Comparison of Methods for Ranking with Ties

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Abstract

We analyse different models used in paired-comparison experiments. The basic Bradley-Terry model is first looked at and the Davidson model is presented as a solution to incorporate draws. The Davidson-Beaver extension to account for multiplicative order effects is also discussed. Maximum likelihood estimation and Bayesian estimation are employed in estimating parameters from the models. These methods are applied to data from round-robin football tournaments.

1 Introduction

The Bradley-Terry (BT) model [1] for paired comparisons provides a means of quantification of the probability of preferring one item above another. The model however does not allow for equal preference (tie) outcomes, which may result from comparison of two items of similar worth. To account for this, extensions of the BT model have been developed and notable among these are the Rao-Kupper model [2] and the Davidson model [3]. We will discuss the latter in this paper. Another issue of interest is the multiplicative order effect, which is due to the order in which the two items to be compared are presented. In football circles for example, this effect is the 'home advantage', which quantifies by how much a particular team's worth is increased when playing on home grounds. A modification of the Davidson model to incorporate order effect, the Davidson-Beaver model [4] is studied in this paper. In section 2, we focus on the theoretical underpinnings of the various models and outline the parameters of interest obtained through maximum likelihood estimation. Section 3 looks at the Bayesian approach to parameter estimation. Section 4 discusses the results obtained from fitting these models to data from the 2017/2018 English Premier league (EPL) and we conclude our findings in Section 5.

2 Methods

2.1 The Bradley-Terry model

The Bradley-Terry model [1] describes the probabilities of the possible outcomes arising from the paired comparisons of individuals in a set. It has been adopted for instance to rank chess players [5], for psychometric studies of comparisons made by different human subjects and in genetic applications where alleles assume the role of the players [6]. A set of K elements (football teams in our case) is considered. We define the vector $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_K$, where $\alpha_k > 0$ indicates the ability or strength of team k. For any two teams $i, j \in 1, \dots, K$,

$$P(i \text{ beats } j) = \frac{\alpha_i}{\alpha_i + \alpha_j}.$$

The odds that i beats j are then given by α_i/α_j. By letting $\lambda_k = \log \alpha_k$ for any team k, we can express the model in the logit form as

$$\text{logit}\left[P(i \text{ beats } j)\right] = \log\left(\frac{P(i \text{ beats } j)}{1 - P(i \text{ beats } j)}\right) = \log\left(\frac{P(i \text{ beats } j)}{P(j \text{ beats } i)}\right) = \lambda_i - \lambda_j.$$

This model can be applied in any problem in which the observed data can be represented by a directed graph with nonnegative weighted edges [7]. In the sports scenario, the weight w_{ij} corresponds to the number of times i beats j. Define the total number of wins of team i by $w_i = \sum_{j=1}^{K} w_{ij}$ and the total number of comparisons between teams i and j by $n_{ij} = w_{ij} + w_{ji}$. We assume $w_{ii} = 0$ and incorporate the constraint $\sum_i \alpha_i = 1$. Assuming independence between different matches, the likelihood and log-likelihood functions are given by

$$L(\boldsymbol{\alpha}) = \prod_{i=1}^{K}\prod_{j=1}^{K}\left(\frac{\alpha_i}{\alpha_i+\alpha_j}\right)^{w_{ij}}$$

$$l(\boldsymbol{\alpha}) = \sum_{i=1}^{K}\sum_{j=1}^{K}\left[w_{ij}\log\alpha_i - w_{ij}\log(\alpha_i+\alpha_j)\right].$$

2.2 The Davidson model

Extensions of the Bradley-Terry model have been proposed to incorporate home advantage [8], draws [2; 3] and multiple comparisons [9; 10]. [3] incorporates ties by introducing a parameter $\delta \geq 0$ corresponding to the prevalence of draws. Under this model, the probability of draw is proportional to the geometric mean of the two wins probabilities:

$$P(i \text{ beats } j) = \frac{\alpha_i}{\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}$$

$$P(j \text{ beats } i) = \frac{\alpha_j}{\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}$$

$$P(\text{draw}) = \frac{\delta\sqrt{\alpha_i\alpha_j}}{\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}.$$

Notice that the case with $\delta = 0$ corresponds to the standard Bradley-Terry model. We define $t_{ij} = t_{ji}$ (with $t_{ii} = 0$) as the number of ties between i and j. By incorporating draws, the number of comparisons for each pair of teams (i, j) is now $n_{ij} = w_{ij} + w_{ji} + t_{ij}$. There are two comparisons for each pair of teams in the Premier League, where one game is played home and one away. The log-likelihood for the Davidson model is

$$l(\boldsymbol{\alpha}, \delta) = \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left[2w_{ij}\log\frac{\alpha_i}{\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}} + t_{ij}\log\frac{\delta\sqrt{\alpha_i\alpha_j}}{\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}\right].$$

Davidson (1970) employs maximum likelihood estimation in obtaining estimates $\hat{\boldsymbol{\alpha}} = \{\hat{\alpha_1}, \hat{\alpha_2}, \ldots, \hat{\alpha_k}\}$ and $\hat{\delta_1}$ of the parameters $\boldsymbol{\alpha}$ and δ respectively. For $K > 2$, the estimates cannot be obtained directly from maximizing the likelihood and an iterative process [3] is used instead. This process yields convergence to the desired estimates under a partitioning assumption [11].

2.3 The Davidson-Beaver model

Davidson and Beaver [4] introduce a multiplicative order effect $\theta_{ij} > 0$ to account for the effect of the order of presentation of the objects in the pair. This can lead to the worth of the object presenting second becoming inflated or the object presented first having an advantage. In the football league application, this order effect corresponds to the location of each game, with an advantage $\theta > 1$ (greater than one since it has a positive effect) given to the team playing home. We will consider the case with a single home-advantage parameter $\theta = \theta_{ij}$. We also consider θ and δ to be structural quantities constant across different seasons of a league, so that they can be determined from historical data [12]. For i being the home team and j the away team, the Davidson-Beaver model is given by:

$$P(i \text{ beats } j) = \frac{\theta\alpha_i}{\theta\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}$$

$$P(j \text{ beats } i) = \frac{\alpha_j}{\theta\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}$$

$$P(\text{draw}) = \frac{\delta\sqrt{\alpha_i\alpha_j}}{\theta\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}.$$

The square root involved in the equation is related to the fact that older football leagues would have a $2-1-0$ point system with 2 points gained per win and 1 per draw. However, the modern "three points for a win" $3-1-0$ system introduced in England in 1981 and adopted by the FIFA in 1995 suggests the use of a cubic root. This is taken into consideration in the alt-3.uk league tables provided by David Firth and Heather Turner [12], where the team strengths are determined using maximum likelihood such that they yield exact agreement between the actual league points to date and the expected points derived by the model. Recall that w_{ij} denotes the number of times i beats j. We now define three new quantities. Let a_{ij} be the number of times i beats j at home. Let b_{ij} be the number of times i loses to j at home. Finally, let c_{ij} be the number of ties between i and j where i was playing home. The log-likelihood is then given by

$$l = \sum_{i=1}^{K}\sum_{j=1}^{K}\left[a_{ij}\log\frac{\theta\alpha_i}{\theta\alpha_i+\alpha_j+\delta\sqrt{\theta\alpha_i\alpha_j}} + b_{ij}\log\frac{\alpha_j}{\theta\alpha_i+\alpha_j+\delta\sqrt{\theta\alpha_i\alpha_j}} + c_{ij}\log\frac{\delta\sqrt{\alpha_i\alpha_j}}{\theta\alpha_i+\alpha_j+\delta\sqrt{\alpha_i\alpha_j}}\right].$$

The parameters in this model are chosen to ensure that the above likelihood is maximized for a given data set and a similar iterative scheme is employed. Let $(\hat{\boldsymbol{\alpha}}^{*}, \hat{\delta_2}, \hat{\theta})$ be the maximum likelihood estimates of $(\boldsymbol{\alpha}, \delta, \theta)$ under the Davidson-Beaver model, where $\hat{\boldsymbol{\alpha}}^{*} = \{\hat{\alpha_1}^{*}, \hat{\alpha_2}^{*}, \ldots, \hat{\alpha_k}^{*}\}$.

2.4 Log-linear Bradley-Terry model with home advantage and draw

We will make use of the Poisson trick [13] in order to express our multinomial outcome as a Poisson response by defining three binary counts: home team won (1 for yes, 0 for no), away team won (1 for yes, 0 for no) or draw (1 for yes, 0 for no). We expand each row (corresponding to one game) into three, one for each of these individual outcomes which will be individually fitted as counts for a Poisson regression. The total count will always be one.

The log strengths λ_i are taken as the estimated coefficients of a matrix X which indicates the distribution of points across games (rows) and teams (columns). For the EPL, 380 games are played in total, each team playing every other team twice (home and away). After expanding the data, we produce the 1140x20 matrix X by recording the number of points awarded to each team after each game. Each row sums to either 1 or 2/3 as 3 points are awarded per win and 1 per draw (to both teams), equivalent to 1 and $2*(1/3)$. The identifiability problem is resolved by setting one of the strengths as a reference with a fixed value. We take West Ham as the reference with a 0 log strength. We can add the home advantage and draw parameters to the model by defining two dummy variables y and z to indicate whether the result was a draw and whether the winning team was playing at home. This will allow us to find out how many times more likely a draw is compared to a win/loss result in the absence of home advantage and how many times more likely the home team is to win compared to the away team by looking at the expected log odds for these parameters. The linear predictor of the model with the binary outcome count being the response can be written as:

$$\eta_{ij} = \log\left[\mathrm{P}(\text{count between } i \text{ and } j = 1)\right] = \lambda_i - \lambda_j + \delta y_{ij} + \theta z_{ij}.$$

3 Bayesian approach to parameter estimation

Davidson models for the EPL typically used a power of either 1/2 (classical Davidson) or 1/3 (used to represent the 3 point per win relative to 1 point for a draw). However, we would like to estimate what power best fits actual EPL results. In order to do that, we look at Davidson models from a Bayesian perspective, and treat the power as an additional parameter, β.

Various Bayesian approaches to Bradley-Terry parameter estimation have been proposed, however tend to be challenging due to the high-dimensional parameter space and intractability of the posterior density. [14] tackle this by using carefully-designed proposal distributions for Metropolis-Hastings (M-H) algorithms, while [5] use latent variables to derive conditional distributions for parameters to be used in Gibbs sampling. Here we take a less efficient, but more approach to Bayesian inference that allows us to incorporate a β parameter. We use adaptive M-H implemented by `rStan` [15] to sample from the posterior.

3.1 The Bayesian model

For all models, we say that for K teams, $i, j \in (1, ..., K)$, team i is the home team and j is the away team, and they meet n_{ij} times with i at home where $r_{ij} \in (1, ..., n_{ij})$. We say that the pairing the r_{ij}th match between team i and team j has outcome X_{ijr}. X_{ijr} has a heirarchical distribution of the following form:

$$\beta \sim \text{Beta}(2,3); \;\; \log(\theta) \sim \text{N}(\log(2), 0.25); \;\; \log(\delta) \sim \text{N}(m, 0.25); \;\; \sigma \sim \Gamma(2K, 2K/\hat{\sigma}^2)$$

$$\log(\boldsymbol{\alpha})|\sigma \sim \text{N}(\mathbf{0}, \sigma^2\mathbf{I}); \;\; \mathbf{p}_{ijr}|\boldsymbol{\alpha}, \delta, \theta, \beta = (Pr(i \text{ beats } j), Pr(j \text{ beats } i), Pr(i \text{ ties } j))$$

$$X_{ijr}|\boldsymbol{\alpha}, \sigma, \delta, \theta, \beta \sim \text{Multinomial}(1, \mathbf{p}_{ijr})$$

We assume that the probability of a specific outcome between teams i and j on their r_{ij}th meeting is independent of previous matches, so $p_{ijr} = p_{ij}$ for all r_{ij}. As in the classical version, $\boldsymbol{\alpha}$ represents a vector of the relative strength parameters of the competing teams. We adopt a log-Normal prior distribution for $\boldsymbol{\alpha}$ recommended by [16], with a Gamma hyperprior for σ, the variance of team strength. The $\hat{\sigma}$ is an estimate of the variance of $\log(\alpha_i)$ which [16] suggests using data from the previous season to determine.

We also specify a log-Normal prior for δ. The log-Normal distribution is relatively uninformative, but restricts the parameter space to non-negative values. In this case, we center the distribution at $m = \log(\hat{\delta}^{\text{MLE}})$. This is not truly Bayesian, as we are using data from the current season to estimate the hyperprior parameters.

In Section 2.3 we introduce the home advantage parameter, $\theta > 0$. We have a strong prior that the home team enjoys an advantage, and anecdotal knowledge that the home team is as much as 2x more likely to win when they're at home, so we center the distribution at log(2). We use β to denote the "power" parameter discussed in Section 2.4. We assume that restrict β to be between 0 and 1, and we have a reasonably strong prior that the true value of β is around 1/3-1/2 ($\beta = 1/2$ recovers classical Davidson). Therefore, we give β a weakly informative prior of $\beta \sim \text{Beta}(2, 3)$. We use `rStan` to estimate the parameters from the Davidson and Davidson-Beaver models with multiplicative effects and the generalized Davidson-Beaver power model. In Section 4, we present these results and compare parameter estimates between Bayesian and MLE approaches.

3.2 Evaluating model fit

In order to evaluate model fit, we look mainly at the models' posterior predictive distributions (PPD) too evaluate how well the model has approximated the data generating process of observed values [17]. To generate the PPD, we first drew a sample from the joint posterior distribution, and then used the sampled parameter values to draw a replicate sample of outcomes, $\mathbf{X}^{\text{rep}}$. This replicate sample can be thought of as a sample that reasonably could have been observed if the model that we've specified is correct (that our null hypothesis is correct). The PPD is then defined as the probability that we observe the replicate sample given that the model is correct and given the outcomes that we actually observed. The PPD functions a reference distribution, against which test statistics based on actual outcomes can be compared. We calculate the classification rate over all $m \in (1, ..., M)$ replicate samples as:

$$\text{Classification Rate}^{\text{PPD}} = \frac{1}{M} \sum_{m=1}^{M} \sum_{i \neq j} \sum_{r=1}^{n_{ij}} \text{I}\{X_{ijr}^{\text{rep}} = X_{ijr}^{\text{obs}}\}$$

4 Numerical Examples and Model Comparison

We analyse and compare parameter estimates from the different model outputs and study the effect of varying input variables (specifically, the power in the Davidson model) on estimates. The analysis uses data from the EPL available at football-data.co.uk [18], involving a total of 20 teams. Each team plays with every other team twice, making a total of 190 distinct pairs and 380 matches in total. The 2017/18 season which is analysed into detail, recorded a total of 198 draws (ties). The models are used to estimate the "worth" parameters of the 20 teams as well as the draw and home-advantage (order-effect) parameters for the season. The different worth estimates serve as references for ranking the teams in order to compare with the actual final league table rankings for the season.

4.1 Davidson-Beaver Model Results

We analyse the 2017/18 data set taking into account the multiplicative order effect as discussed in section 2.3. West Ham is set as the reference: teams with positive estimates have higher strength and those with negative estimates have lower strength. Fig. 1 was obtained using the quasi-standard errors for the estimate, which are calculated such that the standard errors remain constant even when comparison references are changed and allow for the quantification of standard error of the reference item [19]. Non-overlapping intervals represent significant differences. We compare estimates from the Davidson model (with power 1/2 and 1/3) and the Davidson-Beaver model (with power 1/2 and 1/3). We also produce deviance plots to investigate the fit of a different power in the range 0.1 to 0.9.

Season	% Home wins	% Draw	% Away wins	Home advantage estimate	Draw estimate
2015/16	0.41	0.28	0.31	0.37	-0.08
2016/17	0.49	0.22	0.29	0.76	0.20
2017/18	0.46	0.26	0.28	0.64	0.38
2018/19	0.43	0.23	0.34	0.57	0.30

Table 1: Proportion of draws, away team victories and home team victories for the seasons 2015/16 to current 2018/19 (incomplete). Parameter estimates for home advantage and draw in the log-linear BT model with home advantage and power 1/3.

Team	$\hat{\alpha}_i$ (power:1/2)	α_i^* (power:1/2)	Team	$\hat{\alpha}_i$ (power:1/3)	$\hat{\alpha}_i{}^*$ (1/3)
Man City	3.816	4.012	Man City	3.400	3.573
Man United	2.116	2.241	Man United	2.046	2.162
Tottenham	1.907	2.021	Tottenham	1.821	1.926
Liverpool	1.907	2.021	Liverpool	1.712	1.812
Chelsea	1.427	1.515	Chelsea	1.448	1.534
Arsenal	0.988	1.051	Arsenal	1.091	1.157
Burnley	0.654	0.696	Burnley	0.636	0.675
Everton	0.327	0.348	Everton	0.378	0.401
Leicester	0.245	0.261	Leicester	0.273	0.289
Crystal Palace	0.082	0.087	Crystal Palace	0.111	0.118
Bournemouth	0.082	0.087	Newcastle	0.111	0.118
West Ham	0.000	0.00	Bournemouth	0.111	0.118
Newcastle	-0.000	-0.000	West Ham	0.000	0.000
Brighton	-0.083	-0.088	Watford	-0.057	-0.060
Watford	-0.166	-0.176	Brighton	-0.114	-0.121
Southampton	-0.249	-0.266	Huddersfield	-0.290	-0.308
Huddersfield	-0.334	-0.356	Southampton	-0.351	-0.373
Stoke	-0.506	-0.539	Swansea	-0.541	-0.574
West Brom	-0.595	-0.633	Stoke	-0.541	-0.574
Swansea	-0.595	-0.633	West Brom	-0.674	-0.715
$\hat{\delta}_1$ - Davidson draw	-0.122	-	$\hat{\delta}_1$	0.002	-
$\hat{\delta}_2$ - DB draw	-	0.250	$\hat{\delta}_2$	-	0.380
$\hat{\theta}$ - Home adv.	-	0.644	$\hat{\theta}$	-	0.644

Table 2: MLE estimates of the log strengths for models with and without home advantage effect.

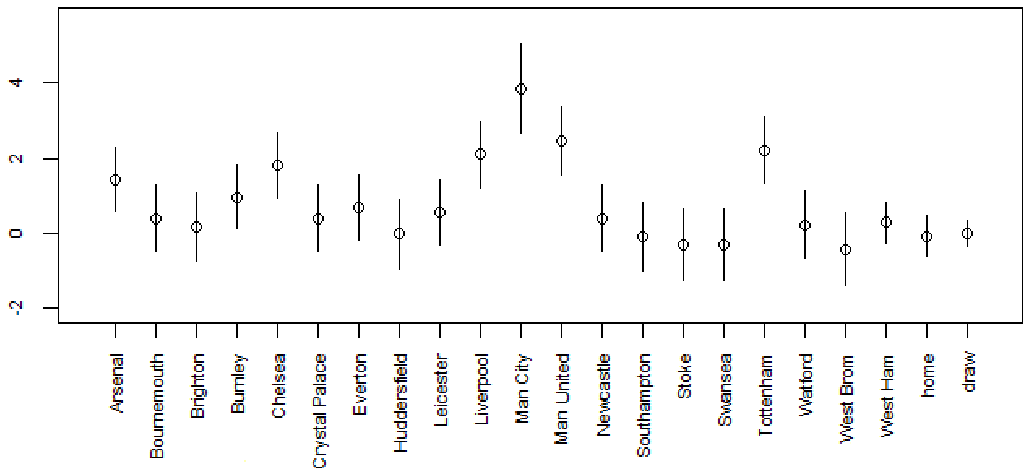

Figure 1: Log strengths for the Davidson-Beaver model with power 1/3 for 2017/2018.

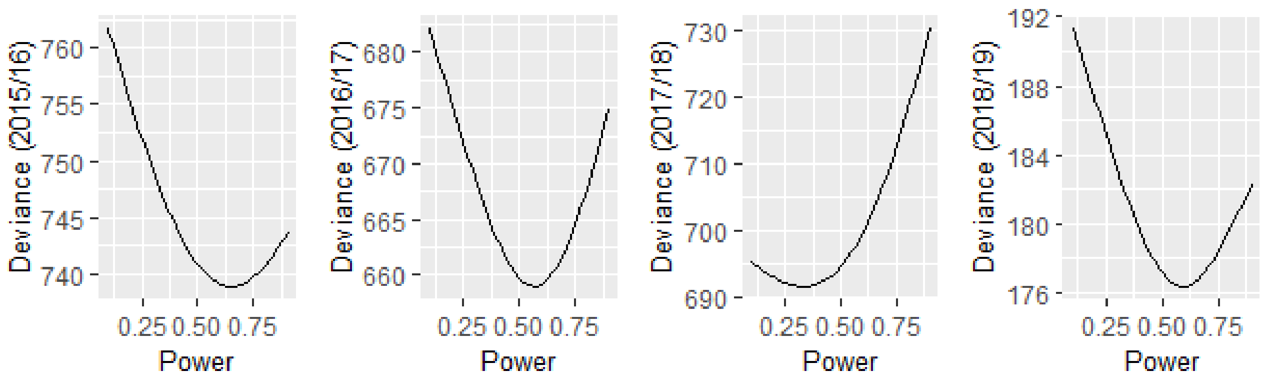

Figure 2: Deviance values for variable power. Minimum deviance found at $\beta = 0.65, 0.57, 0.33, 0.59$.

4.2 Bayesian results

The Classification Rate$^{\text{PPD}}$ for Davidson, Davidson-Beaver and Davidson Power models were 33.8%, 40.5% and 40.7%, respectively. None of the models performs particularly well, especially if we consider that random allocation would result in a correct classification rate of 33%. However relative comparison shows that the the Davidson model with variable power has the highest PPD classification rate, with the Davidson-Beaver model performing second best. It appears that including a home advantage parameter improves model fit, but variable power does not have as large an impact (at least for the 2017/18 season).

Figure 3 shows the posterior distributions of the teams' strength parameters in the 2017/18 season (on the log scale) for Davidson-Beaver and Davidson Power models. The team rankings are largely similar to those derived from classical methods (Table 2), with Man City ranked the strongest and Man United, Tottenham, Liverpool and Arsenal filling out the top 5.

Figure 4 shows the posterior distributions for the additional parameters from Davidson Power model fitted to multiple seasons. The black line denotes the combined distribution across all 4 seasons' distributions. The δ and θ parameters (for prevalence of draws and home advantage) seem to remain relatively consistent from season to season, while the distributions of σ and β are not as consistent. The posterior distributions of β are perhaps the most interesting. The mean of the combined distribution is 0.4, in between the 1/2 and 1/3 values commonly used, however the posterior distribution based on the 2017/18 season has a markedly lower mean than the other seasons ($\beta^{\text{PM}}_{17/18} = 0.2$). This result mirrors the result of the optimal β determined from classical models.

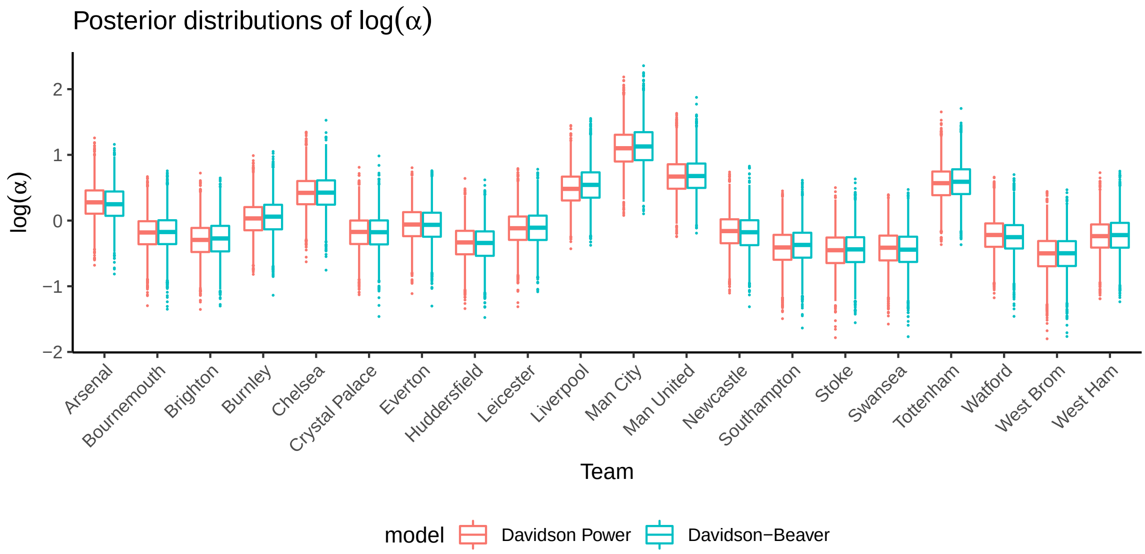

Figure 3: Posterior distributions for the log strength parameters for the EPL 2017/18 season, estimated using Davidson-Beaver and Davidson Power models.

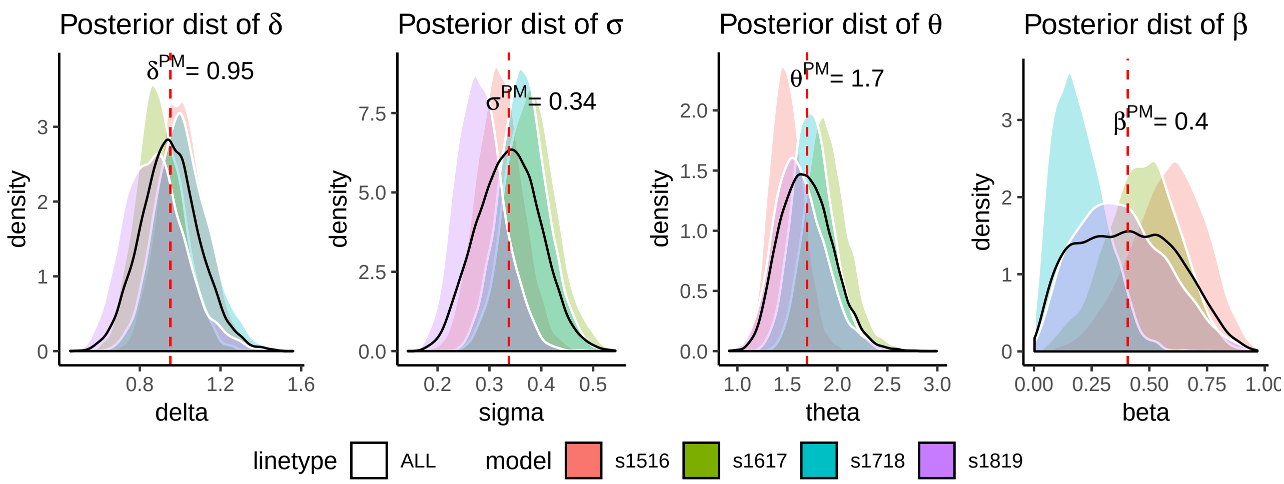

Figure 4: Posterior distributions for Davidson Power model parameters on multiple seasons of data. Black line represents the distributions combined across all seasons.

5 Discussion

In this analysis, we fit generalized Bradley-Terry models to data from the English Premiere League from the past 4 seasons. These generalizations allowed us to incorporate ties, home advantage and variable power. We fit the models using both classical and Bayesian methods, and recovered estimates of the relative strength of teams and additional parameters. Classical and Bayesian models returned similar, though not identical, results. Notably, the Bayesian estimated strength parameters ($\log(\alpha)$) did not have as large a range as those estimated from classical methods. Perhaps the most interesting result were the optimal values of power, β, recovered. The classical results ranged from 0.33 for the 2017/18 season to 0.65 in the 2015/16 season. The Bayesian posterior distributions for β also estimated the lowest value of β for the 2017/18 season and the highest for the 2015/16 season.

A large opportunity for extensions of this work is in examining the appropriateness of the prior distributions for θ, δ and β and the sensitivity of conclusions here to those specifications. Similarly, more work should be done to extend this analysis to more historical data in order to increase the precision of parameter estimates.

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