TD 3 – Inverse problems and approximation

Mathematics of data

02/10/24

Exercise 1.

Part 1. Let $f_0 \in \mathbb{R}^n$ be a discrete signal. We observe the signal $y \coloneqq f_0 \star h + \varepsilon$, where $h \in \mathbb{R}^n$ is a low-pass filter and $\varepsilon \in \mathbb{R}^n$ is noise. For some $\lambda \in \mathbb{R}_+^*$, we look for

$$\operatorname{argmin}_{f\in\mathbb{R}^n} \frac{1}{2}\|f \star h - y\|^2 + \frac{\lambda}{2}\|f\|^2.$$

1. Using the Fourier decomposition, prove that for an optimal $f \in \mathbb{R}^n$,

$$\hat{f}_k = \frac{\overline{\hat{h}_k}}{|\hat{h}_k|^2 + \lambda}\hat{y}_k,$$

   where $\overline{\hat{h}_k}$ denotes the complex conjugate of $\hat{h}_k$.

2. Why does $\lambda > 0$ improve the deconvolution in presence of noise?

Part 2. For $f \in \mathbb{R}^n$, denote $Gf \coloneqq (f_i - f_{i-1})_i$ (considering indexes modulo n).

1. What is the adjoint operator $G^\top : \mathbb{R}^n \to \mathbb{R}^n$ for the canonical inner product? (i.e. $\langle Gf, u\rangle = \langle f, G^\top u\rangle$)
2. Show that G, $G^\top$ and $L \coloneqq GG^\top$ are discrete convolution operators and give their associated filters g, $\tilde{g}$ and ℓ.
3. Compute the discrete Fourier coefficients $\hat{g}$. Express $\hat{\tilde{g}}$ and $\hat{l}$ as a function of $\hat{g}$.

Part 3. We now consider the problem

$$\operatorname{argmin}_{f\in\mathbb{R}^n} \frac{1}{2}\|f \star h - y\|^2 + \frac{\lambda}{2}\|Gf\|^2.$$

1. Using the Fourier decomposition and the results from part 2, prove that for an optimal $f \in \mathbb{R}^n$,

$$\hat{f}_k = \frac{\overline{\hat{h}_k}}{|\hat{h}_k|^2 + \lambda\hat{\ell}_k}\hat{y}_k.$$

2. How does this expression differ from part 1?

Exercise 2. For an arbitrary $y \in [0,1]$, consider $f \coloneqq 1_{[y,1]} \in \mathrm{L}^2([0,1])$, the indicator function of the interval $[y,1]$. Denote $\langle f, g\rangle \coloneqq \int_0^1 fg$ the inner product and $\|f\|^2 \coloneqq \langle f, f\rangle$.

1. For $M \in \mathbb{N}^*$, denote $\theta_k \coloneqq \sqrt{M}1_{[\frac{k}{M},\frac{k+1}{M}[}$ for $0 \le k < M$. Show that $(\theta_k)_k$ is an orthonormal family and give the expression for the linear approximation

$$f_M \coloneqq \sum_k \langle f, \theta_k\rangle\, \theta_k.$$

2. Bound $\|f - f_M\|$ as a function of M, independently of y, using a bound as sharp as possible.
3. Denote $\theta \coloneqq 1_{[0,1]}$ and $\psi \coloneqq 1_{[0,1/2[} - 1_{[1/2,1[}$ (it is called the Haar wavelet). Denote, for $j \le 0$, and $0 \le n < 2^{-j}$ the wavelet functions as $\psi_{j,n} \coloneqq \frac{1}{2^{j/2}}\psi(2^{-j}x - n)$. Draw the wavelets $\psi_{-1,1}$ and $\psi_{-2,3}$.

4. For some $j_{\min} < 0$, show that

$$\{\theta\} \cup \{\psi_{j,n} : 0 \geq j \geq j_{\min} \text{ and } 0 \leq n < 2^{-j}\}$$

is an orthogonal family. What is the space spanned by this family ?

5. For each j, what is the set Σ_j of index n where $\langle f, \psi_{j,n} \rangle$ is non-zero? For these $n \in \Sigma_j$, bound $|\langle f, \psi_{j,n} \rangle|$ as a function of j.

6. For $T > 0$ define the non-linear approximation of f as

$$\hat{f} \coloneqq \langle f, \theta \rangle \theta + \sum_{|\langle f, \psi_{j,n} \rangle| > T} \langle f, \psi_{j,n} \rangle \psi_{j,n}.$$

Bound as a function of T the number M of non-zero coefficients

$$M \coloneqq |\{(j, n) : |\langle f, \psi_{j,n} \rangle| > T\}|.$$

7. Defining $j_0 \coloneqq \lfloor \log_2(T^2) \rfloor$ a cutoff scale, we define an approximation using

$$\tilde{f}_T \coloneqq \langle f, \theta \rangle \theta + \sum_{j \geq j_0, n \in \Sigma_j} \langle f, \psi_{j,n} \rangle \psi_{j,n}.$$

Show that $\left\| f - \hat{f}_T \right\| \leq \| f - \tilde{f}_T \|$.

8. Bound $\| f - \tilde{f}_T \|$ as a function of j_0 and then as a function of M . Compare the decay with M of $\| f - f_M \|$ and $\left\| f - \hat{f}_T \right\|$.

If you have finished all the exercises, you can move on to the TP3 on github.com/vcastin/teaching