# TD 2 – Fourier Transform

Mathematics of data

25/09/24

**Exercise 1.** Compute the discrete Fourier transforms of the following signals $f \in \mathbb{C}^N$.

1. Frequency shift: $f_n = x_n e^{2i\pi n\tau/N}$ (use $\hat{x}_k$ and $\tau$ to express $\hat{f}_k$).
2. Geometric progression: $f_n = a^n$.
3. Window: $f$ is a window of size $W = 2K+1$ centered around 0:

$$f_n = \begin{cases} 1/W & \text{if } \min(n, N-n) \leq K \\ 0 & \text{otherwise.} \end{cases}$$

**Exercise 2.**

1. Let $f \in \mathbb{C}^n$. Bound $|\hat{f}_k|$ using $\|f\|_1$ (the $l^1$ norm of $f$).
2. We define the total variation of $f$ as $\|f\|_{TV} \coloneqq \sum_{i=0}^{n-1} |f_{i+1} - f_i|$. Show that $\|\cdot\|_{TV}$ satisfies the triangular inequality. Is it a norm?
3. Write $\|f\|_{TV}$ as $\|f * h\|_1$ for some $h \in \mathbb{C}^n$.
4. Compute $\hat{h}$ the discrete Fourier transform of $h$, and the norm $|\hat{h}_k|$.
5. Bound $|\hat{f}_k|$ using $\|f\|_{TV}$, $n$, and $k$.

**Exercise 3.**

1. Show that a matrix $P \in \mathbb{R}_+^{n\times n}$ such that $P^\top \mathbf{1}_n = \mathbf{1}_n$ defines a linear map $a \in \Sigma_n \mapsto Pa \in \Sigma_n$ from the simplex to itself.
2. We denote $T_\tau : a \mapsto (a_{i-\tau})_i$ the translation operator for $\tau \in \mathbb{Z}/n\mathbb{Z}$. Show that $P$ commutes with $T_\tau$ for all $\tau$ if and only if there exists $h \in \Sigma_n$ such that $Pa = h * a$.
3. Starting from some $a^{(0)} \in \Sigma_n$, we define recursively $a^{(l+1)} \coloneqq Pa^{(l)}$. Show that if $|\hat{h}_k| < 1$ for all $k \neq 1$, then $a^{(l)}$ converges as $l \to +\infty$. What is its limit?

**Exercise 4.** Denote $G \coloneqq (\mathbb{Z}/2\mathbb{Z})^p$, which is a group with $n \coloneqq 2^p$ elements. Let $\mathbb{R}[G]$ be the vector space of functions $f\colon G \to \mathbb{R}$, endowed with the canonical inner product $\langle f, g\rangle \coloneqq \sum_{x\in G} f(x)g(x)$.

1. For $\omega \in G$, denote $\psi_\omega(x) \coloneqq (-1)^{\sum_i x_i\omega_i}$. Show that $(\psi_\omega)_{\omega\in G}$ is an orthogonal basis of $\mathbb{R}[G]$.
2. Let $\hat{f}(\omega) \coloneqq \langle f, \psi_\omega\rangle$ be the Hadamard-Walsh transform of $f$. Denote $f_0, f_1 : (\mathbb{Z}/2\mathbb{Z})^{p-1} \to \mathbb{R}$ defined as

$$f_0(x_1, \ldots, x_{p-1}) \coloneqq f(x_1, \ldots, x_{p-1}, 0) \quad \text{and} \quad f_1(x_1, \ldots, x_{p-1}) \coloneqq f(x_1, \ldots, x_{p-1}, 1)$$

   Find a relation between $\hat{f}$ and $(\hat{f}_0, \hat{f}_1)$. Write in pseudo-code a fast recursive algorithm to compute $\hat{f}$ from $f$. What is the complexity of this algorithm?

**Exercise 5.** For a function $f \in \mathrm{L}^2(\mathbb{R})$ such that the following quantities exist, we define:

- Time mean $u(f) \coloneqq \frac{1}{\|f\|_2^2} \int_{-\infty}^{+\infty} t|f(t)|^2 \mathrm{d}t$
- Time Variance $\sigma_t^2(f) \coloneqq \frac{1}{\|f\|_2^2} \int_{-\infty}^{+\infty} (t-u)^2 |f(t)|^2 \mathrm{d}t$
- Fourier Mean $\xi(f) \coloneqq \frac{1}{2\pi\|f\|_2^2} \int_{-\infty}^{+\infty} \omega |\hat{f}(\omega)|^2 \mathrm{d}\omega$
- Fourier Variance $\sigma_\omega^2(f) \coloneqq \frac{1}{2\pi\|f\|_2^2} \int_{-\infty}^{+\infty} (\omega-\xi)^2 |\hat{f}(\omega)|^2 \mathrm{d}\omega$

1. Let $s \in \mathbb{R}_+^*$ and $f_s(t) \coloneqq \frac{1}{\sqrt{s}} f(\frac{t}{s})$.
   (a) Show that $\|f_s\|_2^2 = \|f\|_2^2$.
   (b) Derive the time mean and variance of $f_s$ as a function of those of $f$.
   (c) Derive the Fourier mean and variance of $f_s$ as a function of those of $f$.
   (d) How does scaling affect time and Fourier resolution?
2. In the following we assume $u(f) = 0$ and $\xi(f) = 0$. This amounts to replacing $f$ with $g\colon t \mapsto f(t-u)e^{i\xi t}$. Assuming $\lim_{|t|\to+\infty} \sqrt{t} f(t) = 0$, show that the spatial and Fourier variances satisfy the Heisenberg inequality:
$$\sigma_t^2 \sigma_\omega^2 \geq 1/4$$
3. In which case is this an equality?

*If you have finished all the exercises, you can move on to the TP or start looking at the projects for the validation of the exam (the list of projects is on the Google Docs → https://bit.ly/3VnzemW).*