CIS 580, Machine Perception, Spring 2023
Homework 1 PART B MATH
Due: Thursday Feb 09 2023, 11:59pm ET
Version of Wednesday 22$^{\text{nd}}$ February, 2023, 14:19GMT

Instructions

- This is an individual homework and worth 100 points
- You must submit your solutions on Gradescope, the entry code is E7D687, should you not already be autoenrolled via Canvas. We recommend that you use LaTeX, but we will accept scanned solutions as well.
- Start early! Please post your questions on Piazza or come to office hours!

Submission

- For non-coding portion, please submit the answers to the bellow questions via Gradescope

1 Perspective Projection and Projective Geometry (35 points)

A camera and a world coordinate frame are related as follows

$$\begin{pmatrix} X_c \\ Y_c \\ Z_c \end{pmatrix} = R \begin{pmatrix} X_w \\ Y_w \\ Z_w \end{pmatrix} + T.$$

(a) What is the origin of the camera coordinate system in the world frame?

★ **SOLUTION:** $-R^T T$

(b) What is the distance from the camera origin to the plane $Y_w = 0$?

★ **SOLUTION:** $-[0, 1, 0]R^T T$

(c) Assume that pixel coordinates relate to camera coordinates as $u = fX_c/Z_c$ and $v = fY_c/Z_c$ and that the columns of R are $R = (r_1, r_2, r_3)$. Assume the translation vector can be written as (T_x, T_y, T_z). Write the projection of a world point (X_w, Y_w, Z_w) to pixel coordinates (u, v) in matrix-vector form. Show then the projective transformation H from the plane $Y_w = 0$ to the pixel plane.

★ **SOLUTION:** $\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 & r_3 & T \end{bmatrix}$

(d) Is the resulting matrix H always a projective transformation? If not show a camera-world setup as a counterexample.

★ **SOLUTION:** The resulting matrix is not always a projective transformation, inspect the $det([r_1 \quad r_2 \quad T])$=Hence we can find: $det([r_1 \quad r_2 \quad T])$=$(r_1 \times r_3)T$
$det(H) = f^2(r_1 \times r_3)T = f^2(r_2)T = 0$
. Therefore, in the case that the camera world setup satisfies $r_2 T = 0$, the world points will project to a line.

(e) We discussed in class how it is risky to set the (3,3) element of a homography H equal to 1. In the case handled here, is there any camera-world setup where the H_{33} vanishes?

★ **SOLUTION:** Inspect that H_{33} corresponds to T_z, therefore in the case that $T_z = 0$, the determinant of this matrix will be 0,

(f) What are the world coordinates of the point F in $Y_w = 0$ world plane that projects to the point $(f, 0)$?

★ **SOLUTION:** $\lambda \begin{pmatrix} X_w \\ Z_w \\ 1 \end{pmatrix} = \begin{pmatrix} r_1 & r_3 & T \end{pmatrix}^{-1} K^{-1} \begin{pmatrix} f \\ 0 \\ 1 \end{pmatrix}$ Solve for λ, X_w, Z_w.

$$\lambda \begin{pmatrix} X_w \\ Z_w \\ 1 \end{pmatrix} = \begin{pmatrix} r_1 & r_3 & T \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda \begin{pmatrix} r_1 & r_3 & T \end{pmatrix} \begin{pmatrix} X_w \\ Z_w \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda \begin{pmatrix} r_{11}X_w + r_{31}Z_w + T_x \\ r_{12}X_w + r_{32}Z_w + T_y \\ r_{13}X_w + r_{33}Z_w + T_z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$r_{11}X_w + r_{31}Z_w + T_x = r_{13}X_w + r_{33}Z_w + T_z, \quad X_w(r_{11} - r_{13}) + Z_w(r_{31} - r_{33}) + T_x - T_z = 0$

$r_{12}X_w + r_{32}Z_w + T_y = 0$

$$\begin{pmatrix} (r_{11} - r_{13}) & (r_{31} - r_{33}) \\ r_{12} & r_{32} \end{pmatrix} \begin{pmatrix} X_w \\ Z_w \end{pmatrix} = \begin{pmatrix} T_z - T_x \\ -T_y \end{pmatrix}$$

$$\begin{pmatrix} X_w \\ Z_w \end{pmatrix} = \begin{pmatrix} r_{32}(T_z - T_x) - T_y(r_{33} - r_{31}) \\ -r_{12}(T_z - T_x) - T_y(r_{11} - r_{13}) \end{pmatrix} \frac{1}{(r_{11}-r_{13})r_{32}-(r_{31}-r_{33})r_{12}}$$

(g) What is the distance of F from the center of projection?

★ **SOLUTION:** $||RP^W + T||_2$

(h) What are the coordinates of the vanishing point arising from parallel lines parallel to the X_w axis. The answer should be in terms of f and elements of the rotation matrix.

★ **SOLUTION:** $(\frac{fr_{11}}{r_{13}}, f\frac{r_{12}}{r_{13}})$

(i) Repeat the same for parallel lines parallel to the Z_w axis in the world.

★ **SOLUTION:** $(\frac{fr_{31}}{r_{33}}, f\frac{r_{32}}{r_{33}})$

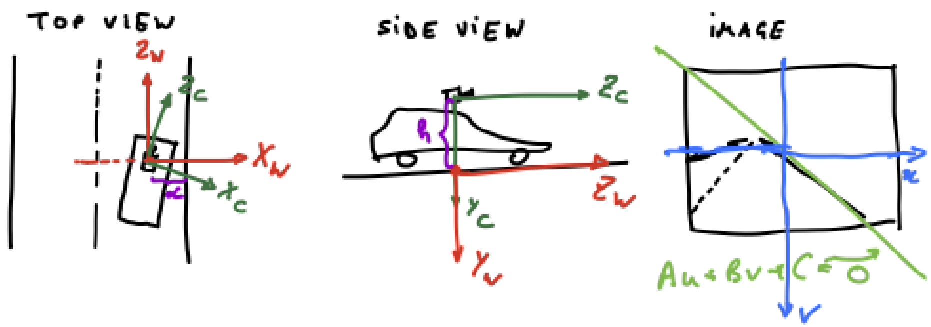
(j) What is the line equation of the horizon arising from the line at infinity in the plane

$Y_w = 0$.

★ **SOLUTION:** $(\frac{fr_{11}}{r_{13}}, f\frac{r_{12}}{r_{13}}, 1) \times (\frac{fr_{31}}{r_{33}}, f\frac{r_{32}}{r_{33}}, 1) = (fr_{11}, fr_{12}, r_{13}) \times (fr_{31}, fr_{32}, r_{33})$
$(fr_{21}, fr_{22}, f^2 r_{23}) = r_{21}x + r_{22}y + fr_{23} = 0$

2 Lane following (15 points)

This is a continuation of the setting above. Assume that the camera is mounted on a car. Here assume a static situation where the car is not moving. The ground plane is still $Y_w = 0$. Assume that $T = (0, h, 0)^T$ is the world's origin in camera coordinates and that the optical axis Z_c of the camera is parallel to the $Y_w = 0$ plane (the image plane is vertical if the ground plane is horizontal). Assume that the yaw angle between Z_c and Z_w is β. Continue assuming that pixel coordinates relate to camera coordinates as $u = fX_c/Z_c$ and $v = fY_c/Z_c$.



(a) Write the projective transformation from ground plane coordinates (X_w, Z_w) to pixel coordinates (u, v). Your matrix should be expressed in terms of β, h, f.

★ **SOLUTION:** If we assume β is the magnitude of the angle, then the rotation is $\begin{bmatrix} cos(\beta) & 0 & -sin(\beta) \\ 0 & 1 & 0 \\ sin(\beta) & 0 & cos(\beta) \end{bmatrix}$. The translation is $\begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}$

So the homography is $\lambda \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim K \begin{bmatrix} r_1 & r_3 & T \end{bmatrix} \begin{pmatrix} X_w \\ Z_w \\ 1 \end{pmatrix} \sim \begin{bmatrix} fcos(\beta) & -fsin(\beta) & 0 \\ 0 & 0 & fh \\ sin(\beta) & cos(\beta) & 0 \end{bmatrix} \begin{pmatrix} X_w \\ Z_w \\ 1 \end{pmatrix}$.

(b) Now assume that a line designating the right lane on the ground plane is parallel to the Z_w axis and passes through $X_w = d$. Compute the projection of this line in the image plane and call it (A, B, C) meaning that the line equation in pixels will read $Au + Bv + C = 0$. Write (A, B, C) in terms of β, h, f, d.

★ **SOLUTION:** We will take two points on the line on the ground plane, project them to the image, and take the cross product of the two projected points to get the line equations.

We can pick the two points to be $P_1 \sim \begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix}$ and $P_2 \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Their projections are $p_1 \sim HP_1 \sim K(dr_1 + T)$ and $p_2 \sim HP_2 \sim Kr_3$.

The line is the cross product, $l \sim p_1 \times p_2 \sim \begin{pmatrix} hcos(\beta) \\ -d \\ fhsin(\beta) \end{pmatrix}$

So we can write $\begin{pmatrix} A \\ B \\ C \end{pmatrix} \sim \begin{pmatrix} hcos(\beta) \\ -d \\ fhsin(\beta) \end{pmatrix}$ or $\lambda \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} hcos(\beta) \\ -d \\ fhsin(\beta) \end{pmatrix}$

(c (A, B, C) as any line coefficients in $\mathbb{P}^2$ (and in high school euclidean geometry) can be computed up to a scalar multiple. Given (A, B, C) up to a scale, compute d and β given f and h.

★ **SOLUTION:** First solve for β by taking the ratio between C and A

$\frac{C}{A} = ftan(\beta)$. So $\beta = tan^{-1}(\frac{C}{fA})$

If $B = 0$, then $b = 0$. Otherwise,

$hcos(\beta) = -d\frac{A}{B}$ and $fhsin(\beta) = d\frac{C}{B}$.

$cos(\beta) = -d\frac{A}{Bh}$ and $sin(\beta) = d\frac{C}{Bfh}$

$(-d\frac{A}{Bh})^2 + (d\frac{C}{Bfh})^2 = 1$

$d^2 = h^2 * \frac{1}{\sqrt{\frac{A^2}{B^2} + \frac{C^2}{B^2 f^2}}}$ and $d = \sqrt{h^2 * \frac{1}{\sqrt{\frac{A^2}{B^2} + \frac{C^2}{B^2 f^2}}}}$