Sequent Calculi for Data-Aware Modal Logics

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This document serves as a companion to the paper of the same title, wherein we introduce a Gentzen-style sequent calculus for $\mathrm{HXPath_D}$. It provides full technical details and proofs from the main paper. As such, it is intended as a reference for readers seeking a deeper understanding of the formal results, including soundness, completeness, invertibility, and cut elimination for the calculus.

1 Hybrid XPath with Data

We assume Prop, Nom, Mod, and Cmp are pairwise disjoint sets of symbols for propositions, nominals, modalities, and data comparisons. We assume Mod and Cmp are finite, and Prop and Nom are countably infinite.

Definition 1. *The language of* $\mathrm{HXPath_D}$ *has* path *expressions (denoted α, β, ...) and* node *expressions (denoted φ, ψ, ...), mutually defined by the grammar:*

$$\alpha,\beta := \mathsf{a} \mid i{:} \mid \varphi? \mid \alpha\beta$$
$$\varphi,\psi := p \mid i \mid \bot \mid \varphi\to\psi \mid @_i\varphi \mid \langle\mathsf{a}\rangle\varphi \mid \langle\alpha =_{\mathsf{c}} \beta\rangle \mid \langle\alpha \neq_{\mathsf{c}} \beta\rangle,$$

where $p\in\mathsf{Prop}$, $i\in\mathsf{Nom}$, $\mathsf{a}\in\mathsf{Mod}$, and $\mathsf{c}\in\mathsf{Cmp}$. For path expressions, we use $\varepsilon := \top?$ to indicate the empty *path. For node expressions, we use standard abbreviations: $\top := \bot\to\bot$, $\neg\varphi := \varphi\to\bot$, $\varphi\vee\psi := \neg\varphi\to\psi$, $\varphi\wedge\psi := \neg(\varphi\to\neg\psi)$, and $\varphi\leftrightarrow\psi := (\varphi\to\psi)\wedge(\psi\to\varphi)$. We also abbreviate: $\langle j{:}\rangle\varphi := @_j\varphi$, $\langle\psi?\rangle\varphi := \psi\wedge\varphi$, $\langle\alpha\beta\rangle\varphi := \langle\alpha\rangle\langle\beta\rangle\varphi$, and $[\alpha]\varphi := \neg\langle\alpha\rangle\neg\varphi$. Finally, we abbreviate $[\alpha \blacktriangle \beta] := \neg\langle\alpha \blacktriangledown \beta\rangle$. In the last abbreviation, we use $\blacktriangle$ when there is no need to distinguish $=_{\mathsf{c}}$ and $\neq_{\mathsf{c}}$, and use $\blacktriangledown$ to indicate $\neq_{\mathsf{c}}$ if $\blacktriangle$ is $=_{\mathsf{c}}$, and to indicate $=_{\mathsf{c}}$ if $\blacktriangle$ is $\neq_{\mathsf{c}}$.*

Path and node expressions are interpreted over hybrid data models.

Definition 2. *A* (hybrid data) model *is a tuple $\mathfrak{M} = \langle \mathrm{N}, \{\mathrm{R_a}\}_{\mathsf{a}\in\mathsf{Mod}}, \{\approx_{\mathsf{c}}\}_{\mathsf{c}\in\mathsf{Cmp}}, g, \mathrm{V}\rangle$, where N is a non-empty set of nodes; each $\mathrm{R_a}$ is a (binary)* accessibility relation *on N; each $\approx_{\mathsf{c}}$ is an equivalence relation on N, called a* comparison*; $g:\mathsf{Nom}\to\mathrm{N}$ is a* nominal assignment*; and $\mathrm{V}:\mathsf{Prop}\to 2^{\mathrm{N}}$ is a* valuation.

The satisfiability relation for path and node expressions is as follows.

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BASIC AXIOMS

(CPL)	all tautologies of CPL
(@-def)	$@_i\varphi \leftrightarrow \langle i{:}\varphi? =_{\mathsf{c}} i{:}\varphi?\rangle$
($\langle \mathsf{a}\rangle$-def)	$\langle\alpha\rangle\varphi \leftrightarrow \langle\alpha\varphi? =_{\mathsf{c}} \alpha\varphi?\rangle$
(K)	$[\alpha](\varphi\to\psi)\to([\alpha]\varphi\to[\alpha]\psi)$
(@K)	$@_i(\varphi\to\psi)\to(@_i\varphi\to @_i\psi)$
(@-self-dual)	$\neg @_i\varphi \leftrightarrow @_i\neg\varphi$
(@-intro)	$i\to(\varphi\leftrightarrow @_i\varphi)$
(@-refl)	$@_i i$

PATH AXIOMS

(comp-assoc)	$\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle \leftrightarrow \langle\alpha(\beta\gamma)\blacktriangle\eta\rangle$
(comp-neutral)†	$\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle \leftrightarrow \langle\alpha\beta\blacktriangle\gamma\rangle$
(comp-dist)	$\langle\alpha\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$

† α or β may be omitted (but not both)

DATA COMPARISON AXIOMS

(equal)	$\langle\varepsilon =_{\mathsf{c}} \varepsilon\rangle$
($\blacktriangle$-comm)	$\langle\alpha\blacktriangle\beta\rangle \leftrightarrow \langle\beta\blacktriangle\alpha\rangle$
(ε-trans)	$(\langle\alpha =_{\mathsf{c}} \varepsilon\rangle \wedge \langle\varepsilon =_{\mathsf{c}} \beta\rangle)\to\langle\alpha =_{\mathsf{c}}\beta\rangle$
(distinct)	$\neg\langle\varepsilon \neq_{\mathsf{c}} \varepsilon\rangle$
(@-data)	$\neg\langle i{:} =_{\mathsf{c}} j{:}\rangle \leftrightarrow \langle i{:} \neq_{\mathsf{c}} j{:}\rangle$
(subpath)	$\langle\alpha\beta\blacktriangle\gamma\rangle\to\langle\alpha\rangle\top$
(@$\blacktriangle$-dist)	$\langle i{:}\alpha\blacktriangle i{:}\beta\rangle \leftrightarrow @_i\langle\alpha\blacktriangle\beta\rangle$
($\blacktriangle$-test)	$\langle\varphi?\alpha\blacktriangle\beta\rangle\leftrightarrow(\varphi\wedge\langle\alpha\blacktriangle\beta\rangle)$
(agree)	$\langle j{:}i{:}\alpha\blacktriangle\beta\rangle\leftrightarrow\langle i{:}\alpha\blacktriangle\beta\rangle$
(back)	$\langle\alpha i{:}\beta\blacktriangle\gamma\rangle\to\langle i{:}\beta\blacktriangle\gamma\rangle$
($\blacktriangle$-comp-dist)	$\langle\alpha\rangle\langle\beta\blacktriangle\gamma\rangle\to\langle\alpha\beta\blacktriangle\alpha\gamma\rangle$

RULES OF INFERENCE

$$\frac{\vdash\varphi \quad \vdash\varphi\to\psi}{\vdash\psi}\,(\mathrm{MP}) \qquad \frac{\vdash\varphi}{\vdash[\alpha]\varphi}\,(\mathrm{Nec}) \qquad \frac{\vdash @_i\varphi}{\vdash\varphi}\,(\mathrm{Name})^\ddagger$$

$$\frac{\vdash @_i\langle\mathsf{a}\rangle j\wedge\langle j{:}\alpha\blacktriangle\beta\rangle\to\varphi}{\vdash\langle i{:}\mathsf{a}\alpha\blacktriangle\beta\rangle\to\varphi}\,(\mathrm{Paste})^\ddagger$$

‡ i and j are different and not in φ, α, β

Figure 1: Axioms System **H** for $\mathrm{HXPath}_{\mathrm{D}}$

Definition 3. *Let $\mathfrak{M}=\langle \mathrm{N},\{\mathrm{R}_{\mathsf{a}}\}_{\mathsf{a}\in\mathsf{Mod}},\{\approx_{\mathsf{c}}\}_{\mathsf{c}\in\mathsf{Cmp}},g,\mathrm{V}\rangle$ be a model, and let $\{n,n'\}\subseteq\mathrm{N}$. The satisfiability relation $\Vdash$ is given by the following conditions:*

$\mathfrak{M},n,n'\Vdash \mathsf{a}$	*iff*	$n\mathrm{R}_{\mathsf{a}}n'$
$\mathfrak{M},n,n'\Vdash i{:}$	*iff*	$g(i)=n'$
$\mathfrak{M},n,n'\Vdash \varphi?$	*iff*	$n=n'$ *and* $\mathfrak{M},n\Vdash\varphi$
$\mathfrak{M},n,n'\Vdash \alpha\beta$	*iff*	*exists* $n''\in\mathrm{N}$ *s.t.* $\mathfrak{M},n,n''\Vdash\alpha$ *and* $\mathfrak{M},n'',n'\Vdash\beta$
$\mathfrak{M},n\Vdash p$	*iff*	$n\in\mathrm{V}(p)$
$\mathfrak{M},n\Vdash i$	*iff*	$g(i)=n$
$\mathfrak{M},n\Vdash\bot$		*never*
$\mathfrak{M},n\Vdash\varphi\to\psi$	*iff*	$\mathfrak{M},n\Vdash\varphi$ *implies* $\mathfrak{M},n\Vdash\psi$
$\mathfrak{M},n\Vdash @_i\varphi$	*iff*	$\mathfrak{M},g(i)\Vdash\varphi$
$\mathfrak{M},n\Vdash\langle\mathsf{a}\rangle\varphi$	*iff*	*exists* $n'\in\mathrm{N}$ *s.t.* $\mathfrak{M},n,n'\Vdash\mathsf{a}$ *and* $\mathfrak{M},n'\Vdash\varphi$
$\mathfrak{M},n\Vdash\langle\alpha =_{\mathsf{c}}\beta\rangle$	*iff*	*exists* $n',n''\in\mathrm{N}$ *s.t.* $\mathfrak{M},n,n'\Vdash\alpha$, $\mathfrak{M},n,n''\Vdash\beta$ *and* $n'\approx_{\mathsf{c}}n''$
$\mathfrak{M},n\Vdash\langle\alpha \neq_{\mathsf{c}}\beta\rangle$	*iff*	*exists* $n',n''\in\mathrm{N}$ *s.t.* $\mathfrak{M},n,n'\Vdash\alpha$, $\mathfrak{M},n,n''\Vdash\beta$ *and* $n'\not\approx_{\mathsf{c}}n''$.

Let Ψ be a set of node expressions, we use $\mathfrak{M},n\Vdash\Psi$ to indicate $\mathfrak{M},n\Vdash\psi$ for all $\psi\in\Psi$. We say Ψ is satisfiable *iff there exists $\mathfrak{M},n$ s.t. $\mathfrak{M},n\Vdash\Psi$. We call a node expression φ a* consequence *of Ψ, written $\Psi\vDash\varphi$, iff $\Psi\cup\{\neg\varphi\}$ is unsatisfiable. If Ψ is the empty set, we write $\vDash\varphi$ and call φ a* tautology.

Proposition 4, immediate from Def. 3, establishes that the abbreviations have their intended meaning.

Proposition 4. *$\mathfrak{M},n\Vdash\langle\alpha\rangle\varphi$ iff exists $n'\in\mathrm{N}$ s.t., $\mathfrak{M},n,n'\Vdash\alpha$ and $\mathfrak{M},n'\Vdash\varphi$. Moreover, $\mathfrak{M},n\Vdash[\alpha =_{\mathsf{c}}\beta]$ iff for all $n',n''\in\mathrm{N}$, $\mathfrak{M},n,n'\Vdash\alpha$, $\mathfrak{M},n,n''\Vdash\beta$ implies $n'\approx_{\mathsf{c}}n''$. Finally, $\mathfrak{M},n\Vdash[\alpha\neq_{\mathsf{c}}\beta]$ iff for all $n',n''\in\mathrm{N}$, $\mathfrak{M},n,n'\Vdash\alpha$, $\mathfrak{M},n,n''\Vdash\beta$ implies $n'\not\approx_{\mathsf{c}}n''$.*

We conclude this section with presenting the Hilbert-style axiom **H** system for $\mathrm{HXPath}_{\mathrm{D}}$ in [1].

Definition 5. *The axioms schemas and rules of* **H** *are summarized in Fig. 1. The notion of a* deduction *of a node expression* φ *in* **H** *is defined as a finite sequence* $\psi_1 \dots \psi_n$ *of node expressions such that* $\psi_n = \varphi$*, and for all* $1 \leq z < n$*,* ψ_z *is either an instantiation of an axiom schema, or it is obtained via (MP), (Nec), (Name) or (Paste). In the cases (Name) and (Paste) it is understood that the nominals used meet the side conditions of the rule. We write* $\vdash_{\mathbf{H}} \varphi$*, and call* φ a theorem (of **H**)*, if there exists a deduction of* φ *in* **H**. *For a set of node expressions* Ψ*, we write* $\Psi \vdash_{\mathbf{H}} \varphi$*, and say that* φ is deducible from Γ (in **H**)*, iff there exists a finite set* $\{\psi_1 \dots \psi_m\} \subseteq \Psi$ *such that* $\vdash_{\mathbf{H}} (\psi_1 \wedge \cdots \wedge \psi_m) \rightarrow \varphi$*. We use the symbol* $\vdash^n_{\mathbf{H}}$ *to indicate that the length of the corresponding deduction is n.*

Remark 1. We note (@-def) and ($\langle \mathsf{a} \rangle$-def) are not part of the axioms for $\mathrm{HXPath_D}$ in [1]. We include them as axioms since we treat @ and $\langle \mathsf{a} \rangle$ as primitive modalities, whereas they are abbreviations in [1].

The following result is immediate from [1].

Theorem 6 (Soundness and Completeness [1])**.** $\Psi \vdash_{\mathbf{H}} \varphi$ *iff* $\Psi \vDash \varphi$.

2 A Sequent Calculus for $\mathrm{HXPath_D}$

Definition 7. *A* sequent *is a pair* $\Gamma \vdash \Delta$*, where* Γ *and* Δ *are finite, possibly empty sets of node expressions of the form* $\langle i{:}\ \blacktriangle\ j{:}\rangle$ *or* $@_i\varphi$.[1] *In a sequent* $\Gamma \vdash \Delta$*, the set* Γ *is called the* antecedent *and the set* Δ *is called the* consequent*. In turn, a* rule *is a pair* $(\Gamma \vdash \Delta, \{\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n\})$*, where* $\Gamma \vdash \Delta$ *is called the* conclusion*, and* $\{\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n\}$ *the set of* premisses *of the rule. If the set of premisses is empty, we say that the rule is an* axiom*. In each rule, certain node expressions in the conclusion sequent are designated as* principal*. These are the node expressions the rule acts upon. The remaining node expressions in the sequent are the* context *of the rule, and are carried unchanged across its application. This distinction enables the rule to isolate and analyze the logical structure of the principal formulas while treating the context uniformly. The rules of* **G** *are shown in Fig. 2, in a standard format—i.e., the conclusion is shown under a line, the premisses are above the line in no particular order, the principal node expressions and the context are identified immediately from the presentation.*

Definition 8. *A* derivation *in* **G** *is a sequent-labeled tree constructed according to the rules in Fig. 2. More precisely, each non-leaf node in the tree is a sequent that is the conclusion of a rule in* **G***, its immediate successors in the tree (going upwards) are the premisses of that rule. The root of the tree is called the* end-sequent *of the derivation. A sequent is* derivable *if it is the end-sequent of some derivation, and it is* provable *if it has a derivation whose leaves are all instances of (Ax) or* $(\bot)$*. We use* $\Gamma \vdash_{\mathbf{G}} \Delta$ *to indicate that* $\Gamma \vdash \Delta$ *is provable. A rule is* derived *iff there is a derivation of the conclusion of the rule whose leaves are either axioms or belong to the premisses of the rule.*

3 Soundness

Definition 9. *A sequent* $\Gamma \vdash \Delta$ *is* valid *iff for all* $\mathfrak{M}$*, it follows that* $\mathfrak{M} \Vdash \Gamma$ *implies* $\mathfrak{M} \Vdash \psi$ *for some* $\psi \in \Delta$*. A rule preserves validity iff the validity of the premisses of the rule implies the validity of the conclusion of the rule.*

Lemma 10 (Soundness)**.** *Every rule in* **G** *preserves validity.*

Proof. We present a selection of representative cases below. The remaining cases use a similar argument and can be verified by routine inspection. In all cases below we reason by contradiction.

[1] Following standard notation, we will not use curly brackets when writing down sequents.

PROPOSITIONAL RULES

$$\frac{}{\varphi,\Gamma\vdash\Delta,\varphi}\,(\mathrm{Ax})^{\ddagger}\qquad \frac{}{@_i\bot,\Gamma\vdash\Delta}\,(\bot)\qquad \frac{\Gamma\vdash\Delta,@_i\varphi\quad @_i\psi,\Gamma\vdash\Delta}{@_i(\varphi\to\psi),\Gamma\vdash\Delta}\,(\to\mathrm{L})\qquad \frac{@_i\varphi,\Gamma\vdash\Delta,@_i\psi}{\Gamma\vdash\Delta,@_i(\varphi\to\psi)}\,(\to\mathrm{R})$$

‡ φ is of the form $@_i p$, $@_i j$, or $\langle i{:}=_{\mathsf{c}} j{:}\rangle$

RULES FOR NOMINALS

$$\frac{@_i i,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\,(@\mathrm{T})\qquad \frac{@_j k,@_i j,@_i k,\Gamma\vdash\Delta}{@_i j,@_i k,\Gamma\vdash\Delta}\,(@5)\qquad \frac{@_i j,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\,(\mathrm{Nom})^{\dagger}$$

$$\frac{@_j\varphi,@_i j,@_i\varphi,\Gamma\vdash\Delta}{@_i j,@_i\varphi,\Gamma\vdash\Delta}\,(\mathrm{S}_1)^{\ddagger}\qquad \frac{@_i\langle\mathsf{a}\rangle k,@_j k,@_i\langle\mathsf{a}\rangle j,\Gamma\vdash\Delta}{@_j k,@_i\langle\mathsf{a}\rangle j,\Gamma\vdash\Delta}\,(\mathrm{S}_2)\qquad \frac{\langle j{:}=_{\mathsf{c}} k{:}\rangle,@_i j,\langle i{:}=_{\mathsf{c}} k{:}\rangle,\Gamma\vdash\Delta}{@_i j,\langle i{:}=_{\mathsf{c}} k{:}\rangle,\Gamma\vdash\Delta}\,(\mathrm{S}_3)$$

† j is not in the conclusion ‡ φ is of the form p, $\bot$, or $\langle\mathsf{a}\rangle k$

RULES FOR MODALITIES

$$\frac{@_i\varphi,\Gamma\vdash\Delta}{@_j@_i\varphi,\Gamma\vdash\Delta}\,(@\mathrm{L})\qquad \frac{\Gamma\vdash\Delta,@_i\varphi}{\Gamma\vdash\Delta,@_j@_i\varphi}\,(@\mathrm{R})\qquad \frac{@_i\langle\mathsf{a}\rangle j,@_j\varphi,\Gamma\vdash\Delta}{@_i\langle\mathsf{a}\rangle\varphi,\Gamma\vdash\Delta}\,(\langle\mathsf{a}\rangle\mathrm{L})^{\dagger}\qquad \frac{@_i\langle\mathsf{a}\rangle j,\Gamma\vdash\Delta,@_i\langle\mathsf{a}\rangle\varphi,@_j\varphi}{@_i\langle\mathsf{a}\rangle j,\Gamma\vdash\Delta,@_i\langle\mathsf{a}\rangle\varphi}\,(\langle\mathsf{a}\rangle\mathrm{R})$$

$$\frac{@_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\langle j{:}\blacktriangle k{:}\rangle,\Gamma\vdash\Delta}{@_i\langle\alpha\blacktriangle\beta\rangle,\Gamma\vdash\Delta}\,(\langle\blacktriangle\rangle\mathrm{L})^{\ddagger}\qquad \frac{@_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\Gamma\vdash\Delta,@_i\langle\alpha\blacktriangle\beta\rangle,\langle j{:}\blacktriangle k{:}\rangle}{@_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\Gamma\vdash\Delta,@_i\langle\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})$$

† j is not in the conclusion ‡ j and k are different and not in the conclusion

RULES FOR DATA COMPARISON

$$\frac{\langle i{:}=_{\mathsf{c}} i{:}\rangle,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\,(\mathrm{EqT})\qquad \frac{\langle j{:}=_{\mathsf{c}} k{:}\rangle,\langle i{:}=_{\mathsf{c}} j{:}\rangle,\langle i{:}=_{\mathsf{c}} k{:}\rangle,\Gamma\vdash\Delta}{\langle i{:}=_{\mathsf{c}} j{:}\rangle,\langle i{:}=_{\mathsf{c}} k{:}\rangle,\Gamma\vdash\Delta}\,(\mathrm{Eq5})\qquad \frac{\Gamma\vdash\Delta,\langle i{:}=_{\mathsf{c}} j{:}\rangle}{\langle i{:}\neq_{\mathsf{c}} j{:}\rangle,\Gamma\vdash\Delta}\,(\mathrm{NEqL})\qquad \frac{\langle i{:}=_{\mathsf{c}} j{:}\rangle,\Gamma\vdash\Delta}{\Gamma\vdash\Delta,\langle i{:}\neq_{\mathsf{c}} j{:}\rangle}\,(\mathrm{NEqR})$$

STRUCTURAL RULES

$$\frac{\Gamma\vdash\Delta,\varphi\quad \varphi,\Gamma'\vdash\Delta'}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(\mathrm{Cut})\qquad \frac{\Gamma\vdash\Delta}{\varphi,\Gamma\vdash\Delta}\,(\mathrm{WL})\qquad \frac{\Gamma\vdash\Delta}{\Gamma\vdash\Delta,\varphi}\,(\mathrm{WR})$$

Figure 2: Sequent Calculus **G** for $\mathrm{HXPath}_{\mathrm{D}}$.

(Nom) Let $\mathfrak{A}$ be a model s.t. $\mathfrak{A}\Vdash\Gamma$ and $\mathfrak{A}\nVdash\psi$ for all $\psi\in\Delta$. Then, introduce a new nominal j that is not in Γ,Δ and build a model $\mathfrak{B}$ that is just like $\mathfrak{A}$ with the exception that $\mathfrak{B}\Vdash @_i j$. We have $\mathfrak{B}\Vdash @_i j,@_i\varphi,\Gamma$ and $\mathfrak{B}\nVdash\psi$ for all $\psi\in\Delta$. This contradicts the validity of the premiss of the rule.

($\langle\blacktriangle\rangle$L) We have: (1) $\blacktriangle$ is $=_{\mathsf{c}}$, or (2) $\blacktriangle$ is $\neq_{\mathsf{c}}$. For (1), take any model $\mathfrak{A}$ s.t.: $\mathfrak{A}\Vdash @_i\langle\alpha=_{\mathsf{c}}\beta\rangle,\Gamma$, and $\mathfrak{A}\nVdash\psi$ for all $\psi\in\Delta$. The semantics of $@_i\langle\alpha=_{\mathsf{c}}\beta\rangle$ tells us there are n and n' in $\mathfrak{A}$ s.t.: $\mathfrak{A},g(i),n\Vdash\alpha$, $\mathfrak{A},g(i),n'\Vdash\beta$, and $(n,n')\in\approx_{\mathsf{c}}$. Then, choose nominals j and k that do not appear in $\Gamma,\Delta,\alpha,\beta$ and build a model $\mathfrak{B}$ that is identical to $\mathfrak{A}$ with the exception that $\mathfrak{B},n\Vdash j$ and $\mathfrak{B},n'\Vdash k$. It is clear that $\mathfrak{B}\Vdash @_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\langle j{:}=_{\mathsf{c}} k{:}\rangle,\Gamma$ and $\mathfrak{B}\nVdash\psi$ for all $\psi\in\Delta$. This contradicts the validity of the premiss of the rule. The case for (2) is similar.

($\langle\blacktriangle\rangle$R) We have: (1) $\blacktriangle$ is $=_{\mathsf{c}}$, or (2) $\blacktriangle$ is $\neq_{\mathsf{c}}$. For (1), take any model $\mathfrak{A}$ s.t.: $\mathfrak{A}\Vdash @_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\Gamma$ and $\mathfrak{B}\nVdash\psi$ for all $\psi\in\Delta,@_i\langle\alpha=_{\blacktriangle}\beta\rangle$. In particular, $\mathfrak{A}\nVdash @_i\langle\alpha=_{\mathsf{c}}\beta\rangle$. This means that for all n and n' in $\mathfrak{A}$, it follows that $\mathfrak{A},g(i),n\Vdash\alpha$, $\mathfrak{A},g(i),n'\Vdash\beta$, and $(n,n')\notin\approx_{\mathsf{c}}$. This contradicts the validity of premiss of the rule. The case for (2) is similar. □

Soundness of **G** follows from Lemma 10 by induction on the structure of a derivation of a sequent.

Theorem 11 (Soundness)**.** *Every provable sequent in* **G** *is valid, i.e.,* $\Gamma\vdash_{\mathbf{G}}\Delta$ *implies* $\Gamma\vDash\Delta$.

4 Invertibility

Definition 12. *A rule* (P) *is* invertible *iff there is a derivation of each premiss of the rule whose leaves are either axioms or the conclusion of the rule. Any such derivation is called an* inverse *of the rule and is denoted by* (P^{-1}).

Theorem 13. *Every rule in* **G** *is invertible.*

Proof. The case for the rules (@T), (@5), (Nom), (S_1), (S_2), (S_3), ($\langle\mathsf{a}\rangle$R), ($\langle\blacktriangle\rangle$R), (EqT), and (Eq5) the result is immediate using weakening.

($\to\mathrm{L}_1$) Given a derivation of $@_i(\varphi\to\psi)$, we can build a derivation of $@_i\varphi$ as follows:

$$\dfrac{\dfrac{\dfrac{}{@_i\varphi,\Gamma\vdash\Delta,@_i\varphi,@_i\psi}\,(\mathrm{Ax})}{\Gamma\vdash\Delta,@_i\varphi,@_i(\varphi\to\psi)}\,(\to\mathrm{R}) \qquad @_i(\varphi\to\psi),\Gamma\vdash\Delta}{\Gamma\vdash\Delta,@_i\varphi}\,(\mathrm{Cut})$$

($\to\mathrm{L}_2$) Given a derivation of $@_i(\varphi\to\psi)$, we can build a derivation of $@_i\varphi$ as follows:

$$\dfrac{\dfrac{\dfrac{}{@_i\varphi,@_i\psi,\Gamma\vdash\Delta,@_i\psi}\,(\mathrm{Ax})}{@_i\psi,\Gamma\vdash\Delta,@_i(\varphi\to\psi)}\,(\to\mathrm{R}) \qquad @_i(\varphi\to\psi),\Gamma\vdash\Delta}{@_i\psi,\Gamma\vdash\Delta}\,(\mathrm{Cut})$$

($\to$R) Given a derivation of $@_i(\varphi\to\psi)$, we can build a derivation of $@_i\varphi$ as follows:

$$\dfrac{\Gamma\vdash\Delta,@_i(\varphi\to\psi) \qquad \dfrac{\dfrac{}{@_i\varphi,\Gamma\vdash\Delta,@_i\psi,@_i\varphi}\,(\mathrm{Ax}) \quad \dfrac{}{@_i\psi,@_i\varphi,\Gamma\vdash\Delta,@_i\psi}\,(\mathrm{Ax})}{@_i(\varphi\to\psi),@_i\varphi,\Gamma\vdash\Delta,@_i\psi}\,(\to\mathrm{L})}{@_i\varphi,\Gamma\vdash\Delta,@_i\psi}\,(\mathrm{Cut})$$

(@L) Given a derivation of $@_j@_i\varphi$, we can build a derivation of $@_i\varphi$ as follows:

$$\dfrac{\dfrac{\dfrac{}{@_i\varphi,\Gamma\vdash\Delta,@_i\varphi}\,(\mathrm{Ax})}{@_i\varphi,\Gamma\vdash\Delta,@_j@_i\varphi}\,(@\mathrm{R}) \qquad @_j@_i\varphi,\Gamma\vdash\Delta}{@_i\varphi,\Gamma\vdash\Delta}\,(\mathrm{Cut})$$

(@R) Given a derivation of $@_j@_i\varphi$, we can build a derivation of $@_i\varphi$ as follows:

$$\dfrac{\Gamma\vdash\Delta,@_j@_i\varphi \qquad \dfrac{\dfrac{}{@_i\varphi,\Gamma\vdash\Delta,@_i\varphi}\,(\mathrm{Ax})}{@_j@_i\varphi,\Gamma\vdash\Delta,@_i\varphi}\,(@\mathrm{L})}{\Gamma\vdash\Delta,@_i\varphi}\,(\mathrm{Cut})$$

($\langle\mathsf{a}\rangle$L) Given a derivation of $@_i\langle\mathsf{a}\rangle\varphi$, we can build a derivation of $@_i\langle\mathsf{a}\rangle j, @_j\varphi$ as follows:

$$\dfrac{\dfrac{\dfrac{}{@_i\langle\mathsf{a}\rangle j,@_j\varphi,\Gamma\vdash\Delta,@_i\langle\mathsf{a}\rangle\varphi,@_j\varphi}\,(\mathrm{Ax})}{@_i\langle\mathsf{a}\rangle j,@_j\varphi,\Gamma\vdash\Delta,@_i\langle\mathsf{a}\rangle\varphi}\,(\langle\mathsf{a}\rangle\mathrm{R}) \qquad @_i\langle\mathsf{a}\rangle\varphi,\Gamma\vdash\Delta}{@_i\langle\mathsf{a}\rangle j,@_j\varphi,\Gamma\vdash\Delta}\,(\mathrm{Cut})$$

(NEqL) Given a derivation of $\langle i\colon\neq_{\mathsf{c}} j\colon\rangle$, we can build a derivation of $\langle i\colon=_{\mathsf{c}} j\colon\rangle$ as follows:

$$\dfrac{\dfrac{\dfrac{}{\langle i\colon=_{\mathsf{c}} j\colon\rangle,\Gamma\vdash\Delta,\langle i\colon=_{\mathsf{c}} j\colon\rangle}\,(\mathrm{Ax})}{\Gamma\vdash\Delta,\langle i\colon=_{\mathsf{c}} j\colon\rangle,\langle i\colon\neq_{\mathsf{c}} j\colon\rangle}\,(\mathrm{NEqR}) \qquad \langle i\colon\neq_{\mathsf{c}} j\colon\rangle,\Gamma\vdash\Delta}{\Gamma\vdash\Delta,\langle i\colon=_{\mathsf{c}} j\colon\rangle}\,(\mathrm{Cut})$$

($\langle\blacktriangle\rangle$L) Given a derivation of $\langle\alpha \blacktriangle \beta\rangle$, we can build a derivation of $@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle$ as follows:

$$\dfrac{\dfrac{\dfrac{}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \Gamma \vdash \Delta, @_i\langle\alpha \blacktriangle \beta\rangle, \langle j{:} \blacktriangle k{:}\rangle}\,(\text{Ax})}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \Gamma \vdash \Delta, @_i\langle\alpha \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\text{R}) \qquad @_i\langle\alpha \blacktriangle \beta\rangle, \Gamma \vdash \Delta}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \Gamma \vdash \Delta}\,(\text{Cut})$$

($\neg$L) Given a derivation of $@_i\neg\varphi$, we can build a derivation of $@_i\varphi$ as follows:

$$\dfrac{\dfrac{\dfrac{}{@_i\varphi \vdash @_i\varphi, @_i\bot}\,(\text{Ax})}{\vdash @_i\varphi, @_i\neg\varphi}\,(\to)\text{R} \qquad @_i\neg\varphi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, @_i\varphi}\,(\text{Cut})$$

($\neg$R) Given a derivation of $@_i\neg\varphi$, we can build a derivation of $@_i\varphi$ as follows:

$$\dfrac{\Gamma \vdash \Delta, @_i\neg\varphi \qquad \dfrac{\dfrac{}{@_i\varphi \vdash @_i\varphi}\,(\text{Ax}) \quad \dfrac{}{@_i\bot, @_i\varphi \vdash}\,(\bot)}{@_i\neg\varphi, @_i\varphi \vdash}\,(\to\text{L})}{@_i\varphi, \Gamma \vdash \Delta}\,(\text{Cut})$$

($\wedge$L) Given a derivation of $@_i(\varphi \wedge \psi)$, we can build a derivation of $@_i\varphi, @_i\psi$ as follows:

$$\dfrac{\dfrac{\dfrac{}{@_i\varphi, @_i\psi \vdash @_i\varphi}\,(\text{Ax}) \quad \dfrac{}{@_i\varphi, @_i\psi \vdash @_i\psi}\,(\text{Ax})}{@_i\varphi, @_i\psi \vdash @_i(\varphi \wedge \psi)}\,(\wedge\text{R}) \qquad @_i(\varphi \wedge \psi), \Gamma \vdash \Delta}{@_i\varphi, @_i\psi, \Gamma \vdash \Delta}\,(\text{Cut})$$

□

5 Derived Rules

Definition 14. *A rule* (P) *is* derived *iff there is a derivation of the conclusion of the rule whose leaves are either axioms or belong to the premiss of the rule.*

The following proofs establish the derivability of the rules in Fig. 3.

- ($\top$L)

$$\dfrac{\dfrac{\dfrac{}{@_i\bot, \Gamma \vdash \Delta, @_i\bot}\,(\bot)}{\Gamma \vdash \Delta, @_i\top}\,(\to\text{R}) \qquad @_i\top, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,(\text{Cut})$$

- ($\neg$L)

$$\dfrac{\Gamma \vdash \Delta, @_i\varphi \qquad \dfrac{}{@_i\bot, \Gamma \vdash \Delta}\,(\bot)}{@_i\neg\varphi, \Gamma \vdash \Delta}\,(\to\text{L})$$

- ($\neg$R)

$$\dfrac{\dfrac{@_i\varphi, \Gamma \vdash \Delta}{@_i\varphi, \Gamma \vdash \Delta, @_i\bot}\,(\text{WR})}{\Gamma \vdash \Delta, @_i\neg\varphi}\,(\to\text{R})$$

- ($\wedge$L)

DERIVED PROPOSITIONAL RULES

$$\frac{}{@_i\varphi,\Gamma\vdash\Delta,@_i\varphi}\,(\mathrm{Ax})\qquad\frac{@_i\top,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\,(\top\mathrm{L})$$

$$\frac{\Gamma\vdash\Delta,@_i\varphi}{@_i\neg\varphi,\Gamma\vdash\Delta}\,(\neg\mathrm{L})\qquad\frac{@_i\varphi,\Gamma\vdash\Delta}{\Gamma\vdash\Delta,@_i\neg\varphi}\,(\neg\mathrm{R})$$

$$\frac{@_i\varphi,@_i\psi,\Gamma\vdash\Delta}{@_i(\varphi\wedge\psi),\Gamma\vdash\Delta}\,(\wedge\mathrm{L})\qquad\frac{\Gamma\vdash\Delta,@_i\varphi\quad\Gamma\vdash\Delta,@_i\psi}{\Gamma\vdash\Delta,@_i(\varphi\wedge\psi)}\,(\wedge\mathrm{R})$$

$$\frac{@_i\varphi,@_i\psi,\Gamma\vdash\Delta\quad\Gamma\vdash\Delta,@_i\varphi,@_i\psi}{@_i(\varphi\leftrightarrow\psi),\Gamma\vdash\Delta}\,(\leftrightarrow\mathrm{L})\qquad\frac{@_i\varphi,\Gamma\vdash\Delta,@_i\psi\quad@_i\psi,\Gamma\vdash\Delta,@_i\varphi}{\Gamma\vdash\Delta,@_i(\varphi\leftrightarrow\psi)}\,(\leftrightarrow\mathrm{R})$$

$$\frac{\Gamma\vdash\Delta,@_i\varphi\quad\Gamma'\vdash\Delta',@_i(\varphi\to\psi)}{\Gamma,\Gamma'\vdash\Delta,\Delta',@_i\psi}\,(\mathrm{MP})$$

DERIVED RULES FOR MODALITIES

$$\frac{@_i\langle\alpha\rangle j,@_j\varphi,\Gamma\vdash\Delta}{@_i\langle\alpha\rangle\varphi,\Gamma\vdash\Delta}\,(\langle\alpha\rangle\mathrm{L})^{\dagger}\qquad\frac{@_i\langle\alpha\rangle j,\Gamma\vdash\Delta,@_i\langle\alpha\rangle\varphi,@_j\varphi}{@_i\langle\alpha\rangle j,\Gamma\vdash\Delta,@_i\langle\alpha\rangle\varphi}\,(\langle\alpha\rangle\mathrm{R})$$

† j does not occur in the conclusion

DERIVED RULES FOR NOMINALS

$$\frac{@_j\varphi,@_ij,@_i\varphi,\Gamma\vdash\Delta}{@_ij,@_i\varphi,\Gamma\vdash\Delta}\,(\mathrm{S}_1)\qquad\frac{@_i\langle\alpha\rangle k,@_jk,@_i\langle\alpha\rangle j,\Gamma\vdash\Delta}{@_jk,@_i\langle\alpha\rangle j,\Gamma\vdash\Delta}\,(\mathrm{S}_2)\qquad\frac{\langle j{:}\blacktriangle k{:}\rangle,@_ij,\langle i{:}\blacktriangle k{:}\rangle,\Gamma\vdash\Delta}{@_ij,\langle i{:}\blacktriangle k{:}\rangle,\Gamma\vdash\Delta}\,(\mathrm{S}_3)$$

$$\frac{@_ji,\Gamma\vdash\Delta}{@_ij,\Gamma\vdash\Delta}\,(@\mathrm{B})\qquad\frac{\langle i{:}\blacktriangle j{:}\rangle,\Gamma\vdash\Delta}{@_k\langle i{:}\blacktriangle j{:}\rangle,\Gamma\vdash\Delta}\,(@\blacktriangle\mathrm{L})\qquad\frac{\Gamma\vdash\Delta,\langle i{:}\blacktriangle j{:}\rangle}{\Gamma\vdash\Delta,@_k\langle i{:}\blacktriangle j{:}\rangle}\,(@\blacktriangle\mathrm{R})$$

DERIVED RULES FOR DATA COMPARISON

$$\frac{\langle j{:}\blacktriangle i{:}\rangle,\Gamma\vdash\Delta}{\langle i{:}\blacktriangle j{:}\rangle,\Gamma\vdash\Delta}\,(\langle\blacktriangle\rangle\mathrm{B})$$

$$\frac{@_i[\alpha\blacktriangle\beta],\langle j{:}\blacktriangle k{:}\rangle,@_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\Gamma\vdash\Delta}{@_i[\alpha\blacktriangle\beta],@_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\Gamma\vdash\Delta}\,([\blacktriangle]\mathrm{L})\qquad\frac{@_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\Gamma\vdash\Delta,\langle j{:}\blacktriangle k{:}\rangle}{\Gamma\vdash\Delta,@_i[\alpha\blacktriangle\beta]}\,([\blacktriangle]\mathrm{R})^{\dagger}$$

† j and k are different and do not occur in the conclusion

Figure 3: Derived Sequent Rules for $\mathrm{HXPath_D}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{@_i\psi, @_i\varphi, \Gamma \vdash \Delta}{@_i\psi, @_i\varphi, \Gamma \vdash \Delta, @_i\bot}\,(\mathrm{WR})}{@_i\varphi, \Gamma \vdash \Delta, @_i(\psi \to \bot)}\,(\to\mathrm{R})}{\Gamma \vdash \Delta, @_i(\varphi \to (\psi \to \bot))}\,(\to\mathrm{R}) \qquad \dfrac{}{@_i\bot, \Gamma \vdash \Delta}\,(\bot)}{@_i(\varphi \wedge \psi), \Gamma \vdash \Delta}\,(\to\mathrm{L})$$

- $(\wedge\mathrm{R})$

$$\dfrac{\dfrac{\Gamma \vdash \Delta, @_i\varphi \qquad \dfrac{\Gamma \vdash \Delta, @_i\psi \qquad \dfrac{}{@_i\bot, \Gamma \vdash \Delta}\,(\bot)}{@_i(\psi \to \bot), \Gamma \vdash \Delta}\,(\to\mathrm{L})}{@_i(\varphi \to (\psi \to \bot)), \Gamma \vdash \Delta, @_i\bot}\,(\mathrm{WR}, \to\mathrm{L})}{\Gamma \vdash \Delta, @_i(\varphi \wedge \psi)}\,(\to\mathrm{R})$$

- $(\leftrightarrow\mathrm{L})$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\varphi, \Gamma \vdash \Delta, @_i\varphi}\,(\mathrm{Ax}) \qquad @_i\psi, @_i\varphi, \Gamma \vdash \Delta}{@_i\varphi, @_i(\varphi \to \psi), \Gamma \vdash \Delta}\,(\to\mathrm{L}) \qquad \dfrac{\Gamma \vdash \Delta, @_i\psi, @_i\varphi \qquad \dfrac{}{@_i\psi, \Gamma \vdash \Delta, @_i\psi}\,(\mathrm{Ax})}{@_i(\varphi \to \psi), \Gamma \vdash \Delta, @_i\psi}\,(\to\mathrm{L})}{@_i(\psi \to \varphi), @_i(\varphi \to \psi), \Gamma \vdash \Delta, @_i\bot}\,(\mathrm{WR}, \to\mathrm{L})}{@_i(\varphi \to \psi), \Gamma \vdash \Delta, @_i((\psi \to \varphi) \to \bot)}\,(\to\mathrm{R})}{\Gamma \vdash \Delta, @_i((\varphi \to \psi) \to ((\psi \to \varphi) \to \bot))}\,(\to\mathrm{R}) \qquad \dfrac{}{@_i\bot, \Gamma \vdash \Delta}\,(\bot)}{@_i(\varphi \leftrightarrow \psi), \Gamma \vdash \Delta}\,(\to\mathrm{L})$$

- $(\leftrightarrow\mathrm{R})$

$$\dfrac{\dfrac{\dfrac{@_i\varphi, \Gamma \vdash \Delta, @_i\psi}{\Gamma \vdash \Delta, @_i(\varphi \to \psi)}\,(\to\mathrm{R}) \qquad \dfrac{\dfrac{@_i\psi, \Gamma \vdash \Delta, @_i\varphi}{\Gamma \vdash \Delta, @_i(\psi \to \varphi)}\,(\to\mathrm{R}) \qquad \dfrac{}{@_i\bot, \Gamma \vdash \Delta}\,(\bot)}{@_i((\psi \to \varphi) \to \bot), \Gamma \vdash \Delta}\,(\to\mathrm{L})}{@_i((\varphi \to \psi) \to ((\psi \to \varphi) \to \bot)), \Gamma \vdash \Delta, @_i\bot}\,(\mathrm{WL}, \to\mathrm{L})}{\Gamma \vdash \Delta, @_i(\varphi \leftrightarrow \psi)}\,(\to\mathrm{R})$$

- (MP)

$$\dfrac{\Gamma \vdash \Delta, @_i\varphi \qquad \dfrac{\Gamma' \vdash \Delta', @_i(\varphi \to \psi)}{@_i\varphi, \Gamma' \vdash \Delta', @_i\psi}\,(\to\mathrm{R}^{-1})}{\Gamma, \Gamma' \vdash \Delta, \Delta', @_i\psi}\,(\mathrm{Cut})$$

- (@B)

$$\dfrac{\dfrac{\dfrac{@_j i, \Gamma \vdash \Delta}{@_j i, @_i j, @_i i, \Gamma \vdash \Delta}\,(\mathrm{WL})}{@_i j, @_i i, \Gamma \vdash \Delta}\,(@5)}{@_i j, \Gamma \vdash \Delta}\,(@\mathrm{T})$$

- $(@\blacktriangle\mathrm{L})$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle i{:}\ \blacktriangle\ j{:}\rangle, \Gamma \vdash \Delta}{\langle i{:}\ \blacktriangle\ j{:}\rangle, @_b j, \langle i{:}\ \blacktriangle\ b{:}\rangle, \Gamma \vdash \Delta}\,(\mathrm{WL})}{\langle i{:}\ \blacktriangle\ b{:}\rangle, @_a i, @_b j, \langle a{:}\ \blacktriangle\ b{:}\rangle, \Gamma \vdash \Delta}\,(\mathrm{WL}, \mathrm{S}_3)}{@_a i, @_b j, \langle a{:}\ \blacktriangle\ b{:}\rangle, \Gamma \vdash \Delta}\,(\mathrm{S}_3)}{@_i a, @_j b, \langle a{:}\ \blacktriangle\ b{:}\rangle, \Gamma \vdash \Delta}\,(@\mathrm{B})}{@_k\langle i{:}\rangle a, @_k\langle j{:}\rangle b, \langle a{:}\ \blacktriangle\ b{:}\rangle, \Gamma \vdash \Delta}\,(@\mathrm{L})}{@_k\langle i{:}\ \blacktriangle\ j{:}\rangle, \Gamma \vdash \Delta}\,(\langle\blacktriangle\rangle\mathrm{L})$$

- $(@\blacktriangle\mathrm{R})$

$$\dfrac{\dfrac{\Gamma \vdash \Delta, \langle i{:} \blacktriangle j{:}\rangle}{\Gamma \vdash \Delta, @_k\langle i{:} \blacktriangle j{:}\rangle, \langle i{:} \blacktriangle j{:}\rangle}\,(\mathrm{WR}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{@_k\langle i{:}\rangle i, @_k\langle j{:}\rangle j, \langle i{:} \blacktriangle j{:}\rangle, \Gamma \vdash \Delta, @_k\langle i{:} \blacktriangle j{:}\rangle, \langle i{:} \blacktriangle j{:}\rangle}\,(\mathrm{Ax})}{@_k\langle i{:}\rangle i, @_k\langle j{:}\rangle j, \langle i{:} \blacktriangle j{:}\rangle, \Gamma \vdash \Delta, @_k\langle i{:} \blacktriangle j{:}\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i i, @_j j, \langle i{:} \blacktriangle j{:}\rangle, \Gamma \vdash \Delta, @_k\langle i{:} \blacktriangle j{:}\rangle}\,(@\mathrm{L}^{-1})}{\langle i{:} \blacktriangle j{:}\rangle, \Gamma \vdash \Delta, @_k\langle i{:} \blacktriangle j{:}\rangle}\,(@\mathrm{T})}{\Gamma \vdash \Delta, @_k\langle i{:} \blacktriangle j{:}\rangle}\,(\mathrm{Cut})$$

- $(\langle\blacktriangle\rangle\mathrm{B})$
  - $\blacktriangle$ is $=_{\mathsf{c}}$

$$\dfrac{\dfrac{\dfrac{\langle j{:} =_{\mathsf{c}} i{:}\rangle, \Gamma \vdash \Delta}{\langle j{:} =_{\mathsf{c}} i{:}\rangle, \langle i{:} =_{\mathsf{c}} j{:}\rangle, \langle i{:} =_{\mathsf{c}} i{:}\rangle, \Gamma \vdash \Delta}\,(\mathrm{WL})}{\langle i{:} =_{\mathsf{c}} j{:}\rangle, \langle i{:} =_{\mathsf{c}} i{:}\rangle, \Gamma \vdash \Delta}\,(\mathrm{Eq5})}{\langle i{:} =_{\mathsf{c}} j{:}\rangle, \Gamma \vdash \Delta}\,(\mathrm{EqT})$$

  - $\blacktriangle$ is $\neq_{\mathsf{c}}$

$$\dfrac{\dfrac{\dfrac{\langle j{:} \neq_{\mathsf{c}} i{:}\rangle, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \langle j{:} =_{\mathsf{c}} i{:}\rangle}\,(\mathrm{NeqL}^{-1}) \qquad \dfrac{\dfrac{}{\langle i{:} =_{\mathsf{c}} j{:}\rangle, \Gamma \vdash \Delta, \langle i{:} =_{\mathsf{c}} j{:}\rangle}\,(\mathrm{Ax})}{\langle j{:} =_{\mathsf{c}} i{:}\rangle, \Gamma \vdash \Delta, \langle i{:} =_{\mathsf{c}} j{:}\rangle}\,(\langle =_{\mathsf{c}}\rangle\mathrm{B})}{\Gamma \vdash \Delta, \langle i{:} =_{\mathsf{c}} j{:}\rangle}\,(\mathrm{Cut})}{\langle i{:} \neq_{\mathsf{c}} j{:}\rangle, \Gamma \vdash \Delta}\,(\mathrm{NEqL})$$

- $(\langle\sigma\rangle\mathrm{L})$. Let $\sigma \in \{\mathsf{a}, \psi?, k{:}\}$. The following is a derived rule:

$$\dfrac{@_i\langle\sigma\rangle j, @_j\varphi, \Gamma \vdash \Delta}{@_i\langle\sigma\rangle\varphi, \Gamma \vdash \Delta}\,(\langle\sigma\rangle\mathrm{L})^{\dagger}$$

† were j is not in the conclusion

The proof is by a case analysis.

  - $\sigma = \mathsf{a}$ is $(\langle\mathsf{a}\rangle\mathrm{L})$.
  - $\sigma = k{:}$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{@_i\langle k{:}\rangle j, @_j\varphi, \Gamma \vdash \Delta}{@_k j, @_j\varphi, \Gamma \vdash \Delta}\,(@\mathrm{L}^{-1})}{@_j\varphi, @_k j, @_k\varphi, \Gamma \vdash \Delta}\,(\mathrm{WL})}{@_k j, @_k\varphi, \Gamma \vdash \Delta}\,(\mathrm{S}_1)}{@_k\varphi, \Gamma \vdash \Delta}\,(\mathrm{Nom})}{@_i\langle k{:}\rangle\varphi, \Gamma \vdash \Delta}\,(@\mathrm{L})$$

  - $\sigma = \psi?$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{@_i\langle\psi\rangle j, @_j\varphi, \Gamma \vdash \Delta}{@_j\varphi, @_i j, @_i\psi, \Gamma \vdash \Delta}\,(\wedge\mathrm{L}^{-1})}{@_j\varphi, @_i j, @_i\psi, @_i\varphi, \Gamma \vdash \Delta}\,(\mathrm{WL})}{@_i j, @_i\psi, @_i\varphi, \Gamma \vdash \Delta}\,(\mathrm{S}_1)}{@_i\psi, @_i\varphi, \Gamma \vdash \Delta}\,(\mathrm{Nom})}{@_i\langle\psi?\rangle\varphi, \Gamma \vdash \Delta}\,(\wedge\mathrm{L})$$

- $(\langle\sigma\rangle\mathrm{R})$. Let $\sigma \in \{\mathsf{a}, \psi?, k{:}\}$. The following is a derived rule:

$$\dfrac{@_i\langle\sigma\rangle j, \Gamma \vdash \Delta, @_i\langle\sigma\rangle\varphi, @_j\varphi}{@_i\langle\sigma\rangle j, \Gamma \vdash \Delta, @_i\langle\sigma\rangle\varphi}\,(\langle\sigma\rangle\mathrm{R})$$

The proof is by a case analysis.

- $\sigma = \mathsf{a}$ is $(\langle \mathsf{a} \rangle \mathrm{L})$.
- $\sigma = k$: is

$$\dfrac{@_i\langle k{:}\rangle j, \Gamma \vdash \Delta, @_i\langle k{:}\rangle\varphi, @_j\varphi \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{@_k\varphi, @_j\varphi, @_jk, \Gamma \vdash @_k\varphi}\,(\mathrm{Ax})}{@_j\varphi, @_jk, \Gamma \vdash @_k\varphi}\,(\mathrm{S}_1)}{@_j\varphi, @_kj, \Gamma \vdash @_k\varphi}\,(@\mathrm{B})}{@_j\varphi, @_i\langle k{:}\rangle j, \Gamma \vdash \Delta, @_i\langle k{:}\rangle\varphi}\,(@\mathrm{L},@\mathrm{R})}{@_i\langle k{:}\rangle j, \Gamma \vdash \Delta, @_i\langle k{:}\rangle\varphi}\,(\mathrm{Cut})$$

- $\sigma = \psi?$ is

$$\mathscr{D} = \dfrac{\dfrac{}{@_j\varphi, @_i\psi, @_ij, \Gamma \vdash \Delta, @_i\psi}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{@_i\varphi, @_j\varphi, @_i\psi, @_ij, \Gamma \vdash \Delta, @_i\varphi}\,(\mathrm{Ax})}{@_j\varphi, @_i\psi, @_ij, \Gamma \vdash \Delta, @_i\varphi}\,(\mathrm{S}_1)}{@_j\varphi, @_i\langle\psi?\rangle j, \Gamma \vdash \Delta, @_i\langle\psi?\rangle\varphi}\,(\wedge\mathrm{L},\wedge\mathrm{R})$$

$$\dfrac{@_i\langle\psi?\rangle j, \Gamma \vdash \Delta, @_i\langle\psi?\rangle\varphi, @_j\varphi \qquad \mathscr{D}}{@_i\langle\psi?\rangle j, \Gamma \vdash \Delta, @_i\langle\psi?\rangle\varphi}\,(\mathrm{Cut})$$

- $(\langle\alpha\rangle\mathrm{L})$. The proof is by induction the definition of $\langle\alpha\rangle\varphi$.
  - Base Case: $\alpha = \sigma$ is $(\langle\sigma\rangle\mathrm{L})$.
  - Inductive Step: $\alpha = \mathsf{a}\alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\sigma\rangle j, @_j\langle\alpha\rangle a, \Gamma \vdash \Delta, @_i\langle\sigma\alpha\rangle a, @_k\langle\alpha\rangle a}\,(\mathrm{Ax})}{@_i\langle\sigma\rangle j, @_j\langle\alpha\rangle a, \Gamma \vdash \Delta, @_i\langle\sigma\alpha\rangle a}\,(\langle\sigma\rangle\mathrm{R}) \qquad @_i\langle\sigma\alpha\rangle a, @_a\varphi, \Gamma \vdash \Delta}{@_i\langle\sigma\rangle j, @_j\langle\alpha\rangle a, @_a\varphi, \Gamma \vdash \Delta}\,(\mathrm{Cut})}{@_i\langle\sigma\rangle j, @_j\langle\alpha\rangle\varphi, \Gamma \vdash \Delta}\,(\mathrm{IH})}{@_i\langle\sigma\alpha\rangle\psi, \Gamma \vdash \Delta}\,(\langle\sigma\rangle\mathrm{L})$$

- $(\langle\alpha\rangle\mathrm{R})$. The proof is by induction the definition of $\langle\alpha\rangle\varphi$.
  - Base Case: $\alpha = \sigma$ is $(\langle\sigma\rangle\mathrm{R})$.
  - Inductive Step: $\alpha = \sigma\alpha$

$$\mathscr{D} = \dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\sigma\rangle j, @_j\langle\alpha\rangle a, @_a\varphi, \Gamma \vdash \Delta @_i\langle\sigma\alpha\rangle\varphi, @_j\langle\alpha\rangle\varphi, @_j\langle\alpha\rangle a, @_a\varphi}\,(\mathrm{Ax})}{@_i\langle\sigma\rangle j, @_j\langle\alpha\rangle a, @_a\varphi, \Gamma \vdash \Delta @_i\langle\sigma\alpha\rangle\varphi, @_j\langle\alpha\rangle\varphi}\,(\mathrm{IH})}{@_i\langle\sigma\rangle j, @_j\langle\alpha\rangle a, @_a\varphi, \Gamma \vdash \Delta @_i\langle\sigma\alpha\rangle\varphi}\,(\langle\sigma\rangle\mathrm{R})}{@_a\varphi, @_i\langle\sigma\alpha\rangle a, \Gamma \vdash \Delta @_i\langle\sigma\alpha\rangle\varphi}\,(\langle\sigma\rangle\mathrm{L})$$

$$\dfrac{@_i\langle\sigma\alpha\rangle a, \Gamma \vdash \Delta @_i\langle\sigma\alpha\rangle\varphi, @_a\varphi \qquad \mathscr{D}}{@_i\langle\sigma\alpha\rangle a, \Gamma \vdash \Delta @_i\langle\sigma\alpha\rangle\varphi}\,(\mathrm{Cut})$$

- (Ax).
  - Base Case: If $\varphi = \bot$, the result is obtained using $(\bot)$. If $\varphi \in \{p, i\}$, the result is obtained using the basic (Ax).
  - Inductive Step:

  * $\varphi = \varphi \to \psi$

$$\dfrac{\dfrac{\dfrac{}{@_i\varphi, \Gamma \vdash \Delta, @_i\psi, @_i\varphi}\,(\text{IH}) \quad \dfrac{}{@_i\varphi, @_i\psi, \Gamma \vdash \Delta, @_i\psi}\,(\text{IH})}{@_i\varphi, @_i(\varphi \to \psi), \Gamma \vdash \Delta, @_i\psi}\,(\to\text{L})}{@_i(\varphi \to \psi), \Gamma \vdash \Delta, @_i(\varphi \to \psi)}\,(\to\text{R})$$

  * $\varphi = \langle\sigma\rangle\varphi$

$$\dfrac{\dfrac{\dfrac{}{@_i\langle\sigma\rangle j, @_j\varphi, \Gamma \vdash \Delta, @_i\langle\sigma\rangle\varphi, @_j\varphi}\,(\text{IH})}{@_i\langle\sigma\rangle j, @_j\varphi, \Gamma \vdash \Delta, @_i\langle\sigma\rangle\varphi}\,(\langle\sigma\rangle\text{R})}{@_i\langle\sigma\rangle\varphi, \Gamma \vdash \Delta, @_i\langle\sigma\rangle\varphi}\,(\langle\sigma\rangle\text{L})$$

  * $\varphi = @_j\varphi$

$$\dfrac{\dfrac{}{@_j\varphi, \Gamma \vdash \Delta, @_j\varphi}\,(\text{IH})}{@_i@_j\varphi, \Gamma \vdash \Delta, @_i@_j\varphi}\,(@\text{L}, @\text{R})$$

  * $\varphi = \langle\alpha \blacktriangle \beta\rangle$

$$\dfrac{\dfrac{\dfrac{}{@_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle, \Gamma \vdash \Delta, @_i\langle\alpha \blacktriangle \beta\rangle, \langle a{:} \blacktriangle b{:}\rangle}\,(\text{Ax})}{@_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle, \Gamma \vdash \Delta, @_i\langle\alpha \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\text{R})}{@_i\langle\alpha \blacktriangle \beta\rangle, \Gamma \vdash \Delta, @_i\langle\alpha \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\text{L})$$

- (S_1)
  - Base Case: If $\varphi = j$, the result is obtained using (@5). If $\varphi \in \{p, \bot\}$, the result is obtained using the basic (S_1).
  - Inductive Step:
    * $\varphi = \varphi \to \psi$

$$\mathscr{D} = \dfrac{\dfrac{\dfrac{\dfrac{}{@_j\psi, @_ij, @_i\psi, \Gamma \vdash \Delta, @_j\psi}\,(\text{Ax})}{@_ij, @_i\psi, \Gamma \vdash \Delta, @_j\psi}\,(\text{IH}) \quad \dfrac{\dfrac{\dfrac{}{@_i\varphi, @_j\varphi, @_ji, \Gamma \vdash \Delta, @_i\varphi}\,(\text{Ax})}{@_j\varphi, @_ji, \Gamma \vdash \Delta, @_i\varphi}\,(\text{IH})}{@_j\varphi, @_ij, \Gamma \vdash \Delta, @_i\varphi}\,(@\text{B})}{@_j\varphi, @_ij, @_i(\varphi \to \psi), \Gamma \vdash \Delta, @_j\psi}\,(\to\text{L,WL})}{@_ij, @_i(\varphi \to \psi), \Gamma \vdash \Delta, @_j(\varphi \to \psi)}\,(\to\text{R})$$

$$\dfrac{\mathscr{D} \quad @_j(\varphi \to \psi), @_ij, @_i(\varphi \to \psi), \Gamma \vdash \Delta}{@_ij, @_i(\varphi \to \psi), \Gamma \vdash \Delta}\,(\text{Cut})$$

    * $\varphi = \langle\sigma\rangle\varphi$

$$\mathscr{D} = \dfrac{\dfrac{\dfrac{\dfrac{}{@_j\langle\sigma\rangle k, @_i\langle\sigma\rangle k, @_k\varphi, @_ij, \Gamma \vdash \Delta, @_j\langle\sigma\rangle\varphi, @_k\varphi}\,(\text{Ax})}{@_j\langle\sigma\rangle k, @_i\langle\sigma\rangle k, @_k\varphi, @_ij, \Gamma \vdash \Delta, @_j\langle\sigma\rangle\varphi}\,(\langle\sigma\rangle\text{R})}{@_i\langle\sigma\rangle k, @_k\varphi, @_ij, \Gamma \vdash \Delta, @_j\langle\sigma\rangle\varphi}\,(\text{S}_1)}{@_ij, @_i\langle\sigma\rangle\varphi, \Gamma \vdash \Delta, @_j\langle\sigma\rangle\varphi}\,(\langle\sigma\rangle\text{L})$$

$$\dfrac{\mathscr{D} \quad @_j\langle\sigma\rangle\varphi, @_ij, @_i\langle\sigma\rangle\varphi, \Gamma \vdash \Delta}{@_ij, @_i\langle\sigma\rangle\varphi, \Gamma \vdash \Delta}\,(\text{Cut})$$

    * $\varphi = @_j\varphi$

$$\dfrac{\dfrac{\dfrac{}{@_ij, @_k\varphi, \Gamma \vdash \Delta, @_k\varphi}\,(\text{Ax})}{@_ij, @_i@_k\varphi, \Gamma \vdash \Delta, @_j@_k\varphi}\,(@\text{L}, @\text{R}) \quad @_j@_k\varphi, @_ij, @_i@_k\varphi, \Gamma \vdash \Delta}{@_ij, @_i@_k\varphi, \Gamma \vdash \Delta}\,(\text{Cut})$$

    * $\varphi = \langle\alpha \blacktriangle \beta\rangle$

$$\mathscr{D} = \dfrac{\dfrac{\dfrac{\dfrac{}{@_j\langle\alpha\rangle a, @_j\langle\beta\rangle b, @_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle, @_i j, \Gamma \vdash \Delta, @_j\langle\alpha \blacktriangle \beta\rangle, \langle a{:} \blacktriangle b{:}\rangle}\,(\text{Ax})}{@_j\langle\alpha\rangle a, @_j\langle\beta\rangle b, @_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle, @_i j, \Gamma \vdash \Delta, @_j\langle\alpha \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\text{R})}{@_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle, @_i j, \Gamma \vdash \Delta, @_j\langle\alpha \blacktriangle \beta\rangle}\,(\text{IH})}{@_i j, @_i\langle\alpha \blacktriangle \beta\rangle, \Gamma \vdash \Delta, @_j\langle\alpha \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\text{L})$$

$$\dfrac{\mathscr{D} \quad @_j\langle\alpha \blacktriangle \beta\rangle, @_i j, @_i\langle\alpha \blacktriangle \beta\rangle, \Gamma \vdash \Delta}{@_i j, @_i\langle\alpha \blacktriangle \beta\rangle, \Gamma \vdash \Delta}\,(\text{Cut})$$

- (S_2)
  - Base Case
    - $\alpha = \mathsf{a}$ corresponds to the (S_2).
    - $\alpha = k{:}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{@_i\langle a{:}\rangle k, @_j k, @_i\langle a{:}\rangle j, \Gamma \vdash \Delta}{@_a k, @_j k, @_a j, \Gamma \vdash \Delta}\,(@\text{L}^{-1})}{@_k a, @_j k, @_j a, \Gamma \vdash \Delta}\,(@\text{B})}{@_j k, @_j a, \Gamma \vdash \Delta}\,(@5)}{@_j k, @_a j, \Gamma \vdash \Delta}\,(@\text{B})}{@_j k, @_i\langle a{:}\rangle j, \Gamma \vdash \Delta}\,(@\text{L})$$

    - $\alpha = \varphi?$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{@_i\langle\varphi?\rangle k, @_j k, @_i\langle\varphi?\rangle j, \Gamma \vdash \Delta}{@_i k, @_j k, @_i\varphi, @_i j, \Gamma \vdash \Delta}\,(\wedge\text{L}^{-1})}{@_k i, @_j k, @_i\varphi, @_j i, \Gamma \vdash \Delta}\,(@\text{B})}{@_j k, @_i\varphi, @_j i, \Gamma \vdash \Delta}\,(@5)}{@_j k, @_i\varphi, @_i j, \Gamma \vdash \Delta}\,(@\text{B})}{@_j k, @_i\langle\varphi?\rangle j, \Gamma \vdash \Delta}\,(\wedge\text{L})$$

  - Inductive Step
    - $\alpha = k{:}\alpha$

$$\dfrac{\dfrac{\dfrac{@_i\langle a{:}\alpha\rangle k, @_j k, @_i\langle a{:}\alpha\rangle j, \Gamma \vdash \Delta}{@_a\langle\alpha\rangle k, @_j k, @_a\langle\alpha\rangle j, \Gamma \vdash \Delta}\,(@\text{L}^{-1})}{@_j k, @_a\langle\alpha\rangle j, \Gamma \vdash \Delta}\,(\text{IH})}{@_j k, @_i\langle a{:}\alpha\rangle j, \Gamma \vdash \Delta}\,(@\text{L})$$

    - $\alpha = \varphi?\alpha$

$$\dfrac{\dfrac{\dfrac{@_i\langle\varphi?\alpha\rangle k, @_j k, @_i\langle\varphi?\alpha\rangle j, \Gamma \vdash \Delta}{@_i\langle\alpha\rangle k, @_j k, @_i\varphi, @_i\langle\alpha\rangle j, \Gamma \vdash \Delta}\,(\wedge\text{L}^{-1})}{@_j k, @_i\varphi, @_i\langle\alpha\rangle j, \Gamma \vdash \Delta}\,(\text{IH})}{@_j k, @_i\langle\varphi?\alpha\rangle j, \Gamma \vdash \Delta}\,(\wedge\text{L})$$

    - $\alpha = \mathsf{a}\alpha$

$$\dfrac{\dfrac{\dfrac{@_i\langle\mathsf{a}\alpha\rangle k, @_j k, @_i\langle\mathsf{a}\alpha\rangle j, \Gamma \vdash \Delta}{@_a\langle\alpha\rangle k, @_j k, @_i\langle\mathsf{a}\rangle a, @_a\langle\alpha\rangle j, \Gamma \vdash \Delta}\,(\langle\mathsf{a}\rangle\text{L}^{-1})}{@_j k, @_i\langle\mathsf{a}\rangle a, @_a\langle\alpha\rangle j, \Gamma \vdash \Delta}\,(\text{IH})}{@_j k, @_i\langle\mathsf{a}\alpha\rangle j, \Gamma \vdash \Delta}\,(\langle\mathsf{a}\rangle\text{L})$$

- (S_3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle i{:} =_{c} k{:}\rangle, \langle j{:} =_{c} k{:}\rangle, @_{j}i, \Gamma \vdash \Delta, \langle i{:} =_{c} k{:}\rangle}\,\text{(Ax)}}{\langle j{:} =_{c} k{:}\rangle, @_{j}i, \Gamma \vdash \Delta, \langle i{:} =_{c} k{:}\rangle}\,(\text{S}_3)}{\langle j{:} =_{c} k{:}\rangle, @_{i}j, \langle i{:} \neq_{c} k{:}\rangle, \Gamma \vdash \Delta}\,\text{(NEqL,@B)}}{@_{i}j, \langle i{:} \neq_{c} k{:}\rangle, \Gamma \vdash \Delta, \langle j{:} \neq_{c} k{:}\rangle}\,\text{(NEqR)} \qquad \langle j{:} \neq_{c} k{:}\rangle, @_{i}j, \langle i{:} \neq_{c} k{:}\rangle, \Gamma \vdash \Delta}{@_{i}j, \langle i{:} \neq_{c} k{:}\rangle, \Gamma \vdash \Delta}\,\text{(Cut)}$$

6 Completeness

Lemma 15. *Let $\vdash^{n}_{\mathbf{H}} \varphi$ indicate that φ is a theorem in $\mathbf{H}$, with a derivation of length n. In addition, let i be a nominal not in φ. It follows that, for any n, $\vdash^{n}_{\mathbf{H}} \varphi$ implies $\vdash_{\mathbf{G}} @_{i}\varphi$ (i.e, $\vdash @_{i}\varphi$ is provable in $\mathbf{G}$).*

Proof. The proof proceeds by (strong) induction on the length of a derivation in $\mathbf{H}$. The base case requires us to show that each axiom φ of the Hilbert system has a corresponding provable sequent $\vdash @_{i}\varphi$. For the inductive step, we must show that $\vdash^{m}_{\mathbf{H}} \psi$ implies $\vdash_{\mathbf{G}} @_{i}\psi$. We proceed by cases, depending on whether $\vdash^{m}_{\mathbf{H}}\psi$ is obtained using (MP), (Nec), (Name), or (Paste).

Base Case.

- Axiom (@-def). We prove $\vdash_{\mathbf{H}} @_{i}\varphi \leftrightarrow \langle i{:}\varphi? =_{c} i{:}\varphi?\rangle$ implies $\vdash_{\mathbf{G}} @_{j}(@_{i}\varphi \leftrightarrow \langle i{:}\varphi? =_{c} i{:}\varphi?\rangle)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle i{:} =_{c} i{:}\rangle, @_{j}\langle i{:}\varphi?\rangle k, @_{j}\langle i{:}\varphi?\rangle k \vdash @_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle, \langle i{:} =_{c} i{:}\rangle}\,\text{(Ax)}}{@_{j}\langle i{:}\varphi?\rangle i, @_{j}\langle i{:}\varphi?\rangle i \vdash @_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle, \langle i{:} =_{c} i{:}\rangle}\,\text{(EqT)}}{@_{j}\langle i{:}\varphi?\rangle i, @_{j}\langle i{:}\varphi?\rangle i \vdash @_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle}\,(\langle\blacktriangle\rangle\text{R})}{@_{i}\langle \varphi? i\rangle \vdash @_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle}\,(@\text{L}^{-1})}{@_{i}i, @_{i}\varphi \vdash @_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle}\,(\wedge\text{L}^{-1})}{@_{i}\varphi \vdash @_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle}\,(@\text{T})}{@_{j}@_{i}\varphi \vdash @_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle}\,(@\text{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_{i}\varphi, @_{i}k, @_{i}\langle\varphi?\rangle l, \langle k{:} =_{c} l{:}\rangle, \Gamma \vdash \Delta, @_{i}\varphi}\,\text{(Ax)}}{@_{i}\langle\varphi?\rangle k, @_{i}\langle\varphi?\rangle l, \langle k{:} =_{c} l{:}\rangle, \Gamma \vdash \Delta, @_{i}\varphi}\,(\wedge\text{L})}{@_{j}\langle i{:}\varphi?\rangle k, @_{j}\langle i{:}\varphi?\rangle l, \langle k{:} =_{c} l{:}\rangle, \Gamma \vdash \Delta, @_{i}\varphi}\,(@\text{L})}{@_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle \vdash @_{i}\varphi}\,(\langle\blacktriangle\rangle\text{L})}{@_{j}\langle i{:}\varphi? =_{c} i{:}\varphi?\rangle \vdash @_{j}@_{i}\varphi}\,(@\text{R})}{\vdash @_{j}(@_{i}\varphi \leftrightarrow \langle i{:}\varphi? =_{c} i{:}\varphi?\rangle)}\,(\leftrightarrow)\text{R}$$

- Axiom ($\langle\alpha\rangle$-def). We prove $\vdash_{\mathbf{H}} \langle\alpha\rangle\varphi \leftrightarrow \langle\alpha\varphi? =_{c} \alpha\varphi?\rangle$ implies $\vdash_{\mathbf{G}} @_{i}(\langle\alpha\rangle\varphi \leftrightarrow \langle\alpha\varphi? =_{c} \alpha\varphi?\rangle)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle j{:} =_{c} j{:}\rangle, @_{i}\langle\alpha\varphi?\rangle j, @_{i}\langle\alpha\varphi?\rangle j, \vdash @_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle, \langle j{:} =_{c} j{:}\rangle}\,\text{(Ax)}}{@_{i}\langle\alpha\varphi?\rangle j, @_{i}\langle\alpha\varphi?\rangle j, \vdash @_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle, \langle j{:} =_{c} j{:}\rangle}\,\text{(EqT)}}{@_{i}\langle\alpha\varphi?\rangle j, @_{i}\langle\alpha\varphi?\rangle j, \vdash @_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle}\,(\langle\blacktriangle\rangle\text{R})}{@_{j}\langle\varphi?\rangle j, @_{i}\langle\alpha\rangle j \vdash @_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle}\,(\langle\alpha\rangle\text{L}^{-1})}{@_{j}j, @_{i}\langle\alpha\rangle j, @_{j}\varphi \vdash @_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle}\,(\wedge\text{L}^{-1})}{@_{i}\langle\alpha\rangle j, @_{j}\varphi \vdash @_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle}\,(@\text{T})}{@_{i}\langle\alpha\rangle\varphi \vdash @_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle}\,(\langle\alpha\rangle\text{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_{i}\langle\alpha\rangle\varphi, @_{k}j, @_{i}\langle\alpha\varphi?\rangle k, \langle j{:} =_{c} k{:}\rangle \vdash @_{i}\langle\alpha\rangle\varphi}\,\text{(Ax)}}{@_{k}\varphi, @_{k}j, @_{i}\langle\alpha\rangle k, @_{i}\langle\alpha\varphi?\rangle k, \langle j{:} =_{c} k{:}\rangle \vdash @_{i}\langle\alpha\rangle\varphi}\,(\langle\alpha\rangle\text{L}^{-1})}{@_{i}\langle\alpha\rangle k, @_{k}\langle\varphi?\rangle j, @_{i}\langle\alpha\varphi?\rangle k, \langle j{:} =_{c} k{:}\rangle \vdash @_{i}\langle\alpha\rangle\varphi}\,(\wedge\text{L})}{@_{i}\langle\alpha\varphi?\rangle j, @_{i}\langle\alpha\varphi?\rangle k, \langle j{:} =_{c} k{:}\rangle \vdash @_{i}\langle\alpha\rangle\varphi}\,(\langle\alpha\rangle\text{L})}{@_{i}\langle\alpha\varphi? =_{c} \alpha\varphi?\rangle \vdash @_{i}\langle\alpha\rangle\varphi}\,(\langle\blacktriangle\rangle\text{L})}{\vdash @_{i}(\langle\alpha\rangle\varphi \leftrightarrow \langle\alpha\varphi? =_{c} \alpha\varphi?\rangle)}\,(\leftrightarrow)$$

- Axiom (K). We prove $\vdash_{\mathbf{H}} [\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$ implies $\vdash_{\mathbf{G}} @_{i}([\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi))$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\varphi \vdash @_j\psi, @_j\varphi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_j\psi \vdash @_j\psi$}
\RightLabel{($\to$R)}
\BinaryInfC{$@_j(\varphi \to \psi), @_j\varphi \vdash @_j\psi$}
\RightLabel{([α]L,WL)}
\UnaryInfC{$@_j\varphi, @_i\langle\alpha\rangle j, @_i[\alpha](\varphi \to \psi) \vdash @_j\psi$}
\RightLabel{([α]L,WL)}
\UnaryInfC{$@_i\langle\alpha\rangle j, {\color{magenta}@_i[\alpha]\varphi}, @_i[\alpha](\varphi \to \psi) \vdash @_j\psi$}
\RightLabel{([α]R)}
\UnaryInfC{$@_i[\alpha]\varphi, @_i[\alpha](\varphi \to \psi) \vdash @_i[\alpha]\psi$}
\RightLabel{($\to$R)}
\UnaryInfC{$@_i[\alpha](\varphi \to \psi) \vdash @_i([\alpha]\varphi \to [\alpha]\psi)$}
\RightLabel{($\to$R)}
\UnaryInfC{$\vdash @_i([\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi))$}
\end{prooftree}
$$

- Axiom (@K). We prove $\vdash_{\mathbf{H}} @_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi)$ implies $\vdash_{\mathbf{G}} @_j(@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi))$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\varphi \vdash @_i\varphi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\psi \vdash @_i\psi$}
\RightLabel{($\to$L,WL,WR)}
\BinaryInfC{$@_i\varphi, @_i(\varphi \to \psi) \vdash @_i\psi$}
\RightLabel{(@L,@R)}
\UnaryInfC{$@_j@_i\varphi, @_i(\varphi \to \psi) \vdash @_j@_i\psi$}
\RightLabel{($\to$R)}
\UnaryInfC{$@_i(\varphi \to \psi) \vdash @_j(@_i\varphi \to @_i\psi)$}
\RightLabel{(@L)}
\UnaryInfC{$@_j@_i(\varphi \to \psi) \vdash @_j(@_i\varphi \to @_i\psi)$}
\RightLabel{($\to$R)}
\UnaryInfC{$\vdash @_j(@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi))$}
\end{prooftree}
$$

- Axiom (@-refl). We prove $\vdash_{\mathbf{H}} @_i i$ implies $\vdash_{\mathbf{G}} @_j@_i i$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i i \vdash @_i i$}
\RightLabel{(Ref)}
\UnaryInfC{$\vdash @_i i$}
\RightLabel{(@R)}
\UnaryInfC{$\vdash @_j@_i i$}
\end{prooftree}
$$

- Axiom (@-self-dual). We prove $\vdash_{\mathbf{H}} \neg @_i\varphi \leftrightarrow @_i\neg\varphi$ implies $\vdash_{\mathbf{G}} @_j(\neg @_i\varphi \leftrightarrow @_i\neg\varphi)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\varphi \vdash @_i\varphi$}
\RightLabel{($\neg$R,@R)}
\UnaryInfC{$\vdash @_j@_i\varphi, @_i\neg\varphi$}
\RightLabel{($\neg$L,@R)}
\UnaryInfC{$@_j\neg @_i\varphi \vdash @_j@_i\neg\varphi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\varphi \vdash @_i\varphi$}
\RightLabel{($\neg$R,@R)}
\UnaryInfC{$@_i\neg\varphi, @_j@_i\varphi \vdash$}
\RightLabel{($\neg$R,@L)}
\UnaryInfC{$@_j@_i\neg\varphi \vdash @_j\neg @_i\varphi$}
\RightLabel{($\leftrightarrow$R)}
\BinaryInfC{$\vdash @_j(\neg @_i\varphi \leftrightarrow @_i\neg\varphi)$}
\end{prooftree}
$$

- Axiom (intro-@). We prove $\vdash_{\mathbf{H}} i \to (\varphi \leftrightarrow @_i\varphi)$ implies $\vdash_{\mathbf{G}} @_j(i \to (\varphi \leftrightarrow @_i\varphi))$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\varphi, @_k\varphi, @_k i \vdash @_i\varphi$}
\RightLabel{(S$_1$)}
\UnaryInfC{$@_k\varphi, @_k i \vdash @_i\varphi$}
\RightLabel{(@R)}
\UnaryInfC{$@_k\varphi, @_k i \vdash @_k@_i\varphi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_k\varphi, @_i\varphi, @_i k \vdash @_k\varphi$}
\RightLabel{(S$_1$)}
\UnaryInfC{$@_i\varphi, @_i k \vdash @_k\varphi$}
\RightLabel{(@B)}
\UnaryInfC{$@_i\varphi, @_k i \vdash @_k\varphi$}
\RightLabel{(@L)}
\UnaryInfC{$@_k@_i\varphi, @_k i \vdash @_k\varphi$}
\RightLabel{($\leftrightarrow$R)}
\BinaryInfC{$@_k i \vdash @_k(\varphi \leftrightarrow @_i\varphi)$}
\RightLabel{($\to$R)}
\UnaryInfC{$\vdash @_k(i \to (\varphi \leftrightarrow @_i\varphi))$}
\end{prooftree}
$$

- Axiom (equal). We prove $\vdash_{\mathbf{H}} \langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle$ implies $\vdash_{\mathbf{G}} @_i(\langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\langle i{:} =_{\mathsf{c}} i{:}\rangle, @_i\langle\varepsilon\rangle i \vdash @_i\langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle, \langle i{:} =_{\mathsf{c}} i{:}\rangle$}
\RightLabel{(EqT)}
\UnaryInfC{$@_i\langle\varepsilon\rangle i \vdash @_i\langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle, \langle i{:} =_{\mathsf{c}} i{:}\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$R)}
\UnaryInfC{$@_i\langle\varepsilon\rangle i \vdash @_i\langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle$}
\RightLabel{($\wedge$L^{-1})}
\UnaryInfC{$@_i\top, @_i i \vdash @_i\langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle$}
\RightLabel{($\top$L)}
\UnaryInfC{$@_i i \vdash @_i\langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle$}
\RightLabel{(@T)}
\UnaryInfC{$\vdash @_i\langle \varepsilon =_{\mathsf{c}} \varepsilon \rangle$}
\end{prooftree}
$$

- Axiom ($\blacktriangle$-comm). We prove $\vdash_{\mathbf{H}} \langle \alpha \blacktriangle \beta \rangle \leftrightarrow \beta \blacktriangle \alpha$ implies $\vdash_{\mathbf{G}} @_i(\langle \alpha \blacktriangle \beta \rangle \leftrightarrow \beta \blacktriangle \alpha)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle k{:} \blacktriangle j{:}\rangle \vdash @_i\langle\beta \blacktriangle \alpha\rangle, \langle k{:} \blacktriangle j{:}\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$B)}
\UnaryInfC{$@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\beta \blacktriangle \alpha\rangle, \langle k{:} \blacktriangle j{:}\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$R)}
\UnaryInfC{$@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\beta \blacktriangle \alpha\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$L)}
\UnaryInfC{$@_i\langle\alpha \blacktriangle \beta\rangle \vdash @_i\langle\beta \blacktriangle \alpha\rangle$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_i\langle\beta\rangle k, @_i\langle\alpha\rangle j, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha \blacktriangle \beta\rangle, \langle j{:} \blacktriangle k{:}\rangle$}
\RightLabel{($@\blacktriangle$B)}
\UnaryInfC{$@_i\langle\beta\rangle k, @_i\langle\alpha\rangle j, \langle k{:} \blacktriangle j{:}\rangle \vdash @_i\langle\alpha \blacktriangle \beta\rangle, \langle j{:} \blacktriangle k{:}\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$R)}
\UnaryInfC{$@_i\langle\beta\rangle k, @_i\langle\alpha\rangle j, \langle k{:} \blacktriangle j{:}\rangle \vdash @_i\langle\alpha \blacktriangle \beta\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$L)}
\UnaryInfC{$@_i\langle\beta \blacktriangle \alpha\rangle \vdash @_i\langle\alpha \blacktriangle \beta\rangle$}
\RightLabel{($\leftrightarrow$R)}
\BinaryInfC{$@_i(\langle\alpha \blacktriangle \beta\rangle \leftrightarrow \langle\beta \blacktriangle \alpha\rangle)$}
\end{prooftree}
$$

- Axiom (ε-trans). $\vdash_{\mathbf{H}} \langle \alpha =_{\mathsf{c}} \varepsilon \rangle \wedge \langle \varepsilon =_{\mathsf{c}} \beta \rangle \to \langle \alpha =_{\mathsf{c}} \beta \rangle$ implies $\vdash_{\mathbf{G}} @_i(\langle \alpha =_{\mathsf{c}} \varepsilon \rangle \wedge \langle \varepsilon =_{\mathsf{c}} \beta \rangle \to \langle \alpha =_{\mathsf{c}} \beta \rangle)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\langle a{:} =_{\mathsf{c}} b{:}\rangle, \langle c{:} =_{\mathsf{c}} a{:}\rangle, \langle c{:} =_{\mathsf{c}} b{:}\rangle \vdash \langle a{:} =_{\mathsf{c}} b{:}\rangle$}
\RightLabel{(Eq5)}
\UnaryInfC{$\langle c{:} =_{\mathsf{c}} a{:}\rangle, \langle c{:} =_{\mathsf{c}} b{:}\rangle \vdash \langle a{:} =_{\mathsf{c}} b{:}\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$B)}
\UnaryInfC{$\langle a{:} =_{\mathsf{c}} c{:}\rangle, \langle c{:} =_{\mathsf{c}} b{:}\rangle \vdash \langle a{:} =_{\mathsf{c}} b{:}\rangle$}
\RightLabel{(S$_3$,WL)}
\UnaryInfC{$@_d c, \langle a{:} =_{\mathsf{c}} c{:}\rangle, \langle d{:} =_{\mathsf{c}} b{:}\rangle \vdash \langle a{:} =_{\mathsf{c}} b{:}\rangle$}
\RightLabel{(@5,WL)}
\UnaryInfC{$@_i c, @_i d, @_i\top, \langle a{:} =_{\mathsf{c}} c{:}\rangle, @_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle d{:} =_{\mathsf{c}} b{:}\rangle \vdash @_i\langle\alpha =_{\mathsf{c}} \beta\rangle, \langle a{:} =_{\mathsf{c}} b{:}\rangle$}
\RightLabel{($\wedge$L,$\langle\blacktriangle\rangle$R)}
\UnaryInfC{$@_i\langle\alpha\rangle a, @_i\langle\varepsilon\rangle c, \langle a{:} =_{\mathsf{c}} c{:}\rangle, @_i\langle\varepsilon\rangle d, @_i\langle\beta\rangle b, \langle d{:} =_{\mathsf{c}} b{:}\rangle \vdash @_i\langle\alpha =_{\mathsf{c}} \beta\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$L)}
\UnaryInfC{$@_i\langle\alpha =_{\mathsf{c}} \varepsilon\rangle, @_i\langle\varepsilon =_{\mathsf{c}} \beta\rangle \vdash @_i\langle\alpha =_{\mathsf{c}} \beta\rangle$}
\RightLabel{($\to$R,$\wedge$L)}
\UnaryInfC{$\vdash @_i(\langle\alpha =_{\mathsf{c}} \varepsilon\rangle \wedge \langle\varepsilon =_{\mathsf{c}} \beta\rangle \to \langle\alpha =_{\mathsf{c}} \beta\rangle)$}
\end{prooftree}
$$

- Axiom (distinct). We prove $\vdash_{\mathbf{H}} \neg\langle \varepsilon \neq_{\mathsf{c}} \varepsilon \rangle$ implies $\vdash_{\mathbf{G}} @_i(\neg\langle \varepsilon \neq_{\mathsf{c}} \varepsilon \rangle)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\langle i{:} =_{\mathsf{c}} i{:}\rangle \vdash \langle i{:} =_{\mathsf{c}} i{:}\rangle$}
\RightLabel{(EqT)}
\UnaryInfC{$\vdash \langle i{:} =_{\mathsf{c}} i{:}\rangle$}
\RightLabel{(NEqL)}
\UnaryInfC{$\langle i{:} \neq_{\mathsf{c}} i{:}\rangle \vdash$}
\RightLabel{(S$_3$,WL)}
\UnaryInfC{$\langle k{:} \neq_{\mathsf{c}} i{:}\rangle, @_k i \vdash$}
\RightLabel{($\langle\blacktriangle\rangle$B,WL)}
\UnaryInfC{$\langle i{:} \neq_{\mathsf{c}} k{:}\rangle, @_k i \vdash$}
\RightLabel{(S$_3$, WL)}
\UnaryInfC{$@_j i, @_k i, \langle j{:} \neq_{\mathsf{c}} k{:}\rangle \vdash$}
\RightLabel{(@B,WL)}
\UnaryInfC{$@_i\top, @_i j, @_i k, \langle j{:} \neq_{\mathsf{c}} k{:}\rangle \vdash$}
\RightLabel{($\wedge$L)}
\UnaryInfC{$@_i\langle\varepsilon\rangle j, @_i\langle\varepsilon\rangle k, \langle j{:} \neq_{\mathsf{c}} k{:}\rangle \vdash$}
\RightLabel{($\langle\blacktriangle\rangle$L)}
\UnaryInfC{$@_i\langle\varepsilon \neq_{\mathsf{c}} \varepsilon\rangle \vdash$}
\RightLabel{($\neg$R)}
\UnaryInfC{$\vdash @_i\neg\langle\varepsilon \neq_{\mathsf{c}} \varepsilon\rangle$}
\end{prooftree}
$$

- Axiom (@-data). We prove $\vdash_{\mathbf{H}} \neg\langle i{:} = j{:}\rangle \leftrightarrow \langle i{:} \neq j{:}\rangle$ implies $\vdash_{\mathbf{G}} @_k(\neg\langle i{:} = j{:}\rangle \leftrightarrow \langle i{:} \neq j{:}\rangle)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\langle i{:} = j{:}\rangle \vdash \langle i{:} = j{:}\rangle$}
\RightLabel{(NEqR)}
\UnaryInfC{$\vdash \langle i{:} \neq j{:}\rangle, \langle i{:} = j{:}\rangle$}
\RightLabel{($@\blacktriangle$R)}
\UnaryInfC{$\vdash @_k\langle i{:} \neq j{:}\rangle, @_k\langle i{:} = j{:}\rangle$}
\RightLabel{($\neg$L)}
\UnaryInfC{$@_k\neg\langle i{:} = j{:}\rangle \vdash @_k\langle i{:} \neq j{:}\rangle$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\langle i{:} = j{:}\rangle \vdash \langle i{:} = j{:}\rangle$}
\RightLabel{(NEqL)}
\UnaryInfC{$\langle i{:} = j{:}\rangle, \langle i{:} \neq j{:}\rangle \vdash$}
\RightLabel{($@\blacktriangle$L)}
\UnaryInfC{$@_k\langle i{:} = j{:}\rangle, @_k\langle i{:} \neq j{:}\rangle \vdash$}
\RightLabel{($\neg$R)}
\UnaryInfC{$@_k\langle i{:} \neq j{:}\rangle \vdash @_k\neg\langle i{:} = j{:}\rangle$}
\RightLabel{($\leftrightarrow$L)}
\BinaryInfC{$\vdash @_k(\neg\langle i{:} = j{:}\rangle \leftrightarrow \langle i{:} \neq j{:}\rangle)$}
\end{prooftree}
$$

- Axiom (Subpath). We prove $\vdash_{\mathbf{H}} \langle \alpha \blacktriangle \beta \rangle \to \langle \alpha \rangle\top$ implies $\vdash_{\mathbf{G}} @_i(\langle \alpha \blacktriangle \beta \rangle \to \langle \alpha \rangle\top)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_j\top, @_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\langle\alpha\rangle\top, @_j\top}\,(\mathrm{Ax})}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\langle\alpha\rangle\top, @_j\top}\,(\top\mathrm{L})}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\langle\alpha\rangle\top}\,(\langle\alpha\rangle\mathrm{R})}{@_i\langle\alpha\blacktriangle\beta\rangle \vdash @_i\langle\alpha\rangle\top}\,(\langle\blacktriangle\rangle\mathrm{L})}{\vdash @_i(\langle\alpha\blacktriangle\beta\rangle \to \langle\alpha\rangle\top)}\,(\to\mathrm{R})$$

- Axiom (@ $\blacktriangle$-dist). We prove $\vdash_{\mathbf{H}} \langle i{:}\alpha \blacktriangle i{:}\beta\rangle \leftrightarrow @_i\langle\alpha\blacktriangle\beta\rangle$ implies $\vdash_{\mathbf{G}} @_j(\langle i{:}\alpha \blacktriangle i{:}\beta\rangle \leftrightarrow @_i\langle\alpha\blacktriangle\beta\rangle)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_i\langle\alpha\blacktriangle\beta\rangle, \langle a{:}\,\blacktriangle\, b{:}\rangle}\,(\mathrm{Ax})}{@_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_i\langle\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_j\langle i{:}\alpha\rangle a, @_j\langle i{:}\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_i\langle\alpha\blacktriangle\beta\rangle}\,(@\mathrm{L})}{@_j\langle i{:}\alpha\blacktriangle i{:}\beta\rangle \vdash @_i\langle\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})}{@_j\langle i{:}\alpha\blacktriangle i{:}\beta\rangle \vdash @_j@_i\langle\alpha\blacktriangle\beta\rangle}\,(@\mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_j\langle i{:}\alpha\rangle a, @_j\langle i{:}\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_j\langle i{:}\alpha\blacktriangle i{:}\beta\rangle, \langle a{:}\,\blacktriangle\, b{:}\rangle}\,(\mathrm{Ax})}{@_j\langle i{:}\alpha\rangle a, @_j\langle i{:}\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_j\langle i{:}\alpha\blacktriangle i{:}\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_j\langle i{:}\alpha\blacktriangle i{:}\beta\rangle}\,(@\mathrm{L}^{-1})}{@_i\langle\alpha\blacktriangle\beta\rangle \vdash @_j\langle i{:}\alpha\blacktriangle i{:}\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})}{@_j@_i\langle\alpha\blacktriangle\beta\rangle \vdash @_j\langle i{:}\alpha\blacktriangle i{:}\beta\rangle}\,(@\mathrm{L})}{\vdash @_j(\langle i{:}\alpha\blacktriangle i{:}\beta\rangle \leftrightarrow @_i\langle\alpha\blacktriangle\beta\rangle)}\,(\leftrightarrow\mathrm{R})$$

- Axiom ($\blacktriangle$-test). We prove $\vdash_{\mathbf{H}} \langle\varphi?\alpha\blacktriangle\beta\rangle \leftrightarrow (\varphi\wedge\langle\alpha\blacktriangle\beta\rangle)$ implies $\vdash_{\mathbf{G}} @_i(\langle\varphi?\alpha\blacktriangle\beta\rangle \leftrightarrow (\varphi\wedge\langle\alpha\blacktriangle\beta\rangle))$.

$$\mathscr{D} = \dfrac{\dfrac{\dfrac{\dfrac{}{@_i\varphi, @_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\varphi}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{@_i\varphi, @_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\langle\alpha\blacktriangle\beta\rangle, \langle j{:}\,\blacktriangle\, k{:}\rangle}\,(\mathrm{Ax})}{@_i\varphi, @_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\langle\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\varphi, @_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i(\varphi\wedge\langle\alpha\blacktriangle\beta\rangle)}\,(\wedge\mathrm{R})}{@_i\langle\varphi?\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i(\varphi\wedge\langle\alpha\blacktriangle\beta\rangle)}\,(\wedge\mathrm{L})}{@_i\langle\varphi?\alpha\blacktriangle\beta\rangle \vdash @_i(\varphi\wedge\langle\alpha\blacktriangle\beta\rangle)}\,(\langle\blacktriangle\rangle\mathrm{L})$$

$$\dfrac{\mathscr{D} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\varphi?\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\langle\varphi?\alpha\blacktriangle\beta\rangle, \langle j{:}\,\blacktriangle\, k{:}\rangle}\,(\mathrm{Ax})}{@_i\langle\varphi?\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle \vdash @_i\langle\varphi?\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:}\,\blacktriangle\, k{:}\rangle, @_i\varphi \vdash @_i\langle\varphi?\alpha\blacktriangle\beta\rangle}\,(\wedge\mathrm{L}^{-1})}{@_i\varphi, @_i\langle\alpha\blacktriangle\beta\rangle \vdash @_i\langle\varphi?\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})}{@_i(\varphi\wedge\langle\alpha\blacktriangle\beta\rangle) \vdash @_i\langle\varphi?\alpha\blacktriangle\beta\rangle}\,(\wedge\mathrm{L})}{\vdash @_i(\langle\varphi?\alpha\blacktriangle\beta\rangle \leftrightarrow (\varphi\wedge\langle\alpha\blacktriangle\beta\rangle))}\,(\leftrightarrow\mathrm{R})$$

- Axiom (Agree). We prove $\vdash_{\mathbf{H}} \langle i{:}\alpha\blacktriangle\beta\rangle \to \langle j{:}i{:}\alpha\blacktriangle\beta\rangle$ implies $\vdash_{\mathbf{G}} @_k(\langle i{:}\alpha\blacktriangle\beta\rangle \to \langle j{:}i{:}\alpha\blacktriangle\beta\rangle)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_k\langle j{:}i{:}\alpha\blacktriangle\beta\rangle \vdash @_k\langle j{:}i{:}\alpha\blacktriangle\beta\rangle}\,(\mathrm{Ax})}{@_k\langle j{:}i{:}\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_k\langle j{:}i{:}\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L}^{-1})}{@_j\langle i{:}\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_k\langle j{:}i{:}\alpha\blacktriangle\beta\rangle}\,(@\mathrm{L}^{-1})}{@_i\langle\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_k\langle j{:}i{:}\alpha\blacktriangle\beta\rangle}\,(@\mathrm{L}^{-1})}{@_k\langle i{:}\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:}\,\blacktriangle\, b{:}\rangle \vdash @_k\langle j{:}i{:}\alpha\blacktriangle\beta\rangle}\,(@\mathrm{L})}{@_k\langle i{:}\alpha\blacktriangle\beta\rangle \vdash @_k\langle j{:}i{:}\alpha\blacktriangle\beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})}{\vdash @_k(\langle i{:}\alpha\blacktriangle\beta\rangle \to \langle j{:}i{:}\alpha\blacktriangle\beta\rangle)}\,(\to\mathrm{R})$$

- Axiom (Back). We prove $\vdash_{\mathbf{H}} \langle\gamma i{:}\alpha\blacktriangle\beta\rangle \to \langle i{:}\alpha\blacktriangle\beta\rangle$ implies $\vdash_{\mathbf{G}} @_j(\langle\gamma i{:}\alpha\blacktriangle\beta\rangle \to \langle i{:}\alpha\blacktriangle\beta\rangle)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_k\langle i{:}\alpha \blacktriangle \beta\rangle \vdash @_k\langle i{:}\alpha \blacktriangle \beta\rangle}\,(\mathrm{Ax})}{@_k\langle i{:}\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle \vdash @_k\langle i{:}\alpha \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L}^{-1})}{@_i\langle\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle \vdash @_k\langle i{:}\alpha \blacktriangle \beta\rangle}\,(@\mathrm{L}^{-1})}{@_k\langle\gamma\rangle c, @_c\langle i{:}\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle \vdash @_k\langle i{:}\alpha \blacktriangle \beta\rangle}\,(@\mathrm{L,WL})}{@_k\langle\gamma i{:}\alpha\rangle a, @_k\langle\beta\rangle b, \langle a{:} \blacktriangle b{:}\rangle \vdash @_k\langle i{:}\alpha \blacktriangle \beta\rangle}\,(\langle\alpha\rangle\mathrm{L})}{@_k\langle\gamma i{:}\alpha \blacktriangle \beta\rangle \vdash @_k\langle i{:}\alpha \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})}{\vdash @_k(\langle\gamma i{:}\alpha \blacktriangle \beta\rangle \to \langle i{:}\alpha \blacktriangle \beta\rangle)}\,(\to\mathrm{R})
$$

- Axiom ($\blacktriangle$-comp-dist). We prove $\vdash_{\mathbf{H}} \langle\alpha\rangle\langle\beta \blacktriangle \gamma\rangle \to \langle\alpha\beta \blacktriangle \alpha\gamma\rangle$ implies $\vdash_{\mathbf{G}} @_i(\langle\alpha\rangle\langle\beta \blacktriangle \gamma\rangle \to \langle\alpha\beta \blacktriangle \alpha\gamma\rangle)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\alpha\beta \blacktriangle \alpha\gamma\rangle \vdash @_i\langle\alpha\beta \blacktriangle \alpha\gamma\rangle}\,(\mathrm{Ax})}{@_i\langle\alpha\beta\rangle b, @_i\langle\alpha\gamma\rangle c, \langle b{:} \blacktriangle c{:}\rangle \vdash @_i\langle\alpha\beta \blacktriangle \alpha\gamma\rangle}\,(\langle\blacktriangle\rangle\mathrm{L}^{-1})}{@_a\langle\beta\rangle b, @_a\langle\gamma\rangle c, \langle b{:} \blacktriangle c{:}\rangle, @_i\langle\alpha\rangle a \vdash @_i\langle\alpha\beta \blacktriangle \alpha\gamma\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1})}{@_i\langle\alpha\rangle a, @_a\langle\beta \blacktriangle \gamma\rangle \vdash @_i\langle\alpha\beta \blacktriangle \alpha\gamma\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})}{@_i\langle\alpha\rangle\langle\beta \blacktriangle \gamma\rangle \vdash @_i\langle\alpha\beta \blacktriangle \alpha\gamma\rangle}\,(\langle\alpha\rangle\mathrm{L})}{\vdash @_i(\langle\alpha\rangle\langle\beta \blacktriangle \gamma\rangle \to \langle\alpha\beta \blacktriangle \alpha\gamma\rangle)}\,(\to\mathrm{R})
$$

- Axiom (comp-assoc). $\vdash_{\mathbf{H}} \langle(\alpha\beta)\gamma \blacktriangle \eta\rangle \leftrightarrow \langle\alpha(\beta\gamma) \blacktriangle \eta\rangle$ implies $\vdash_{\mathbf{G}} @_i(\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle \leftrightarrow \langle\alpha(\beta\gamma) \blacktriangle \eta\rangle)$.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\alpha(\beta\gamma)\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle, \langle j{:} \blacktriangle k{:}\rangle}\,(\mathrm{Ax})}{@_i\langle\alpha(\beta\gamma)\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle\alpha\rangle a, @_a\langle\beta\gamma\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1})}{@_i\langle\alpha\rangle a, @_a\langle\beta\rangle b, @_b\langle\gamma\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1})}{@_i\langle\alpha\beta\rangle b, @_b\langle\gamma\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L})}{@_i\langle(\alpha\beta)\gamma\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L})}{@_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle \vdash @_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle(\alpha\beta)\gamma\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle, \langle j{:} \blacktriangle k{:}\rangle}\,(\mathrm{Ax})}{@_i\langle(\alpha\beta)\gamma\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle\alpha\beta\rangle b, @_b\gamma j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1})}{@_i\langle\alpha\rangle a, @_a\beta b, @_b\gamma j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1})}{@_i\langle\alpha\rangle a, @_a(\beta\gamma) j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L})}{@_i\langle\alpha(\beta\gamma)\rangle j, @_i\langle\eta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \vdash @_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle}\,(\langle\alpha\rangle\mathrm{L})}{@_i\langle\alpha(\beta\gamma) \blacktriangle \eta\rangle \vdash @_i\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})
}{\vdash @_i(\langle(\alpha\beta)\gamma \blacktriangle \eta\rangle \leftrightarrow \langle\alpha(\beta\gamma) \blacktriangle \eta\rangle)}\,(\leftrightarrow\mathrm{R})
$$

- Axiom (comp-neutral). We prove $\vdash_{\mathbf{H}} \langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle \leftrightarrow \langle\alpha\beta \blacktriangle \gamma\rangle$ implies $\vdash_{\mathbf{G}} @_i(\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle \leftrightarrow \langle\alpha\beta \blacktriangle \gamma\rangle)$.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\alpha\beta\rangle j, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\beta \blacktriangle \gamma\rangle, \langle j{:} \blacktriangle k{:}\rangle}\,(\mathrm{Ax})}{@_i\langle\alpha\beta\rangle j, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\beta \blacktriangle \gamma\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_a\top, @_a\langle\beta\rangle j, @_i\langle\alpha\rangle a, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\beta \blacktriangle \gamma\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1},\mathrm{WL})}{@_i\langle\alpha\rangle a, @_a\langle\varepsilon\beta\rangle j, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\beta \blacktriangle \gamma\rangle}\,(\wedge\mathrm{L})}{@_i\langle\alpha\varepsilon\beta\rangle j, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\beta \blacktriangle \gamma\rangle}\,(\langle\alpha\rangle\mathrm{L})}{@_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle \vdash @_i\langle\alpha\beta \blacktriangle \gamma\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\langle\alpha\varepsilon\beta\rangle b, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle, \langle j{:} \blacktriangle k{:}\rangle}\,(\mathrm{Ax})}{@_i\langle\alpha\varepsilon\beta\rangle b, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle\alpha\varepsilon\rangle a, @_a\langle\beta\rangle b, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1})}{@_a\langle\varepsilon\rangle a, @_i\langle\alpha\rangle a, @_a\langle\beta\rangle b, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle}\,(\langle\alpha\rangle\mathrm{L}^{-1})}{@_a\top, @_a a, @_i\langle\alpha\rangle a, @_a\langle\beta\rangle b, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle}\,(\wedge\mathrm{L}^{-1})}{@_i\langle\alpha\rangle a, @_a\langle\beta\rangle b, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle}\,(\top\mathrm{L},\ @\mathrm{T})}{@_i\langle\alpha\beta\rangle j, @_i\langle\gamma\rangle k, \langle j{:} \blacktriangle k{:}\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle}\,(\langle\alpha\rangle\mathrm{L})}{@_i\langle\alpha\beta \blacktriangle \gamma\rangle \vdash @_i\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle}\,(\langle\blacktriangle\rangle\mathrm{L})
}{\vdash @_i(\langle\alpha\varepsilon\beta \blacktriangle \gamma\rangle \leftrightarrow \langle\alpha\beta \blacktriangle \gamma\rangle)}\,(\leftrightarrow\mathrm{R})
$$

- Axiom (comp-dist). We prove $\vdash_{\mathbf{H}} \langle\alpha\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$ implies $\vdash_{\mathbf{G}} @_i(\langle\alpha\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi)$.

$$
\dfrac{\dfrac{}{@_i\langle\alpha\beta\rangle\varphi \vdash @_i\langle\alpha\rangle\langle\beta\rangle\varphi}\,(\mathrm{Ax}) \quad \dfrac{}{@_i\langle\alpha\rangle\langle\beta\rangle\varphi \vdash @_i\langle\alpha\beta\rangle\varphi}\,(\mathrm{Ax})}{\vdash @_i(\langle\alpha\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi)}\,(\leftrightarrow\mathrm{R})
$$

Inductive Case. The inductive hypothesis (IH) is: for all $1 \leq n < m$, if $\vdash^n_{\mathbf{H}} \varphi$, then $\vdash_{\mathbf{G}} @_i\varphi$, for some nominal i not occurring in φ.

- (MP). Suppose that $\psi = \chi \to \varphi$ and there are $n, n' < m$ s.t. $\vdash^n_{\mathbf{H}} \chi$ and $\vdash^{n'}_{\mathbf{H}} \chi \to \varphi$. From the IH, $\vdash_{\mathbf{G}} @_i\chi$ and $\vdash_{\mathbf{G}} @_i(\chi \to \varphi)$. The derived rule of (MP) gives us $\vdash_{\mathbf{G}} @_i\psi$ as required.

$$\dfrac{\vdash_{\mathbf{G}} @_i\chi \quad \vdash_{\mathbf{G}} @_i(\chi\to\varphi)}{\vdash @_i\varphi}\,\text{(MP)}$$

- (Nec). Suppose that $\psi = [\alpha]\chi$ and there is $n < m$ s.t. $\vdash^n_{\mathbf{H}} \chi$. From the IH, $\vdash_{\mathbf{G}} @_j\chi$. Using (WL) we obtain $@_i\langle\alpha\rangle j \vdash_{\mathbf{G}} @_j\chi$. Finally, ($[\alpha]$R) establishes $\vdash_{\mathbf{G}} @_i\psi$ as required.

$$\dfrac{\dfrac{\vdash_{\mathbf{G}} @_j\chi}{@_i\langle\alpha\rangle j \vdash @_j\chi}\,\text{(WL)}}{\vdash @_i[\alpha]\chi}\,([\alpha]\text{R})$$

- (Name). Suppose that $\psi = \varphi$ and there is $n < m$ s.t. $\vdash^n_{\mathbf{H}} @_i\psi$. From the IH, $\vdash_{\mathbf{G}} @_j @_i\psi$. From (@R^{-1}), $\vdash_{\mathbf{G}} @_i\psi$.

$$\dfrac{\vdash_{\mathbf{G}} @_j @_i\psi}{\vdash @_i\psi}\,(@\text{R}^{-1})$$

- (Paste). Suppose that $\psi = \langle j{:}\mathsf{a}\alpha \blacktriangle \beta\rangle \to \chi$, and there is $n < m$ such that $\vdash^n_{\mathbf{H}} (@_j\langle\mathsf{a}\rangle k \wedge \langle k{:}\alpha \blacktriangle \beta\rangle) \to \chi$. By the IH, it follows that $\vdash_{\mathbf{G}} @_i((@_j\langle\mathsf{a}\rangle k \wedge \langle k{:}\alpha\blacktriangle\beta\rangle) \to \chi)$. The following derivation establishes $\vdash_{\mathbf{G}} @_i\psi$, as required.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$@_j\langle\mathsf{a}\rangle k \vdash @_j\langle\mathsf{a}\rangle k$}
\RightLabel{(@R)}
\UnaryInfC{$@_j\langle\mathsf{a}\rangle k \vdash @_i @_j\langle\mathsf{a}\rangle k$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\langle a{:}\blacktriangle b{:}\rangle \vdash \langle a{:}\blacktriangle b{:}\rangle$}
\RightLabel{($\langle\blacktriangle\rangle$R,W$*$)}
\UnaryInfC{$@_i\langle k{:}\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\blacktriangle b{:}\rangle \vdash @_i\langle k{:}\alpha\blacktriangle\beta\rangle$}
\RightLabel{(@L^{-1})}
\UnaryInfC{$@_k\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\blacktriangle b{:}\rangle \vdash @_i\langle k{:}\alpha\blacktriangle\beta\rangle$}
\RightLabel{($\wedge$R,W$*$)}
\BinaryInfC{$@_j\langle\mathsf{a}\rangle k, @_k\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\blacktriangle b{:}\rangle \vdash @_i(@_j\langle\mathsf{a}\rangle k \wedge \langle k{:}\alpha\blacktriangle\beta\rangle)$}
\AxiomC{$\vdash_{\mathbf{G}} @_i(@_j\langle\mathsf{a}\rangle k \wedge \langle k{:}\alpha\blacktriangle\beta\rangle \to \chi)$}
\RightLabel{($\to$R^{-1})}
\UnaryInfC{$@_i(@_j\langle\mathsf{a}\rangle k \wedge \langle k{:}\alpha\blacktriangle\beta\rangle) \vdash @_i\chi$}
\RightLabel{(Cut)}
\BinaryInfC{$@_j\langle\mathsf{a}\rangle k, @_k\langle\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\blacktriangle b{:}\rangle \vdash @_i\chi$}
\RightLabel{($\langle\mathsf{a}\rangle$L)}
\UnaryInfC{$@_j\langle\mathsf{a}\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\blacktriangle b{:}\rangle \vdash @_i\chi$}
\RightLabel{(@L)}
\UnaryInfC{$@_i\langle j{:}\mathsf{a}\alpha\rangle a, @_i\langle\beta\rangle b, \langle a{:}\blacktriangle b{:}\rangle \vdash @_i\chi$}
\RightLabel{($\langle\blacktriangle\rangle$L)}
\UnaryInfC{$@_i\langle j{:}\mathsf{a}\alpha\blacktriangle\beta\rangle \vdash @_i\chi$}
\RightLabel{($\to$R)}
\UnaryInfC{$\vdash_{\mathbf{G}} @_i(\langle j{:}\mathsf{a}\alpha\blacktriangle\beta\rangle \to \chi)$}
\end{prooftree}
$$

In the derivation above, (W$*$) indicates the simultaneous application of the rules (WL) and (WR) and (AxG) is a generation of the rule (Ax).

$\square$

Theorem 16 (Completeness)**.** *Every valid sequent is provable.*

Proof. Suppose $\gamma_1, \ldots, \gamma_n \vdash \delta_1, \ldots, \delta_m$ is valid. From the completeness result in [1], we know there is k such that $\vdash^k_{\mathbf{H}} \bigwedge_{1\leq i\leq n}\gamma_i \to \bigvee_{1\leq j\leq m}\delta_j$. From Lemma 15, we get $\vdash_{\mathbf{G}} @_i(\bigwedge_{1\leq i\leq n}\gamma_i \to \bigvee_{1\leq j\leq m}\delta_j)$. This implies $@_i\gamma_1, \ldots, @_i\gamma_n \vdash_{\mathbf{G}} @_i\delta_1, \ldots, @_i\delta_m$, and so $\gamma_1, \ldots, \gamma_n \vdash_{\mathbf{G}} \delta_1, \ldots, \delta_m$ as required. $\square$

7 Cut Elimination

Definition 17. *The* size *of a node expression is defined by mutual recursion as:*

$$\begin{array}{rlr} \mathrm{size}(p)=1 & \mathrm{size}(\varphi\to\psi)=1+\mathrm{size}(\varphi)+\mathrm{size}(\psi) & \mathrm{size}(\mathsf{a})=1 \\ \mathrm{size}(i)=1 & \mathrm{size}(@_i\varphi)=1+\mathrm{size}(\varphi) & \mathrm{size}(i{:})=1 \\ \mathrm{size}(\bot)=1 & \mathrm{size}(\langle\mathsf{a}\rangle\varphi)=1+\mathrm{size}(\varphi) & \mathrm{size}(\varphi?)=1+\mathrm{size}(\varphi) \\ & \mathrm{size}(\langle\alpha\blacktriangle\beta\rangle)=1+\mathrm{size}(\alpha)+\mathrm{size}(\beta) & \mathrm{size}(\alpha\beta)=\mathrm{size}(\alpha)+\mathrm{size}(\beta) \end{array}$$

The function size *induces a well-founded partial order over the set of node expressions.*

Definition 18. *The* height *of a derivation is the length of its longest branch (e.g., a derivation consisting of only an application of (Ax) has height 1). If $\Gamma\vdash\Delta$ is the end-sequent in a derivation, we will use $\Gamma\vdash^n\Delta$ to indicate that the derivation has height n. The* cut height *of an application of (Cut) in a given derivation is the sum of the heights of the derivations of the premisses of the rule; i.e., if we have derivations $\Gamma\vdash^n\Delta,\varphi$, and $\varphi,\Gamma'\vdash^m\Delta'$, using (Cut), we obtain a derivation $\Gamma,\Gamma'\vdash^{(\max(n,m)+1)}\Delta,\Delta'$, in this case, the cut height is $n+m$. In such an application of (Cut), we call φ the* active cut *expression.*

Theorem 19. *Every use of (Cut) in the derivation of a provable sequent can be eliminated.*

Proof. The proof is by induction on two measures: the size of the active cut expression, and the cut height. To this end, we associate with each application of (Cut) in a derivation a pair (k,h), called *cut complexity*, where: k corresponds to the size of the active cut expression, and h corresponds to the cut height. The induction is on the lexicographic order of the pairs (k,h).

Base Case. The base cases of the induction involve derivations in which (Cut) is applied only once, with axiom rules applied to its premisses, i.e., the premisses of (Cut) must be instances of (Ax) or $(\bot)$. Eliminating (Cut) in such configurations is unproblematic. We illustrate one representative case below. Suppose $\mathscr{C}$ and $\mathscr{A}$ are derivations with the following structure:

$$\mathscr{C}=\dfrac{\dfrac{}{\Gamma\vdash\Delta,\varphi}\,(\mathrm{Ax})\qquad\dfrac{}{\varphi,\Gamma'\vdash\Delta'}\,(\bot)}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(\mathrm{Cut})\qquad\qquad \mathscr{A}=\dfrac{}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(\mathrm{P})$$

$\mathscr{C}$ represents a case in which one of the premisses of (Cut) is (Ax) and the other is $(\bot)$. This derivation can be transformed into the cut-free derivation $\mathscr{A}$ in which: (P) is (Ax) if $\varphi\notin\Gamma$; and it is $(\bot)$ otherwise. The remaining cases use a similar argument.

Inductive Step. We identify an application of (Cut) that is minimal according to the cut height, and eliminate it. We proceed by a case analysis based on the syntactic form of the active cut, and on whether the active cut is principal in either premiss of the application of (Cut).

Non-principal Cases. First, let us cover the case where the active cut is not principal in the right premiss of (Cut). In this case, the application of (Cut) is permuted up.

Case ($\langle\blacktriangle\rangle$R). Precisely, suppose that we find a derivation

$$\dfrac{\Gamma\vdash^n\Delta,\varphi\qquad\dfrac{@_i\langle\alpha\rangle j,@_i\langle\beta\rangle k,\langle j{:}\blacktriangle k{:}\rangle,\varphi,\Gamma'\vdash^m\Delta}{\varphi,@_i\langle\alpha\blacktriangle\beta\rangle,\Gamma'\vdash\Delta}\,(\langle\blacktriangle\rangle\mathrm{L})}{@_i\langle\alpha\blacktriangle\beta\rangle,\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(\mathrm{Cut})$$

and that in this derivation the use of (Cut) has a cut complexity $(\mathrm{size}(\varphi),n+(m+1))$, where

$n+(m+1)$ is minimal. W.l.g., assume that j and k do not appear in $\Gamma \vdash \Delta$.[2] We transform this derivation into:

$$\dfrac{\dfrac{\Gamma \vdash^n \Delta, \varphi \qquad \varphi, @_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j\!: \blacktriangle\, k\!:\rangle, \Gamma' \vdash^m \Delta}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j\!: \blacktriangle\, k\!:\rangle, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut})}{@_i\langle\alpha \blacktriangle \beta\rangle, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\langle\blacktriangle\rangle\mathrm{L})$$

The (Cut) in the transformed derivation has complexity $(\mathrm{size}(\varphi), n+m)$, so it can be eliminated by the inductive hypothesis.

Remaining Cases. The remaining cases where the active cut is not principal in the right premiss follow the same strategy. To see why, note that all such derivations have the following general structure:

$$\dfrac{\Gamma \vdash^n \Delta, \varphi \qquad \dfrac{\varphi, \Phi', \Gamma' \vdash^m \Delta', \Sigma'}{\varphi, \Phi, \Gamma' \vdash \Delta', \Sigma}\,(\mathrm{P})}{\Phi, \Gamma, \Gamma' \vdash \Delta, \Delta', \Sigma}\,(\mathrm{Cut})$$

In this derivation, we are considering (P) is a single premiss rule of **G**, the set $\varphi, \Gamma', \Delta'$ is the context for the rule, and the set $\Phi', \Phi, \Sigma', \Sigma$ are the node expressions the rule acts upon. Again, we assume that (Cut) has complexity $(\mathrm{size}(\varphi), n+(m+1))$ where $n+(m+1)$ is minimal; i.e., where (Cut) is not used in $\Gamma \vdash^n \Delta, \varphi$, nor in $\varphi, \Phi', \Gamma' \vdash^m \Delta', \Sigma'$. Modulo a possible renaming of nominals, we transform this derivation into:

$$\dfrac{\dfrac{\Gamma \vdash^n \Delta, \varphi \quad \varphi, \Phi', \Gamma' \vdash^m \Delta', \Sigma'}{\Phi', \Gamma', \Gamma \vdash \Delta, \Delta', \Sigma'}\,(\mathrm{Cut})}{\Phi, \Gamma', \Gamma \vdash \Delta, \Delta', \Sigma}\,(\mathrm{P})$$

The use of (Cut) in the transformed derivation has complexity $(\mathrm{size}(\varphi), n+m)$, using the inductive hypothesis, we obtain a cut-free derivation of $\Phi, \Gamma', \Gamma \vdash \Delta, \Delta', \Sigma$.

The cases where the active cut is not principal in the left premiss is symmetric. The ($\rightarrow$L) case—the only two-premise rule in **G**—is handled similarly, and is well known in the literature.

Principal Cases. Let us now turn our attention to derivations where the active cut is principal in both premisses. We examine all combinations of rules with a possibly matching active cut.

Interaction between ($\rightarrow$L) and ($\rightarrow$R). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{@_i\varphi, \Gamma \vdash^n \Delta, @_i\psi}{\Gamma \vdash \Delta, @_i(\varphi \rightarrow \psi)}\,(\rightarrow\mathrm{R}) \qquad \dfrac{\Gamma' \vdash^{m_1} \Delta', @_i\varphi \quad @_i\psi, \Gamma' \vdash^{m_2} \Delta'}{@_i(\varphi \rightarrow \psi), \Gamma' \vdash \Delta'}\,(\rightarrow\mathrm{L})}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut})$$

We transform this derivation into:

$$\dfrac{\Gamma' \vdash^{m_1} \Delta', @_i\varphi \qquad \dfrac{@_i\varphi, \Gamma, \vdash^n \Delta, @_i\psi \quad @_i\psi, \Gamma' \vdash^{m_2} \Delta'}{@_i\varphi, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut}_1)}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut}_2)$$

The original use of (Cut) has complexity $(\mathrm{size}(@_i(\varphi \rightarrow \psi)), (n+1)+(\max(m_1, m_2)+1))$. In the transformed derivation, (Cut_1) has complexity $(\mathrm{size}(@_i\psi), n+m_2)$. In turn, the complexity of

[2] If j or k do appear in $\Gamma \vdash \Delta$, we can simply choose different nominals, and rewrite the derivation of $\varphi, @_i\langle\alpha \blacktriangle \beta\rangle, \Gamma' \vdash \Delta$ using the new selection of nominals.

(Cut_2) is such that $\mathrm{size}(@_i\varphi) < \mathrm{size}(@_i(\varphi \to \psi))$. Both applications of (Cut) in the transformed derivation can be eliminated by the inductive hypothesis.

Interaction between ($\langle \mathsf{a}\rangle$R) and ($\langle \mathsf{a}\rangle$L). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{@_i\langle \mathsf{a}\rangle j, \Gamma \vdash^n \Delta, @_i\langle \mathsf{a}\rangle\varphi, @_j\varphi}{@_i\langle \mathsf{a}\rangle j, \Gamma \vdash \Delta, @_i\langle \mathsf{a}\rangle\varphi}\,(\langle \mathsf{a}\rangle\mathrm{R}) \quad \dfrac{@_i\langle \mathsf{a}\rangle j, @_j\varphi, \Gamma' \vdash^m \Delta'}{@_i\langle \mathsf{a}\rangle\varphi, \Gamma' \vdash \Delta'}\,(\langle \mathsf{a}\rangle\mathrm{L})}{@_i\langle \mathsf{a}\rangle j, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut})$$

We transform this derivation into:

$$\dfrac{\dfrac{@_i\langle \mathsf{a}\rangle j, \Gamma, \vdash^n \Delta, @_j\varphi, @_i\langle \mathsf{a}\rangle\varphi \quad \dfrac{@_i\langle \mathsf{a}\rangle j, @_j\varphi, \Gamma' \vdash^m \Delta'}{@_i\langle \mathsf{a}\rangle\varphi, \Gamma' \vdash \Delta'}\,(\langle \mathsf{a}\rangle\mathrm{L})}{@_i\langle \mathsf{a}\rangle j, \Gamma, \Gamma' \vdash \Delta, \Delta', @_j\varphi}\,(\mathrm{Cut}_1) \quad @_j\varphi, @_i\langle \mathsf{a}\rangle j, \Gamma' \vdash^m \Delta'}{@_i\langle \mathsf{a}\rangle j, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut}_2)$$

The original use of (Cut) has complexity $(\mathrm{size}(@_i\langle \mathsf{a}\rangle\varphi), (n+1)+(m+1))$. In the transformed derivation, (Cut_1) has complexity $(\mathrm{size}(@_i\langle \mathsf{a}\rangle\varphi), n+(m+1))$. In turn, the complexity of (Cut_2) is such that $\mathrm{size}(@_j\varphi) < \mathrm{size}(@_i\langle \mathsf{a}\rangle\varphi)$. Both applications of (Cut) in the transformed derivation can be eliminated by the inductive hypothesis.

Interaction between ($\langle \mathsf{a}\rangle$R) and ($\langle\blacktriangle\rangle$R). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{@_i\langle \mathsf{a}\rangle j, \Gamma \vdash^n \Delta, @_i\langle \mathsf{a}\rangle a, @_j a}{@_i\langle \mathsf{a}\rangle j, \Gamma \vdash \Delta, @_i\langle \mathsf{a}\rangle a}\,(\langle \mathsf{a}\rangle\mathrm{R}) \quad \dfrac{@_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash^m \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle, \langle a{:} \blacktriangle b{:}\rangle}{@_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle \mathsf{a}\rangle j, @_i\langle\beta\rangle b, \Gamma, \Gamma' \vdash \Delta, \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\mathrm{Cut})$$

We transform this derivation into:

$$\mathscr{D} = @_i\langle \mathsf{a}\rangle j, \Gamma \vdash^n \Delta, @_j a, @_i\langle \mathsf{a}\rangle a$$

$$\dfrac{\dfrac{\mathscr{D} \quad \dfrac{@_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash^m \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle, \langle a{:} \blacktriangle b{:}\rangle}{@_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle \mathsf{a}\rangle j, @_i\langle\beta\rangle b, \Gamma, \Gamma' \vdash \Delta, \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle, @_j a}\,(\mathrm{Cut}_1) \quad \dfrac{\dfrac{\dfrac{\dfrac{@_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash^m \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle, \langle a{:} \blacktriangle b{:}\rangle}{@_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\langle\blacktriangle\rangle\mathrm{R})}{@_i\langle \mathsf{a}\rangle j, @_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\mathrm{WL})}{@_j a, @_i\langle \mathsf{a}\rangle j, @_i\langle \mathsf{a}\rangle a, @_i\langle\beta\rangle b, \Gamma' \vdash \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\mathrm{WL})}{@_j a, @_i\langle \mathsf{a}\rangle j, @_i\langle\beta\rangle b, \Gamma' \vdash \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\mathrm{S}_2)}{@_i\langle \mathsf{a}\rangle j, @_i\langle\beta\rangle b, \Gamma, \Gamma' \vdash \Delta, \Delta', @_i\langle \mathsf{a} \blacktriangle \beta\rangle}\,(\mathrm{Cut}_2)$$

The original use of (Cut) has complexity $(\mathrm{size}(@_i\langle \mathsf{a}\rangle a), (n+1)+(m+1))$. In the transformed derivation, (Cut_1) has complexity $(\mathrm{size}(@_i\langle \mathsf{a}\rangle a), n+(m+1))$. In turn, the complexity of (Cut_2) is such that $\mathrm{size}(@_j a) < \mathrm{size}(@_i\langle \mathsf{a}\rangle a)$. Both applications of (Cut) in the transformed derivation can be eliminated by the inductive hypothesis.

Interaction between (@L) and (@R). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{\Gamma \vdash^n \Delta, @_i\varphi}{\Gamma \vdash \Delta, @_j@_i\varphi}\,(@\mathrm{R}) \quad \dfrac{@_i\varphi, \Gamma' \vdash^m \Delta'}{@_j@_i\varphi, \Gamma' \vdash \Delta'}\,(@\mathrm{L})}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut})$$

We transform this derivation into:

$$\dfrac{\Gamma \vdash^n \Delta, @_i\varphi \quad @_i\varphi, \Gamma' \vdash^m \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{(Cut)}$$

The original use of (Cut) has complexity $(\text{size}(@_j @_i \varphi), (n+1)+(m+1))$. In the transformed derivation, the new application of (Cut) has complexity $(\text{size}(@_i \varphi), n+m)$. And can thus be eliminated by the inductive hypothesis.

Interaction between ($\langle\blacktriangle\rangle$R) and ($\langle\blacktriangle\rangle$L). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \Gamma \vdash^n \Delta, @_i\langle\alpha \blacktriangle \beta\rangle, \langle j{:} \blacktriangle k{:}\rangle}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \Gamma \vdash \Delta, @_i\langle\alpha \blacktriangle \beta\rangle}(\langle\blacktriangle\rangle\text{R}) \quad \dfrac{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \Gamma' \vdash^m \Delta'}{@_i\langle\alpha \blacktriangle \beta\rangle, \Gamma' \vdash \Delta'}(\langle\blacktriangle\rangle\text{L})}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{(Cut)}$$

We transform this derivation into:

$$\dfrac{\dfrac{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \Gamma \vdash^n \Delta, \langle j{:} \blacktriangle k{:}\rangle, @_i\langle\alpha \blacktriangle \beta\rangle \quad \dfrac{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \langle j{:} \blacktriangle k{:}\rangle, \Gamma' \vdash^m \Delta'}{@_i\langle\alpha \blacktriangle \beta\rangle, \Gamma' \vdash \Delta'}(\langle\blacktriangle\rangle\text{L})}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \Gamma, \Gamma' \vdash \Delta, \Delta', \langle j{:} \blacktriangle k{:}\rangle}(\text{Cut}_1) \quad \langle j{:} \blacktriangle k{:}\rangle, @_i\langle\alpha\rangle j, @_i\langle\beta\rangle j, \Gamma' \vdash^m \Delta'}{@_i\langle\alpha\rangle j, @_i\langle\beta\rangle k, \Gamma, \Gamma' \vdash \Delta, \Delta'}(\text{Cut}_2)$$

The original use of (Cut) has complexity $(\text{size}(@_i\langle\alpha \blacktriangle \beta\rangle), (n+1)+(m+1))$. In the transformed derivation, (Cut_1) has complexity $(\text{size}(@_i\langle\alpha \blacktriangle \beta\rangle), n+(m+1))$. In turn, the complexity of (Cut_2) is such that $\text{size}(\langle j{:} \blacktriangle k{:}\rangle) < \text{size}(@_i\langle\alpha \blacktriangle \beta\rangle)$. Both applications of (Cut) in the transformed derivation can be eliminated by the inductive hypothesis.

Interaction between (NEqL) and (NEqR). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{\langle i{:} =_{\mathsf{c}} j{:}\rangle, \Gamma \vdash^n \Delta}{\Gamma \vdash \Delta, \langle i{:} \neq_{\mathsf{c}} j{:}\rangle}(\text{NEqR}) \quad \dfrac{\Gamma' \vdash^m \Delta', \langle i{:} =_{\mathsf{c}} j{:}\rangle}{\langle i{:} \neq_{\mathsf{c}} j{:}\rangle, \Gamma' \vdash \Delta'}(\text{NEqL})}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{(Cut)}$$

We transform this derivation into:

$$\dfrac{\Gamma' \vdash^m \Delta', \langle i{:} =_{\mathsf{c}} j{:}\rangle \quad \langle i{:} =_{\mathsf{c}} j{:}\rangle, \Gamma \vdash^n \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{(Cut)}$$

The original use of (Cut) has complexity $(\text{size}(\langle i{:} \neq_{\mathsf{c}} j{:}\rangle), (n+1)+(m+1))$. The use of (Cut) in the transformed derivation has complexity $(\text{size}(\langle i{:} =_{\mathsf{c}} j{:}\rangle), n+m)$. This second use can be eliminated applying the inductive hypothesis.

Interaction between ($\langle\mathsf{a}\rangle$R) and (S_1). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{@_i\langle\mathsf{a}\rangle j, \Gamma \vdash^n \Delta, @_i\langle\mathsf{a}\rangle a, @_j a}{@_i\langle\mathsf{a}\rangle j, \Gamma \vdash \Delta, @_i\langle\mathsf{a}\rangle a}(\langle\mathsf{a}\rangle\text{R}) \quad \dfrac{@_k\langle\mathsf{a}\rangle a, @_i\langle\mathsf{a}\rangle a, @_i k, \Gamma' \vdash^m \Delta'}{@_i\langle\mathsf{a}\rangle a, @_i k, \Gamma' \vdash \Delta'}(\text{S}_1)}{@_i\langle\mathsf{a}\rangle j, @_i k, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{(Cut)}$$

We transform this derivation into:

$$\dfrac{\dfrac{@_i\langle\mathsf{a}\rangle j, \Gamma \vdash^n \Delta, @_j a, @_i\langle\mathsf{a}\rangle a \qquad \dfrac{@_k\langle\mathsf{a}\rangle a, @_i\langle\mathsf{a}\rangle a, @_i k, \Gamma' \vdash^m \Delta'}{@_i\langle\mathsf{a}\rangle a, @_i k, \Gamma' \vdash \Delta'}\,(\mathrm{S}_1)}{@_i\langle\mathsf{a}\rangle j, @_i k, \Gamma, \Gamma' \vdash \Delta, \Delta', @_j a}\,(\mathrm{Cut}_1) \qquad \dfrac{\dfrac{\dfrac{@_i\langle\mathsf{a}\rangle a, @_i k, \Gamma' \vdash^{(m+1)} \Delta'}{@_i\langle\mathsf{a}\rangle j, @_i\langle\mathsf{a}\rangle a, @_i k, \Gamma' \vdash \Delta'}\,(\mathrm{WL})}{@_j a, @_i\langle\mathsf{a}\rangle j, @_i\langle\mathsf{a}\rangle a, @_i k, \Gamma' \vdash \Delta'}\,(\mathrm{WL})}{@_j a, @_i\langle\mathsf{a}\rangle j, @_i k, \Gamma' \vdash \Delta'}\,(\mathrm{S}_2)}{@_i\langle\mathsf{a}\rangle j, @_i k, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut}_2)$$

The original use of (Cut) has complexity $(\mathrm{size}(@_i\langle\mathsf{a}\rangle a), (n+1)+(m+1))$. In the transformed derivation, (Cut_1) has complexity $(\mathrm{size}(@_i\langle\mathsf{a}\rangle a), n+(m+1))$. In turn, the complexity of (Cut_2) is such that $\mathrm{size}(@_j a) < \mathrm{size}(@_i\langle\mathsf{a}\rangle a)$. Both applications of (Cut) in the transformed derivation can be eliminated using the inductive hypothesis.

Interaction between ($\langle\mathsf{a}\rangle$R) and (S_2). Suppose that in a derivation we encounter an application of (Cut) of minimal height of the form:

$$\dfrac{\dfrac{@_i\langle\mathsf{a}\rangle j, \Gamma \vdash^n \Delta, @_i\langle\mathsf{a}\rangle a, @_j a}{@_i\langle\mathsf{a}\rangle j, \Gamma \vdash \Delta, @_i\langle\mathsf{a}\rangle a}\,(\langle\mathsf{a}\rangle\mathrm{R}) \qquad \dfrac{@_i\langle\mathsf{a}\rangle k, @_i\langle\mathsf{a}\rangle a, @_a k, \Gamma' \vdash^m \Delta'}{@_i\langle\mathsf{a}\rangle a, @_a k, \Gamma' \vdash \Delta'}\,(\mathrm{S}_2)}{@_a k, @_i\langle\mathsf{a}\rangle j, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut})$$

We transform this derivation into:

$$\dfrac{\dfrac{@_i\langle\mathsf{a}\rangle j, \Gamma \vdash^n \Delta, @_j a, @_i\langle\mathsf{a}\rangle a \qquad @_i\langle\mathsf{a}\rangle a, @_a k, \Gamma' \vdash^{(m+1)} \Delta'}{@_a k, @_i\langle\mathsf{a}\rangle j, \Gamma, \Gamma' \vdash \Delta, \Delta', @_j a}\,(\mathrm{Cut}_1) \qquad \dfrac{\dfrac{@_i\langle\mathsf{a}\rangle a, @_i\langle\mathsf{a}\rangle k, @_a k, \Gamma' \vdash^m \Delta'}{@_i\langle\mathsf{a}\rangle a, @_j a, @_a k, @_i\langle\mathsf{a}\rangle j, \Gamma' \vdash \Delta'}\,(\mathrm{S}_2,\mathrm{WL})}{@_j a, @_a k, @_i\langle\mathsf{a}\rangle j, \Gamma' \vdash \Delta'}\,(\mathrm{S}_2)}{@_a k, @_i\langle\mathsf{a}\rangle j, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{Cut}_2)$$

The original use of (Cut) has complexity $(\mathrm{size}(@_i\langle\mathsf{a}\rangle a), (n+1)+(m+1))$. In the transformed derivation, (Cut_1) has complexity $(\mathrm{size}(@_i\langle\mathsf{a}\rangle a), n+(m+1))$. In turn, the complexity of (Cut_2) is such that $\mathrm{size}(@_j a) < \mathrm{size}(@_i\langle\mathsf{a}\rangle a)$. Both uses of (Cut) can be eliminated using the inductive hypothesis.

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References

[1] C. Areces & R. Fervari (2021): *Axiomatizing Hybrid XPath with Data*. *Logical Methods in Computer Science* 17(3).