King Abdullah University of Science and Technology

KAUST

CS 247 – Scientific Visualization
Lecture 13: Scalar Fields, Pt.9 [preview]

Markus Hadwiger, KAUST

Reading Assignment #7 (until Mar 14)

Read (required):

- Real-Time Volume Graphics, Chapter 1
  (*Theoretical Background and Basic Approaches*),
  from beginning to 1.4.4 (inclusive)

Read (optional):

- Paper:
  Nelson Max, Optical Models for Direct Volume Rendering,
  IEEE Transactions on Visualization and Computer Graphics, 1995
  `http://dx.doi.org/10.1109/2945.468400`

Interlude: Tensor Calculus

In tensor calculus, first-order tensors can be

- Contravariant $\mathbf{v} = v^i \, \mathbf{e}_i$
- Covariant $\boldsymbol{\omega} = v_i \, \boldsymbol{\omega}^i$

The gradient vector is a contravariant vector $\mathbf{v} = v^i \, \boldsymbol{\partial}_i$

The gradient 1-form is a covariant vector (a covector) $df = \frac{\partial f}{\partial x^i} dx^i$

Very powerful; necessary for non-Cartesian coordinate systems

On (intrinsically) curved manifolds (sphere, ...):
Cartesian coordinates not even possible

Interlude: Tensor Calculus

In tensor calculus, first-order tensors can be

- Contravariant $\mathbf{v} = v^i \, \mathbf{e}_i$
- Covariant $\boldsymbol{\omega} = v_i \, \boldsymbol{\omega}^i$

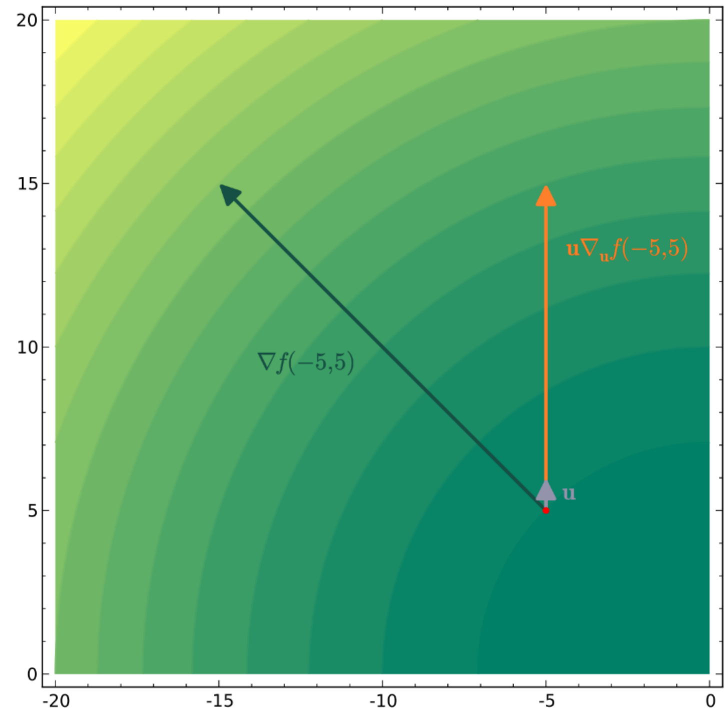
The gradient vector is a contravariant vector $\mathbf{v} = v^i \, \boldsymbol{\partial}_i$

The gradient 1-form is a covariant vector (a covector) $df = \frac{\partial f}{\partial x^i} dx^i$

This is also the fundamental reason why in graphics a normal vector transforms differently: as a covector, not as a vector!

(typical graphics rule: **n** transforms with transpose of inverse matrix)

Gradient Vectors and Differential 1-Forms



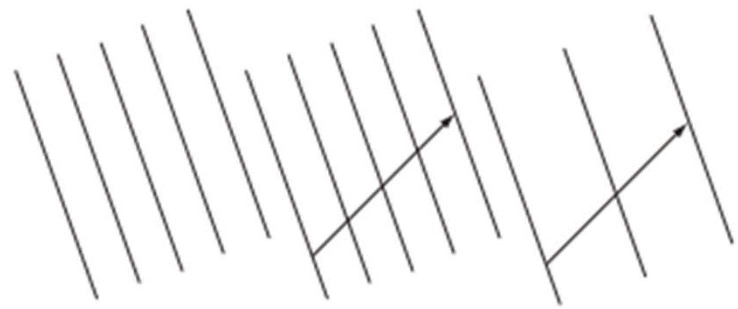
from Wikipedia (for **u** a unit vector),

the function here is $f(r,\theta) = r^2$

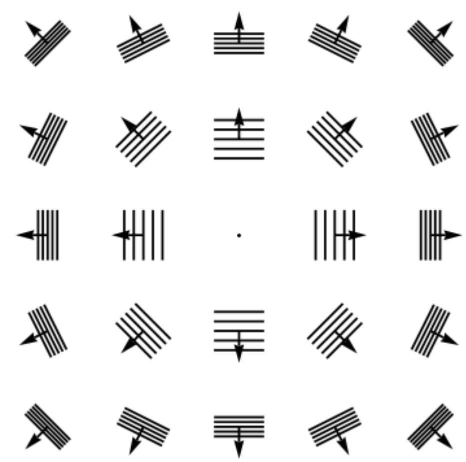
$$\nabla f(r,\theta) = 2r\,\mathbf{e}_r + 0\frac{1}{r^2}\,\mathbf{e}_\theta = 2r\,\mathbf{e}_r$$

$$df(r,\theta) = 2r\,dr + 0\,d\theta = 2r\,dr$$

different 1-forms
evaluated in some direction



1-form (field) df



Gradient Vector from Differential 1-Form

The metric (and inverse metric) *lower* or *raise* indices
(i.e., convert between covariant and contravariant tensors)

$$\boxed{\begin{aligned} v^i &= g^{ij} v_j \\ v_i &= g_{ij} v^j \end{aligned}}$$

$$v^i \mathbf{e}_i = g^{ij} v_j \mathbf{e}_i$$

$$v_i \boldsymbol{\omega}^i = g_{ij} v^j \boldsymbol{\omega}^i$$

Inverse metric (contravariant)

$$\boxed{[g^{ij}] = [g_{ij}]^{-1}}$$

$$g^{ik} g_{kj} = \delta^i_j$$

Kronecker delta behaves
like identity matrix

Gradient Vector from Differential 1-Form

So the gradient vector is

$$\boxed{\nabla f = \left(g^{ij} \frac{\partial f}{\partial x^j} \right) \mathbf{e}_i}$$

Vector-valued 1-form

$$d\mathbf{r} = dx^i \, \mathbf{e}_i$$

$$d\mathbf{r}(\cdot) = dx^i(\cdot) \, \mathbf{e}_i$$

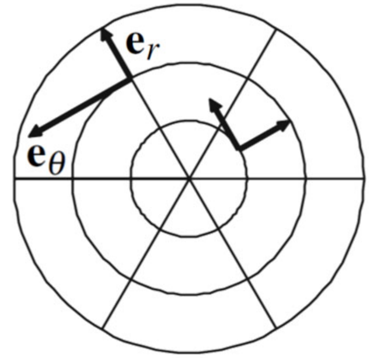
Directional derivative via inner product:

$$\begin{aligned} \langle \nabla f, \cdot \rangle &= g_{kj} g^{ik} \frac{\partial f}{\partial x^i} dx^j(\cdot) \\ &= \delta^i{}_j \frac{\partial f}{\partial x^i} dx^j(\cdot) \\ &= \frac{\partial f}{\partial x^i} dx^i(\cdot) \end{aligned}$$

$$\begin{aligned} \nabla f \cdot d\mathbf{r} &= g_{kj} g^{ik} \frac{\partial f}{\partial x^i} dx^j \\ &= \delta^i{}_j \frac{\partial f}{\partial x^i} dx^j \\ &= \frac{\partial f}{\partial x^i} dx^i \end{aligned}$$

Example: Polar Coordinates

Metric tensor and inverse metric for polar coordinates



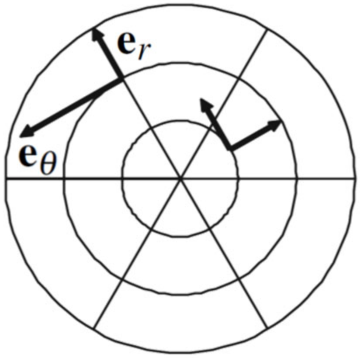
$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \qquad g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}$$

Gradient vector from 1-form: raise index with inverse metric

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{bmatrix} = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$$

Example: Polar Coordinates

Metric tensor and inverse metric for polar coordinates



$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \qquad g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}$$

Gradient vector from 1-form: raise index with inverse metric

$$\nabla f(r,\theta) = \frac{\partial f(r,\theta)}{\partial r}\,\mathbf{e}_r(r,\theta) + \frac{1}{r^2}\,\frac{\partial f(r,\theta)}{\partial \theta}\,\mathbf{e}_\theta(r,\theta)$$

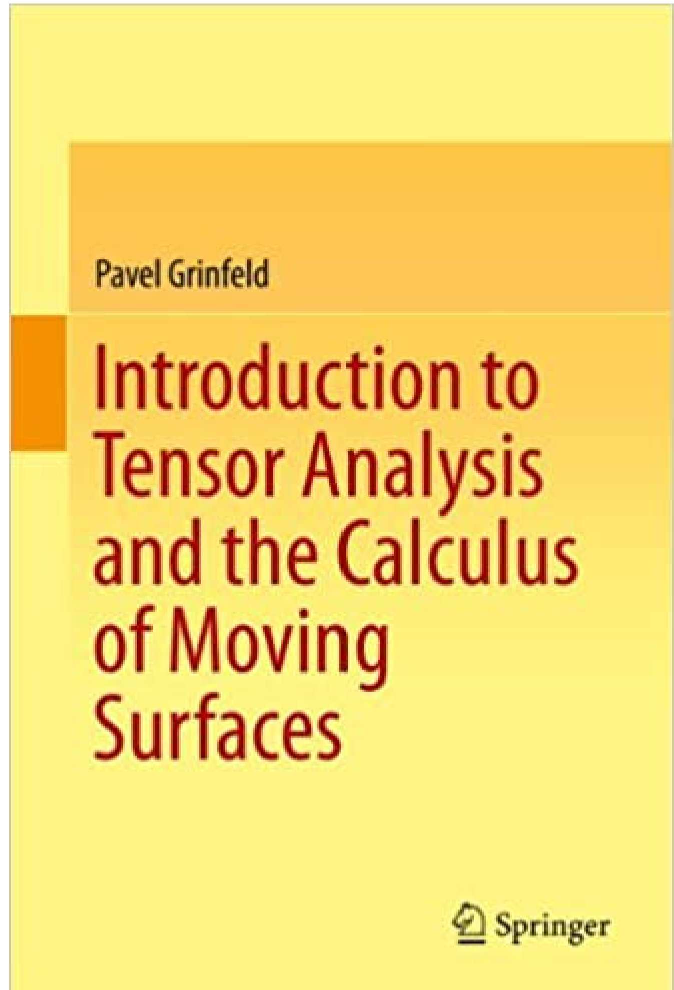
don’t forget that all of this is position-dependent!

Tensor Calculus

Highly recommended:

Very nice book,

complete lecture on Youtube!



Multi-Linear Interpolation

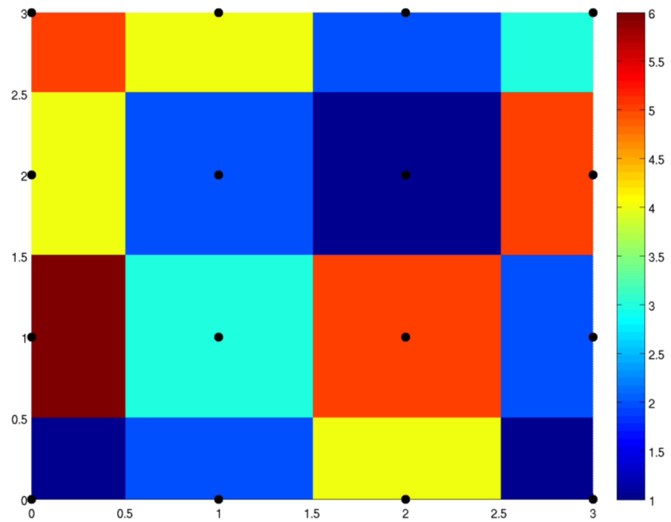
Bi-linear Filtering Example (Magnification)

Original image

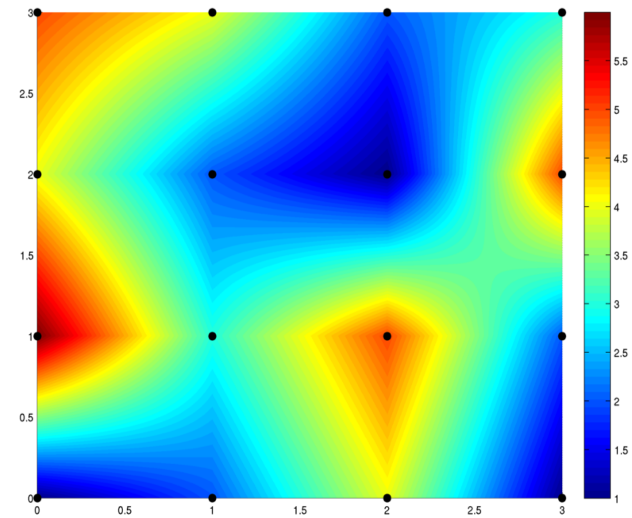
Nearest neighbor

Bi-linear filtering

Bi-Linear Interpolation vs. Nearest Neighbor



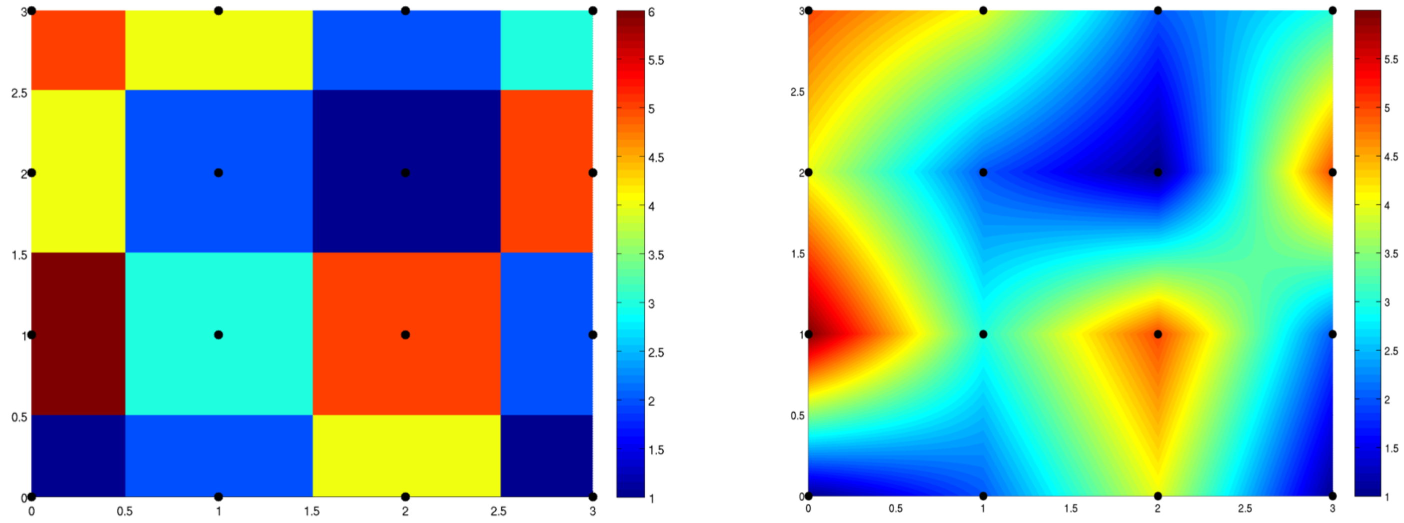
nearest-neighbor



bi-linear

wikipedia

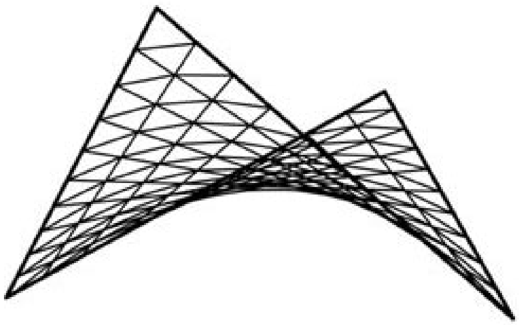
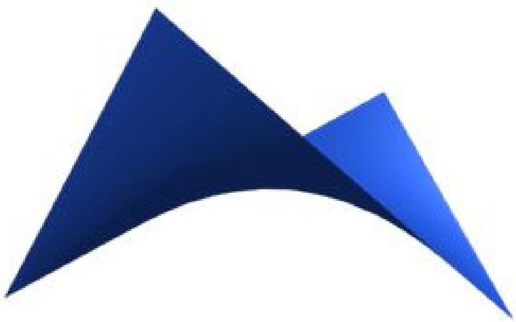
Bi-Linear Interpolation vs. Nearest Neighbor



nearest-neighbor bi-linear

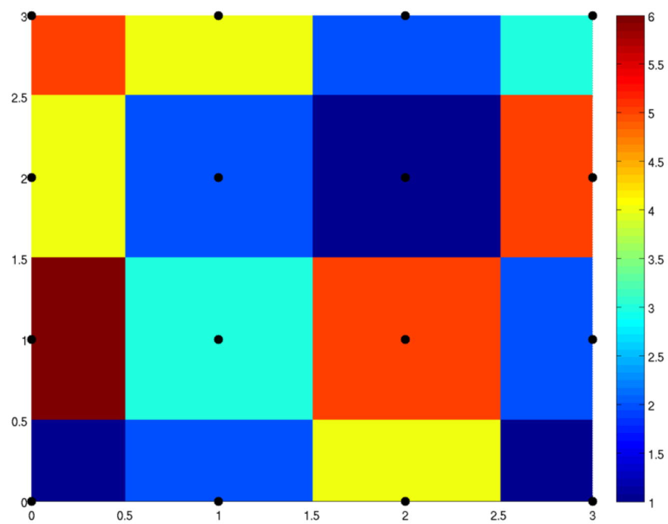
wikipedia

for surfaces,
height interpolation:

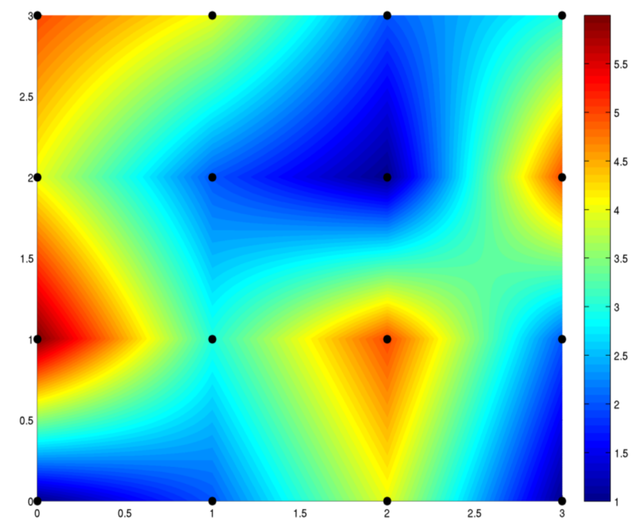



Bilinear patch (courtesy J. Han)

Bi-Linear Interpolation vs. Nearest Neighbor



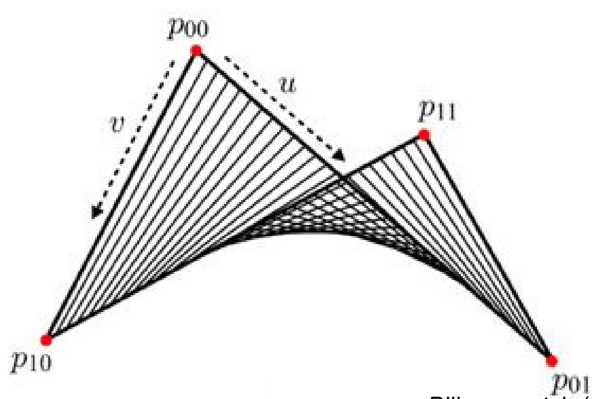
nearest-neighbor

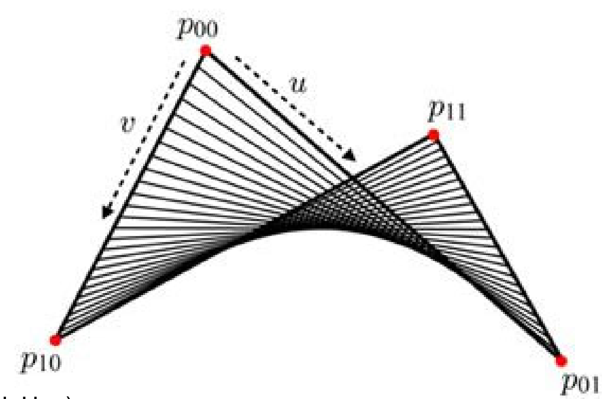


bi-linear

wikipedia

for surfaces,
height interpolation:



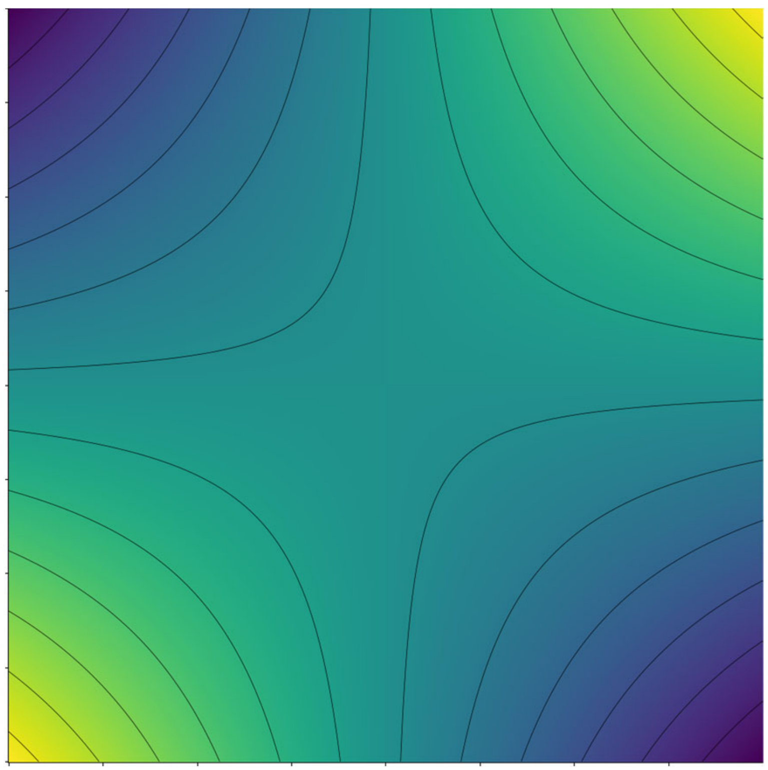


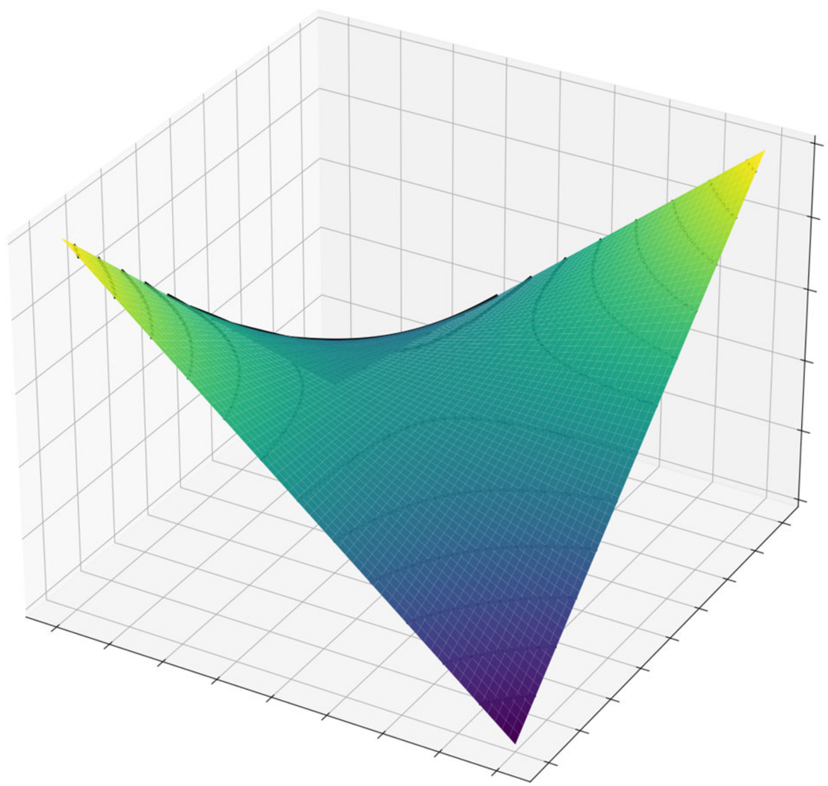
Bilinear patch (courtesy J. Han)

Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers)

Example #1: 1 at bottom-left and top-right, 0 at top-left and bottom-right

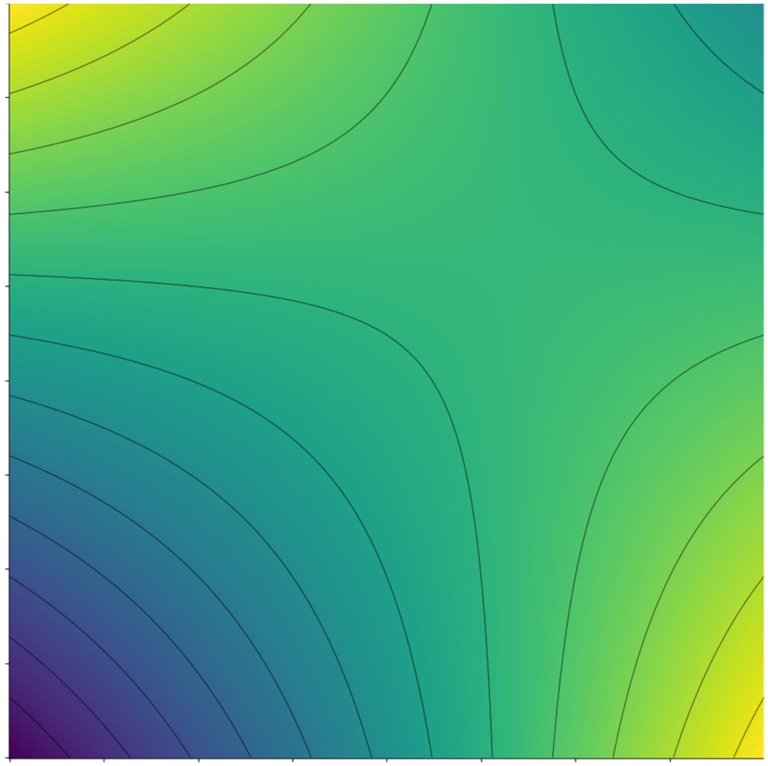


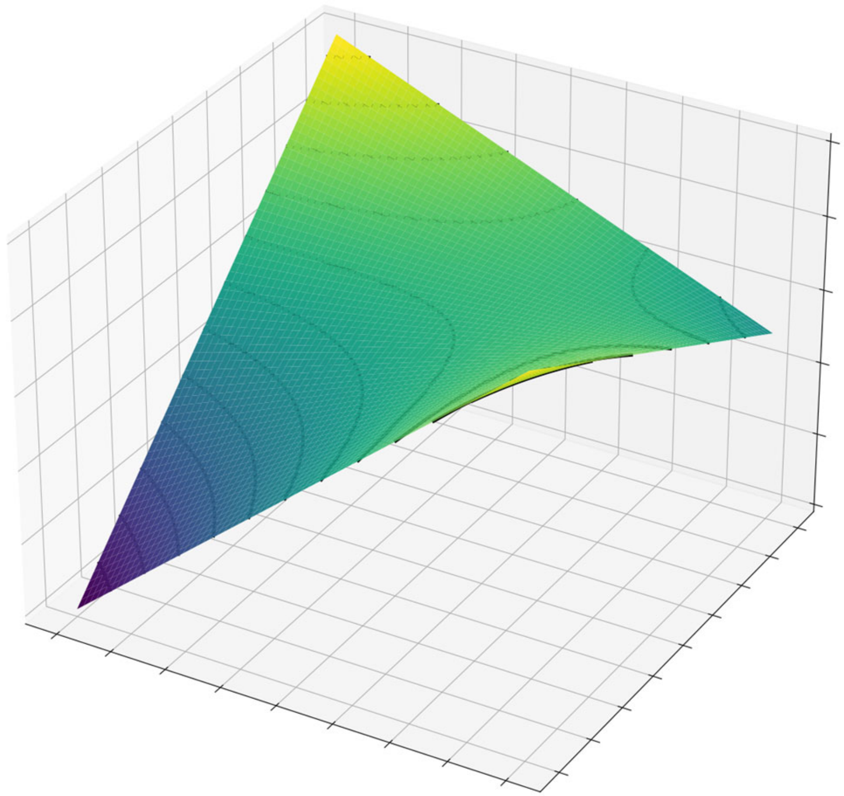


Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers)

Example #2: 1 at top-left and bottom-right, 0 at bottom-left, 0.5 at top-right





Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers):

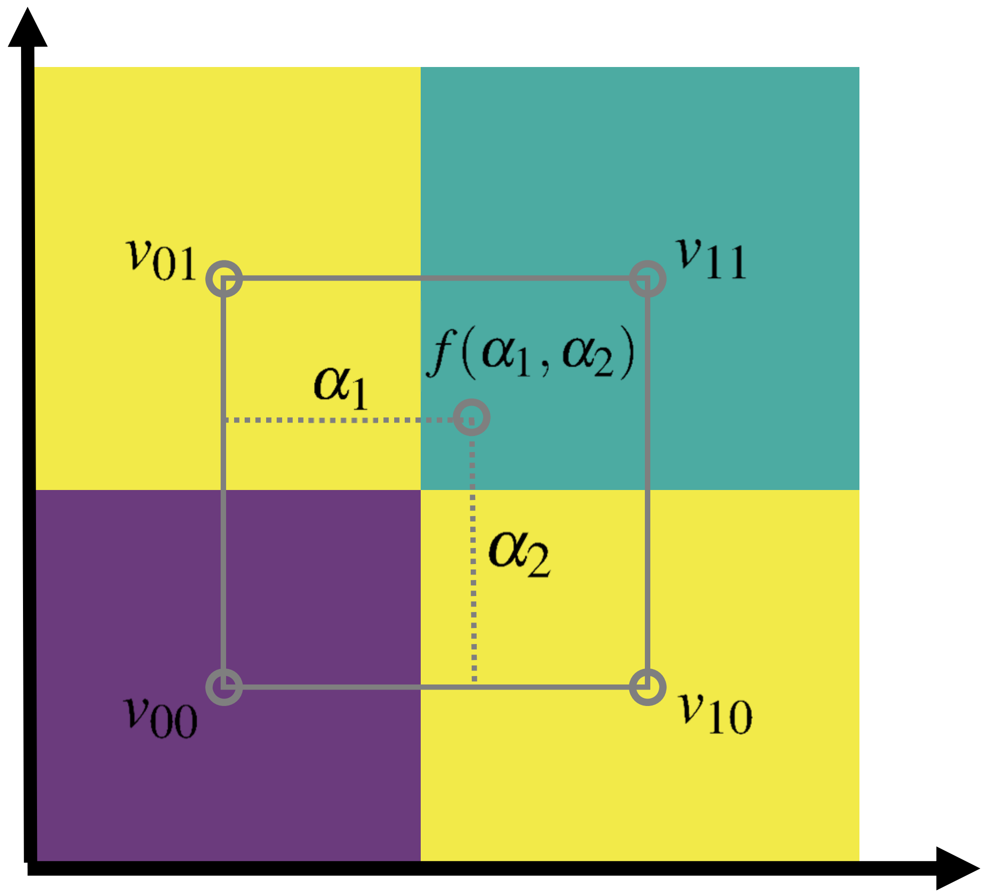
Given any (fractional) position

$$\alpha_1 := x_1 - \lfloor x_1 \rfloor \qquad \alpha_1 \in [0.0, 1.0)$$
$$\alpha_2 := x_2 - \lfloor x_2 \rfloor \qquad \alpha_2 \in [0.0, 1.0)$$

and 2x2 sample values

$$\begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix}$$

Compute: $f(\alpha_1, \alpha_2)$



Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers):

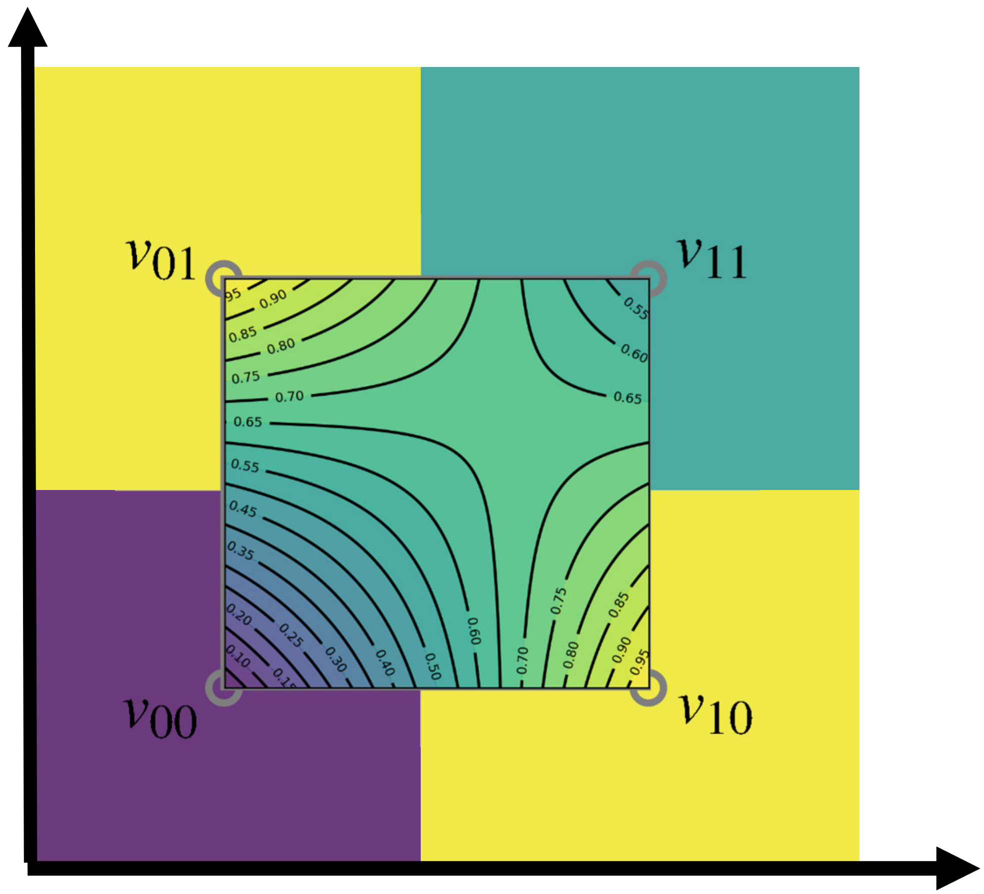
Given any (fractional) position

$$\alpha_1 := x_1 - \lfloor x_1 \rfloor \qquad \alpha_1 \in [0.0, 1.0)$$
$$\alpha_2 := x_2 - \lfloor x_2 \rfloor \qquad \alpha_2 \in [0.0, 1.0)$$

and 2x2 sample values

$$\begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix}$$

Compute: $f(\alpha_1, \alpha_2)$



Bi-Linear Interpolation

Interpolate function at (fractional) position (α_1, α_2) :

$$f(\alpha_1, \alpha_2) = \begin{bmatrix} \alpha_2 & (1-\alpha_2) \end{bmatrix} \begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix} \begin{bmatrix} (1-\alpha_1) \\ \alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_2 & (1-\alpha_2) \end{bmatrix} \begin{bmatrix} (1-\alpha_1)v_{01} + \alpha_1 v_{11} \\ (1-\alpha_1)v_{00} + \alpha_1 v_{10} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_2 v_{01} + (1-\alpha_2)v_{00} & \alpha_2 v_{11} + (1-\alpha_2)v_{10} \end{bmatrix} \begin{bmatrix} (1-\alpha_1) \\ \alpha_1 \end{bmatrix}$$

Bi-Linear Interpolation

Interpolate function at (fractional) position (α_1, α_2) :

$$f(\alpha_1, \alpha_2) = \begin{bmatrix} \alpha_2 & (1-\alpha_2) \end{bmatrix} \begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix} \begin{bmatrix} (1-\alpha_1) \\ \alpha_1 \end{bmatrix}$$

$$= (1-\alpha_1)(1-\alpha_2)v_{00} + \alpha_1(1-\alpha_2)v_{10} + (1-\alpha_1)\alpha_2 v_{01} + \alpha_1\alpha_2 v_{11}$$

$$= v_{00} + \alpha_1(v_{10} - v_{00}) + \alpha_2(v_{01} - v_{00}) + \alpha_1\alpha_2(v_{00} + v_{11} - v_{10} - v_{01})$$

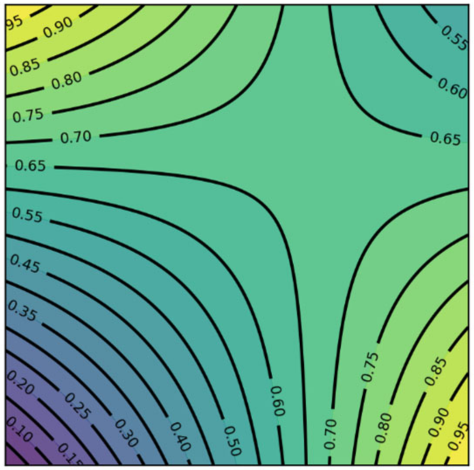
Bi-Linear Interpolation: Contours

Find one specific iso-contour (can of course do this for any/all isovalues):

$$f(\alpha_1, \alpha_2) = c$$

Find all (α_1, α_2) where:

$$v_{00} + \alpha_1 (v_{10} - v_{00}) + \alpha_2 (v_{01} - v_{00}) + \alpha_1 \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01}) = c$$



Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

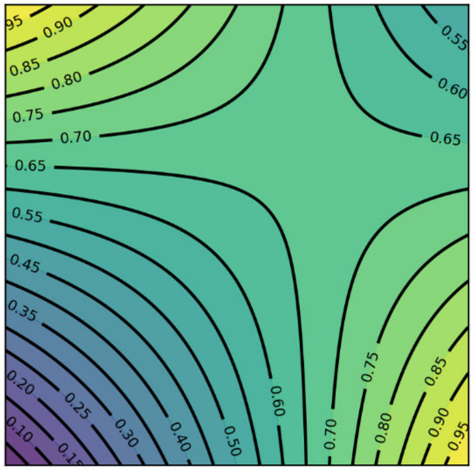
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

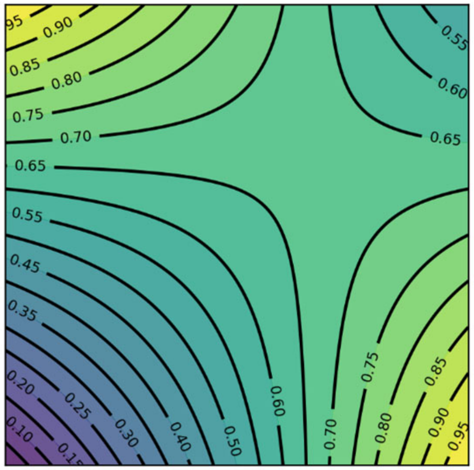
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



if denominator is zero, bi-linear interpolation has degenerated to linear interpolation (or const)! (also means: no isolated critical points!)

Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

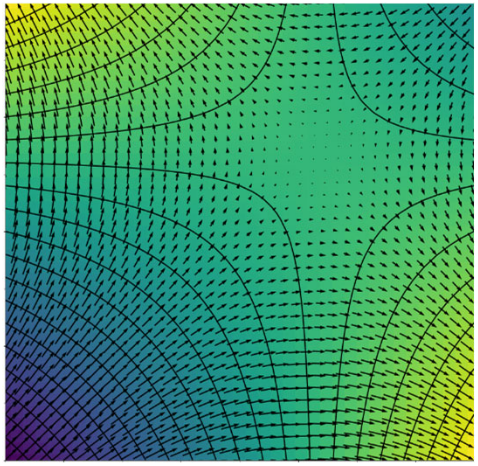
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



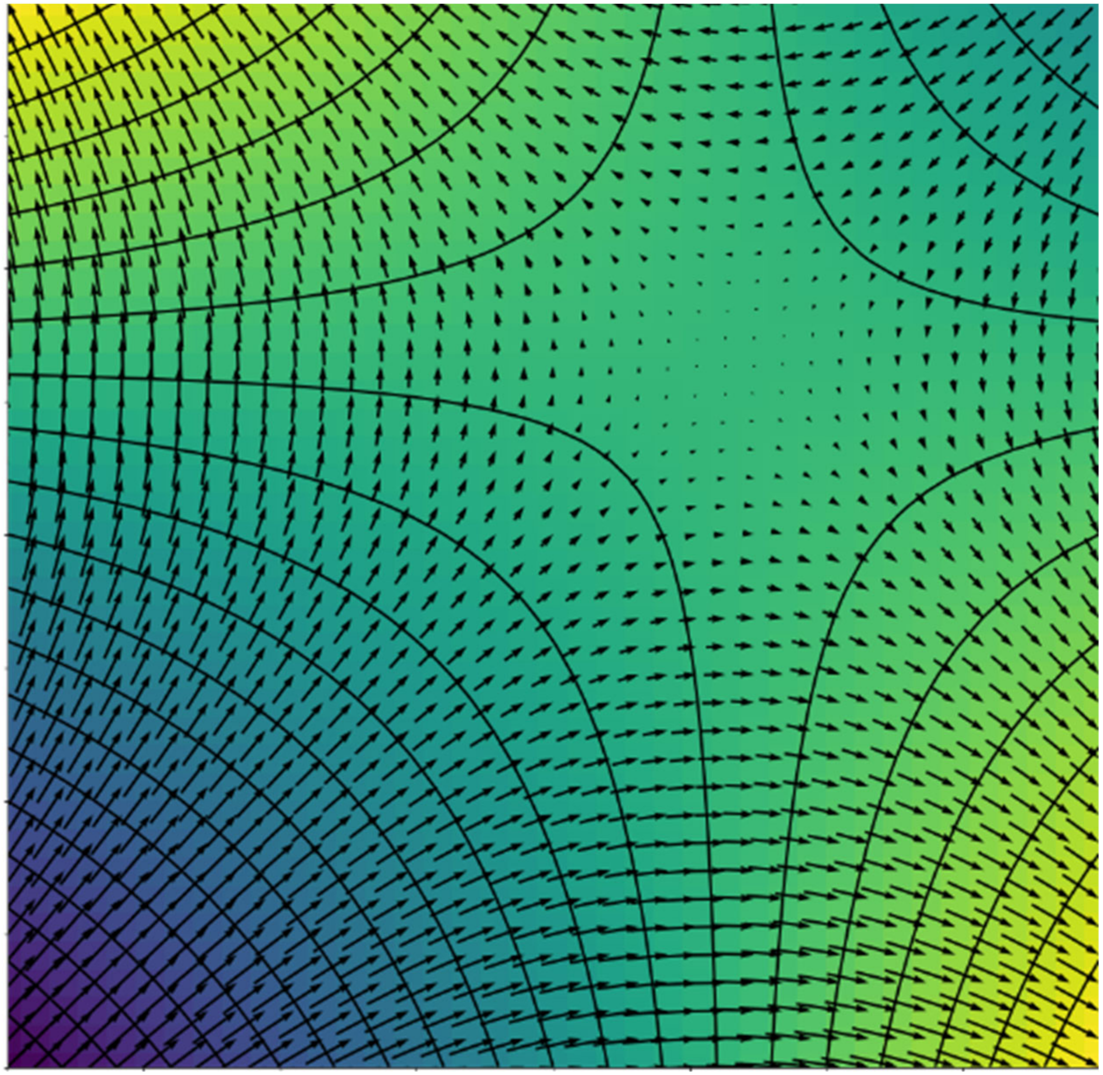
Bi-Linear Interpolation: Critical Points

Compute gradient

Note that isolines are
farther apart where
gradient is smaller

Note the horizontal and
vertical lines where
gradient becomes
vertical/horizontal

Note the critical point



Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

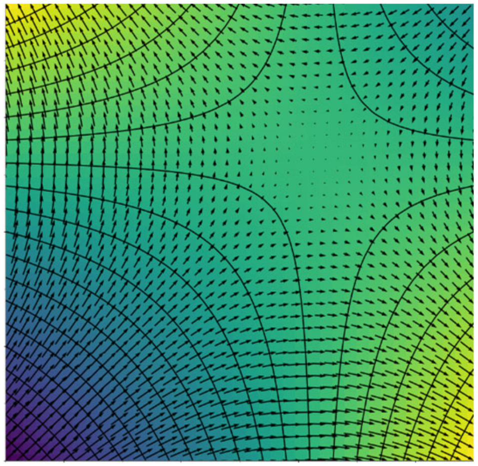
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \quad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \quad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

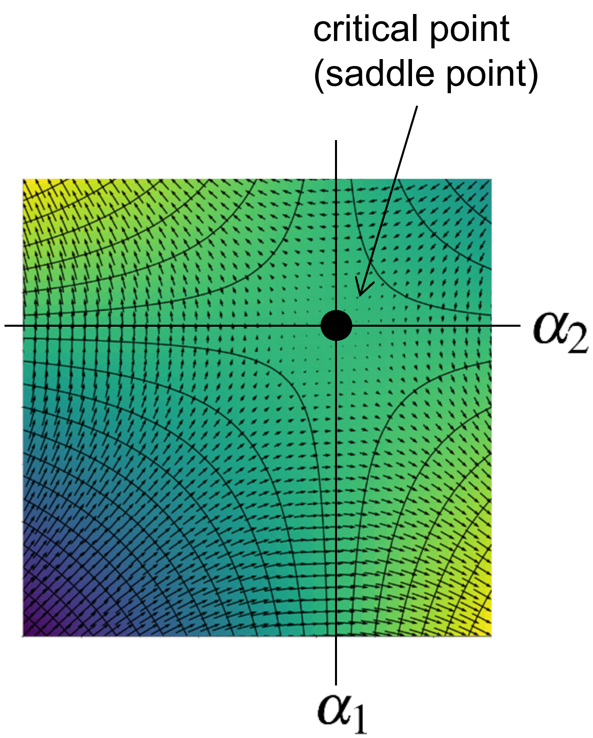
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

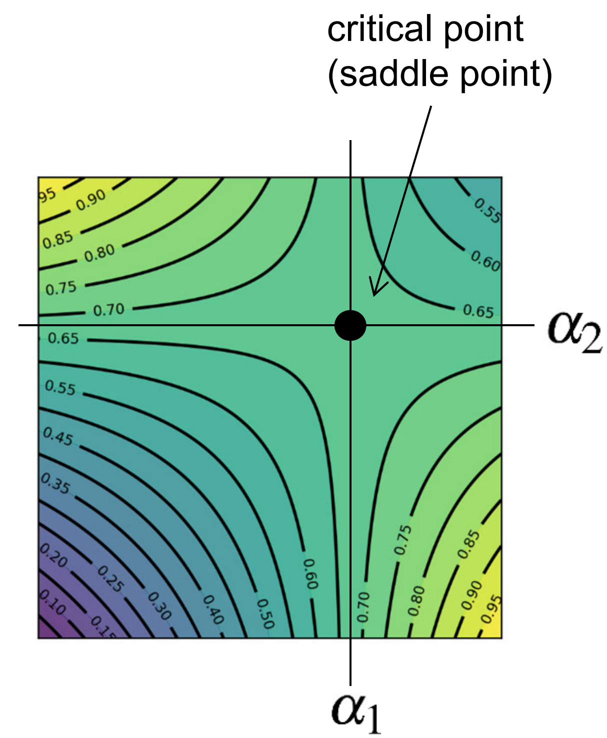
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



Bi-Linear Interpolation: Critical Points

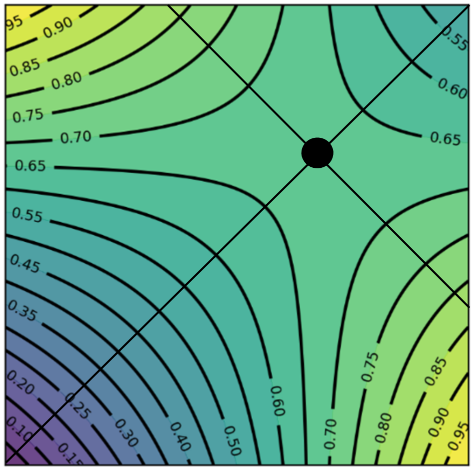
Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \qquad a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



(here also: principal curvature magnitudes and directions
of this function’s graph == surface embedded in 3D)

Bi-Linear Interpolation: Critical Points

Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

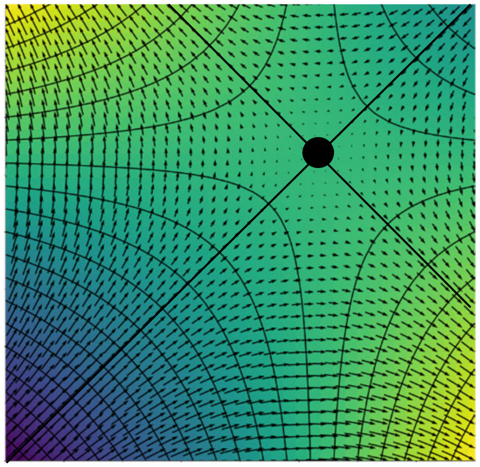
$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

$$a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



(here also: principal curvature magnitudes and directions
of this function’s graph == surface embedded in 3D)

Bi-Linear Interpolation: Critical Points

Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

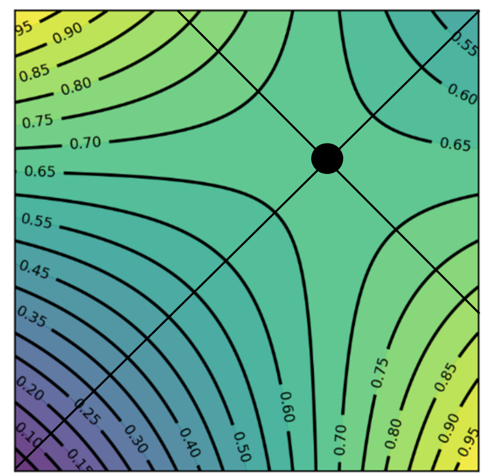
$$a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

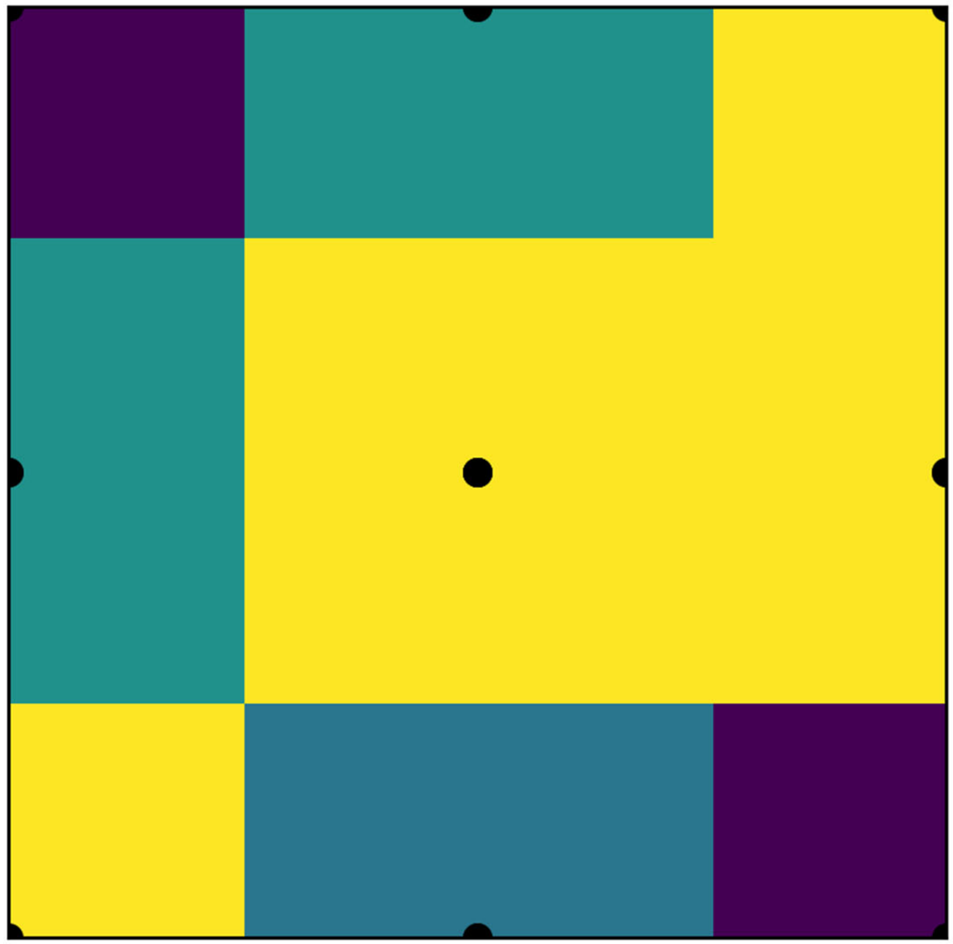
$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

degenerate means determinant = 0 (at least one eigenvalue = 0);
bi-linear is simple: a = 0 means degenerated to
linear anyway: no critical point at all! (except constant function)
(but with more than one cell: can have max or min at vertices)



Bi-Linear Interpolation: Comparisons

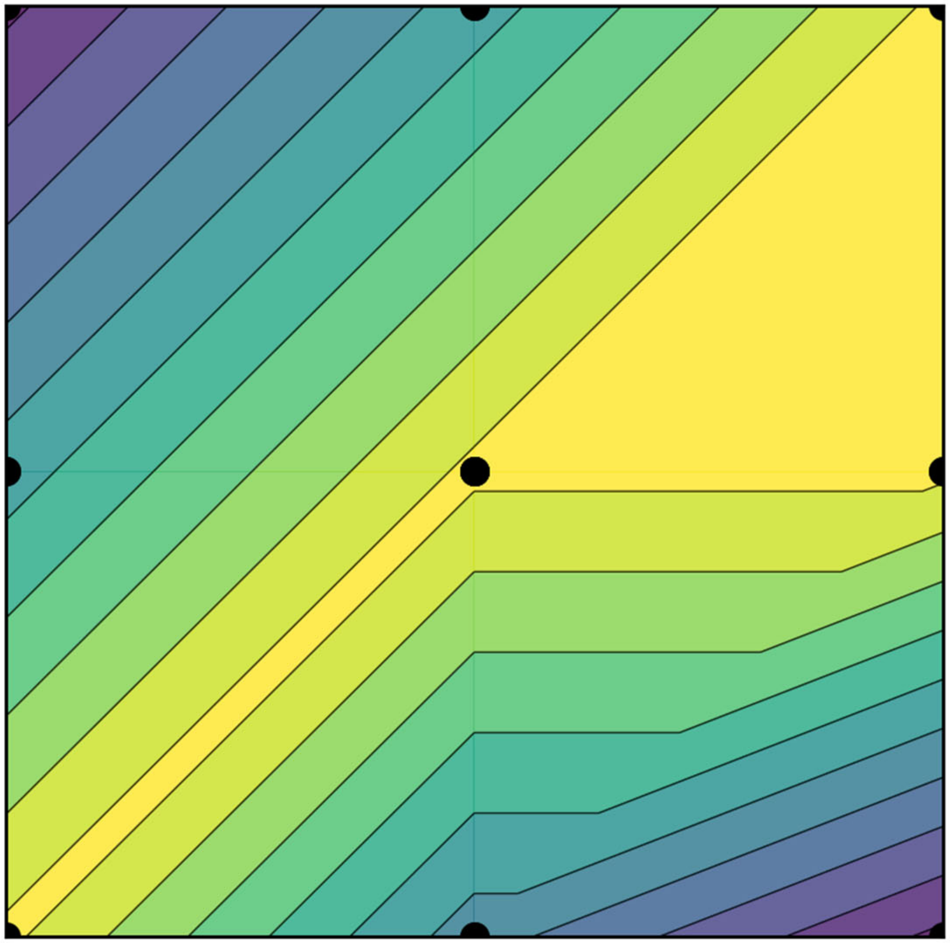
nearest-neighbor



Bi-Linear Interpolation: Comparisons

linear

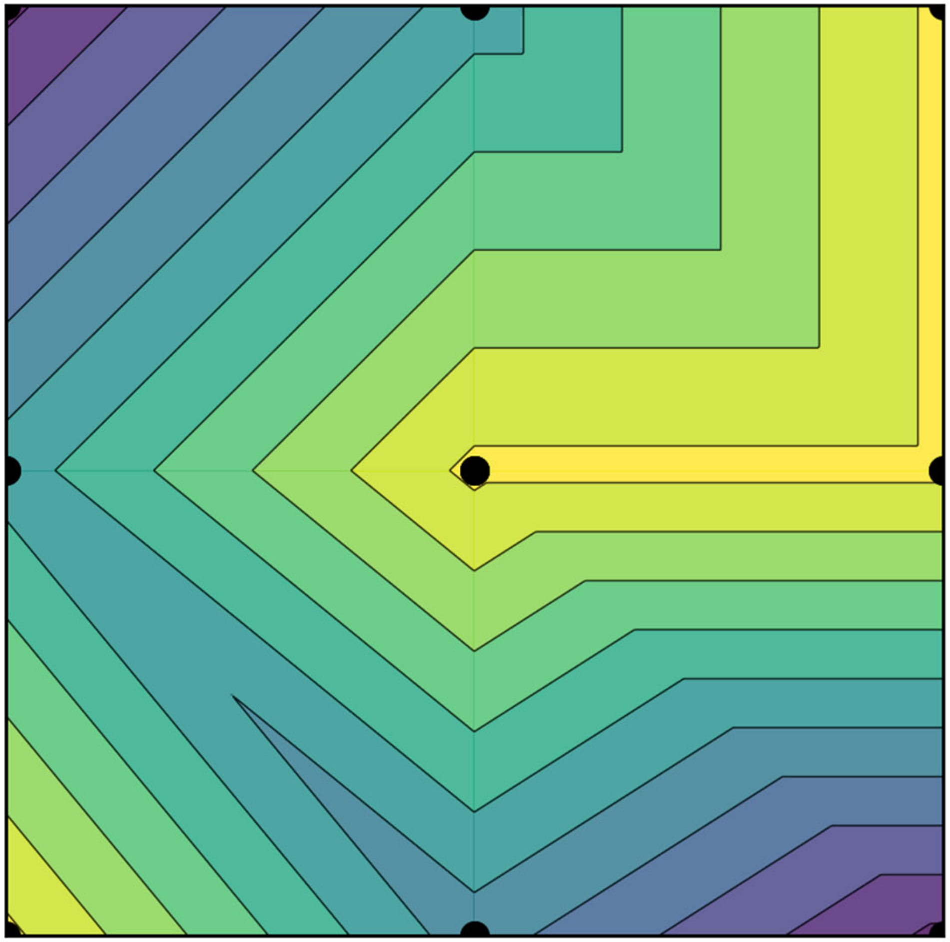
(2 triangles per quad;
diagonal:
bottom-left,
top-right)



Bi-Linear Interpolation: Comparisons

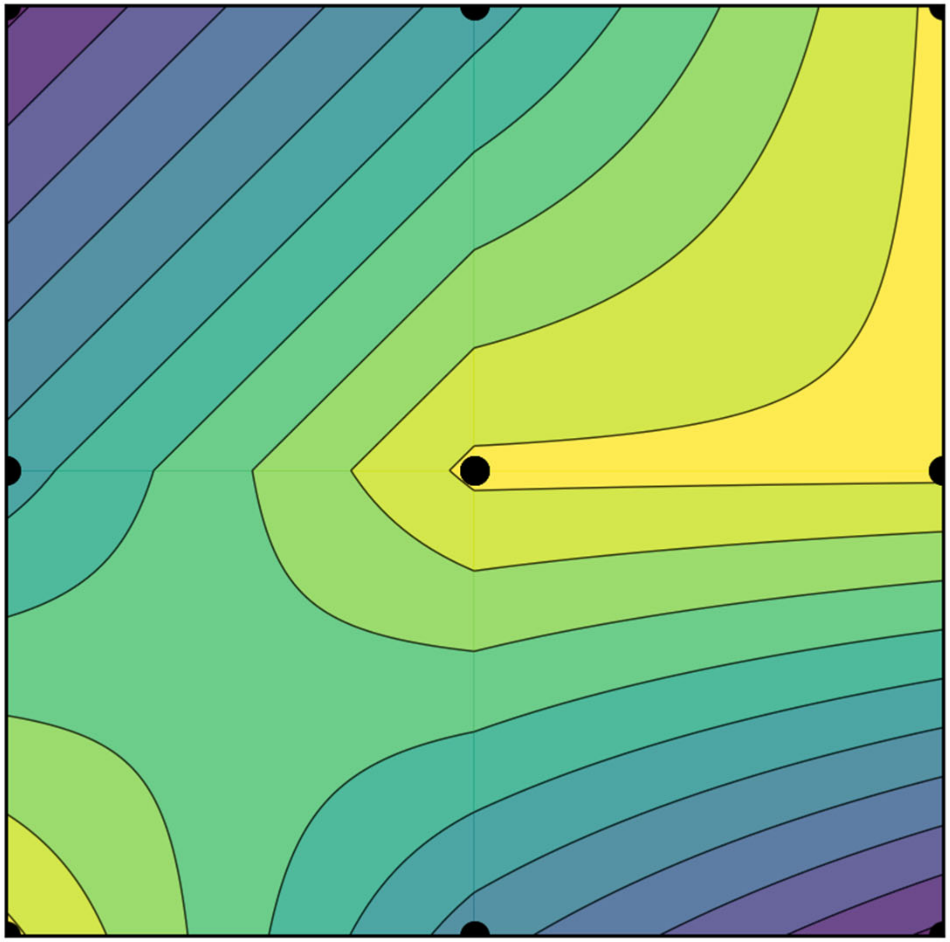
linear

(2 triangles per quad;
diagonal:
top-left,
bottom-right)



Bi-Linear Interpolation: Comparisons

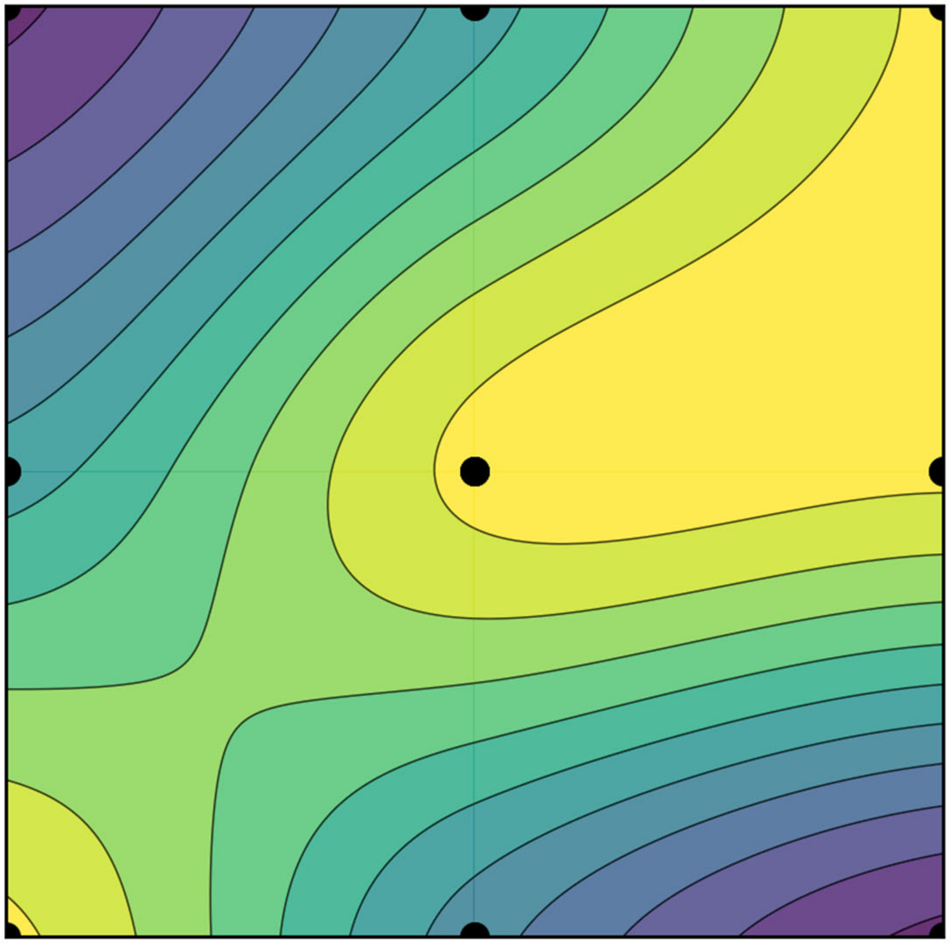
bi-linear



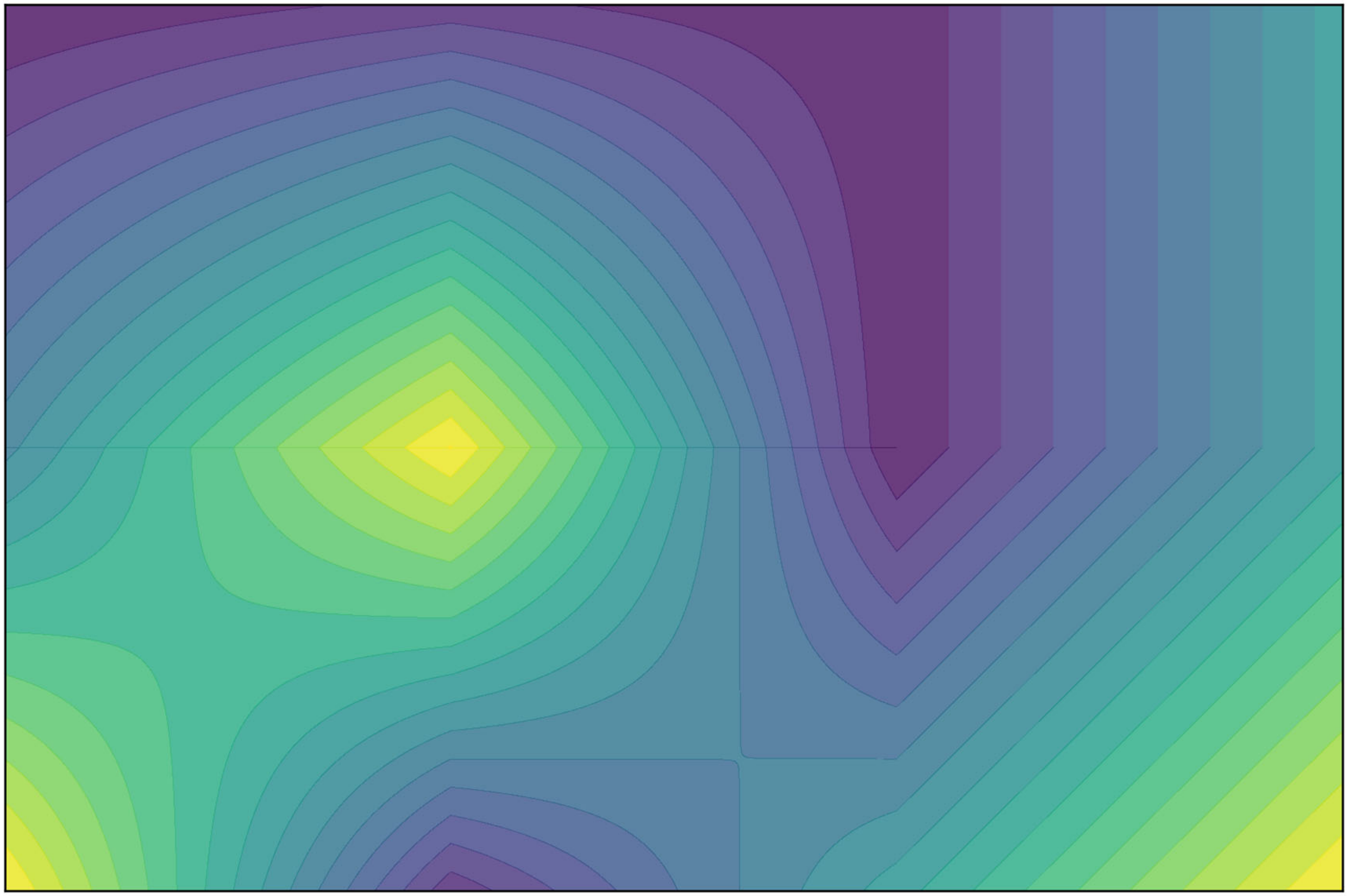
Bi-Linear Interpolation: Comparisons

bi-cubic

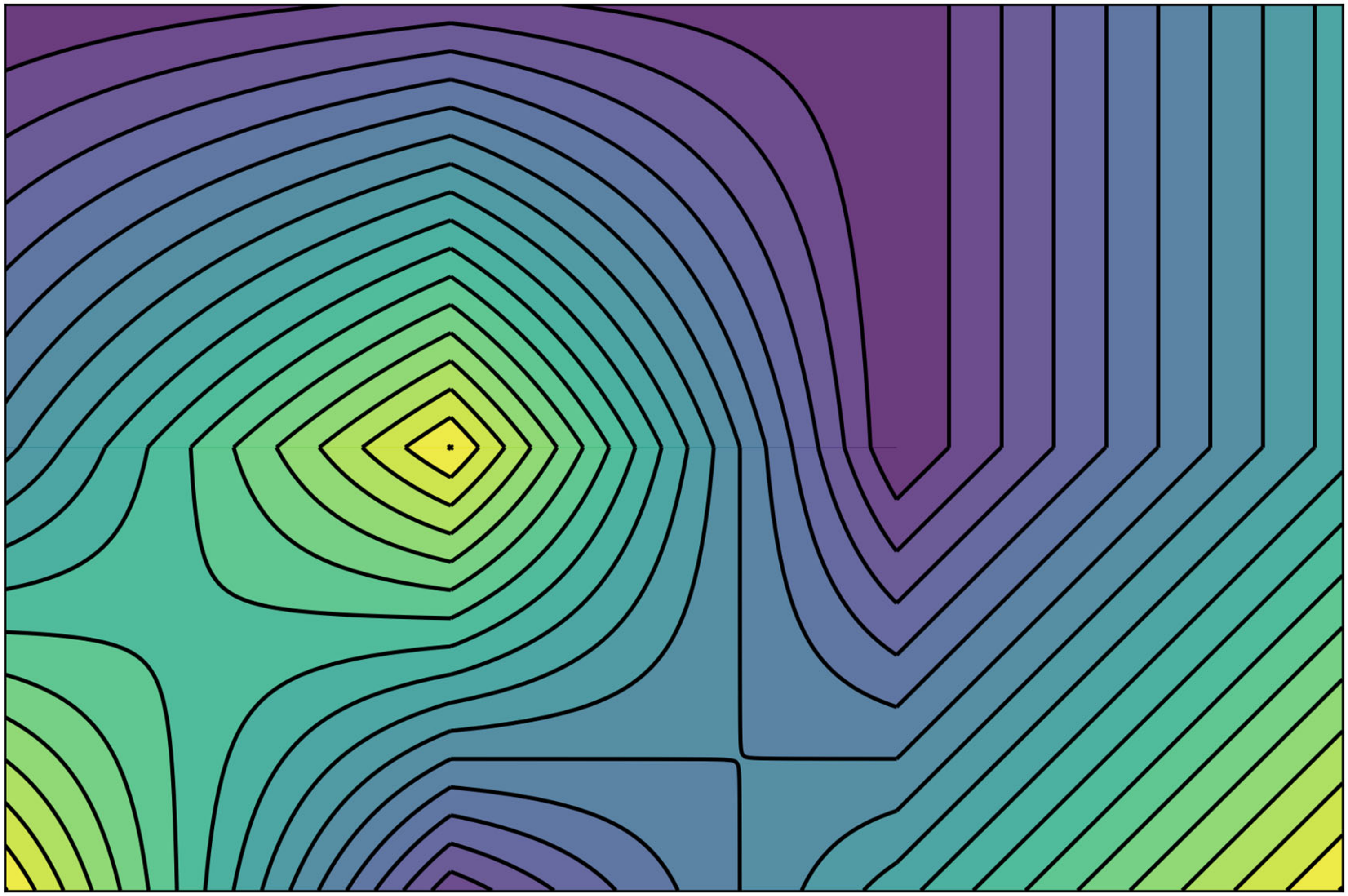
(Catmull-Rom spline)



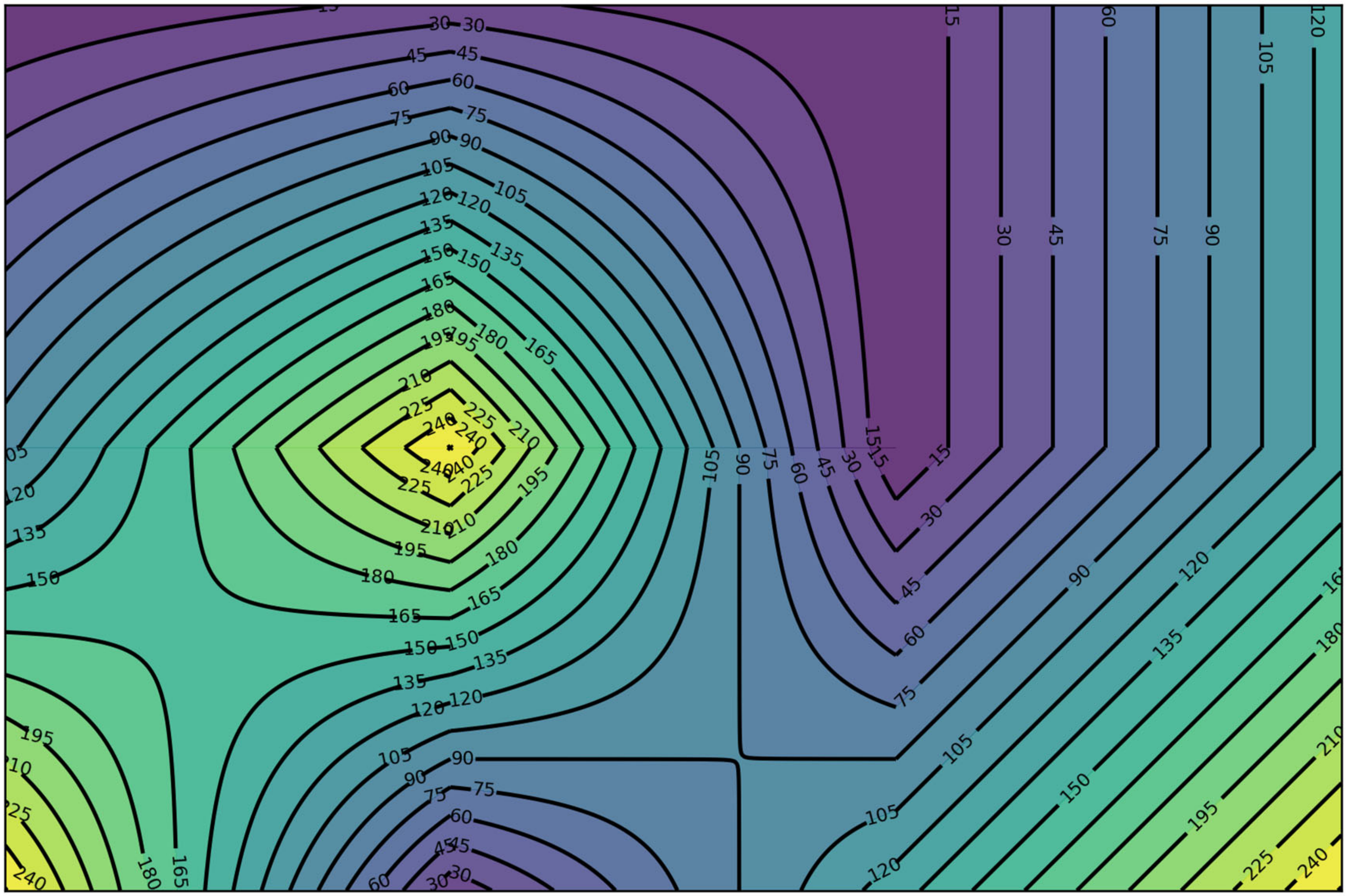
Piecewise Bi-Linear (Example: 3x2 Cells)



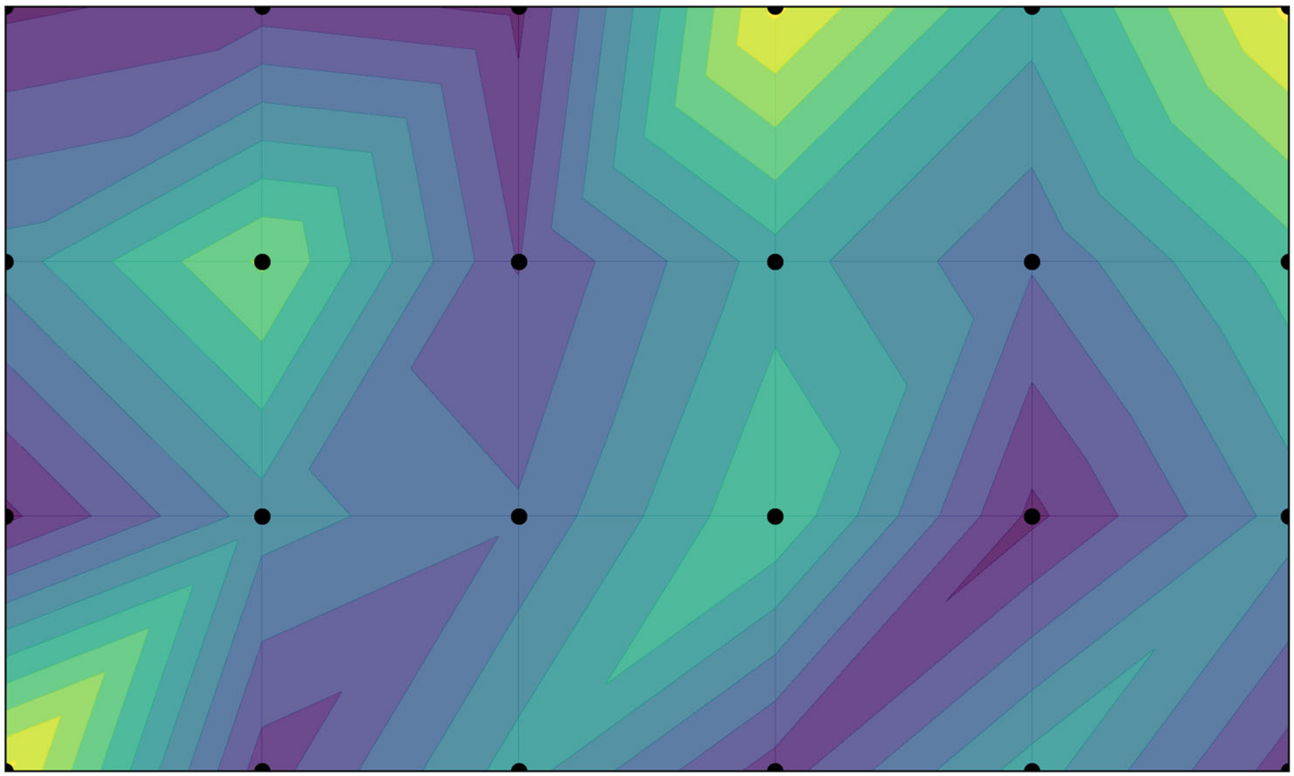
Piecewise Bi-Linear (Example: 3x2 Cells)



Piecewise Bi-Linear (Example: 3x2 Cells)

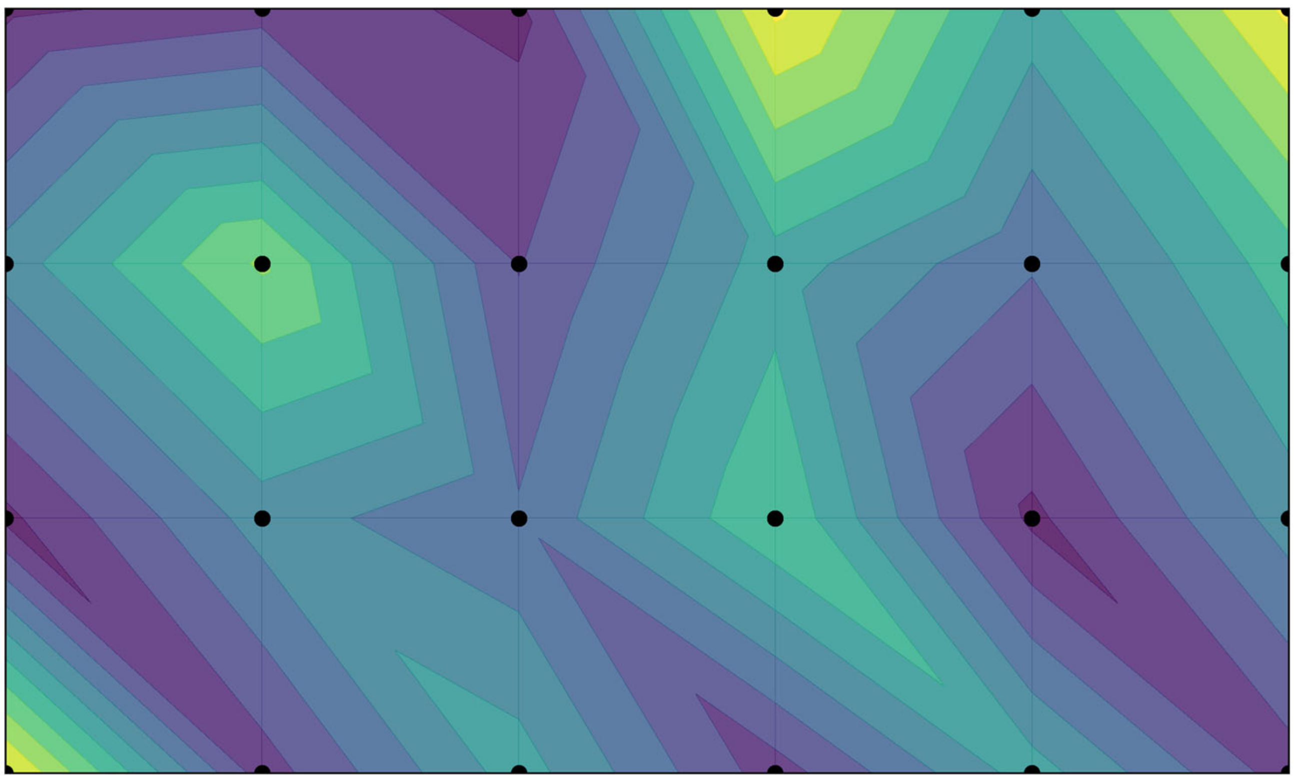


Bi-Linear Interpolation: Comparisons



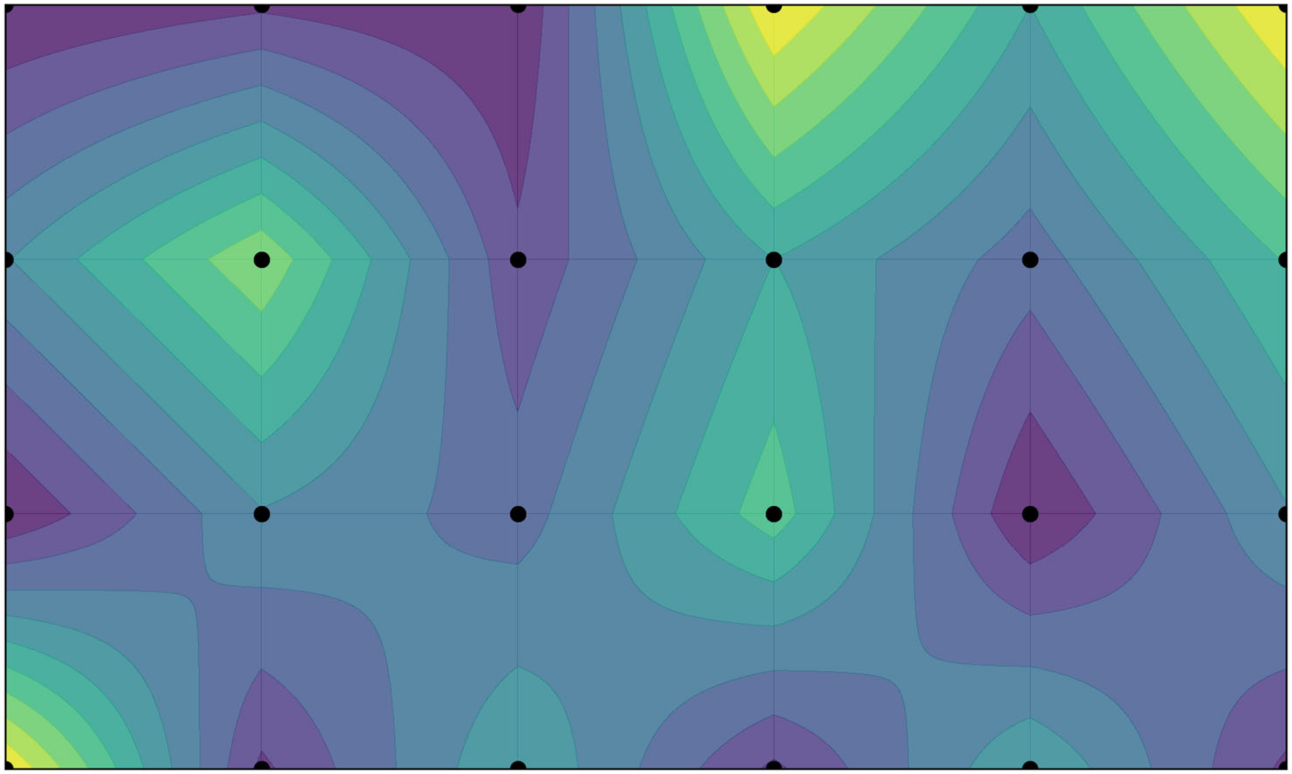
linear (diagonal 1)

Bi-Linear Interpolation: Comparisons



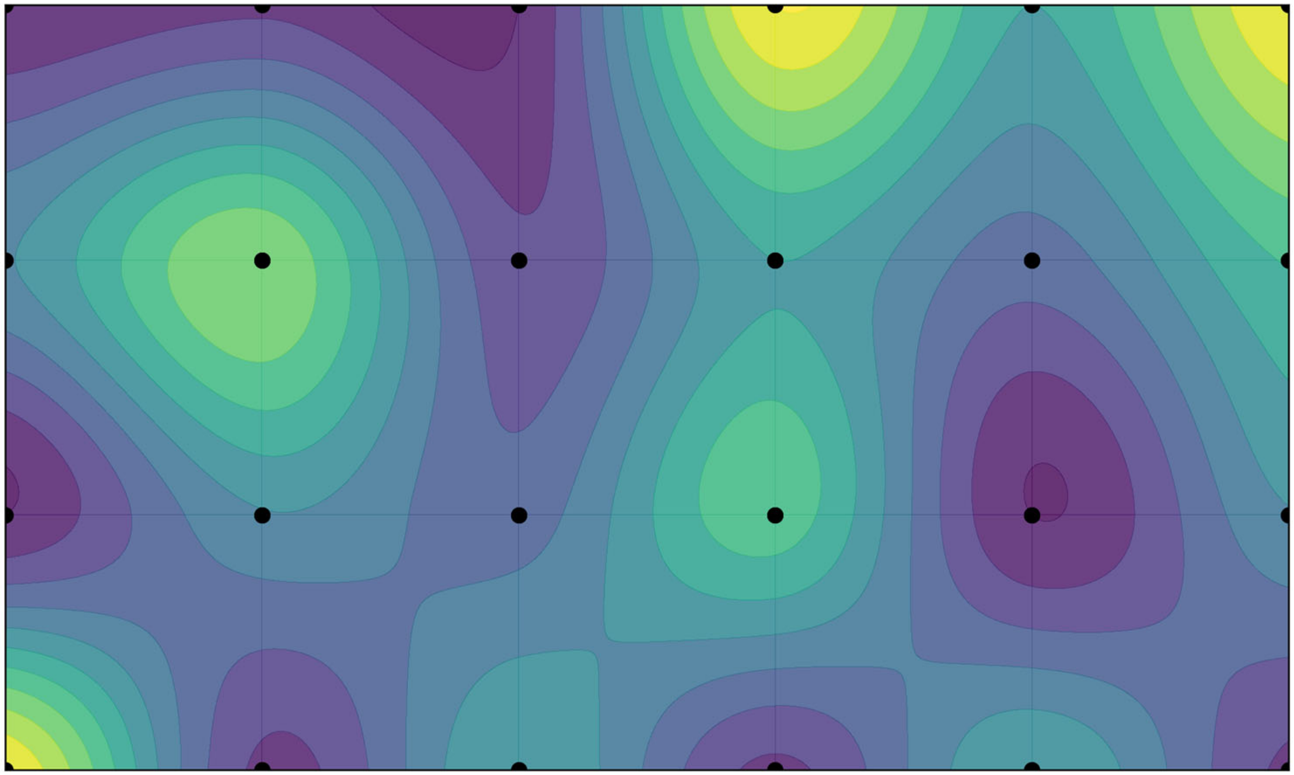
linear (diagonal 2)

Bi-Linear Interpolation: Comparisons



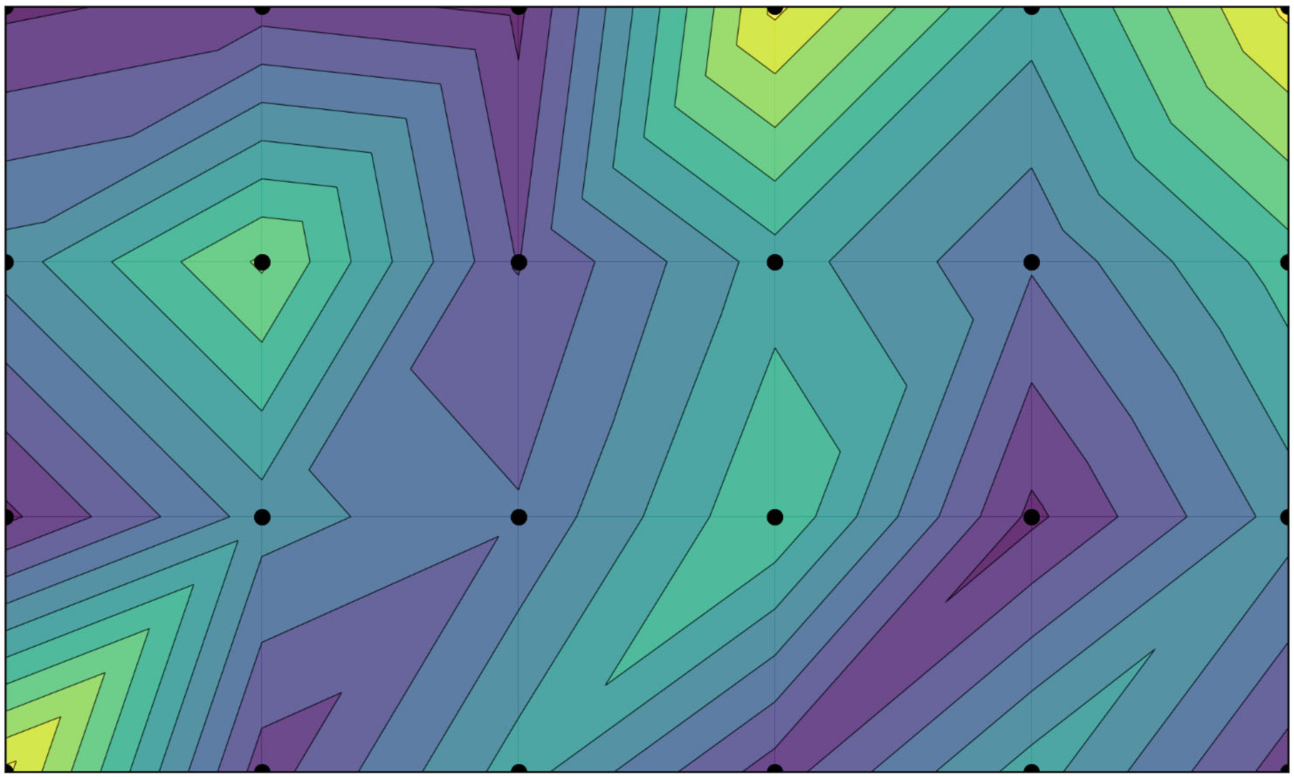
bi-linear (in 3D: tri-linear)

Bi-Linear Interpolation: Comparisons



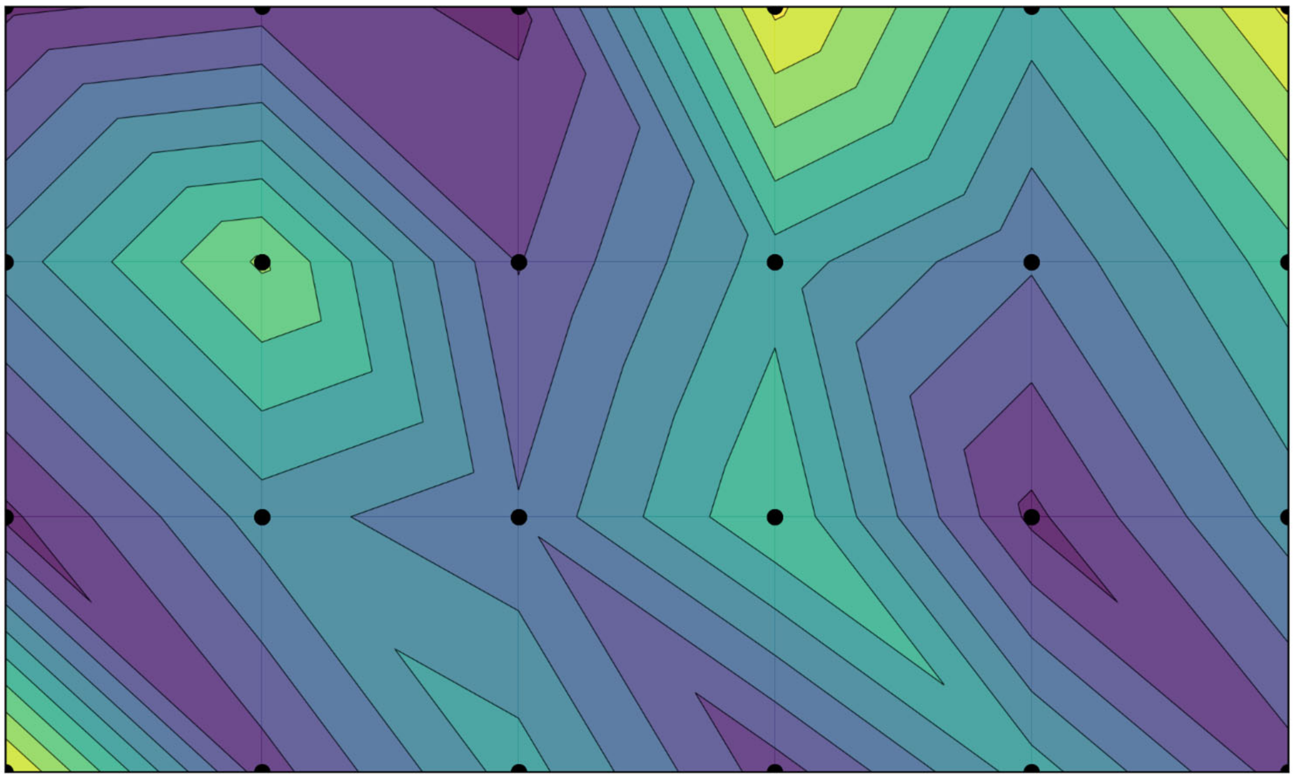
bi-cubic (in 3D: tri-cubic)

Bi-Linear Interpolation: Comparisons



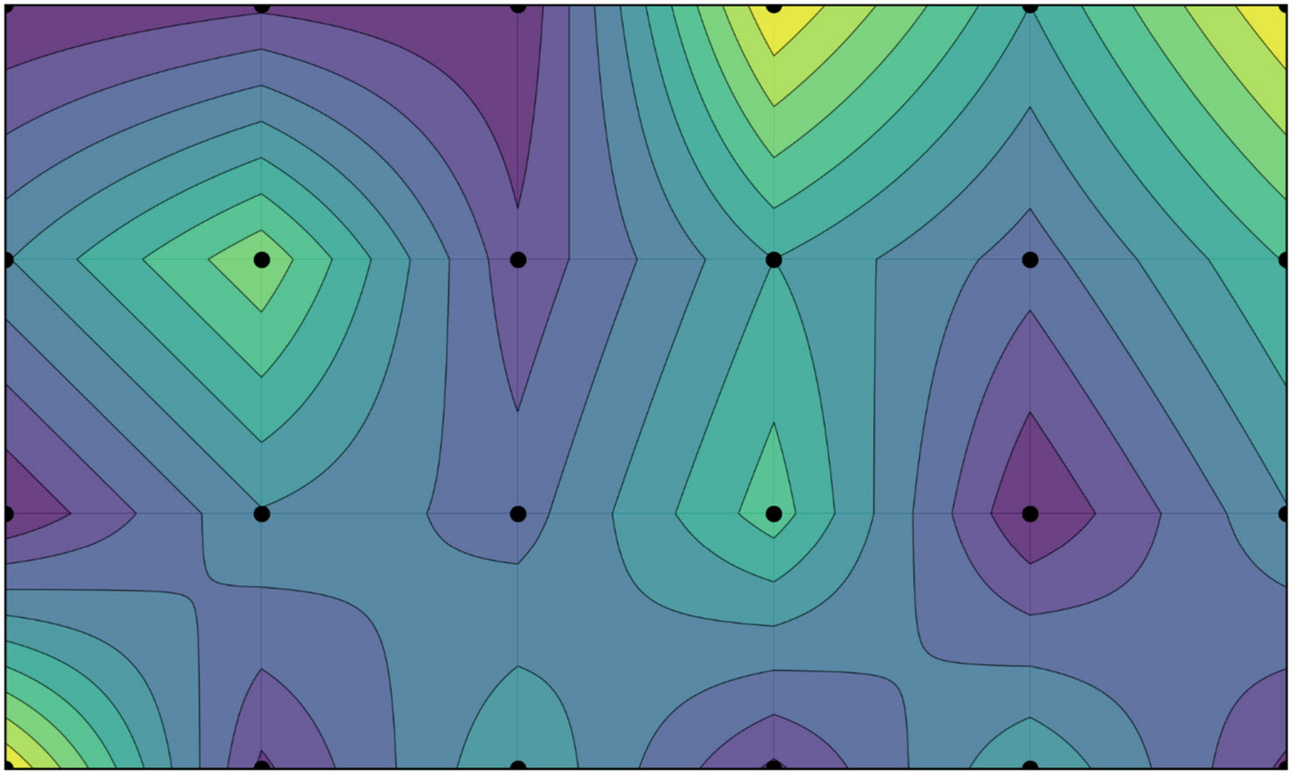
linear (diagonal 1)

Bi-Linear Interpolation: Comparisons



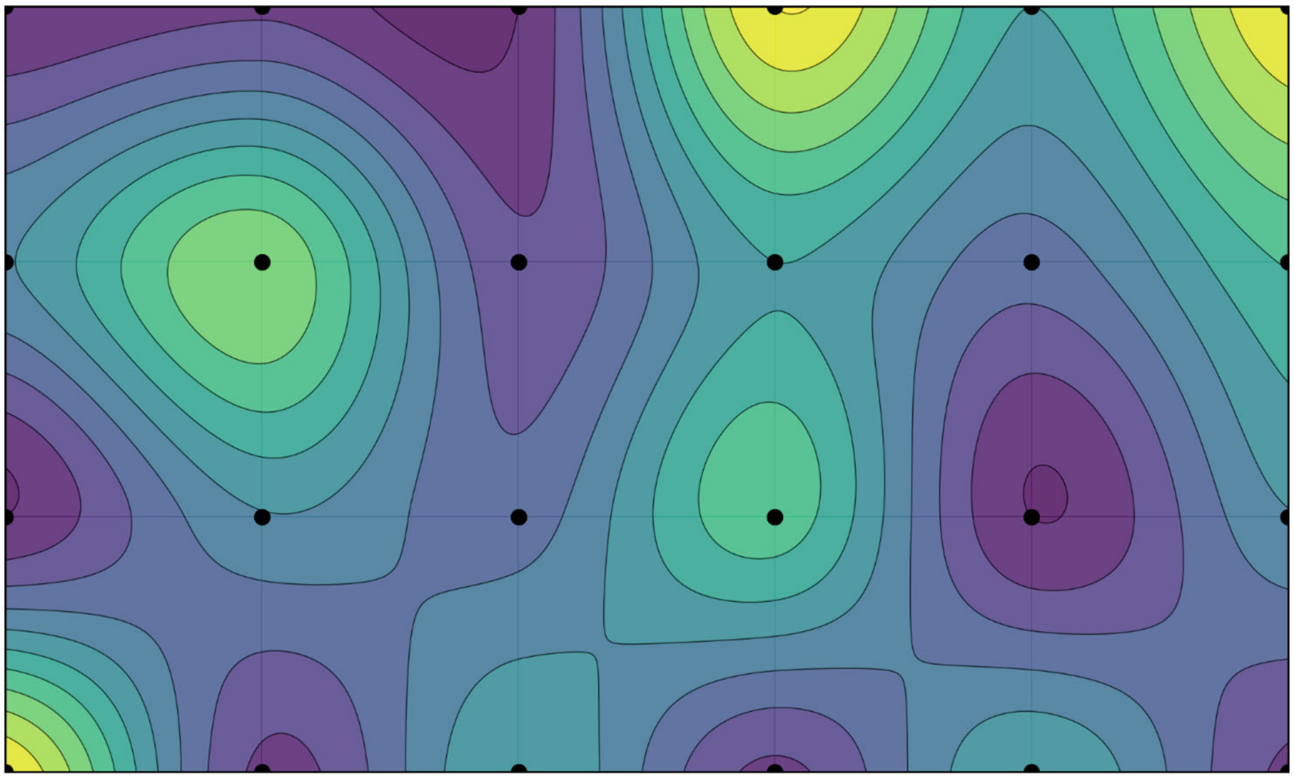
linear (diagonal 2)

Bi-Linear Interpolation: Comparisons



bi-linear (in 3D: tri-linear)

Bi-Linear Interpolation: Comparisons

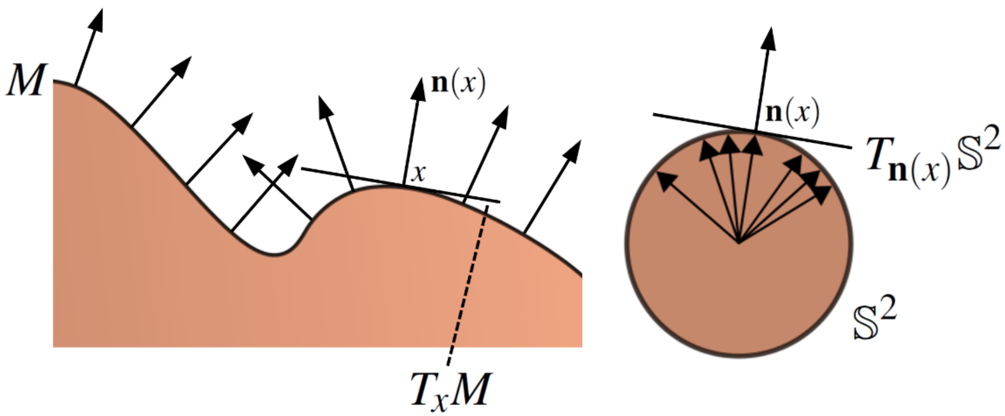


bi-cubic (in 3D: tri-cubic)

Interlude: Curvature and Shape Operator

Gauss map

$$\begin{aligned} \mathbf{n} \colon M &\to \mathbb{S}^2 \\ x &\mapsto \mathbf{n}(x) \end{aligned}$$



Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_x M$$

Differential of Gauss map

$$\begin{aligned} d\mathbf{n} \colon TM &\to T\mathbb{S}^2 \\ \mathbf{v} &\mapsto d\mathbf{n}(\mathbf{v}) \end{aligned}$$

$$\begin{aligned} (d\mathbf{n})_x \colon T_x M &\to T_{\mathbf{n}(x)}\mathbb{S}^2 \\ \mathbf{v} &\mapsto d\mathbf{n}(\mathbf{v}) \end{aligned}$$

Shape operator (Weingarten map)

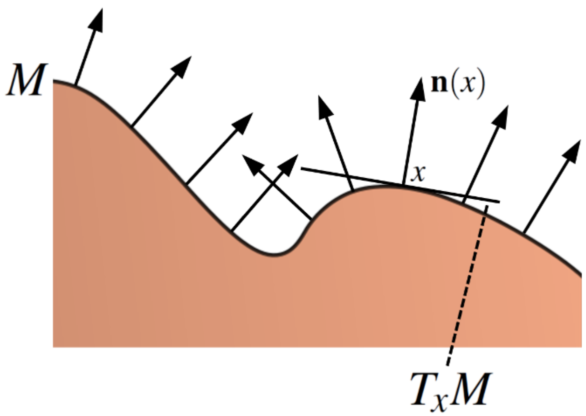
$$\mathbf{S} \colon TM \to TM$$

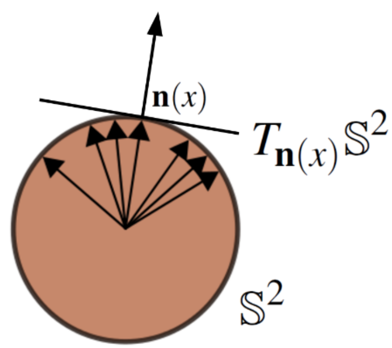
$$\boxed{\begin{aligned} \mathbf{S}_x \colon T_x M &\to T_x M \\ \mathbf{v} &\mapsto \mathbf{S}_x(\mathbf{v}) = d\mathbf{n}(\mathbf{v}) \end{aligned}}$$

Interlude: Curvature and Shape Operator

Gauss map

$$\mathbf{n} \colon M \to \mathbb{S}^2$$
$$x \mapsto \mathbf{n}(x)$$





Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_x M$$

Differential of Gauss map

$$d\mathbf{n} \colon TM \to T\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

$$(d\mathbf{n})_x \colon T_x M \to T_{\mathbf{n}(x)}\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

Shape operator (Weingarten map)

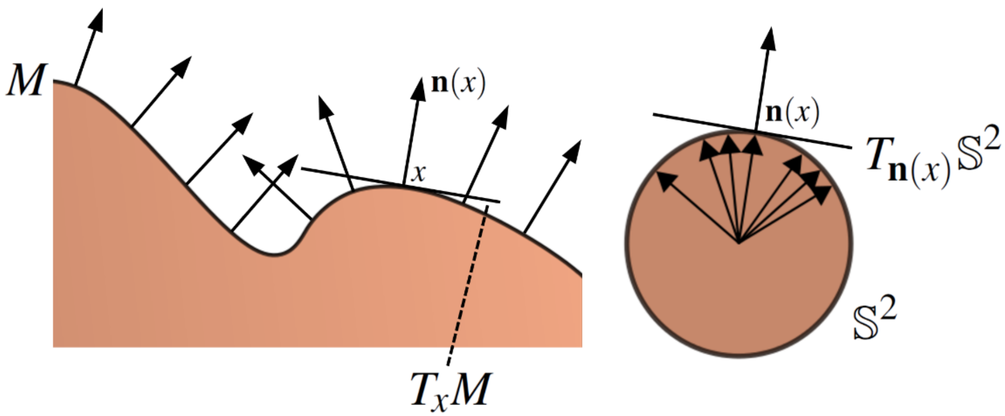
$$\mathbf{S} \colon TM \to TM$$

$$\boxed{\begin{aligned} \mathbf{S}_x \colon T_x M &\to T_x M \\ \mathbf{v} &\mapsto \mathbf{S}_x(\mathbf{v}) = \nabla_{\mathbf{v}}\mathbf{n} \end{aligned}}$$

Interlude: Curvature and Shape Operator

Gauss map

$$\mathbf{n}\colon M \to \mathbb{S}^2$$
$$x \mapsto \mathbf{n}(x)$$



Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_xM$$

Differential of Gauss map

$$d\mathbf{n}\colon TM \to T\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

$$(d\mathbf{n})_x\colon T_xM \to T_{\mathbf{n}(x)}\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

Shape operator (Weingarten map)

$$\mathbf{S}\colon TM \to TM$$

$$\boxed{\begin{aligned} \mathbf{S}_x\colon T_xM &\to T_xM \\ \mathbf{v} &\mapsto \mathbf{S}_x(\mathbf{v}) = -\nabla_{\mathbf{v}}\mathbf{n} \end{aligned}}$$

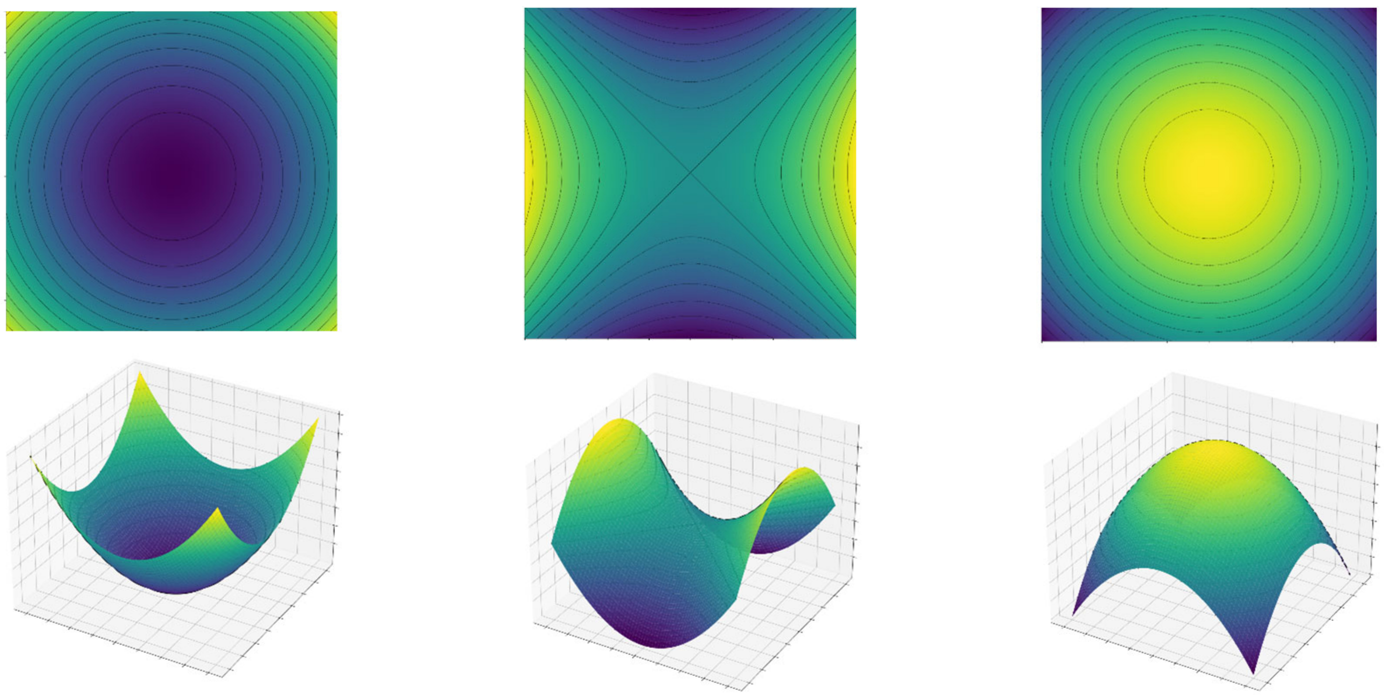
(sign is convention)

General Case (2D Scalar Fields)

In 2D scalar fields, only *three types* of (isolated, non-degenerate) critical points

Index of critical point: dimension of eigenspace with negative-definite Hessian

minimum (index 0)	saddle point (index 1)	maximum (index 2)



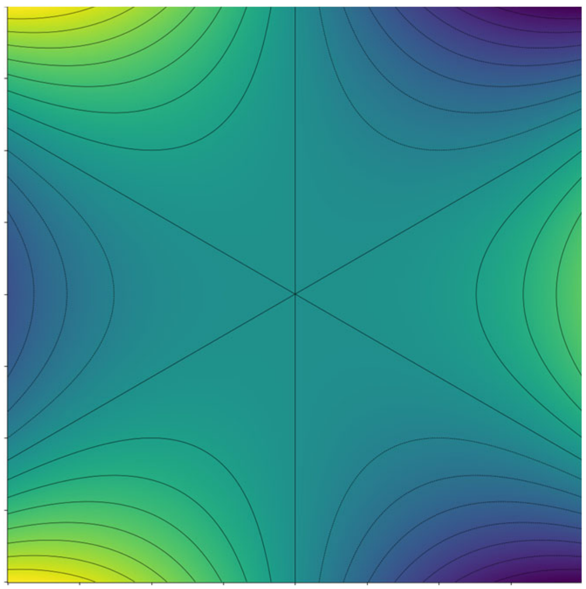
Interesting Degenerate Critical Points?

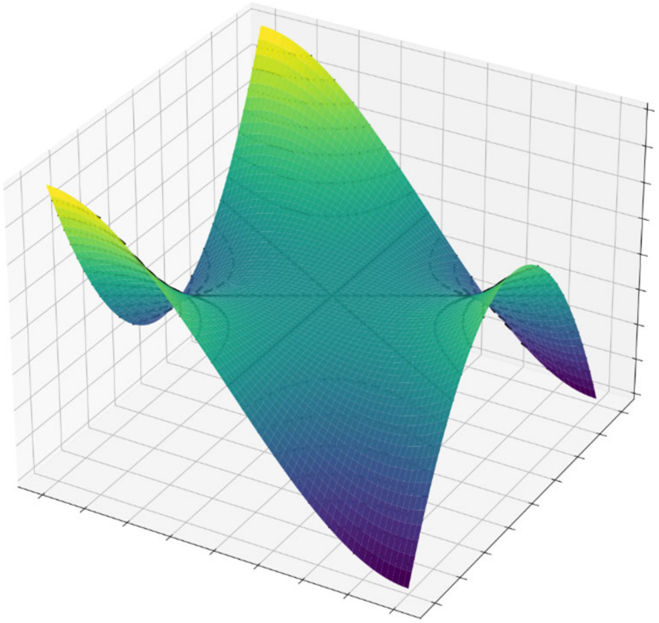
Hessian matrix is singular (determinant = 0)

- Cannot say what happens: need higher-order derivatives, …

Interesting example: monkey saddle $z = x^3 - 3xy^2$ ('third-order saddle')

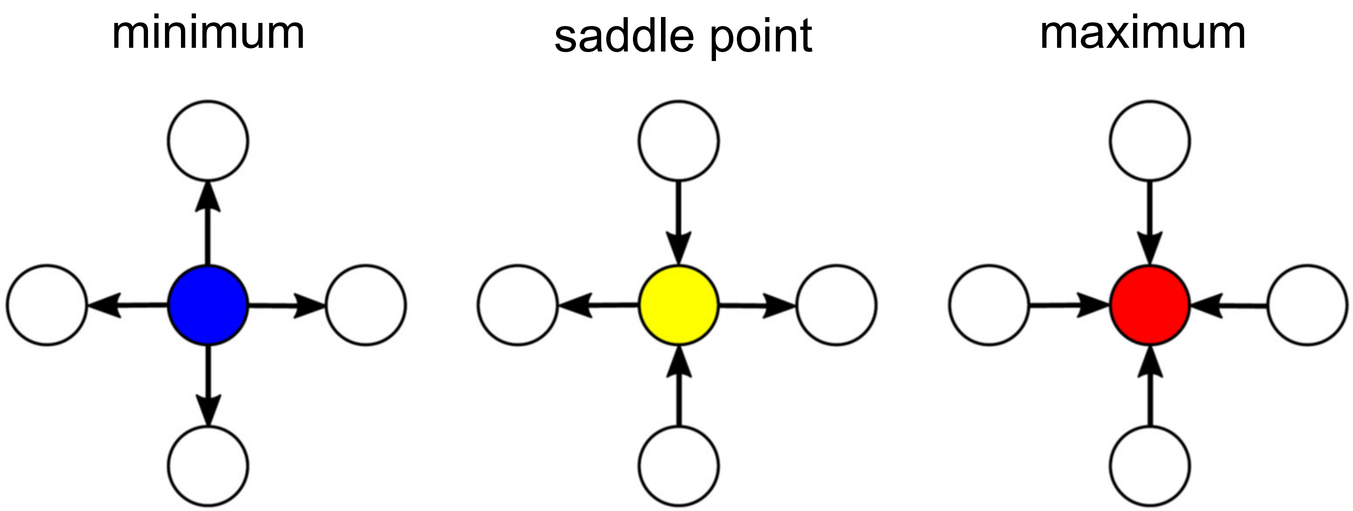
- Point (0,0) in center: Hessian = 0; Gaussian curvature = 0 (umbilical point)





Discrete Classification of Critical Points

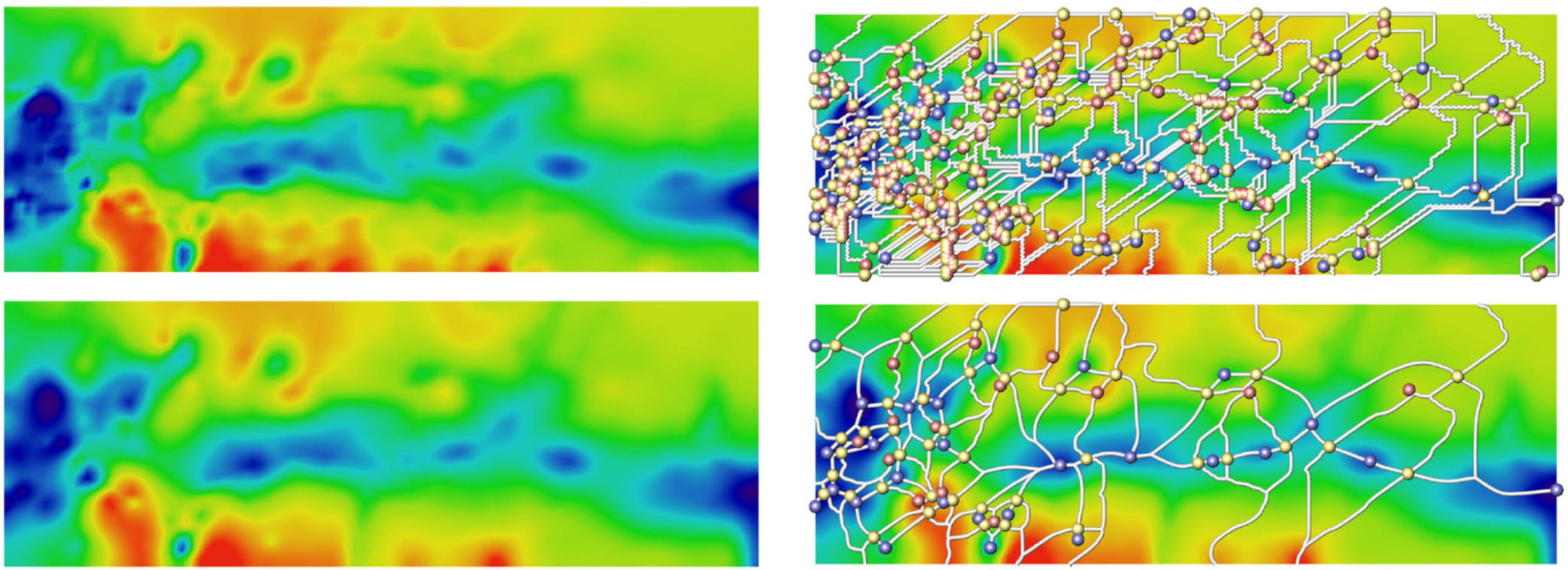
Combinatorial classification (looking at and comparing neighbors)
instead of looking at derivatives
(i.e., derivatives of the smooth function that is not known)



…toward scalar field topology, discrete Morse theory, Morse-Smale complex, …

Example: Scalar Field Simplification

Topology-based smoothing of 2D scalar fields, Weinkauf et al., 2010



Example: Differential Topology

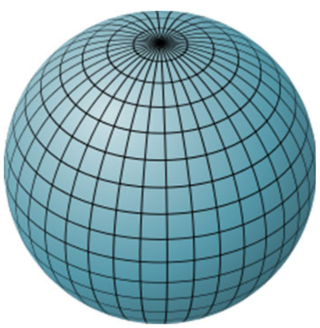
Morse theory

- Morse function: scalar function where all critical points are non-degenerate and have different critical value

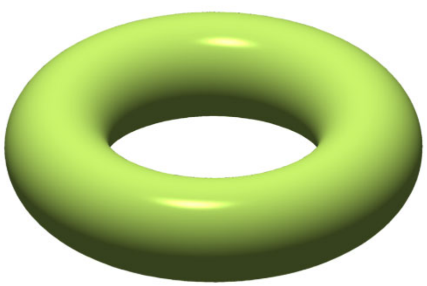
Topological invariant: Euler characteristic $\chi(M)$ of manifold M

(for 2-manifold mesh: $\chi(M) = V - E + F$)

$\chi = 2 - 2g$ (orientable)



genus $g = 0$
Euler characteristic $\chi = 2$



genus $g = 1$
Euler characteristic $\chi = 0$

genus $g = 2$
Euler characteristic $\chi = -2$

Example: Differential Topology

Morse theory

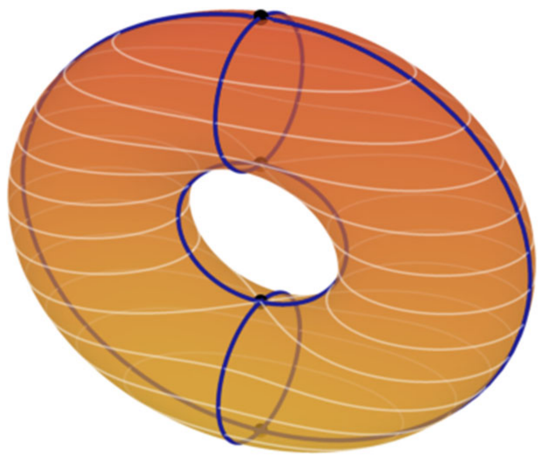
- Morse function: scalar function where all critical points are non-degenerate and have different critical value

Topological invariant: Euler characteristic $\chi(M)$ of manifold M

$$\chi(M) = \sum_{i=0}^{n} (-1)^i m_i$$

m_i: number of critical points with index i

n: dimensionality of M



genus $g(M) = 1$

Euler characteristic $\chi(M) = 0 \; (= 1 - 2 + 1)$

scalar function on torus is height function $f(x,y,z) = z$:

1 min, 1 max, 2 saddles

critical points are where

$$df(x,y,z) = 0$$

(tangent plane horizontal)

Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama