King Abdullah University of Science and Technology

KAUST

CS 247 – Scientific Visualization
Lecture 14: Scalar Fields, Pt. 10; Volume Rendering, Pt. 1 [preview]

Markus Hadwiger, KAUST

Reading Assignment #8 (until Mar 21)

Read (required):

- Real-Time Volume Graphics, Chapter 1 (*Theoretical Background and Basic Approaches*), from beginning to 1.4.4 (inclusive)
- Real-Time Volume Graphics, Chapter 4 (Transfer Functions) until Sec. 4.4 (inclusive)

- Look at:
  Nelson Max, Optical Models for Direct Volume Rendering, IEEE Transactions on Visualization and Computer Graphics, 1995
  `http://dx.doi.org/10.1109/2945.468400`

wrapping up the previous part…

Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

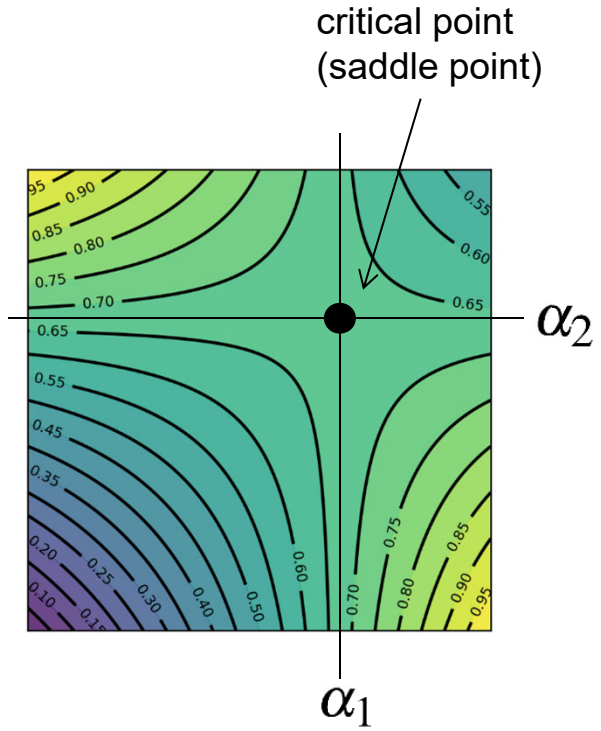
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10}-v_{00}) + \alpha_2(v_{00}+v_{11}-v_{10}-v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01}-v_{00}) + \alpha_1(v_{00}+v_{11}-v_{10}-v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00}-v_{10}}{v_{00}+v_{11}-v_{10}-v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00}-v_{01}}{v_{00}+v_{11}-v_{10}-v_{01}}$$



Bi-Linear Interpolation: Critical Points

Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

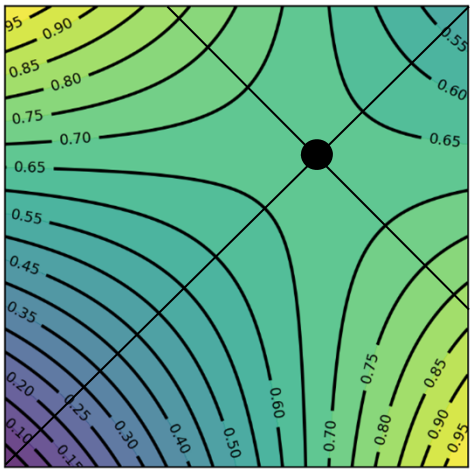
$$a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(here also: principal curvature magnitudes and directions of this function's graph == surface embedded in 3D)



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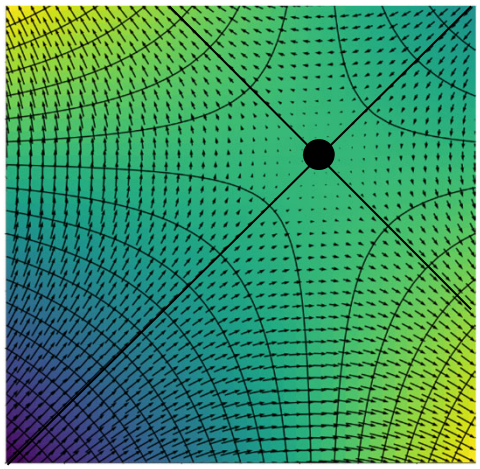

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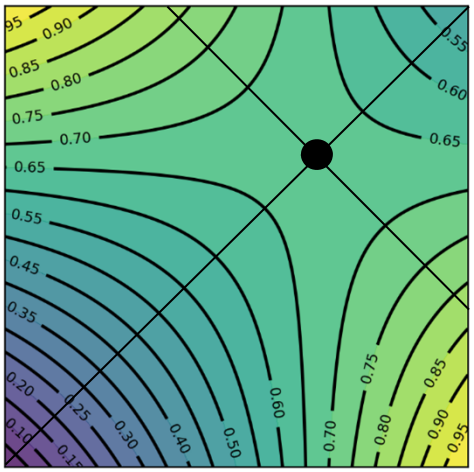
$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

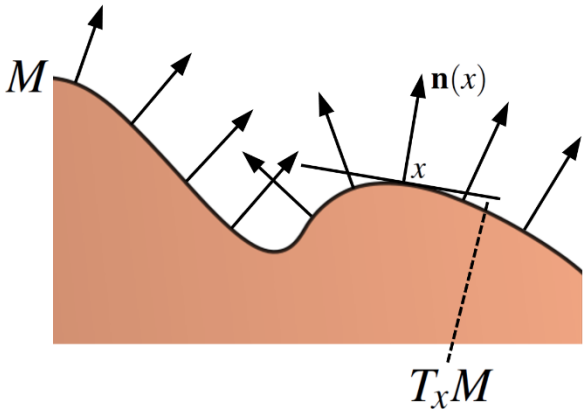
degenerate means determinant = 0 (at least one eigenvalue = 0);
bi-linear is simple: a = 0 means degenerated to
linear anyway: no critical point at all! (except constant function)
(but with more than one cell: can have max or min at vertices)

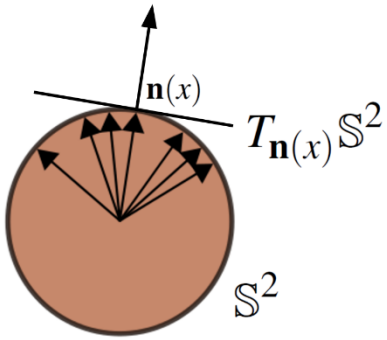


Interlude: Curvature and Shape Operator

Gauss map

$$\mathbf{n}\colon M \to \mathbb{S}^2$$
$$x \mapsto \mathbf{n}(x)$$





Differential of Gauss map

$$d\mathbf{n}\colon TM \to T\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

$$(d\mathbf{n})_x\colon T_xM \to T_{\mathbf{n}(x)}\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

Shape operator (Weingarten map)

$$\mathbf{S}\colon TM \to TM$$

$$\mathbf{S}_x\colon T_xM \to T_xM$$
$$\mathbf{v} \mapsto \mathbf{S}_x(\mathbf{v}) = d\mathbf{n}(\mathbf{v})$$

Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_xM$$

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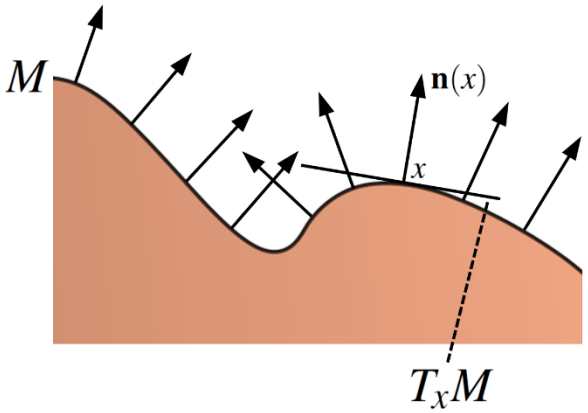

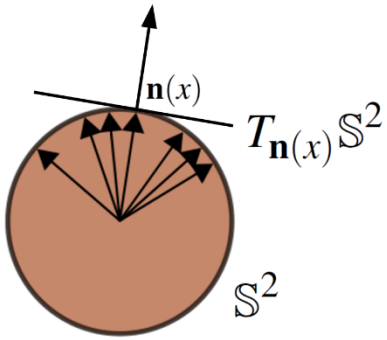

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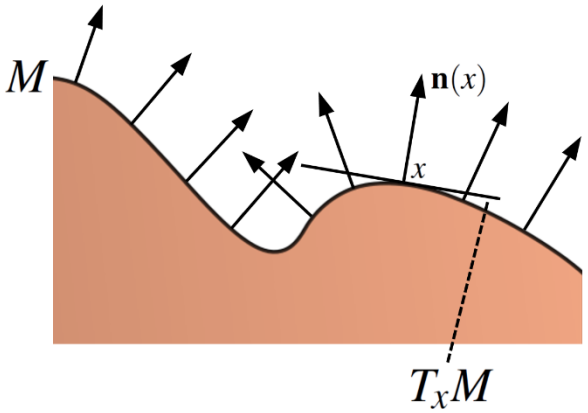

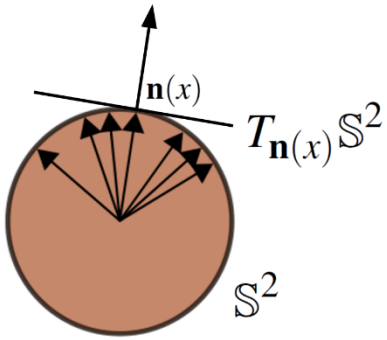

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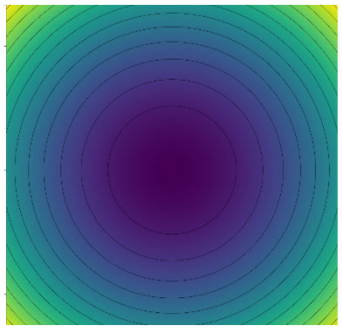
(sign is convention)

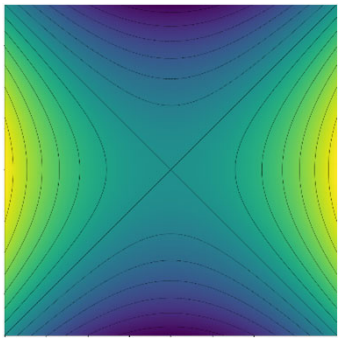
General Case (2D Scalar Fields)

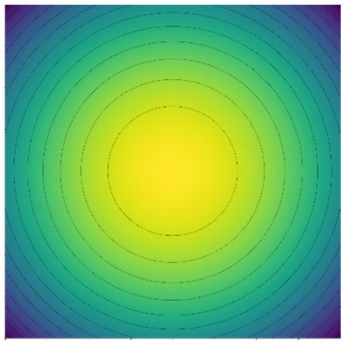
In 2D scalar fields, only *three types* of (isolated, non-degenerate) critical points

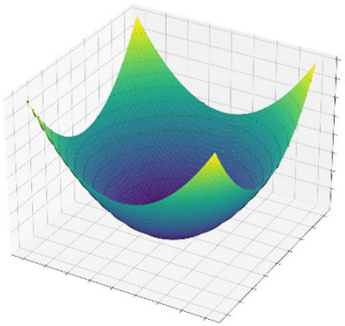
Index of critical point: dimension of eigenspace with negative-definite Hessian

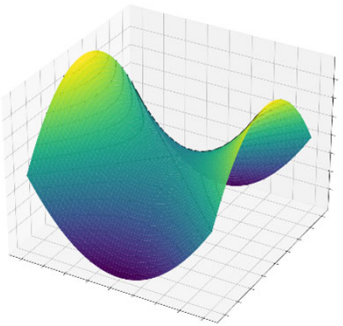
minimum (index 0)	saddle point (index 1)	maximum (index 2)

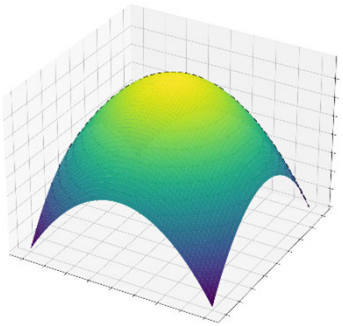












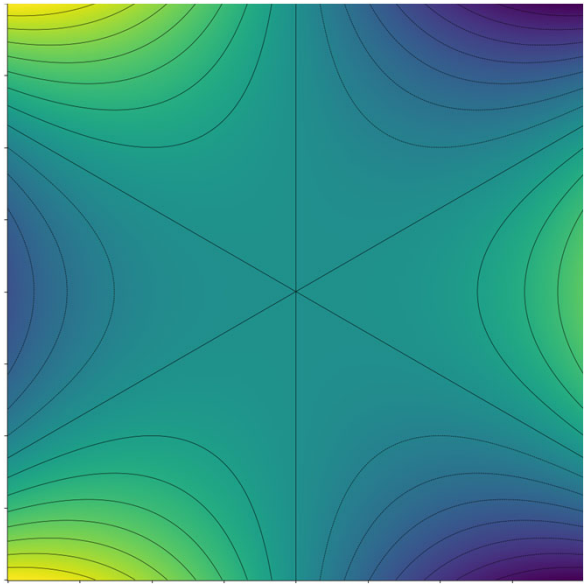
Interesting Degenerate Critical Points?

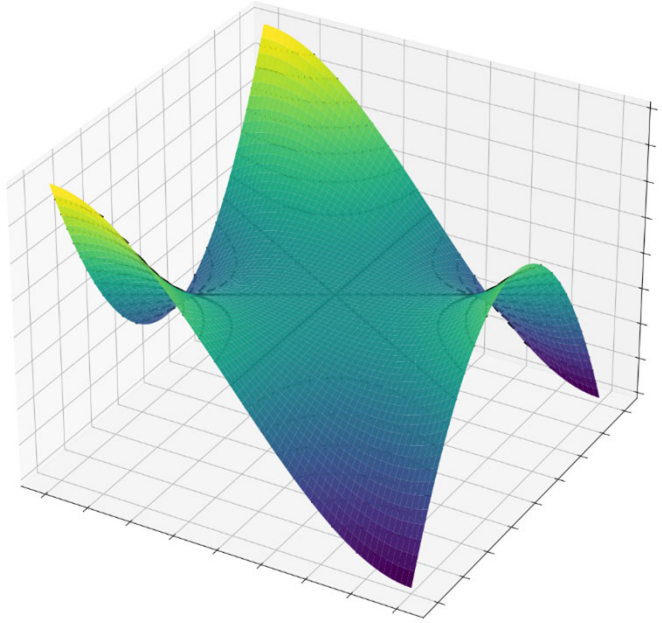
Hessian matrix is singular (determinant = 0)

- Cannot say what happens: need higher-order derivatives, …

Interesting example: monkey saddle $z = x^3 - 3xy^2$ (‘third-order saddle’)

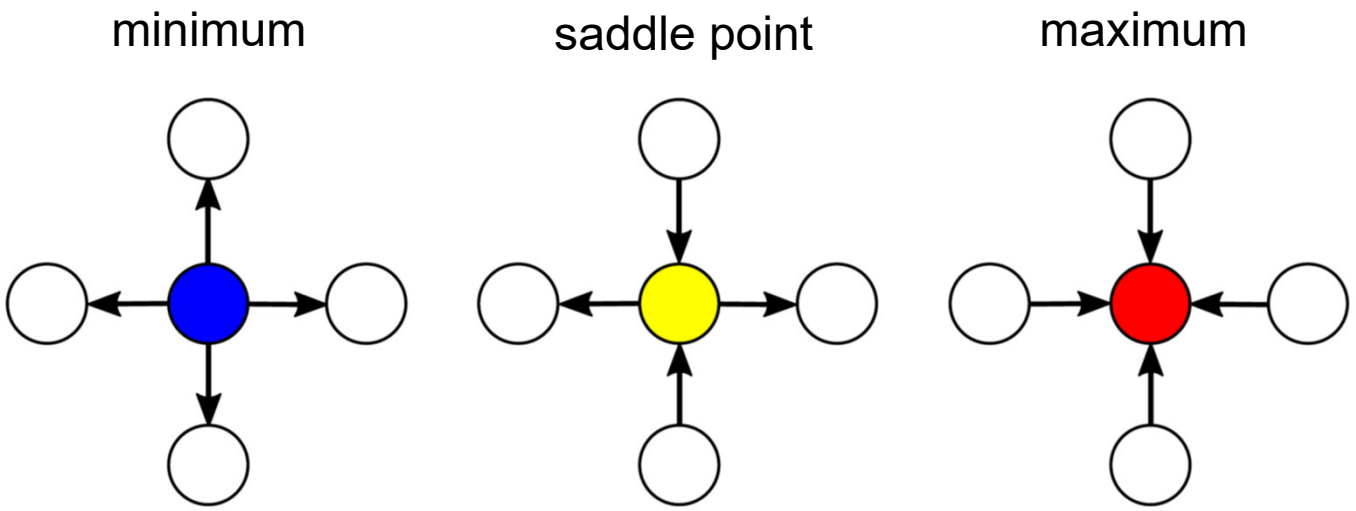
- Point (0,0) in center: Hessian = 0; Gaussian curvature = 0 (umbilical point)





Discrete Classification of Critical Points

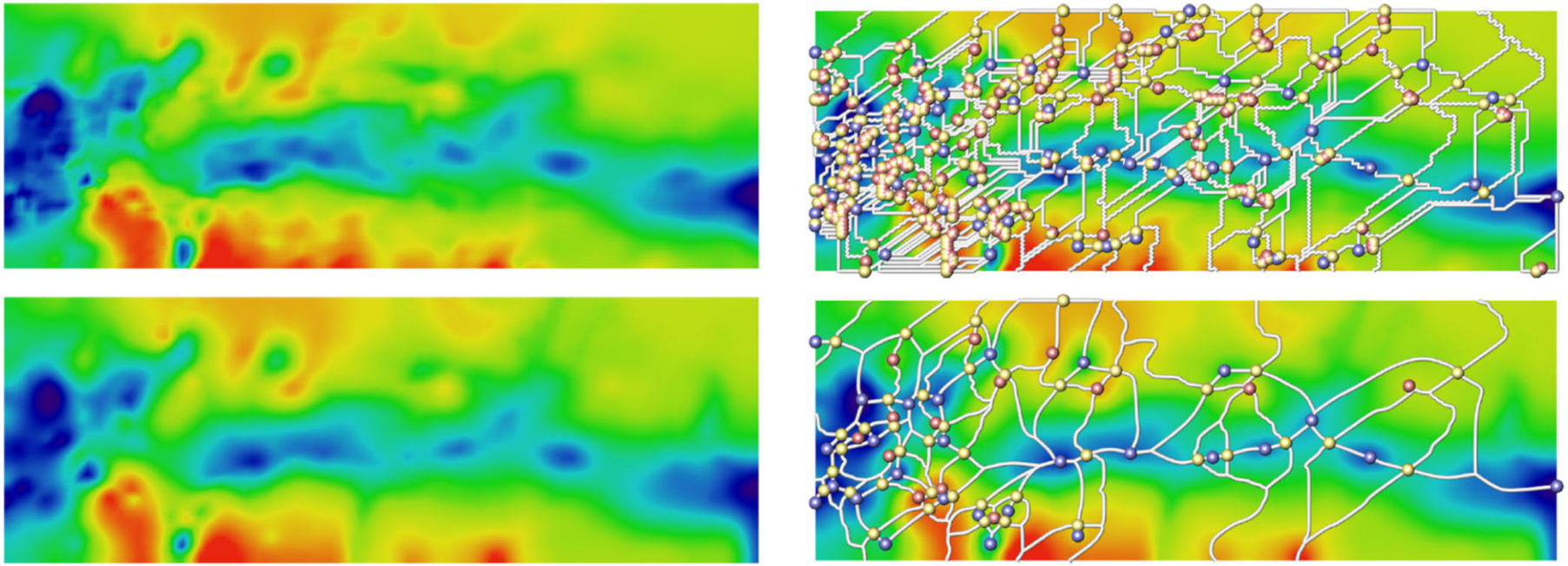
Combinatorial classification (looking at and comparing neighbors)
instead of looking at derivatives
(i.e., derivatives of the smooth function that is not known)



…toward scalar field topology, discrete Morse theory, Morse-Smale complex, …

Example: Scalar Field Simplification

Topology-based smoothing of 2D scalar fields, Weinkauf et al., 2010



Example: Differential Topology

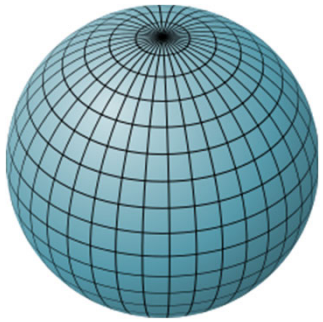
Morse theory

- Morse function: scalar function where all critical points are non-degenerate and have different critical value

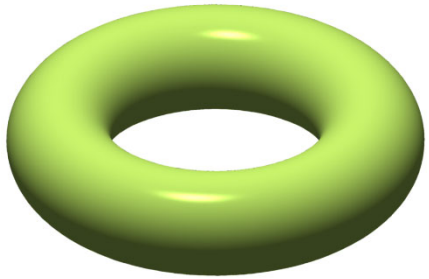
Topological invariant: Euler characteristic $\chi(M)$ of manifold M

(for 2-manifold mesh: $\chi(M) = V - E + F$)

$\chi = 2 - 2g$ (orientable)



genus $g = 0$
Euler characteristic $\chi = 2$



genus $g = 1$
Euler characteristic $\chi = 0$

genus $g = 2$
Euler characteristic $\chi = -2$

Example: Differential Topology

Morse theory

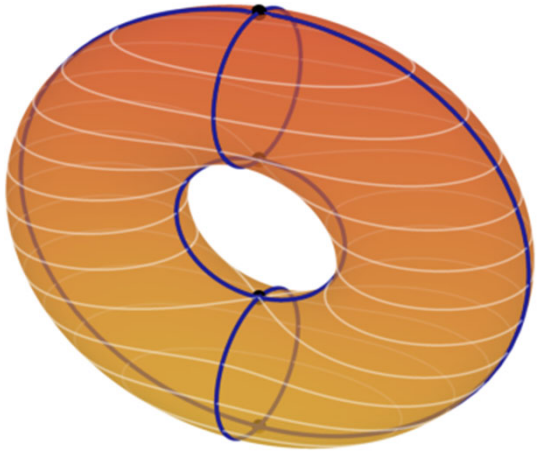
- Morse function: scalar function where all critical points are non-degenerate and have different critical value

Topological invariant: Euler characteristic $\chi(M)$ of manifold M

$$\chi(M) = \sum_{i=0}^{n} (-1)^i m_i$$

m_i: number of critical points with index i

n: dimensionality of M



genus $g(M) = 1$

Euler characteristic $\chi(M) = 0 \; (= 1 - 2 + 1)$

scalar function on torus is height function $f(x,y,z) = z$:

1 min, 1 max, 2 saddles

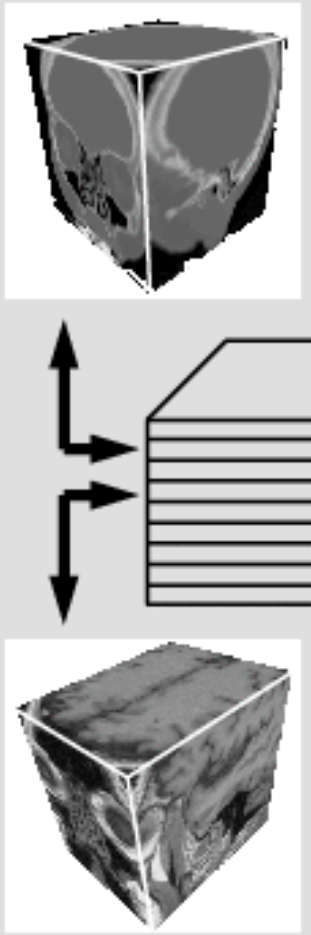
critical points are where

$$df(x,y,z) = 0$$

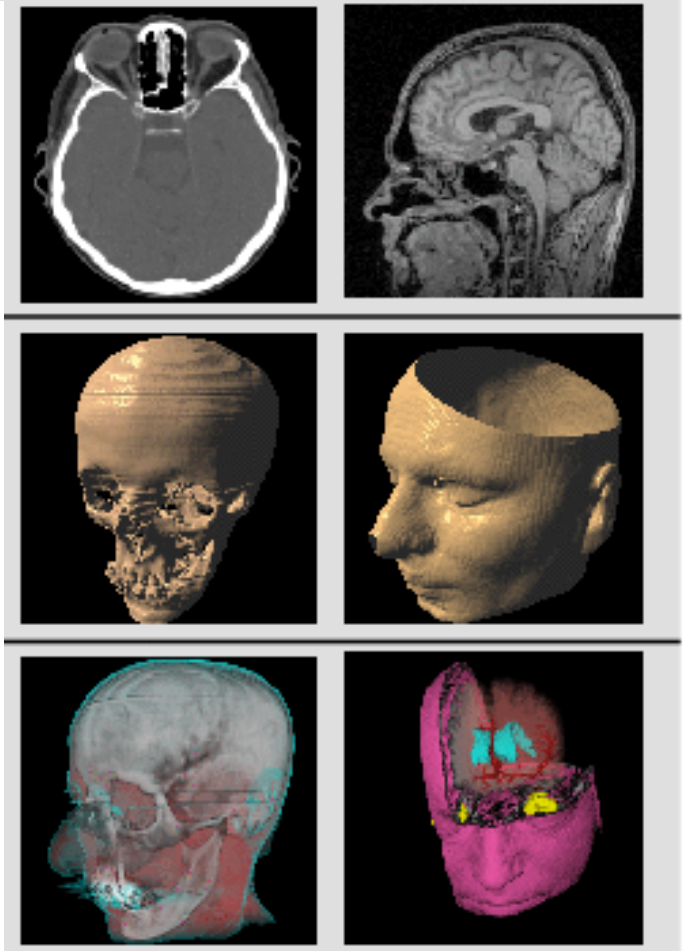
(tangent plane horizontal)

Volume Visualization

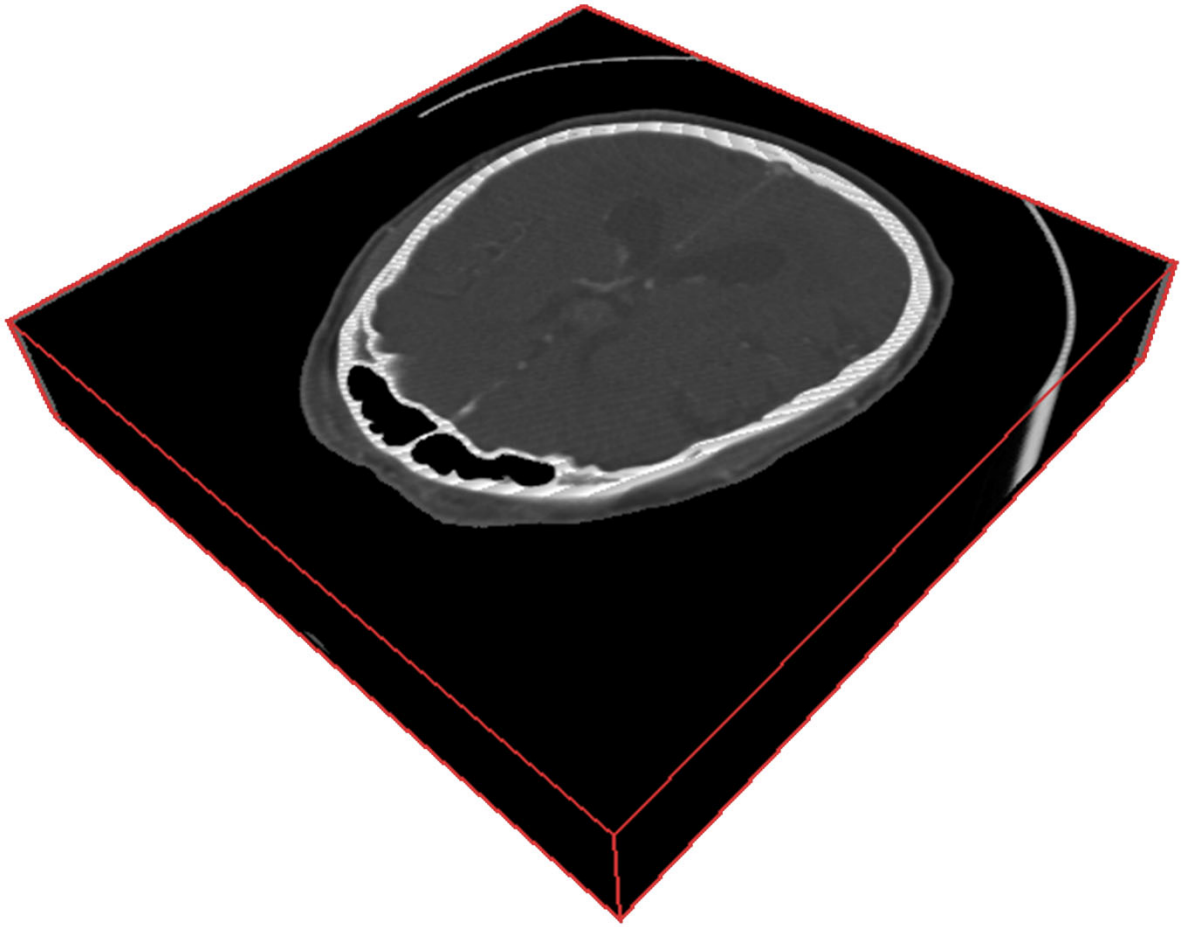
Volume Visualization



- 2D visualization
  slice images
  (or multi-planar
  reformatting MPR)
- *Indirect*
  3D visualization
  isosurfaces
  (or surface-shaded
  display: SSD)
- *Direct*
  3D visualization
  (direct volume
  rendering: DVR)



