King Abdullah University of Science and Technology KAUST

CS 247 – Scientific Visualization
Lecture 24: Vector / Flow Visualization, Pt. 3

Markus Hadwiger, KAUST

Reading Assignment #13 (until Apr 25)

Read (required):

- Data Visualization book
  - Chapter 6.1 (Divergence and Vorticity)
- Diffeomorphisms / smooth deformations

  https://en.wikipedia.org/wiki/Diffeomorphism
- Integral curves: Stream lines, path lines, streak lines

  https://en.wikipedia.org/wiki/Integral_curve

  https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines
- Paper:
  Bruno Jobard and Wilfrid Lefer
  Creating Evenly-Spaced Streamlines of Arbitrary Density,

  http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.29.9498

Quiz #3: Apr 25

Organization

- First 30 min of lecture
- No material (book, notes, ...) allowed

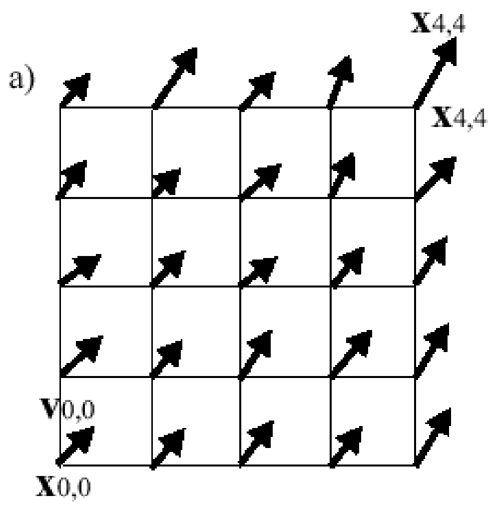
Content of questions

- Lectures (both actual lectures and slides)
- Reading assignments (except optional ones)
- Programming assignments (algorithms, methods)
- Solve short practical examples

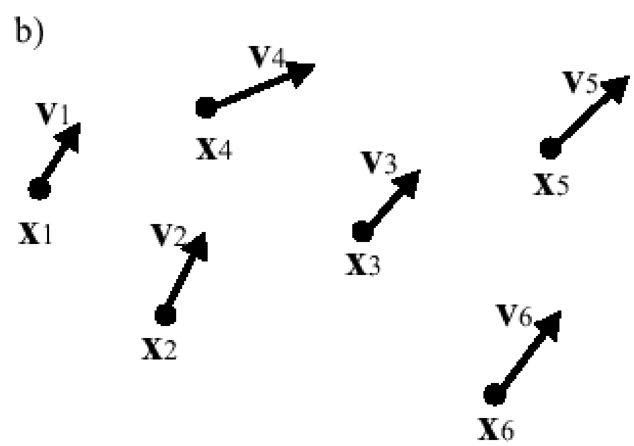
Vector Fields

Each vector is usually thought of as a velocity vector

- Example for actual velocity: fluid flow
- But also force fields, etc. (e.g., electrostatic field)



vectors given at grid points



vectors given at particle positions

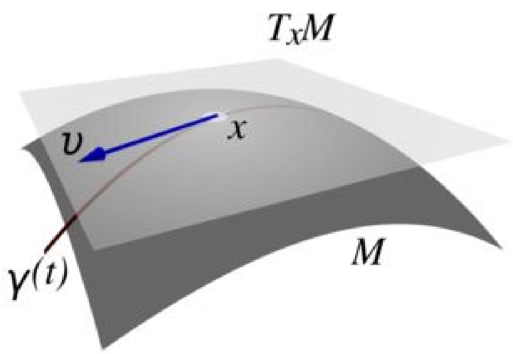
Vector Fields

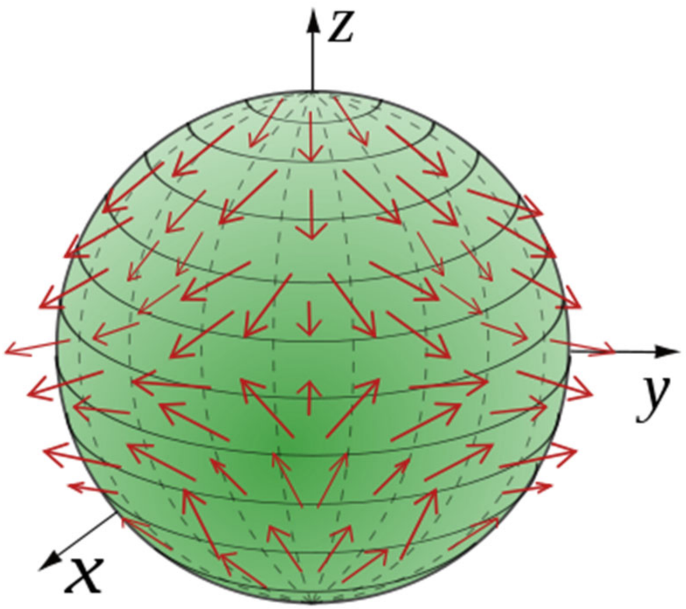
Each vector is usually thought of as a velocity vector

- Example for actual velocity: fluid flow
- But also force fields, etc. (e.g., electrostatic field)

Each vector in a vector field
lives in the **tangent space**
of the manifold at that point:

Each vector is a **tangent vector**





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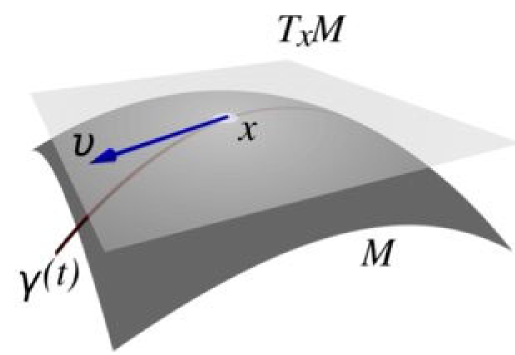
Vector Fields

Vector fields on general manifolds M (not just Euclidean space)

Tangent space at a point $x \in M$:

$$T_x M$$

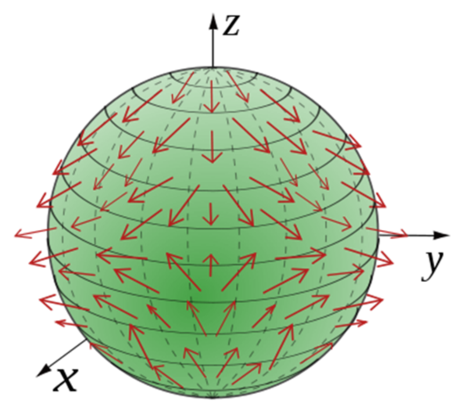


Tangent bundle: Manifold of all tangent spaces over base manifold

$$\pi : TM \to M$$

Vector field: *Section* of tangent bundle

$$s : M \to TM, \quad x \mapsto s(x). \qquad \pi\left(s(x)\right) = x$$



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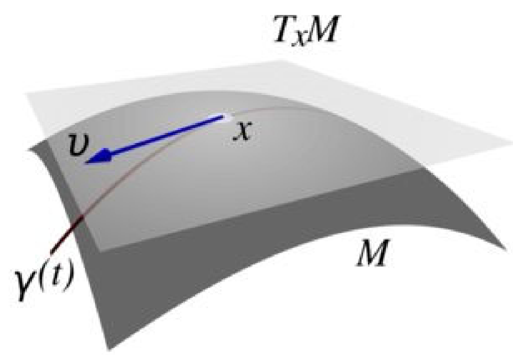
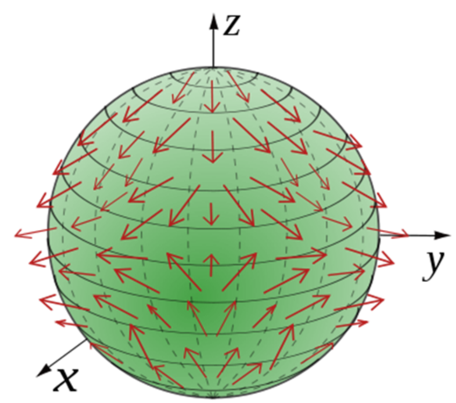


Tangent bundle: Manifold of all tangent spaces over base manifold

$$\pi : TM \to M$$

Vector field: *Section* of tangent bundle

$$\begin{aligned} \mathbf{v} : M &\to TM, \\ x &\mapsto \mathbf{v}(x). \end{aligned} \qquad \mathbf{v}(x) \in T_x M$$



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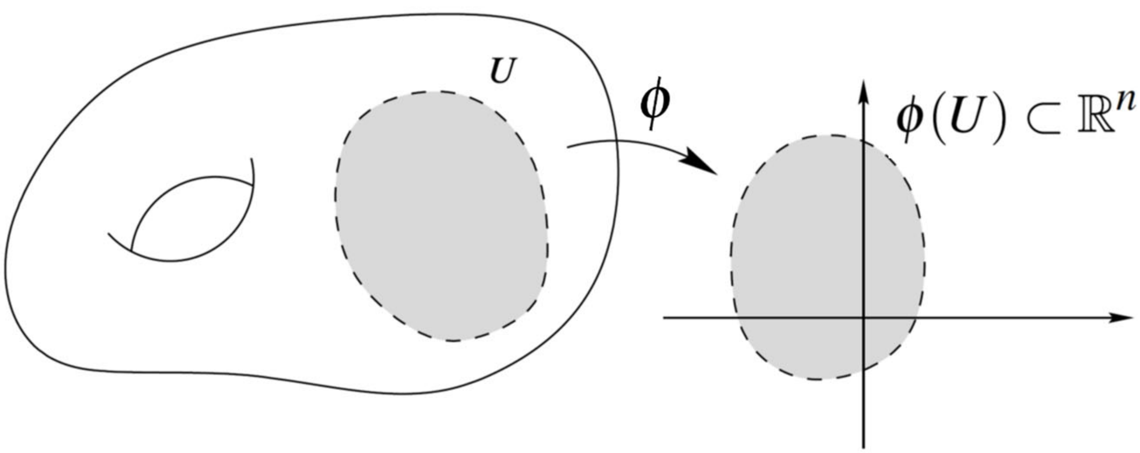
Interlude: Coordinate Charts

Interlude: Coordinate Charts

Coordinate chart

$$\phi : U \subset M \to \mathbb{R}^n,$$
$$x \mapsto (x^1, x^2, \ldots, x^n).$$



Interlude: Coordinate Charts

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$$\phi : U \subset M \to \mathbb{R}^n,$$
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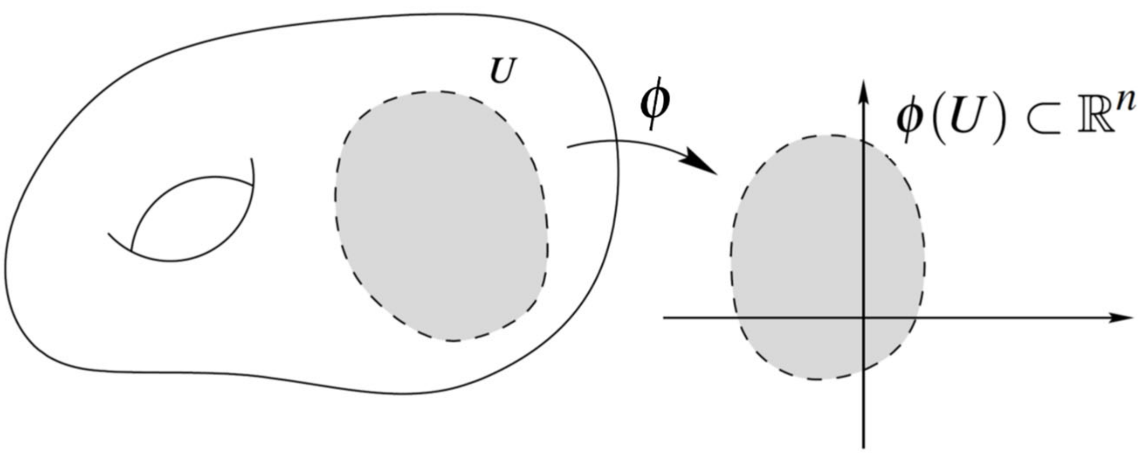
Coordinate functions

$$x^i : U \subset M \to \mathbb{R},$$
$$x \mapsto x^i(x).$$



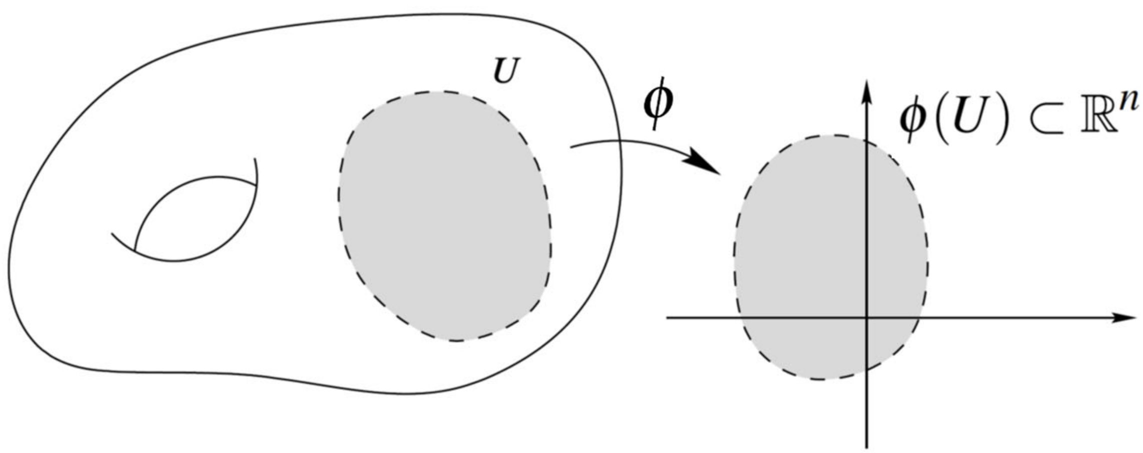
Interlude: Coordinate Charts

Coordinate charts

$$\phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n,$$
$$x \mapsto (x^1, x^2, \ldots, x^n).$$

Atlas

$$\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$$



Interlude: Coordinate Charts

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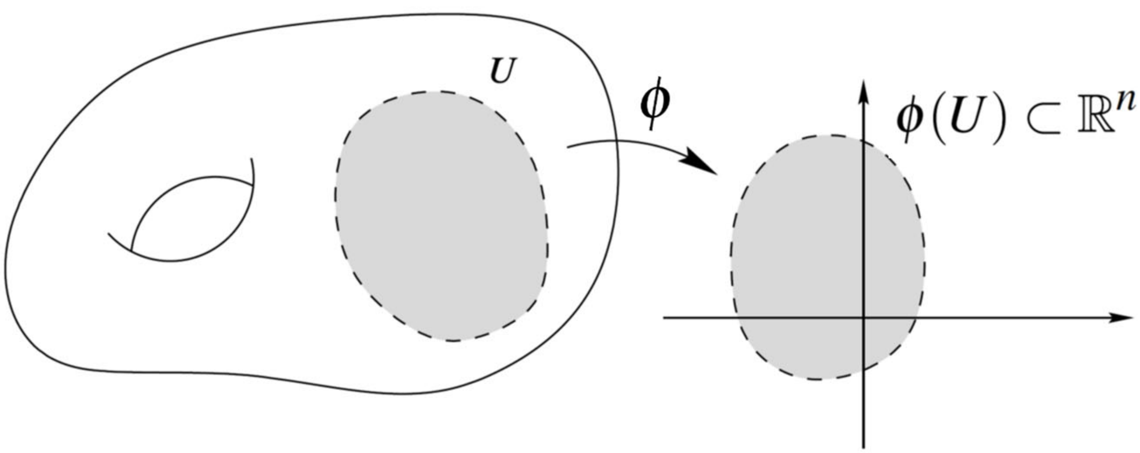
$$\phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n, \\ x \mapsto (x^1(x), x^2(x), \ldots, x^n(x)).$$

Atlas

$$\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$$



Vector Fields vs. Vectors in Components

Because Euclidean space is most common, often slightly sloppy notation

$$\begin{aligned}\mathbf{v}: U \subset \mathbb{R}^2 &\to \mathbb{R}^2, \\ (x, y) &\mapsto \begin{bmatrix} u \\ v \end{bmatrix}.\end{aligned}$$

Vector Fields vs. Vectors in Components

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$$\mathbf{v}: U \subset \mathbb{R}^2 \to \mathbb{R}^2,$$
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Vector Fields vs. Vectors in Components

Because Euclidean space is most common, often slightly sloppy notation

$$\mathbf{v}: U \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto \begin{bmatrix} u \\ v \end{bmatrix}.$$

$$\mathbf{v}: U \subset \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,y,z) \mapsto \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

$$\mathbf{v}: U \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

$$\mathbf{v}: U \subset \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,y,z) \mapsto \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}.$$

Vector Fields vs. Vectors in Components

$$\mathbf{v}: U \subset \mathbb{R}^n \to \mathbb{R}^n,$$

$$(x^1, x^2, \ldots, x^n) \mapsto \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}.$$

$$\mathbf{v}: U \subset \mathbb{R}^n \to \mathbb{R}^n,$$

$$(x^1, x^2, \ldots, x^n) \mapsto \begin{pmatrix} v^1(x^1, x^2, \ldots, x^n) \\ v^2(x^1, x^2, \ldots, x^n) \\ \vdots \\ v^n(x^1, x^2, \ldots, x^n) \end{pmatrix}.$$

Vector Fields vs. Vectors in Components

$$\mathbf{v} : U \subset \mathbb{R}^n \to \mathbb{R}^n,$$

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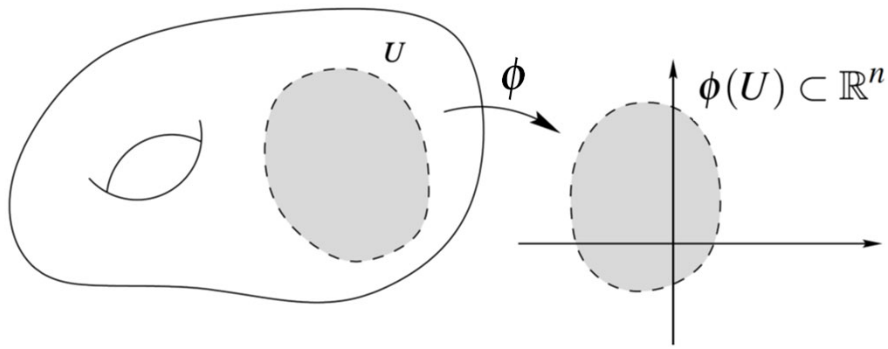

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$$\mathbf{v}\big|_U : \phi(U) \subset \mathbb{R}^n \to \mathbb{R}^n,$$

$$(x^1, x^2, \ldots, x^n) \mapsto \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}.$$

Vector Fields vs. Vectors in Components

Need basis vector fields

$$\mathbf{e}_i : U \subset M \to TM, \\ x \mapsto \mathbf{e}_i(x)$$

$$\{\mathbf{e}_i(x)\}_{i=1}^{n} \quad \text{basis for } T_x M$$

Vector Fields vs. Vectors in Components

Need basis vector fields

$$\mathbf{e}_i : U \subset M \to TM, \quad x \mapsto \mathbf{e}_i(x) \qquad \{\mathbf{e}_i(x)\}_{i=1}^n \quad \text{basis for } T_xM$$

$$\mathbf{v} : U \subset M \to TM, \quad x \mapsto v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + \ldots + v^n\mathbf{e}_n.$$

$$\mathbf{v} : U \subset M \to TM, \quad x \mapsto v^1(x)\,\mathbf{e}_1(x) + v^2(x)\,\mathbf{e}_2(x) + \ldots + v^n(x)\,\mathbf{e}_n(x).$$

Vector Fields vs. Vectors in Components

Need basis vector fields

$$\mathbf{e}_i : U \subset M \to TM, \quad x \mapsto \mathbf{e}_i(x)$$

$$\{\mathbf{e}_i(x)\}_{i=1}^{n} \quad \text{basis for } T_xM$$

Coordinate basis:

$$\mathbf{e}_i := \frac{\partial}{\partial x^i}$$

$$\mathbf{v} : U \subset M \to TM, \quad x \mapsto v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + \ldots + v^n\mathbf{e}_n.$$

$$\mathbf{v} : U \subset M \to TM, \quad x \mapsto v^1(x)\,\mathbf{e}_1(x) + v^2(x)\,\mathbf{e}_2(x) + \ldots + v^n(x)\,\mathbf{e}_n(x).$$

Examples of Coordinate Curves and Bases

Coordinate functions, coordinate curves, bases

- Coordinate functions are real-valued ("scalar") functions on the domain
- On each coordinate curve, *one* coordinate changes, *all others stay constant*
- Basis: n linearly independent vectors at each point of domain

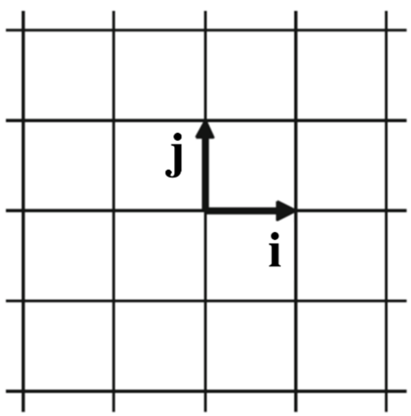
Cartesian coordinates

$$x^1 = x$$

$$x^2 = y$$

$$\mathbf{e}_1 = \frac{\partial}{\partial x} = \mathbf{i}$$

$$\mathbf{e}_2 = \frac{\partial}{\partial y} = \mathbf{j}$$



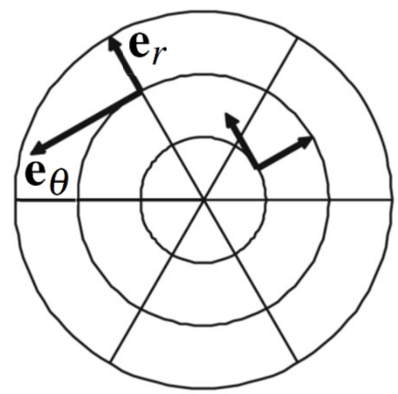
polar coordinates

$$x^1 = r$$

$$x^2 = \theta$$

$$\mathbf{e}_1 = \frac{\partial}{\partial r} = \mathbf{e}_r$$

$$\mathbf{e}_2 = \frac{\partial}{\partial \theta} = \mathbf{e}_\theta$$

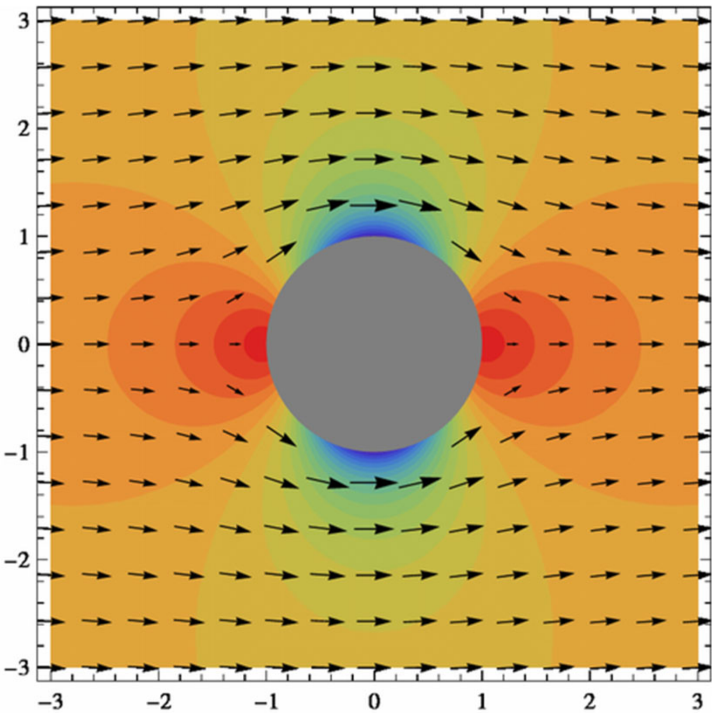


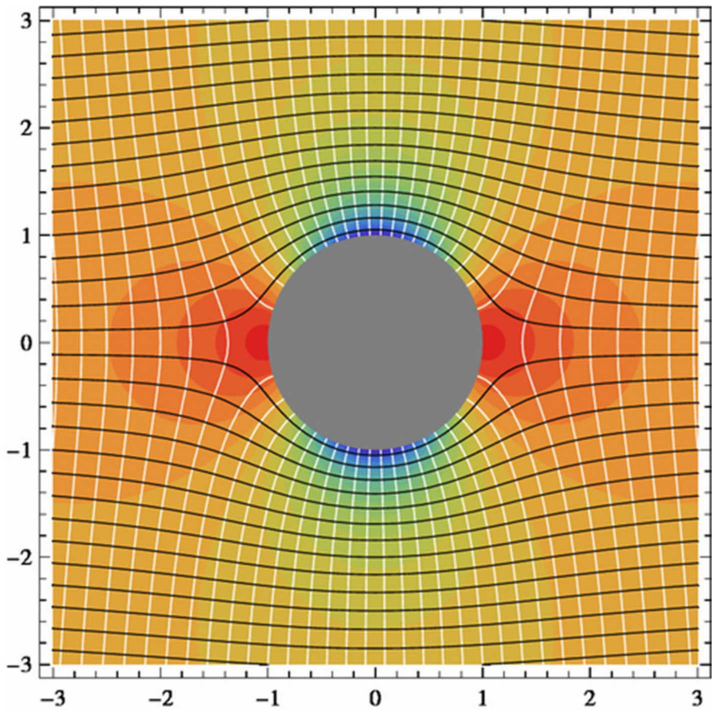
Flow Field Example (1)

Potential flow around a circular cylinder

`https://en.wikipedia.org/wiki/Potential_flow_around_a_circular_cylinder`

Inviscid, incompressible flow that is irrotational (curl-free) and can be modeled as the gradient of a scalar function called the (scalar) velocity potential





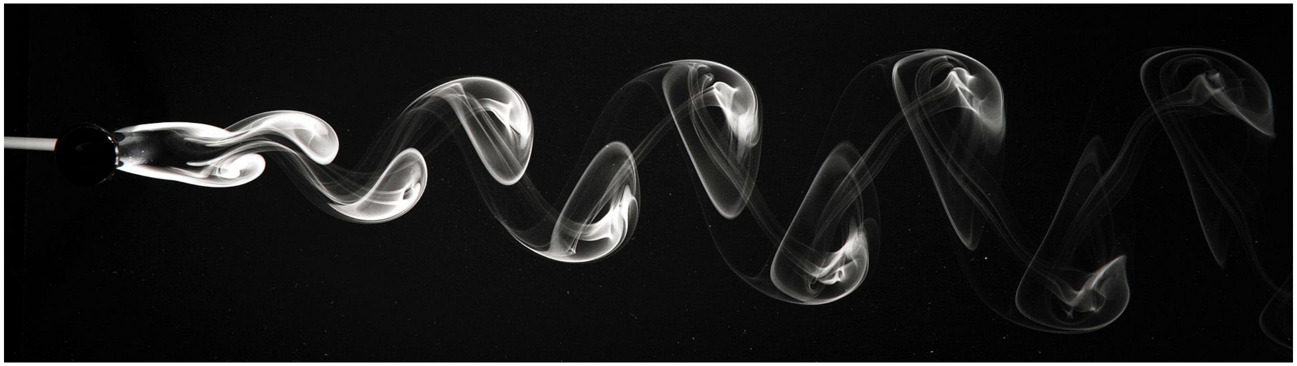
images from wikipedia

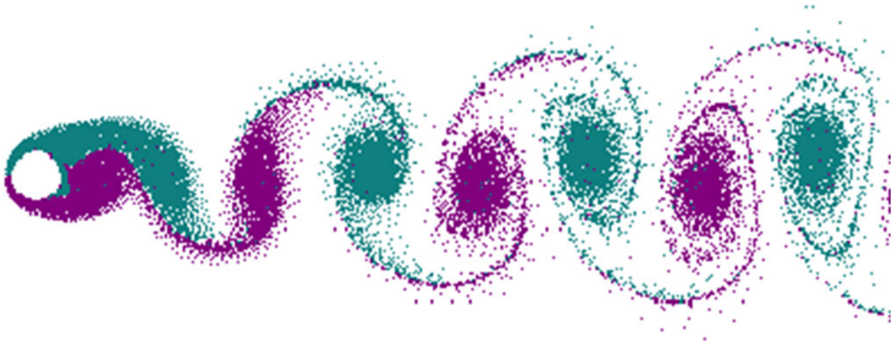
Flow Field Example (2)

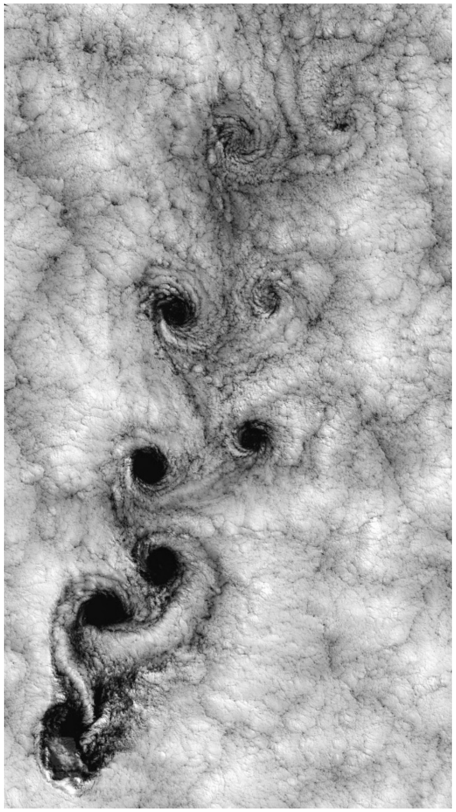
Depending on Reynolds number, turbulence will develop

Example: von Kármán vortex street: vortex shedding

`https://en.wikipedia.org/wiki/Karman_vortex_street`







images from wikipedia

Steady vs. Unsteady Flow

- Steady flow: time-independent
  - Flow itself is static over time: $\mathbf{v}(\mathbf{x})$
  - Example: laminar flows

$$\mathbf{v}: \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \mathbf{v}(x).$$

- Unsteady flow: time-dependent
  - Flow itself changes over time: $\mathbf{v}(\mathbf{x}, t)$
  - Example: turbulent flows

$$\mathbf{v}: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad x \mapsto \mathbf{v}(x, t).$$

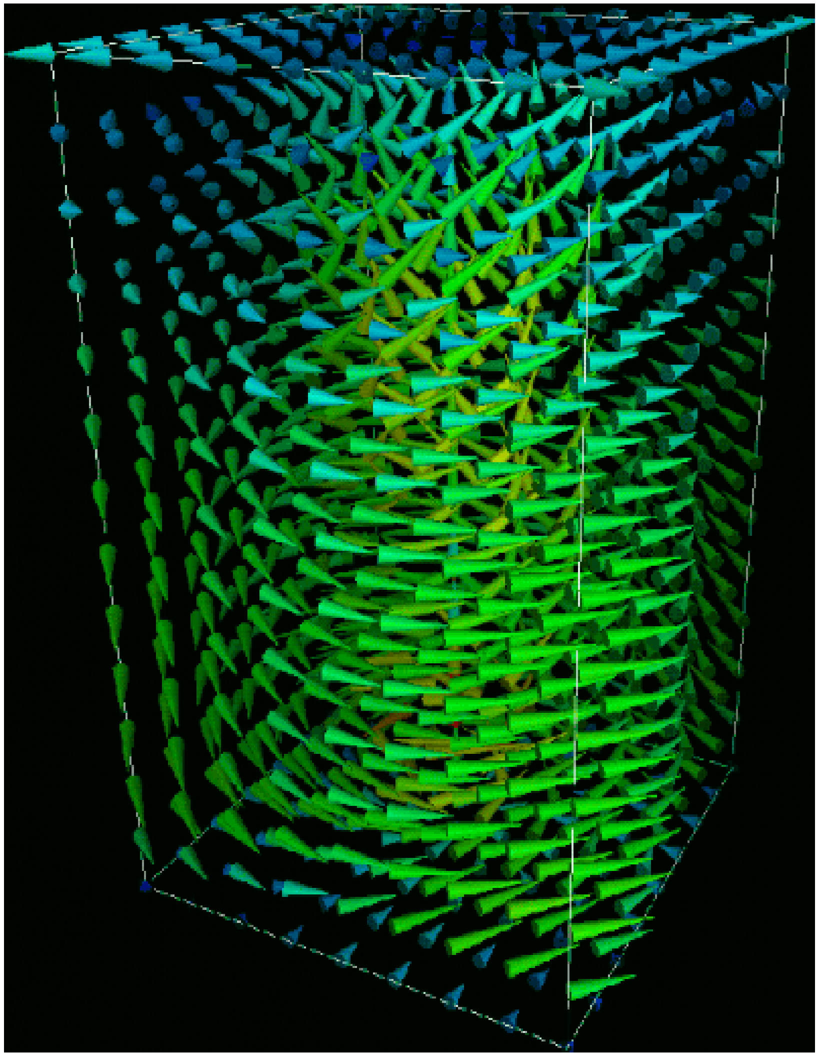
(here just for Euclidean domain; analogous on general manifolds)

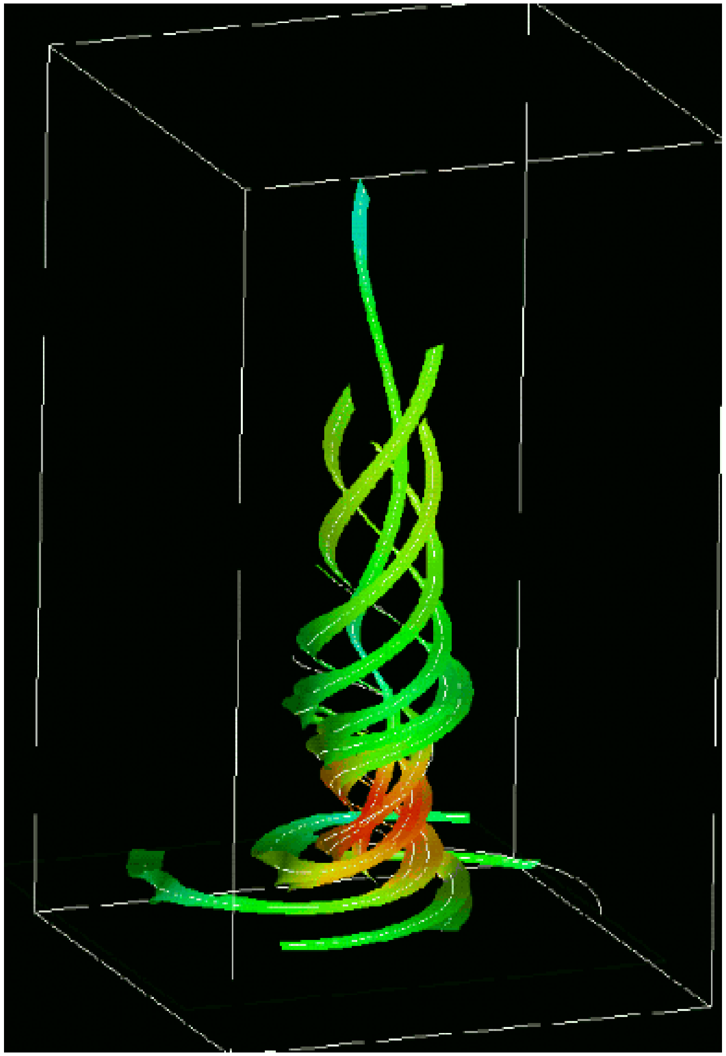
Direct vs. Indirect Flow Visualization

- Direct flow visualization
  - Overview of current flow state
  - Visualization of vectors: arrow plots (“hedgehog” plots)

- Indirect flow visualization
  - Use intermediate representation: vector field integration over time
  - Visualization of temporal evolution
  - Integral curves: streamlines, pathlines, streaklines, timelines
  - Integral surfaces: streamsurfaces, pathsurfaces, streaksurfaces

Direct vs. Indirect Flow Visualization

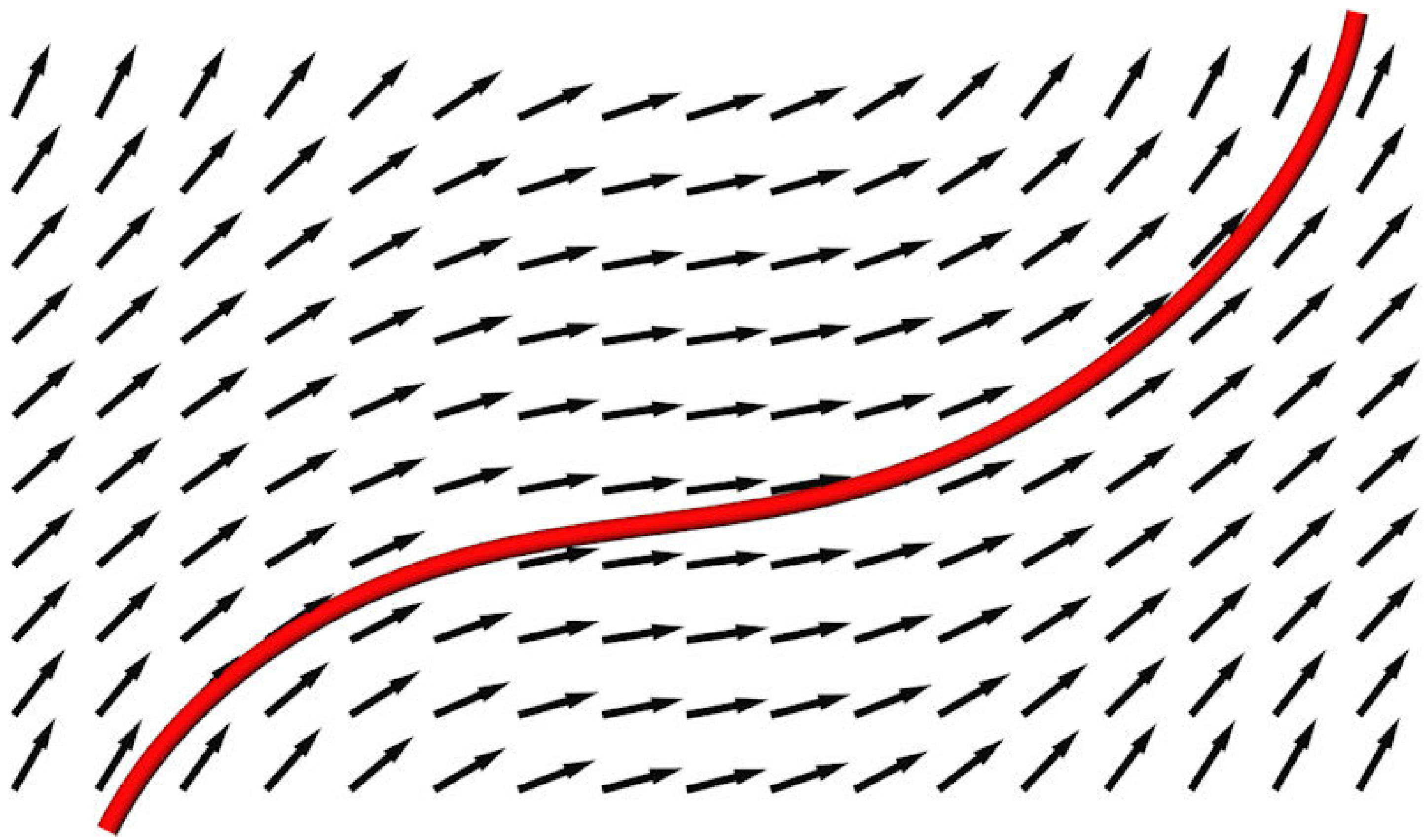




Integral Curves: Intro

Integral Curves / Stream Objects

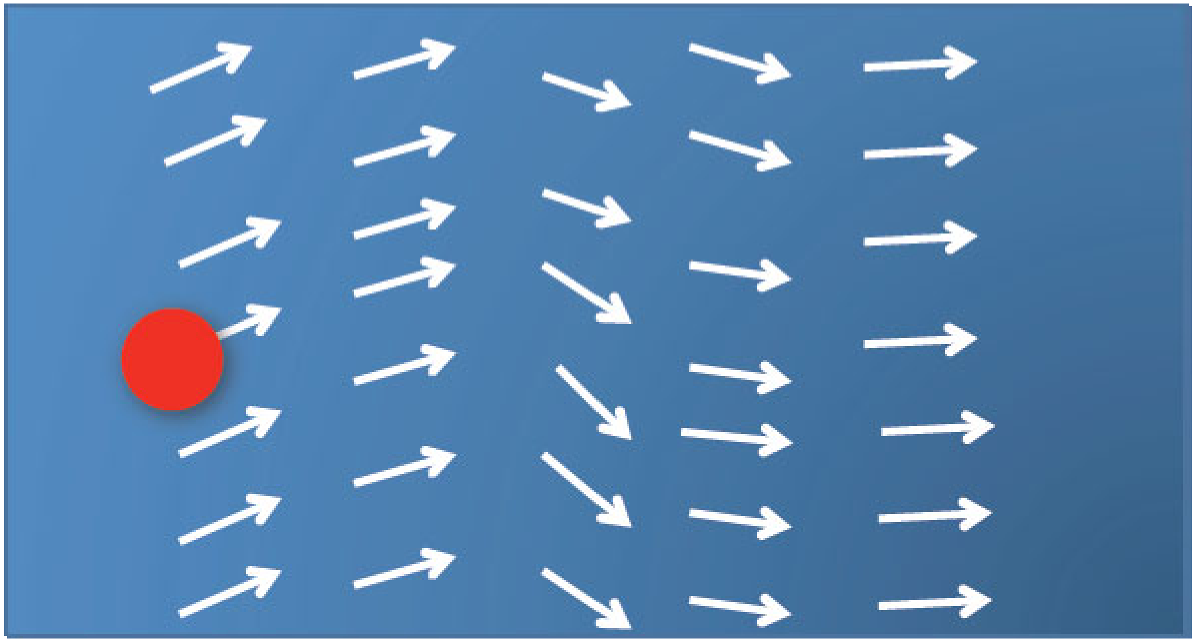
Integrating velocity over time yields spatial motion



Particle Trajectories

Courtesy Jens Krüger

Particle Trajectories



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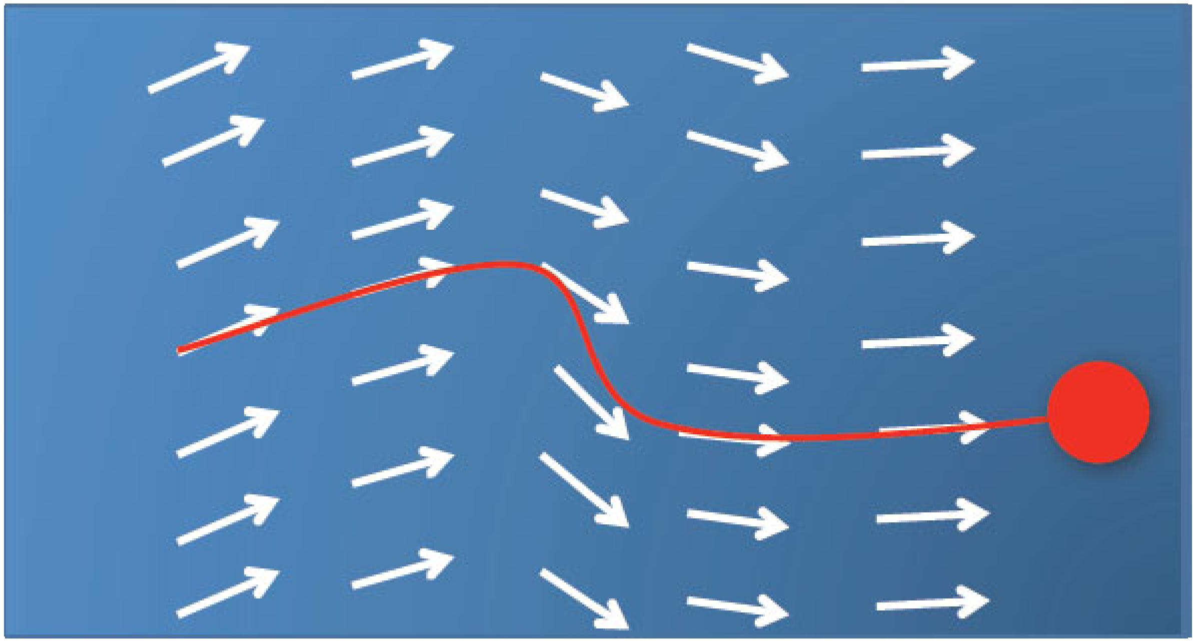
Particle Trajectories



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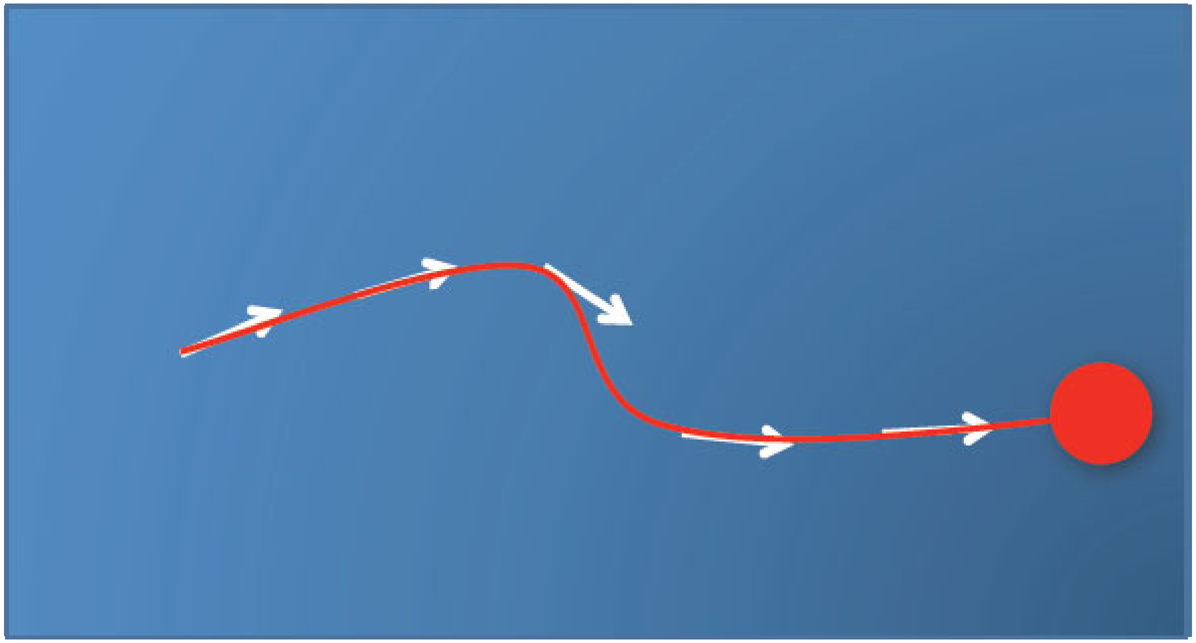
Particle Trajectories



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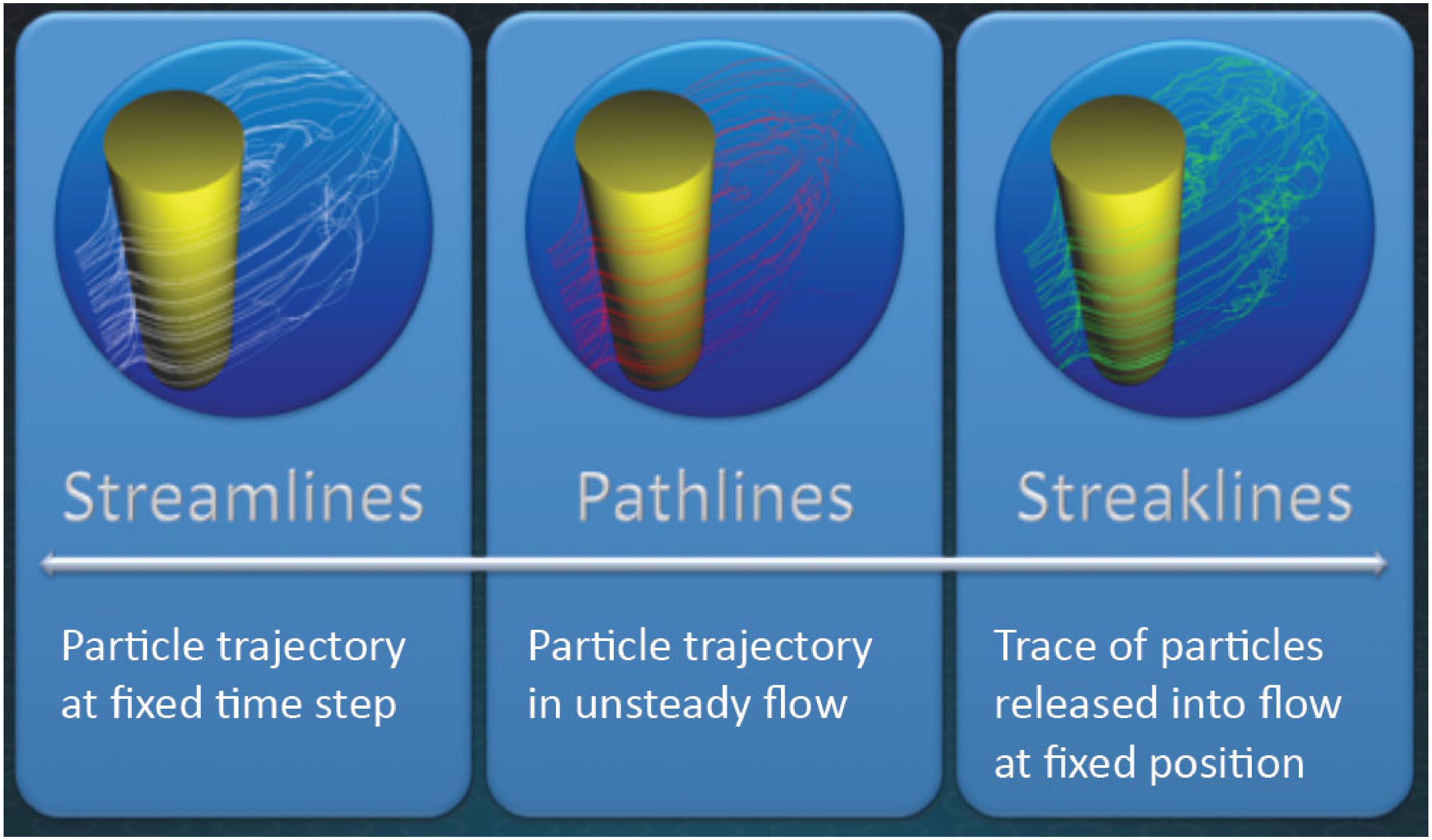
Integral Curves



Streamline

- Curve parallel to the vector field in each point for a fixed time

Pathline

- Describes motion of a massless particle over time

Streakline

- Location of all particles released at a *fixed position* over time

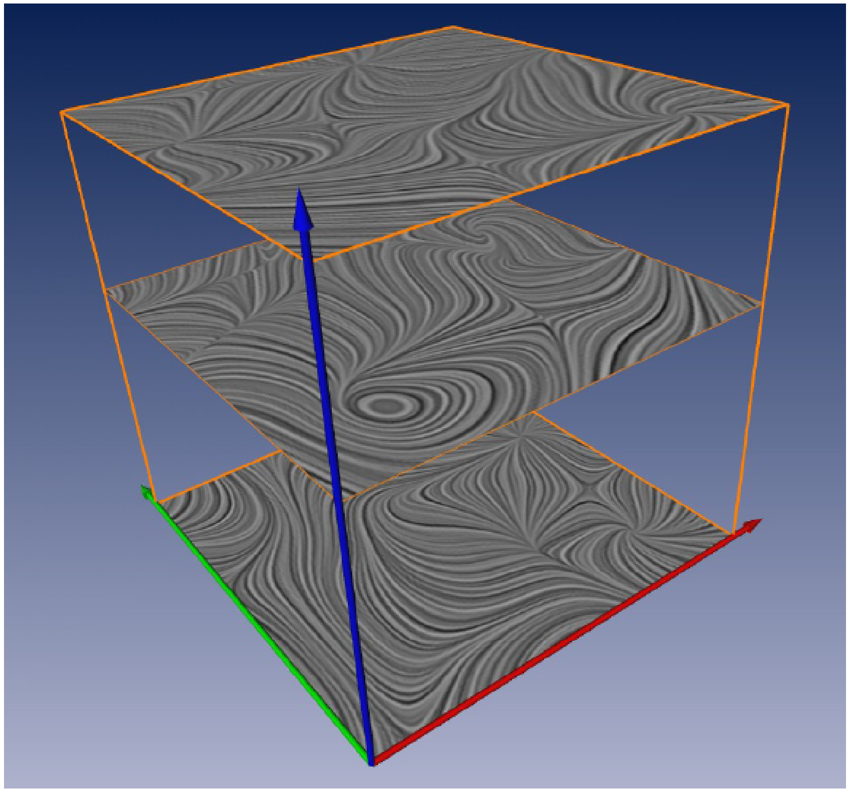
Timeline

- Location of all particles released along a line at a *fixed time*

Streamlines Over Time

Defined only for steady flow or for a fixed time step (of unsteady flow)

Different tangent curves in every time step for time-dependent vector fields (unsteady flow)

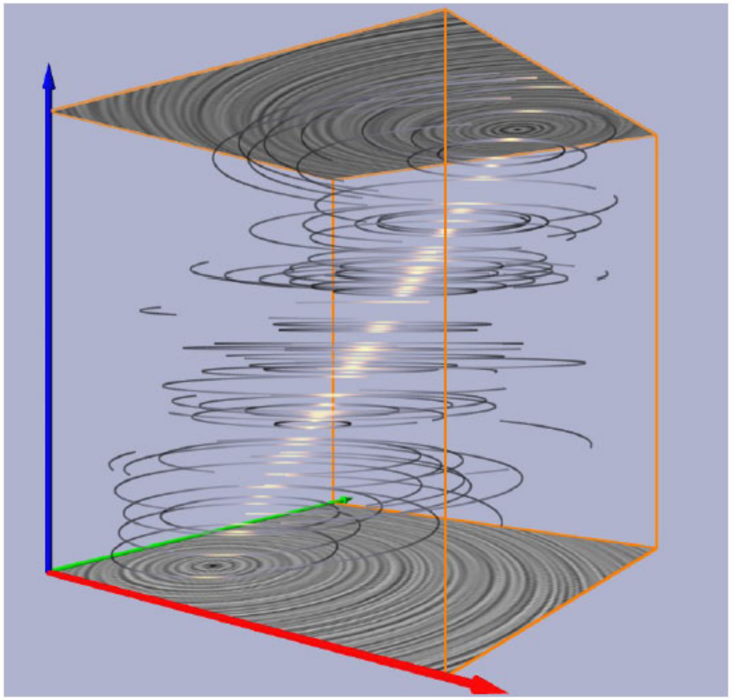


Tino Weinkauf

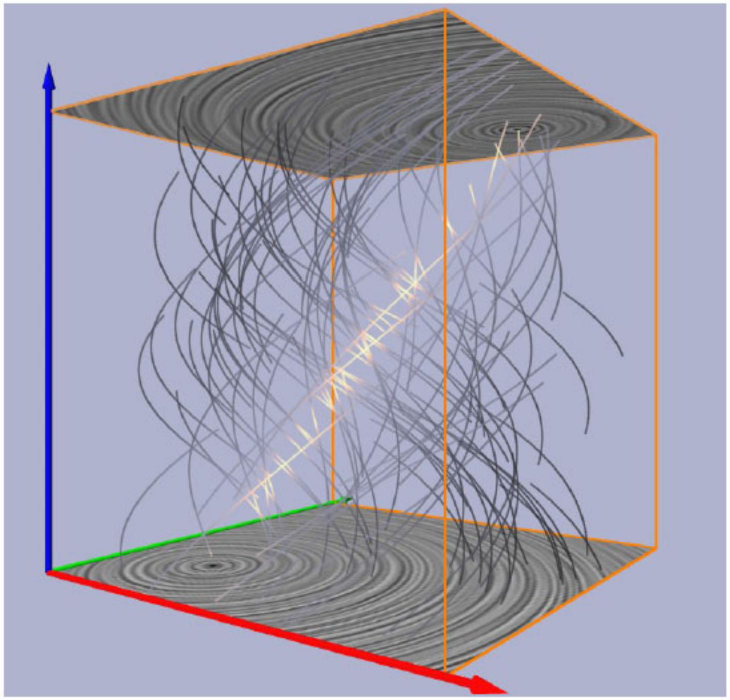
Stream Lines vs. Path Lines Viewed Over Time

Plotted with time as third dimension

- Tangent curves to a (n + 1)-dimensional vector field



Stream Lines



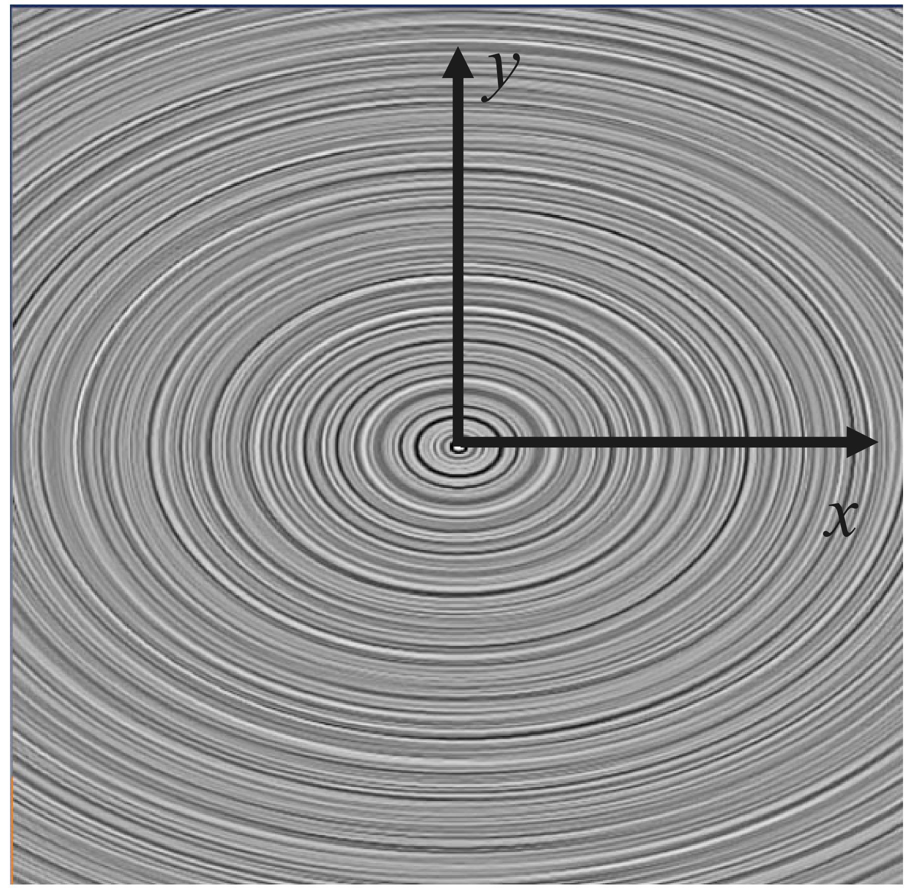
Path Lines

Numerical Integration

- **Numerical integration of stream lines:**

- approximate streamline by polygon $\mathbf{x}_i$

- **Testing example:**
  - $\mathbf{v}(x,y) = (-y, x/2)$^$T$
  - exact solution: ellipses
  - starting integration from (0,-1)



Streamlines – Practice

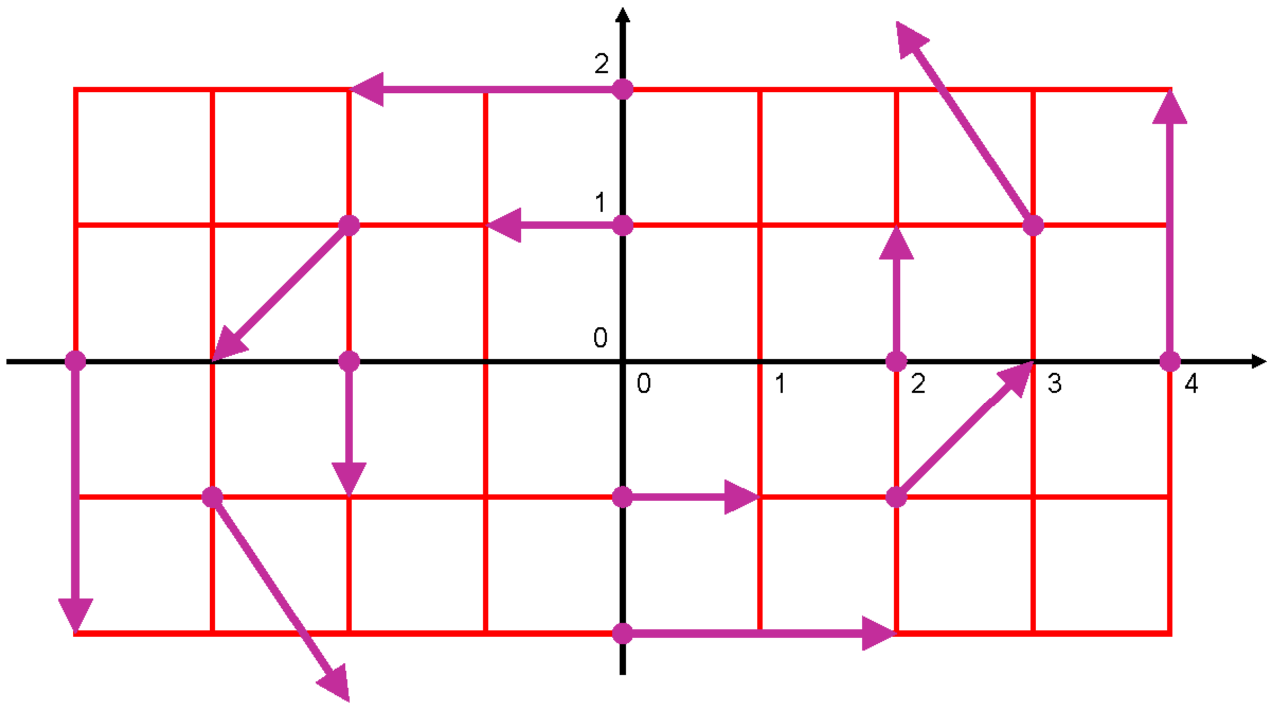
- Basic approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{s}(u))\,\mathrm{d}u$
  - practice: numerical integration
  - idea:
    (very) locally, the solution is (approx.) linear
  - Euler integration:
    follow the current flow vector $\mathbf{v}(\mathbf{s}_i)$ from the current streamline point $\mathbf{s}_i$ for a very small time ($\mathrm{d}t$) and therefore distance
  - Euler integration: $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i)$,
    integration of small steps ($\mathrm{d}t$ very small)

Euler Integration – Example

- 2D model data:

$$v_x = dx/dt = -y$$
$$v_y = dy/dt = x/2$$

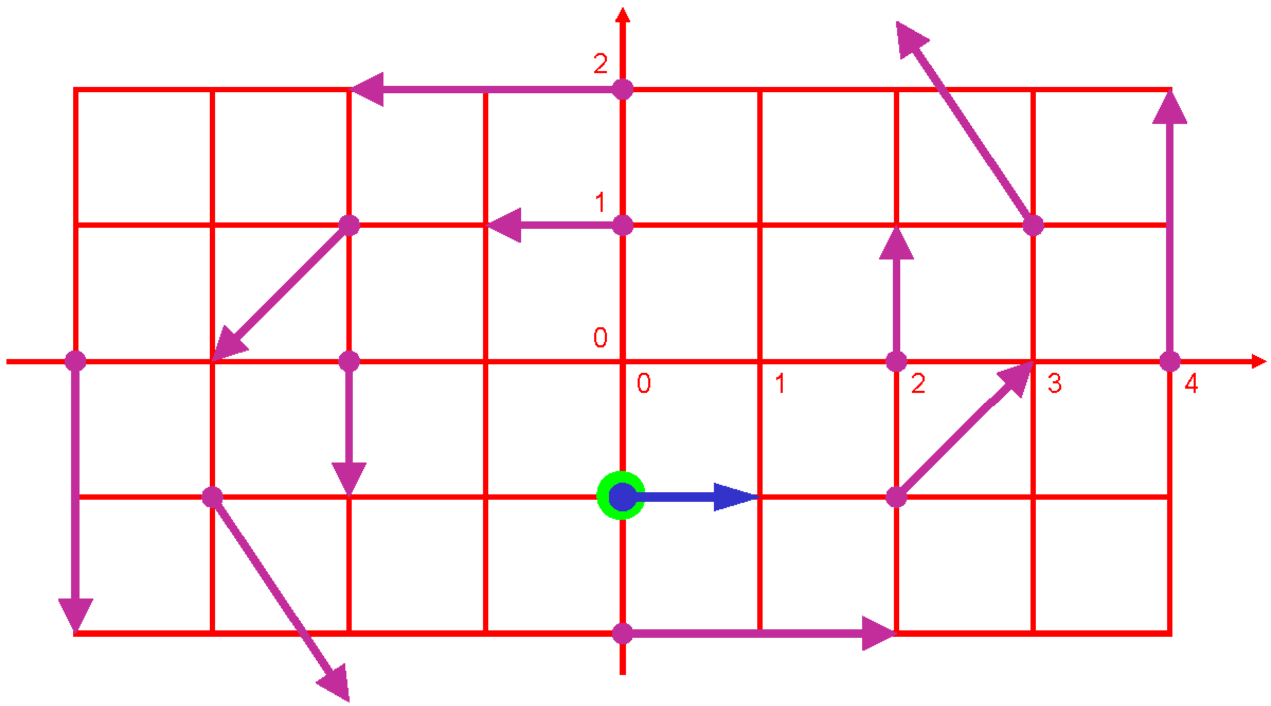
- Sample arrows:



- True solution: ellipses!

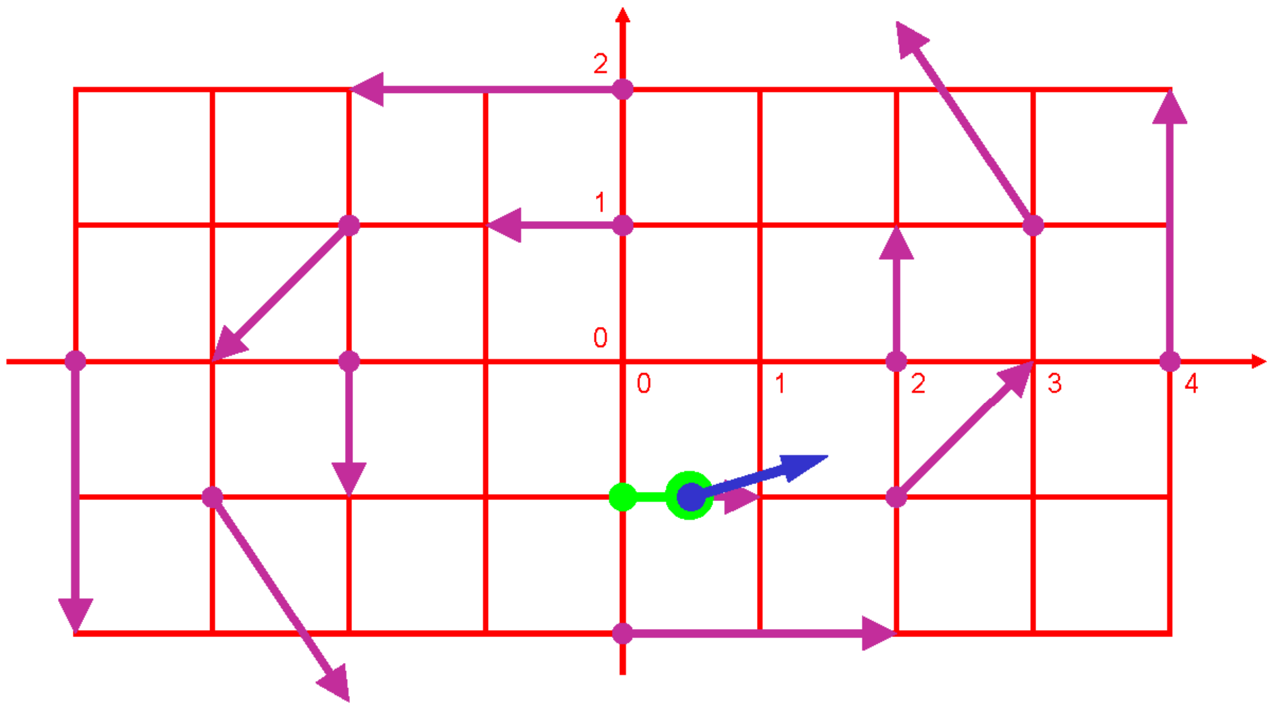
Euler Integration – Example

- Seed point $\mathbf{s}_0 = (0|-1)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (1|0)^T$;
  $dt = 1/2$



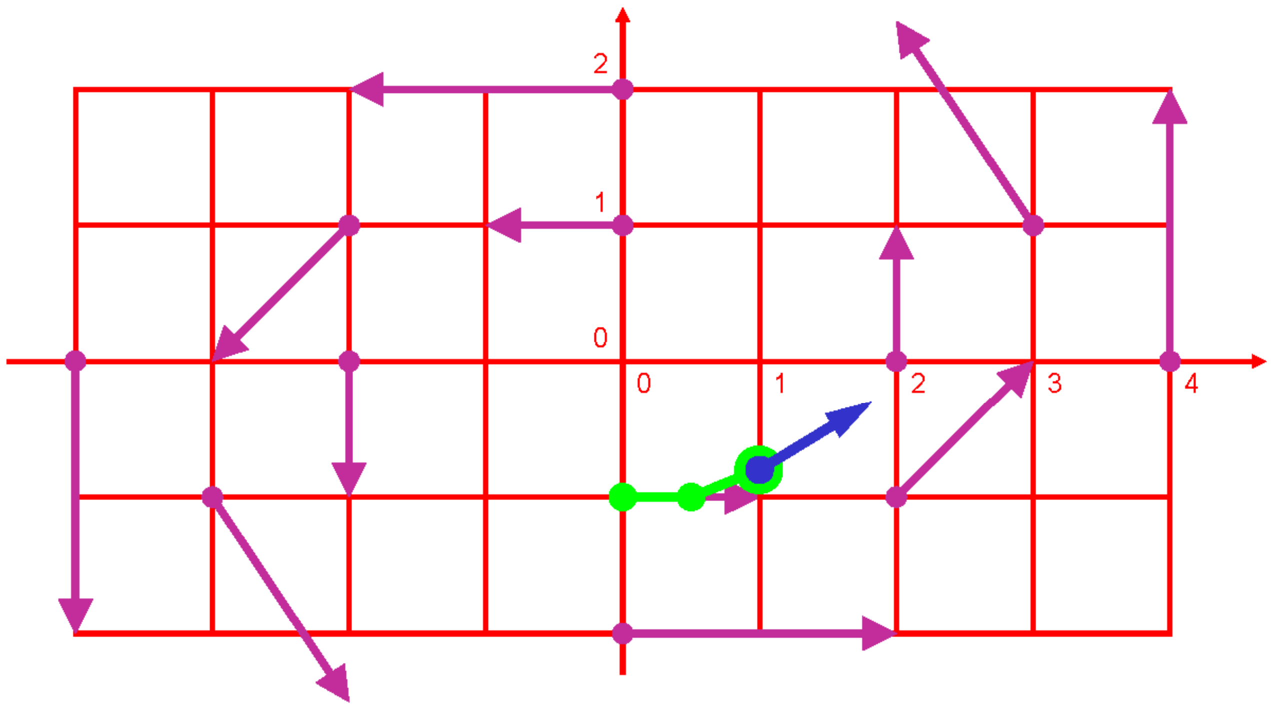
Euler Integration – Example

- New point $\mathbf{s}_1 = \mathbf{s}_0 + \mathbf{v}(\mathbf{s}_0) \cdot dt = (1/2 \,|\, -1)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (1 \,|\, 1/4)^T$;



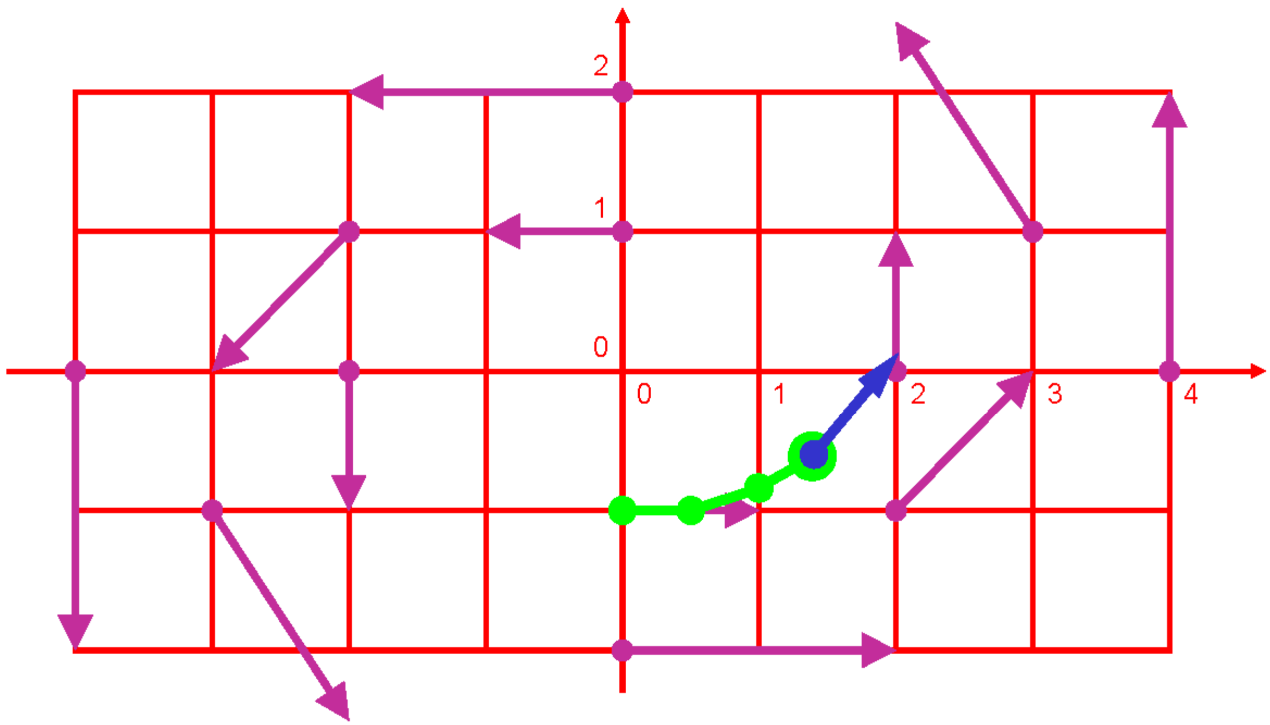
Euler Integration – Example

- New point $\mathbf{s}_2 = \mathbf{s}_1 + \mathbf{v}(\mathbf{s}_1) \cdot dt = (1\,|\,-7/8)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_2) = (7/8\,|\,1/2)^T$;



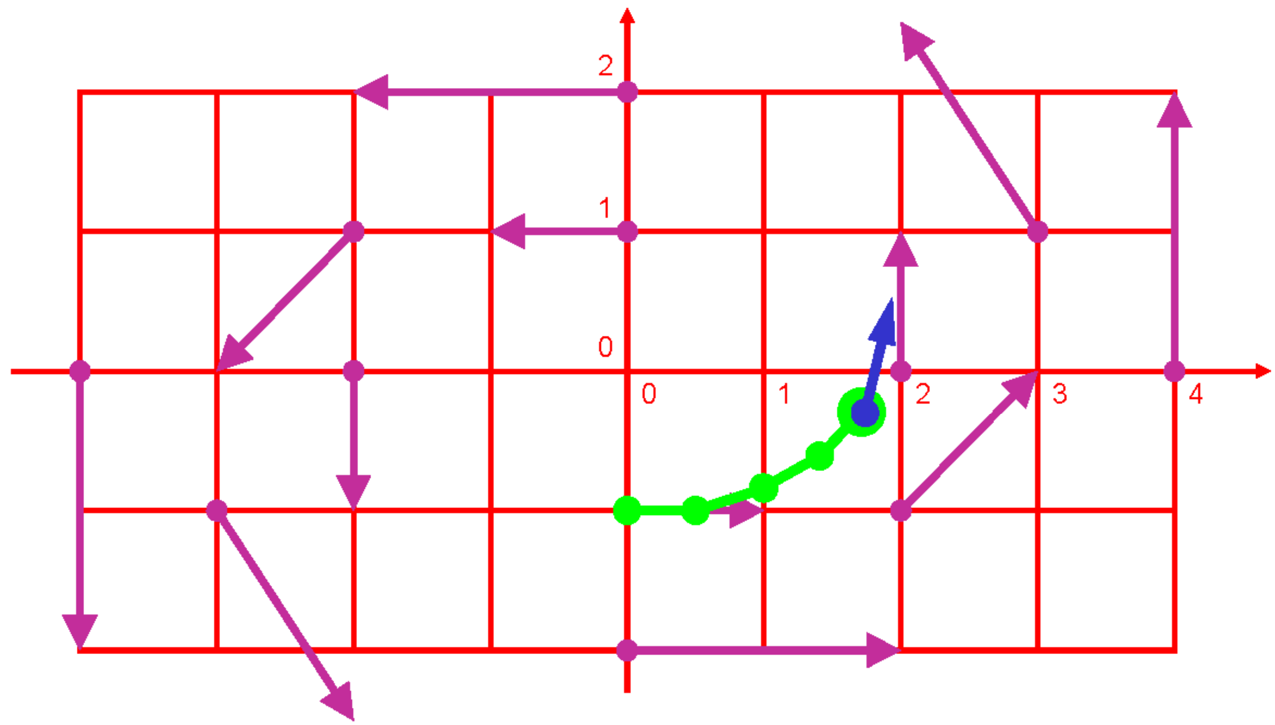
Euler Integration – Example

- $\mathbf{s}_3 \quad = (23/16 \,|\, -5/8)^T \quad \approx (1.44 \,|\, -0.63)^T;$
  $\mathbf{v}(\mathbf{s}_3) \quad = (5/8 \,|\, 23/32)^T \quad \approx (0.63 \,|\, 0.72)^T;$



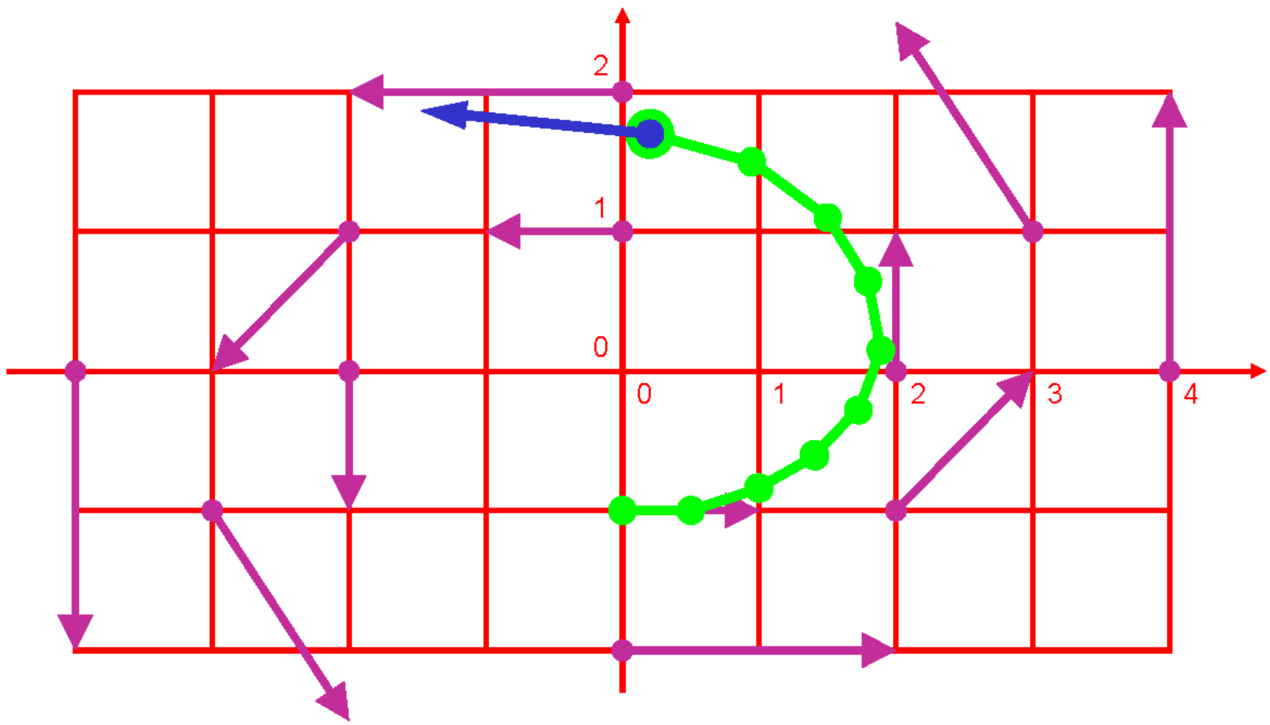
Euler Integration – Example

- $s_4 = (7/4 \mid -17/64)^T \approx (1.75 \mid -0.27)^T$;
  $v(s_4) = (17/64 \mid 7/8)^T \approx (0.27 \mid 0.88)^T$;



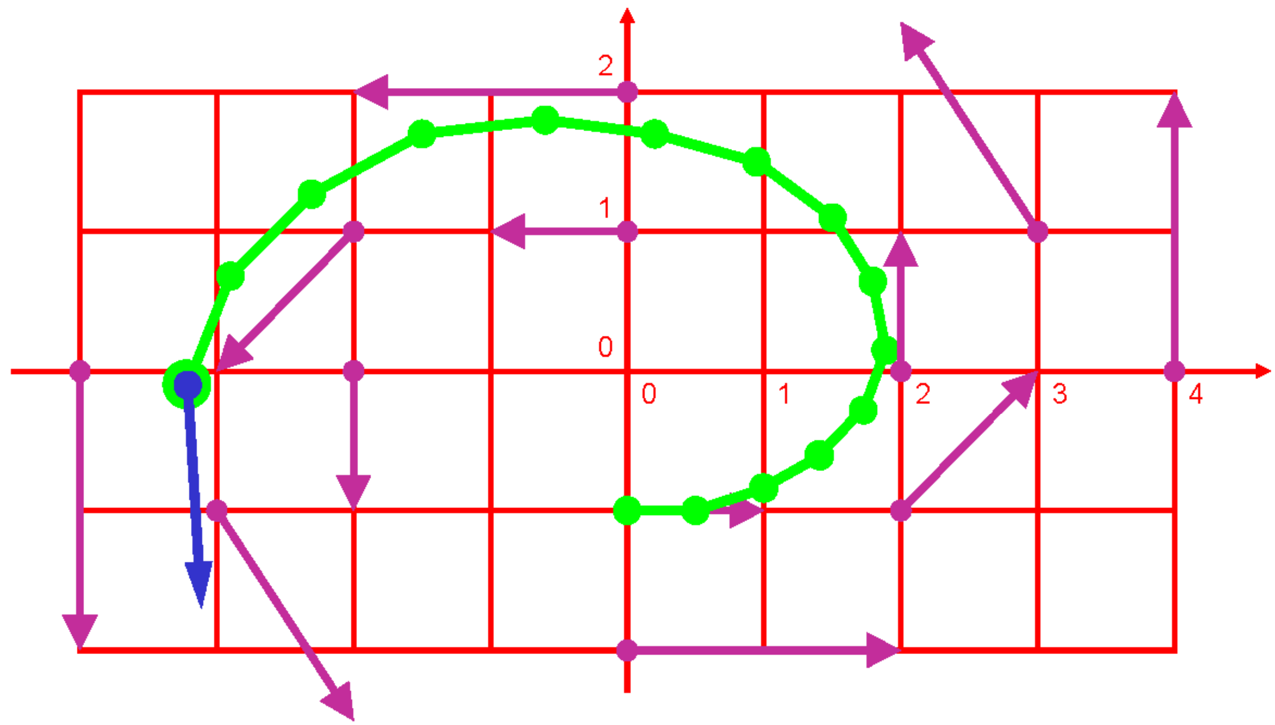
Euler Integration – Example

- $\mathbf{s}_9 \approx (0.20\,|\,1.69)^T;$
  $\mathbf{v}(\mathbf{s}_9) \approx (-1.69\,|\,0.10)^T;$



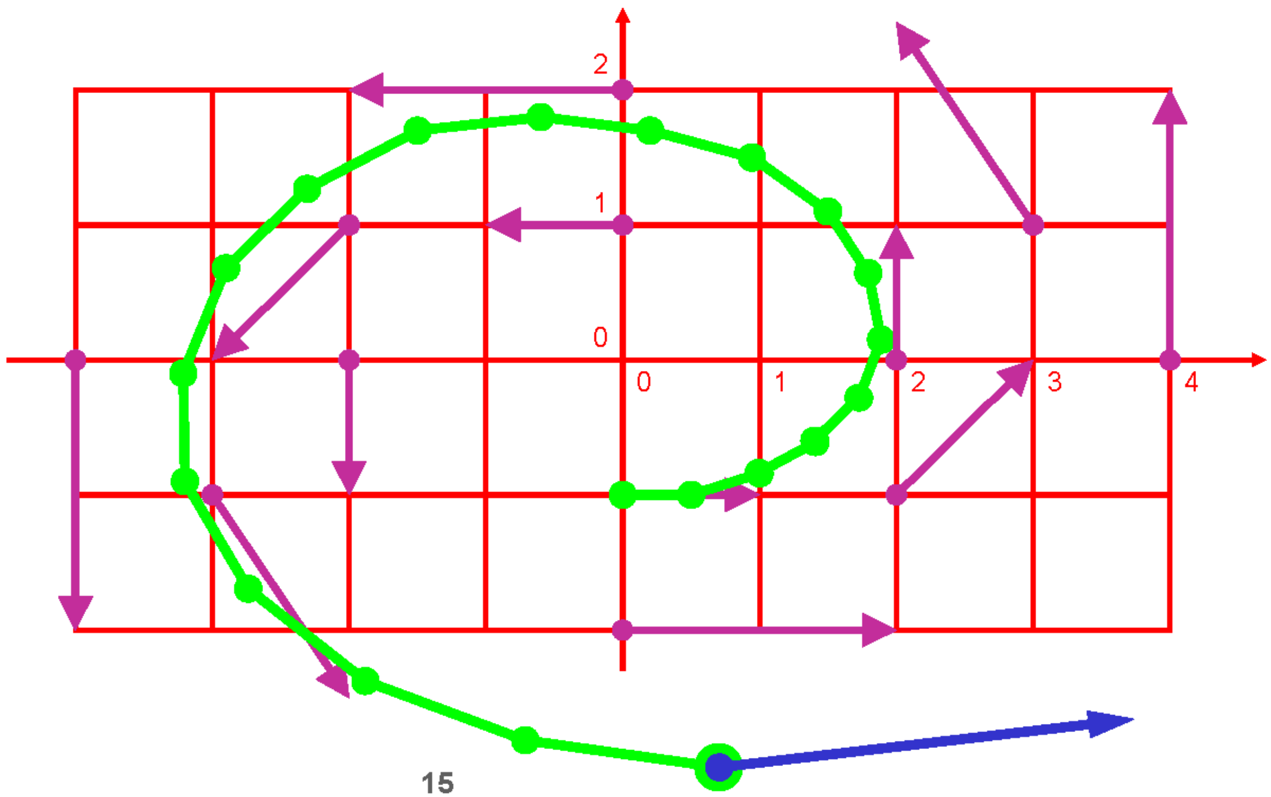
Euler Integration – Example

- $\mathbf{s}_{14} \approx (-3.22|-0.10)^T$;
  $\mathbf{v}(\mathbf{s}_{14}) \approx (0.10|-1.61)^T$;



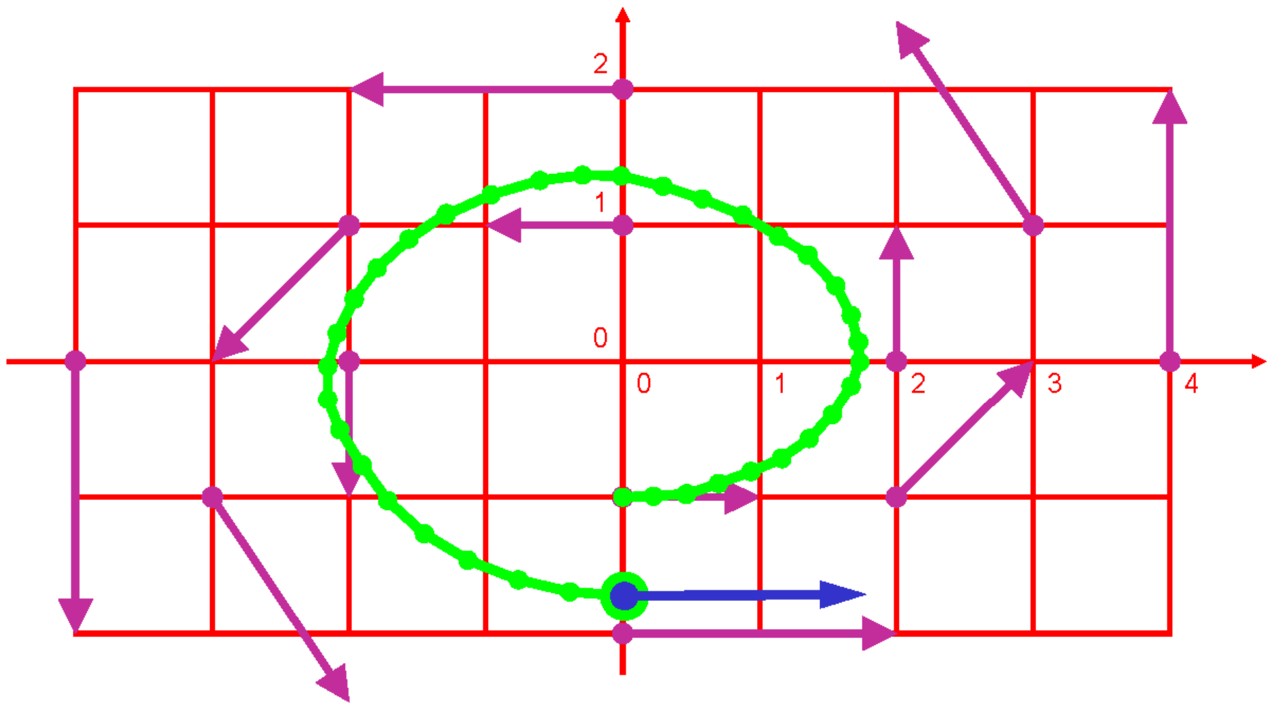
Euler Integration – Example

- $\mathbf{s}_{19} \approx (0.75|-3.02)^T$; $\mathbf{v}(\mathbf{s}_{19}) \approx (3.02|0.37)^T$;
  clearly: large integration error, dt too large!
  19 steps



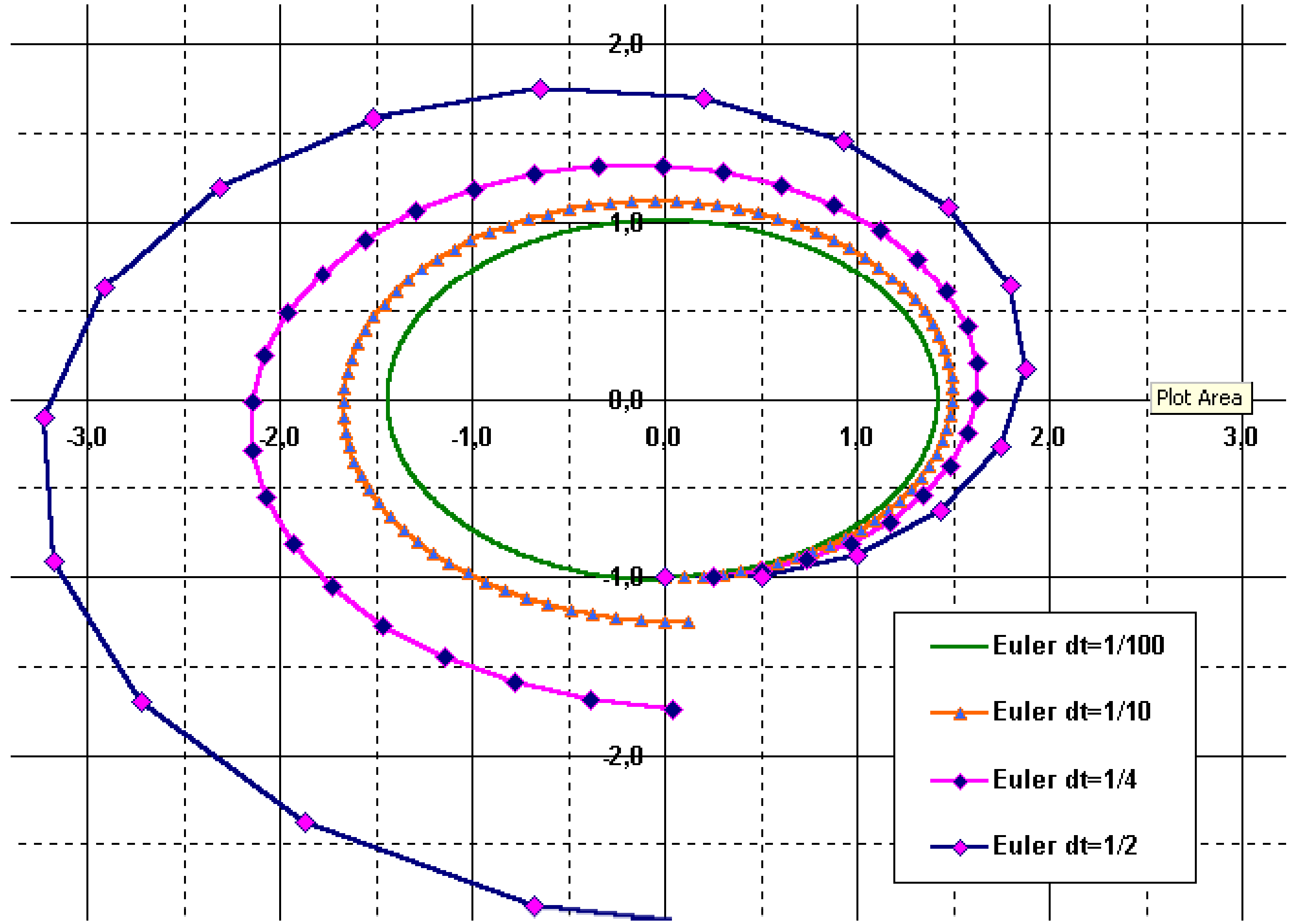
Euler Integration – Example

- d*t* smaller (1/4): more steps, more exact!
  $\mathbf{s}_{36} \approx (0.04 \,|\, -1.74)^T$; $\mathbf{v}(\mathbf{s}_{36}) \approx (1.74 \,|\, 0.02)^T$;
- 36 steps



Comparison Euler, Step Sizes

Euler
is getting
better
propor-
tionally
to d*t*

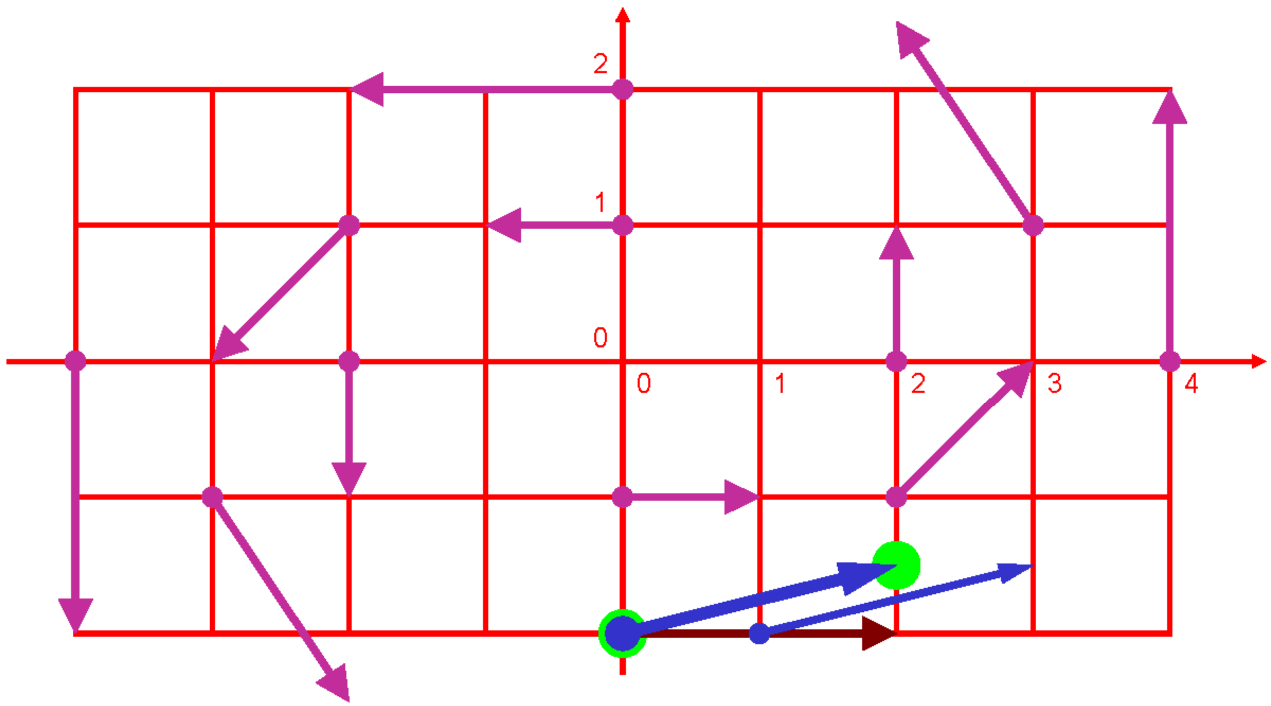


Better than Euler Integr.: RK

- Runge-Kutta Approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{s}(u))\, du$
  - Euler: $\mathbf{s}_i = \mathbf{s}_0 + \sum_{0 \le u < i} \mathbf{v}(\mathbf{s}_u) \cdot dt$
  - Runge-Kutta integration:
    - idea: cut short the curve arc
    - RK-2 (second order RK):
      1.: do half a Euler step
      2.: evaluate flow vector there
      3.: use it in the origin
    - RK-2 (two evaluations of **v** per step):
      $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathbf{v}(\mathbf{s}_i + \mathbf{v}(\mathbf{s}_i) \cdot dt/2) \cdot dt$

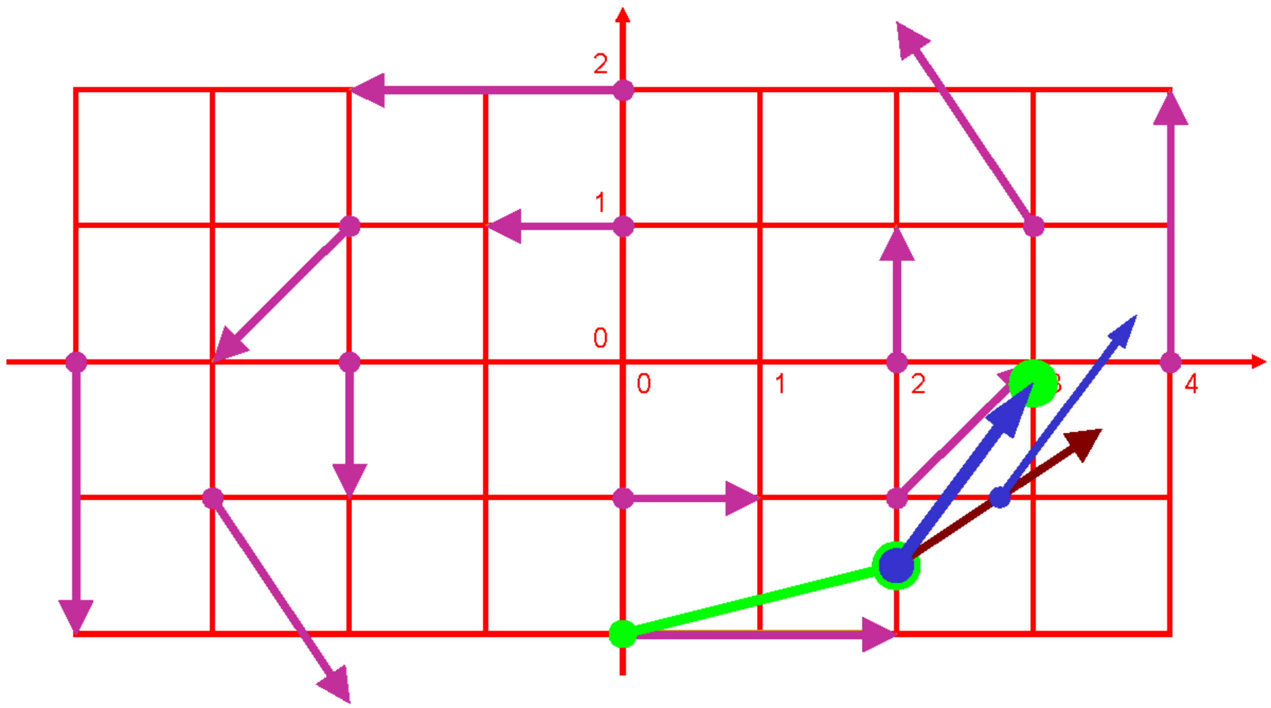
RK-2 Integration – One Step

- Seed point $\mathbf{s}_0 = (0|-2)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (2|0)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_0+\mathbf{v}(\mathbf{s}_0)\cdot dt/2) = (2|0.5)^T$;
  $dt = 1$



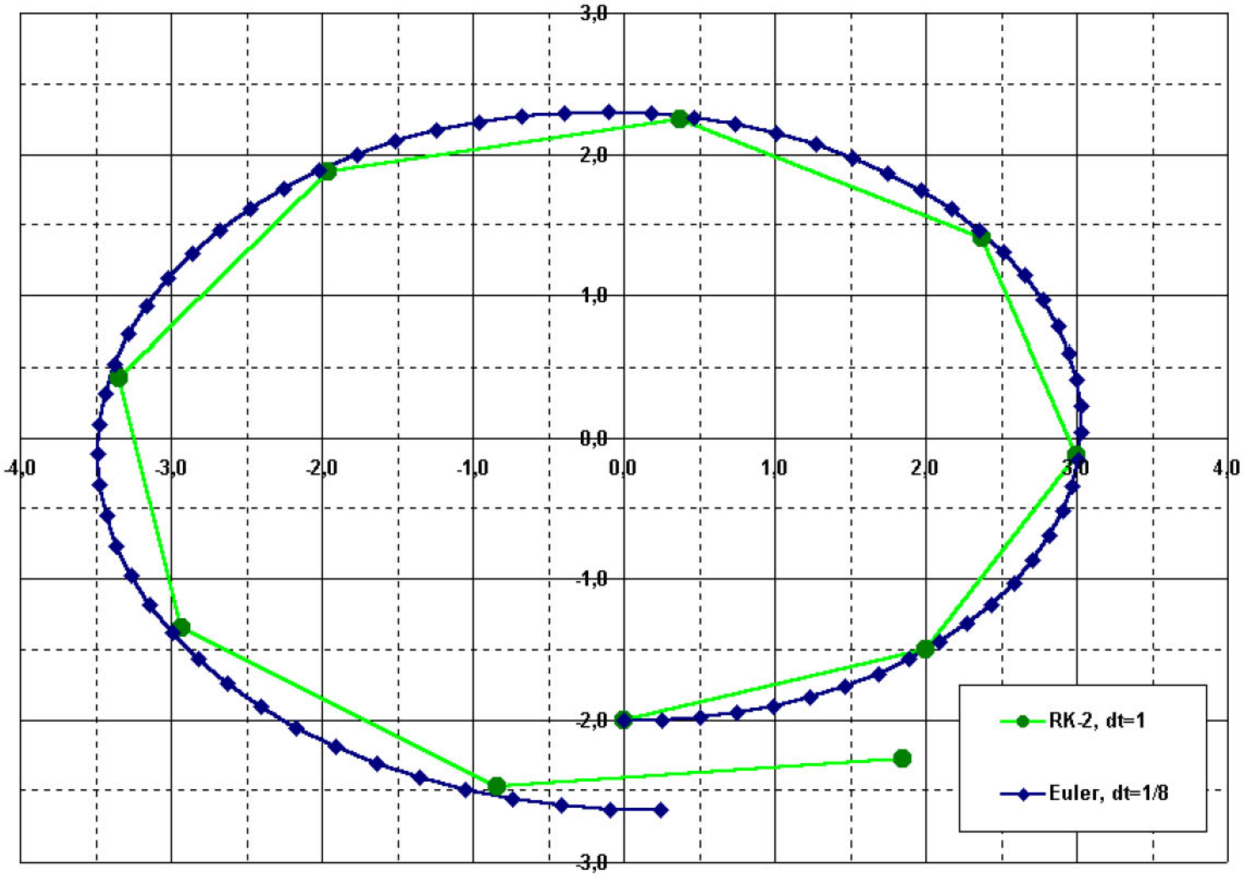
RK-2 – One more step

- Seed point $\mathbf{s}_1 = (2|-1.5)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (1.5|1)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_1+\mathbf{v}(\mathbf{s}_1)\cdot dt/2) \approx (1|1.4)^T$;
  $dt = 1$



RK-2 – A Quick Round

- RK-2: even with dt=1 (9 steps) better than Euler with dt=1/8 (72 steps)

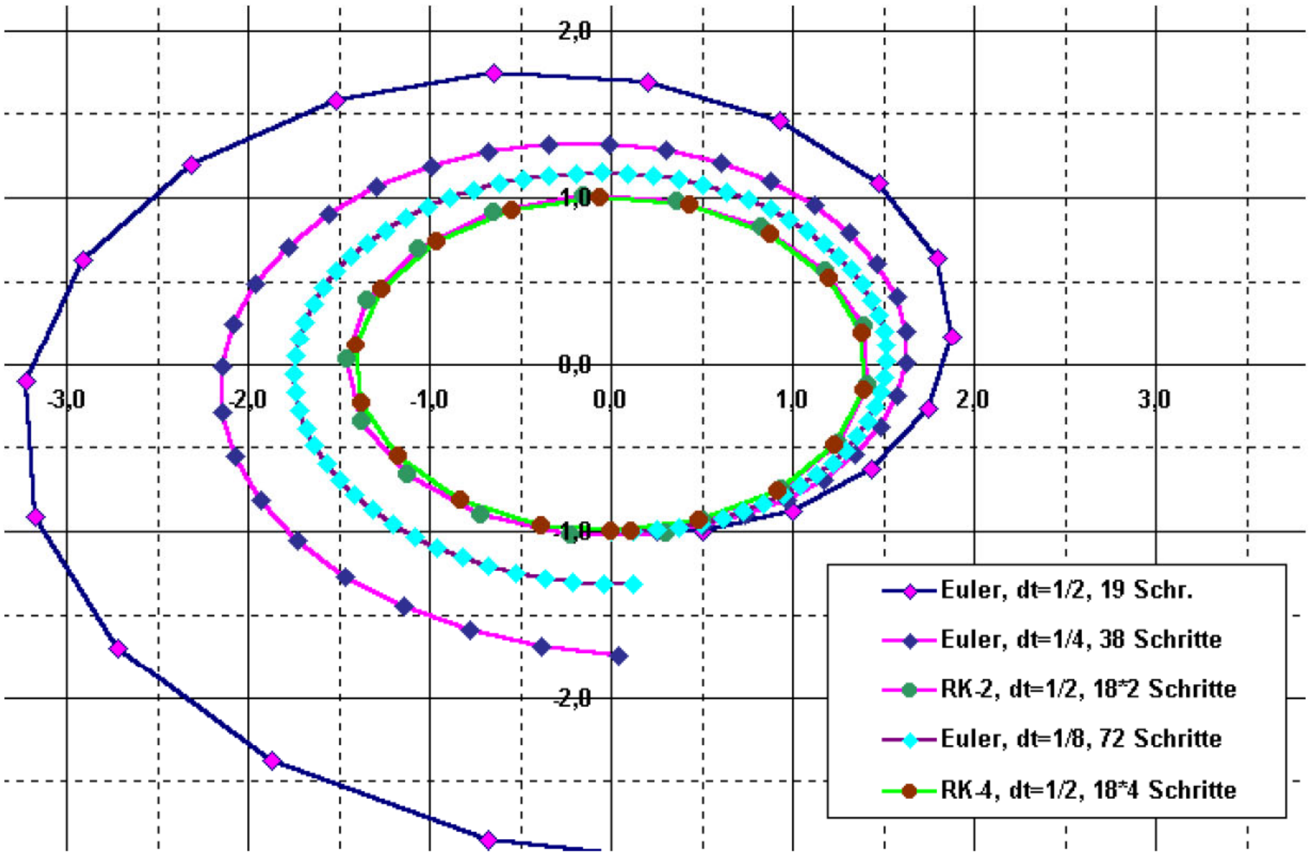


RK-4 vs. Euler, RK-2

- Even better: fourth order RK:
  - four vectors **a**, **b**, **c**, **d**
  - one step is a convex combination: $\mathbf{s}_{i+1} = \mathbf{s}_i + (\mathbf{a} + 2 \cdot \mathbf{b} + 2 \cdot \mathbf{c} + \mathbf{d})/6$
  - vectors:
    - $\mathbf{a} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i)$ … original vector
    - $\mathbf{b} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{a}/2)$ … RK-2 vector
    - $\mathbf{c} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{b}/2)$ … use RK-2 …
    - $\mathbf{d} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{c})$ … and again!

Euler vs. Runge-Kutta

- RK-4: pays off only with complex flows
- Here approx. like RK-2



Integration, Conclusions

- Summary:
  - analytic determination of streamlines usually not possible
  - hence: numerical integration
  - several methods available (Euler, Runge-Kutta, etc.)
  - Euler: simple, imprecise, esp. with small dt
  - RK: more accurate in higher orders
  - furthermore: adaptive methods, implicit methods, etc.

Bonus Slides:
Vectors as Derivative Operators

Vectors as Derivative Operators

A vector applied to a (real) function on the manifold gives the *directional derivative* in that direction

- From this viewpoint, the vector is a derivative operator (actually, a *derivation*)
- Can be used as *definition* of a vector (must fulfill props. of a derivation; esp. Leibniz rule)

$$\begin{aligned} f : M &\to \mathbb{R}, \\ x &\mapsto f(x). \end{aligned} \qquad \mathbf{v} f$$

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Kronecker delta
("identity matrix")

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For vector field: obtain directional derivative at each point

$$\mathbf{v} f : M \to \mathbb{R}, \quad x \mapsto \mathbf{v}(x) f = df(\mathbf{v}(x)).$$

(remember that this just *looks* scary (maybe) ...)

Thank you.

Thanks for material

- Helwig Hauser
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- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama