangulated categories]

Let T be a triangulated category with small coproducts.

We say that T $\boxed{\text{satisfies Brown representability}}$ if any functor $F : T^{op} \to \mathbf{Ab}$ is representable iff the following conditions are satisfied:

(i) The functor F is <u>cohomological</u>: it sends a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

to an exact sequence of abelian groups

$$F(Z) \to F(Y) \to F(X);$$

(ii) The functor F sends coproducts in T to products in $\mathbf{Ab}$: for any $\coprod_I X_i \in T$, there is an isomorphism of abelian groups

$$F(\coprod_I X_i) \cong \prod_I F(X_i).$$

Theorem 3.2.58. [Nee01, Prop. 8.4.2]

If T is a triangulated category that is well-generated (3.2.51), then it satisfies Brown representability (3.2.57).

Definition 3.2.59. [NRS20, Def. 5.1]

[representable functors on ∞-categories]

Let $\mathcal{C}$ be an ∞-category. For any object $x \in \mathcal{C}$, let

$$h\mathcal{C}(-,x) : \mathcal{C} \xrightarrow{j_x} \mathcal{S} \xrightarrow{\pi_0} N(\mathbf{Set})$$

be the functor which sends an object $y \in \mathcal{C}$ to the set $\pi_0 \operatorname{Map}_{\mathcal{C}}(y,x) = \operatorname{Hom}_{h\mathcal{C}}(y,x)$

A functor $f : \mathcal{C}^{op} \to N(\mathbf{Set})$ is called $\boxed{\text{representable}}$ if there is an object $x \in \mathcal{C}$ and a natural isomorphism $f \simeq h\mathcal{C}(-,x)$.

Sanity check 3.2.60. A functor $f : \mathcal{C}^{op} \to N(\mathbf{Set})$ is representable in the ∞-sense (3.2.59) iff it is representable in the ordinary sense (3.2.56).

Proof. A functor $f : \mathcal{C}^{op} \to N(\mathbf{Set})$ is representable iff there is a natural isomorphism $f \simeq h\mathcal{C}(-,x)$ in the functor category $\operatorname{Fun}(\mathcal{C}^{op}, N(\mathbf{Set}))$. This corresponds to an isomorphism in the homotopy category

$$h\operatorname{Fun}(\mathcal{C}^{op}, N(\mathbf{Set})) \simeq \mathbf{Cat}(h\mathcal{C}^{op}, \mathbf{Set})$$

(2.8.6) between

$$hf \simeq h\mathcal{C}(-,x).$$

$\square$

Definition 3.2.61. [NRS20, Def. 5.1.1]

[Brown representability in ∞-categories]

Let $\mathcal{C}$ be an ∞-category with small colimits.

We say that $\mathcal{C}$ $\boxed{\text{satisfies Brown representability}}$ if any functor $F : \mathcal{C}^{op} \to N(\mathbf{Set})$ is representable (3.2.59) iff the following conditions are satisfied:

- For any pushout diagram in $\mathcal{C}$ (2.9.41):

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

the induced map of sets

$$F(D) \to F(B) \times_{F(A)} F(C)$$

is surjective.

- For any small coproduct $\coprod_I X_i$ in $\mathcal{C}$, the induced map

$$F(\coprod_I X_i) \to \prod_I F(X_i)$$

is an isomorphism.

Theorem 3.2.62. [NRS20, Thm. 5.2.7]

Every ∞-category that is compactly generated (2.13.22) satisfies Brown representability.

Theorem 3.2.63. [NRS20, Thm. 5.3.2]

Every presentable stable ∞-category (2.13.21, 3.1.10) satisfies Brown representability.

Proposition 3.2.64. [NRS20, Cor. 5.3.3]

Let $\mathcal{C}$ be a stable presentable ∞-category, and $h\mathcal{C}$ its homotopy category.

Then a cohomological functor $H : h\mathcal{C}^{op} \to \mathbf{Ab}$ is representable iff it sends coproducts to products.

A corollary of Brown representability is of the form of an adjoint functor theorem.

Proposition 3.2.65. [Nee96, Thm. 4.1]

[representability as an adjoint functor theorem]

Let S and T be triangulated categories, and suppose S is well-generated (3.2.51). Let $f : S \to T$ be a triangulated functor (3.2.20) that respects coproducts.

Then f has a right adjoint.

Definition 3.2.66. Let C be an ordinary category.

A $\boxed{\text{weak pushout}}$ in C is a diagram $[1]\times[1] \to C$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

which satisfies the existence, but not the uniqueness property of an ordinary pushout. That is, for any object $X \in C$ with maps $B \to X$ and $C \to X$ making the square commute, there exists at least one map $D \to X$ making the diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dotted, "\exists"] & \\ & & X \end{tikzcd}$$

Proposition 3.2.67. [NRS20, Prop. 5.1.3]

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. Suppose that $\mathcal{C}$ has all small colimits and satisfies Brown representability (3.2.61).

Then the induced functor $hF : h\mathcal{C} \to h\mathcal{D}$ admits a right adjoint iff F satisfies the following:

- F sends small coproducts in $\mathcal{C}$ to coproducts in $h\mathcal{D}$.

- F sends a pushout square (2.9.41) in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

to a <u>weak pushout</u> (3.2.66) in $h\mathcal{D}$:

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(B) \arrow[d] \\ F(C) \arrow[r] & F(D) \end{tikzcd}$$

Remark 3.2.68. Given a functor $F : \mathcal{C} \to \mathcal{D}$, right adjoints on the homotopy-level (that is, a right adjoint of $hF : h\mathcal{C} \to h\mathcal{D}$) do not, in general, lift to right adjoints on the ∞-level, unless the functor on the ∞-level is known already to admit an (∞-) adjoint (2.10.9). But if $\mathcal{C}$ satisfies Brown representability, we can say a bit more.

Proposition 3.2.69. [NRS20, Cor. 5.1.5]

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, suppose $\mathcal{C}$ has small colimits and satisfies Brown representability (3.2.61).

Then a functor $F : \mathcal{C} \to \mathcal{D}$ admits a right adjoint iff it preserves small colimits.

3.2.7 Localization

Definition 3.2.70. [kernel of an exact functor]

Let $f : T \to T'$ be an triangulated functor (3.2.20) of triangulated categories.

The $\boxed{\text{kernel of } f}$ is the collection of objects:

$$\ker(f) := \{x \in T \;:\; f(x) \cong 0_{T'}\}.$$

Lemma 3.2.71. [Nee01, Lem. 2.1.4]

Let $f : T \to T'$ be an exact functor of triangulated categories (3.2.20).

The kernel $\ker(f) \subseteq T$ is a thick triangulated subcategory (3.2.26, 3.2.30). That is, it's closed under suspensions, triangles, and direct summands.

Definition 3.2.72. [Verdier localization (triangulated)]

Let T be a triangulated category and $S \subseteq T$ a triangulated subcategory.

The $\boxed{\text{Verdier localization of } T \text{ by } S}$ is a triangulated functor $\gamma : T \to T/S$, such that $S \subseteq \ker(\gamma)$, and γ is universal with this property; that is, any exact functor $f : T \to T'$ such that $S \subseteq \ker(f)$ factors uniquely through γ:

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & T' \\ & T/S \arrow[ur, dotted] & \end{tikzcd}$$

The triangulated category T/S is called the $\boxed{\text{Verdier quotient}}$ of T by S and the map $\gamma : T \to T/S$ is called the $\boxed{\text{Verdier localization functor}}$.

Theorem 3.2.73. [Nee01, Thm. 2.1.8]

Let T be a triangulated category, and let $S \subseteq T$ be a triangulated subcategory (3.2.26).

Then there exists a Verdier localization $\gamma : T \to T/S$ (3.2.72) with $S \subseteq \ker(\gamma)$.

Remark 3.2.74. If $\gamma : T \to T/S$ is a Verdier localization, then $\ker(\gamma) \subseteq T$ is the smallest thick subcategory containing S.

Remark 3.2.75. The definition above is reminiscent of the universal property of a categorical localization $T \to T[W^{-1}]$ (4.0.1), where W consists of morphisms f in T that can be completed to an exact triangle of the form:

$$\bullet \xrightarrow{f} \bullet \to s \to \bullet$$

where f forms the base of the triangle and $s \in S$. In other words, W consists of morphisms whose cone lies in S. One can define a Verdier quotient in this way [Kra09, Def. 4.6]. This will look like our ∞- interpretation below.

In certain cases, Verdier quotients can be realized as part of a nicer notion of Bousfield/reflective localization:

Definition 3.2.76. [Bousfield/reflective localization (triangulated)]

Let $S \subseteq T$ be a thick subcategory of a triangulated category. Let $\gamma : T \to T/S$ be the Verdier localization functor of T at S (3.2.72).

The $\boxed{\text{Bousfield localization}}$ (if it exists) is a fully-faithful right adjoint to the map $\gamma : T \to T/S$.

Given a Bousfield localization $L : T \to T$, its kernel forms a thick subcategory, and one can take a Verdier quotient $T \to T/\ker(L)$. One can ask: which thick subcategories of T arise in this way? This was the same question we asked of Bousfield localizations of ∞-categories (2.14.4).

Proposition 3.2.77. [Kra09, Prop. 4.9.1]

Let $S \subseteq T$ be a thick subcategory (3.2.30) of a triangulated category. The following are equivalent.

(i) There is a Bousfield localization $L : T \to T$ with $S = \ker(L)$.

(ii) The inclusion $S \subseteq T$ admits a right adjoint.

(iii) For each $X \in T$, there is an exact triangle

$$X_S \to X \to X_\perp \to \Sigma X$$

where $X_S \in S$, and $X_\perp$ is "orthogonal to S":

$$X_\perp \in S^\perp = \{t \in T \ : \ \mathrm{Hom}_T(s,t) = 0 \text{ for all } s \in S\}.$$

Equivalently, $X_\perp$ is W-local, where W is the collection of morphisms in T whose cone lives in S. (This will look like our ∞- version below.)

Definition 3.2.78. [BGT13, Def. 5.4]

[Verdier localization (stable ∞-)]

Let $f : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor (2.8.13) between presentable stable ∞-categories (2.13.21, 3.1.10).

The $\boxed{\text{Verdier quotient of } \mathcal{D} \text{ by } \mathcal{C}}$ is the cofiber $\mathcal{D}/\mathcal{C} := \mathrm{cofib}(f)$, where the cofiber (3.1.9) is taken in the ∞-category Pr^L_{st} of presentable stable ∞-categories and left-exact functors between them.

In [BGT13], they show that this agrees with ordinary Verdier localization, if the triangulated categories come from presentable stable ∞-categories.

Proposition 3.2.79. [BGT13, Prop. 5.9]

Let $\mathcal{C}$ and $\mathcal{D}$ be presentable stable ∞-categories (2.13.21, 3.1.10).

Given a fully faithful functor (2.8.13) $\mathcal{C} \to \mathcal{D}$, there is an equivalence

$$h\mathcal{D}/h\mathcal{C} \xrightarrow{\sim} h(\mathcal{D}/\mathcal{C}),$$

where the left is the ordinary Verdier localization of triangulated categories (3.2.72) and the right is the homotopy category of the Verdier localization of stable ∞-categories (3.2.78).

They also show that forming a Verdier quotient (3.2.78) agrees with Bousfield localizing (2.14.4):

Proposition 3.2.80. [BGT13, Prop. 5.6]

Let $\mathcal{C} \to \mathcal{D}$ be a fully faithful functor (2.8.13) between presentable stable ∞-categories (2.13.21, 3.1.10). Let

$$W := \{f \in \mathcal{D}_1 \ : \ \mathrm{cone}(f) \in \mathcal{C}\}.$$

Then W is strongly saturated (2.14.10) and of small generation (2.14.11), so we can Bousfield localize (2.14.4) $\mathcal{D} \to W^{-1}\mathcal{D}$.

Then there is an equivalence:

$$\mathcal{D}/\mathcal{C} \simeq W^{-1}\mathcal{D},$$

where the left is the (∞-) Verdier quotient (3.2.78), and the right is Bousfield localization (2.14.4).

Sanity check 3.2.81. With $\mathcal{C}, \mathcal{D}$ as above, we can describe this equivalence in a diagram:

$$\begin{tikzcd} \mathcal{D} \arrow[d] \arrow[dr, "L"] & \\ \mathcal{D}/\mathcal{C} \arrow[r, "\sim"'] & W^{-1}\mathcal{D} \end{tikzcd}$$

where $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is the Verdier quotient (3.2.78), and L is Bousfield localization at the collection

$$W = \{f \in \mathcal{D}_1 \ : \ \mathrm{cone}(f) \in \mathcal{C}\}.$$

Taking homotopy categories, we get a diagram of categories:

$$\begin{tikzcd} h\mathcal{D} \arrow[d] \arrow[dr, "hL"] & \\ h\mathcal{D}/h\mathcal{C} \arrow[r, "\sim"'] & h(W^{-1}\mathcal{D}) \end{tikzcd}$$

using the fact that $h(\mathcal{D}/\mathcal{C}) \simeq h\mathcal{D}/h\mathcal{C}$ (3.2.79). The functor $h\mathcal{D} \to h\mathcal{D}/h\mathcal{C}$ in the diagram is the ordinary Verdier quotient (3.2.72) of $h\mathcal{D}$ at $h\mathcal{C}$.

Since an adjunction on the ∞-level forms an adjunction on the homotopy level (2.10.8), and a fully faithful functor of ∞-categories is fully faithful on the homotopy-level (3.1.35), then the functor $hL : h\mathcal{D} \to h(W^{-1}\mathcal{D})$ is a Bousfield localization of $h\mathcal{D}$ at

$$W_1 := \{[f] \in \mathrm{Hom}(h\mathcal{D}) \ : \ f \in W\}.$$

This forms an equivalence $h(W^{-1}\mathcal{D}) \simeq (W_1)^{-1}(h\mathcal{D})$, which composes with the equivalence $h\mathcal{D}/h\mathcal{C} \simeq h(W^{-1}\mathcal{D})$:

$$\begin{tikzcd} h\mathcal{D} \arrow[d] \arrow[dr, "hL"] & & \\ h\mathcal{D}/h\mathcal{C} \arrow[r, "\sim"'] & h(W^{-1}\mathcal{D}) \arrow[r, "\sim"'] & (W_1)^{-1}h\mathcal{D} \end{tikzcd}$$

to form an equivalence

$$h\mathcal{D}/h\mathcal{C} \simeq (W_1)^{-1}(h\mathcal{D})$$

where $h\mathcal{D}/h\mathcal{C}$ is ordinary Verdier localization (3.2.72) and the right is Bousfield localization at the set of maps in $h\mathcal{D}$ whose cone lies in $h\mathcal{C}$ (3.2.76). This recovers the result from triangulated categories.

3.2.8 t-structures

A reference for this section is [HA, 1.2.1].

Definition 3.2.82. [t-structures]

(on triangulated categories): Let T be a triangulated category with a suspension $\Sigma : T \to T$.

A $\boxed{\text{t-structure}}$ on T is a pair of full, replete subcategories

$$T_{\geq 0}, T_{\leq 0} \subseteq T$$

such that:

(i) The subcategory $T_{\geq 0}$ is stable under suspensions:

$$\Sigma(T_{\geq 0}) \subseteq T_{\geq 0},$$

and $T_{\leq 0}$ is stable under desuspensions:

$$\Sigma^{-1}(T_{\leq 0}) \subseteq T_{\leq 0}.$$

(ii) For $X \in T_{\geq 0}, Y \in T_{\leq 0}$,

$$T(X, \Sigma^{-1} Y) = 0.$$

(iii) For any $A \in T$, there exists a distinguished triangle

$$X \to A \to \Sigma^{-1} Y \to \Sigma X$$

where $X \in T_{\geq 0}, Y \in T_{\leq 0}$.

(Equivalently, there exists a triangle $X \to A \to Y \to \Sigma X$ where $X \in T_{\geq 0}, Y \in T_{\leq 1}$.)

(on stable ∞-categories): Let $\mathcal{C}$ be a stable ∞-category. A $\boxed{\text{t-structure}}$ on $\mathcal{C}$ is a t-structure on $h\mathcal{C}$ in the sense above. The subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ are defined to be the full subcategories spanned by $(h\mathcal{C})_{\geq 0}$ and $(h\mathcal{C})_{\leq 0}$ respectively.

Remark 3.2.83. The above definition uses homological indexing. One can similarly define a notion of a t-structure with <u>cohomological</u> indexing, with subcategories usually labelled $\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0} \subseteq \mathcal{C}$ and the conditions re-indexed appropriately.

Remark 3.2.84. For a stable ∞-category $\mathcal{C}$, let

$$\mathcal{C}_{\geq n} := \Sigma^n(\mathcal{C}_{\geq 0})$$
$$\mathcal{C}_{\leq n} := \Sigma^n(\mathcal{C}_{\leq 0}).$$

Example 3.2.85. A motivating example is the derived category of an abelian category A. A t-structure lets one consider objects with homology in non-negative or non-positive degrees. Suspension is given by shifting, so the subcategory $D(A)_{\geq n} \subseteq D(A)$ consists of objects with homology in degrees $\geq n$. The inclusion $D(A)_{\geq n} \subseteq D(A)$ admits a right adjoint

$$\tau_{\geq n} : D(A) \to D(A)_{\geq n}$$

given by "truncating" – taking a chain complex in $D(A)$

$$X = (\cdots \leftarrow X_{i-1} \xleftarrow{d_i} X_i \leftarrow \dots)_{i \in \mathbb{Z}}$$

to the chain complex:

$$\tau_{\geq n}X = (\cdots \leftarrow 0 \leftarrow \ker(d_n) \leftarrow X_{n+1} \leftarrow \dots),$$

ie. killing homology groups in degrees $< n$.

Proposition 3.2.86. [HA, 1.2.1.5]

Let $\mathcal{C}$ be a stable ∞-category, with a t-structure given by $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. We can define, for any $n \in \mathbb{N}$, full subcategory inclusions:

$$\mathcal{C}_{\geq n} \hookrightarrow \mathcal{C}$$

$$\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C},$$

with adjoints:

$$\begin{tikzcd} \mathcal{C}_{\geq n} \arrow[d, hook, shift right] \\ \mathcal{C} \arrow[u, shift right, "\tau_{\geq n}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{\leq n} \arrow[d, hook, shift left] \\ \mathcal{C} \arrow[u, shift left, "\tau_{\leq n}"] \end{tikzcd}$$

(left adjoints on the left).

The truncations $\tau_{\geq n}, \tau_{\leq n}$ are fully-faithful; ie. the subcategories $\mathcal{C}_{\geq n}, \mathcal{C}_{\leq n}$ are localizations of $\mathcal{C}$ (2.14.4).

Definition 3.2.87. [the heart]

Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. The $\boxed{\text{heart}}$ of the t-structure is a subcategory $\mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$ defined as:

$$\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}.$$

Remark 3.2.88. We can generalize homology functors to homotopy functors

$\pi_n : \mathcal{C} \to \mathcal{C}^\heartsuit$ for all $n \in \mathbb{Z}$:

$$\pi_0 := \tau_{\leq 0} \circ \tau_{\geq 0} \simeq \tau_{\geq 0} \circ \tau_{\leq 0},$$
$$\pi_n := \pi_0 \circ \Sigma^{-n}$$

Remark 3.2.89. The heart $T^\heartsuit$ of a t-structure on a triangulated category is an abelian category, and there is an equivalence $\mathcal{C}^\heartsuit \simeq N(h\mathcal{C}^\heartsuit)$.

Example 3.2.90. (a) In the case $T = D(A) = h\mathcal{D}(A)$, the derived category of an abelian category A, then for any $n \in \mathbb{Z}$ there is a t-structure given by:

$$D(A)_{\geq n} = \{\text{objects with homology only in degrees} \geq n\},$$
$$D(A)_{\leq n} = \{\text{objects with homology only in degrees} \leq n\}.$$

(b) The analagous t-structure in the case $\mathcal{C} = \mathbf{Sp}$ the category of spectra:

$$\mathbf{Sp}_{\geq n} = \{\text{spectra whose nonzero homotopy groups are only in degree} \geq n\}$$
$$\mathbf{Sp}_{\leq n} = \{\text{spectra whose nonzero homotopy groups are only in degree} \leq n\}$$

Remark 3.2.91. We saw that every t-structure determines certain localizations (3.2.86).

Not every localization arises in this way, and not every localization determines a t-strucutre. But we can characterize those localizations that do form t-structures as <u>t-localizations</u> (3.2.93).

For a stable ∞-category $\mathcal{C}$, there is a bijection:

$$\Big\{\text{t-structures on } h\mathcal{C}\Big\} \quad \leftrightarrow \quad \Big\{\text{t-localizations of } \mathcal{C}\Big\}$$

This is mentioned briefly in the introduction to [HA, 1.2], but he doesn't fully commit to using the terminology of "t-localizations". Nonetheless he exhibits some equivalent conditions describing a t-localization in [HA, 1.2.1.16].

First a preliminary definition.

Definition 3.2.92. [quasisaturated sets]

Let $\mathcal{C}$ be an ∞-category. A set $S \subseteq \mathcal{C}_1$ is called $\boxed{\text{quasisaturated}}$ if it:

(i) contains all equivalences of $\mathcal{C}$;

(ii) satisfies a 2-out-of-3 property: given a 2-simplex in $\mathcal{C}$:

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"] & & \bullet \end{tikzcd}$$

if any two of $\{f, g, h\}$ are in S, then so is the third;

(iii) is closed under pushouts: given a pushout square in $\mathcal{C}$:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "f'"] & \bullet \end{tikzcd}$$

if $f \in S$, then $f' \in S$.

For any set $S \subseteq \mathcal{C}_1$, the set

$$\overline{S} := \{\text{the smallest quasisaturated set containing } S\}$$

is called the $\boxed{\text{quasisaturated set generated by } S}$.

Definition 3.2.93. [t-localizations]

Let $\mathcal{C}$ be a stable ∞-category and $L : \mathcal{C} \to \mathcal{C}$ a localization (2.14.4).

We call $L : \mathcal{C} \to \mathcal{C}$ a $\boxed{\text{t-localization}}$ if either of the following (equivalent) conditions are satisfied:

(1) The image of L is closed under extension: given a fiber sequence (3.1.7) $X \to Y \to Z$ (a triangle in $h\mathcal{C}$), if $X, Z \in L(\mathcal{C})$, then $Y \in L(\mathcal{C})$.

(2) The full subcategories

$$\mathcal{C}_{\geq 0} = \{X \ : \ LX \simeq 0\}$$
$$\mathcal{C}_{\leq -1} = \{X \ : \ LX \simeq X\}$$

describe a t-structure (3.2.82) on $\mathcal{C}$.

(3) The class

$$S := \{f \in \mathcal{C}_1 \ : \ Lf \text{ is an equivalence}\}$$

is generated as a quasisaturated set by some collection of morphisms of the form $\{0 \to X\}$.

(4) The class S as above is generated as a quasisaturated set by the collection

$$\{0 \to X \ : \ LX \simeq 0\}.$$

4 Appendix (Model categories)

> *I find some of this exchange truly depressing. There is a subject of "brave new algebra"and there are myriads of past and present constructions and calculations that depend on having concrete and specific constructions. People who actually compute anything do not use $(\infty, 1)$ categories when doing so. To lay down a challenge, they would be of little or no use there. One can sometimes use $(\infty, 1)$ categories to construct things not easily constructed otherwise, and then one can compute things about them (e.g. work of Behrens and Lawson). But the tools of computation are not $(\infty, 1)$ categorical, and often not even model categorical. People should learn some serious computations, do some themselves, before totally immersing themselves in the formal theory. Note that $(\infty, 1)$ categories are in principle intermediate between the point-set level and the homotopy category level. It is easy to translate into $(\infty, 1)$ categories from the point-set level, whether from model categories or from something weaker. Then one can work in $(\infty, 1)$ categories. But the translation back out to the "old-fashioned" world that some writers seem to imagine expendable lands in homotopy categories. That is fine if that is all that one needs, but one often needs a good deal more. One must be eclectic. Just one old man's view.*
>
> Peter May
> (Mathoverflow post) [MM]

Model categories were introduced by Daniel Quillen in [Qui67] as a way to formalize the notion of an abstract homotopy theory. The typical situation of a homotopy theory is this: one has a category C with a class $W \subseteq \mathrm{Hom}(C)$ of "weak equivalences" which one would like to formally invert (eg. weak homotopy equivalences in topology, or quasi-isomorphisms in homological algebra). One wants to form a "homotopy category" – one in which the weak equivalences have been formally inverted.

For us, model categories are relevant in two ways:

1. Model categories and ∞-categories are both ways of modelling homotopy theories. Underlying any model category is an ∞-category which models the same homotopy theory (2.11).[13]

2. We turn homotopy theory on itself to study a "homotopy theory of homotopy theories," or in other words, a homotopy theory of ∞-categories. For example, an equivalence of ∞-categories (eg. as simplicial sets or as simplicially enriched categories) does not want to be an isomorphism, which turns out to be too strong. Instead we would like a weaker equivalence (resp. "weak categorical equivalence" or "Dwyer-Kan equivalence"). The homotopy theory of ∞-categories can be described in model-categorical terms. This is useful for example when travelling between models of ∞-categories.

Definition 4.0.1. [localization of a category with weak equivalences]

Let (C, W) be a pair where C is a category and $W \subseteq \mathrm{Hom}(C)$ is a collection of morphisms that one would like to formally invert. In particular, they should contain all the isomorphisms of C.

The $\boxed{\text{localization}}$ of C at W is a functor $\gamma : C \to C[W^{-1}]$ (we may sometimes also refer to the category $C[W^{-1}]$ as the localization) such that for any morphism $w \in W$, the image $\gamma(w) \in \mathrm{Hom}(C[W^{-1}])$ is an isomorphism, and which satisfies the universal property that given any functor $F : C \to D$ such that for

[13] Not all ∞-categories arise in this way.

all $w \in W$ the image $F(w) \in D$ is an isomorphism, then F factors through γ:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\gamma"'] & D \\ C[W^{-1}] \arrow[ur, dotted] & \end{tikzcd}$$

Definition 4.0.2. [homotopy category as a localization]

Let (C, W) be a category C with weak equivalences.

The $\boxed{\text{homotopy category}}$ $ho(C) = C[W^{-1}]$, which localizes C at its weak equivalences. In other words, it turns weak equivalences into isomorphisms.

One can construct a localization of a category with weak equivalences by hand [Bal21, Def. 2.2.9] by first forming a free category $F(C, W^{-1})$, whose objects are objects of C, and in which there is a morphism $x \to y$ for every zig-zag: a string in C of the form

$$x \to \bullet \xleftarrow{\sim} \bullet \to \cdots \to \bullet \xleftarrow{\sim} \bullet \to y,$$

where each backwards map (marked with a $\sim$) is a weak equivalence.

Then $C[W^{-1}]$ is a quotient category of $F(C, W^{-1})$ by the relations:

- $\mathrm{id}_x = (x \xrightarrow{\mathrm{id}} x)$ for all $x \in C$;
- $(x \xrightarrow{f} y \xleftarrow{\mathrm{id}_y} y \xrightarrow{g} z) = (x \xrightarrow{gf} z)$ for all $(f : x \to y)$ and $(g : y \to z)$ in C;
- $\mathrm{id}_x = (x \xrightarrow{w} y \xleftarrow{w} x)$ for all $(w : x \to y) \in W$;
- $\mathrm{id}_y = (y \xleftarrow{w} x \xrightarrow{w} y)$ for all $(w : x \to y) \in W$.

Here the functor $\gamma : C \to C[W^{-1}]$ sends a morphism $f : x \to y$ to a string of length 1. Notice that by the latter two relations above, for every $w \in W$, the map $\gamma(w)$ is made an isomorphism. In fact there is an isomorphism in $C[W^{-1}]$ for any string in C where all the morphisms are in W:

$$\bullet \xrightarrow{\sim} \bullet \xleftarrow{\sim} \cdots \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet.$$

The problem with defining a homotopy category in this way is that $C[W^{-1}]$ may not form an actual category: since zig-zags are defined with arbitrary length, it may happen that $\mathrm{Hom}_{C[W^{-1}]}(x,y)$ may not form sets for all objects.

If C has some extra structure of a model category, there is an alternate description of the homotopy category, which is equivalent as a category to the above construction, and involves zig-zags only of length 3. This will mean that localizing a model category forms an honest category (4.0.31).

Definition 4.0.3. [model categories]

A $\boxed{\text{model category}}$ M is a category with three distinguished classes of morphisms:

- weak equivalences: W_M (denoted $\xrightarrow{\sim}$),
- cofibrations: Cof_M (denoted $\hookrightarrow$ or $\rightarrowtail$),
- fibrations: Fib_M (denoted $\twoheadrightarrow$).

These are allowed to overlap, and we call maps in $W \cap Fib$ acyclic fibrations (or trivial fibrations) and maps in $W \cap Cof$ acyclic cofibrations (or trivial cofibrations).

The category M along with the classes $W, Fib, Cof \subseteq \mathrm{Hom}(M)$ should satisfy the following axioms (MC1-5):

(MC1) (completeness/co-completeness): M has all small limits and colimits.

(MC2) (2-out-of-3): Weak equivalences satisfy the "2-out-of-3" property: given composable maps f and g in M, if any two maps in $\{f, g, gf\}$ are in W, then so is the third.

(MC3) (retracts): Each of the three classes $\{W, Fib, Cof\}$ is closed under retracts: given a diagram in M of the form:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "g"'] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"'] \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

then $f \in W$ (resp. $\in Fib, Cof$) implies $g \in W$ (resp. $\in Fib, Cof$).

(MC4) (weak factorization): Any map f in M can be factored as either an acylic cofibration followed by a fibration, or a cofibration followed by an acyclic fibration:

$$\begin{tikzcd}
\bullet \arrow[rr, "f"] \arrow[dr, tail, "\sim"'] & & \bullet \\
& \bullet \arrow[ur, two heads] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\bullet \arrow[rr, "f"] \arrow[dr, tail] & & \bullet \\
& \bullet \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

(MC5) (weak factorization lifting): Given a commutative square in M, a "lift" (the dotted diagonal maps below) exists if the vertical maps are either an acyclic cofibration on the left and a fibration on the right, or a cofibration on the left and an acyclic fibration on the right:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, tail, "\sim"'] & \bullet \arrow[d, two heads] \\
\bullet \arrow[r] \arrow[ur, dotted] & \bullet
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, tail] & \bullet \arrow[d, two heads, "\sim"] \\
\bullet \arrow[r] \arrow[ur, dotted] & \bullet
\end{tikzcd}$$

Remark 4.0.4. Quillen's (MC1) [Qui67] actually only requires M to have finite limits and colimits, but in practice it helps to have all small (co-) limits. Most authors now choose to require completeness and co-completeness as part of the axioms.

Remark 4.0.5. Most people [Bal21, Rk. 2.1.4] assume that the factorizations of (MC4) are functorial.

Definition 4.0.6. [lifting properties] Let M be a category.

A $\boxed{\text{lifting problem}}$ for a pair (i, j) of morphisms in M is a commutative square in M of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "j"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

A solution or lift to the lifting problem is a morphism (the dotted arrow above) making the diagram commute.

If such a lift exists we say that i has the $\boxed{\text{left lifting property}}$ (LLP) with respect to j, or equivalently that j has the $\boxed{\text{right lifting property}}$ (RLP) with respect to i.

We can write this succinctly as

$$i \boxslash j,$$

which may be read as "i has the LLP with respect to j and (equivalently) j has the RLP with respect to i". Given classes of maps $I, J \subseteq \mathrm{Hom}(M)$, we write

$$i \boxslash J$$

to mean that "i has the LLP with respect to any map $j \in J$" ($I \boxslash j$ is defined similarly). We denote by RLP(I) and LLP(J) the following:

$$\mathrm{RLP}(I) := \{j \in \mathrm{Hom}(M) \ : \ I \boxslash j\}$$

$$\mathrm{LLP}(J) := \{i \in \mathrm{Hom}(M) \ : \ i \boxslash J\}.$$

Definition 4.0.7. [weak factorization systems]

Let M be a category, and let $A, B \subseteq \mathrm{Hom}(M)$.

The pair (A, B) forms a $\boxed{\text{weak factorization system}}$ on M if

- Any map $f \in \mathrm{Hom}(M)$ factors as a map in A followed by a map in B.

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "A\ni"'] & & \bullet \\ & \bullet \arrow[ur, "\in B"'] & \end{tikzcd}$$

- $A = \mathrm{LLP}(B)$ (equivalently $B = \mathrm{RLP}(A)$). That is, a lift exists in any square of the form:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "A\ni"'] & \bullet \arrow[d, "\in B"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

Remark 4.0.8. Consider a collection

$$(M, W, Cof, Fib)$$

where M is a category and $W, Cof, Fib \subseteq \mathrm{Hom}(M)$ are collections of morphisms. Such a collection is a model category (4.0.3) iff it has all small limits and colimits, and has a <u>model structure</u>, which is defined as the following:

- The collection of weak equivalences W is closed under retracts and 2-out-of-3.

- The pairs $(\mathrm{Cof} \cap W, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ are both weak factorization systems on $\mathcal{M}$.

Remark 4.0.9. A category may have more than one model structure. Sometimes one takes a weaker/stronger kind of weak equivalence. More interestingly, there can even be more than one model structure on a category with the same weak equivalences.

Example 4.0.10. Often model structures are named after people, and we denote categories with a certain model structure by a subcript. For example **Top** with the Quillen model structure will be written $\mathbf{Top}_{Quillen}$.

(i) $\mathbf{Top}_{Quillen}$ [Bal21, §7.1]:

- $W = \{\text{weak homotopy equivalences}\}$

  A map $f : X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on homotopy groups $f_* : \pi_n(X) \xrightarrow{\sim} \pi_n(Y)$ for all $n \in \mathbb{N}$.

- $Fib = \{\text{Serre fibrations}\} = \mathrm{RLP}\left(\{D^n \times \{0\} \hookrightarrow D^n \times I\}_{n\in\mathbb{N}}\right)$

  Here D^n is the n-disk, and so a Serre fibration is a map with the right lifting property against the inclusion of any n-disk to one end of a cylinder.

- $Cof = \mathrm{LLP}(Fib \cap W)$.

(ii) $\mathbf{Top}_{Strøm}$ (or sometimes $\mathbf{Top}_{Hurewicz}$) [Bal21, §7.2]:

- $W = \{\text{(strong) homotopy equivalences}\}$

  A map $f : X \to y$ is a homotopy equivalence of spaces if there is a map $g : Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$, where $\sim$ is the relation of homotopy.

- $Fib = \{\text{Hurewicz fibrations}\} = \mathrm{RLP}(\{X \times \{0\} \hookrightarrow X \times I\}_{X \in \mathbf{Top}})$

  A Hurewicz fibration is a map with right lifting against the inclusion of any space into a cylinder. (In particular, a Hurewicz fibration is a Serre fibration.)

- $Cof = \{\text{closed Hurewicz cofibrations}\} = \mathrm{LLP}(Fib \cap W)$.

  [Bal21, Rk. 7.2.4].

(iii) $\mathbf{sSet}_{Quillen}$ (or $\mathbf{sSet}_{Kan-Quillen}$) [Bal21, §6.1]:

- $W = \{\text{weak homotopy equivalences, a.k.a. Kan equivalences}\}$

  A Kan equivalence is a map of simplicial sets $F : X \to Y$ whose geometric realization $|F| : |X| \to |Y|$ (2.1.16) is a weak homotopy equivalence of spaces (a weak equivalence in $\mathbf{Top}_{Quillen}$).

- $Fib = \{\text{Kan fibrations (2.7.1)}\} = \mathrm{RLP}\left(\left\{\Lambda^n_i \hookrightarrow \Delta^n \ : \ 0 \le i \le n\right\}\right)$.

- $Cof = \{\text{monomorphisms}\}$:

  A simplicial map $f : X \to Y$ is a monomorphism iff it is levelwise injective; ie. each function $f_n : X_n \to Y_n$ is injective.

(iv) $\mathbf{Cat}_{canonical}$ [Bal21, §9.1]:

- $W = \{\text{equivalences of categories}\}$

- $Fib = \{\text{isofibrations}\} = \mathrm{RLP}([0] \to J)$

  Here J is the walking isomorphism: a category consisting of two objects and an isomorphism between them. Drawing out the lifting diagram,

$$\begin{tikzcd} {[0]} \arrow[r] \arrow[d] & C \arrow[d, "F"] \\ J \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

this means that a functor $F : C \to D$ is an isofibration iff any isomorphism in D whose source or target has pre-image in C lifts to an isomorphism in C.

- Cof = {functors that are injective on objects}

(v) $\mathbf{sCat}_{Bergner}$ [Bal21, §11.1]:

- W = {Dwyer-Kan equivalences}

  A Dwyer-Kan equivalence [DK80b] is a map $f : \mathcal{C} \to \mathcal{D}$ between simplicially enriched categories such that (1) the induced functor $hf : h\mathcal{C} \to h\mathcal{D}$ (2.4.6) is an equivalence of categories, and (2) for every hom-sset $\mathcal{C}(c,c')_\bullet$, the induced map

  $$\mathcal{C}(c,c')_\bullet \to \mathcal{D}(fc,fc')_\bullet$$

  is a Kan equivalence of simplicial sets (a weak equivalence in the Quillen model structure $\mathbf{sSet}_{Quillen}$).

- Fib: A map $f : \mathcal{C} \to \mathcal{D}$ of simplicially enriched categories is a fibration in $\mathbf{sCat}_{Bergner}$ if (1) $hf : h\mathcal{C} \to h\mathcal{D}$ (2.4.6) is an isofibration (a weak equivalence in $\mathbf{Cat}_{canonical}$), and (2) for every hom-sset $\mathcal{C}(c,c')_\bullet$, the map

  $$\mathcal{C}(c,c')_\bullet \to \mathcal{D}(fc,fc')_\bullet$$

  is a Kan fibration (a fibration in $\mathbf{sSet}_{Quillen}$).

(vi) $\mathbf{sSet}_{Joyal}$ [Bal21, §6.2]:

- W = {weak categorical equivalences}

A weak categorical equivalence is a map between simplicial sets $f : X \to Y$ such that for any ∞-category $\mathcal{C}$, the induced map

$$h\mathrm{Fun}(Y, \mathcal{C}) \to h\mathrm{Fun}(X, \mathcal{C})$$

is an ordinary equivalence of categories, where $\mathrm{Fun}(\bullet, \bullet)$ is the functor ∞-category (2.8.2) and h is the homotopy construction (2.4.9). Equivalently, f is a weak categorical equivalence iff the functor $\mathfrak{C}f : \mathfrak{C}X \to \mathfrak{C}Y$ formed by the functor $\mathfrak{C} : \mathbf{sSet} \to \mathbf{Cat}$ (2.2.12) is a Dwyer-Kan equivalence (a weak equivalence in $\mathbf{sCat}_{Bergner}$).

- $Cof = \{\text{monomorphisms of ssets}\}$
- $Fib = \mathrm{RLP}(Cof \cap W)$.

Remark 4.0.11. Given a class of weak equivalences and any one of the other two classes (either fibrations or cofibrations) completely determines the other via lifting properties. For example, if we are given a category M with classes W, Fib, then cofibrations are completely determined as being those maps with the left lifting property against acyclic fibrations: $Cof = \mathrm{LLP}(W \cap Fib)$.

It's sometimes the case that only one of the classes of either fibrations or cofibrations is actually easily describable. It's often the case that one will define only weak equivalences and either fibrations or cofibrations, and then define the rest in terms of lifting properties.

Remark 4.0.12. Model structures present homotopy theories.

- The model structures $\mathbf{Top}_{Quillen}$ and $\mathbf{sSet}_{Quillen}$ both present the homotopy theory of spaces.

- The model structures $\mathbf{sCat}_{Bergner}$ and $\mathbf{sSet}_{Joyal}$ both present the homotopy theory of $\underline{\infty\text{-categories}}$.

Armed with notions of fibrations and cofibrations in a model category M, we identify particularly nice objects that we call “fibrant” or “cofibrant”.

Definition 4.0.13. [fibrant and cofibrant objects]

Let M be a model category. Let $0 \in M$ be an initial object, and $1 \in M$ be a final object.

We call an object $X \in M$...

- $\boxed{\text{fibrant}}$ if the unique map $X \to 1$ is a fibration;
- $\boxed{\text{cofibrant}}$ if the unique map $0 \to X$ is a cofibration.

If X is both fibrant and cofibrant, we will call it $\boxed{\text{fibrant-cofibrant}}$ or bifibrant.

Definition 4.0.14. [fibrant/cofibrant replacement]

For any object $X \in M$, we can factor the maps $0 \to X$ and $X \to 1$ using (MC4):

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, tail] & & X \\ & X^c \arrow[ur, two heads, "\sim"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[rr] \arrow[dr, tail, "\sim"'] & & 1 \\ & X^f \arrow[ur, two heads] & \end{tikzcd}$$

resulting in an object X^c that we call a $\boxed{\text{cofibrant replacement}}$ of X, and an object X^f that we call a $\boxed{\text{fibrant replacement}}$ of X.[14]

Remark 4.0.15. Cofibrant and fibrant replacements of an object $X \in M$ are not strictly unique, but unique $\underline{\text{up to homotopy}}$. That is, any two cofibrant replacements X^c and $(X^c)'$ (resp. fibrant replacements X^f and $(X^f)'$) are connected by

[14]Some authors use the notation QX for cofibrant replacement and RX for fibrant replacement, but we won't.

a zig-zag of weak equivalences:

$$X^c \xrightarrow{\sim} X \xleftarrow{\sim} (X^c)' \qquad \text{(resp. } X^f \xleftarrow{\sim} X \xrightarrow{\sim} (X^f)'\text{)}$$

In the homotopy category (4.0.30), such zig-zags form isomorphisms, so cofibrant and fibrant replacements are unique when considered as objects in the homotopy category.

Remark 4.0.16. Any morphism between objects in a model category induces a morphism between (co-) fibrant replacements. For example, given a map $\varphi : X \to Y$ in M, it sits inside a lifting problem:

$$\begin{tikzcd}
0 \arrow[rr, rightarrowtail] \arrow[d, rightarrowtail] & & Y^c \arrow[d, two heads, "\sim"] \\
X^c \arrow[r, two heads, "\sim"'] \arrow[urr, dotted, "\varphi^c"] & X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

which admits a solution $\varphi^c : X^c \to Y^c$. (Similarly we get a map $\varphi^f : X^f \to Y^f$.)

Assuming functorial factorization (4.0.5), we can form cofibrant and fibrant replacement functors

$$(-)^c : M \to M$$
$$(-)^f : M \to M.$$

Remark 4.0.17. We can further factor $X^c \to 1$ (resp. $0 \to X^f$) to get an object

$(X^c)^f$, a fibrant replacement of the cofibrant replacement of X (resp. $(X^f)^c$).

$$
\begin{tikzcd}
 & X & \\
0 \arrow[ur] \arrow[r, tail] & X^c \arrow[u, two heads, "\sim"'] \arrow[d, tail, "\sim"'] \arrow[r] & 1 \\
 & (X^c)^f \arrow[ur, two heads] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[d, tail, "\sim"'] \arrow[dr] & \\
0 \arrow[r] \arrow[dr, tail] & X^f \arrow[r, two heads] & 1 \\
 & (X^f)^c \arrow[u, two heads, "\sim"'] &
\end{tikzcd}
$$

Both $(X^c)^f$ and $(X^f)^c$ are fibrant-cofibrant, and both are connected via zig-zags of weak equivalences to X. We call such objects fibrant-cofibrant replacements of X. Since $(X^c)^f$ and $(X^f)^c$ are connected to each other via a zig-zag of weak equivalences, these fibrant-cofibrant replacements are unique up to homotopy – ie. unique in the homotopy category.

Remark 4.0.18. Restricting to fibrant/cofibrant objects of M forms full subcategories:

- $M^c \subseteq M$ is the subcategory of cofibrant objects.
- $M^f \subseteq M$ is the subcategory of fibrant objects.
- $M^{cf} \subseteq M$ is the subcategory of fibrant/cofibrant objects.

Remark 4.0.19. These cofibrant and fibrant replacements represent particularly nice versions of objects in a given model category. For example they facilitate the calculations of homotopy limits and colimits, as well as derived functors.

By the way that lifting problems are organized, we can think of cofibrations as morphisms that are nice to map along, and fibrations as morphisms that are nice to factor through. In particular, cofibrant objects are nice to map out of, and fibrant objects are nice to map into.

Example 4.0.20. [examples of fibrant and cofibrant objects]

- $\mathbf{Top}_{Quillen}$: Every object is fibrant and the cofibrant objects are retracts of cell complexes [Bal21, Prop. 7.1.3].

  In particular, CW-complexes are fibrant-cofibrant, and CW approximation can be thought of as a cofibrant replacement.

- $\mathbf{sSet}_{Quillen}$: Every object is cofibrant and the fibrant objects are precisely the Kan complexes [Bal21, Def. 6.1.4].

- $\mathbf{sSet}_{Joyal}$: Every object is cofibrant and the fibrant objects are precisely the quasi-categories, ie. ∞-categories [Bal21, Prop. 6.2.4].

- $\mathbf{sCat}_{Bergner}$: Fibrant objects are Kan-enriched categories, ie. ∞-categories [Bal21, Prop. 11.1.6].

Remark 4.0.21. The homotopy theory of topological spaces hinges on the unit interval $I = [0,1]$. We define a homotopy between two maps $X \rightrightarrows Y$ as a map out of a cylinder object ($X \times I \to Y$), or alternatively as a map into a path object ($X \to Y^I$), both of which are mediated by the interval I. We have corresponding constructions more generally in a model category.

Definition 4.0.22. [cylinder and path objects]

Let X be an object in a model category M.

- A $\boxed{\text{cylinder object}}$ of X is a factorization of the codiagonal $\begin{pmatrix}\mathrm{id}\\ \mathrm{id}\end{pmatrix} : X \sqcup X \to$

X:

$$\begin{tikzcd}
X \sqcup X \arrow[rr, "\begin{pmatrix}\mathrm{id}\\ \mathrm{id}\end{pmatrix}"] \arrow[dr, tail, "\begin{pmatrix}i_0\\ i_1\end{pmatrix}"'] & & X \\
& \mathrm{Cyl}(X) \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

- A $\boxed{\text{path object}}$ of X is a factorization of the diagonal $(\mathrm{id}, \mathrm{id}) : X \to X \times X$:

$$\begin{tikzcd}
X \arrow[rr, "(\mathrm{id},\mathrm{id})"] \arrow[dr, tail, "\sim"'] & & X \times X \\
& \mathrm{Path}(X) \arrow[ur, two heads, "(p_0,p_1)"'] &
\end{tikzcd}$$

Example 4.0.23. [examples of cylinder and path objects]:

- In $\mathbf{Top}_{Quillen}$, $\mathrm{Cyl}(X) = X \times I$ and $\mathrm{Path}(X) = X^I = \mathbf{Top}(I, X)$.
- In $\mathbf{sSet}_{Quillen}$, $\mathrm{Cyl}(X) = X \times \Delta^1$ and $\mathrm{Path}(X) = X^{\Delta^1} = \mathbf{sSet}(\Delta^1, X)$.

Remark 4.0.24. Path and cylinder objects are, like fibrant/cofibrant replacements, unique up to homotopy. Any two choices of a cylinder/path object for a given object are connected by zig-zags:

$$\mathrm{Cyl}(X) \xrightarrow{\sim} X \xleftarrow{\sim} \mathrm{Cyl}(X)', \qquad \text{and} \qquad \mathrm{Path}(X) \xleftarrow{\sim} X \xrightarrow{\sim} \mathrm{Path}(X)'.$$

Definition 4.0.25. [left and right homotopy]

Let $f, g : X \rightrightarrows Y$ be maps in a model category M.

- A $\boxed{\text{left homotopy}}$ between f and g is a map $H : \mathrm{Cyl}(X) \to Y$ so that the following commutes:

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, bend left, "f"] & \\ \mathrm{Cyl}(X) \arrow[r, "H"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, bend right, "g"'] & \end{tikzcd}$$

In this case we say "f is left-homotopic to g" and write $f \sim_\ell g$. Note that although we have defined this as a map H, there is a choice of cylinder object involved.

- A $\boxed{\text{right homotopy}}$ between f and g is a map $H : X \to \mathrm{Path}(Y)$ so that the following commutes:

$$\begin{tikzcd} & Y \\ X \arrow[ur, bend left, "f"] \arrow[r, "H"] \arrow[dr, bend right, "g"'] & \mathrm{Path}(Y) \arrow[u, "p_0"'] \arrow[d, "p_1"] \\ & Y \end{tikzcd}$$

In this case we say "f is right-homotopic to g" and write $f \sim_r g$. Note that although we have defined this as a map H, there is a choice of path object involved.

If f is both right- and left- homotopic to g, then we say that f is $\boxed{\text{homotopic}}$ to g, and write $f \sim g$.

Proposition 4.0.26. Let $f, g : X \rightrightarrows Y$ be maps in a model category M.

- If X is cofibrant, then $(f \sim_\ell g) \implies (f \sim_r g)$.
- If Y is fibrant, then $(f \sim_r g) \implies (f \sim_\ell g)$.
- If X is cofibrant <u>and</u> Y is fibrant, then $(f \sim_\ell g)$ iff $(f \sim_r g)$ iff $(f \sim g)$. In this case, homotopy forms an equivalence relation on $\mathrm{Hom}_M(X, Y)$.

Definition 4.0.27. [homotopy equivalence]

Let $f : X \to Y$ be a morphism in a model category.

We call f a $\boxed{\text{homotopy equivalence}}$ if there exists a map $f' : Y \to X$ such that $f'f \sim \mathrm{id}_X$ and $ff' \sim \mathrm{id}_Y$.

Proposition 4.0.28. [Whitehead's Theorem][Bal21, Prop. 2.2.7]

Let $f : X \to Y$ be a map between fibrant-cofibrant objects in a model category. Then f is a weak equivalence iff it's a homotopy equivalence.

Remark 4.0.29. In the specific case of the model category $\mathbf{Top}_{Quillen}$, this recovers the original Whitehead theorem: a weak homotopy equivalence between bifibrant objects (in particular between CW-complexes) is a weak homotopy equivalence iff it is a (strong) homotopy equivalence.

Definition 4.0.30. [homotopy category of a model category]

Let M be a model category.

The $\boxed{\text{homotopy category}}$ $ho(M)$ is determined, up to equivalence of categories, as the category defined by:

- (objects) : Fibrant-cofibrant objects of M, ie. $\mathrm{ob}(ho(M)) := \mathrm{ob}(M^{cf})$;
- (morphisms): For any $X, Y \in ho(M)$,

$$\mathrm{Hom}_{ho(M)}(X, Y) := \mathrm{Hom}_M(X, Y)/\sim,$$

  where $\sim$ is the equivalence relation of homotopy (4.0.25).

Theorem 4.0.31. [Bal21, Thm. 2.2.10]

Let M be a model category, and $\gamma : M \to M[W^{-1}]$ the localization at its weak equivalences (4.0.1).

- There is an equivalence of categories $ho(M) \simeq M[W^{-1}]$ given by:

$$ho(M) \simeq M^{cf}[W^{-1}] \hookrightarrow M[W^{-1}]$$

- There are natural isomorphisms for all $X, Y \in M$:

$$\begin{aligned} \mathrm{Hom}_M(X^{cf}, Y^{cf})/\sim &\cong \mathrm{Hom}_M(X^{fc}, Y^{fc})/\sim \\ &\cong \mathrm{Hom}_{M[W^{-1}]}(\gamma(X), \gamma(Y)) \end{aligned}$$

  where $\sim$ is the relation of homotopy (4.0.25).

  In particular, this shows that the localization $M[W^{-1}]$ of a model category forms an honest category (without size issues).

- The functor $\gamma : M \to M[W^{-1}]$ identifies left- or right- homotopic morphisms.

- Given a morphism $f \in \mathrm{Hom}(M)$ such that $\gamma(f) \in \mathrm{Hom}(M[W^{-1}])$ is an isomorphism, then f is a weak equivalence.

4.1 Quillen adjunctions and Quillen equivalences

Remark 4.1.1. In practice, functors of model categories come in adjoint pairs. To see why, notice that in an adjunction $(F, G) : C \rightleftarrows D$, given maps $(A \xrightarrow{f} B) \in C$ and $(X \xrightarrow{g} Y) \in D$, we have a correspondence between lifting problems and their solutions:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & GX \arrow[d, "Gg"] \\ B \arrow[r] \arrow[ur, dotted] & GY \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} FA \arrow[r] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ FB \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

(in C) (in D)

When C and D are model categories, such lifts exist if f and Ff are cofibrations and g and Gg are fibrations (and one of the maps in either lifting problem is a weak equivalence).

Definition 4.1.2. [Quillen adjunctions, Quillen functors]

Let M and N be model categories and suppose there is an adjunction:

$$F : M \rightleftarrows N : G.$$

We say that (F, G) is a $\boxed{\text{Quillen adjunction}}$ if either of the following (equivalent) conditions are satisfied:

- F preserves cofibrations and acyclic cofibrations;
- G preserves fibrations and acyclic fibrations;
- F preserves cofibrations and G preserves fibrations;
- F preserves acyclic cofibrations and G preserves acyclic fibrations.

In this case, we call F a $\boxed{\text{left Quillen}}$ functor and G a $\boxed{\text{right Quillen}}$ functor.

Definition 4.1.3. [derived functors]

Let $F : M \rightleftarrows N : G$ be a Quillen adjunction.

- The $\boxed{\text{left derived functor}}$ of F is the composite:

$$\mathbb{L}F : ho(M) \xrightarrow{ho((-)^c)} ho(M) \xrightarrow{ho(F)} ho(N).$$

- The $\boxed{\text{right derived functor}}$ of G is the composite:

$$\mathbb{R}G : ho(N) \xrightarrow{ho((-)^f)} ho(N) \xrightarrow{ho(G)} ho(M).$$

Remark 4.1.4. A Quillen adjunction

$$F : M \rightleftarrows N : G$$

forms an adjunction on homotopy categories:

$$\mathbb{L}F : ho(M) \rightleftarrows ho(N) : \mathbb{R}G.$$

Definition 4.1.5. [Quillen equivalences]

A Quillen adjunction (F, G) is called a $\boxed{\text{Quillen equivalence}}$ if the derived adjunction $(\mathbb{L}F, \mathbb{R}G)$ is an equivalence of homotopy categories.

Example 4.1.6. [Quillen equivalences]

- $(|-|, \mathrm{Sing}) : \mathbf{sSet}_{Quillen} \rightleftarrows \mathbf{Top}_{Quillen}$ is a Quillen equivalence [Qui67].

  These model the homotopy theory of <u>spaces</u> up to weak equivalence.

- $(\mathfrak{C}, N_\Delta) : \mathbf{sSet}_{Joyal} \rightleftarrows \mathbf{sCat}_{Bergner}$ is a Quillen equivalence [HTT 2.2.5.1].

  These model the homotopy theory of <u>∞-categories</u> up to weak categorical equivalence.

4.2 Homotopy limits and colimits

Definition 4.2.1. [homotopy limits/colimits in a model category]

Let M be a model category and let I be a diagram category. Let $M^I = \mathbf{Cat}(I, M)$ be the category whose objects are functors $I \to M$ (I-shaped diagrams in M), and whose morphisms are natural transformations.

The constant functor $c : M \to M^I$ sends any object to its constant diagram. Just as ordinary colimits can be described with left and right adjoints:

$$\begin{tikzcd} M \arrow[d, "c"] \\ M^I \arrow[u, bend left=60, "\mathrm{colim}_I"] \arrow[u, bend right=60, "\lim_I"'] \end{tikzcd}$$

We can consider M^I as a category with weak equivalences by taking the weak equivalences $W_{M^I} \subseteq \mathrm{Hom}(M^I)$ to be those natural transformations whose objectwise maps are in W_M. That is a weak equivalence in M^I is a natural transformation $\alpha : F \to G$ such that $(Fm \xrightarrow{\alpha_m} Gm) \in W_M$ for all $m \in M$.

The $\boxed{\text{homotopy colimit and limit}}$ then (if they exist) are the left- and right-derived functors respectively:

$$\begin{tikzcd} M^I \arrow[r, "\mathrm{colim}"] \arrow[d] & M \arrow[d] \\ \mathrm{ho}(M^I) \arrow[r, "\mathrm{hocolim}=\mathbb{L}\,\mathrm{colim}"'] & \mathrm{ho}(M) \end{tikzcd} \qquad \begin{tikzcd} M^I \arrow[r, "\lim"] \arrow[d] & M \arrow[d] \\ \mathrm{ho}(M^I) \arrow[r, "\mathrm{holim}=\mathbb{R}\lim"'] & \mathrm{ho}(M) \end{tikzcd}$$

4.3 Stable model categories

[SS03]

Among homotopy theories, there are certain special ones that are <u>stable</u>.

These are presented by stable ∞-categories (3.1.10), or their model-categorical analogue, stable model categories.

Definition 4.3.1. [stable model categories]

A model category is called $\boxed{\text{stable}}$ if:

- it is pointed (has a zero object);
- suspension and loop functors are self-equivalences on the homotopy category.

Remark 4.3.2. This definition corresponds precisely to one of the equivalent conditions defining a stable ∞-category (3.1.31).

Proposition 4.3.3. If M is a stable model category (4.3.1), then its underlying ∞-category M^∞ (2.11.5)is a stable ∞-category (3.1.10).

Proof. A stable ∞-category is one which admits cofibers and on which suspension was an equivalence (3.1.31). An equivalence of stable ∞-categories is detectable on the level of homotopy categories [BGT13, Cor. 5.11]. Taking $\mathcal{C} = \mathcal{D} = M^\infty$, if suspension forms an equivalence $\Sigma : ho(M) = hM^\infty \to hM^\infty$, this lifts to an equivalence of stable ∞-categories $M^\infty \to M^\infty$. □

4.4 Combinatorial model categories

A combinatorial model category is one that is determined by a sufficiently small amount of data – in particular it is generated from a small set of cofibrations between small objects. This allows one a certain amount of control with them.

They are closely related to presentable ∞-categories (2.13) which, similarly, are ∞-categories determined by a sufficiently small amount of data.

Definition 4.4.1. [combinatorial model categories]

A model category M is called $\boxed{\text{combinatorial}}$ if the following are satisfied:

(i) M is locally presentable (2.13.12) – that is:

- M has all small colimits (this is baked into our definition of a model category);
- M is accessible (2.13.11) – that is, it is locally small (this is also baked into our definition of a model category), and there exists a regular cardinal κ (2.13.1) and a set $G \subseteq \mathrm{ob}(M)$ of κ-small objects that generate M under κ-filtered colimits (2.13.5).

(ii) M is cofibrantly generated – that is:

- There exists a set of cofibrations $I \subseteq Cof_M$ that generate all cofibrations in M in the following way:

$$Cof_M = \mathrm{LLP}(\mathrm{RLP}(I));$$

- There exists a set of acyclic cofibrations $J \subseteq W_M \cap \mathrm{Cof}_M$ that generate all acyclic cofibrations in M in the following way:

$$Cof_M \cap W_M = \mathrm{LLP}(\mathrm{RLP}(J)).$$

Note that this is equivalent to saying that $Fib = \mathrm{RLP}(J)$.

Proposition 4.4.2. [Dugger's theorem][Dug00, Cor. 1.2]

Any combinatorial model category is Quillen equivalent to a simplicial combinatorial model category.

In particular, this means that the underlying ∞-category of a combinatorial model category is presentable.

4.5 Homotopy limits/colimits

Limits and colimits in an ∞-category (2.9.39) do not, in general, correspond to those in the corresponding homotopy category (2.4.13) (where limits and colimits often don't exist). There are related notions of homotopy limits and homotopy colimits in the context of model categories (4.2.1) as well as in simplicially enriched categories [BK72].

This section will explore some of the ways in which these homotopy limits/colimits are related to limits/colimits in ∞-categories.[15]

Proposition 4.5.1. [HA, 1.3.4.23, 1.3.4.24]

Let M be a combinatorial model category (4.4.1), $F : I \to M^c$ a small diagram of cofibrant objects (4.0.14, 4.0.18). Since a model category is assumed to be complete, the limit $\lim_I F \in M$ exists. Let $\varphi : x \to \lim_I F$ be a map in M.

Then x is a homotopy limit of F (in the model-categorical sense) iff the induced map

$$(NI)^{\triangleleft} \to N(M^c) \to N(M^c)[W^{-1}] = M^{\infty}$$

is a (∞-) limit in the underlying ∞-category M^{∞} (2.11.5).

Dually, a map $\operatorname{colim}_I F \to x$ exhibits x as homotopy colimit of F iff the induced map $(NI)^{\triangleright} \to M^{\infty}$ is a (∞-) colimit in the underlying.

Proposition 4.5.2. [Cis19, Thm. 7.9.8, Rk. 7.9.10]

[15] Some people will even refer to limits/colimits in an ∞-category as "homotopy limits/colimits" to distinguish them from ordinary limits/colimits in the homotopy category (eg. [HTT, 1.2.13]). We will not use this terminology, opting instead to say "(∞-) limits/colimits," or simply "limits/colimits in an ∞-category".

[homotopy limits in a model category vs. (∞-) limits in the underlying]

Let M be a model category and M^∞ its underlying ∞-category (2.11.5). Let I be a diagram category.

If it exists, the homotopy limit functor (4.2.1)

$$\mathrm{holim}_I = \mathbb{R}\lim_I : ho(M^I) \to ho(M)$$

can be recovered from the (∞-) limit functor (2.9.44)

$$\lim_{NI} : (M^\infty)^{NI} \to M^\infty$$

by taking its image under formation of homotopy categories (taking the image via $h : \mathbf{sSet} \to \mathbf{Cat}$).

In other words

$$\mathrm{holim}_I = h(\lim_{NI}) : h((M^\infty)^{NI}) \to h(M^\infty).$$

Proof. The homotopy limit of a model category was defined as a right adjoint of the (derived) constant functor (4.2.1):

$$\begin{tikzcd} ho(M) \arrow[d, "c"'] \\ ho(M^I) \arrow[u, bend right, "\mathrm{holim}_I"'] \end{tikzcd}$$

On the other hand we can look at the constant functor on the underlying ∞-category M^∞:

$$\begin{tikzcd} M^\infty \arrow[d, "c"'] \\ (M^\infty)^{NI} \arrow[u, bend right, dotted] \end{tikzcd}$$

where $(M^\infty)^{NI} = \mathrm{Fun}(NI, M^\infty)$ is the ∞-category of functors $NI \to M^\infty$.

There is an equivalence (2.14.3):

$$\begin{aligned}
\mathrm{Fun}(NI, M^\infty) &= \mathrm{Fun}(NI, NM[W^{-1}]) \\
&\xleftarrow{\sim} \mathrm{Fun}(NI, NM)[(\overline{W})^{-1}] \\
&= N(M^I)[(\overline{W})^{-1}] \\
&= (M^I)^\infty
\end{aligned}$$

where $\overline{W} \subseteq \mathrm{Fun}(NI, NM)_1$ is the set of fiberwise weak equivalences. That is, the ∞-category of functors $NI \to M^\infty$ is equivalent to the underlying of the model category M^I.

The underlying M^∞ and M have the same homotopy category: $h(M^\infty) \simeq ho(M)$. And the homotopy category of

$$(M^\infty)^{NI} \simeq N(M^I)[W^{-1}]$$

is equivalent to the homotopy category $ho(M^I)$.

So the image of the constant

$$c : M^\infty \to (M^\infty)^{NI}$$

under forming homotopy categories $h : \mathbf{sSet} \to \mathbf{Cat}$ is a map:

$$h(c) = \mathrm{const} : ho(M) \to ho(M^I)$$

the usual constant map. And since adjoints on ∞-categories induce adjoints on homotopy categories, we can consider a right adjoint

$$\mathrm{holim}_I : \underbrace{ho(M^I)}_{=h(M^\infty)^{NI}} \to \underbrace{ho(M)}_{=hM^\infty}$$

as the image $h(\lim_I)$ of the limit functor in ∞-land.

$$\begin{tikzcd} M^\infty \arrow[d, "c"'] \\ \mathrm{Fun}(NI, M^\infty) \arrow[u, bend right, dotted, "\lim_I"'] \end{tikzcd} \qquad \begin{tikzcd} ho(M) \arrow[d] \\ ho(M^I) \arrow[u, bend right, dotted, "h(\lim_I)=\mathrm{holim}_I"'] \end{tikzcd}$$

□

In the case that our model category is <u>combinatorial</u> (4.4.1), we can compute limits/colimits in the underlying ∞-category by homotopy limits/colimits in the model category.

Proposition 4.5.3. [Hin15, Cor. 1.5.2]

Let M be a combinatorial model category (4.4.1), and let M^∞ be its underlying ∞-category (2.11.5).

Then limits/colimits in M^∞ can be calculated as homotopy limits/colimits in M. That is, the Quillen adjunctions

$$\mathrm{colim}_I : M^I \rightleftarrows M : c$$
$$c : M \rightleftarrows M^I : \lim_I$$

(if they exist) induce adjunctions on the underlying ∞-categories, and since

$(M^I)^\infty \simeq (M^\infty)^{NI}$, we can write these as:

$$\operatorname{colim}_{NI} : (M^\infty)^{NI} \rightleftarrows M^\infty : c$$
$$c : M^\infty \rightleftarrows (M^\infty)^{NI} : \lim_{NI}$$

Proposition 4.5.4. [HTT, 4.2.4.1]

[(∞-) limits vs. homotopy limits of simplicial categories]

Let C, I be fibrant simplicial categories (ie. Kan-enriched), and $F : I \to C$ be a functor. Let X be an object of C and say we have a family of maps $\alpha = \{\alpha_i : F(i) \to X\}_{i \in I}$.

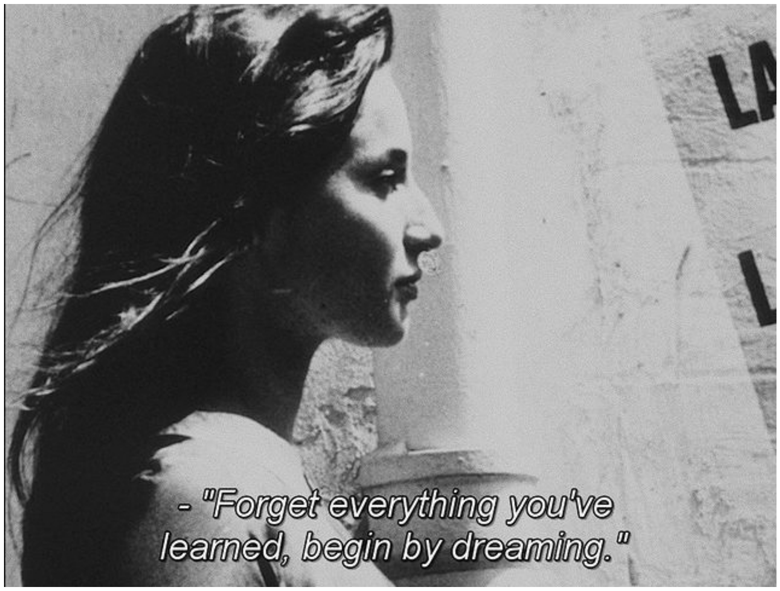
Then the following are equivalent:

(1) The maps $\alpha = \{\alpha_i\}_{i \in I}$ describe a natural transformation exhibiting $X = \operatorname{hocolim}_I F$, as a homotopy colimit of simplicial categories.

(2) The maps $\alpha = \{\alpha_i\}_{i \in I}$ induce a map $\overline{F} : (NI)^{\triangleright} \to NC$:

$$\begin{tikzcd}
NI \arrow[d, hook] \arrow[dr, bend left, "NF"] & \\
(NI)^{\triangleright} \arrow[r, "\overline{F}"] & NC \\
\Delta^0 \arrow[u, hook] \arrow[ur, bend right, "X"'] &
\end{tikzcd}$$

exhibiting $X = \operatorname{colim}_{NI} NF \in NC$ (a (∞-) colimit).



from “L’Ete” (1968) – Marcel Hanoun

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