Direct Volume Rendering



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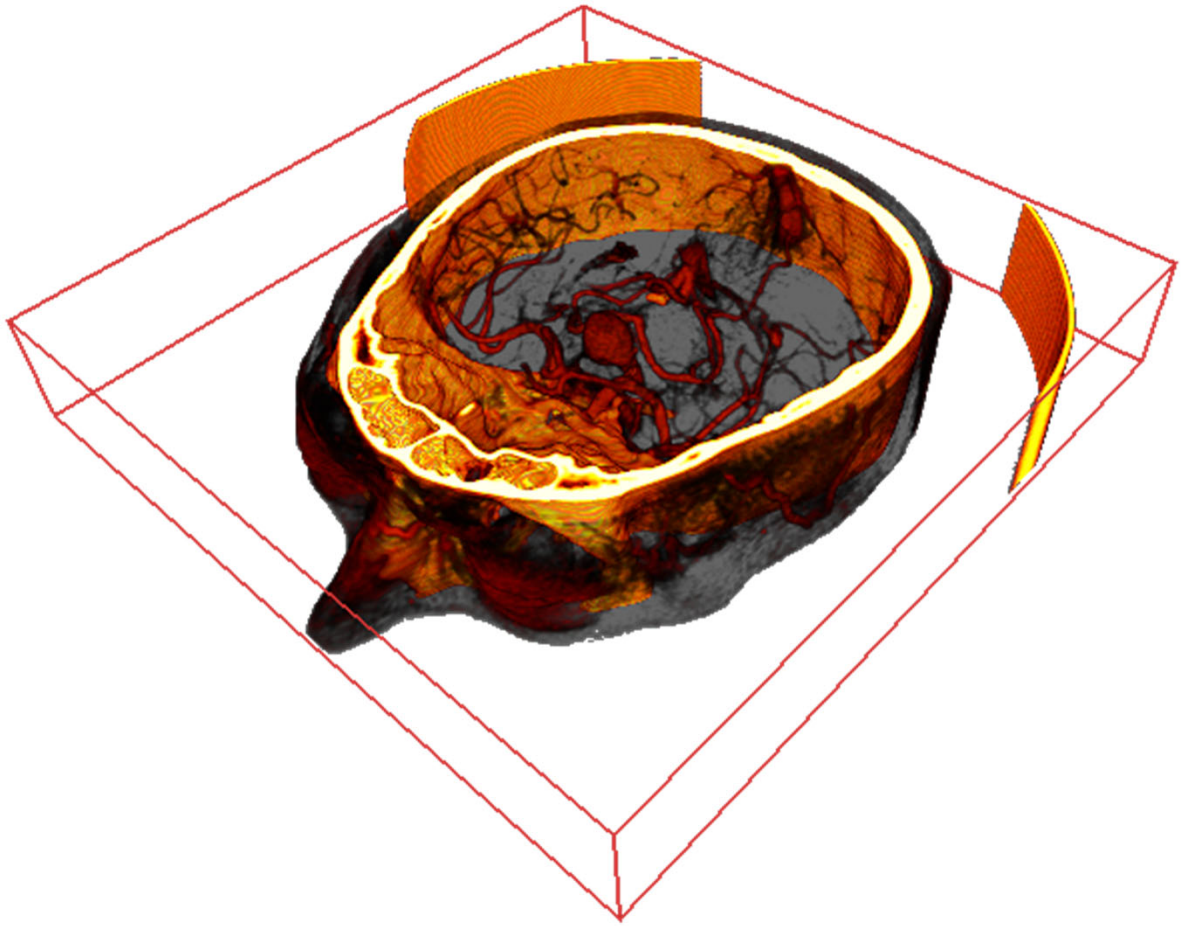
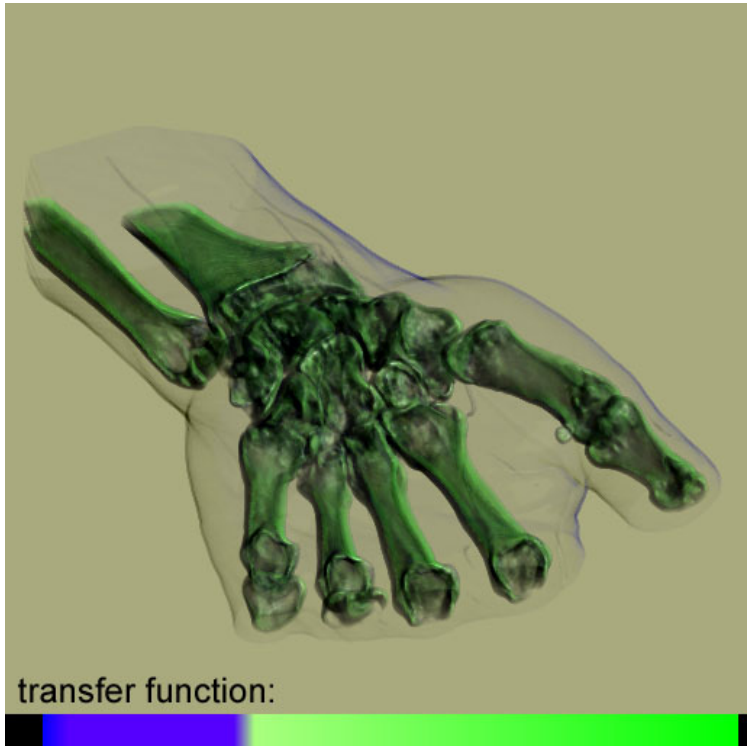


Transparent Volumes vs. Isosurfaces

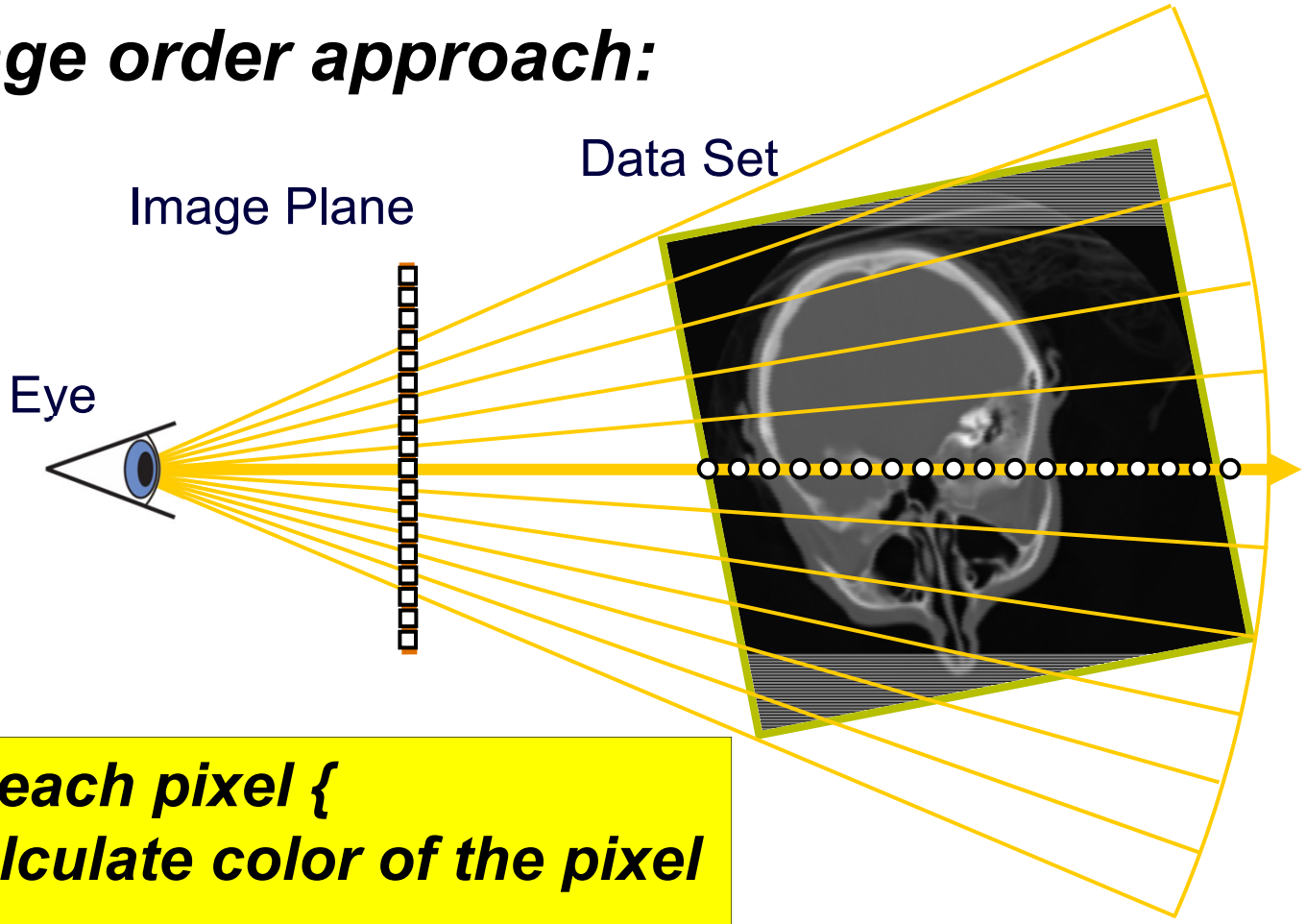
The *transfer function* assigns *optical properties* to data

- Translucent volumes
- But also: isosurface rendering using step function as transfer function



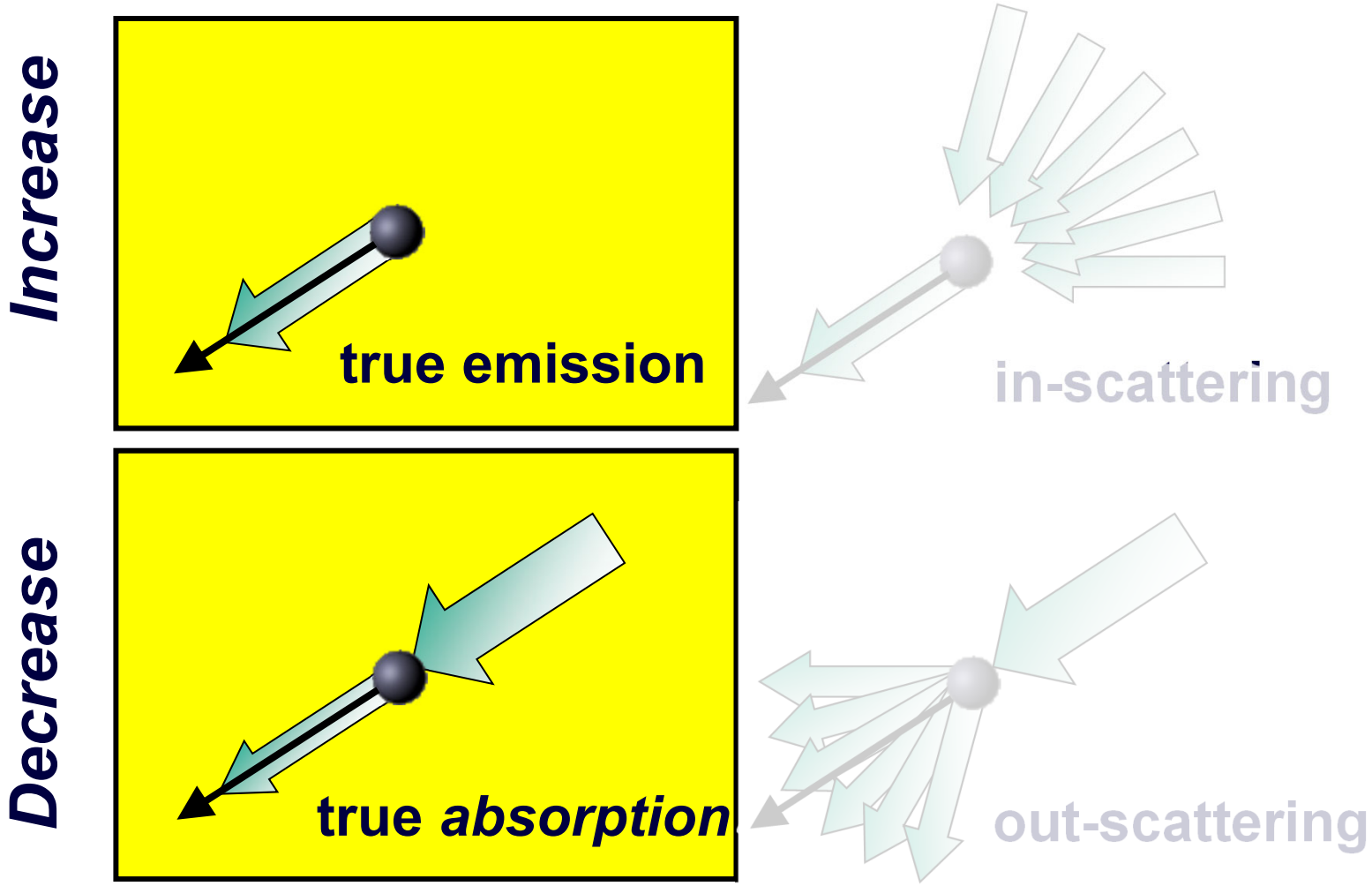
Direct Volume Rendering

Image order approach:




```
For each pixel {
   calculate color of the pixel
}
```

Physical Model of Radiative Transfer



Volume Rendering Integral

Volume rendering integral
for *Emission Absorption* model

true emission **true *absorption***

$$I(s) = I(s_0)\, e^{-\tau(s_0,s)} + \int_{s_0}^{s} q(\tilde{s})\, e^{-\tau(\tilde{s},s)}\, \mathrm{d}\tilde{s}$$

Numerical solutions:

Back-to-front compositing

$$C'_i = C_i + (1 - A_i)C'_{i-1}$$

Front-to-back compositing

$$C'_i = C'_{i+1} + (1 - A'_{i+1})C_i$$
$$A'_i = A'_{i+1} + (1 - A'_{i+1})A_i$$

Volume Rendering Integral

How do we determine the radiant energy along the ray?

Physical model: emission and absorption, no scattering

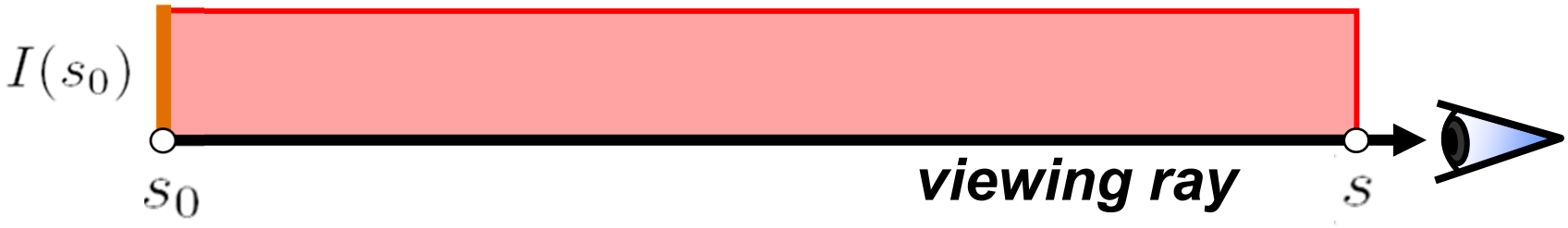
Initial intensity at s_0

$$I(s) = I(s_0)$$

Volume Rendering Integral

How do we determine the radiant energy along the ray?

Physical model: emission and absorption, no scattering



Initial intensity at s_0

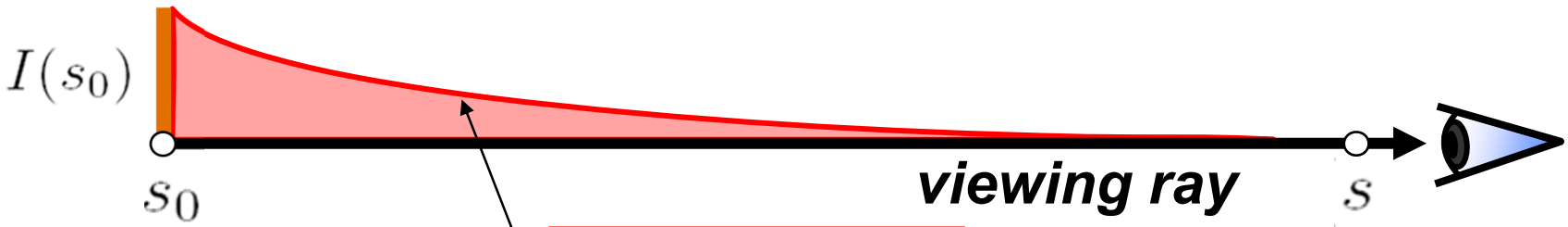
$$I(s) = I(s_0)$$

Without absorption all the initial radiant energy would reach the point ***s.***

Volume Rendering Integral

How do we determine the radiant energy along the ray?

Physical model: emission and absorption, no scattering



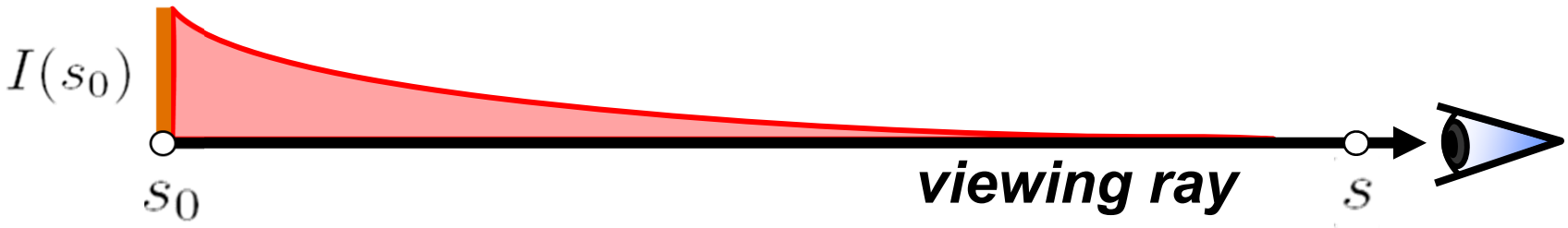
Absorption along the ray segment s_0 - s

$$I(s) = I(s_0)\, e^{-\tau(s_0,s)}$$

Volume Rendering Integral

How do we determine the radiant energy along the ray?

Physical model: emission and absorption, no scattering



$$I(s) = I(s_0)\, e^{-\tau(s_0,s)}$$

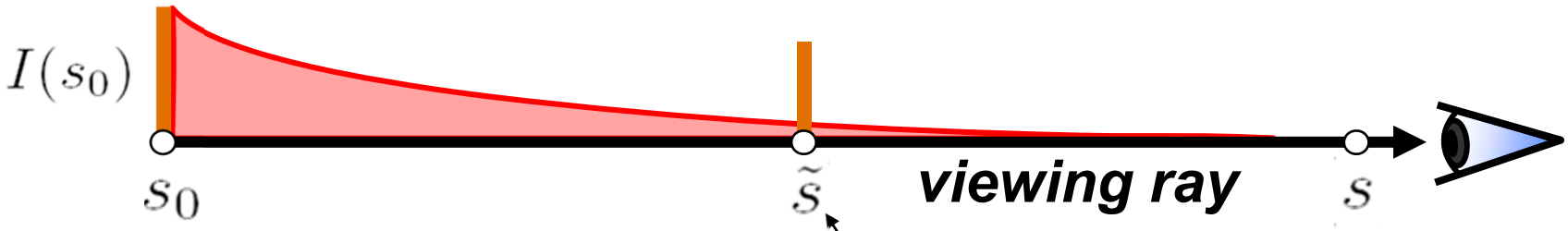
Optical depth τ
Absorption κ

$$\tau(s_1, s_2) = \int_{s_1}^{s_2} \kappa(s)\, ds.$$

Volume Rendering Integral

How do we determine the radiant energy along the ray?

Physical model: emission and absorption, no scattering



One point $\tilde{s}$ along the viewing ray emits additional radiant energy.

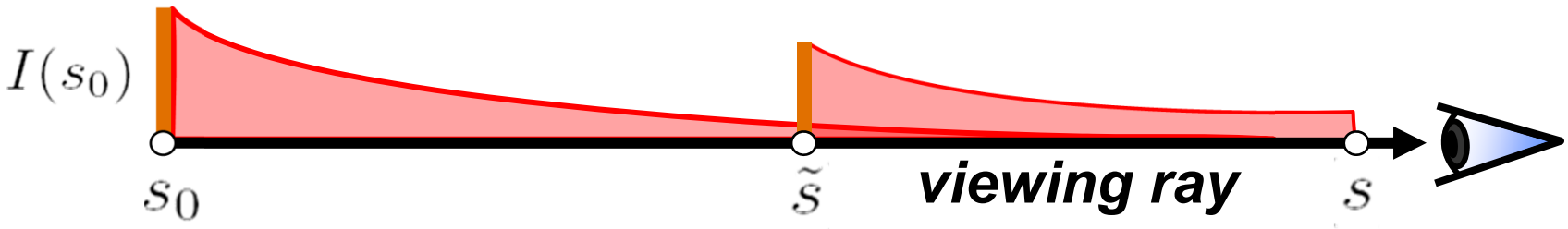
Active emission at point $\tilde{s}$

$$I(s) = I(s_0)\, e^{-\tau(s_0,s)} + q(\tilde{s})$$

Volume Rendering Integral

How do we determine the radiant energy along the ray?

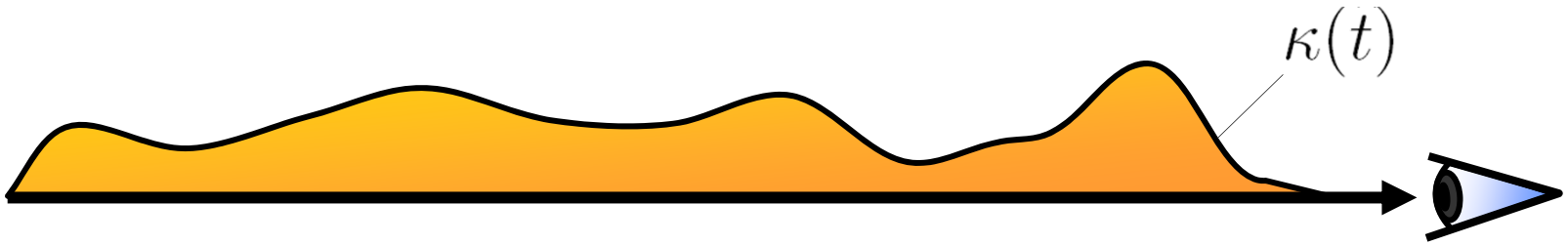
Physical model: emission and absorption, no scattering



Every point $\tilde{s}$ along the viewing ray emits additional radiant energy

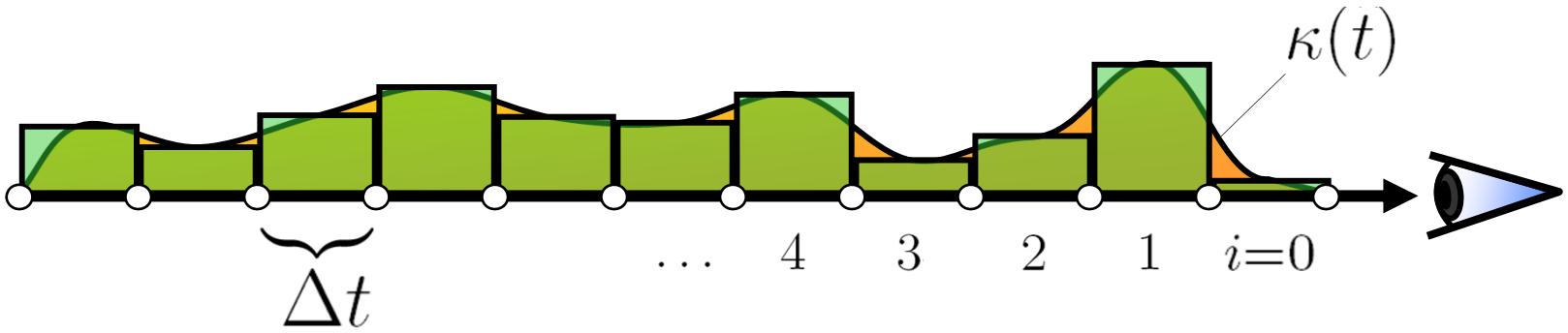
$$I(s) = I(s_0)\, e^{-\tau(s_0,s)} + \int_{s_0}^{s} q(\tilde{s})\, e^{-\tau(\tilde{s},s)}\, d\tilde{s}$$

Volume Rendering Integral: Numerical Solution



Optical depth: $\tau(0,t) = \int_0^t \kappa(\hat{t})\, d\hat{t}$

Volume Rendering Integral: Numerical Solution

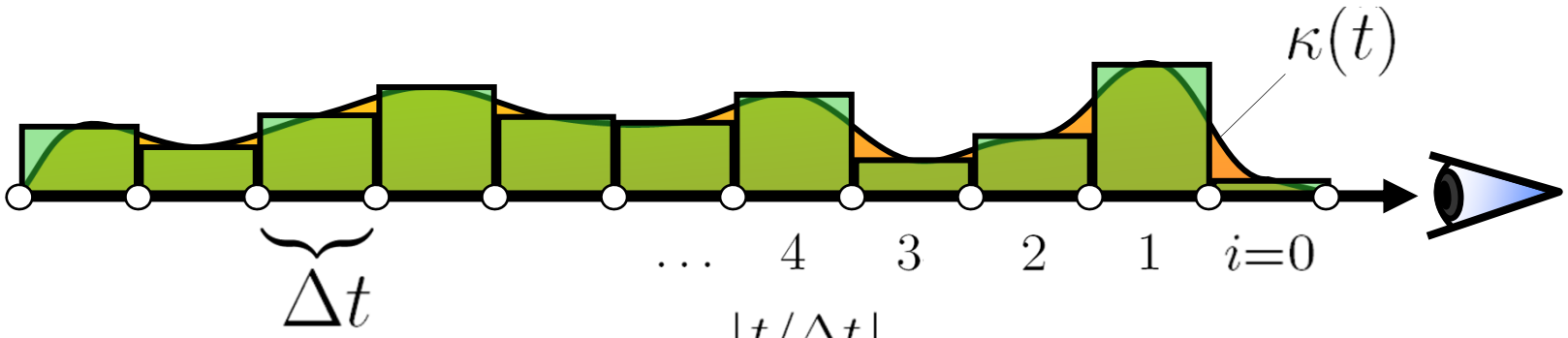


Optical depth: $\tau(0,t) = \int_0^t \kappa(\hat{t})\, d\hat{t}$

Approximate Integral by Riemann sum:

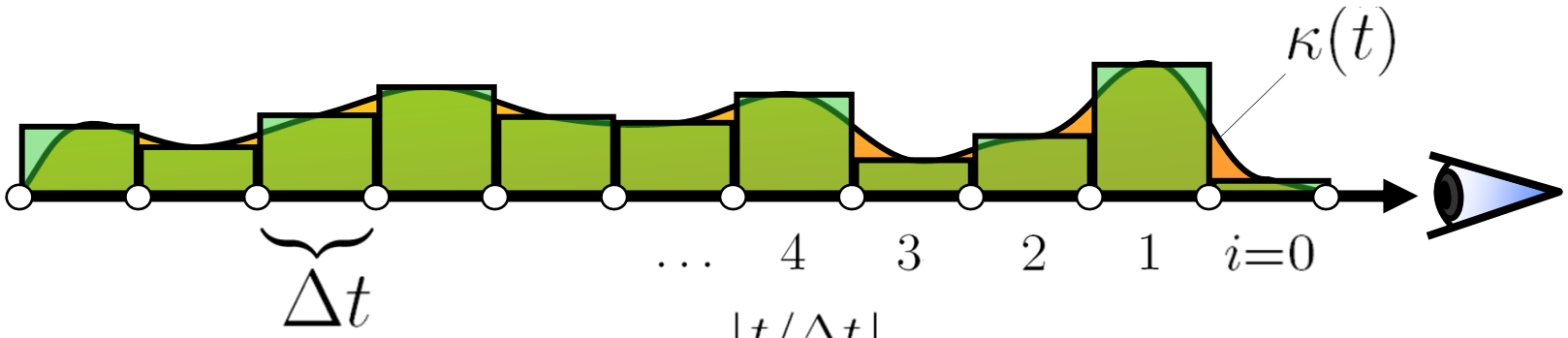
$$\tau(0,t) \approx \sum_{i=0}^{\lfloor t/\Delta t \rfloor} \kappa(i \cdot \Delta t)\, \Delta t$$

Volume Rendering Integral: Numerical Solution



$$\tau(0,t) \approx \tilde{\tau}(0,t) = \sum_{i=0}^{\lfloor t/\Delta t \rfloor} \kappa(i \cdot \Delta t)\, \Delta t$$

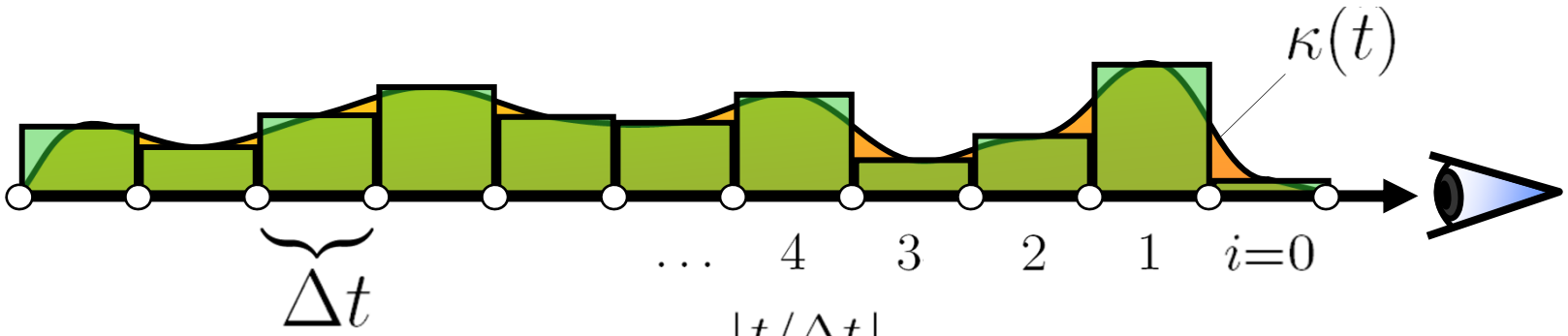
Volume Rendering Integral: Numerical Solution



$$\tau(0,t) \approx \tilde{\tau}(0,t) = \sum_{i=0}^{\lfloor t/\Delta t \rfloor} \kappa(i \cdot \Delta t)\, \Delta t$$

$$e^{-\tilde{\tau}(0,t)} = e^{-\sum_{i=0}^{\lfloor t/\Delta t \rfloor} \kappa(i \cdot \Delta t)\, \Delta t}$$

Volume Rendering Integral: Numerical Solution



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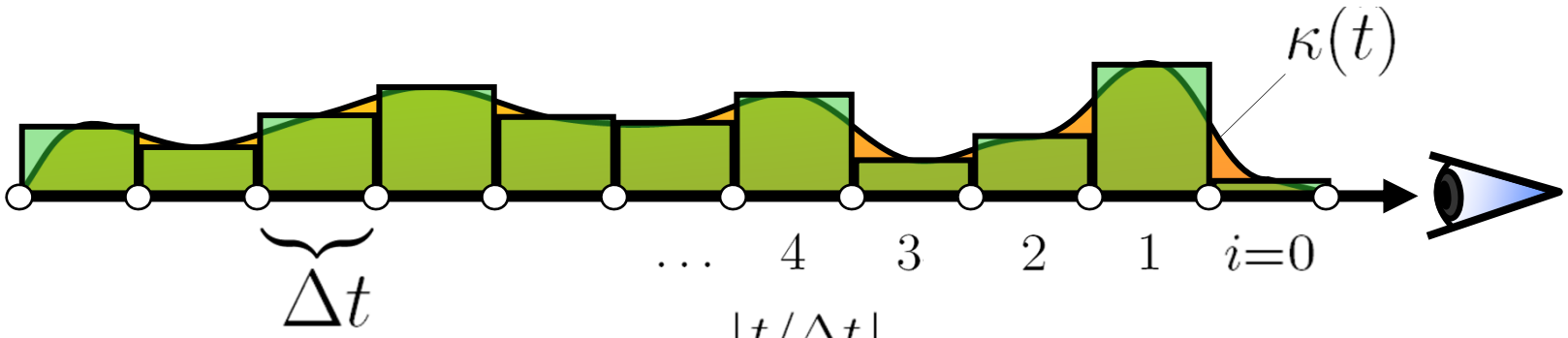
Volume Rendering Integral: Numerical Solution



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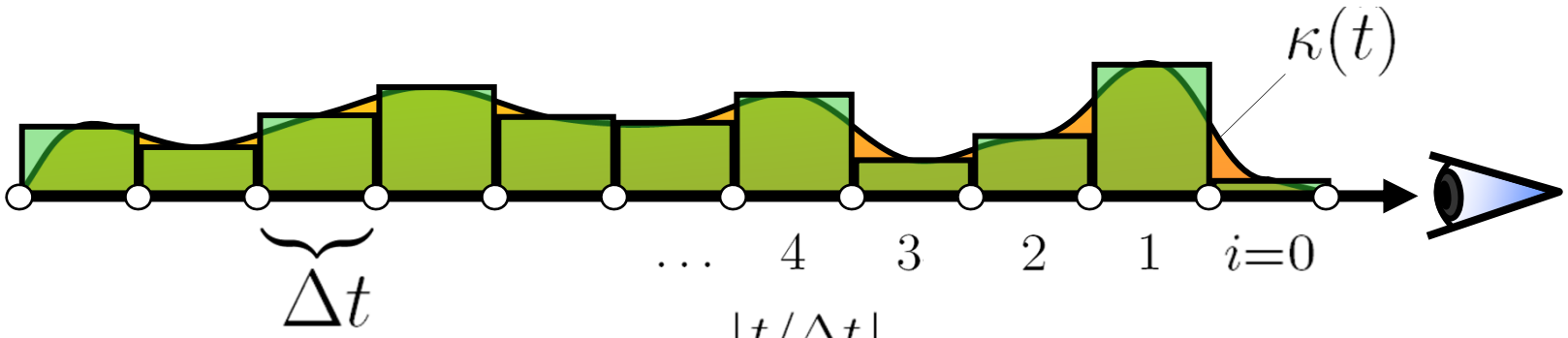
Now we introduce *opacity*:

$$A_i = 1 - e^{-\kappa(i \cdot \Delta t)\, \Delta t}$$

Volume Rendering Integral: Numerical Solution



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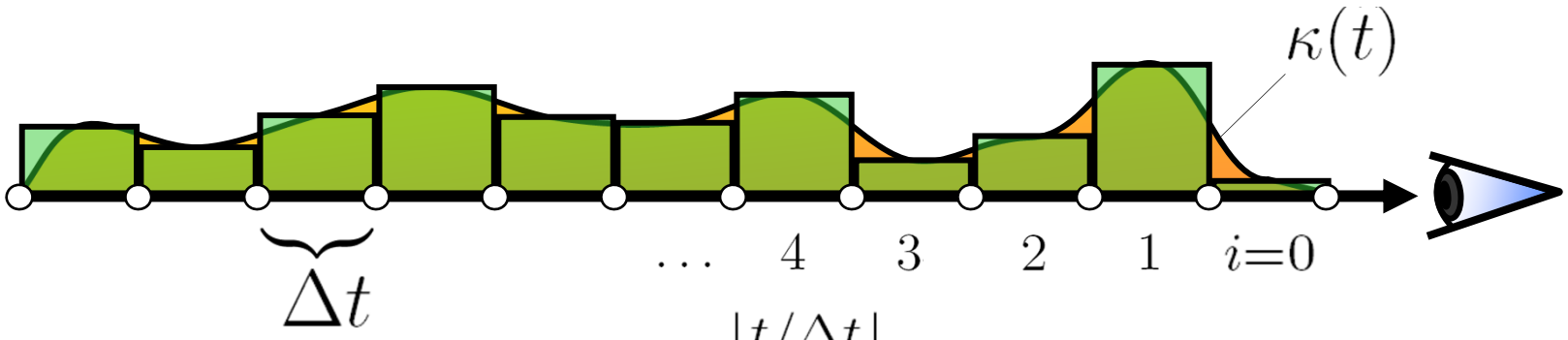
Volume Rendering Integral: Numerical Solution



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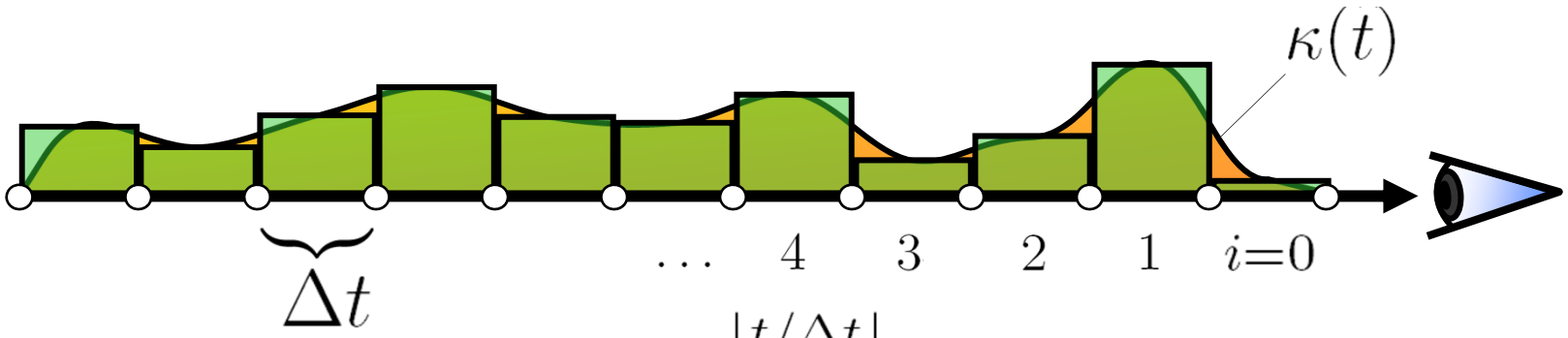

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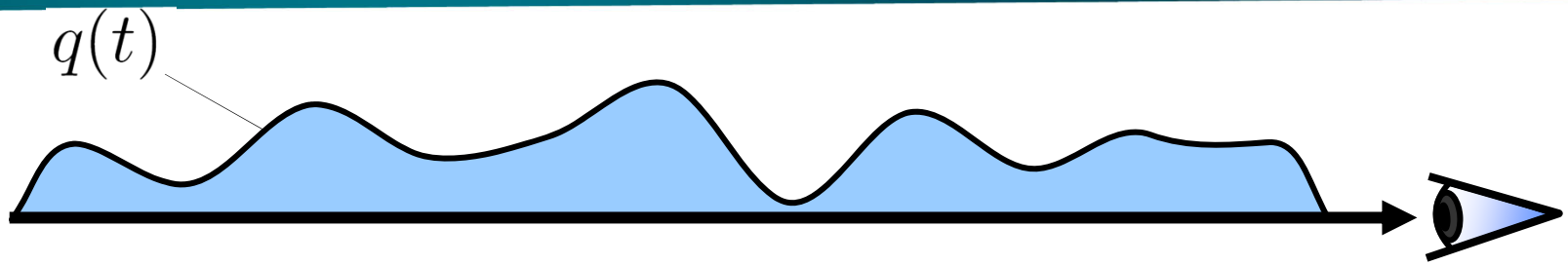
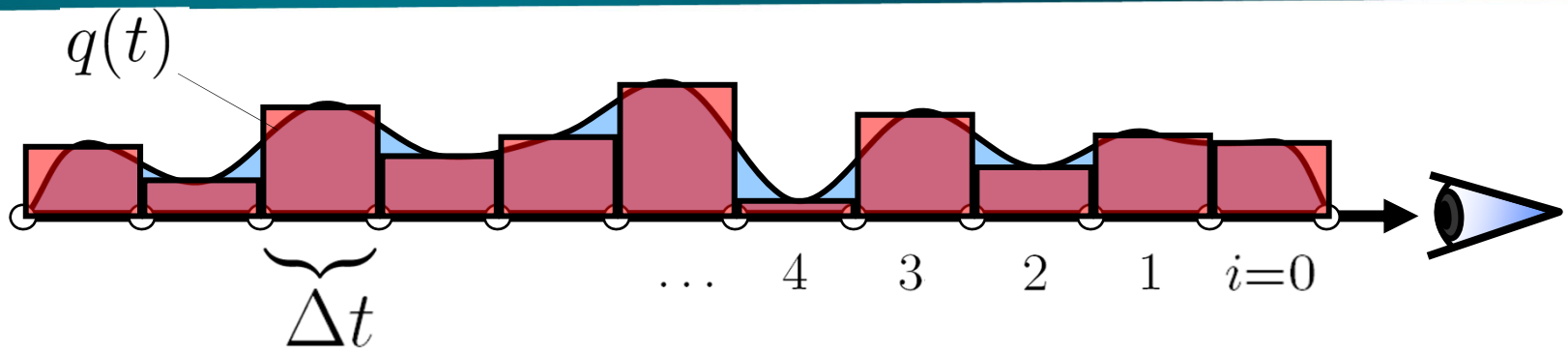
Volume Rendering Integral: Numerical Solution



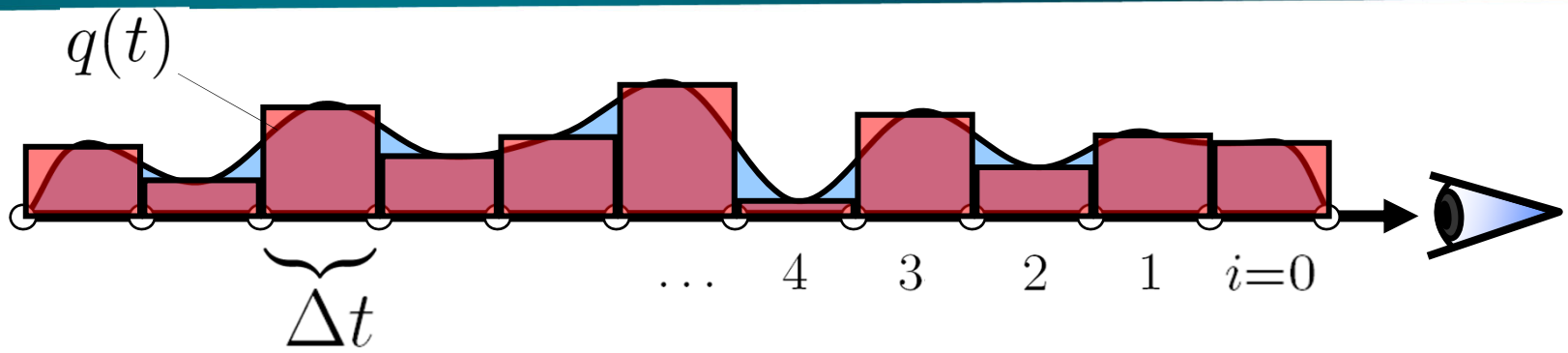
Volume Rendering Integral: Numerical Solution



$$e^{-\tilde{\tau}(0,t)} = \prod_{i=0}^{\lfloor t/\Delta t \rfloor} (1 - A_i)$$

$$q(t) \approx C_i = c(i \cdot \Delta t)\,\Delta t$$

Volume Rendering Integral: Numerical Solution



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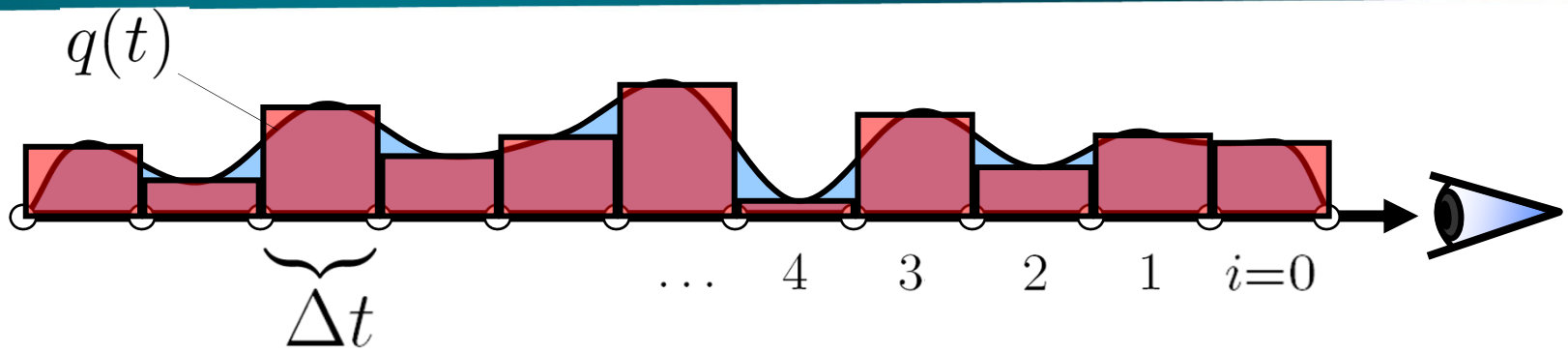
$$\tilde{C} = \sum_{i=0}^{\lfloor T/\Delta t \rfloor} C_i\, e^{-\tilde{\tau}(0,t)}$$

Volume Rendering Integral: Numerical Solution



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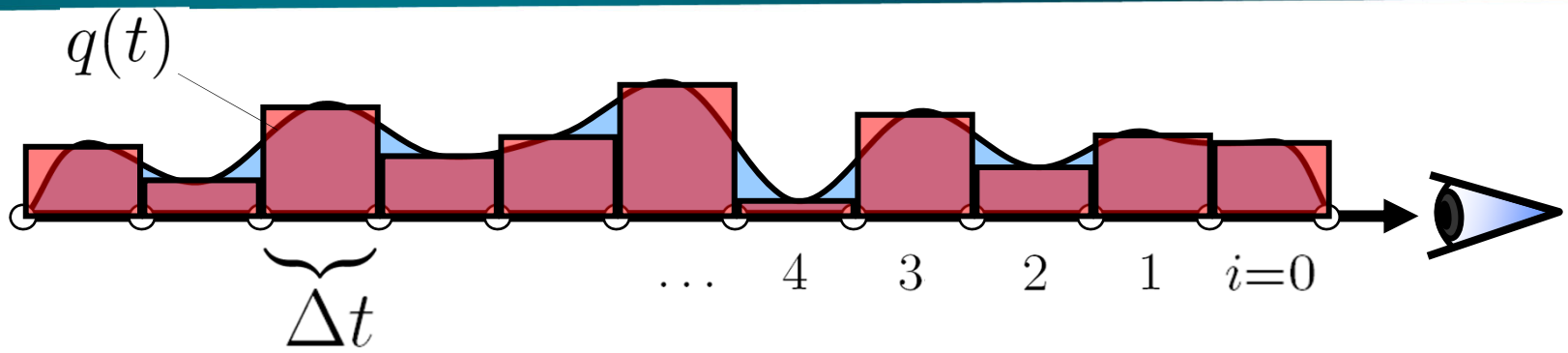
Volume Rendering Integral: Numerical Solution



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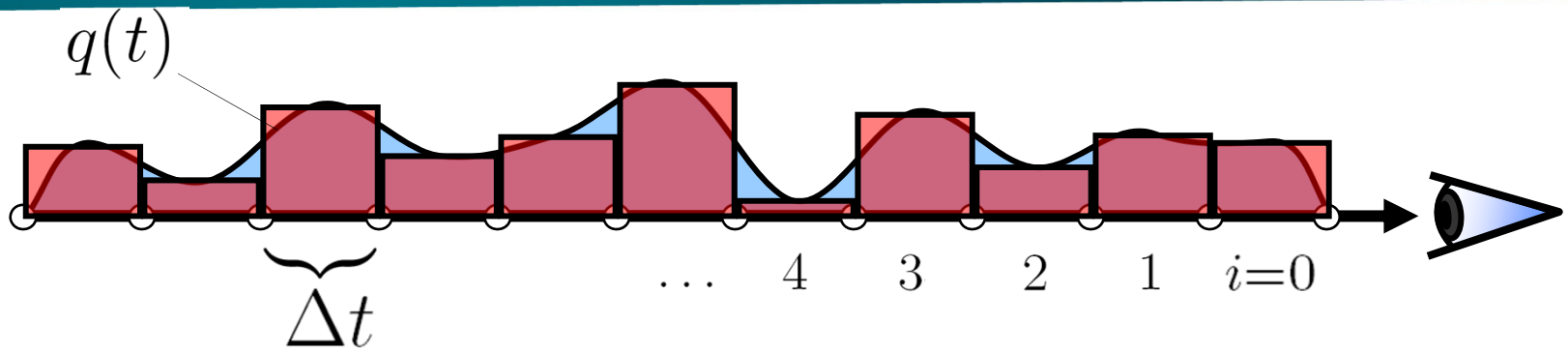
Volume Rendering Integral: Numerical Solution



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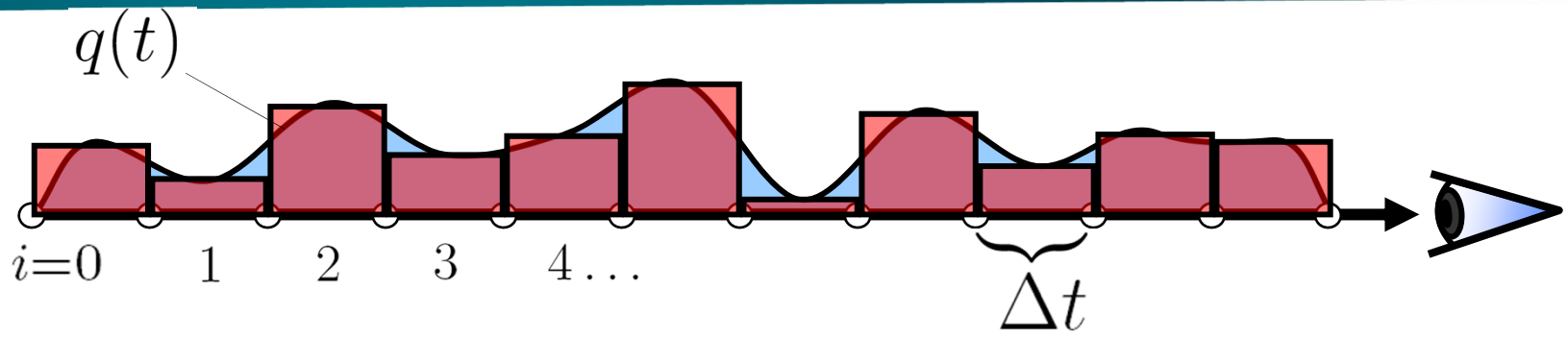
$$\tilde{C} = \sum_{i=0}^{\lfloor T/\Delta t \rfloor} C_i \prod_{j=0}^{i-1} (1 - A_j)$$

can be computed recursively/iteratively!

Volume Rendering Integral: Numerical Solution



Note: we just changed the convention from i=0 is at the front of the volume (previous slides) to i=0 is at the back of the volume !

can be computed recursively/iteratively:

$$C'_i = C_i + (1 - A_i)C'_{i-1}$$

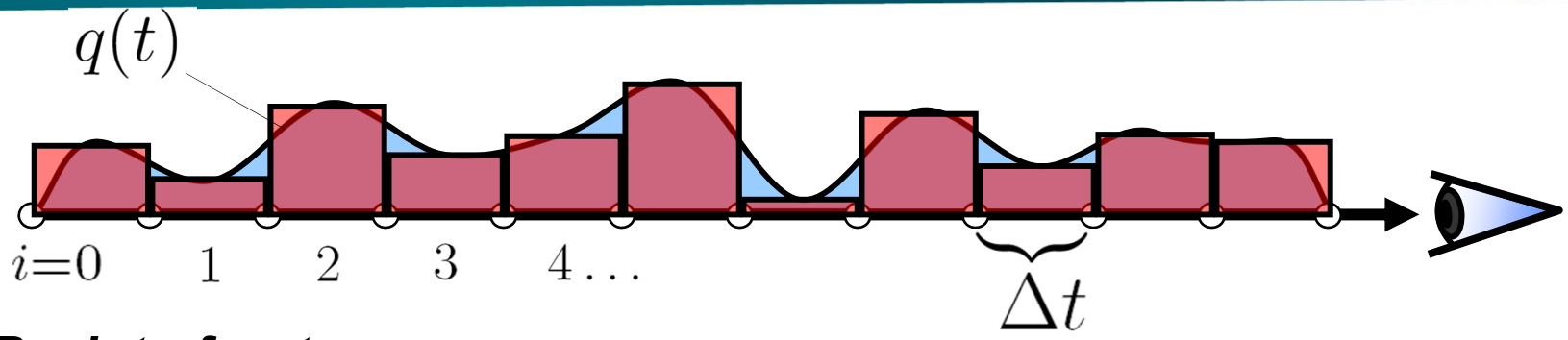
Radiant energy observed at position ***i***

Radiant energy emitted at position ***i***

Absorption at position ***i***

Radiant energy observed at position ***i–1***

Volume Rendering Integral: Numerical Solution



Back-to-front compositing

$$C'_i = C_i + (1 - A_i)C'_{i-1}$$

Iterate from *i*=0 (back) to *i*=max (front): *i* increases

Front-to-back compositing

$$C'_i = C'_{i+1} + (1 - A'_{i+1})C_i$$
$$A'_i = A'_{i+1} + (1 - A'_{i+1})A_i$$

Iterate from *i*=max (front) to *i*=0 (back) : *i* decreases

Volume Rendering Integral Summary

Volume rendering integral
for *Emission Absorption* model

true emission **true *absorption***

$$I(s) = I(s_0)\, e^{-\tau(s_0,s)} + \int_{s_0}^{s} q(\tilde{s})\, e^{-\tau(\tilde{s},s)}\, \mathrm{d}\tilde{s}$$

Numerical solutions:

Back-to-front compositing

$$C'_i = C_i + (1 - A_i)C'_{i-1}$$

Front-to-back compositing

$$C'_i = C'_{i+1} + (1 - A'_{i+1})C_i$$
$$A'_i = A'_{i+1} + (1 - A'_{i+1})A_i$$

Opacity Correction

Simple compositing only works as far as the opacity values are correct… and they depend on the sample distance!

$$T_i = e^{-\int_{s_i}^{s_i+\Delta x} \kappa \, \mathrm{d}s} \approx e^{-\kappa(s_i)\Delta x} \qquad \tilde{T} = T^{\left(\frac{\Delta \tilde{x}}{\Delta x}\right)}$$

Opacity correction formula:

$$A_i = 1 - e^{-\kappa(s_i)\Delta x}$$

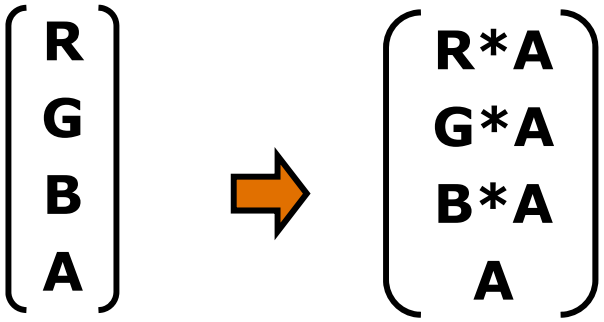
$$\tilde{A}_i = 1 - (1 - A_i)^{\left(\frac{\Delta \tilde{x}}{\Delta x}\right)}$$

Beware that usually this is done *for each different scalar value* (every transfer function entry), not actually at spatial positions/intervals *i*

Associated Colors

Associated (or “opacity-weighted” colors) are often used in compositing equations

Every color is *pre-multiplied* by its corresponding opacity



Our compositing equations assume associated colors!

Important:

After opacity-correction, all associated colors must be updated!

Associated Colors in Volume Rendering

(Standard) emission-absorption optical model

- Only one kind of particle: the same particles that absorb light, emit light
- Aha! Therefore lower absorption means lower emission as well

Light observed from (in front of) segment i:

$$\frac{q_i}{\kappa_i}\left(1 - e^{-\kappa_i \Delta t}\right)$$

$$\lim_{\kappa \to 0}(1 - e^{-\kappa \Delta t})/\kappa = \Delta t$$

$$\lim_{\kappa \to \infty}(1 - e^{-\kappa \Delta t})/\kappa = 0$$

$$= C_i A_i \qquad A_i := 1 - e^{-\kappa_i \Delta t}$$

$$q_i := C_i \kappa_i$$

Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama