$(\infty,2)$-categories Workshop.

"2-Comical Sets"

Organization.

{ Definitions + Motivation
- cubical sets
- marked cubical sets + comical sets.
- the 2-comical model structure.

~20 mins

{ Relations to other constructions
- 2-cats.
- 1-comical sets & quasicats
- 2-complicial sets.

~10 mins

{ OK ... Why?.
Gray $\otimes$

~10 mins.

References: [CKM] = "A Cubical Model for $(\infty,n)$-categories" – Campion, Kapulkin, Maehara

[DKM] = "Equivalence of Cubical & Simplicial Models of $(\infty,n)$-Categories"
– Doherty, Kapulkin, Maehara.

## §1 – Definitions & motivation.

### §1.1 – Cubical sets.

| use ↓ to model → | $(\infty,1)$-cats. | $(\infty,2)$-cats. |
|---|---|---|
| $sSet = \hat{\Delta}$ | quasicategories. = 1-complicial sets | 2-complicial sets. |
| $cSet = \hat{\square}$ | cubical quasicats (1-comical sets) | 2-comical sets. = COMplicial + cubICAL |

**Def.** The cube category $\square$ is a subcategory $\square \subseteq Poset$ w/:

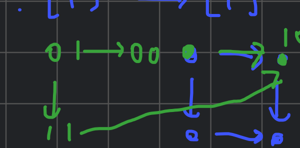
- objects = $[1]^n = \{0 \to 1\}^n$
- morphisms = generated under composition by:

$[1]^0 = 0$
$[1]^1 = (0 \to 1)$
$[1]^2 = \begin{matrix} 00 & \to & 10 \\ \downarrow & & \downarrow \\ 01 & \to & 11 \end{matrix}$

– faces: $\partial_{i,\varepsilon} : [1]^{n-1} \longrightarrow [1]^n$ $\quad i = 1, \ldots, n$, $\varepsilon = 0,1$

$(x_1, \ldots, x_{n-1}) \longmapsto (x_1, \ldots, \varepsilon, \ldots, x_{n-1})$ $\quad$ $i^{th}$ place.

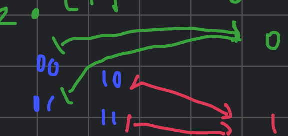
$i=2$, $\varepsilon=0$ $\quad \partial_{2,0} : [1]^1 \to [1]^2$



– degeneracies: $\sigma_i : [1]^n \to [1]^{n-1}$ $\quad i = 1, \ldots, n$

$(x_1, \ldots, x_n) \longmapsto (\hat{x_i})$

$\sigma_2 : [1]^2 \to [1]^1$



$\gamma_{2,0} : [1]^2 \to [1]^1$

– connections. $\gamma_{i,\varepsilon} : [1]^n \to [1]^{n-1}$ $\quad i = 1, \ldots, n-1$, $\varepsilon = 0,1$ $\quad$ $i^{th}$ place.

$\gamma_{i,0} : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, \max(x_i, x_{i+1}), \ldots, x_n)$

$\gamma_{i,1} : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, \min(x_i, x_{i+1}), \ldots, x_n)$

**Rk.** These maps $\partial, \sigma, \gamma$ satisfy "cubical identities" which are similar to the simplicial analogs ... (+ rules describing connections).

[omitted]

**Def.** A **cubical set** is a presheaf on $\square$, i.e. a functor $\square^{op} \to Set$

A map between them is a nat'l transf. of such functors.

convention.
$$\begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "2"'] \arrow[dr, "3"] & {} \\ {} & {} \end{tikzcd}$$

**Eg.** • For $n = 0, 1, 2, \dots$ the standard $n$-cube:

$$\square^n := よ([1]^n) = \mathrm{Hom}_{\square}(-, [1]^n)$$

$$\square^0 = 0 \qquad \square^1 = 0 \to 1$$

$$\square^2 = \begin{tikzcd} (0,0) \arrow[r, "\partial_{2,0}"] \arrow[d, "\partial_{1,0}"'] & (1,0) \arrow[d, "\partial_{1,1}"] \\ (0,1) \arrow[r, "\partial_{2,1}"'] & (1,1) \end{tikzcd}$$

direction

• For $n = 0, 1, \dots$ the boundary of $\square^n$

$$\partial \square^n := \bigcup_{\substack{i = 1, \dots, n \\ \varepsilon = 0, 1}} \partial_{i,\varepsilon} \square^{n-1}$$

$$\partial \square^2 = \begin{tikzcd} (0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\Rightarrow"] & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd}$$

• For $n \geq 1$, $i = 1, 2, \dots$, $\varepsilon = 0, 1$ the $(i,\varepsilon)^{th}$ open box

$$\sqcap^n_{i,\varepsilon} = \partial \square^n \setminus \partial_{i,\varepsilon} \square^{n-1}$$

$$\sqcap^2_{2,1} = \begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\ (0,1) \arrow[r, phantom, "\not\to"] \arrow[r, phantom, "\partial_{2,1}"'] & (1,1) \end{tikzcd}$$

**Rk:** • Degeneracies of $\square^1 = (0 \xrightarrow{f} 1)$:

$$f\sigma_2 : \begin{tikzcd} 0 \arrow[r, "f"] \arrow[d, equal] & 1 \arrow[d, equal] \\ 0 \arrow[r, "f"'] & 1 \end{tikzcd} \qquad f\sigma_1 : \begin{tikzcd} 0 \arrow[r, equal] \arrow[d, "f"'] & 0 \arrow[d, "f"] \\ 1 \arrow[r, equal] & 1 \end{tikzcd}$$

convention: $=$ is a degenerate edge.

• Connections of $\square^1 = (0 \to 1)$:

$$\gamma^1_{1,0}(0 \to 1) = \begin{tikzcd} 0 \arrow[r, "f"] \arrow[d, "f"'] & 1 \arrow[d, equal] \\ 1 \arrow[r, equal] & 1 \end{tikzcd} \qquad \gamma^1_{1,1} = \begin{tikzcd} 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[d, "f"] \\ 0 \arrow[r, "f"'] & 1 \end{tikzcd}$$

**Rk.** Note that connections are an additional type of degeneracy that shows up in this setting. In the cubical setting, $n$-cubes are called deg. if they arise as the face or the connection of another $n$-cube.

**Rk.** Any map in $\square$ can be written as a composite of the form:

(face maps)(connections)(degeneracies)

in a specific order.

We'll call this the "standard form" of the map.

⚠ **Rk.** Sometimes people define cubical sets w/o some or any connections

• $\square_\varnothing = \square - \{\text{all connections}\}$ ⇝ $cSet_\varnothing = \widehat{\square_\varnothing}$ "minimal cubical sets"

• $\square_0 = \square - \{\text{positive connections}\}$ (resp. $\square_1$) (resp. negative) ⇝ $cSet_0 = \widehat{\square_0}$ "cubical sets w/ negative connections" positive

1.2 – Marked cubical sets + comical sets.

Def. The marked cube category $\square^+$ consists of ...

- objects: $[1]^n$ $\Big\}$ $n = 0, 1, 2, \ldots$
  $[1]^n_e$ — will represent marked cubes (e for ... "equivalence?")

- morphisms: generated by the maps:
  - faces $\partial$
  - degeneracies $\sigma$
  - connections $\gamma$

  $\Big\}$ same as before

  - $\varphi^n : [1]^n \to [1]^n_e$
  - $\zeta^n_i : [1]^n_e \to [1]^{n-1}$
  - $\xi^n_{i,\varepsilon} : [1]^n_e \to [1]^{n-1}$

  $n = 1, 2, \ldots$
  $i = 1, 2, \ldots, n$
  $\varepsilon = 0, 1$

  will represent markings ($\varphi$) & how markings should interact w/ degeneracies ($\zeta$) or connections ($\xi$)

  subject to the cubical identities
  +
  additional relations relating $\varphi, \zeta, \xi$
  [omitted]

(pre-)
Def. A marked comical set is a presheaf on $\square^+$.
i.e. a functor $X : (\square^+)^{op} \to Set$

Rk. Similarly as in comp'l sets, marked simplices are meant to be the "weak equivalences" in the corresponding $(\infty, n)$-cat.

Rk. What do these new objects & maps represent?

- $[1]^n_e$ : represents a marked n-cube.
  - underlying $< \square^n$
  - making: only top-dimensional face.

  $\rightsquigarrow$ $\mathrm{Hom}_{\square^+}(-,[1]^n_e) = \square^n_e$

  for a marked cset $X : (\square^+)^{op} \to Set$

  $eX_n := X([1]^n_e)$ = marked n-cubes of X.

- $\varphi^* : \square^n_e \longrightarrow \square^n$ represents a marking

  $\rightsquigarrow$ $eX_n \longrightarrow X_n$ marked n-cubes of X

- $\zeta_i^*$ :

$$\begin{tikzcd} X_{n-1} \arrow[r, "\zeta_i^*"] \arrow[dr, "\sigma_i"'] & eX_n \arrow[d, "\varphi^*"] \arrow[dl, phantom, "="] \\ & X_n \end{tikzcd}$$

i.e. degenerate n-cubes are marked

- $\xi_{i,\varepsilon}^*$ :

$$\begin{tikzcd} X_{n-1} \arrow[r, "\xi_{i,\varepsilon}^*"] \arrow[dr, "\gamma_{i,\varepsilon}^*"'] & eX_n \arrow[d, "\varphi^*"] \arrow[dl, phantom, "="] \\ & X_n \end{tikzcd}$$

i.e. connections are marked.

**Def.** A (structurally) marked cubical set is a presheaf on $\square^+$, i.e. a functor:

$$X : (\square^+)^{op} \longrightarrow Set.$$

such that the maps $\varphi^* : eX_n \longrightarrow X_n$ are monomorphisms.

**Rk.** Remember $\varphi^*$ represents markings, so saying these are mono's is saying that an $n$-cube can be marked at most once.
This makes marking a property rather than structure.

**e.g.**

- the standard marked $n$-cube $\square^n = Hom_{\square^+}(-,[1]^n)$ w/ only degenerate cubes marked.
- the standard $(i,\varepsilon)$-open boxes w/ same underlying w/ same marking
- the $(i,\varepsilon)$-comical $n$-cube $\square^n_{i,\varepsilon}$ w/ underlying cubical set $\square^n$ & everything marked except non-deg $\square^k \xrightarrow{\alpha} \square^n_{i,\varepsilon}$ s.t. either:
  - the standard form of $\alpha$ contains $\partial_{i,\varepsilon}$ or $\partial_{i,\varepsilon-1}$
  - for some $j > i$, the stand. form of $\alpha$ contains $\partial_{j,\varepsilon}$ & $\partial_{m,1-\varepsilon}$ $\forall j > m > i$
  - for some $j < i$ the stand. form of $\alpha$ contains $\partial_{j,\varepsilon}$ & $\partial_{m,1-\varepsilon}$ $\forall j < m < i$

$$\square^2_{1,0} = \begin{tikzcd} (0,0) \arrow[r, "\partial_{2,0}\ \times"] \arrow[d, "\times\ \partial_{1,0}"'] \arrow[dr, phantom, "\sim"] & (1,0) \arrow[d, "\partial_{1,1}\ \times"] \\ (0,1) \arrow[r, "\partial_{2,1}"', "\sim"] & (1,1) \end{tikzcd}$$

- the $(i,\varepsilon)$-comical open box $\sqcap^n_{i,\varepsilon}$ — underlying is $\sqcap^n_{i,\varepsilon}$ w/ marking inherited from $\square^n_{i,\varepsilon}$

$$\sqcap^2_{1,0} = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] \\ \bullet \arrow[r, "\sim"'] & \bullet \end{tikzcd}$$

- $(\square^n_{i,\varepsilon})' = \square^n_{i,\varepsilon}$ w/ all $(n-1)$-cubes marked except $\partial_{i,\varepsilon}$

$$(\square^2_{1,0})' = \begin{tikzcd} \bullet \arrow[r, "\sim"] \arrow[d, "\times\ \partial_{1,0}"'] \arrow[dr, phantom, "\sim"] & \bullet \arrow[d, "\sim"] \\ \bullet \arrow[r, "\sim"'] & \bullet \end{tikzcd}$$

- $\tau_{n-2}(\square^n_{i,\varepsilon})' = (\square^n_{i,\varepsilon})'$ w/ all $k_{\geq n-2}$-cubes marked

$$\tau_0(\square^2_{1,0})' = \begin{tikzcd} \bullet \arrow[r, "\sim"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\sim"] & \bullet \arrow[d, "\sim"] \\ \bullet \arrow[r, "\sim"'] & \bullet \end{tikzcd}$$

- $\tilde{\square}^n = \tau_{n-1}\square^n = \square^n$ w/ all $k_{\geq n-1}$-cubes marked
  $= \delta([1]'_e)$

$$\tilde{\square}^2 = \tau_1\square^2 = \begin{tikzcd} \bullet \arrow[r, "\times"] \arrow[d, "\times"'] \arrow[dr, phantom, "\sim"] & \bullet \arrow[d, "\times"] \\ \bullet \arrow[r, "\times"'] & \bullet \end{tikzcd}$$

# Comical Sets

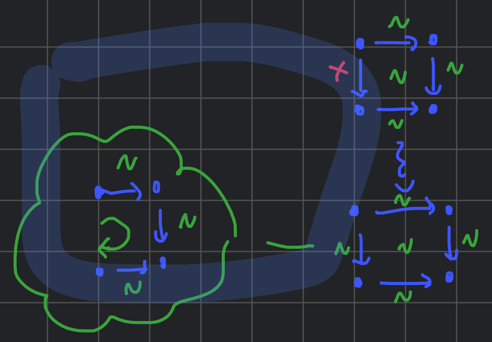
**Def**: A comical set is a marked cubical set w/ the RLP w.r.t. . . .

(i) all comical open-box fillings

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d, hook] & X \\ \square^n_{i,\varepsilon} \arrow[ur, dashed] & \end{tikzcd}$$

(ii) elementary comical marking ext's.

$$\begin{tikzcd} (\square^n_{i,\varepsilon})' \arrow[r] \arrow[d] & X \\ \tau_{n-2}\square^n_{i,\varepsilon} \arrow[ur, dashed] & \end{tikzcd}$$



"2-out-of-3" property.

**Def.** Additionally, a comical set is called . . .

(iii) 2-trivial if it has RLP

$$\begin{tikzcd} \square^k \arrow[r] \arrow[d, hook] & X \\ \tilde{\square}^k \arrow[ur, dashed] & \end{tikzcd}$$

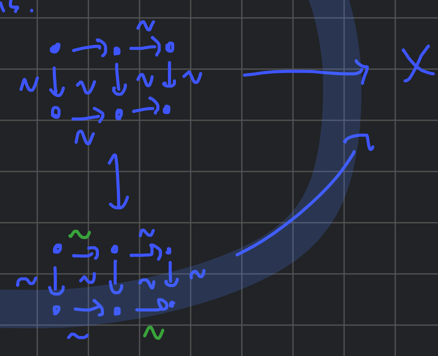
for $k > 2$.

represents markings of cubes of dim > 2.

(iv) saturated if it has RLP w.r.t. Rezk maps.

represents a "2-out-of-6" property for w.eq's in all dimensions.

e.g.:



# Comical Sets.

| | $\Delta$ | $\square$ |
|---|---|---|
| an $(\infty,2)$-cat. | $X \in sSet^+$ | $Y : (\square^+)^{op} \to Set$ |
| • n-morphisms $(n=0,1,2,\ldots)$ | $X_n = X(\Delta^n)$ | $Y_n = Y(\square^n)$ |
| • composition of n-morphisms | $\begin{array}{ccc} \Lambda & \to & X \\ \downarrow & \nearrow & \\ \Delta & & \end{array}$ inner horn fillers | $\begin{array}{ccc} \sqcap^n_{i,\varepsilon} & \to & Y \\ \downarrow & \nearrow & \\ \square^n_{i,\varepsilon} & & \end{array}$ inner box fillings. |
| • "weak equivalences" | markings $\tilde{\Delta}^n \longrightarrow X$ | markings $\square^n \longrightarrow Y$ |
| • comp of weak eq's are weak eq's. | $\begin{array}{ccc} (\Delta^n_i)' & \to & X \\ \downarrow & \nearrow & \\ (\Delta^n_i)'' & & \end{array}$ | elementary comical marking ext's $\begin{array}{ccc} (\square^n_{i,\varepsilon})' & \longrightarrow & Y \\ \downarrow & \nearrow & \\ \tau_{n-2}\,\square^n_{i,\varepsilon} & & \end{array}$ |
| • all $k$-morphisms $k \geq 2$ are w.e's. | 2-triviality | 2-triviality: $\begin{array}{ccc} \square^k_{i,\varepsilon} & \longrightarrow & Y \\ \downarrow & \nearrow & \\ \tilde{\square}^k_{i,\varepsilon} & & \end{array}$ for $k \geq 2$. |
| • isomorphisms in all levels. | saturatedness | saturated: $Y \in RLP\{\text{Rezk maps}\}$ (marked cubes) = (invertible morphisms in all levels) |

# The model structure ..

**Def.** [DKM]: There's a model structure on $cSet^+$ in which 2-comical sets are the fibrant objects..

- cofibs = monomorphisms of csets.
- fibs w/ fibrant domain = RLP $\left\{ \begin{array}{l} \bullet\ \sqcap^n_{i,\varepsilon} \hookrightarrow \square^n_{i,\varepsilon} \\ \bullet \text{ comp'l marking ext's} \\ \bullet \text{ markings } \sqcap^n \longrightarrow \tilde{\sqcap}^n \text{ for } n \geq 2 \\ \bullet \text{ Rezk maps} \end{array} \right\}$

**Thm** [DKM]: There are Quillen equivalences..

triangulation

$[1]^n \mapsto (\Delta^1)^n$

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & sSet \\ cSet \arrow[ur, "T"'] & \end{tikzcd}$$

{marked version

$$\begin{tikzcd} \square^+ \arrow[r] \arrow[d] & sSet^+ \arrow[dl, shift left, "u"] \\ cSet^+ \arrow[ur, shift left, "T"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

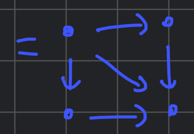
$$\begin{tikzcd} cSet^+_{2\text{-comical}} \arrow[d, bend right, "T"'] \arrow[d, phantom, "\dashv"] \\ sSet^+_{2\text{-complicial}} \arrow[u, bend right, "u"'] \end{tikzcd}$$

$sSet^+ \longrightarrow cSet^+$

$X \longmapsto \{ uX_n = \mathrm{Hom}_{cSet^+}(\square^n, uX) \cong \mathrm{Hom}_{sSet^+}(T\square^n, X) \}$

eg. $uX_2 = \mathrm{Hom}_{sSet^+}(T\square^2, X)$

$T\square^2 \cong \Delta^1 \times \Delta^1$

$=$ 

cubification

$\Delta \longrightarrow cSet$

$[n] \longmapsto Q^n$

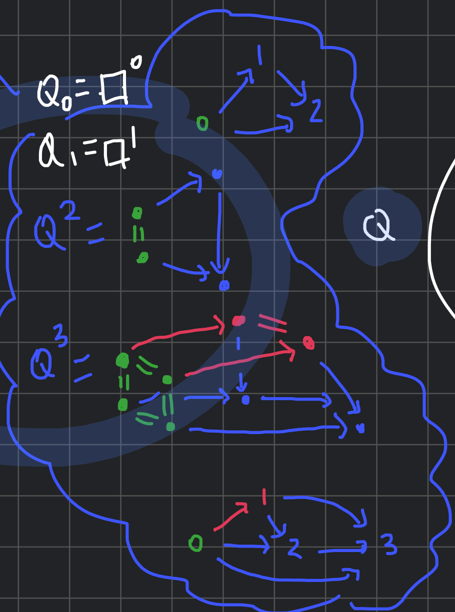
$$\begin{tikzcd} \Delta^+ \arrow[r] \arrow[d, "\int"'] & cSet^+ \arrow[dl, shift left, "\int"] \\ sSet^+ \arrow[ur, shift left, "Q"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

$Q_0 = \square^0$

$Q_1 = \square^1$

$Q^2 =$

$Q^3 =$




$$\begin{tikzcd} sSet^+_{2\text{-compl'e}} \arrow[d, bend right, "Q"'] \arrow[d, phantom, "\dashv"] \\ cSet^+_{2\text{-com'l}} \arrow[u, bend right, "\int"'] \end{tikzcd}$$

$sSet^+(\Delta^n, \int X) \cong cSet^+(Q\Delta^n, X)$, $Q\Delta^n = Q^n$

eg. $\int X_2 = cSet^+(Q^2, X)$

$= \{$  $\}$ in $X$.

Since 2-complicial sets model $(\infty,2)$-cats this exhibits 2-comical sets as an equivalent model for $(\infty,2)$-cats.

**Thm.** [DKM]. The derived functors

$$\begin{tikzcd} ho(cSet^+) \arrow[d, bend right, "T"'] \\ ho(sSet^+) \arrow[u, bend right, "Q"'] \end{tikzcd}$$

form an equivalence of 1-categories.

## $(\infty,2)$-cats.

- (2-cats): $$\begin{array}{ccccc} 2\text{-Cat} & \xrightarrow{N_{\text{street}}?} & sSet^+ & \xrightarrow{u} & cSet^+ \\ C & \longmapsto & NC_0 = ob(C) & \longmapsto & UNC \\ & & NC_1 = 1\text{-mor}(C) & & \\ & & NC_2 = 2\text{-mor}(C) & & \end{array}$$
  w/ a suitable marking.

I don't think this has been written down carefully.

- (1-comical sets): There's a similar Quillen equivalence for

$$\{1\text{-comical sets}\} \leftarrow\!\sim\!\rightarrow \{1\text{-complicial sets}\}$$

1-comical sets are a trivial kind of 2-comical set
$(\infty,1)$ $(\infty,2)$
in which all 2-cubes are marked (made into weak equiv's).

$$\begin{array}{ccccc} \{q\text{-cats}\} = \{1\text{-compl'l sets}\} & \longleftrightarrow & \{1\text{-comical sets}\}. \\ \downarrow & & \downarrow \\ \{2\text{-compl'l sets}\} & \longleftrightarrow & \{2\text{-comical sets}\}. \end{array}$$

- (quasicategories):

$$\begin{array}{ccccc} \{\text{quasicats}\} & \leftarrow\!\sim\!\rightarrow & \{1\text{-complicial sets}\} & \xrightarrow{\sim\, Q\, \sim} & \{1\text{-comical sets}\} \\ & & \downarrow & & \downarrow \\ q\text{Cat} & \hookrightarrow & sSet^+ & \xrightarrow{Q} & cSet^+ \end{array}$$

- (cubical quasicats):

$$\{\square\text{'l quasicats}\} \xrightarrow[\sim]{\text{mark equivalences} \cdot \text{all } n\text{-cubes}_{>1}} \{1\text{-comical set}\}$$

also hasn't been written down yet?

— wasn't in the thesis

# Gray ⊗.

Recall: The Gray ⊗ is the ⊗ that gives a ⊗-Hom adjunction for 2-cats.

$$\mathrm{Hom}_{2\text{-Cat}}(X\otimes Y, Z) \cong \mathrm{Hom}_{2\text{-Cat}}(X, \mathrm{Hom}_{2\text{-Cat}}(Y, Z))$$

In 1-cats this is just cartesian × ... but in 2-cats we need something else..

eg. $(\cdot\to\cdot)\times(\cdot\to\cdot) = $
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$
(in 2-cats we want this to be a weak eq. (invertible 2-mor)

$\underset{\text{lax}}{\otimes}$ : $X, Y \in$ 2-Cat ... $\{0\to1\}\underset{\text{lax}}{\otimes}\{0,1\} = $
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

$\underset{\text{pseudo}}{\otimes}$ : $\{0\to1\}\underset{\text{ps}}{\otimes}\{0,1\} = $
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\sim}{\nearrow}"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

We'd want an analogous construction in 2-comical sets..

**Def.** The geometric product of cubical sets $X, Y$ is a cubical set $X\underset{\text{geo.}}{\otimes}Y$ w/..

$$(X\underset{\text{geo.}}{\otimes}Y)_n := \{x\otimes y : x\in X_k,\ y\in Y_\ell,\ k+\ell=n,\ (x\sigma_{k+1})\otimes y = x\otimes(y\sigma_1)\}.$$

w/ face & degeneracy maps... [omitted].

(In particular an n-cube $x\otimes y\in X\otimes Y$ is non-deg. iff both $x\in X, y\in Y$ are non-deg.)

eg. $\square^1\underset{\text{geo}}{\otimes}\square^1 = (0\xrightarrow{f}1)\otimes(0'\xrightarrow{g}1') = $
$$\begin{tikzcd} {[0\,0']} \arrow[r, "f\otimes0'"] \arrow[d, "0\otimes g"'] \arrow[dr, phantom, "f\otimes g"] & {[1\,0']} \arrow[d, "1\otimes g"] \\ {[0\,1']} \arrow[r, "f\otimes1'"'] & {[1\,1']} \end{tikzcd}$$

**Def.** The (lax/pseudo) Gray ⊗ of marked □'l sets $X, Y$ is a marked □'l set $X\underset{\text{lax/ps.}}{\otimes}Y$ ...

- (lax): $X\underset{\text{lax}}{\otimes}Y = \begin{cases} \cdot\ \text{underlying □'l set is } X\underset{\text{geo.}}{\otimes}Y \\ \cdot\ \text{an n-cube } x\otimes y \text{ is marked iff either } x\in X \text{ or } y\in Y \text{ is marked.}\end{cases}$
- (pseudo): $X\underset{\text{ps.}}{\otimes}Y = \begin{cases} \cdot\ \text{underlying □'l set is } X\otimes_{\text{geo}}Y \\ \cdot\ \text{an n-cube } x\otimes y \text{ is unmarked iff either } \cdot\, x\in X_0 \text{ \& } y \text{ is unmarked} \\ \qquad\qquad \cdot\, y\in Y_0 \text{ \& } x \text{ is unmarked.}\end{cases}$

$\square^1\underset{\text{lax}}{\otimes}\square^1 = $
$$\begin{tikzcd} {[0\,0']} \arrow[r, "f\otimes0'"] \arrow[d, "0\otimes g"'] \arrow[dr, phantom, "\overset{f\otimes g}{\Rightarrow}\times"] & {[1\,0']} \arrow[d, "1\otimes g"] \\ {[0\,1']} \arrow[r, "f\otimes1'"'] & {[1\,1']} \end{tikzcd}$$

$\square^1\underset{\text{ps.}}{\otimes}\square^1 = $
$$\begin{tikzcd} {[0\,0']} \arrow[r, "f\otimes0'"] \arrow[d, "0\otimes g"'] \arrow[dr, phantom, "\overset{f\otimes g}{\Rightarrow}\sim"] & {[1\,0']} \arrow[d, "1\otimes g"] \\ {[0\,1']} \arrow[r, "f\otimes1'"'] & {[1\,1']} \end{tikzcd}$$

**Rk.** Compare this to the Gray ⊗ in comp'l sets, which was more complicated (involved unwieldy "i-cloven" conditions...)

⚠

**Thm** [CKM]: The $\Delta^{\text{tion}}$ functor $T: c\mathrm{Set}^+\to\mathrm{PreComp}$ is strong monoidal w.r.t. both lax & pseudo Gray ⊗'s.

$$T(X\otimes Y) = TX\otimes TY$$

$$\begin{tikzcd} & \mathrm{PreComp} \arrow[d] \\ c\mathrm{Set} \arrow[ur] \arrow[r] & s\mathrm{Set} \end{tikzcd}$$

• Talk w/ Udit re: Gray ⊗

if he's not gonna talk about it then mention that $T: c\mathrm{Set}^+\to s\mathrm{Set}^+$ is not strong mon'l but up to w.eq. $T(X\otimes Y)\simeq TX\otimes TY$.