

Nerves, ∞ -categories and the Boardman-Vogt construction.

∞ -category theory reading group, nfpyserv

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13 - 08-2022

Plan

If time
permits

- Segal category

- The Nerve construction
 - Def and Examples
 - Gromthendieck Segal condition

- Categories

- Def and Examples

- α -groupoids

- Op - categories construction

Boardman - Vogt - construction

- The realization functor

- Joyal's whiteness lemma and
the Homotopy cat. const.

Def: The nerve associated to a category
is the S-set •

$$NC_1 := \lim_{\text{Cat-}} ([\mathbb{I}, C])$$

→ category elements
category

We can think as follows

$NC_1 = \{ \text{strings of } n\text{-cells or alike arrows in } C \}$

The degeneracy map $s_i : NC_n \rightarrow NC_{n+1}$
takes a string of n -cells or alike arrows

to a $n+1$ -string by attaching an id-map at the i^{th} place.

$$c_0 \xrightarrow{b_1} c_1 \xrightarrow{b_2} \cdots \xrightarrow{b_i} c_i \xrightarrow{\text{id}} c_{i+1} \xrightarrow{b_{i+1}} c_{i+2} \cdots \xrightarrow{b_n} c_n$$

$\downarrow s_i$

$$c_0 \xrightarrow{b_1} c_1 \xrightarrow{b_2} \cdots \xrightarrow{b_i} c_i \xrightarrow{\text{id}} c_i \xrightarrow{b_{i+1}} c_{i+1} \xrightarrow{\cdots} c_n$$

and the face map $d_i: NC_n \rightarrow NC_{n-1}$ composed
 the i^{th} and the $(i+1)^{\text{th}}$ arrows from $\partial C_i C_n$
 and if $i=0, 1$, delete i^{th} object.

$$c_0 \xrightarrow{b_1} c_1 \xrightarrow{b_2} c_2 \rightarrow \dots \rightarrow c_{i-1} \xrightarrow{b_i} c_i \xrightarrow{b_{i+1}} c_{i+1} \rightarrow \dots \xrightarrow{b_n} c_n$$

$\downarrow d_i \quad (0 \leq i < n)$

$$c_0 \xrightarrow{b_1} c_1 \xrightarrow{b_2} c_2 \rightarrow \dots \rightarrow c_{i-1} \xrightarrow{b_{i+1} \circ b_i} c_i \xrightarrow{b_{i+2}} \dots \xrightarrow{b_n} c_n$$

We've a fully faithful functor

$$i : \Delta \longrightarrow \mathbf{Set}$$

The nerve functor is just evaluation along

$$N = i^* : \mathbf{Cat} \xrightarrow{\sim} \mathbf{Set}$$

By Kan's theorem, it has a left adjoint.

$\tau: \text{sSet} \rightarrow \text{Cat}$

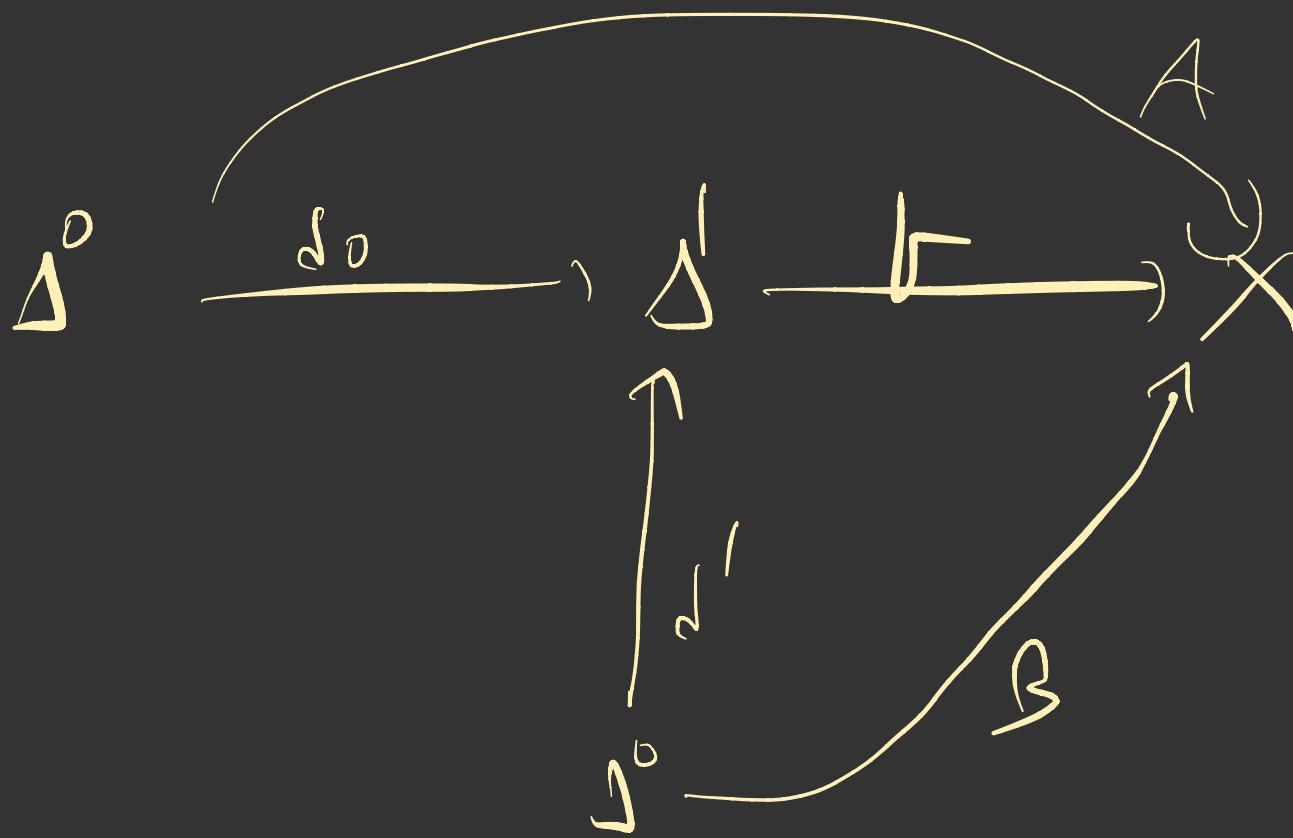
$(\tau, N): \text{sSet} \rightleftarrows \text{Cat}$

Def: For a s-set X , the objects are 0-simplices
(i.e. $\Delta^0 \rightarrow X$) and morphisms are

1-simplices (i.e., $\Delta^1 \rightarrow X$)

$A \xrightarrow{f} B$ (^{an} edge in X)

the faces of f are given by
forget to $f \subseteq B$ & now & diff = A



for an object of X , the dependent edge
 $J_0(x)$: in the morphism $X \rightarrow$

Recall: For $n \in \mathbb{Z}^{>0}$ Δ^n is the set which is

given by the following construction

$$([u] \in \Delta) \mapsto (\text{Term}_\Delta([u], r))$$

Lemma For $n \in \mathbb{Z}^{>0}$ $N(n)$ can be identified with

$$\Delta^n.$$

For a fin. totally ordered set E we

define $\Delta^E = N(E)$

Now $\partial \Delta^n = \bigcup_{E \subsetneq \{1\}} \Delta^E \subset \Delta^{n-1}$

$$\Delta^{\leq n} = \bigcup_{k \in \mathbb{N}} \Delta^k \subset \Delta^n \quad \begin{matrix} k \geq 1 \\ 0 \leq k \leq n \end{matrix}$$

Definition: Spine(τ_n) is a subset of Δ^n where κ -simplices are monotone maps
 $\tau: [\kappa] \rightarrow [n]$ with the condition $\kappa \leq \ell(0) + 1$

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \dots \longrightarrow n$$

$$\text{Spine } \tau_n = \bigcup_{0 \leq j \leq n} \Delta^{\{i, j+1\}} \subset \Delta^n$$

Def: A simp. obj. X in a category \mathcal{C} is
a s.set internal to \mathcal{C} .

E.g (Čech Nerve)

Let's consider a category \mathcal{C} with pullbacks

and $V \rightarrow Z \in \text{Ob}(\mathcal{C})$

The Čech nerve is a simp. obj. in \mathcal{C}

$$\begin{aligned} C(V) := & \left[\begin{array}{c} V \times_Z V \times_Z V \\ \downarrow \quad \downarrow \quad \downarrow \\ V \times_Z V \\ \downarrow \end{array} \right] \\ & \rightarrow V \end{aligned}$$

Def. A triangle in a set X is a map

$$f: \partial\Delta^2 \rightarrow X$$

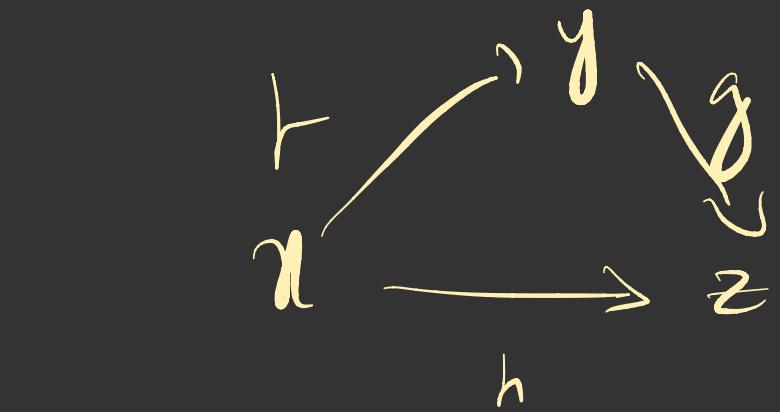
It can be visualized (high) with

$$f, g, h \in \text{Mor}(X), \text{ with}$$

target of f coincides with the source of g ,
source of f and h are the same,
 g and h have the same target

$$\partial\Delta^2 := \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,2\}}$$

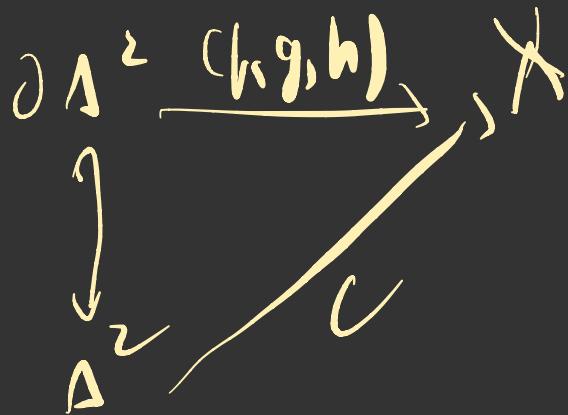
↓ correspond to the map $\Delta' \cong \Delta^{20,11} \subset \partial \Delta^1 \rightarrow Y$
 $\delta \cong \Delta^{21,23} \subset \partial \Delta^2 \rightarrow X$



$$\text{J} \quad \Lambda_{\mu}^{21} = \text{Spin}(2) \rightarrow X$$



Def: A triangle (b, g, h) is said to commute if
a morphism $c: \Delta^2 \rightarrow X$ which restricted to
boundary withers with (b, g, h)



Grothendieck condition

A s-set X satisfies the Grothendieck -
Ssegel condition if the restriction along
the inclusion $\text{Spine}(n) \subset \Delta^n$ induces a

bijection

$$\mathrm{Hom}(\Delta^1, X) \xrightarrow{\sim} \mathrm{Hom}(S^1, X)$$

for $\in \mathbb{Z}^{\geq 2}$

Consequence If $X \in NC$

$$\mathrm{Hom}(\Delta^n, NC) \xrightarrow{\sim} \mathrm{Hom}(S^n, NC)$$

Prop. The new functor is fully faithful.

Prop. $h : \mathrm{Hom}_{\mathrm{Cat}}(C, D) \rightarrow \mathrm{Hom}_{\mathrm{Set}}(NC, ND)$

All sketch This also follows from the

fact that the nerve construction
gives a 2-coskeletal s-set.

Brick form

If in the simplex set Δ we have a
subset - $\Delta_{\leq n} \subset \Delta_n$

→ object in $\mathbf{Set}_-, \mathbf{h}\mathbf{h}$

Then $\Delta|_{\leq n} \hookrightarrow \Delta$ induces a transformation

functor

$f_{n_1}: \mathbf{SSet} \longrightarrow \mathbf{Set}_{\leq n} = (\Delta_{\leq n}^{\text{op}}, \mathbf{Set})$

If han fully birthful left adj

$$\text{sk}_n : \text{sSet}_{\leq n} \rightarrow \text{sSet}$$

& fully birthful right adj.

$$\text{cosk}_n : \text{sSet}_{\leq n} \rightarrow \text{sSet} \quad \text{rank of}$$

Def: Ssets isomorphy to object in the n -Verks.
are n -skeletal

$$\text{Hom}_{\text{sSet}}(N_C, N_D) \cong \text{Hom}_{\text{sSet}_{\leq 2}}(N_C|_{S^1_{\leq 2}}, N_D|_{S^1_{\leq 2}})$$

Prop: The following conditions are equivalent:

\times_{essel}

(i) \exists a small cat C and an iso-
 $\times \in N(C)$

(ii) $X \rightarrow N(\tau(X))$ is invertible

(iii) X satisfies the bottomless
-synd -unsh.

Defining α -cat

Def!: A set is a Ran-complex if every

horn $\Lambda_u^{\cap} \rightarrow X$ can extend to the
($0 \leq k \leq n$) n-complex



Fact: The category of Kan complexes is Cartesian closed.

(fin. prod.
 $a \in \text{Obj}(\mathcal{C})$
 $\rightarrow a^* : \mathcal{C} \rightarrow \text{Kan}$
 with adj.)

Def: An ∞ -category is a weak Kan-complex

i.e. an extension of $A_k^n \rightarrow X$
from ~~open~~

to $\Delta^n \rightarrow X$ exist

Equivalent: Inner A_k^n (cells) have
bases
horn

Further Equivalent: An α -cat is a set with
all right lifting property for inner
horn factorizations $\Delta^n \hookrightarrow \Delta^m$ such

Conseq: $\text{fun}(\Delta^1 X) \longrightarrow \text{fun}(\Delta_k^1, X)$
is surjective if X is
an ∞ -cat

- For an n -category C objects are vertices
 $x \in C_0$, morphism are 1-simplices in C ,

If $y \in C_1$ then $y = d_1(x)$ or $y = d_0(x)$.
 $x \rightarrow y$

- A. 1-ger. edge be $x \in C_0$, $s(x)$
is identity morphism for x .

Def: (∞ -groupoid)

An ∞ -groupoid is an ∞ -category (where
every morphism is invertible)

$$\forall \gamma: x \rightarrow y \in \text{Mor}(C)$$

$\exists g: \gamma \rightarrow x$, $h: \gamma \rightarrow x$ s.t

$$\begin{array}{ccc} \gamma & \xrightarrow{g} & x \\ \downarrow & & \downarrow \\ x & = & x \end{array}$$

$$\begin{array}{ccc} & \nearrow h & \\ \gamma & = & \gamma \\ & \searrow & \end{array}$$

Examples of ∞ -category

(1) For a small cat C , NC is an ∞ -category

Obj \rightarrow Obj of C
Mor \rightarrow Mor of C

(2) For $X \in \text{Top}$, $\text{Sing}(X)$ is an ∞ -cat.

Identify $(\Delta^n)^1$ with the hypercube $[0,1]^n$

Obj \rightarrow points of X

Morph + cont. path $f: [0,1] \rightarrow X$
source $\uparrow^{(0)}$
target $\downarrow^{(1)}$

$\text{Id}^{\text{Nex}}_{\mathcal{S}(0,1)} \rightarrow X$ taking value at X

Faith we've the "Wells" adjunction formulation.

$$\text{Hom}((\mathbb{K}), X) \cong \text{Hom}(\mathbb{K}, \text{sing}(X))$$

Surjection along $N_{\mathbb{K}} \hookrightarrow S^1$

$$\text{Hom}(S^1, \text{sing}(X)) \longrightarrow \text{Hom}(N_{\mathbb{K}}, \text{sing } X)$$

Prop: Every Kan complex is an ∞ -groupoid

Proof: Let $f: A \rightarrow B \in \text{Mon}(X)$ $\xleftarrow{\quad \text{Kan} \quad}$

→ a unique morphism

$$\wedge^2 \rightarrow A$$

Since non-degen. I sum of $\Delta^{20,13} \cup \Delta^{20,11}$ to $\Delta^{20,12}$

$$1 \quad \quad \quad 1$$



Let Grpd be the category of groups.
Then we've the following comm.
diagram

$$\begin{array}{ccc} \text{Grpd} & \longrightarrow & \mathcal{A} \\ \downarrow N & & \downarrow N \\ \text{Kan} & \longrightarrow & \text{Set} \end{array}$$

constant we'll define a set $X := \Delta^{\text{op}} \rightarrow \text{Set}$

$$\Delta^{\text{op}} \xrightarrow{\text{OP}} \Delta^{\text{op}} \xrightarrow{X} \text{Set}$$

For $\alpha : [m] \rightarrow [n]$ in Δ , the morphism
 $\text{OP}(\alpha) : [n] \rightarrow [m]$
given by $\text{OP}(\alpha)(j) = n - \alpha(m-j)$

Prf.: For a small category C ,

$$N(C^{\text{op}}) = N(C)^{\text{op}}$$

λ -sum in of NC on

Diagram

$$c_0 \xrightarrow{h_1} c_1 \rightarrow \dots \xrightarrow{h_n} c_n \text{ in } \mathcal{C}$$

Then Diagram gives a universion of

$$c_0 \xrightarrow{h_1} c_1 \rightarrow \dots \xrightarrow{h_n} c_n \text{ in } \mathcal{C}^{\text{op}}$$

$$N^{(0)} = N(c)^{\text{op}} \text{ up to } \text{on } 1/\text{id}$$

Boardman-Vogt construction

Def: For any $X \in \text{Top}$ define
 $\pi_{\leq 1}(X)$ to be the fundamental shd
of X with

obj -> points in X

For any X , if $m_1, m_2 \in \pi_{\leq 1}(X)$ (m_1, m_2 can be identical)

with homotopy class of cont. paths

$P - P_{0,1} \rightarrow X$ with $P(0) = m_1$, $P(1) = m_2$

- composition is by concatenation of paths

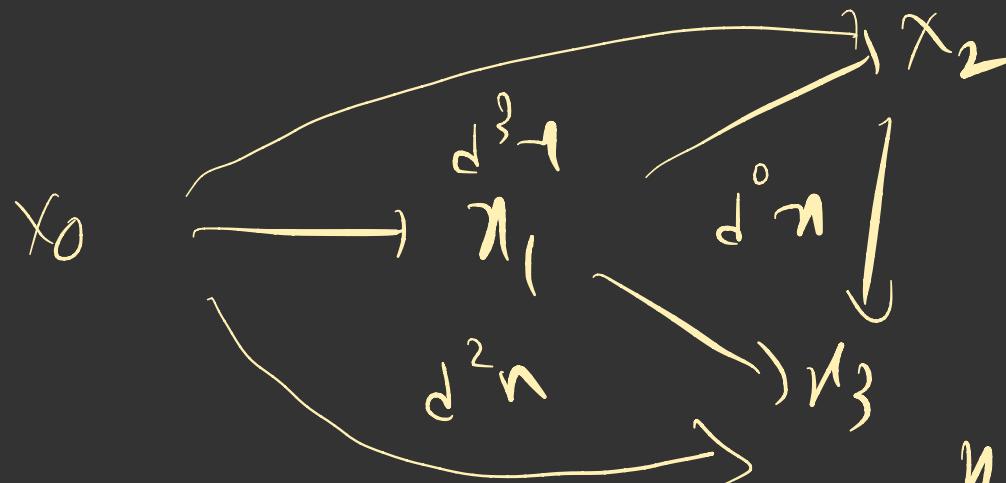
Refinement

object \rightarrow point w/ \times

morphisms \rightarrow paths w/ \times

2-morphisms \rightarrow commutes between
higher morphism paths \rightarrow high level 2

$\lambda: \text{Sk}_1(\Delta^3) \rightarrow X$



$d^n: \partial\Delta^3 \rightarrow X$
corresponding to
restrictions of λ

to subcategory of $\text{Sk}_1(\Delta^k)$

$$n = \{0, 1, 2, 3\} - \{k\}$$

Joyal's theorem

\vdash $\frac{d^0 u, d^3 v \text{ and}}{d^3 v}$

$\frac{d^1 u \text{ commutes to } d^2 v \text{ and}}{d^2 v}$

Skein

$$\partial\Delta^2 \xrightarrow{d^0 u} X \\ \downarrow \delta^2 \quad \gamma_0$$

$$\partial\Delta^2 \xrightarrow{d^3 v} X \\ \downarrow \delta^2 \quad \gamma_3$$

$$\partial\Delta^2 \xrightarrow{d^1 \alpha} X \\ \downarrow \delta^2 \quad \gamma_1$$

$d^1 u$ and
with $\gamma_2 =$

We get a map $\gamma = (\gamma_0, \gamma_1, \gamma_2) : \Delta^3 \rightarrow X$

$$\begin{array}{ccc} \gamma : \Delta^3 & \longrightarrow & X \\ \downarrow \gamma_2 & \nearrow \gamma_1 & \\ \Delta^2 & \xrightarrow{\gamma} & X \end{array}$$

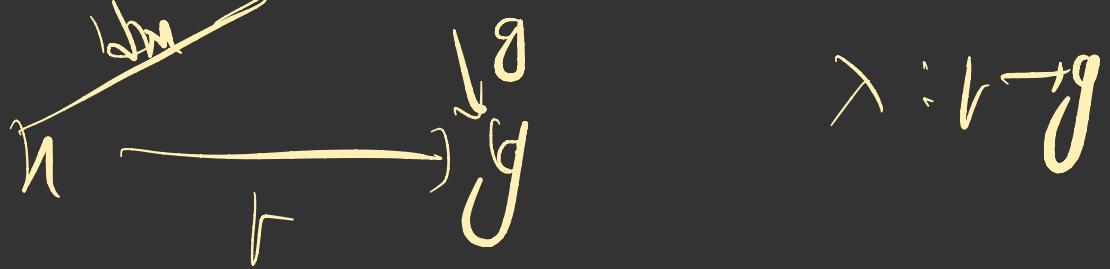
$$\gamma_L = \gamma \cdot \delta_2$$

HOM

Def Two morphism $f, g : \underline{n} \rightarrow \underline{m}$ in an category

are homotopic ($f \simeq g$) if \exists a 2-simplex

$\gamma : \Delta^2 \rightarrow C$, with boundary (g, h, i)



Lemma The forth relations are equivalent to

$$r \succeq g \quad \text{if} \quad g^{-1}r = r$$

$$r \succeq_1 g \quad \text{if} \quad r^{-1}r = g$$

$$r \succeq_2 g \quad \text{if} \quad 1_g r = g$$

$$r \succeq_3 g \quad \text{if} \quad 1_g g = r$$

Def.: \mathcal{D} is an equivalent relation. $\text{Hom}(X, Y)$ for $X, Y \in \mathcal{C}$

(homotopy class of $f : X \rightarrow Y$) is denoted by $[f]$.

Def.: Let \mathcal{C} be an ∞ -category with same obj. as \mathcal{A} .
There is a $\text{Ho}(\mathcal{C})$ with same obj. as \mathcal{C} and morphisms homotopy classes of morphisms in \mathcal{C} .
(composition)

$$[fg] \circ [h] := [g \circ h] \quad ([x] := [f]) \\ = [s_0(x)]$$

The (Boardman
Vogt) \rightarrow a unique map

$$X \xrightarrow{\quad} N(\text{hol}(X))$$

which is identity on objects

and which sends $f : x \rightarrow y$ in X with

[v]

$$\tau(X) \xrightarrow{\quad} \text{Ho}(X)$$

Commutative ∞ -category X is an ∞ -groupoid if

$$\tau(X) \cong \text{Ho}(X) \text{ is a groupoid.}$$

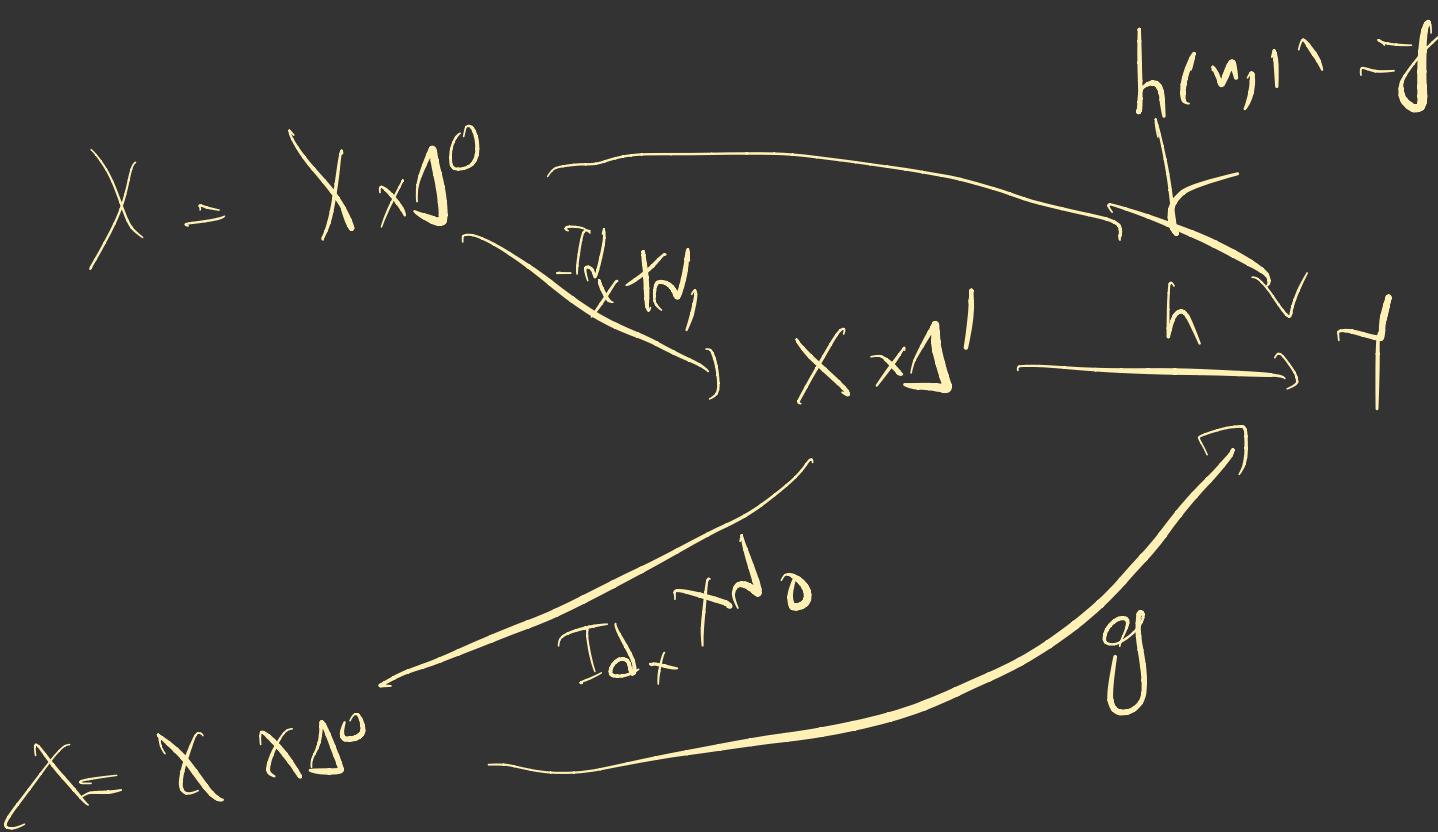
Def: A functor between Δ -category $X \dashv Y$
 is morphism between Δ -sets $X \dashv Y$.

$$h, g : X \dashv Y$$

$$h : X \times \Delta^1 \dashv Y$$

$$h(n, 0) = v(n)$$

$$h(n, 1) = f(n)$$



E.g. For $X \in \text{Top}$ \dashv CAlg

$$F: \underline{\text{Sug}}(X) \rightarrow N(C)$$

Def. A morphism $f: u \rightarrow y$ in an \mathcal{A} -category C
is an epivariel if $\text{Sug}(f): u \rightarrow y$ is an
isomorphism in $\text{HO}(C)$