

# CS 247 – Scientific Visualization

## Lecture 24: Vector / Flow Visualization, Pt. 3 [preview]

Markus Hadwiger, KAUST

# Reading Assignment #13 (until Apr 25)



Read (required):

- Data Visualization book
  - Chapter 6.1 (Divergence and Vorticity)
- Diffeomorphisms / smooth deformations

<https://en.wikipedia.org/wiki/Diffeomorphism>

- Integral curves: Stream lines, path lines, streak lines

[https://en.wikipedia.org/wiki/Integral\\_curve](https://en.wikipedia.org/wiki/Integral_curve)

[https://en.wikipedia.org/wiki/Streamlines,\\_streaklines,\\_and\\_pathlines](https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines)

- Paper:

Bruno Jobard and Wilfrid Lefer

*Creating Evenly-Spaced Streamlines of Arbitrary Density,*

<http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.29.9498>



# Quiz #3: Apr 25

## Organization

- First 30 min of lecture
- No material (book, notes, ...) allowed

## Content of questions

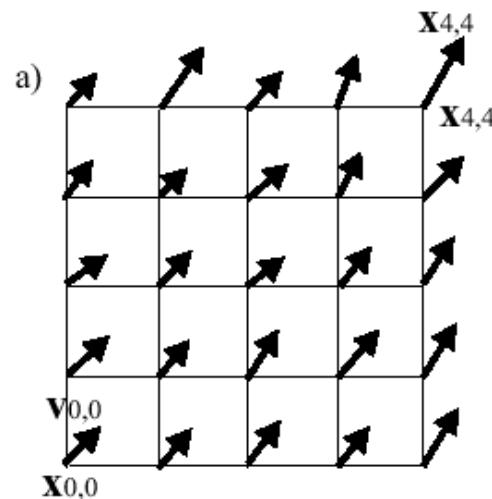
- Lectures (both actual lectures and slides)
- Reading assignments (except optional ones)
- Programming assignments (algorithms, methods)
- Solve short practical examples

# Vector Fields

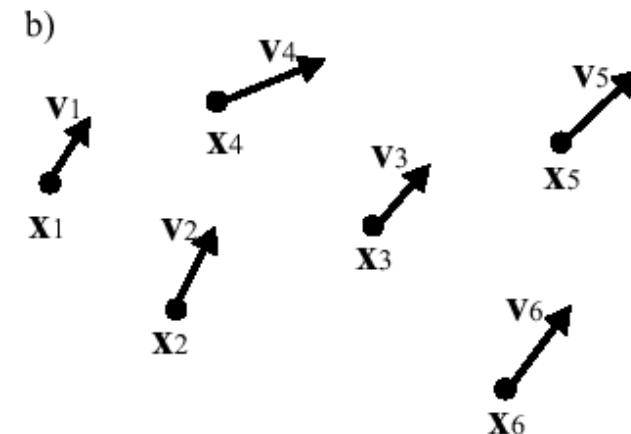


Each vector is usually thought of as a velocity vector

- Example for actual velocity: fluid flow
- But also force fields, etc. (e.g., electrostatic field)



vectors given at grid points



vectors given at particle positions



# Vector Fields

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Each vector in a vector field  
lives in the **tangent space**  
of the manifold at that point:

Each vector is a **tangent vector**

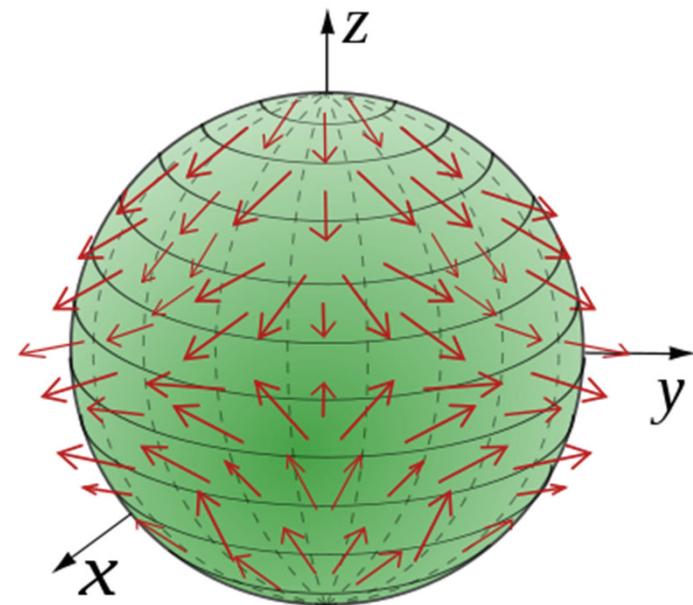
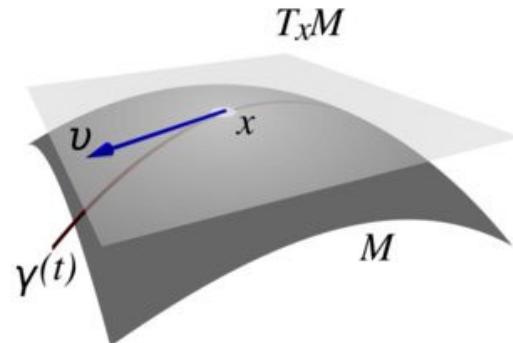


image from wikipedia

# Vector Fields



Vector fields on general manifolds  $M$  (not just Euclidean space)

*Tangent space at a point  $x \in M$ :*

$$T_x M$$

*Tangent bundle:* Manifold of all tangent spaces over base manifold

$$\pi: TM \rightarrow M$$

Vector field: *Section of tangent bundle*

$$s: M \rightarrow TM,$$

$$x \mapsto s(x). \quad \pi(s(x)) = x$$

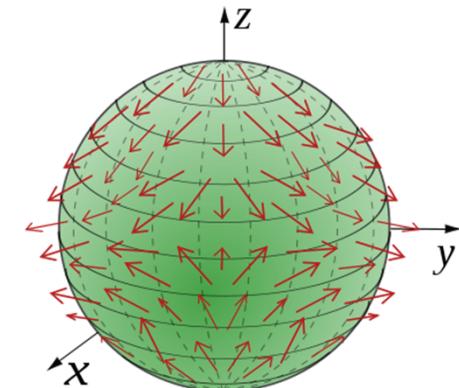
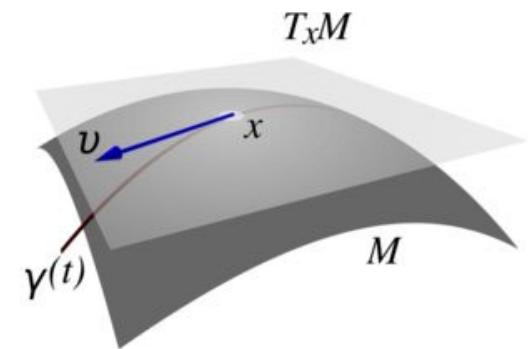


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Vector field: *Section of tangent bundle*

$$\mathbf{v}: M \rightarrow TM,$$

$$x \mapsto \mathbf{v}(x).$$

$$\mathbf{v}(x) \in T_x M$$

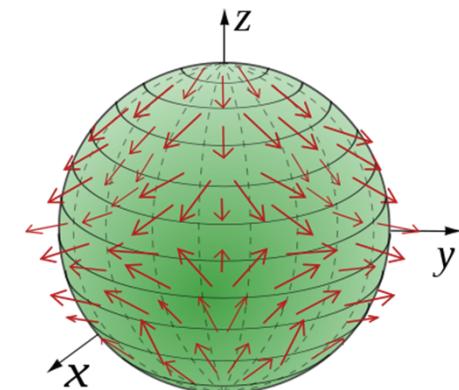
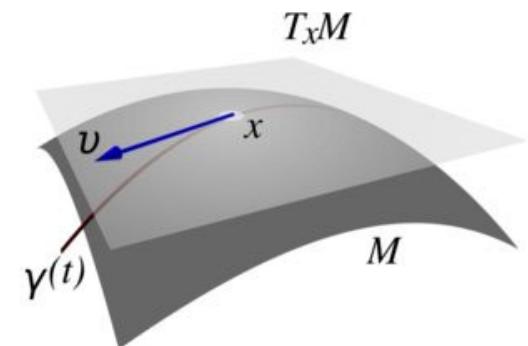


image from wikipedia

# Interlude: Coordinate Charts

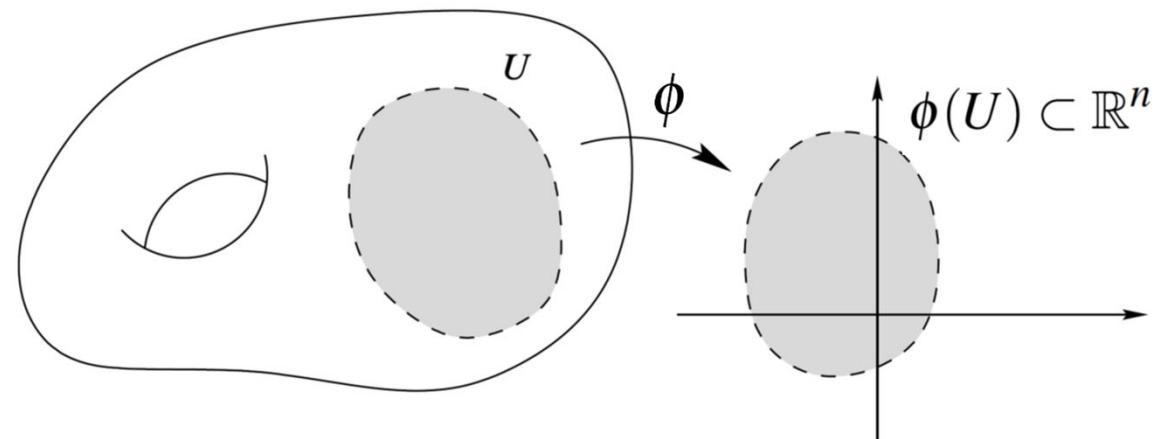


# Interlude: Coordinate Charts

Coordinate chart

$$\phi: U \subset M \rightarrow \mathbb{R}^n,$$

$$x \mapsto (x^1, x^2, \dots, x^n).$$





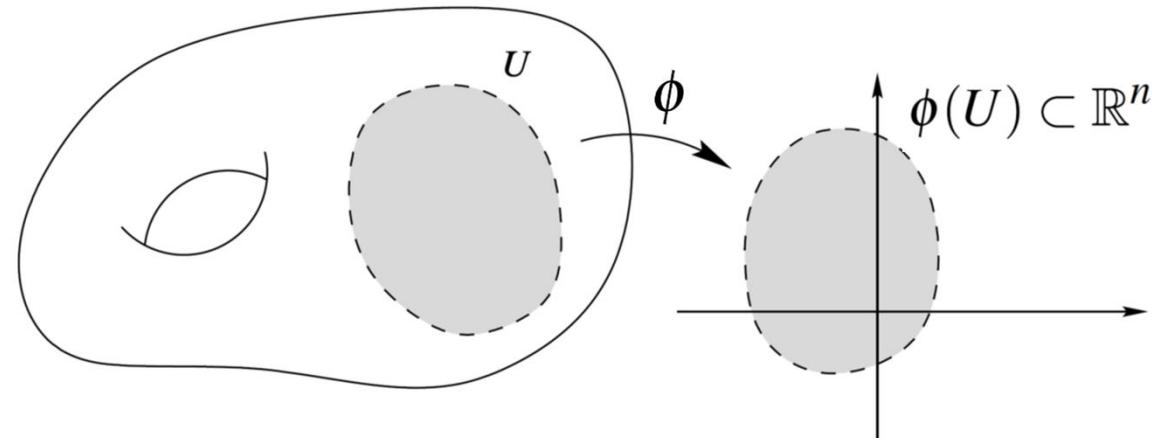
# Interlude: Coordinate Charts

Coordinate chart

$$\begin{aligned}\phi: U \subset M &\rightarrow \mathbb{R}^n, \\ x &\mapsto (x^1, x^2, \dots, x^n).\end{aligned}$$

Coordinate functions

$$\begin{aligned}x^i: U \subset M &\rightarrow \mathbb{R}, \\ x &\mapsto x^i(x).\end{aligned}$$





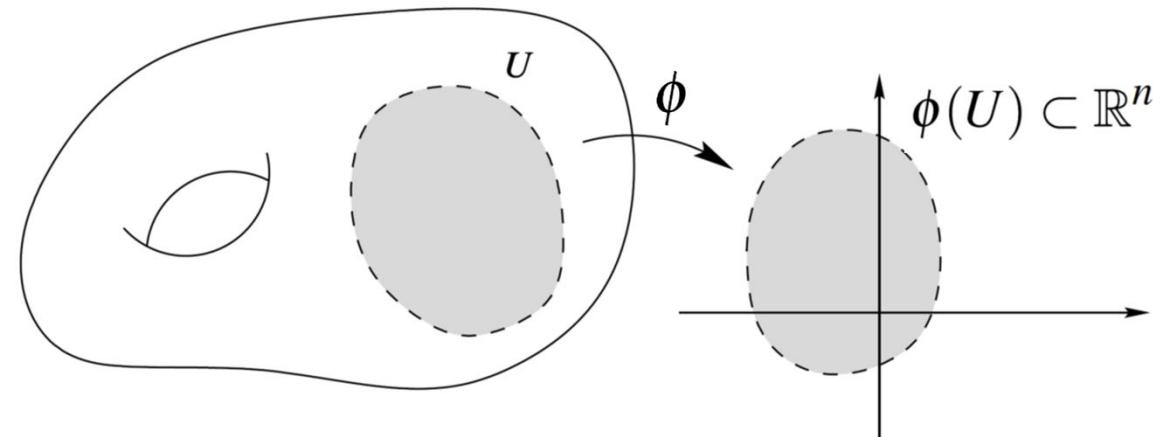
# Interlude: Coordinate Charts

Coordinate charts

$$\begin{aligned}\phi_\alpha: U_\alpha \subset M &\rightarrow \mathbb{R}^n, \\ x &\mapsto (x^1, x^2, \dots, x^n).\end{aligned}$$

Atlas

$$\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$$





# Interlude: Coordinate Charts

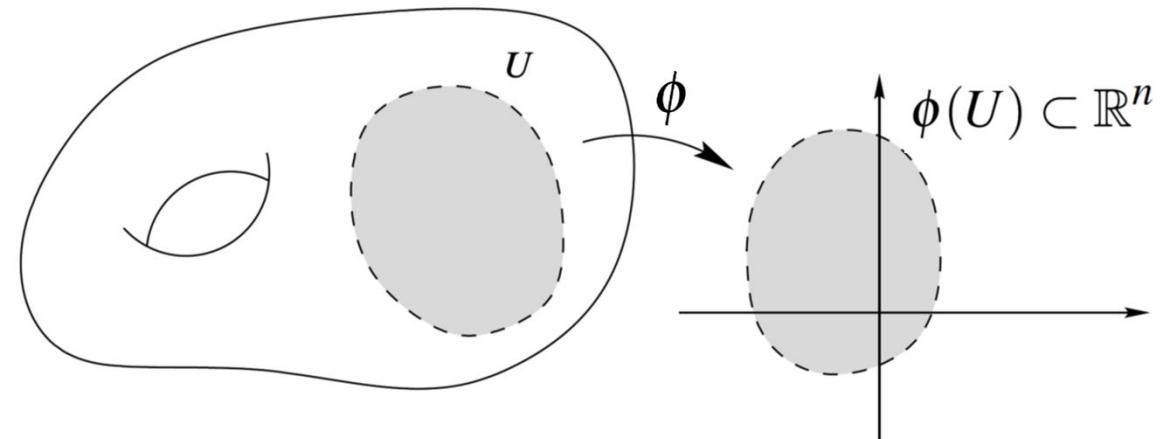
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Atlas

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$$\begin{aligned}\phi_\alpha : U_\alpha \subset M &\rightarrow \mathbb{R}^n, \\ x &\mapsto (x^1(x), x^2(x), \dots, x^n(x)).\end{aligned}$$





# Vector Fields vs. Vectors in Components

Because Euclidean space is most common, often slightly sloppy notation

$$\mathbf{v}: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

$$(x, y) \mapsto \begin{bmatrix} u \\ v \end{bmatrix}.$$



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$$\mathbf{v}: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

$$(x, y, z) \mapsto \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

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# Vector Fields vs. Vectors in Components

$\mathbf{v}: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n,$

$$(x^1, x^2, \dots, x^n) \mapsto \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}.$$

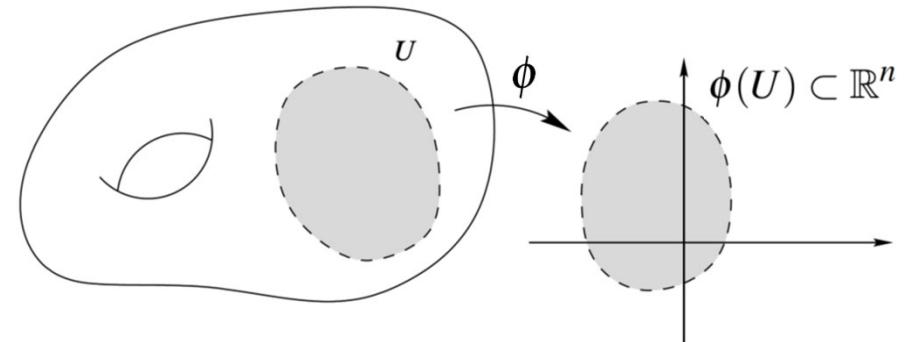
$\mathbf{v}: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n,$

$$(x^1, x^2, \dots, x^n) \mapsto \begin{pmatrix} v^1(x^1, x^2, \dots, x^n) \\ v^2(x^1, x^2, \dots, x^n) \\ \vdots \\ v^n(x^1, x^2, \dots, x^n) \end{pmatrix}.$$



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$$\mathbf{v}|_U: \phi(U) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n,$$
$$(x^1, x^2, \dots, x^n) \mapsto \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}.$$



# Vector Fields vs. Vectors in Components

Need basis vector fields

$$\begin{aligned}\mathbf{e}_i : U \subset M &\rightarrow TM, \\ x &\mapsto \mathbf{e}_i(x)\end{aligned}\quad \left\{\mathbf{e}_i(x)\right\}_{i=1}^n \text{ basis for } T_x M$$



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$$\begin{aligned}\mathbf{v}: U \subset M &\rightarrow TM, \\ x &\mapsto v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + \dots + v^n \mathbf{e}_n.\end{aligned}$$

$$\begin{aligned}\mathbf{v}: U \subset M &\rightarrow TM, \\ x &\mapsto v^1(x) \mathbf{e}_1(x) + v^2(x) \mathbf{e}_2(x) + \dots + v^n(x) \mathbf{e}_n(x).\end{aligned}$$



# Vector Fields vs. Vectors in Components

Need basis vector fields

$$\mathbf{e}_i : U \subset M \rightarrow TM, \quad x \mapsto \mathbf{e}_i(x) \quad \{\mathbf{e}_i(x)\}_{i=1}^n \text{ basis for } T_x M$$

Coordinate basis:  
 $\mathbf{e}_i := \frac{\partial}{\partial x^i}$

$$\mathbf{v} : U \subset M \rightarrow TM, \quad x \mapsto v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + \dots + v^n \mathbf{e}_n.$$

$$\mathbf{v} : U \subset M \rightarrow TM, \quad x \mapsto v^1(x) \mathbf{e}_1(x) + v^2(x) \mathbf{e}_2(x) + \dots + v^n(x) \mathbf{e}_n(x).$$

# Examples of Coordinate Curves and Bases



Coordinate functions, coordinate curves, bases

- Coordinate functions are real-valued (“scalar”) functions on the domain
- On each coordinate curve, *one* coordinate changes, *all others stay constant*
- Basis: n linearly independent vectors at each point of domain

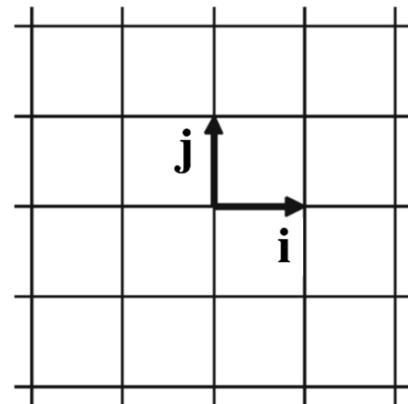
Cartesian coordinates

$$x^1 = x$$

$$x^2 = y$$

$$\mathbf{e}_1 = \frac{\partial}{\partial x} = \mathbf{i}$$

$$\mathbf{e}_2 = \frac{\partial}{\partial y} = \mathbf{j}$$



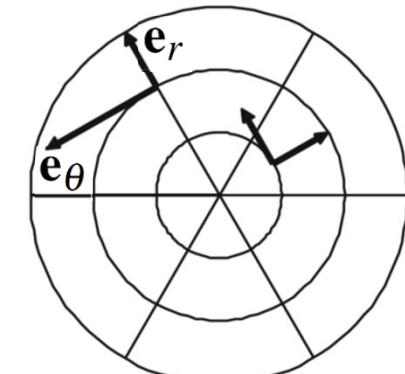
polar coordinates

$$x^1 = r$$

$$x^2 = \theta$$

$$\mathbf{e}_1 = \frac{\partial}{\partial r} = \mathbf{e}_r$$

$$\mathbf{e}_2 = \frac{\partial}{\partial \theta} = \mathbf{e}_\theta$$



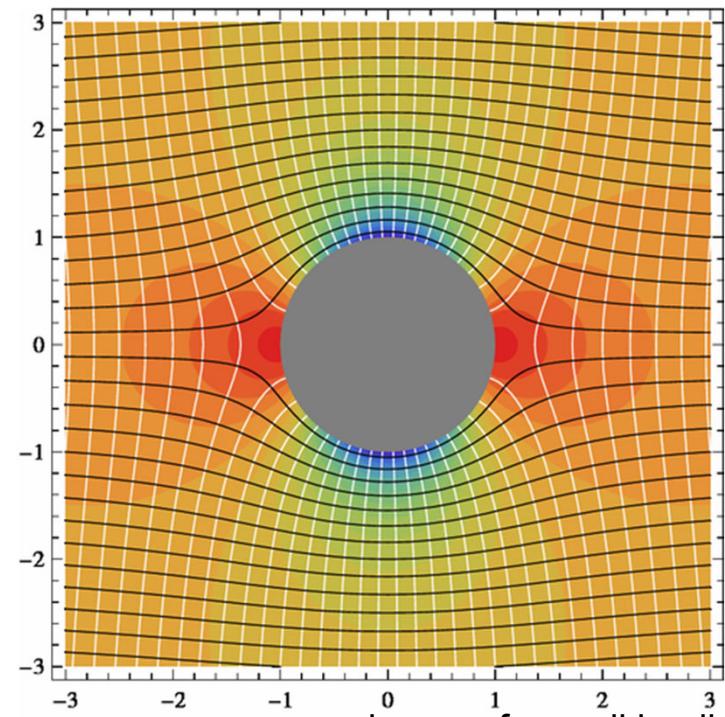
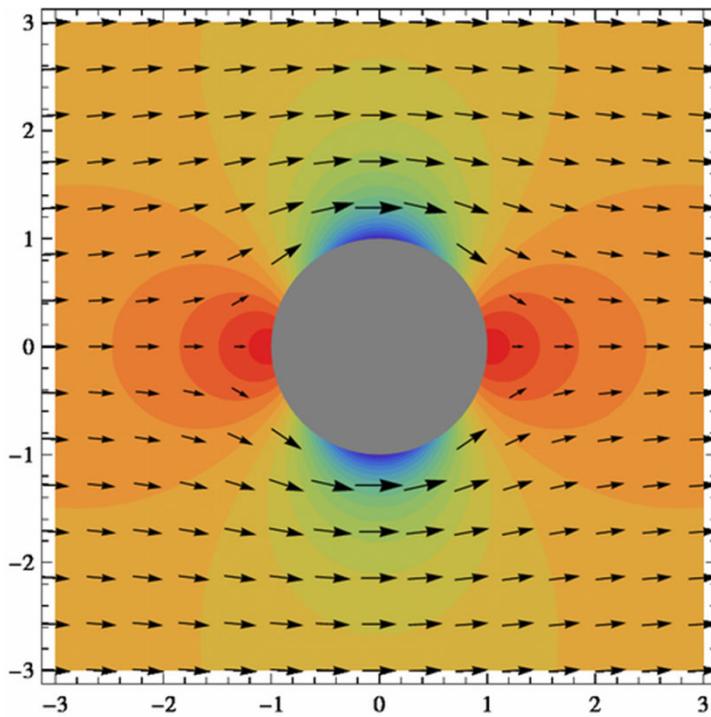


# Flow Field Example (1)

## Potential flow around a circular cylinder

[https://en.wikipedia.org/wiki/Potential\\_flow\\_around\\_a\\_circular\\_cylinder](https://en.wikipedia.org/wiki/Potential_flow_around_a_circular_cylinder)

Inviscid, incompressible flow that is irrotational (curl-free) and can be modeled as the gradient of a scalar function called the (scalar) velocity potential



images from wikipedia

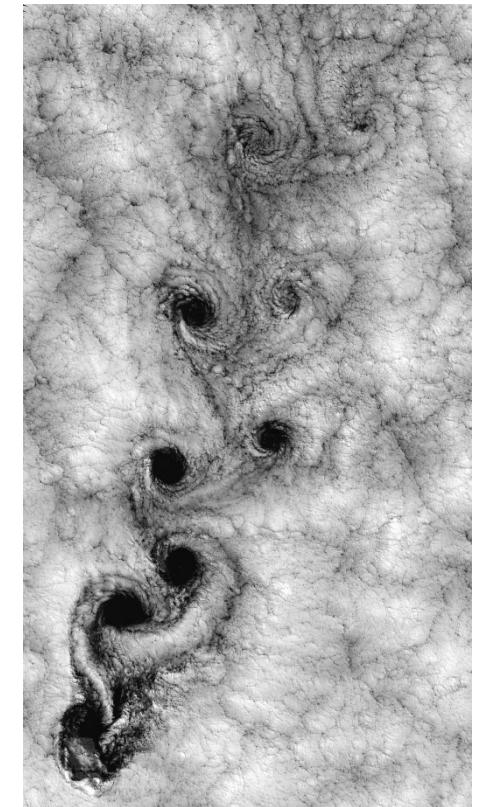
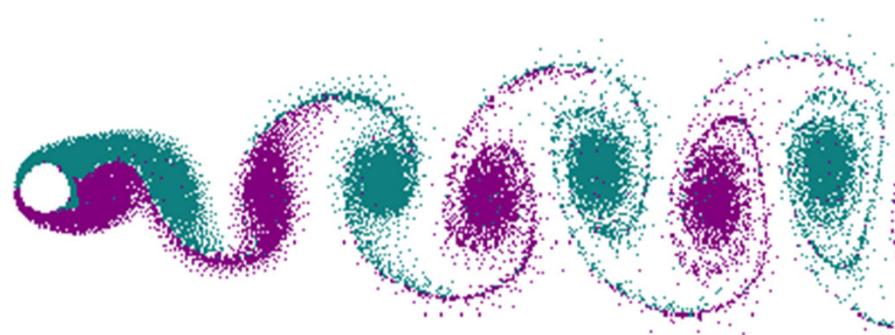


## Flow Field Example (2)

Depending on Reynolds number, turbulence will develop

Example: von Kármán vortex street: vortex shedding

[https://en.wikipedia.org/wiki/Karman\\_vortex\\_street](https://en.wikipedia.org/wiki/Karman_vortex_street)



images from wikipedia



# Steady vs. Unsteady Flow

- Steady flow: time-independent
  - Flow itself is static over time:  $\mathbf{v}(\mathbf{x})$   $\mathbf{v}: \mathbb{R}^n \rightarrow \mathbb{R}^n,$
  - Example: laminar flows  $x \mapsto \mathbf{v}(x).$
- Unsteady flow: time-dependent
  - Flow itself changes over time:  $\mathbf{v}(\mathbf{x}, t)$   $\mathbf{v}: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n,$
  - Example: turbulent flows  $x \mapsto \mathbf{v}(x, t).$

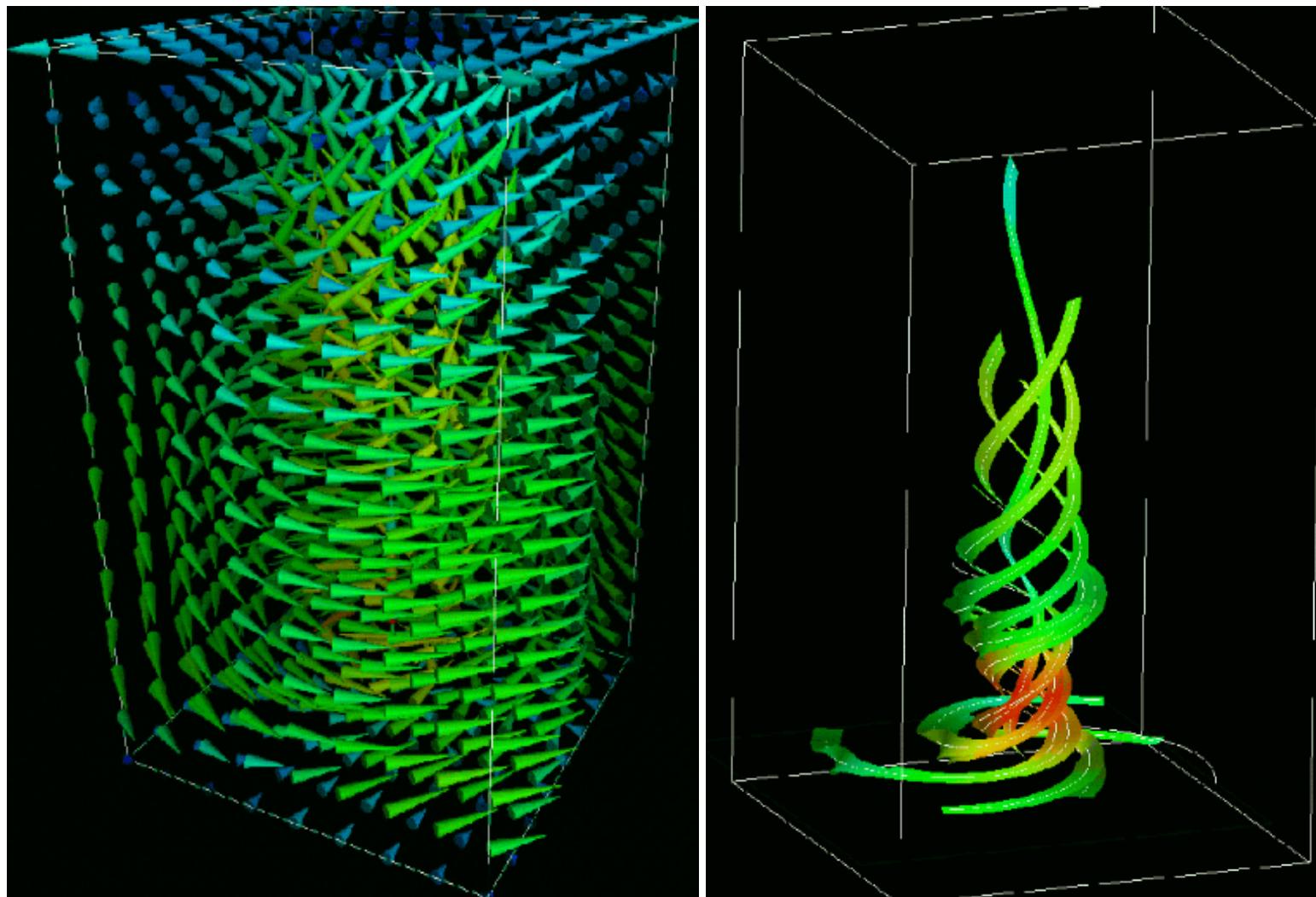
(here just for Euclidean domain; analogous on general manifolds)



# Direct vs. Indirect Flow Visualization

- Direct flow visualization
  - Overview of current flow state
  - Visualization of vectors: arrow plots (“hedgehog” plots)
- Indirect flow visualization
  - Use intermediate representation: vector field integration over time
  - Visualization of temporal evolution
  - Integral curves: streamlines, pathlines, streaklines, timelines
  - Integral surfaces: streamsurfaces, pathsurfaces, streaksurfaces

# Direct vs. Indirect Flow Visualization

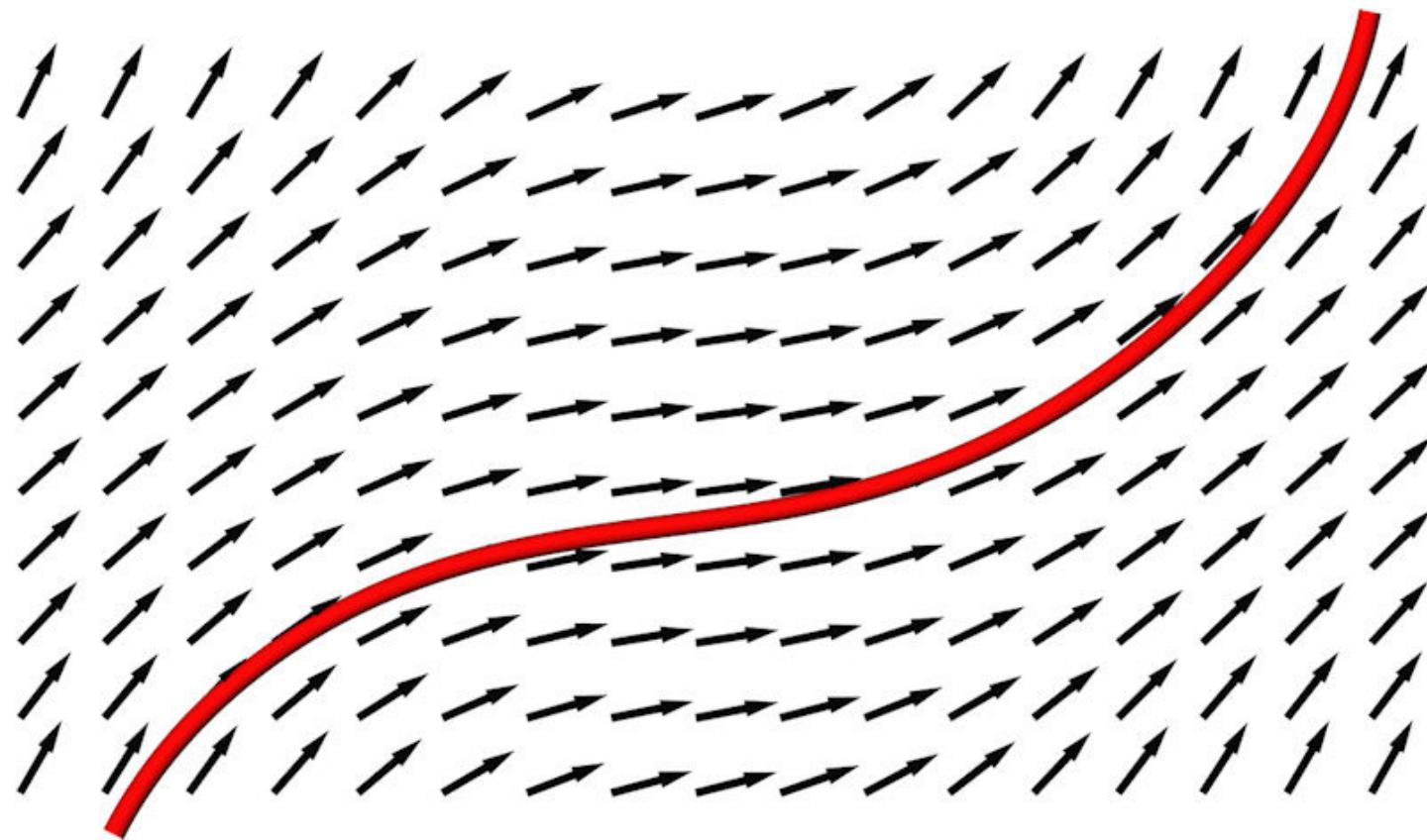


# Integral Curves: Intro

# Integral Curves / Stream Objects



Integrating velocity over time yields spatial motion



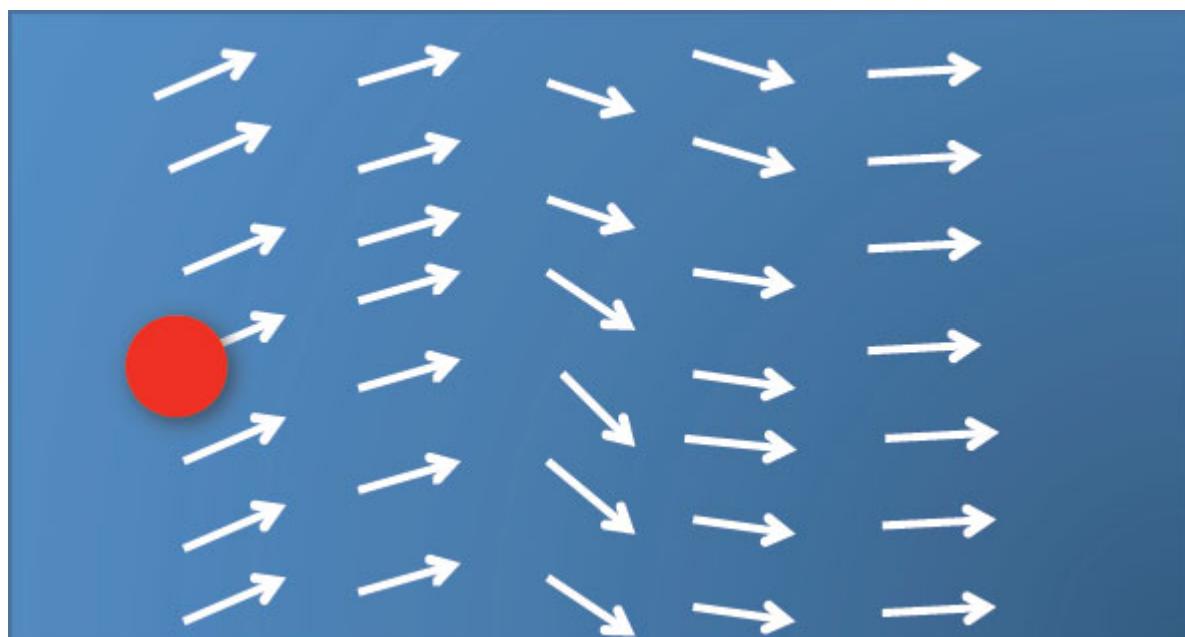
# Particle Trajectories



Courtesy Jens Krüger



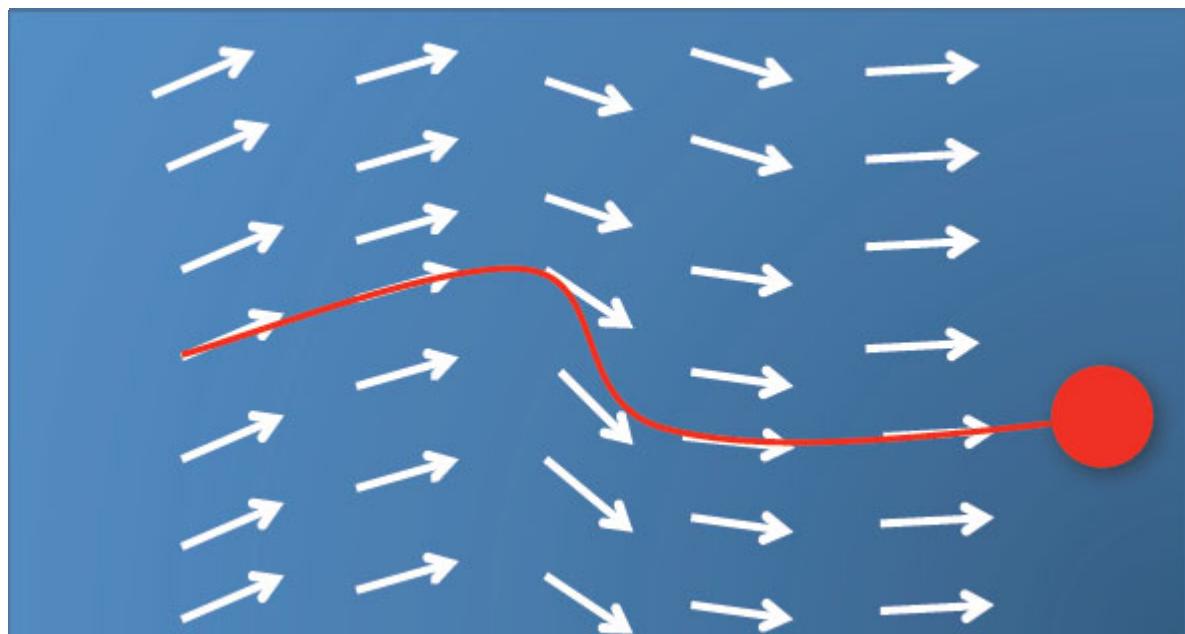
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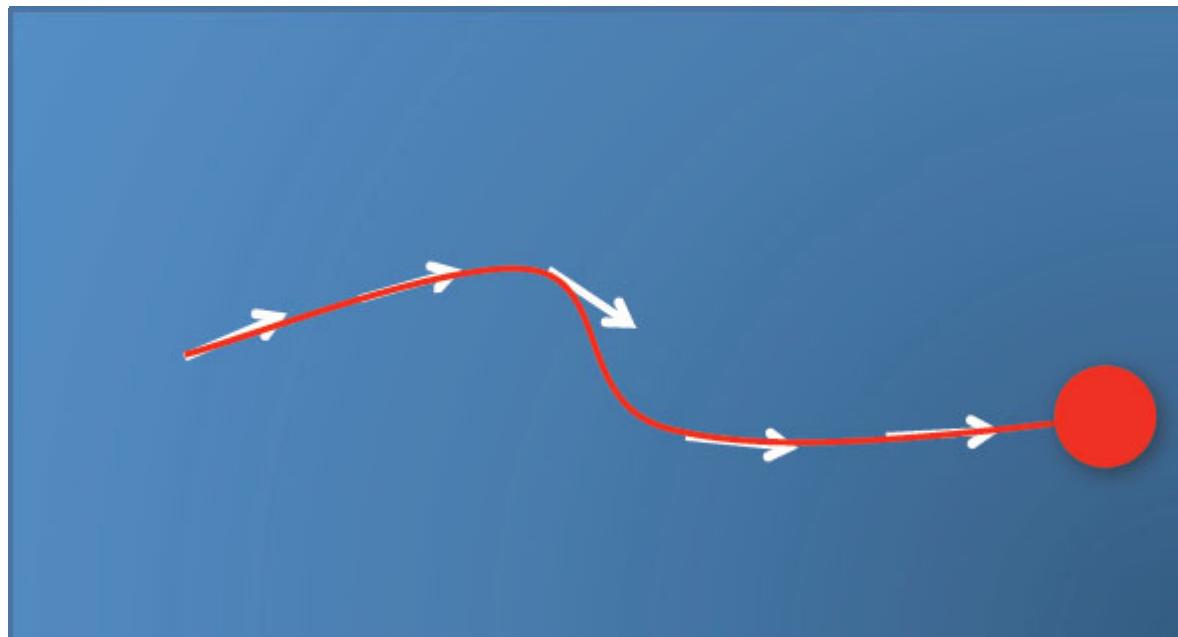


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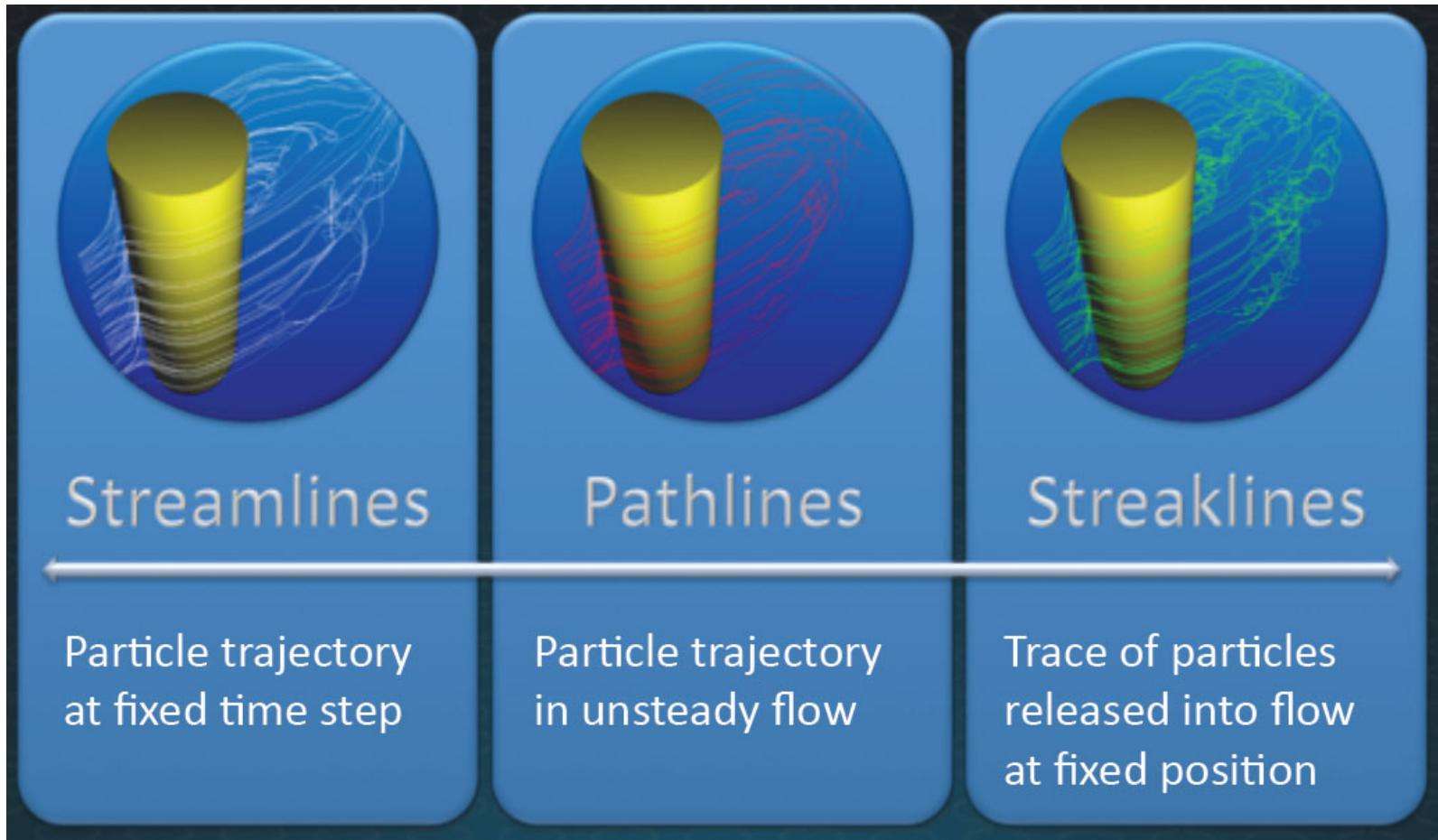
Courtesy Jens Krüger

# Particle Trajectories



Courtesy Jens Krüger

# Integral Curves



## **Streamline**

- Curve parallel to the vector field in each point for a fixed time

## **Pathline**

- Describes motion of a massless particle over time

## **Streakline**

- Location of all particles released at a *fixed position* over time

## **Timeline**

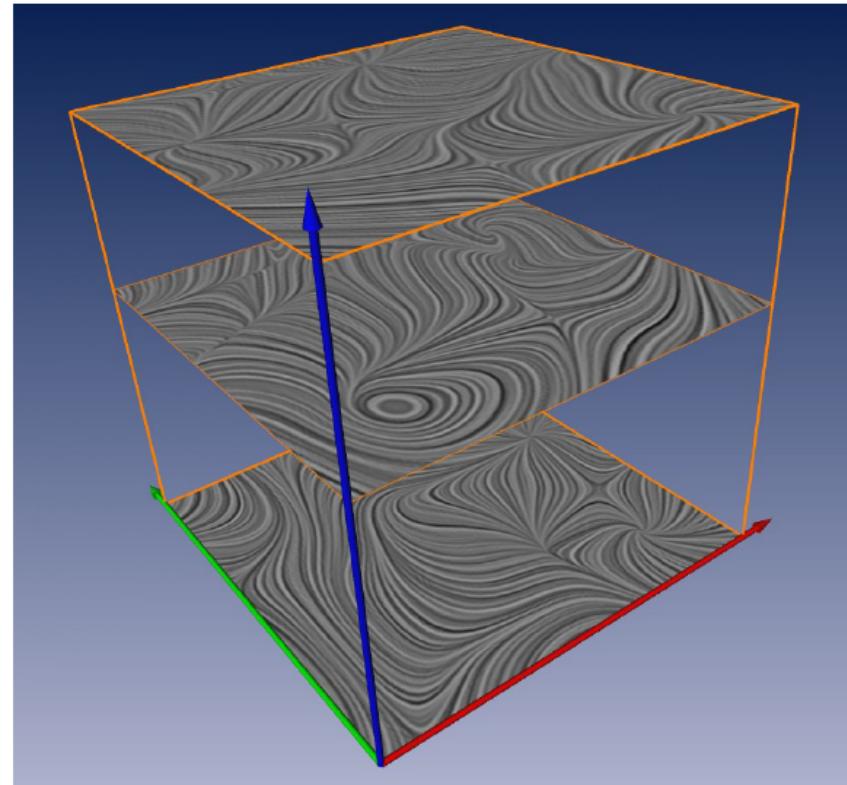
- Location of all particles released along a line at a *fixed time*



# Streamlines Over Time

Defined only for steady flow or for a fixed time step (of unsteady flow)

Different tangent curves in every time step for time-dependent vector fields (unsteady flow)

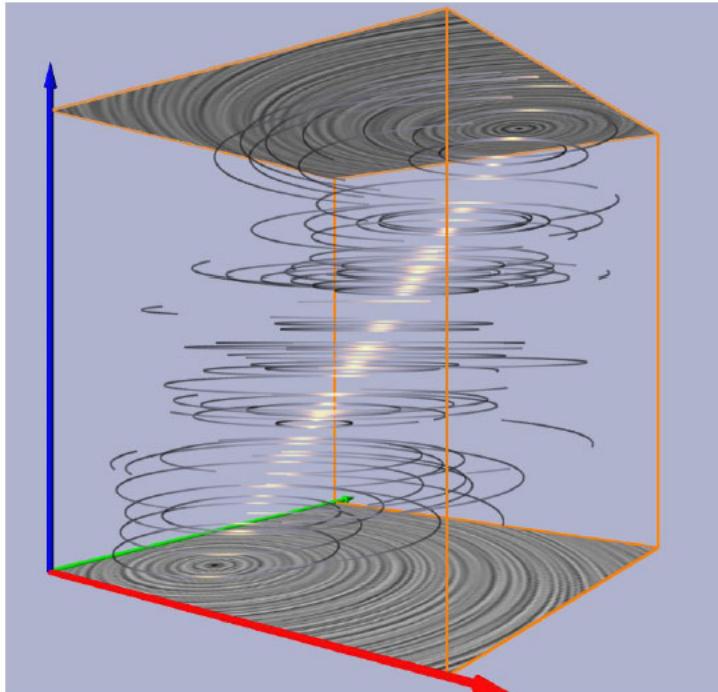


# Stream Lines vs. Path Lines Viewed Over Time

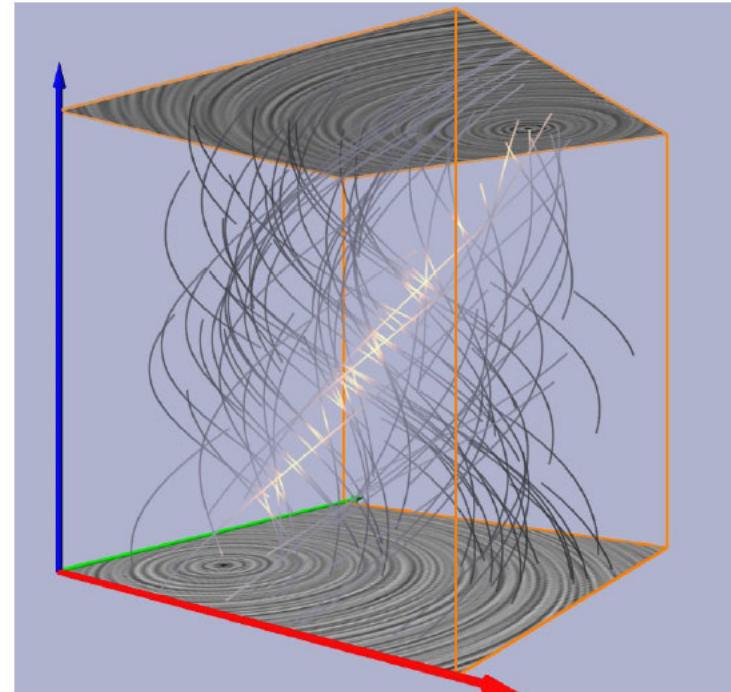


Plotted with time as third dimension

- Tangent curves to a  $(n + 1)$ -dimensional vector field



Stream Lines



Path Lines

## Vector fields

A static vector field  $\mathbf{v}(\mathbf{x})$  is a vector-valued function of space.

A time-dependent vector field  $\mathbf{v}(\mathbf{x}, t)$  depends also on time.

In the case of velocity fields, the terms steady and unsteady flow are used.

The dimensions of  $\mathbf{x}$  and  $\mathbf{v}$  are equal, often 2 or 3, and we denote components by  $x, y, z$  and  $u, v, w$ :

$$\mathbf{x} = (x, y, z), \quad \mathbf{v} = (u, v, w)$$

Sometimes a vector field is defined on a surface  $\mathbf{x}(i, j)$ . The vector field is then a function of parameters and time:

$$\mathbf{v}(i, j, t)$$

## Vector fields as ODEs

For simplicity, the vector field is now interpreted as a **velocity** field.

Then the field  $\mathbf{v}(\mathbf{x}, t)$  describes the connection between location and velocity of a (massless) particle.

It can equivalently be expressed as an **ordinary differential equation**

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t), t)$$

This ODE, together with an **initial condition**

$$\mathbf{x}(t_0) = \mathbf{x}_0 ,$$

is a so-called **initial value problem** (IVP).

Its solution is the **integral curve** (or **trajectory**)

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{v}(\mathbf{x}(\tau), \tau) d\tau$$

## Vector fields as ODEs

The integral curve is a **pathline**, describing the **path** of a massless **particle** which was released at time  $t_0$  at position  $x_0$ .

Remark:  $t < t_0$  is allowed.

For static fields, the ODE is **autonomous**:

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$$

and its integral curves

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{v}(\mathbf{x}(\tau)) d\tau$$

are called **field lines**, or (in the case of velocity fields) **streamlines**.

## *Vector fields as ODEs*

In **static** vector fields, pathlines and streamlines are **identical**.

In **time-dependent** vector fields, **instantaneous streamlines** can be computed from a "snapshot" at a fixed time  $T$  (which is a static vector field)

$$\mathbf{v}_T(\mathbf{x}) = \mathbf{v}(\mathbf{x}, T)$$

In practice, time-dependent fields are often given as a dataset per time step. Each dataset is then a snapshot.

## *Streamline integration*

Outline of algorithm for numerical streamline integration  
(with obvious extension to pathlines):

Inputs:

- static vector field  $\mathbf{v}(\mathbf{x})$
- seed points with time of release  $(\mathbf{x}_0, t_0)$
- control parameters:
  - step size (temporal, spatial, or in local coordinates)
  - step count limit, time limit, etc.
  - order of integration scheme

Output:

- streamlines as "polylines", with possible attributes  
(interpolated field values, time, speed, arc length, etc.)

## *Streamline integration*

Preprocessing:

- set up search structure for point location
- for each seed point:
  - **global point location**: Given a point  $\mathbf{x}$ ,  
find the cell containing  $\mathbf{x}$  and the local coordinates  $(\xi, \eta, \zeta)$   
or if the grid is structured:  
find the computational space coordinates  $(i + \xi, j + \eta, k + \zeta)$
  - If  $\mathbf{x}$  is not found in a cell, remove seed point

## *Streamline integration*

Integration loop, for each seed point  $\mathbf{x}$ :

- interpolate  $\mathbf{v}$  trilinearly to local coordinates  $(\xi, \eta, \zeta)$
- do an integration step, producing a new point  $\mathbf{x}'$
- **incremental point location**: For position  $\mathbf{x}'$  find cell and local coordinates  $(\xi', \eta', \zeta')$  making use of information (coordinates, local coordinates, cell) of old point  $\mathbf{x}$

Termination criteria:

- grid boundary reached
- step count limit reached
- optional: velocity close to zero
- optional: time limit reached
- optional: arc length limit reached

## *Streamline integration*

Integration step: widely used integration methods:

- **Euler** (used only in special speed-optimized techniques, e.g. GPU-based texture advection)

$$\mathbf{x}_{new} = \mathbf{x} + \mathbf{v}(\mathbf{x}, t) \cdot \Delta t$$

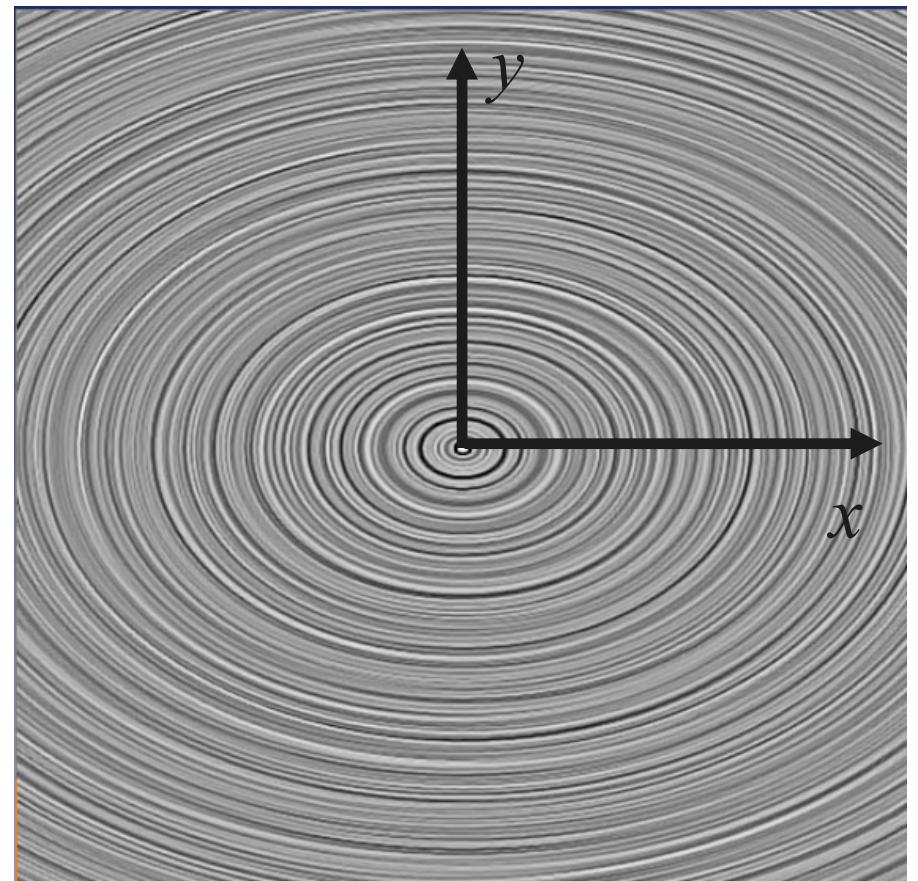
- **Runge-Kutta**, 2<sup>nd</sup> or 4<sup>th</sup> order

Higher order than 4<sup>th</sup>?

- often too slow for visualization
- study (Yeung/Pope 1987) shows that, when using standard trilinear interpolation, **interpolation errors** dominate **integration errors**.

# Numerical Integration

- Numerical integration of stream lines:
- approximate streamline by polygon  $\mathbf{x}_i$
- Testing example:
  - $\mathbf{v}(x,y) = (-y, x/2)^T$
  - exact solution: ellipses
  - starting integration from  $(0,-1)$





# Streamlines – Practice

## ■ Basic approach:

- theory:  $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \leq u \leq t} \mathbf{v}(\mathbf{s}(u)) du$
- practice: numerical integration
- idea:  
(very) locally, the solution is (approx.) linear
- Euler integration:  
follow the current flow vector  $\mathbf{v}(\mathbf{s}_i)$  from the current streamline point  $\mathbf{s}_i$  for a very small time ( $dt$ ) and therefore distance
- Euler integration:  $\mathbf{s}_{i+1} = \mathbf{s}_i + dt \cdot \mathbf{v}(\mathbf{s}_i)$ ,  
integration of small steps ( $dt$  very small)

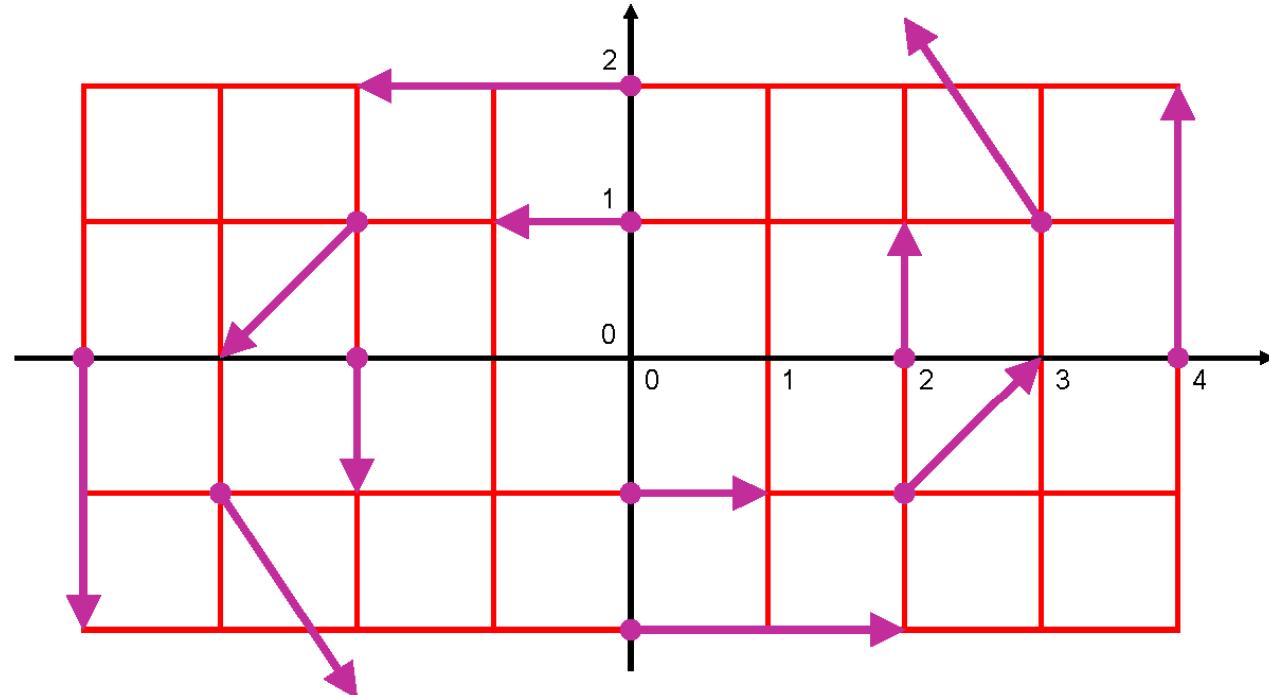
# Euler Integration – Example

- 2D model data:

$$v_x = dx/dt = -y$$
$$v_y = dy/dt = x/2$$

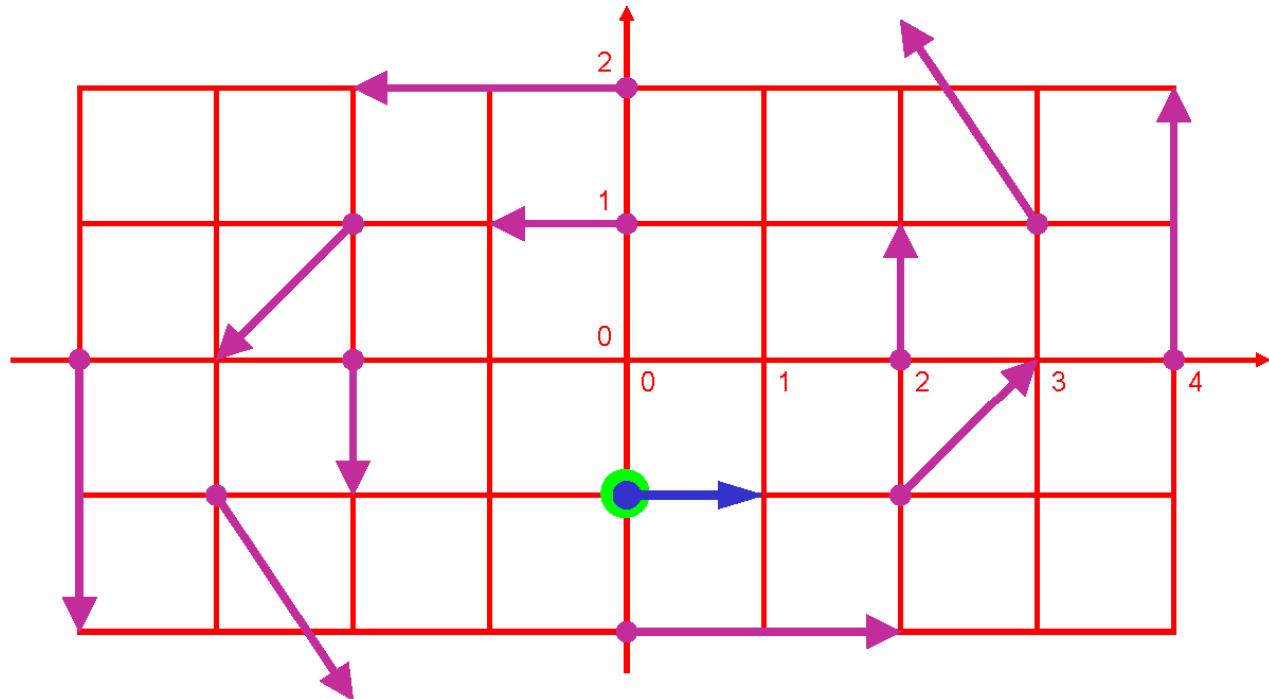
- Sample arrows:

- True solution: ellipses!



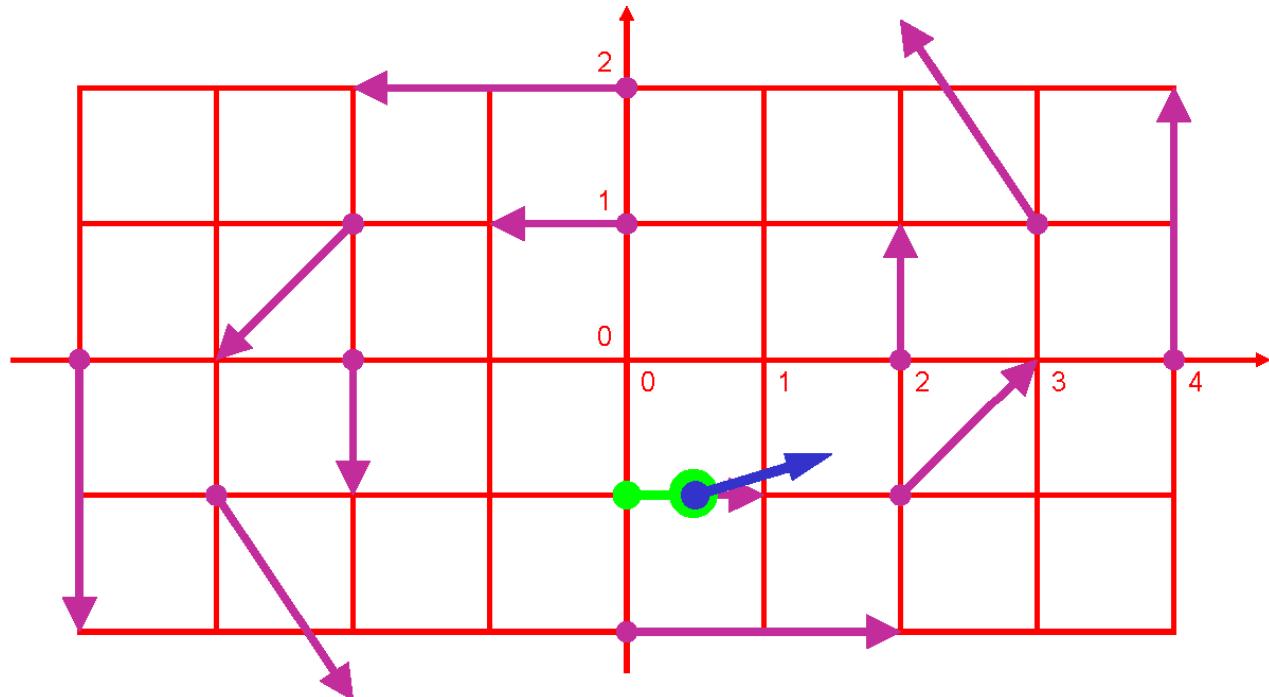
# Euler Integration – Example

- Seed point  $s_0 = (0|-1)^T$ ;  
current flow vector  $v(s_0) = (1|0)^T$ ;  
 $dt = 1/2$



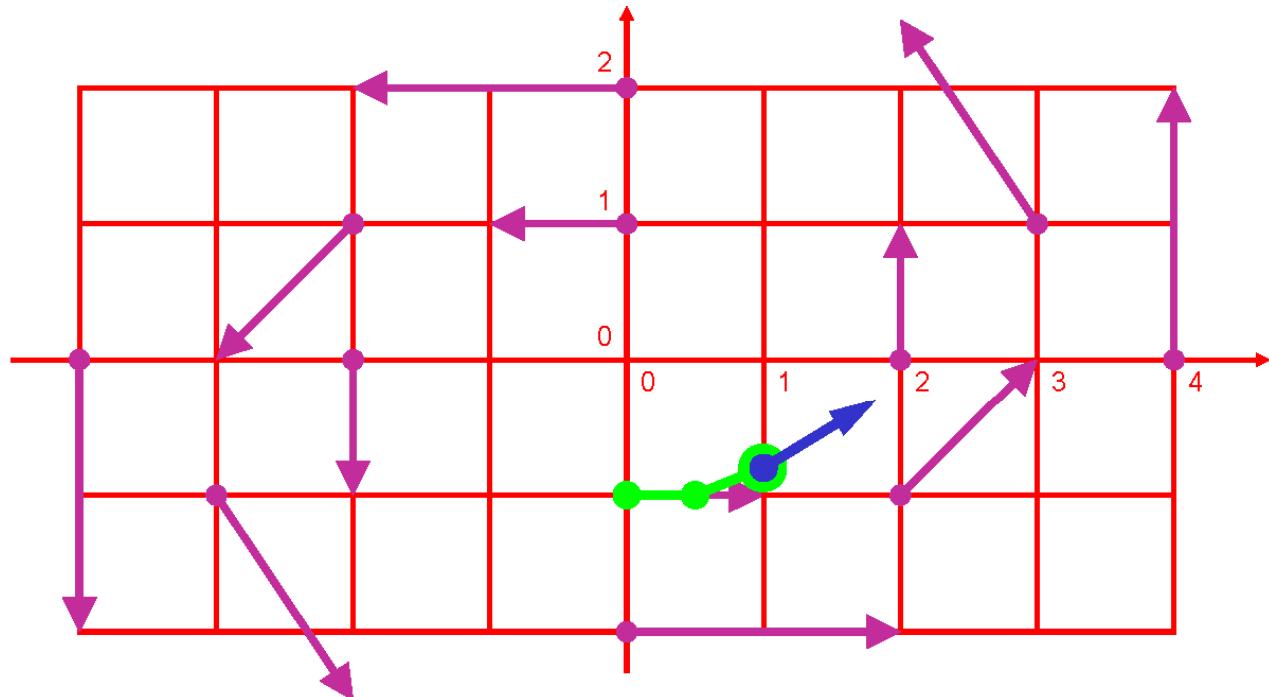
# Euler Integration – Example

- New point  $\mathbf{s}_1 = \mathbf{s}_0 + \mathbf{v}(\mathbf{s}_0) \cdot dt = (1/2 | -1)^T$ ;  
current flow vector  $\mathbf{v}(\mathbf{s}_1) = (1 | 1/4)^T$ ;



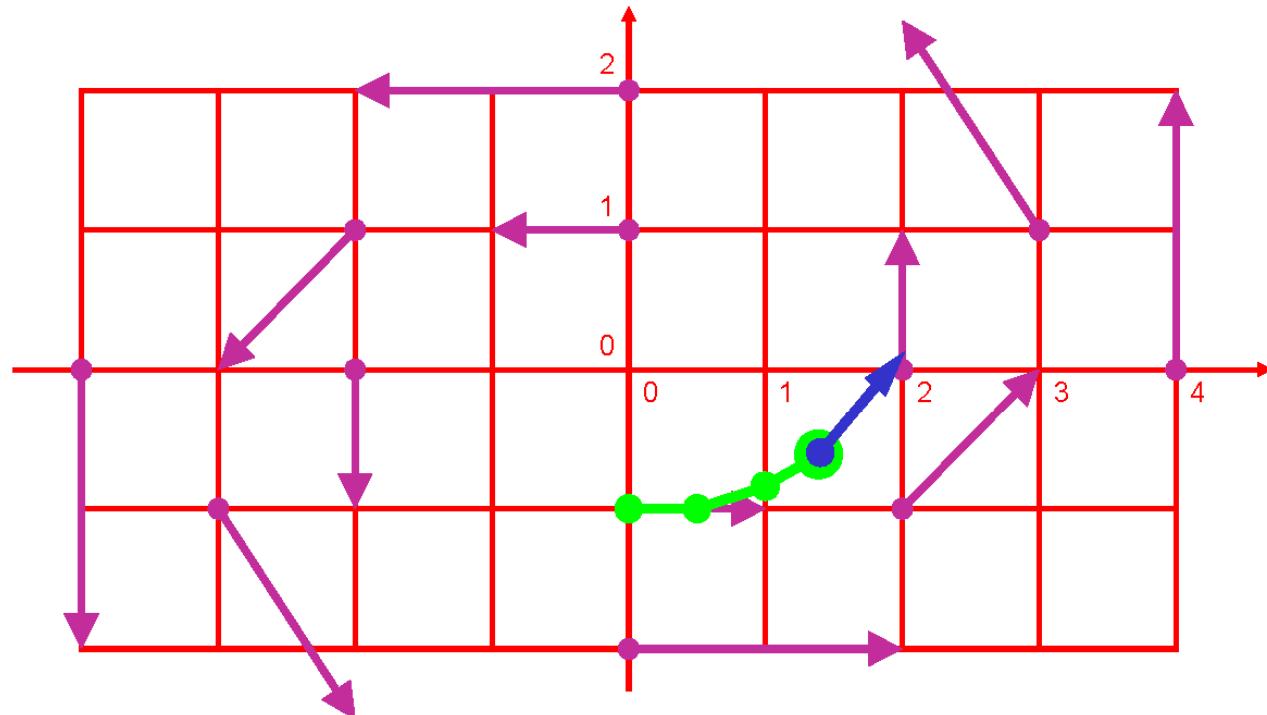
# Euler Integration – Example

- New point  $\mathbf{s}_2 = \mathbf{s}_1 + \mathbf{v}(\mathbf{s}_1) \cdot dt = (1 | -7/8)^T$ ;  
current flow vector  $\mathbf{v}(\mathbf{s}_2) = (7/8 | 1/2)^T$ ;



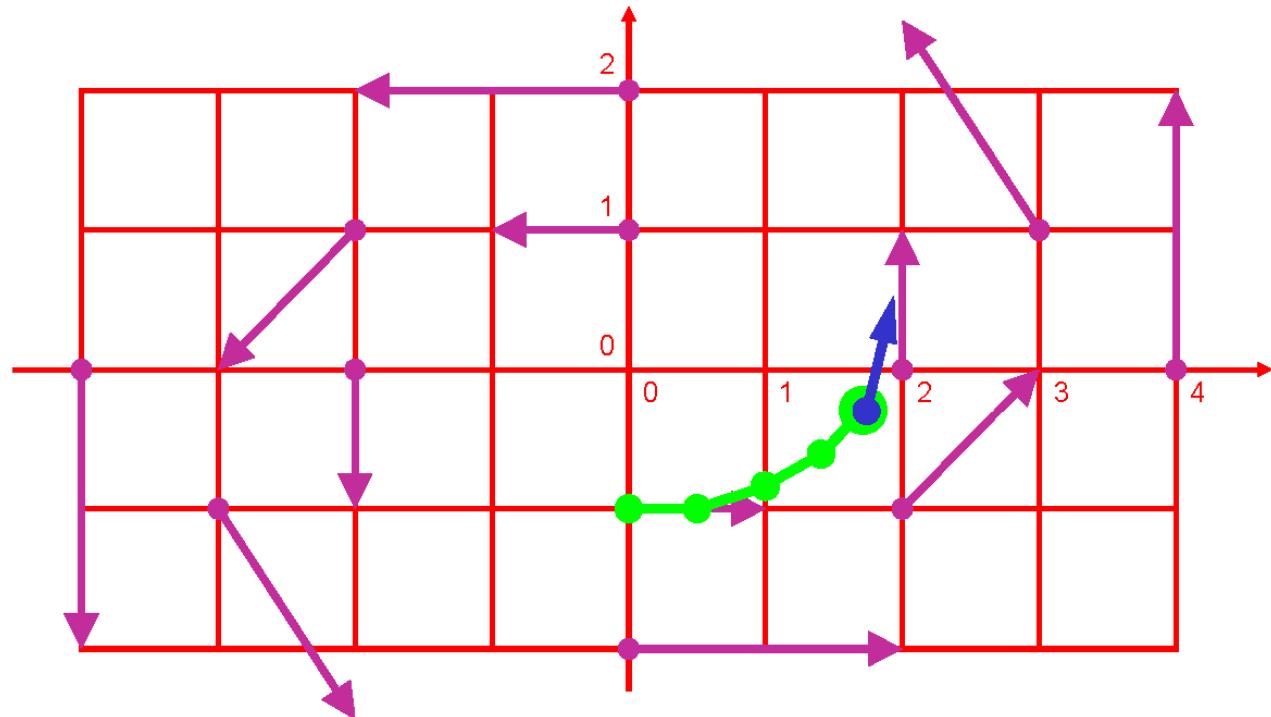
# Euler Integration – Example

- $\mathbf{s}_3 = (23/16 | -5/8)^T \approx (1.44 | -0.63)^T;$
- $\mathbf{v}(\mathbf{s}_3) = (5/8 | 23/32)^T \approx (0.63 | 0.72)^T;$



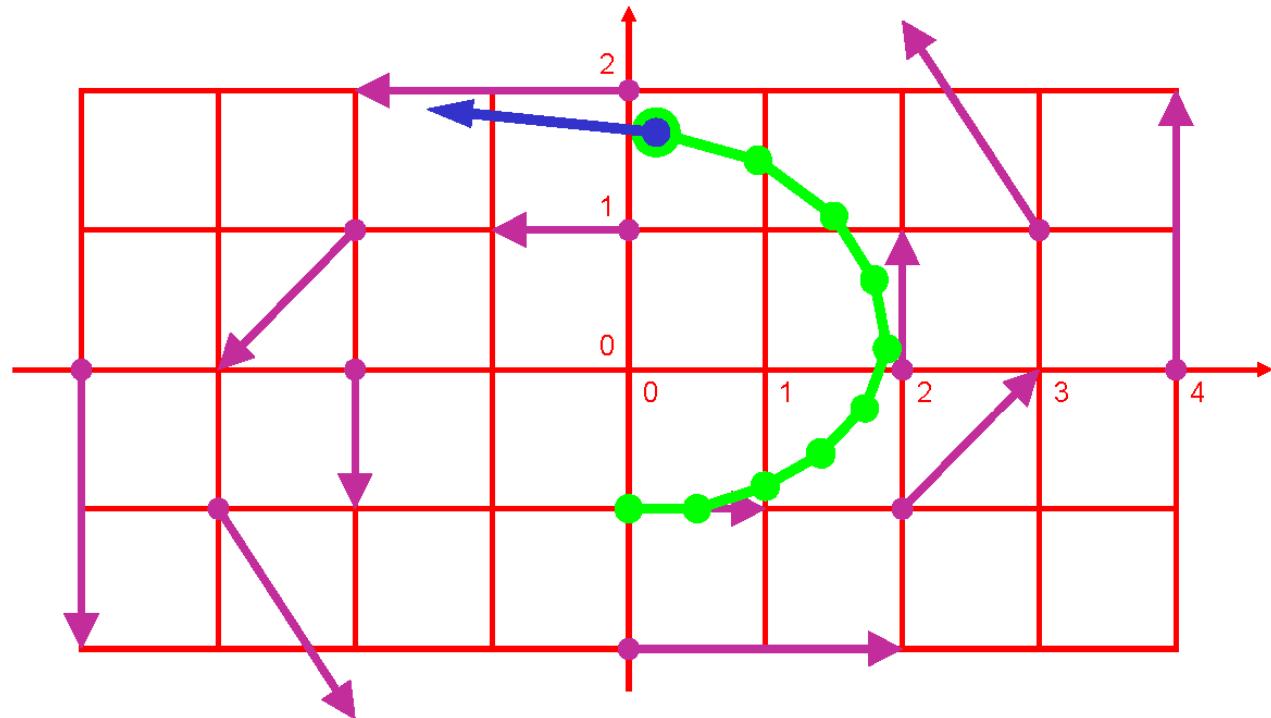
# Euler Integration – Example

- $s_4 = (7/4 | -17/64)^T \approx (1.75 | -0.27)^T;$
- $v(s_4) = (17/64 | 7/8)^T \approx (0.27 | 0.88)^T;$



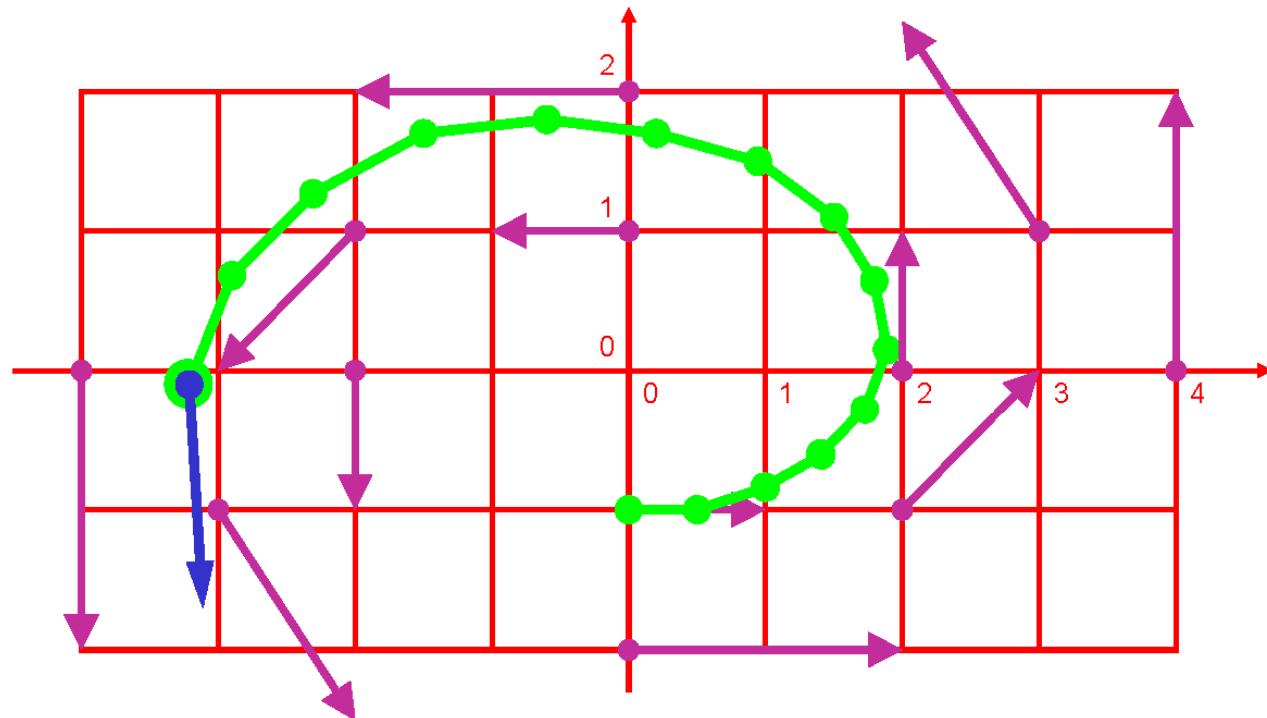
# Euler Integration – Example

- $s_9 \approx (0.20 | 1.69)^T;$   
 $v(s_9) \approx (-1.69 | 0.10)^T;$



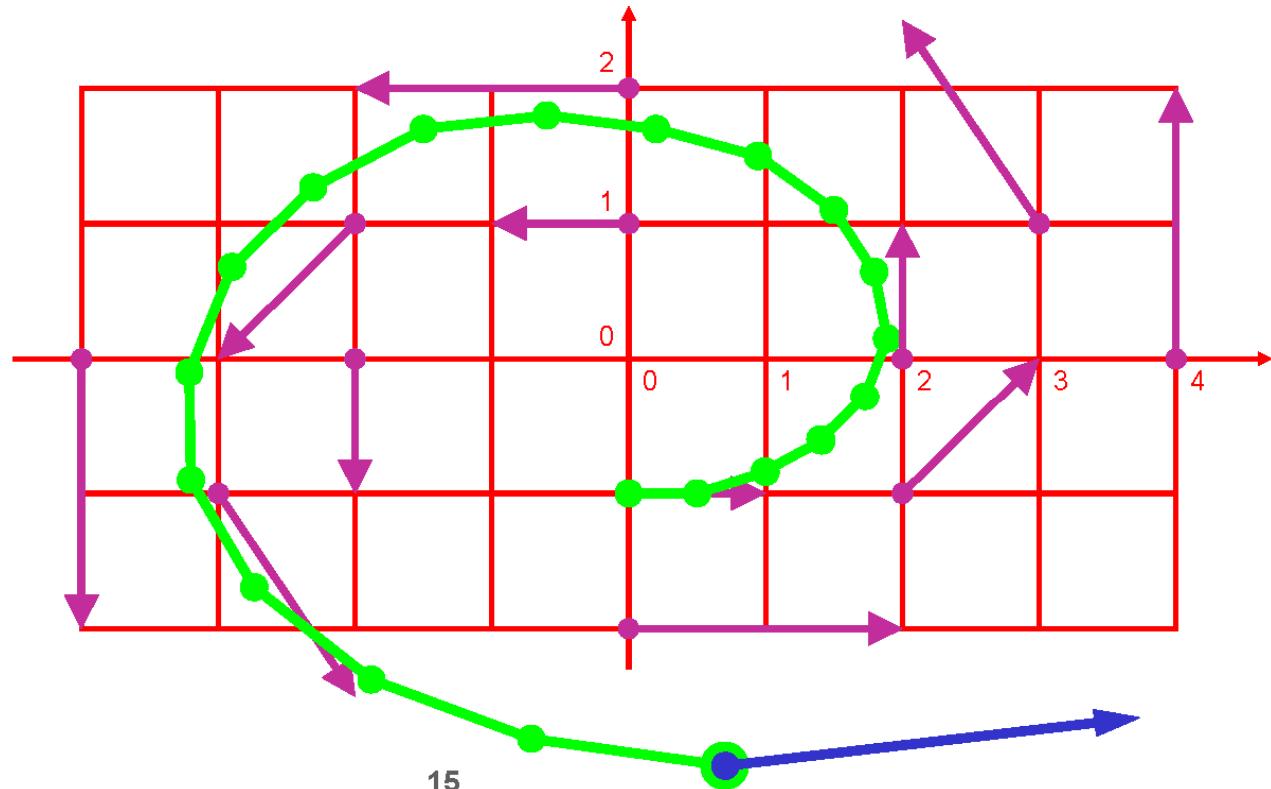
# Euler Integration – Example

- $s_{14} \approx (-3.22 | -0.10)^T;$   
 $v(s_{14}) \approx (0.10 | -1.61)^T;$



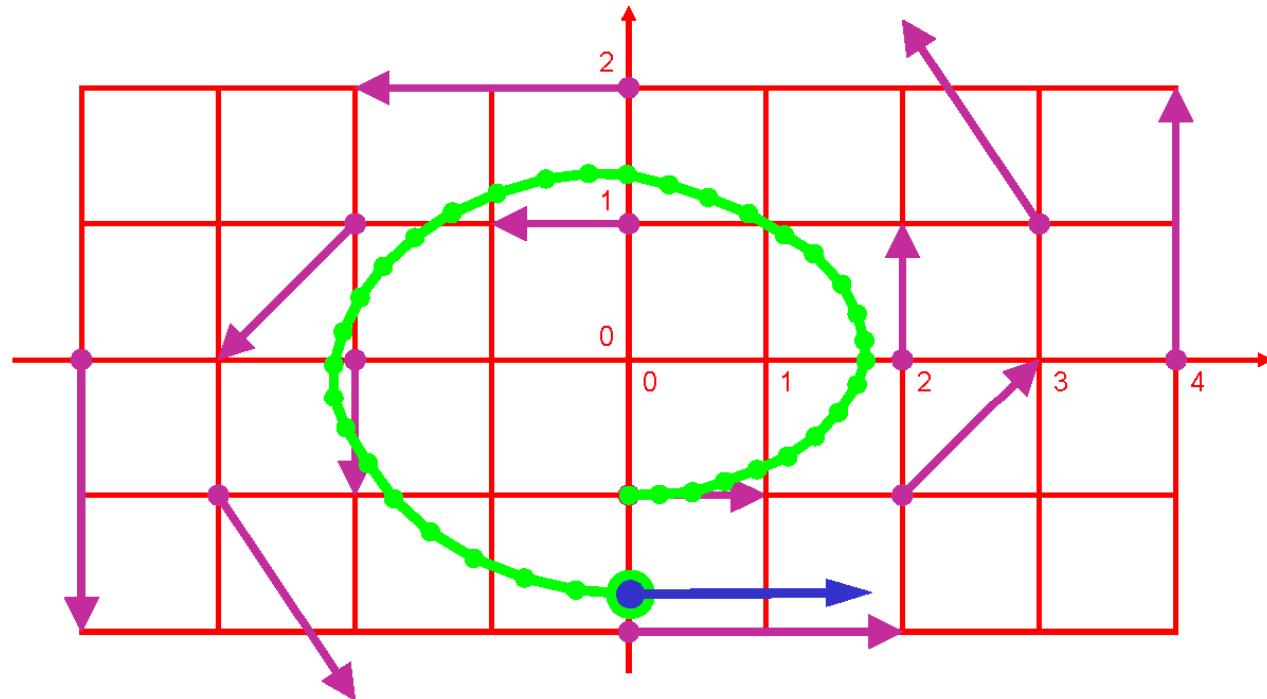
# Euler Integration – Example

- $\mathbf{s}_{19} \approx (0.75|-3.02)^T$ ;  $\mathbf{v}(\mathbf{s}_{19}) \approx (3.02|0.37)^T$ ;  
clearly: large integration error,  $dt$  too large!  
19 steps



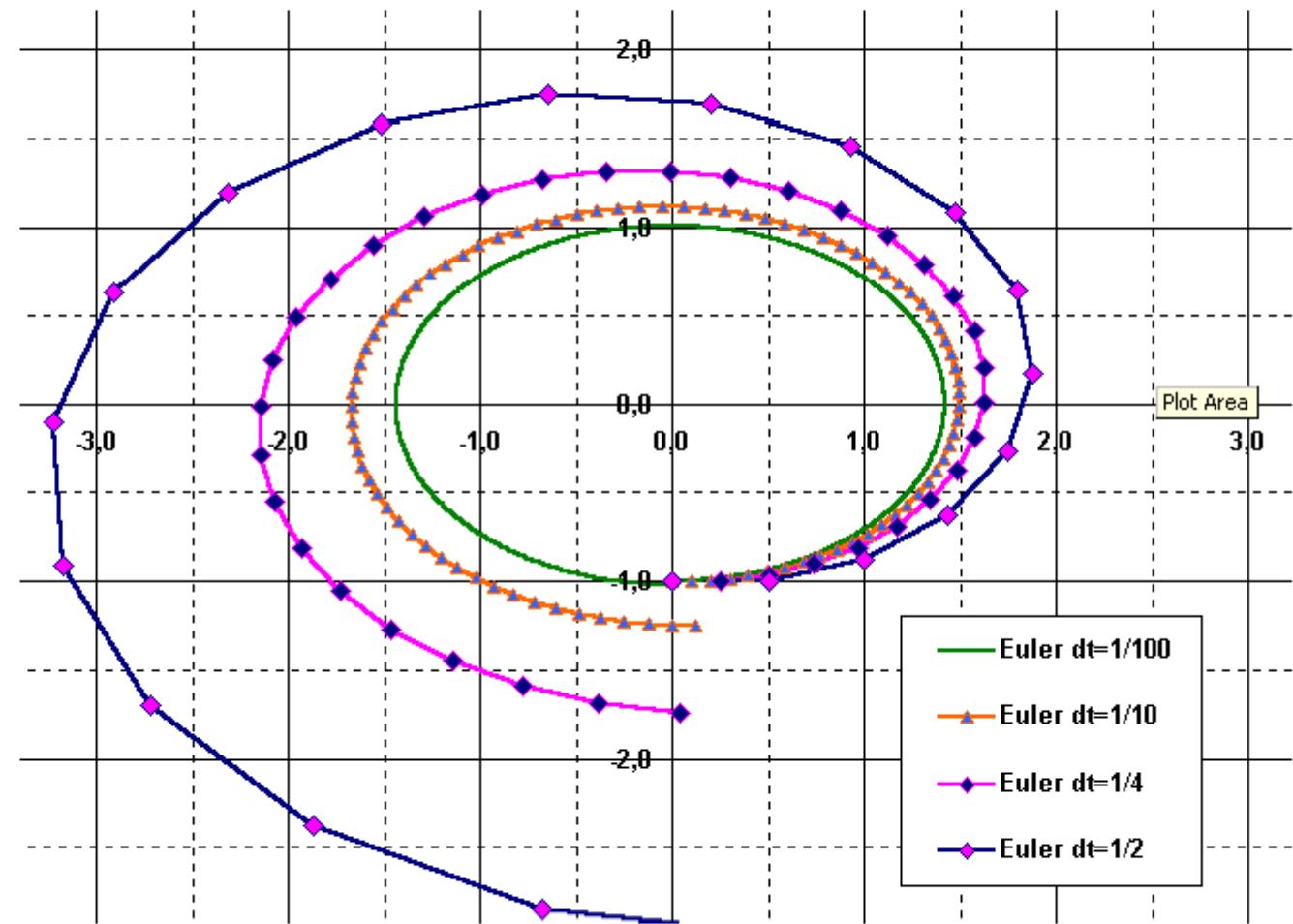
# Euler Integration – Example

- $dt$  smaller ( $1/4$ ): more steps, more exact!  
 $\mathbf{s}_{36} \approx (0.04|-1.74)^T$ ;  $\mathbf{v}(\mathbf{s}_{36}) \approx (1.74|0.02)^T$ ;
- 36 steps



# Comparison Euler, Step Sizes

Euler  
is getting  
better  
proportionally  
to  $dt$





# Better than Euler Integr.: RK

## ■ Runge-Kutta Approach:

- theory:  $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \leq u \leq t} \mathbf{v}(\mathbf{s}(u)) du$

- Euler:  $\mathbf{s}_i = \mathbf{s}_0 + \sum_{0 \leq u < i} \mathbf{v}(\mathbf{s}_u) \cdot dt$

- Runge-Kutta integration:

  - idea: cut short the curve arc

  - RK-2 (second order RK):

    - 1.: do half a Euler step

    - 2.: evaluate flow vector there

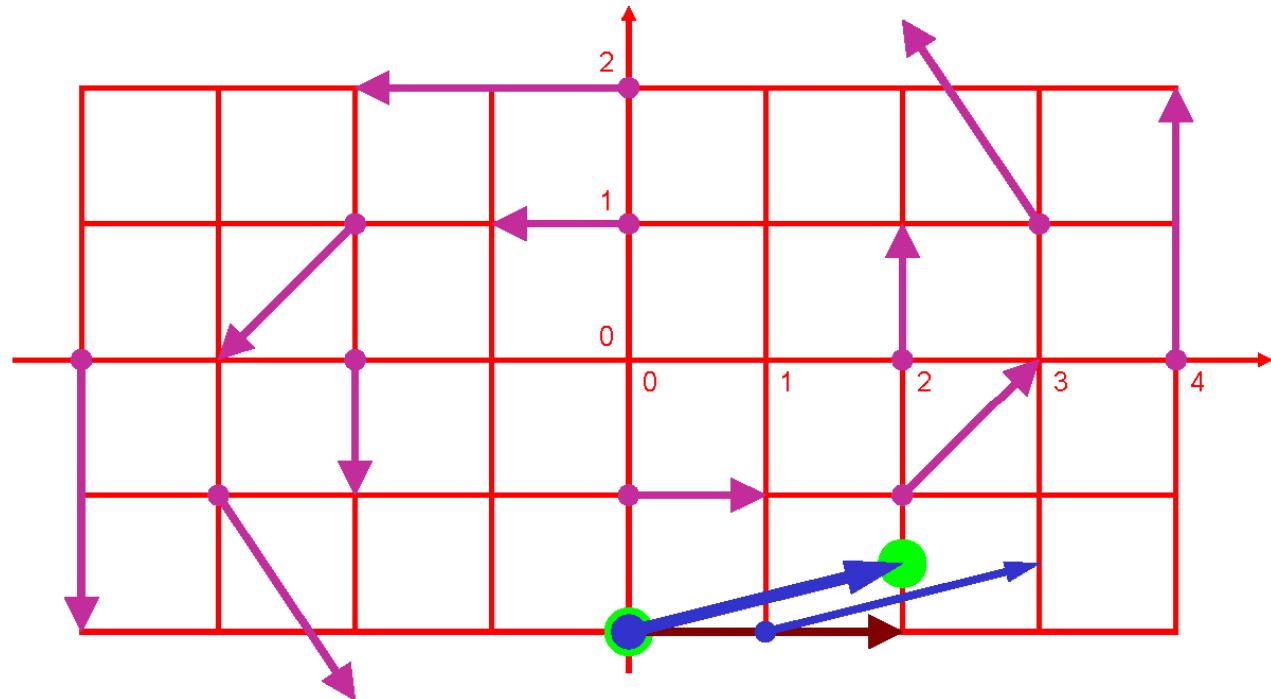
    - 3.: use it in the origin

  - RK-2 (two evaluations of  $\mathbf{v}$  per step):

$$\mathbf{s}_{i+1} = \mathbf{s}_i + \mathbf{v}(\mathbf{s}_i + \mathbf{v}(\mathbf{s}_i) \cdot dt/2) \cdot dt$$

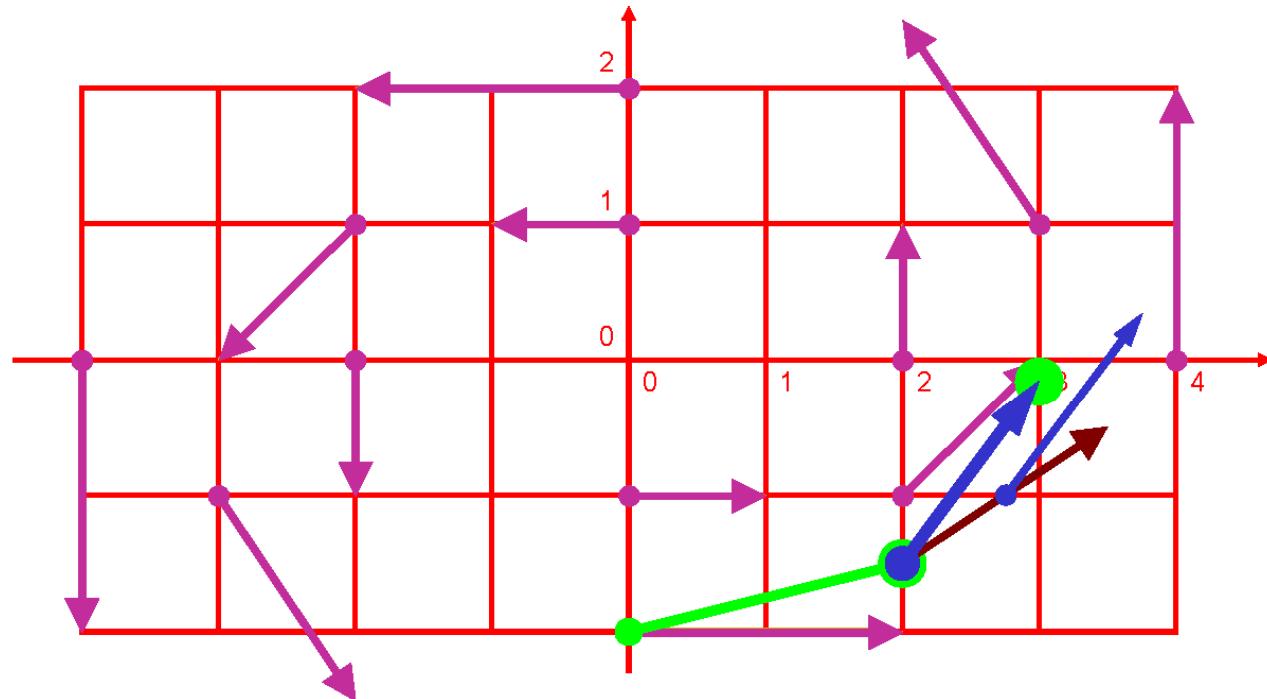
# RK-2 Integration – One Step

- Seed point  $s_0 = (0|-2)^T$ ;  
 current flow vector  $v(s_0) = (2|0)^T$ ;  
 preview vector  $v(s_0 + v(s_0) \cdot dt/2) = (2|0.5)^T$ ;  
 $dt = 1$



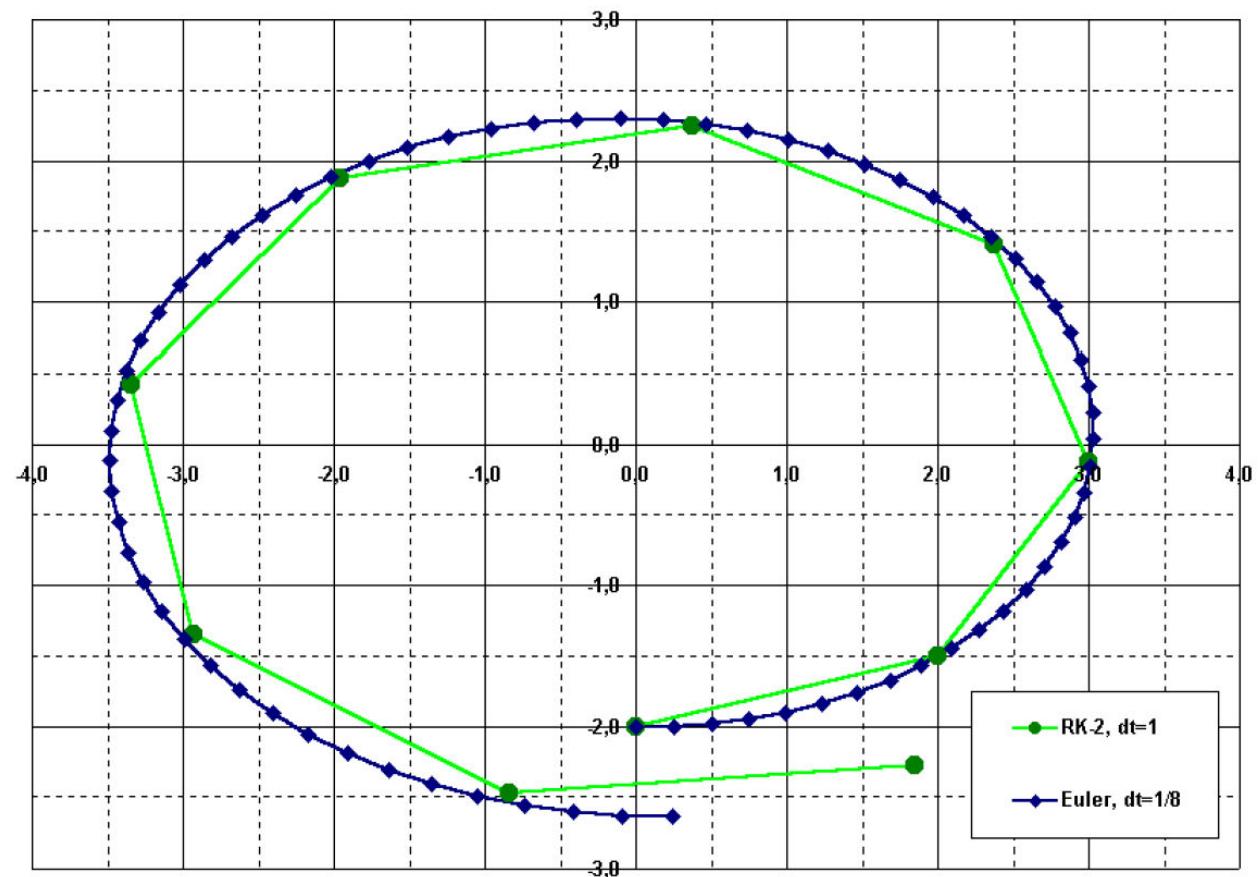
# RK-2 – One more step

- Seed point  $s_1 = (2|-1.5)^T$ ;  
 current flow vector  $v(s_1) = (1.5|1)^T$ ;  
 preview vector  $v(s_1 + v(s_1) \cdot dt/2) \approx (1|1.4)^T$ ;  
 $dt = 1$



# RK-2 – A Quick Round

- RK-2: even with  $dt=1$  (9 steps)  
better  
than Euler  
with  $dt=1/8$   
(72 steps)



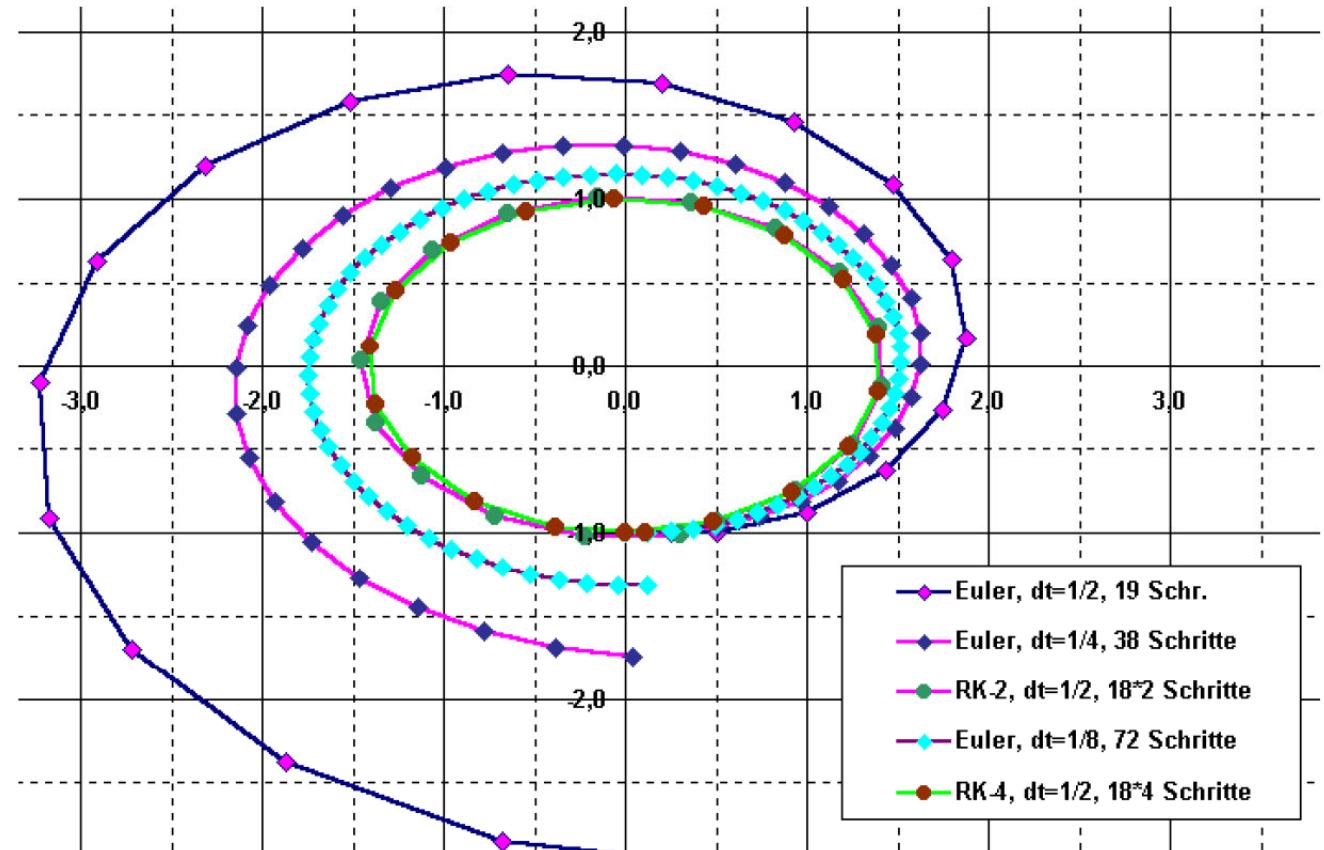


## RK-4 vs. Euler, RK-2

- Even better: fourth order RK:
  - four vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$
  - one step is a convex combination:  
$$\mathbf{s}_{i+1} = \mathbf{s}_i + (\mathbf{a} + 2\cdot\mathbf{b} + 2\cdot\mathbf{c} + \mathbf{d})/6$$
  - vectors:
    - $\mathbf{a} = dt \cdot \mathbf{v}(\mathbf{s}_i)$  ... original vector
    - $\mathbf{b} = dt \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{a}/2)$  ... RK-2 vector
    - $\mathbf{c} = dt \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{b}/2)$  ... use RK-2 ...
    - $\mathbf{d} = dt \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{c})$  ... and again!

# Euler vs. Runge-Kutta

- RK-4: pays off only with complex flows
- Here approx. like RK-2



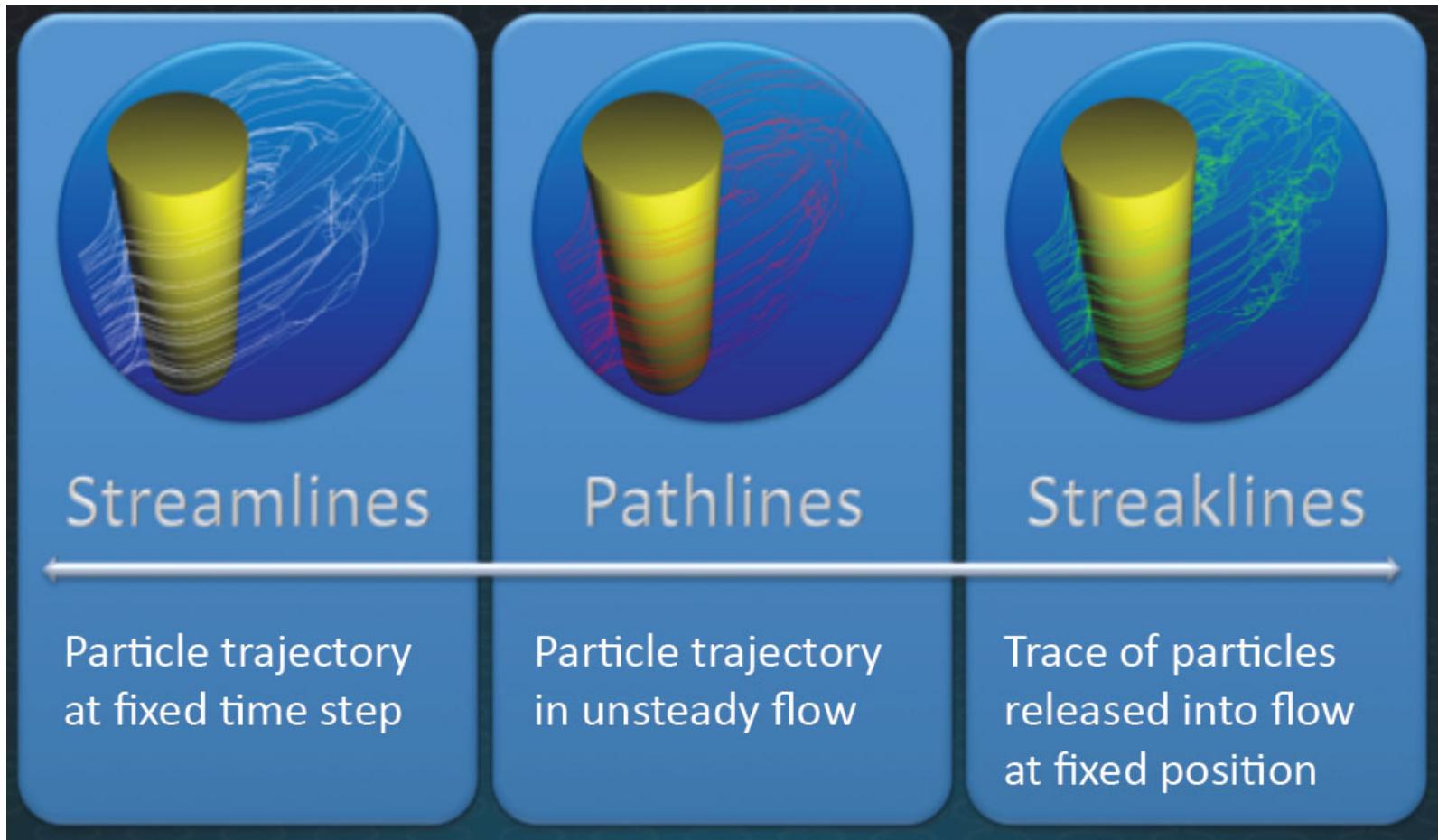


## ■ Summary:

- analytic determination of streamlines  
usually not possible
- hence: numerical integration
- several methods available  
(Euler, Runge-Kutta, etc.)
- Euler: simple, imprecise, esp. with small  $dt$
- RK: more accurate in higher orders
- furthermore: adaptive methods, implicit methods, etc.

# Integral Curves, Pt. 2

# Integral Curves



## **Streamline**

- Curve parallel to the vector field in each point for a fixed time

## **Pathline**

- Describes motion of a massless particle over time

## **Streakline**

- Location of all particles released at a *fixed position* over time

## **Timeline**

- Location of all particles released along a line at a *fixed time*

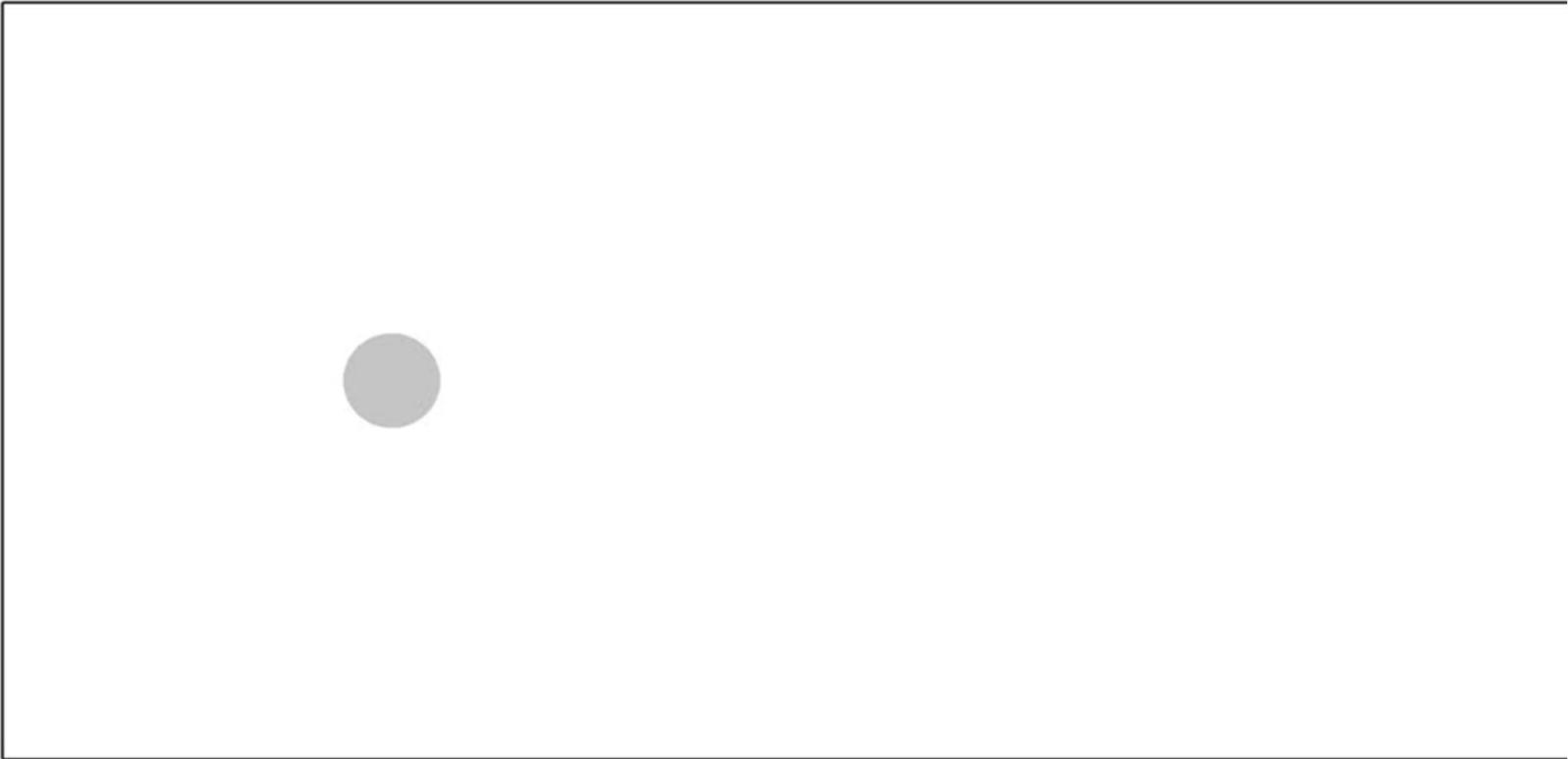


**Time**



### **streak line**

location of all particles set out at a fixed point at different times



### **Particle visualization**

**2D time-dependent flow around a cylinder**

**time line**

location of all particles set out on a certain line at a fixed time



# The Flow / Flow Map of a Vector Field (1)

Flow of a *steady* (*time-independent*) vector field

- Map source position  $x$  “forward” ( $t>0$ ) or “backward” ( $t<0$ ) by time  $t$

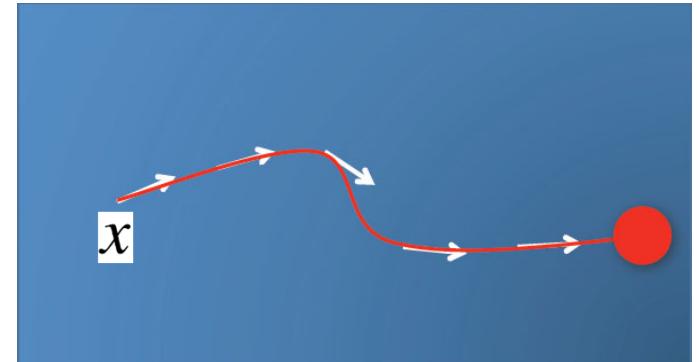
$$\boxed{\phi(x,t)}$$

$$\boxed{\phi_t(x)}$$

with

$$\phi_0(x) = x$$

$$\phi: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, \quad \phi_t: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \phi_s(\phi_t(x)) = \phi_{s+t}(x)$$
$$(x, t) \mapsto \phi(x, t). \quad x \mapsto \phi_t(x).$$





# The Flow / Flow Map of a Vector Field (1)

Flow of a *steady* (*time-independent*) vector field

- Map source position  $x$  “forward” ( $t>0$ ) or “backward” ( $t<0$ ) by time  $t$

$$\boxed{\phi(x, t)}$$

$$\boxed{\phi_t(x)}$$

with

$$\phi_0(x) = x$$

$$\begin{aligned} \phi : M \times \mathbb{R} &\rightarrow M, & \phi_t : M &\rightarrow M, \\ (x, t) &\mapsto \phi(x, t). & x &\mapsto \phi_t(x). \end{aligned}$$





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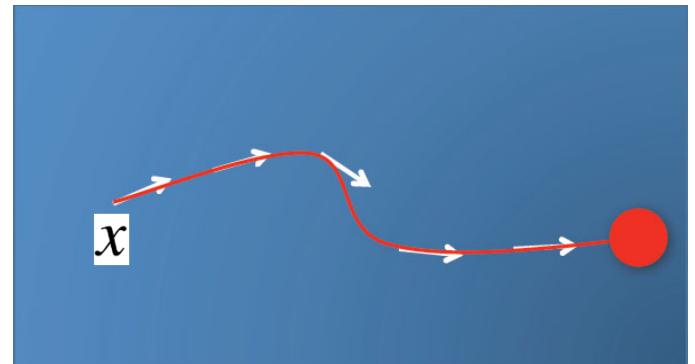
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$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

$$\phi(x,t) = x + \int_0^t \mathbf{v}(\phi(x,\tau)) d\tau$$

(on a general manifold  $M$ , integration  
is performed in coordinate charts)





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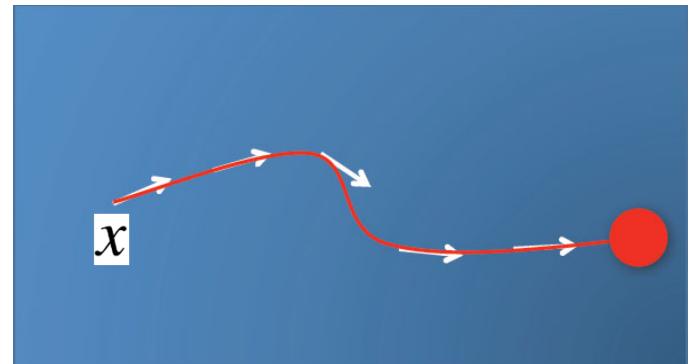
$$\begin{aligned}\phi : M \times \mathbb{R} &\rightarrow M, & \phi_t : M &\rightarrow M, \\ (x,t) &\mapsto \phi(x,t). & x &\mapsto \phi_t(x).\end{aligned}$$

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

- Unsteady flow? Just fix arbitrary time  $T$

$$\phi(x,t) = x + \int_0^t \mathbf{v}(\phi(x,\tau), T) d\tau$$

(on a general manifold  $M$ , integration  
is performed in coordinate charts)





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Flow of a *steady* (*time-independent*) vector field

- Map source position  $x$  “forward” ( $t>0$ ) or “backward” ( $t<0$ ) by time  $t$

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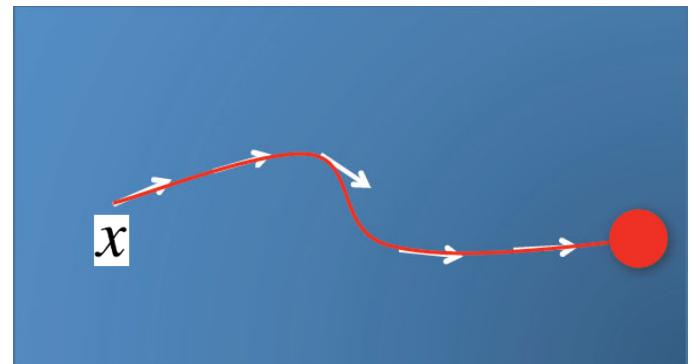
$$\begin{aligned}\phi : M \times \mathbb{R} &\rightarrow M, & \phi_t : M &\rightarrow M, \\ (x,t) &\mapsto \phi(x,t). & x &\mapsto \phi_t(x).\end{aligned}$$

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

Can write explicitly as function of independent variable  $t$ , with *position  $x$  fixed*

$$t \mapsto \phi(x,t) \qquad t \mapsto \phi_t(x)$$

= stream line going through point  $x$





# The Flow / Flow Map of a Vector Field (2)

Flow of an *unsteady (time-dependent)* vector field

- Map source position  $x$  from time  $s$  to destination position at time  $t$   
( $t < s$  is allowed: map forward or backward in time)

$$\psi_{t,s}(x)$$

with

$$\psi_{t,s}(x) = x + \int_s^t \mathbf{v}(\psi_{\tau,s}(x), \tau) d\tau$$

$$\psi_{s,s}(x) = x$$

$$\psi_{t,r}(\psi_{r,s}(x)) = \psi_{t,s}(x)$$

# The Flow / Flow Map of a Vector Field (3)



Flow of an *unsteady (time-dependent)* vector field

- Map source position  $x$  from time  $s$  to destination position at time  $t$   
( $t < s$  is allowed: map forward or backward in time)

$$\boxed{\psi_{t,s}(x)} \quad \psi_{t,s}(x) = x + \int_s^t \mathbf{v}(\psi_{\tau,s}(x), \tau) d\tau$$

Can write explicitly as function of  $t$ , *with  $s$  and  $x$  fixed*

$$t \mapsto \psi_{t,s}(x) \quad \rightarrow \text{path line}$$

Can write explicitly as function of  $s$ , *with  $t$  and  $x$  fixed*

$$s \mapsto \psi_{t,s}(x) \quad \rightarrow \text{streak line}$$

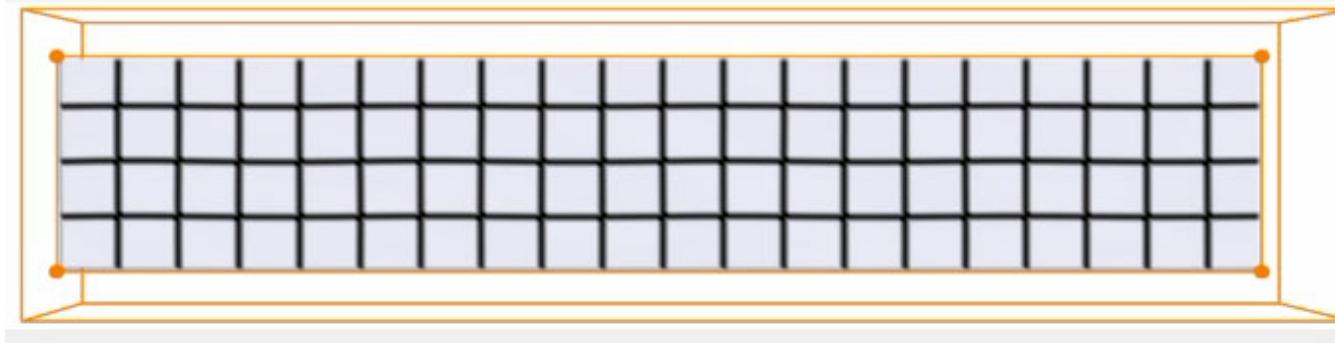
$\psi_{t,s}(x)$  is also often written as **flow map**  $\phi_t^\tau(x)$  (with  $t:=s$  and either  $\tau:=t$  or  $\tau:=t-s$ )

# The Flow / Flow Map of a Vector Field (4)



Can map a whole set of points (or the entire domain) through the flow map (this map is a *diffeomorphism*):

$$t \mapsto \psi_{t,s}(U)$$



$U$

$$= \psi_{s,s}(U)$$



$$\psi_{t,s}(U)$$

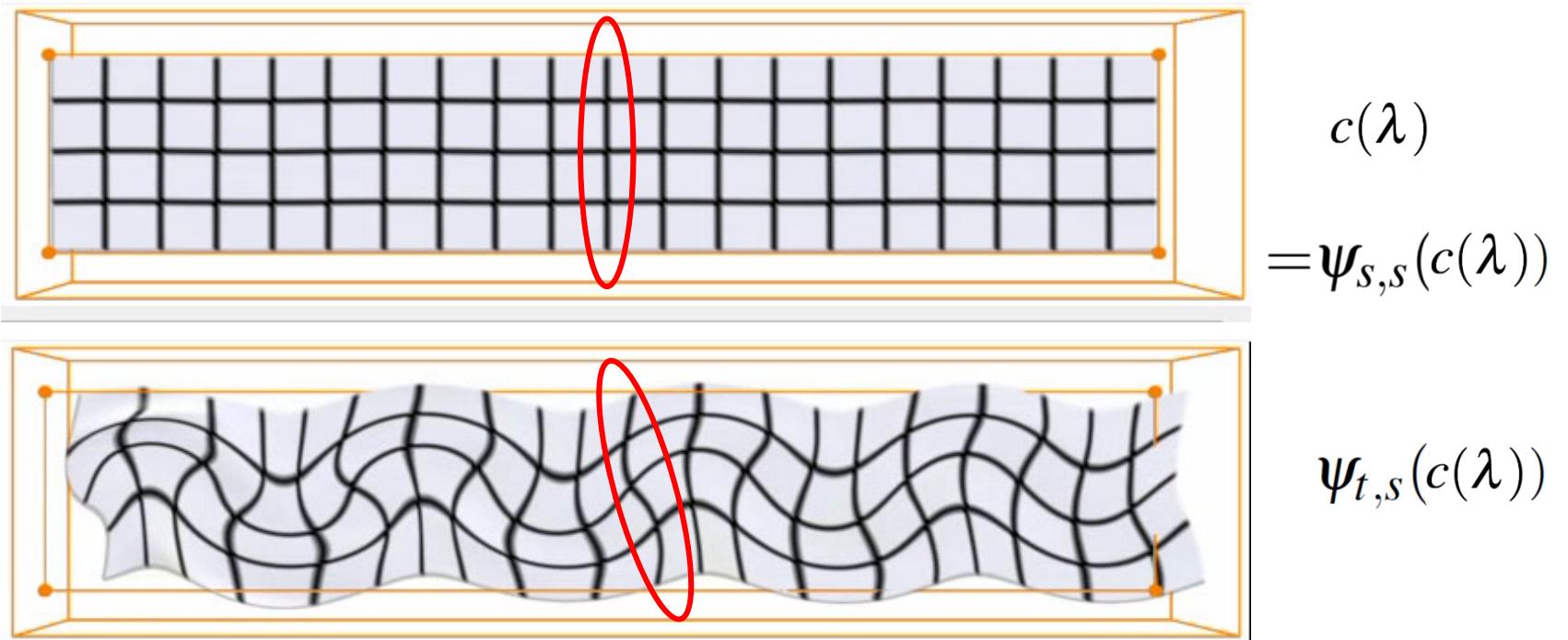
(this is a  
*time surface!*)

# The Flow / Flow Map of a Vector Field (5)



Time line: Map a whole curve from one fixed time ( $s$ ) to another time ( $t$ )

$$t \mapsto \psi_{t,s}(c(\lambda))$$

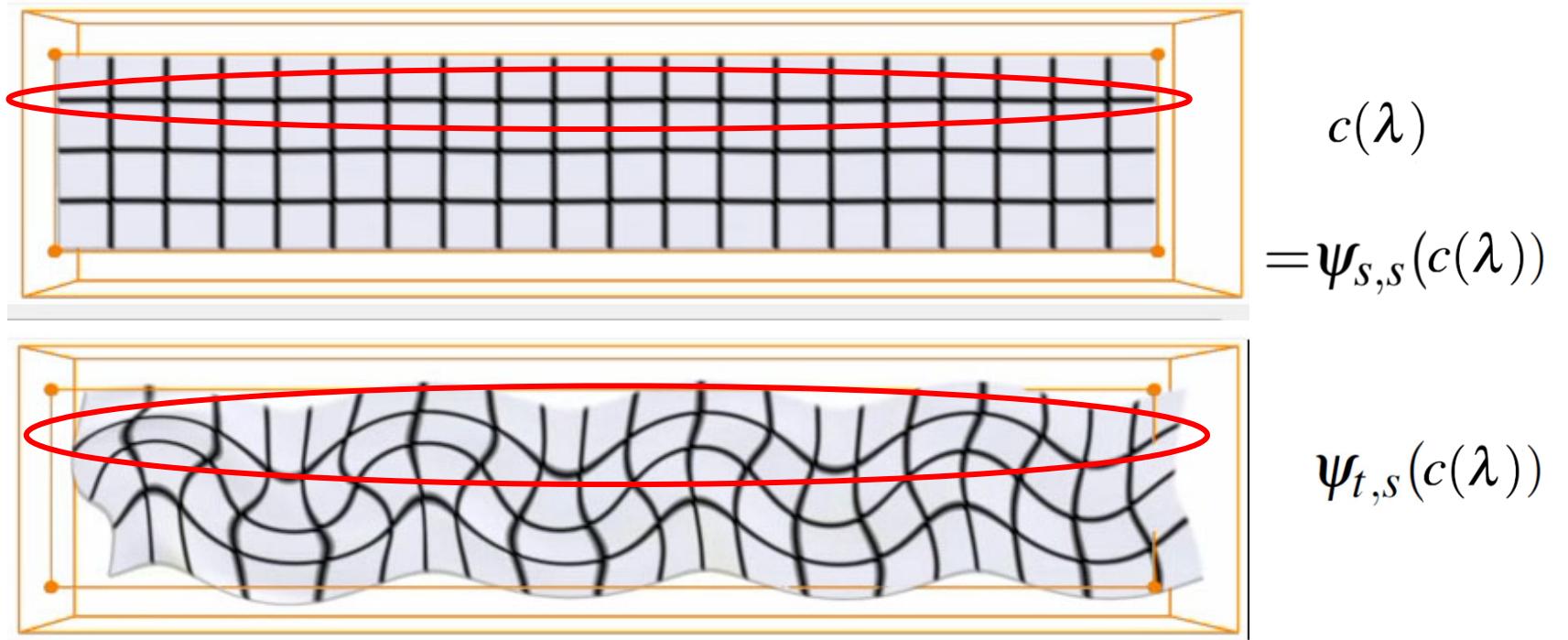




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Time line: Map a whole curve from one fixed time ( $s$ ) to another time ( $t$ )

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## **Streamline**

- Curve parallel to the vector field in each point for a fixed time

## **Pathline**

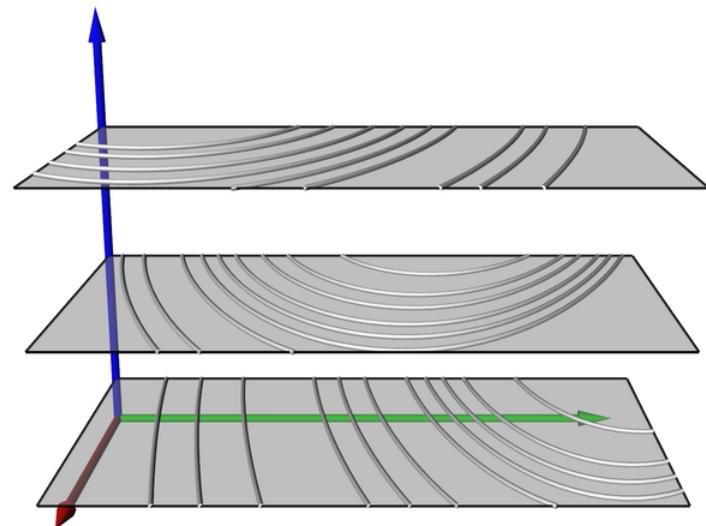
- Describes motion of a massless particle over time

## **Streakline**

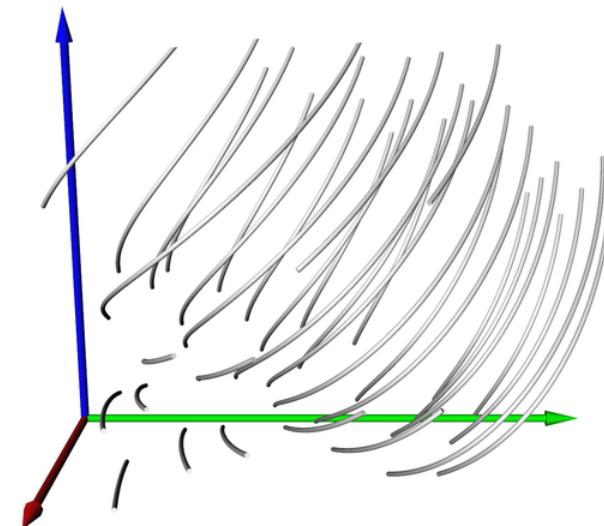
- Location of all particles released at a *fixed position* over time

## **Timeline**

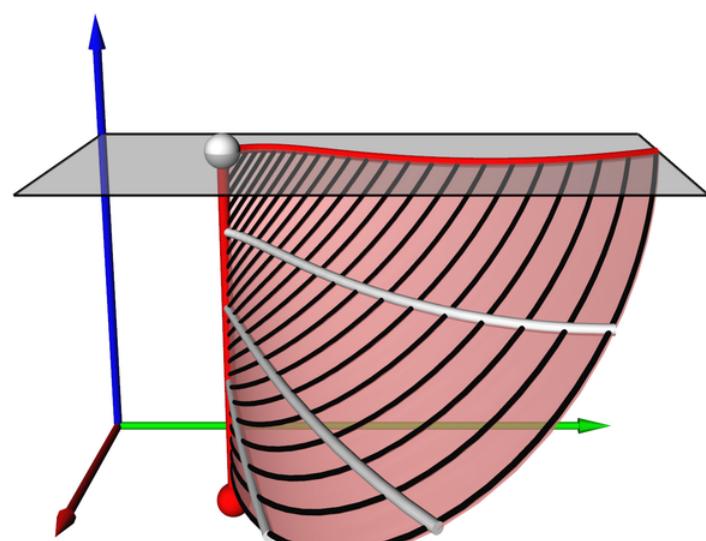
- Location of all particles released along a line at a *fixed time*



stream lines

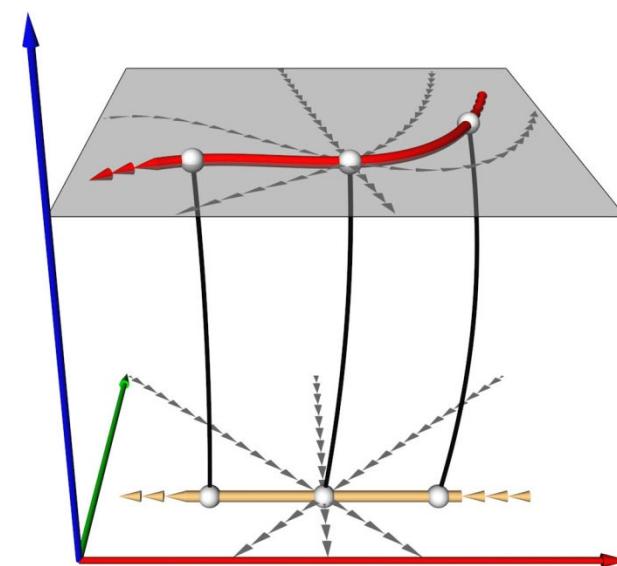


path lines



streak lines

time lines



## *Streamlines, pathlines, streaklines, timelines*

Comparison of techniques:

(1) Pathlines:

- are physically meaningful
- allow comparison with experiment (observe marked particles)
- are well suited for dynamic visualization (of particles)

(2) Streamlines:

- are only geometrically, not physically meaningful
- are easiest to compute (no temporal interpolation, single IVP)
- are better suited for static visualization (prints)
- don't intersect (under reasonable assumptions)

## *Streamlines, pathlines, streaklines, timelines*

### (3) Streaklines:

- are physically meaningful
- allow comparison with experiment (dye injection)
- are well suited for static and dynamic visualization
- good choice for fast moving vortices
- can be approximated by set of disconnected particles

### (4) Timelines:

- are physically meaningful
- are well suited for static and dynamic visualization
- can be approximated by set of disconnected particles



# Bonus Slides: Vectors as Derivative Operators



# Vectors as Derivative Operators

A vector applied to a (real) function on the manifold gives the *directional derivative* in that direction

- From this viewpoint, the vector is a derivative operator (actually, a *derivation*)
- Can be used as *definition* of a vector (must fulfill props. of a derivation; esp. Leibniz rule)

$$f: M \rightarrow \mathbb{R}, \quad \mathbf{v} f \\ x \mapsto f(x).$$



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$$f: M \rightarrow \mathbb{R}, \quad \mathbf{v} f := df(\mathbf{v}) \\ x \mapsto f(x).$$



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$$\frac{\partial}{\partial x^i} f = df \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial f}{\partial x^i}$$



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$$\frac{\partial}{\partial x^i} f = df \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial f}{\partial x^i}$$

$$\frac{\partial}{\partial x^i} x^j = dx^j \left( \frac{\partial}{\partial x^i} \right) = \delta_i^j$$

Kronecker delta  
("identity matrix")  




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$$\mathbf{v}f := df(\mathbf{v})$$

$$\mathbf{e}_i f := df(\mathbf{e}_i)$$

$$\frac{\partial}{\partial x^i} f = df \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial f}{\partial x^i} \quad \frac{\partial}{\partial x^i} x^j = dx^j \left( \frac{\partial}{\partial x^i} \right) = \delta_i^j$$

For vector field: obtain directional derivative at each point

Kronecker delta  
("identity matrix")

$$\mathbf{v}f: M \rightarrow \mathbb{R},$$

$$x \mapsto \mathbf{v}(x) f = df(\mathbf{v}(x)).$$

(remember that this just looks scary (maybe) ...)

# Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama