CS 247 – Scientific Visualization
Lecture 10: Scalar Fields, Pt.6 [preview]

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Reading Assignment #5 (until Feb 28)

Read (required):

- Gradients of scalar-valued functions

  `https://en.wikipedia.org/wiki/Gradient`

- Critical points

  `https://en.wikipedia.org/wiki/Critical_point_(mathematics)`

- Multivariable derivatives and differentials

  `https://en.wikipedia.org/wiki/Total_derivative`

  `https://en.wikipedia.org/wiki/Differential_of_a_function#Differentials_in_several_variables`

  `https://en.wikipedia.org/wiki/Hessian_matrix`

- Dot product, inner product (more general)

  `https://en.wikipedia.org/wiki/Dot_product`

  `https://en.wikipedia.org/wiki/Inner_product_space`

From 2D to 3D (Domain)

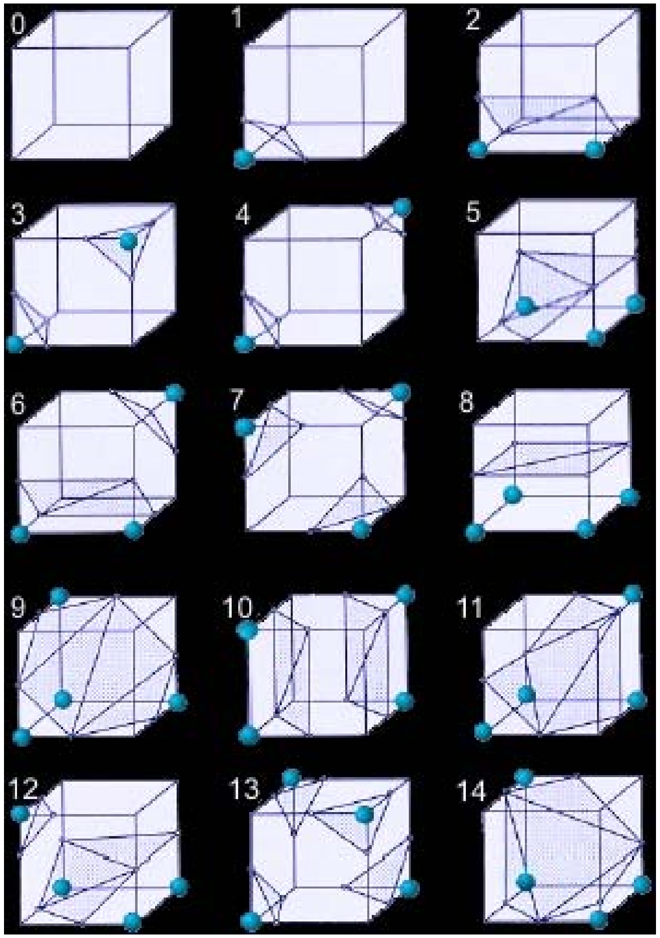
2D - Marching Squares Algorithm:

1. Locate the contour corresponding to a user-specified iso value
2. Create lines

3D - Marching Cubes Algorithm:

1. Locate the surface corresponding to a user-specified iso value
2. Create triangles
3. Calculate normals to the surface at each vertex
4. Draw shaded triangles

Marching Cubes



- For each cell, we have 8 vertices with 2 possible states each (inside or outside).
- This gives us 2^8 possible patterns = 256 cases.
- Enumerate cases to create a LUT
- Use symmetries to reduce problem from 256 to 15 cases.

Explanations

- Data Visualization book, 5.3.2
- Marching Cubes: A high resolution 3D surface construction algorithm, Lorensen & Cline, ACM SIGGRAPH 1987

The marching cubes algorithm

Contours of 3D scalar fields are known as isosurfaces.

Before 1987, isosurfaces were computed as

- contours on planar slices, followed by
- "contour stitching".

The marching cubes algorithm computes contours directly in 3D.

- Pieces of the isosurfaces are generated on a cell-by-cell basis.
- Similar to marching squares, a 8-bit number is computed from the 8 signs of $\tilde{f}(x_i)$ on the corners of a hexahedral cell.
- The isosurface piece is looked up in a table with 256 entries.

How to build up the table of 256 cases?

Lorensen and Cline (1987) exploited 3 types of symmetries:

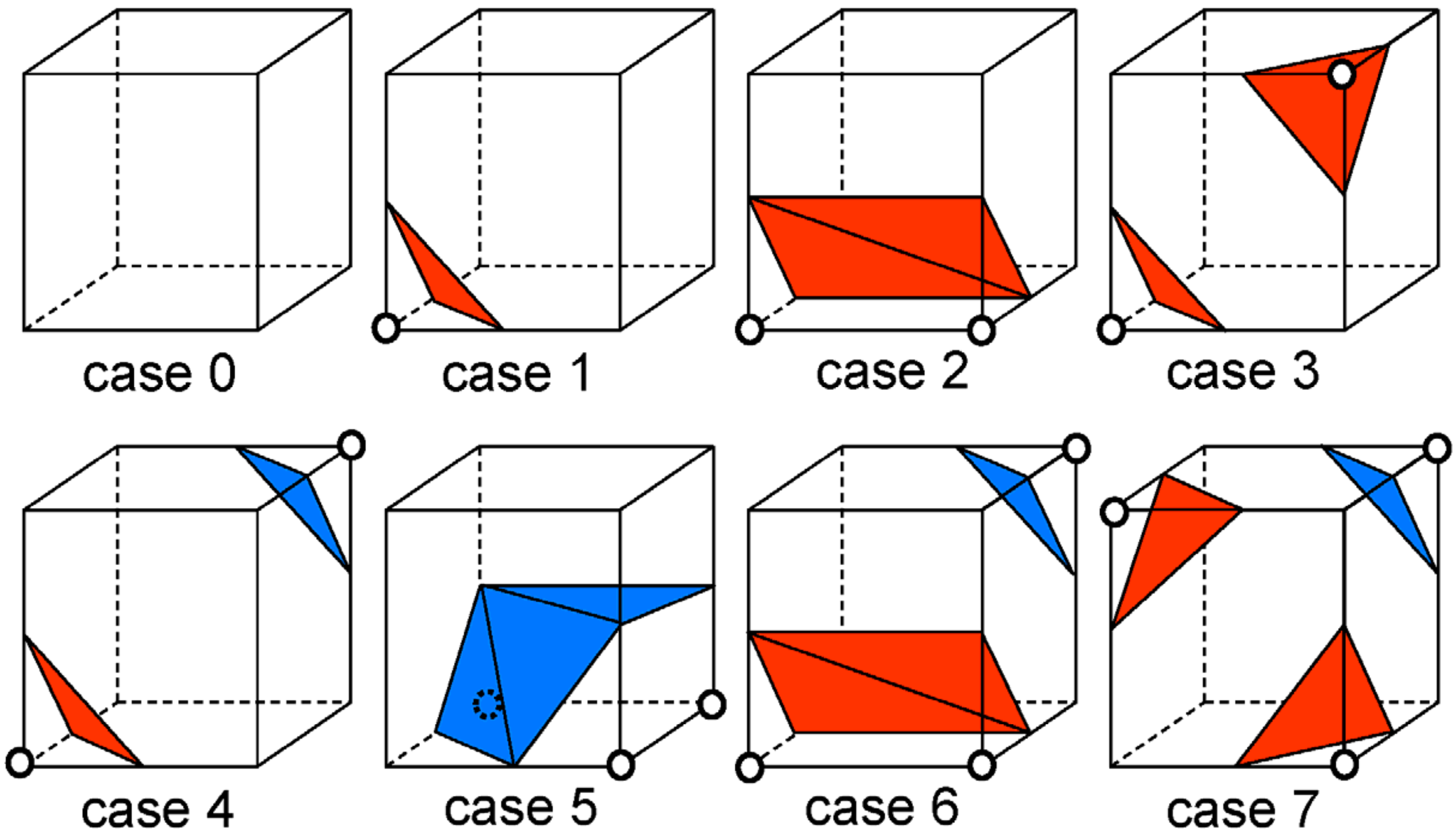
- rotational symmetries of the cube
- reflective symmetries of the cube
- sign changes of $\tilde{f}(x_i)$

They published a reduced set of 14*) cases shown on the next slides where

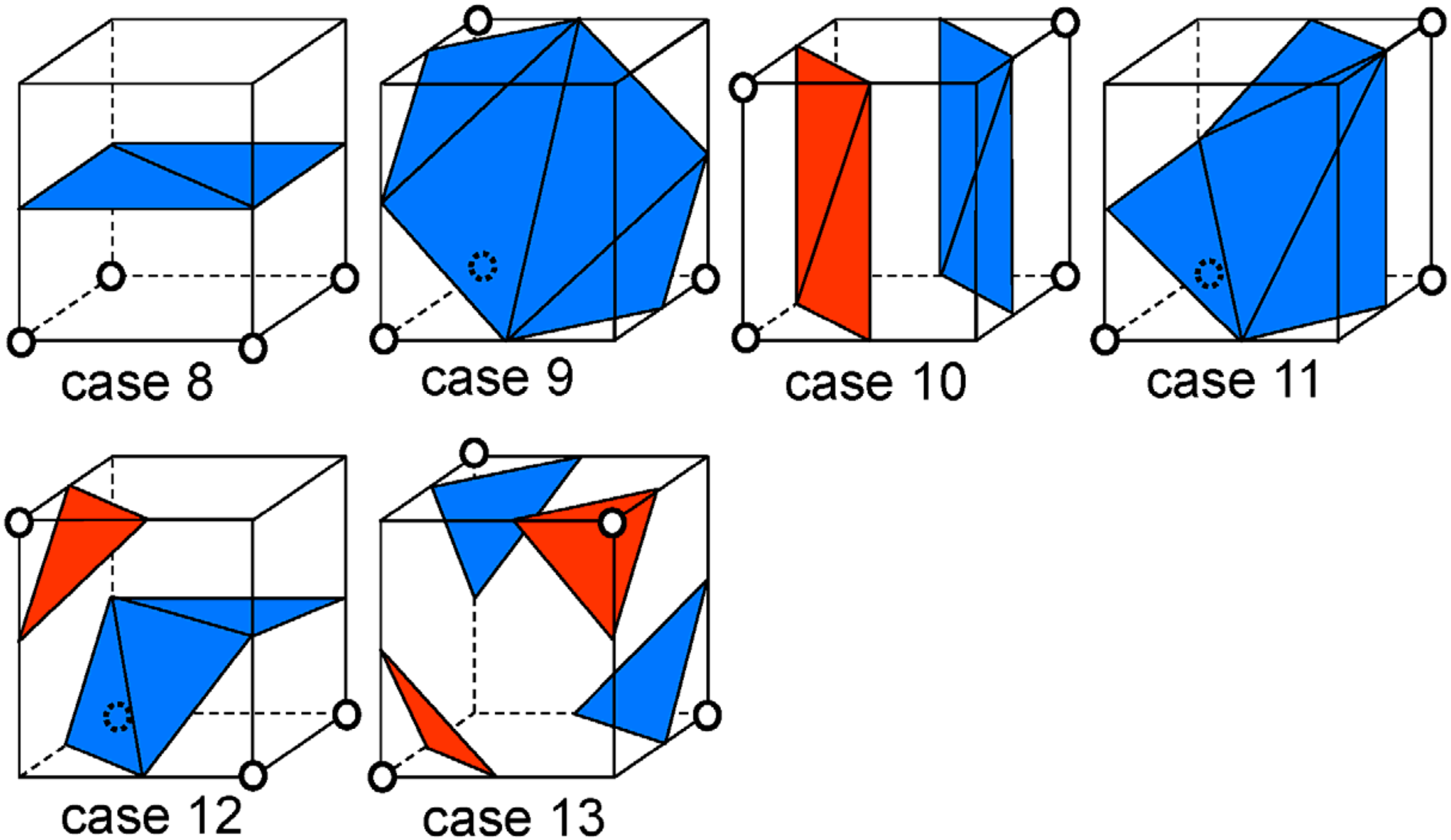
- white circles indicate positive signs of $\tilde{f}(x_i)$
- the positive side of the isosurface is drawn in red, the negative side in blue.

*) plus an unnecessary "case 14" which is a symmetric image of case 11.

The marching cubes algorithm

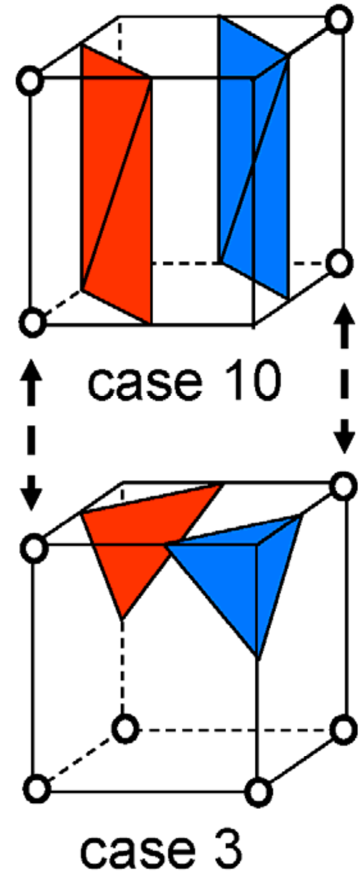


The marching cubes algorithm



Do the pieces fit together?

- The correct isosurfaces of the trilinear interpolant would fit (trilinear reduces to bilinear on the cell interfaces)
- but the marching cubes polygons don't necessarily fit.

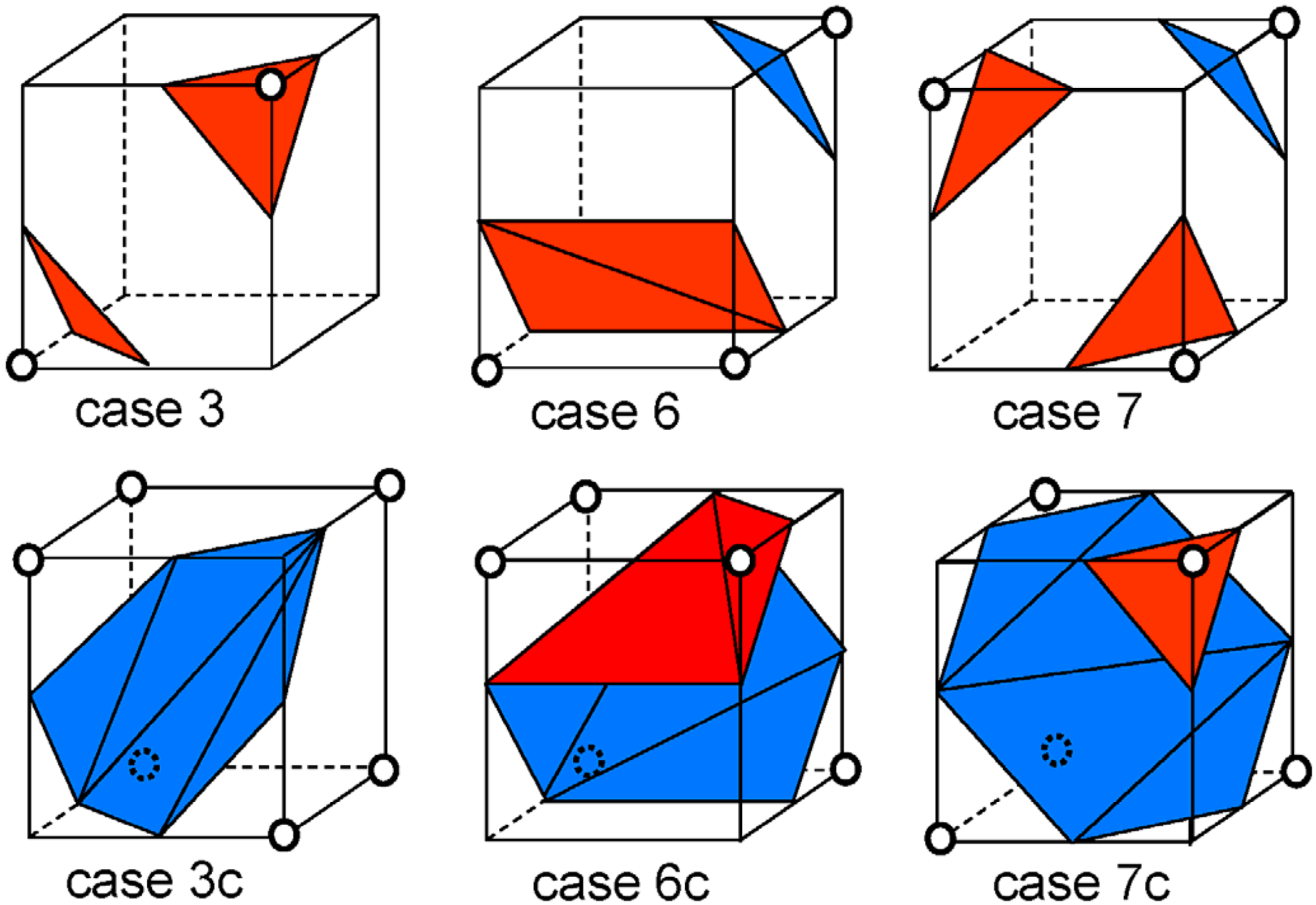


Example

- case 10, on top of
- case 3 (rotated, signs changed)

have matching signs at nodes but polygons don't fit.

The marching cubes algorithm



Summary of marching cubes algorithm:

Pre-processing steps:

- build a table of the 28 cases
- derive a table of the 256 cases, containing info on
  - intersected cell edges, e.g. for case 3/256 (see case 2/28): (0,2), (0,4), (1,3), (1,5)
  - triangles based on these points, e.g. for case 3/256: (0,2,1), (1,3,2).

The marching cubes algorithm

Loop over cells:

- find sign of $\tilde{f}(x_i)$ for the 8 corner nodes, giving 8-bit integer
- use as index into (256 case) table
- find intersection points on edges listed in table, using linear interpolation
- generate triangles according to table

Post-processing steps:

- connect triangles (share vertices)
- compute normal vectors
  - by averaging triangle normals (problem: thin triangles!)
  - by estimating the gradient of the field $f(x_i)$ (better)

Triangle Mesh Data Structure (1)

Store list of vertices; vertices shared by triangles are replicated

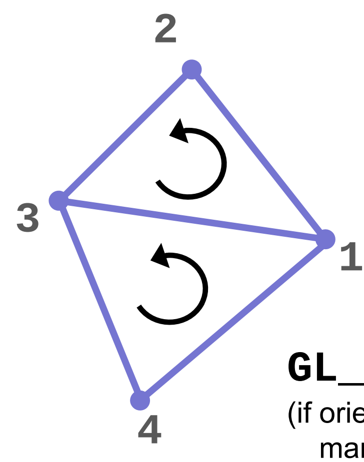
Render, e.g., with OpenGL immediate mode, …

coords for vertex 1 →

```
x1,y1,z1 ┐
x2,y2,z2 │ face 1
x3,y3,z3 ┘
x1,y1,z1 ┐
x3,y3,z3 │ face 2
x4,y4,z4 ┘

...
```

```
struct face
  float verts[3][3]
  DataType val;
```



GL_CCW

(if orientable manifold)

Redundant, large storage size, cannot modify shared vertices easily

Store data values per face, or separately

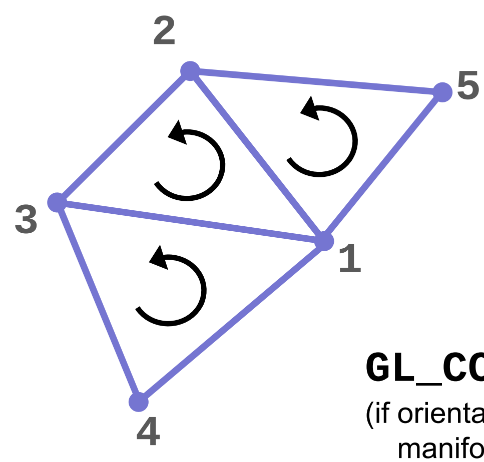
Triangle Mesh Data Structure (2)

Indexed face set: store list of vertices; store triangles as indexes

Render using separate vertex and index arrays / buffers

coords for vertex 1

vertex list	face list
`x1,y1,(z1)`	`1,2,3`
`x2,y2,(z2)`	`1,3,4`
`x3,y3,(z3)`	`2,1,5`
`x4,y4,(z4)`	`...`
`...`	



`GL_CCW`
(if orientable manifold)

Less redundancy, more efficient in terms of memory

Easy to change vertex positions; still have to do (global) search for shared edges (local information)

Orientability (2-manifold embedded in 3D)

Orientability of 2-manifold:

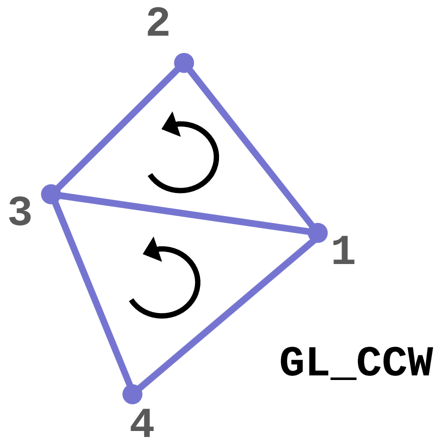
Possible to assign consistent normal vector orientation

not orientable

Moebius strip
(only one side!)

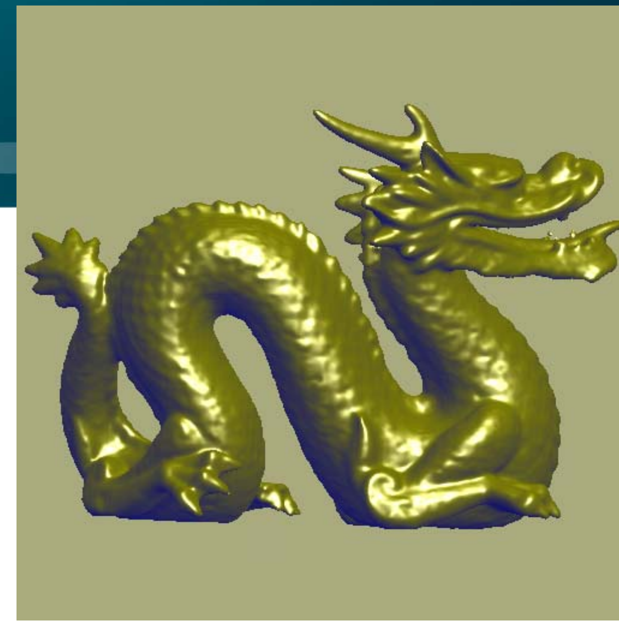
Triangle meshes

- Edges
  - Consistent ordering of vertices: CCW (counter-clockwise) or CW (clockwise) (e.g., (3,1,2) on one side of edge, (1,3,4) on the other side)
- Triangles
  - Consistent front side vs. back side
  - Normal vector; or ordering of vertices (CCW/CW)
  - See also: “right-hand rule”

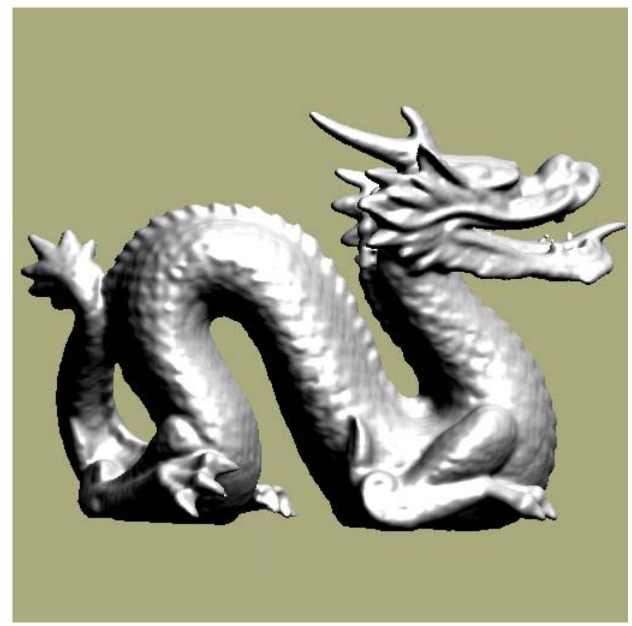


Iso-Surface / Volume Illumination

What About Volume Illumination?



Crucial for perceiving shape and depth relationships



this is a scalar volume (3D distance field)!

Local Illumination in Volumes

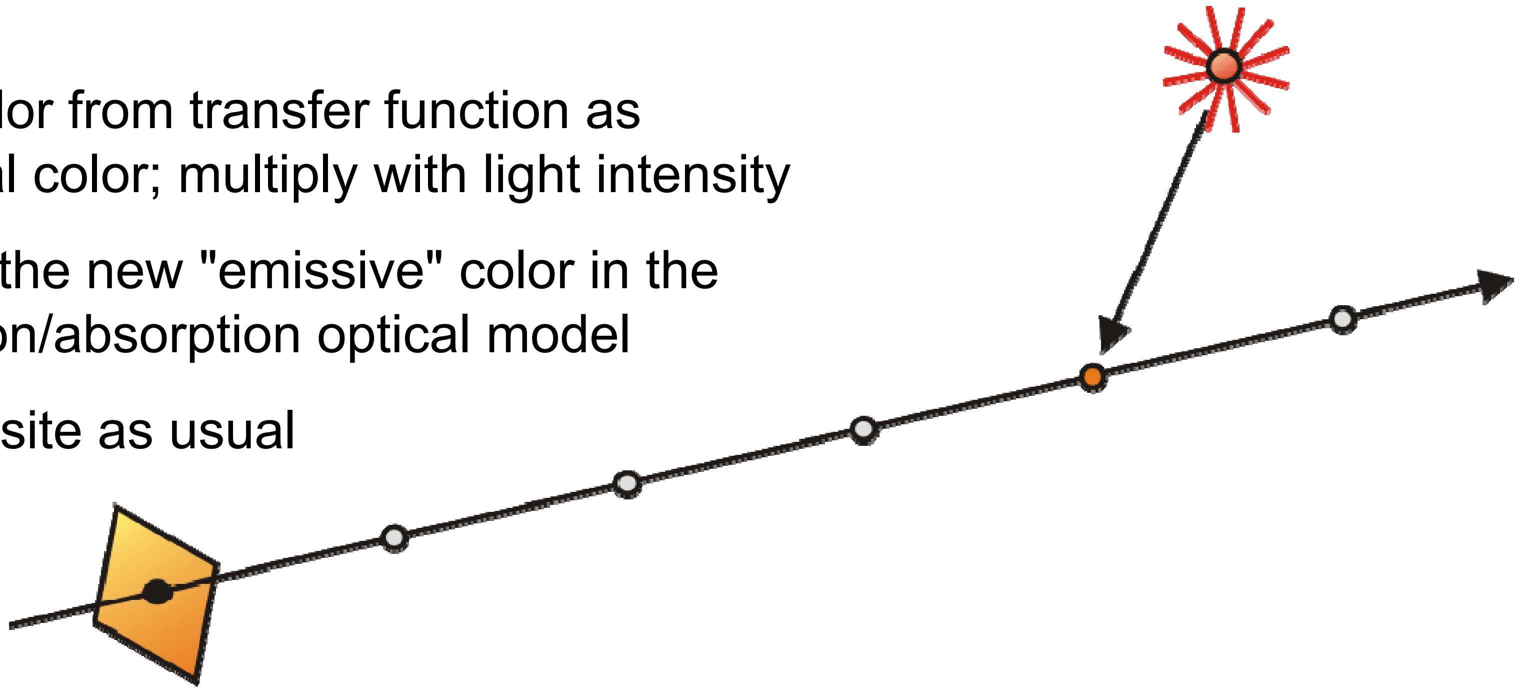
Interaction between light source and point in the volume

Local shading equation; evaluate at each point along a ray

Use color from transfer function as material color; multiply with light intensity

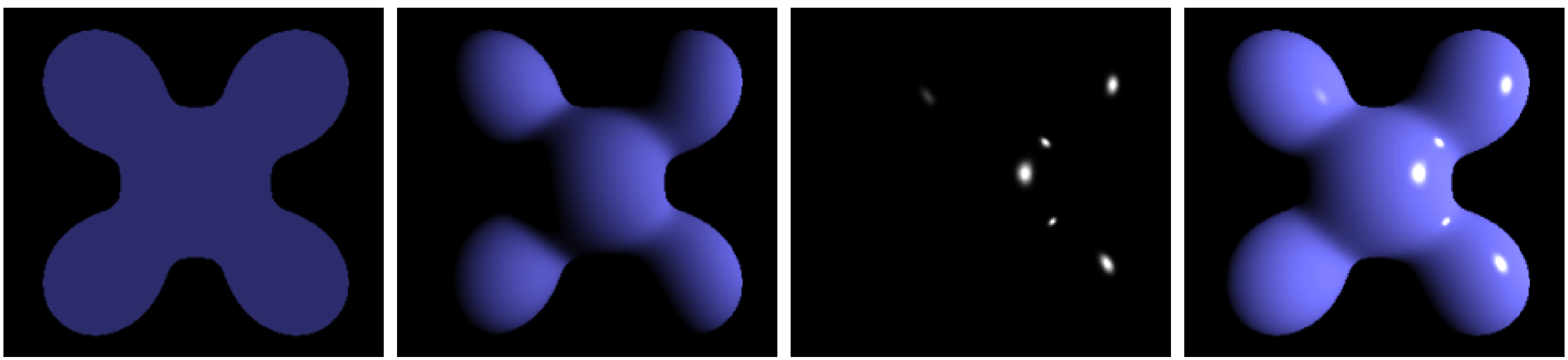
This is the new "emissive" color in the emission/absorption optical model

Composite as usual



Local Illumination Model: Phong Lighting Model

$$\mathbf{I}_{\text{Phong}} = \mathbf{I}_{\text{ambient}} + \mathbf{I}_{\text{diffuse}} + \mathbf{I}_{\text{specular}}$$



Ambient + Diffuse + Specular = Phong Reflection

Local Illumination Model: Phong Lighting Model

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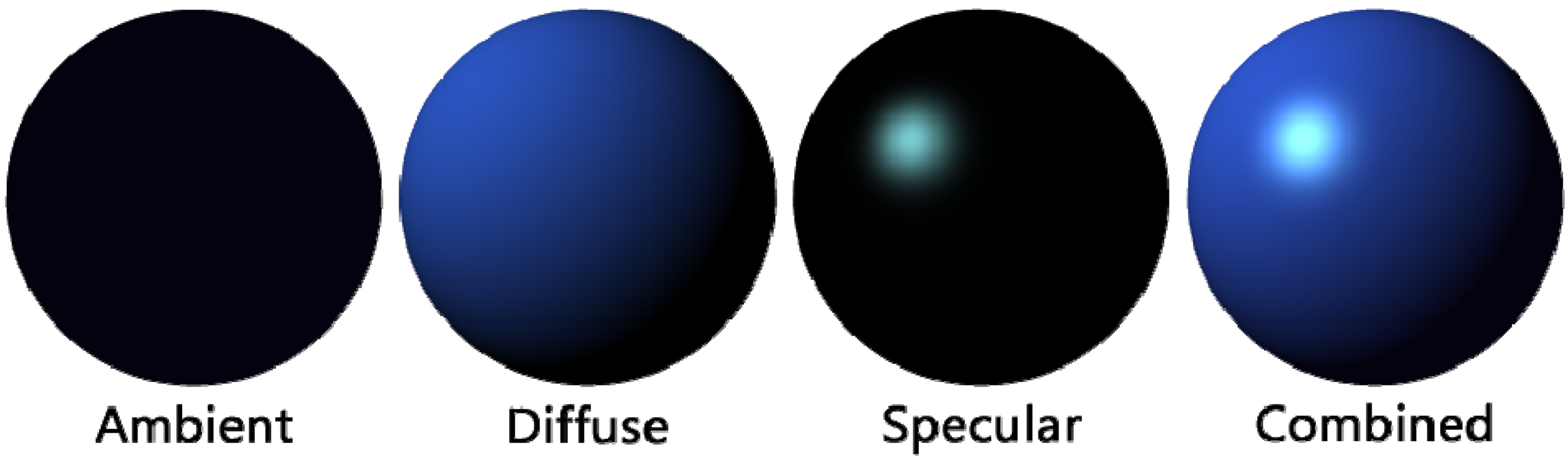
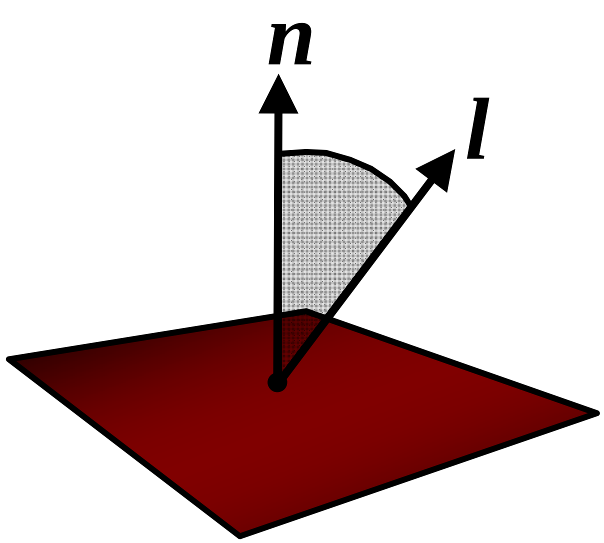


Local Shading Equations

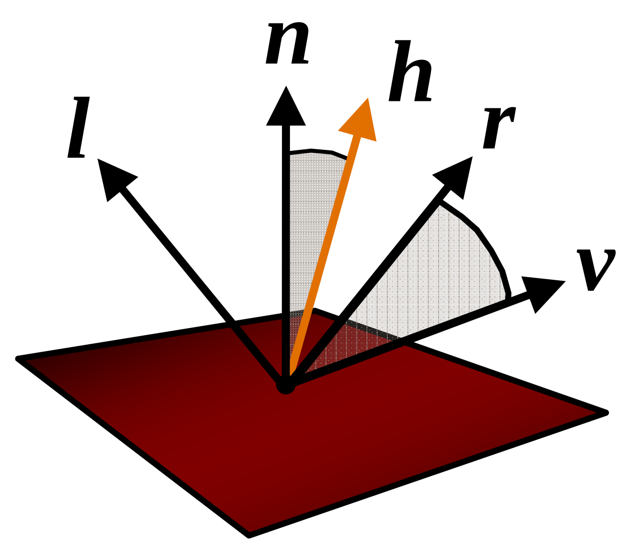
Standard volume shading adapts surface shading

Most commonly Blinn/Phong model

But what about the "surface" normal vector?



diffuse reflection



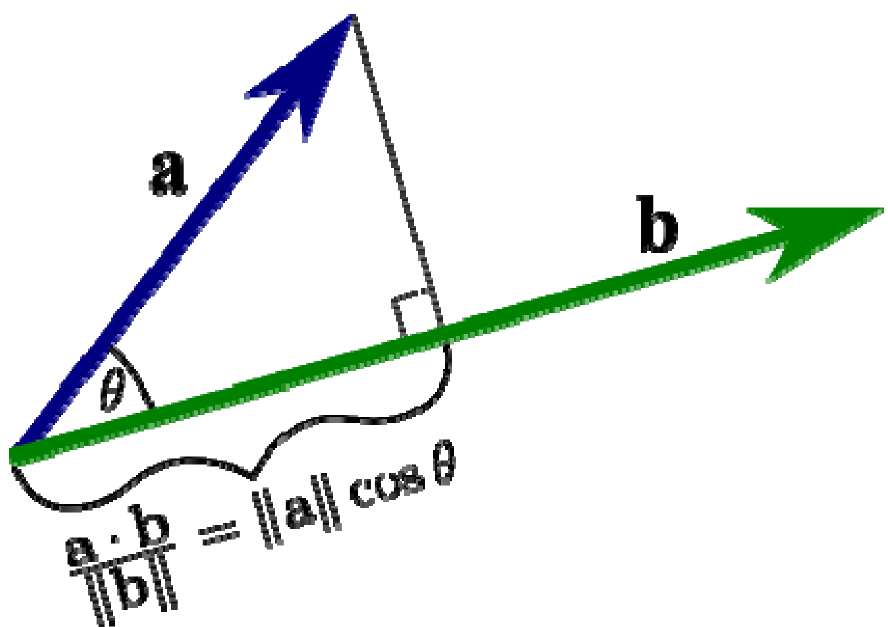
specular reflection

The Dot Product (Scalar / Inner Product)

Cosine of angle between two vectors times their lengths

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \qquad \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

(standard inner product in Cartesian coordinates)



Many uses:

- Project vector onto another vector,
  project into basis,
  project into tangent plane,
  ...

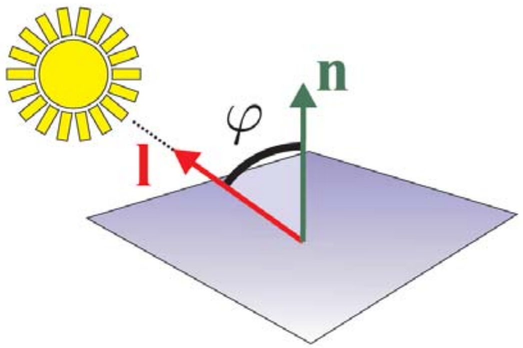
Local Illumination Model: Phong Lighting Model

$$\mathbf{I}_{\text{Phong}} = \mathbf{I}_{\text{ambient}} + \mathbf{I}_{\text{diffuse}} + \mathbf{I}_{\text{specular}}$$

$$\mathbf{I}_{\text{ambient}} = k_a \, \mathbf{M}_a \, \mathbf{I}_a$$

Local Illumination Model: Phong Lighting Model

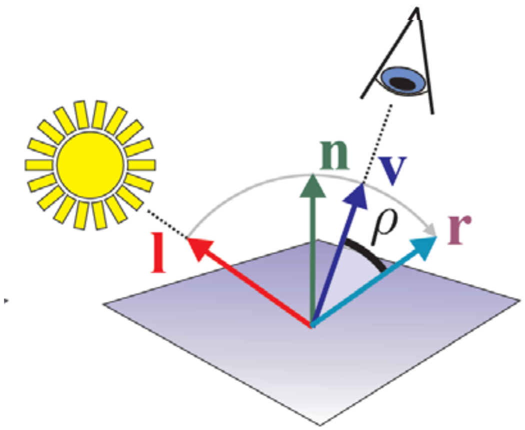
$$\mathbf{I}_{\text{Phong}} = \mathbf{I}_{\text{ambient}} + \mathbf{I}_{\text{diffuse}} + \mathbf{I}_{\text{specular}}$$



$$\begin{aligned}\mathbf{I}_{\text{diffuse}} &= k_d\,\mathbf{M}_d\,\mathbf{I}_d\cos\varphi \qquad \text{if } \varphi \le \frac{\pi}{2}\\ &= k_d\,\mathbf{M}_d\,\mathbf{I}_d\max((\mathbf{n}\cdot\mathbf{l}),0)\end{aligned}$$

Local Illumination Model: Phong Lighting Model

$$\mathbf{I}_{\text{Phong}} = \mathbf{I}_{\text{ambient}} + \mathbf{I}_{\text{diffuse}} + \mathbf{I}_{\text{specular}}$$

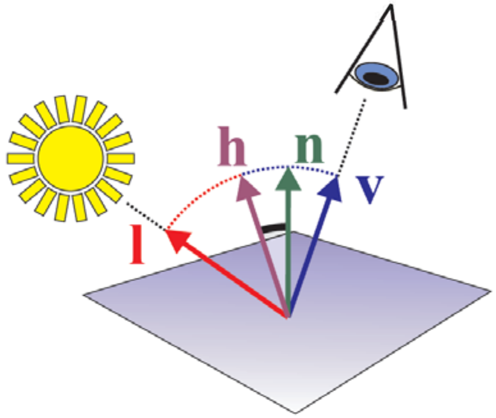


$$\begin{aligned}\mathbf{I}_{\text{specular}} &= k_s\, \mathbf{M}_s\, \mathbf{I}_s \cos^n \rho\,, \quad \text{if } \rho \leq \frac{\pi}{2} \\ &= k_s\, \mathbf{M}_s\, \mathbf{I}_s\, (\mathbf{r} \cdot \mathbf{v})^n\end{aligned}$$

must also clamp!

Local Illumination Model: Phong Lighting Model

$$\mathbf{I}_{\text{Phong}} = \mathbf{I}_{\text{ambient}} + \mathbf{I}_{\text{diffuse}} + \mathbf{I}_{\text{specular}}$$



$$\mathbf{I}_{\text{specular}} \approx k_s\, \mathbf{M}_s\, \mathbf{I}_s\, (\mathbf{h} \cdot \mathbf{n})^n$$

must also clamp!

$$\mathbf{h} = \frac{\mathbf{v} + \mathbf{l}}{\|\mathbf{v} + \mathbf{l}\|}$$

half-way vector

The Gradient as Normal Vector

Gradient of the scalar field gives direction+magnitude of fastest change

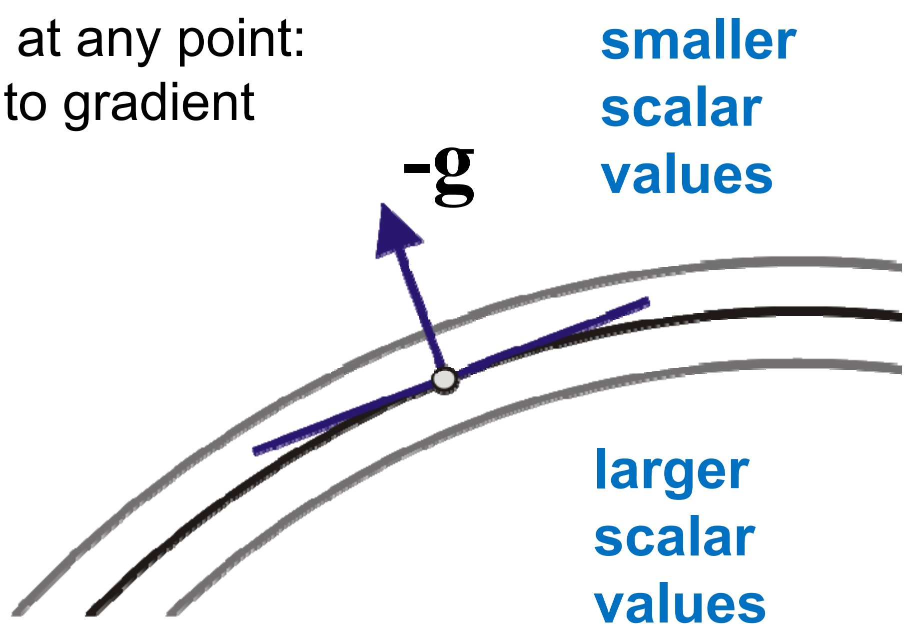
$$\mathbf{g} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^{\mathbf{T}}$$

(only correct in Cartesian coordinates [see later lectures])

Local approximation to isosurface at any point:
tangent plane = plane orthogonal to gradient

Normal of this isosurface:
normalized gradient vector
(negation is common convention)

$$\mathbf{n} = -\mathbf{g}/|\mathbf{g}|$$



Gradient and Directional Derivative

Gradient $\nabla f(x,y,z)$ of scalar function $f(x,y,z)$: (in Cartesian coordinates)

$$\nabla f(x,y,z) = \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z} \right)^T$$

Directional derivative in direction $\mathbf{u}$:

$$D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$$

And therefore also:

$$D_{\mathbf{u}} f(x,y,z) = ||\nabla f|| \; ||\mathbf{u}|| \; \cos\theta$$

Gradient and Directional Derivative

Gradient $\nabla f(x,y,z)$ of scalar function $f\left(x,y,z\right)$: (in Cartesian coordinates)

$$\nabla f(x,y,z) = \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z} \right)^T$$

(Cartesian vector components; basis vectors not shown)

But: always need **basis vectors**! With Cartesian basis:

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\,\mathbf{i} + \frac{\partial f(x,y,z)}{\partial y}\,\mathbf{j} + \frac{\partial f(x,y,z)}{\partial z}\,\mathbf{k}$$

What about the Basis?

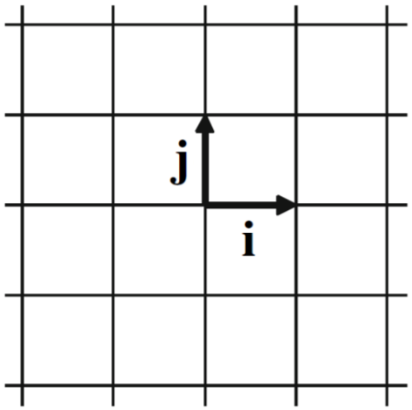
On the previous slide, this actually meant

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\,\mathbf{i}(x,y,z) + \frac{\partial f(x,y,z)}{\partial y}\,\mathbf{j}(x,y,z) + \frac{\partial f(x,y,z)}{\partial z}\,\mathbf{k}(x,y,z)$$

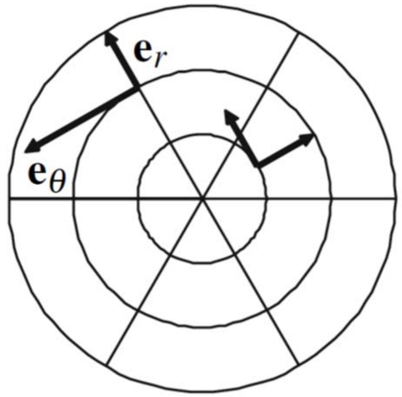
It's just that the Cartesian basis vectors are the same everywhere...

But this is not true for many other coordinate systems:

Cartesian coordinates



polar coordinates



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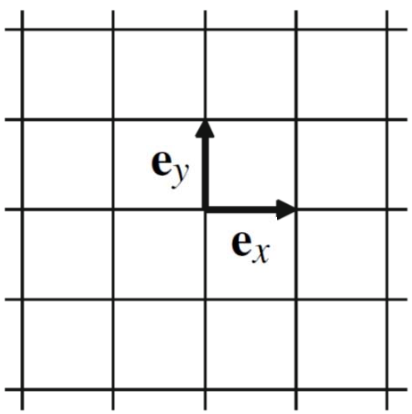

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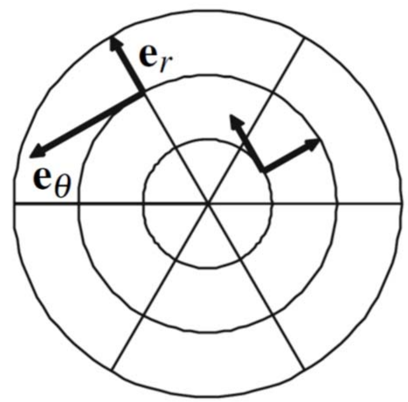


The Gradient as a Differential Form

The gradient as a *differential* (differential 1-form) is the "primary" concept (also "total differential" or "total derivative")

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

A differential 1-form is a scalar-valued linear function that takes a (direction) vector as input, and gives a scalar as output

Each of the 1-forms df, dx, dy, dz takes direction vector as input, gives scalar output

In the expression of the gradient df above, all 1-forms on the right-hand side get the same vector as input

df is simply a linear combination of the coordinate differentials dx, dy, dz

The Gradient as a Differential Form

The gradient as a *differential* (differential 1-form) is the "primary" concept (also "total differential" or "total derivative")

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

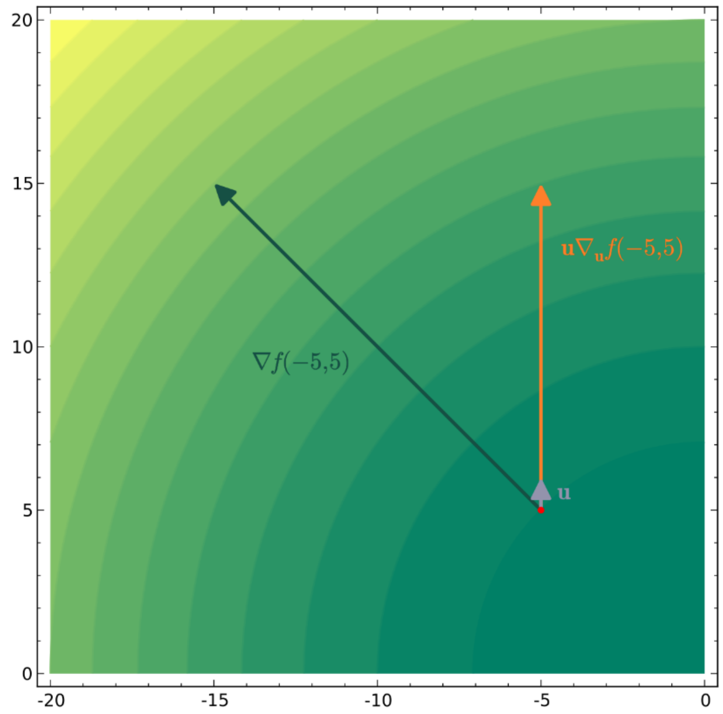
The directional derivative and the gradient vector

$$D_{\mathbf{u}} f = df(\mathbf{u})$$
$$df(\mathbf{u}) = \nabla f \cdot \mathbf{u}$$

The gradient vector is then *defined*, such that:

$$\nabla f \cdot \mathbf{u} := df(\mathbf{u})$$

Gradient Vectors and Differential 1-Forms

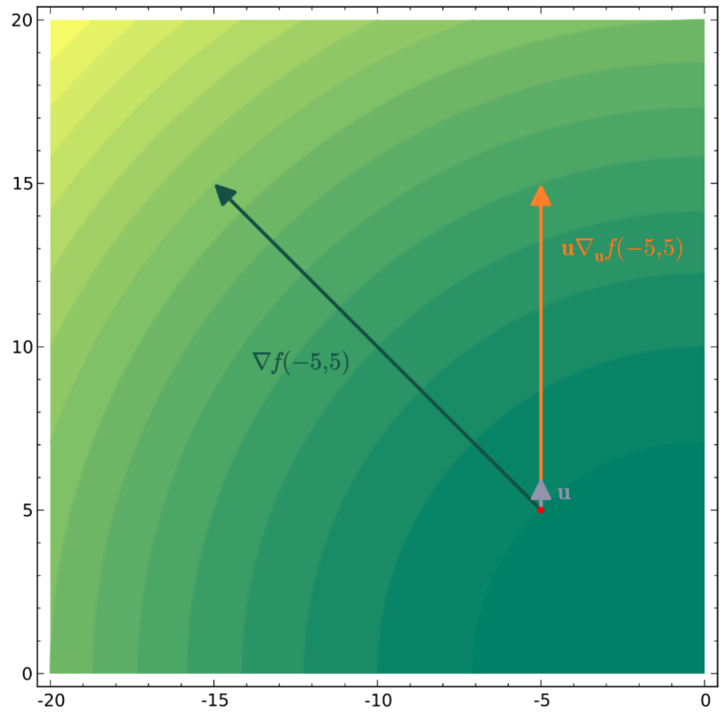


from Wikipedia (for **u** a unit vector),

the function here is $f(x,y) = x^2 + y^2$

$$\nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

Gradient Vectors and Differential 1-Forms



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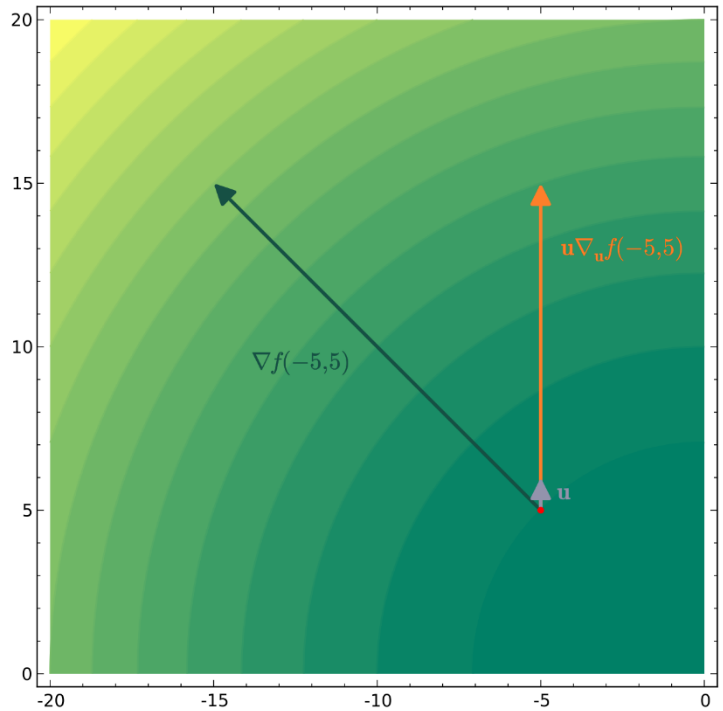
$$\nabla f(x,y) = 2x\,\mathbf{e}_x + 2y\,\mathbf{e}_y$$

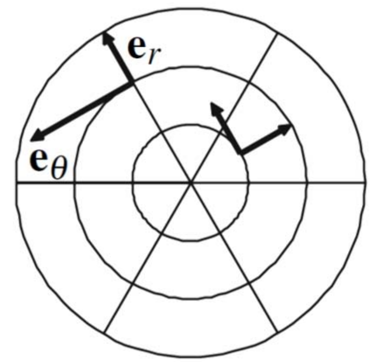
$$df(x,y) = 2x\,dx + 2y\,dy$$

Gradient Vectors and Differential 1-Forms



from Wikipedia (for **u** a unit vector),

how about in polar coordinates?

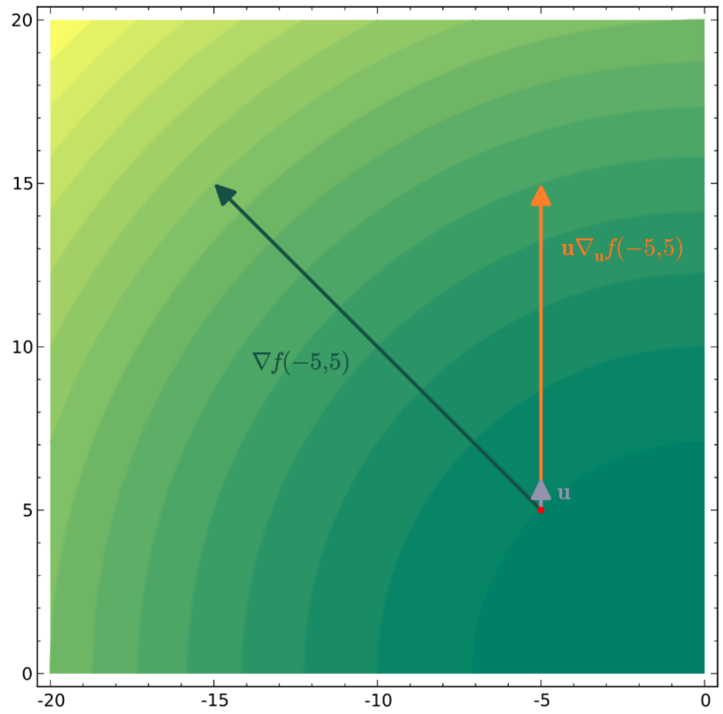


the function here is $f(r,\theta) = r^2$

$$\nabla f(r,\theta) = 2r\,\mathbf{e}_r + 0\frac{1}{r^2}\,\mathbf{e}_\theta = 2r\,\mathbf{e}_r$$

$$df(r,\theta) = 2r\,dr + 0\,d\theta = 2r\,dr$$

Gradient Vectors and Differential 1-Forms



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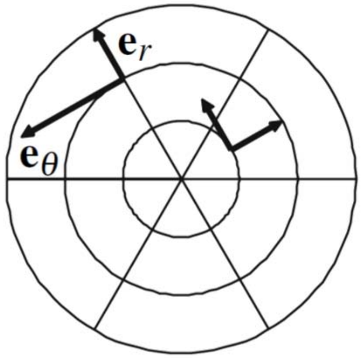
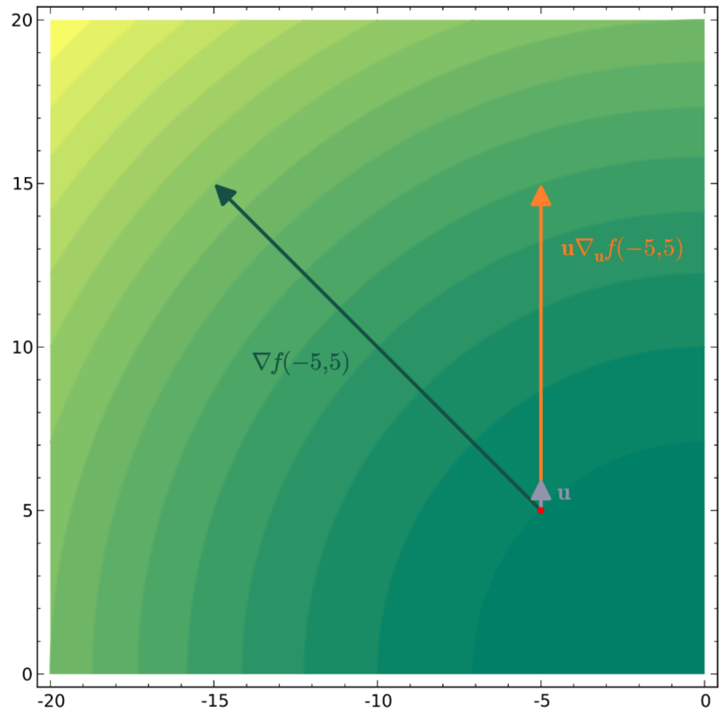
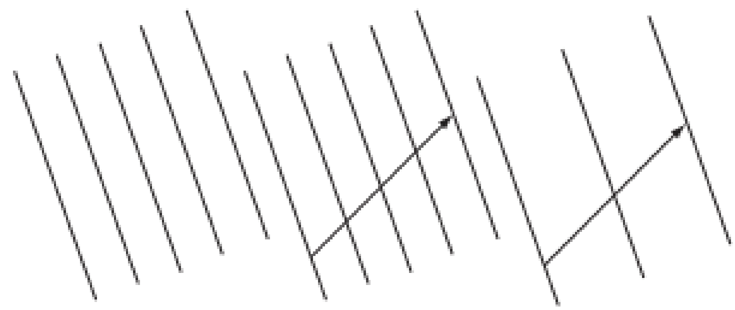
how about in polar coordinates?



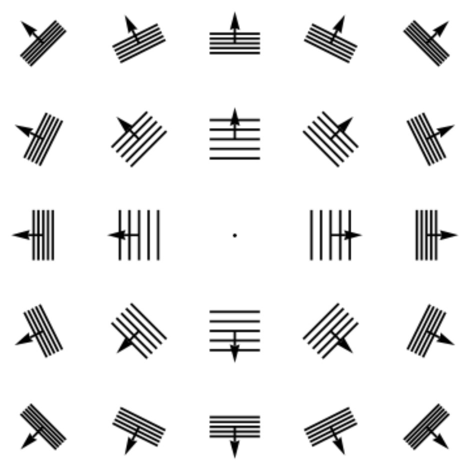
Gradient Vectors and Differential 1-Forms



different 1-forms
evaluated in some direction



1-form (field) df



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$$df(r,\theta) = 2r\,dr + 0\,d\theta = 2r\,dr$$

Interlude: Tensor Calculus

In tensor calculus, first-order tensors can be

- Contravariant $\mathbf{v} = v^i \, \mathbf{e}_i$
- Covariant $\boldsymbol{\omega} = v_i \, \boldsymbol{\omega}^i$

The gradient vector is a contravariant vector $\mathbf{v} = v^i \, \boldsymbol{\partial}_i$

The gradient 1-form is a covariant vector (a covector) $df = \dfrac{\partial f}{\partial x^i} \, dx^i$

Very powerful; necessary for non-Cartesian coordinate systems

On (intrinsically) curved manifolds (sphere, ...):
Cartesian coordinates not even possible

Interlude: Tensor Calculus

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The gradient 1-form is a covariant vector (a covector) $$df = \frac{\partial f}{\partial x^i}\, dx^i$$

This is also the fundamental reason why in graphics a normal vector transforms differently: as a covector, not as a vector!

(typical graphics rule: **n** transforms with transpose of inverse matrix)

Metric Tensor (Field)

Symmetric second-order tensor field: *Defines* inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{g}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta$$

$$\begin{aligned} \|\mathbf{v}\|^2 &= \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \mathbf{g}(\mathbf{v}, \mathbf{v}) \\ &= g_{ij} v^i v^j \\ &= \mathbf{v}^T \mathbf{g} \, \mathbf{v} \end{aligned}$$

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$$\mathbf{g} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$\|\mathbf{v}\|^2 = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

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Cartesian coordinates:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\|\mathbf{v}\|^2 = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \mathbf{v}^T \mathbf{v}$$

Metric Tensor (Field)

Components referred to coordinates

$$g_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle$$

A second-order tensor field is bi-linear, i.e.,
linear in each (vector) argument separately

Therefore, we immediately get:

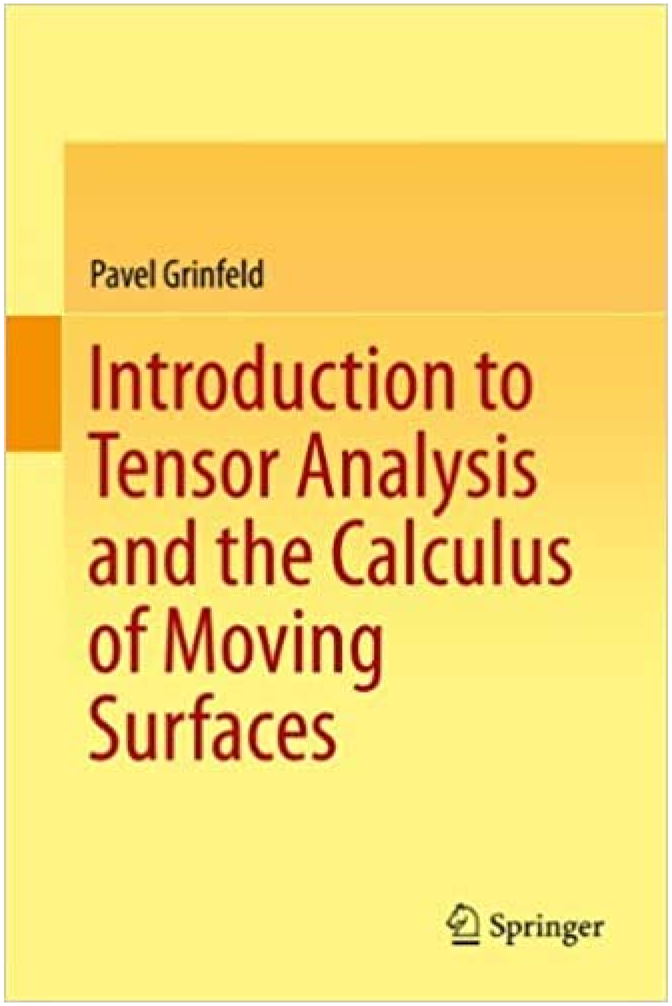
$$\begin{aligned} \mathbf{g}(\mathbf{v},\mathbf{v}) &= \mathbf{g}\left(v^i \mathbf{e}_i, v^j \mathbf{e}_j\right) \\ &= v^i v^j \mathbf{g}\left(\mathbf{e}_i, \mathbf{e}_j\right) \\ &= g_{ij} v^i v^j \end{aligned}$$

Tensor Calculus

Highly recommended:

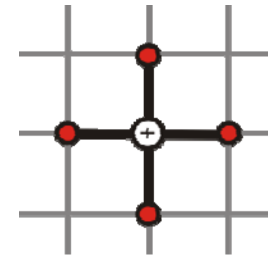
Very nice book,

complete lecture on Youtube!



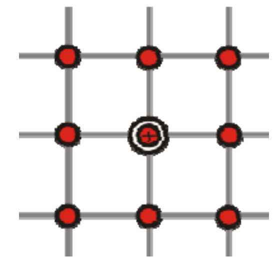
(Numerical) Gradient Reconstruction

We need to reconstruct the derivatives of a continuous function given as discrete samples



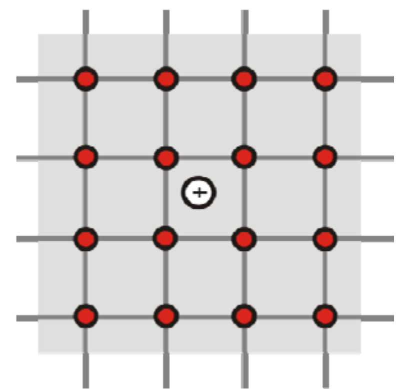
Central differences

- Cheap and quality often sufficient (2*3 neighbors in 3D)



Discrete convolution filters on grid

- Image processing filters; e.g. Sobel (3^3 neighbors in 3D)



Continuous convolution filters

- Derived continuous reconstruction filters
- E.g., the cubic B-spline and its derivatives (4^3 neighbors)

Finite Differences

Obtain first derivative from Taylor expansion

$$
\begin{aligned}
f(x_0+h) &= f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \ldots \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}h^n .
\end{aligned}
$$

Forward differences / backward differences

$$
\begin{aligned}
f(x_0)' &= \frac{f(x_0+h) - f(x_0)}{h} + o(h) \\
f(x_0)' &= \frac{f(x_0) - f(x_0-h)}{h} + o(h)
\end{aligned}
$$

Finite Differences

Central differences

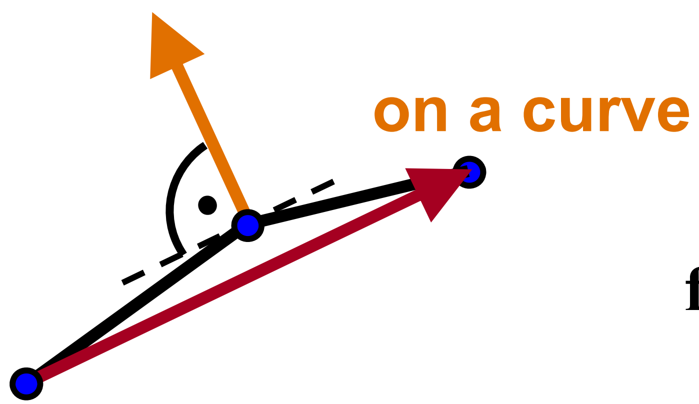
$$
\begin{aligned}
f(x_0+h) &= f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + o(h^3) \\
f(x_0-h) &= f(x_0) - \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + o(h^3)
\end{aligned}
$$

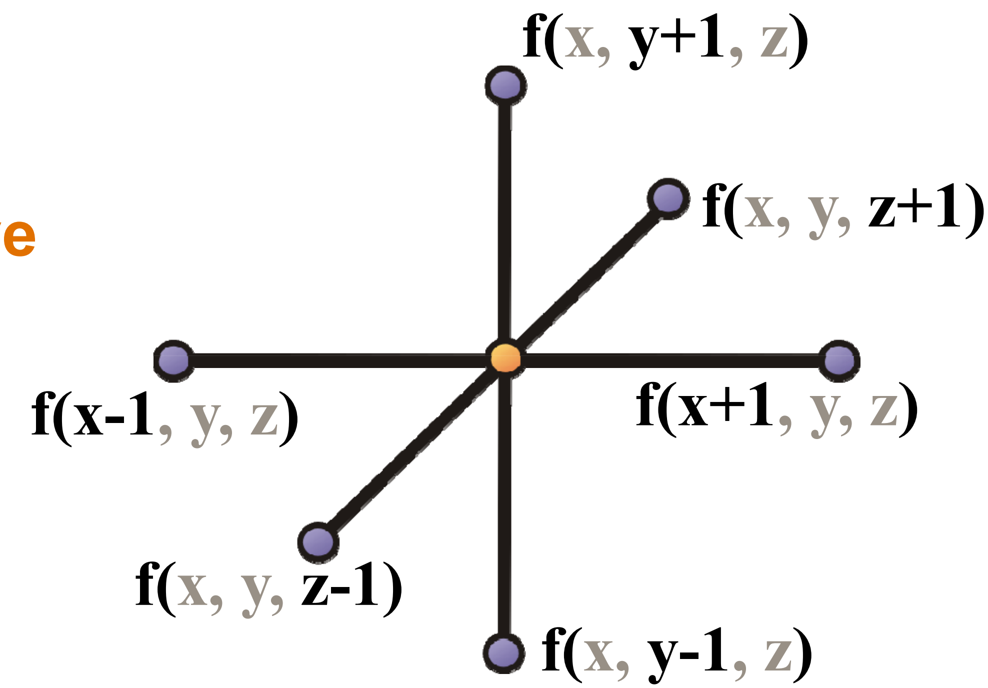
$$
f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + o(h^2)
$$

Central Differences

Need only two neighboring voxels per derivative

Most common method





$$g_x = 0.5(\ f(x+1,\ y,\ z) - f(x-1,\ y,\ z)\ )$$

$$g_y = 0.5(\ f(x,\ y+1,\ z) - f(x,\ y-1,\ z)\ )$$

$$g_z = 0.5(\ f(x,\ y,\ z+1) - f(x,\ y,\ z-1)\ )$$

in a volume

Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama