CS 247 – Scientific Visualization
Lecture 28: Vector / Flow Visualization, Pt. 7 [preview]

Markus Hadwiger, KAUST

Reading Assignment #15++ (1)

Read (required):

- Data Visualization book, Chapter 6.7
- J. van Wijk: *Image-Based Flow Visualization*,
  ACM SIGGRAPH 2002

  `http://www.win.tue.nl/~vanwijk/ibfv/ibfv.pdf`

Read (optional):

- T. Günther, A. Horvath, W. Bresky, J. Daniels, S. A. Buehler:
  Lagrangian Coherent Structures and Vortex Formation in High Spatiotemporal-Resolution Satellite Winds of an Atmospheric Karman Vortex Street, 2021

  `https://www.essoar.org/doi/10.1002/essoar.10506682.2`

- H. Bhatia, G. Norgard, V. Pascucci, P.-T. Bremer:
  The Helmholtz-Hodge Decomposition – A Survey, TVCG 19(8), 2013

  `https://doi.org/10.1109/TVCG.2012.316`

- Work through online tutorials of multi-variable partial derivatives, grad, div, curl, Laplacian:

  `https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives`
  `https://www.youtube.com/watch?v=rB83DpBJQsE` (3Blue1Brown)

- Matrix exponentials:

  `https://www.youtube.com/watch?v=O85OWBJ2ayo` (3Blue1Brown)

Reading Assignment #15++ (2)

Read (optional):

- Tobias Günther, Irene Baeza Rojo:

  Introduction to Vector Field Topology

  `https://cgl.ethz.ch/Downloads/Publications/Papers/2020/Gun20b/Gun20b.pdf`

- Roxana Bujack, Lin Yan, Ingrid Hotz, Christoph Garth, Bei Wang:

  State of the Art in Time-Dependent Flow Topology: Interpreting Physical Meaningfulness Through Mathematical Properties

  `https://onlinelibrary.wiley.com/doi/epdf/10.1111/cgf.14037`

- B. Jobard, G. Erlebacher, M. Y. Hussaini:

  Lagrangian-Eulerian Advection of Noise and Dye Textures for Unsteady Flow Visualization

  `http://dx.doi.org/10.1109/TVCG.2002.1021575`

- Anna Vilanova, S. Zhang, Gordon Kindlmann, David Laidlaw:

  An Introduction to Visualization of Diffusion Tensor Imaging and Its Applications

  `http://vis.cs.brown.edu/docs/pdf/Vilanova-2005-IVD.pdf`

Lagrangian vs. Eulerian

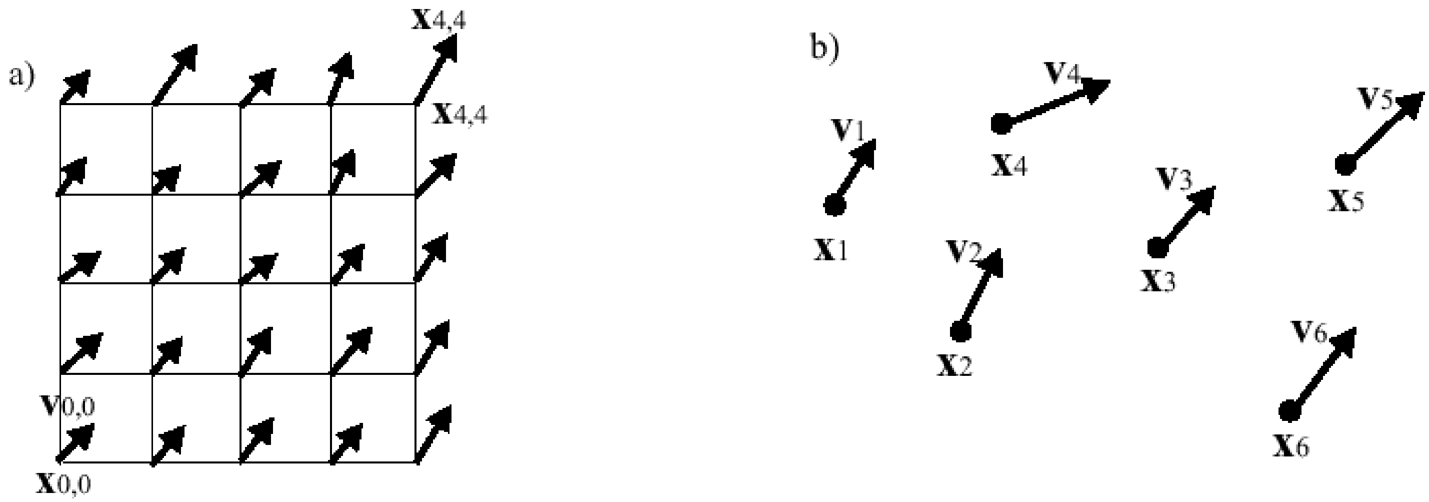
Lagrangian vs. Eulerian

Eulerian

- Flow properties given at fixed spatial positions (grid points)
- Partial time derivative

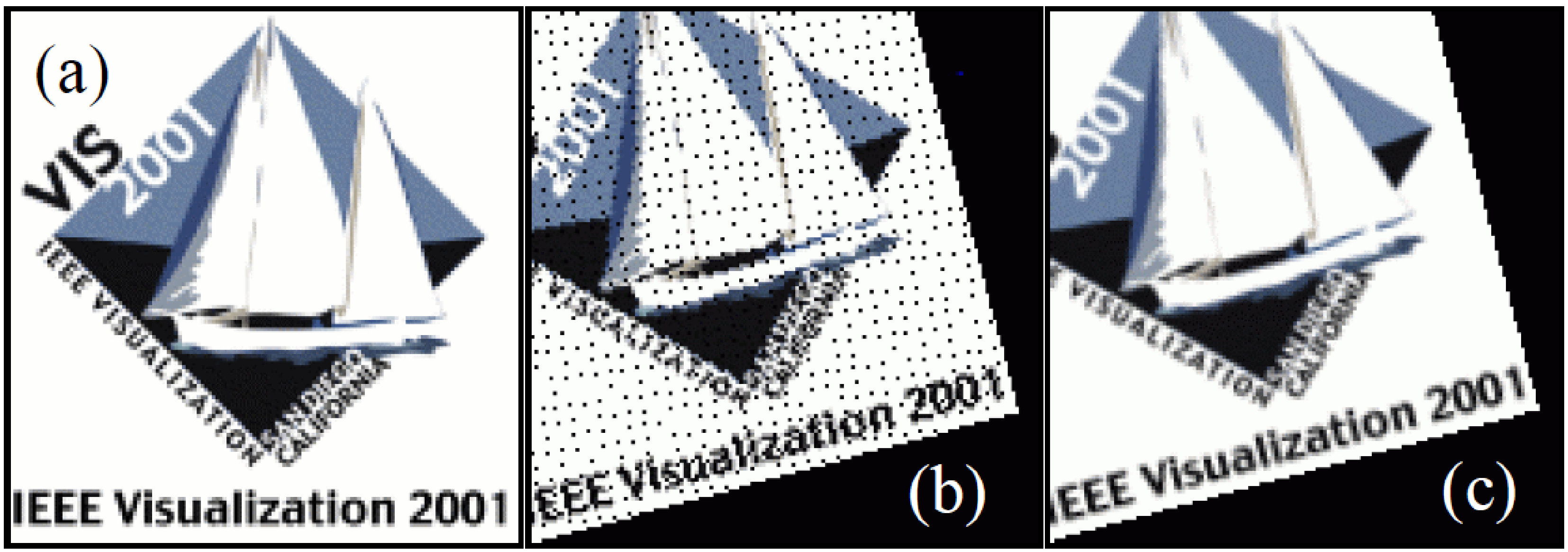
Lagrangian

- Flow properties given for each particle (particles are moving)
- Material time derivative



Lagrangian vs. Eulerian

- Lagrangian: move along with the particle
- Eulerian: consider fixed point in space, look at particles moving through



- Example for pixels: rotate image (a),
  Lagrangian: move pixels forward (b),
  Eulerian: fetch pixels from backward direction (c)

Material Derivative (1)

The material time derivative (convective derivative) gives the rate of change when following a particle in the flow

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

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$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T$$

Material Derivative (1)

The material time derivative (convective derivative) gives the rate of change when following a particle in the flow

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}$$

Material Derivative (2)

Actually, nothing else than application of the multi-variable chain rule:

$$dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

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$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

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$$u := \frac{dx}{dt}, \qquad v := \frac{dy}{dt}, \qquad w := \frac{dz}{dt}$$

Material Derivative (2)

Actually, nothing else than application of the multi-variable chain rule:

We are given $T(x, y, z, t)$ with four independent variables;

But now we want to go along a parameterized path with parameter t, so x, y, z become dependent variables: $x(t),\ y(t),\ z(t)$

Along this path, our goal is now to compute the derivative of the function

$T\left(x(t), y(t), z(t), t\right)$ with t as only independent variable:

$$\frac{d}{dt} T\left(x(t), y(t), z(t), t\right) =$$

$$\frac{\partial}{\partial t} T(x,y,z,t) + \frac{\partial}{\partial x} T(x,y,z,t) \frac{d}{dt} x(t) + \frac{\partial}{\partial y} T(x,y,z,t) \frac{d}{dt} y(t) + \frac{\partial}{\partial z} T(x,y,z,t) \frac{d}{dt} z(t)$$

$$u(t) := \frac{dx(t)}{dt}, \qquad v(t) := \frac{dy(t)}{dt}, \qquad w(t) := \frac{dz(t)}{dt}$$

Advection

Advection equation; velocity field **u(** *x, y, z, t* **)**,
no change following particle, just advection:
set material derivative = 0:

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = 0$$

In the Navier-Stokes equations: "self-advection" of velocity

- Advect scalar components of velocity field individually (actually two equations in 2D, three equations in 3D)

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right) \mathbf{u}$$

this is equivalent to saying that the acceleration is zero!

Fluid Simulation Basics

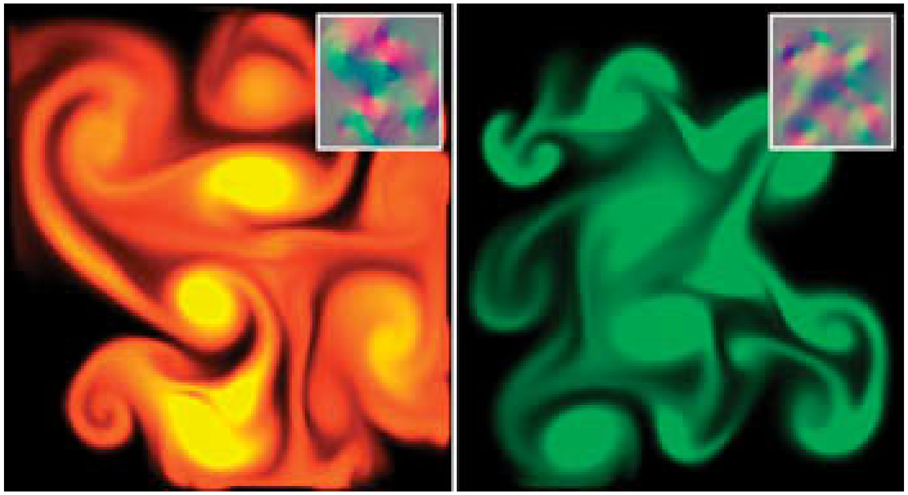
Fluid Simulation and Rendering

Compute advection of fluid

- (Incompressible or compressible) Navier-Stokes solvers
- Lattice Boltzmann Method (LBM)

Discretized domain

- Velocity, pressure
- Dye, smoke density, vorticity, …



Courtesy Mark Harris

Fluid Simulation: Navier Stokes (1)

Incompressible (divergence-free) Navier Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F},$$

$$\nabla \cdot \mathbf{u} = 0,$$

Components:

- Self-advection of velocity (i.e., advection of velocity according to velocity)
- Pressure gradient (force due to pressure differences)
- Diffusion of velocity due to viscosity (for viscous fluids, i.e., not inviscid)
- Application of (arbitrary) external forces, e.g., gravity, user input, etc.

Fluid Simulation: Navier Stokes (1)

Incompressible (divergence-free) Navier Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F},$$

$$\nabla \cdot \mathbf{u} = 0,$$

this is the velocity gradient tensor!

Components:

- Self-advection of velocity (i.e., advection of velocity according to velocity)
- Pressure gradient (force due to pressure differences)
- Diffusion of velocity due to viscosity (for viscous fluids, i.e., not inviscid)
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Fluid Simulation: Navier Stokes (2)

Given a (Cartesian) coordinate system, the momentum equation can be seen as a system of equations (2 equations in 2D, 3 equations in 3D)

For 2D (Cartesian):

$$\frac{\partial u}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right) u - \frac{1}{\rho}(\nabla p)_x + \nu \nabla^2 u + f_x ,$$

$$\frac{\partial v}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right) v - \frac{1}{\rho}(\nabla p)_y + \nu \nabla^2 v + f_y .$$

these are PDEs!

Vector Fields, Vector Calculus, and Dynamical Systems

Some Vector Calculus (1)

Gradient (scalar field → vector field)

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \right)$$

- Direction of steepest ascent; magnitude = rate
- *Conservative* vector field: gradient of some scalar (potential) function

Divergence (vector field → scalar field)

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

- Volume density of outward flux:
  "exit rate: source? sink?"
- *Incompressible/solenoidal/divergence-free vector field*: div u = 0
  can express as curl (next slide) of some vector (potential) function

Laplacian (scalar field → scalar field)

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}$$

- Divergence of gradient
- Measure for difference between point and its neighborhood

Some Vector Calculus (2)

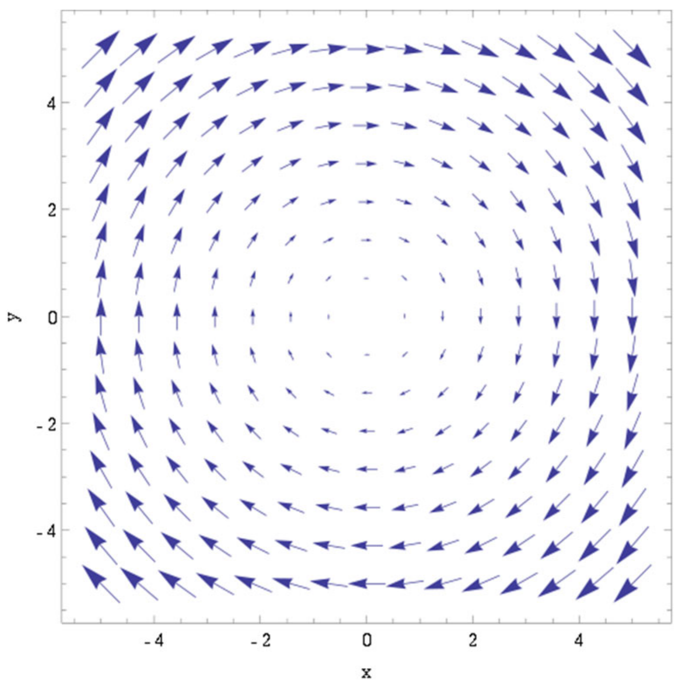
Curl (vector field → vector field)

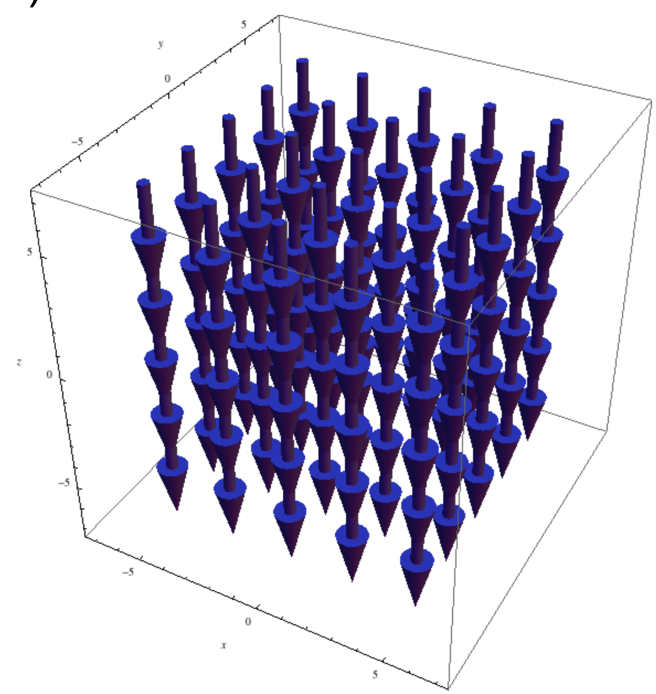
$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

- Circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative (path-independent) field is irrotational (and vice versa if domain is simply connected)

Example:
curl = const
everywhere





Some Vector Calculus (3)

Curl (vector field → vector field)

$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

- Circulation density at a point (*vorticity*)
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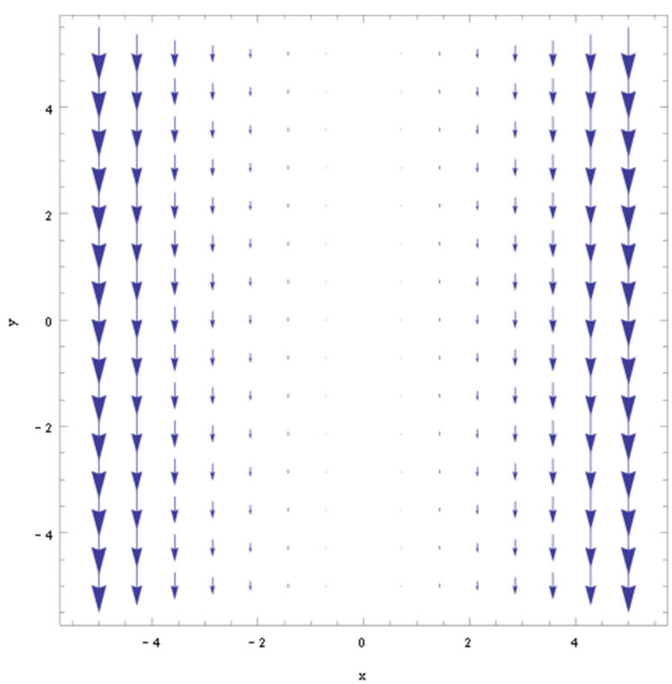
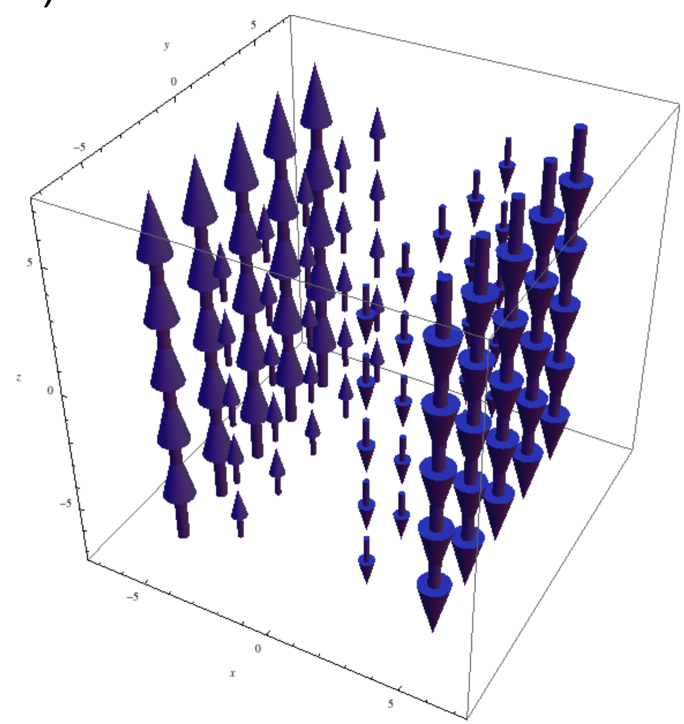
Example: curl not always “obviously rotational”





Some Vector Calculus (4)

Curl (vector field → vector field)

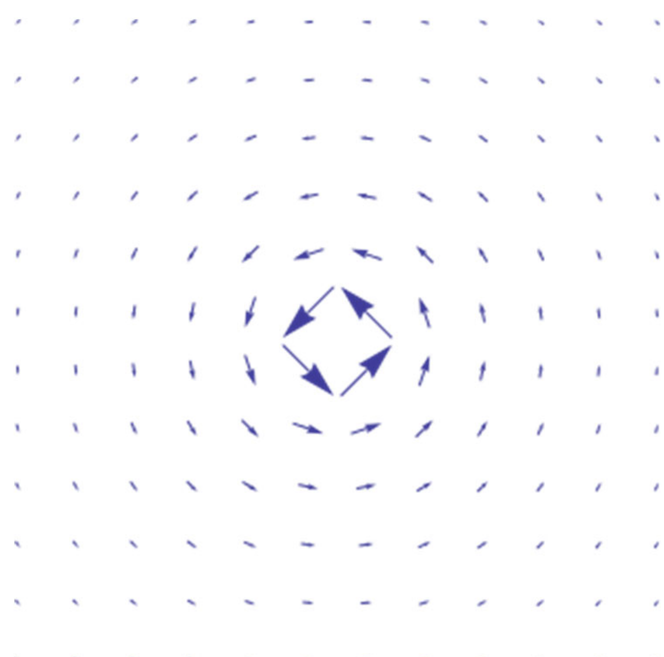
$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

- Circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative (path-independent) field is irrotational (and vice versa if domain is simply connected)

Example:
non-obvious
curl-free field

[this domain is **not** simply connected! it is the “punctured plane”, i.e., the point (0,0) is not in the domain]



$$\mathbf{v}(x, y, z) = \frac{(-y, x, 0)}{x^2 + y^2}$$

not defined at (x,y) = (0,0)

$$v_x = u_y \qquad \nabla \times \mathbf{v} = 0$$

velocity gradient $\nabla \mathbf{v}$ is symmetric (see later)

Some Vector Calculus (5)

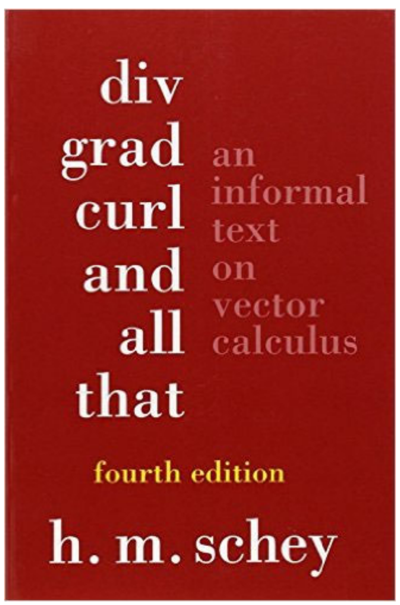
Curl (vector field → vector field)

- Circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative (path-independent) field is irrotational (and vice versa if domain is simply connected)

$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

Book:



Interactive tutorial on curl:
`http://mathinsight.org/curl_idea`

Fundamental theorem of vector calculus:
Helmholtz decomposition: Any vector field can be expressed as the sum of a solenoidal (*divergence-free*) vector field and an irrotational (*curl-free*) vector field (Helmholtz-Hodge: plus *harmonic* vector field)

Vector Fields and Dynamical Systems (1)

Velocity gradient tensor, (vector field → tensor field)

- Gradient of vector field: how does the vector field change?
- In Cartesian coordinates: *spatial partial derivatives (Jacobian matrix)*

$$\nabla \mathbf{v}\,(x,y,z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

these are partial derivatives!

- Can be decomposed into *symmetric* part + *anti-symmetric* part

$$\nabla \mathbf{v} = \mathbf{D} + \mathbf{S}$$ *velocity gradient tensor*

sym.: $\mathbf{D} = \frac{1}{2}\,(\,\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}\,)$ deform.: *rate-of-strain tensor*

skew-sym.: $\mathbf{S} = \frac{1}{2}\,(\,\nabla \mathbf{v} - (\nabla \mathbf{v})^{\mathrm{T}}\,)$ rotation: *vorticity/spin tensor*

Vector Fields and Dynamical Systems (2)

Vorticity/spin/angular velocity tensor

- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor ½)

$$\mathbf{S} = \tfrac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathrm{T}} \right)$$

these are partial derivatives!

$$\mathbf{S} = \tfrac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

$\mathbf{S}$ acts on vector like cross product with ω : $\mathbf{S}\, \bullet \; = \tfrac{1}{2}\,\omega \times$

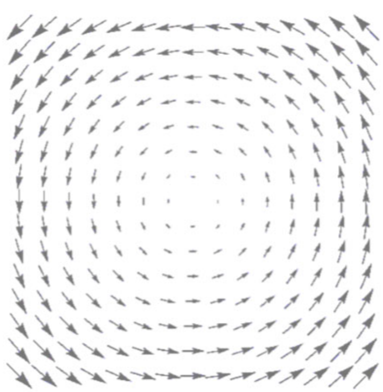
$$\mathbf{v}^{(r)} = \mathbf{S} \cdot d\mathbf{r} = \frac{1}{2}\left[\nabla\mathbf{v} - (\nabla\mathbf{v})^T\right] \cdot d\mathbf{r} = \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r}$$

Angular Velocity of Rigid Body Rotation

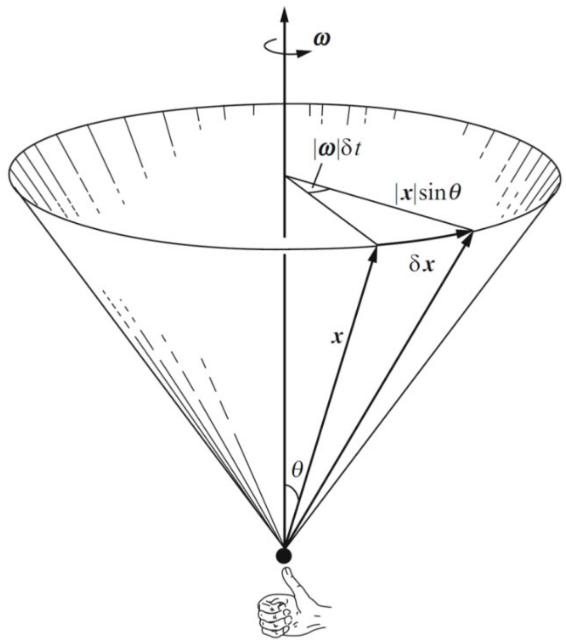
Rate of rotation

- Scalar ω: angular displacement per unit time (rad s^{-1})
  - Angle Θ at time t is $\Theta(t) = \omega t$; $\omega = 2\pi f$ where f is the frequency ($f = 1/T$; s^{-1})
- Vector $\boldsymbol{\omega}$: axis of rotation; magnitude is angular speed (if $\boldsymbol{\omega}$ is curl: speed x2)
  - Beware of different conventions that differ by a factor of ½ !

Cross product of ½$\boldsymbol{\omega}$ with vector to center of rotation ($\mathbf{r}$) gives linear velocity vector $\mathbf{v}$ (tangent)



$$\mathbf{v}^{(r)} = \frac{1}{2}\,\boldsymbol{\omega} \times d\mathbf{r}$$



Velocity Gradient Tensor and Components (1)

Velocity gradient tensor

(here: in Cartesian coordinates)

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial}{\partial x} v^x & \frac{\partial}{\partial y} v^x & \frac{\partial}{\partial z} v^x \\ \frac{\partial}{\partial x} v^y & \frac{\partial}{\partial y} v^y & \frac{\partial}{\partial z} v^y \\ \frac{\partial}{\partial x} v^z & \frac{\partial}{\partial y} v^z & \frac{\partial}{\partial z} v^z \end{bmatrix}$$

these are the same partial derivatives as before!

$$\nabla \mathbf{v} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right) + \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T\right)$$

Velocity Gradient Tensor and Components (2)

Rate-of-strain (rate-of-deformation) tensor

(symmetric part; here: in Cartesian coordinates)

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} 2\frac{\partial}{\partial x}v^x & \frac{\partial}{\partial y}v^x + \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x + \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y + \frac{\partial}{\partial y}v^x & 2\frac{\partial}{\partial y}v^y & \frac{\partial}{\partial z}v^y + \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z + \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z + \frac{\partial}{\partial z}v^y & 2\frac{\partial}{\partial z}v^z \end{bmatrix}$$

$$tr(\mathbf{D}) = \nabla \cdot \mathbf{v}$$

Velocity Gradient Tensor and Components (3)

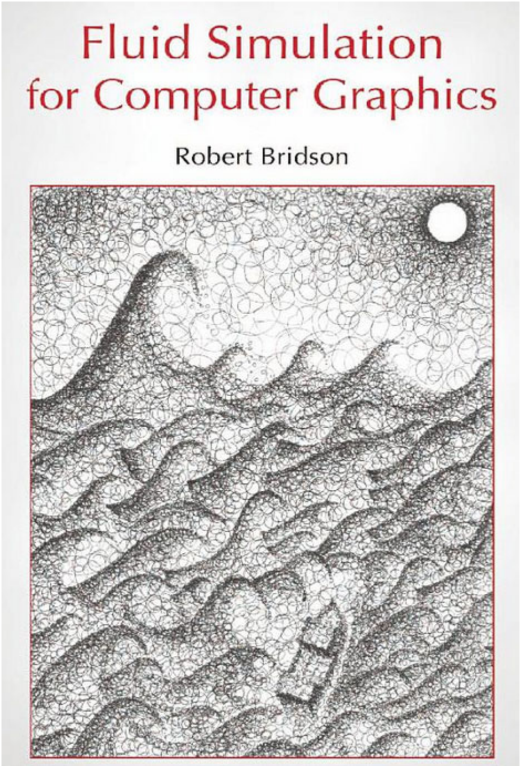
Vorticity tensor (spin tensor)

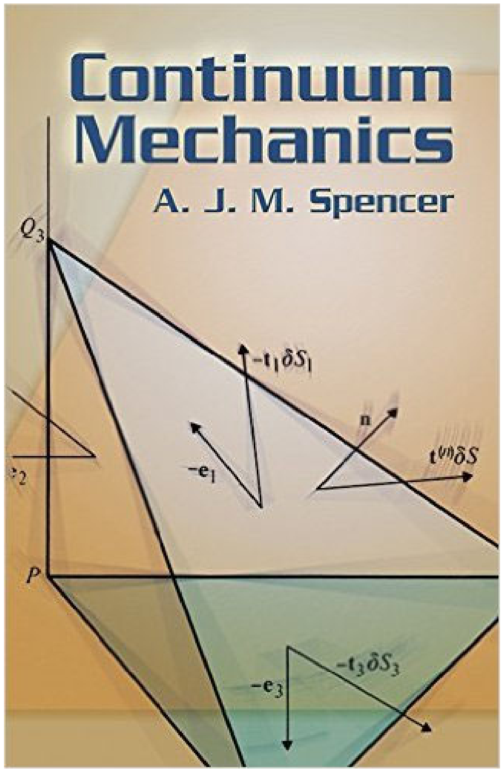
(skew-symmetric part; here: in Cartesian coordinates)

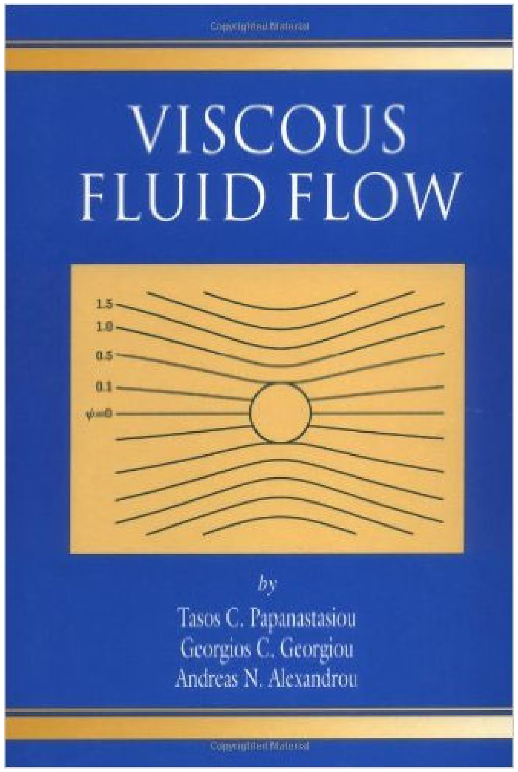
$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial}{\partial y}v^x - \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x - \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y - \frac{\partial}{\partial y}v^x & 0 & \frac{\partial}{\partial z}v^y - \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z - \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z - \frac{\partial}{\partial z}v^y & 0 \end{bmatrix}$$

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \qquad \boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$$

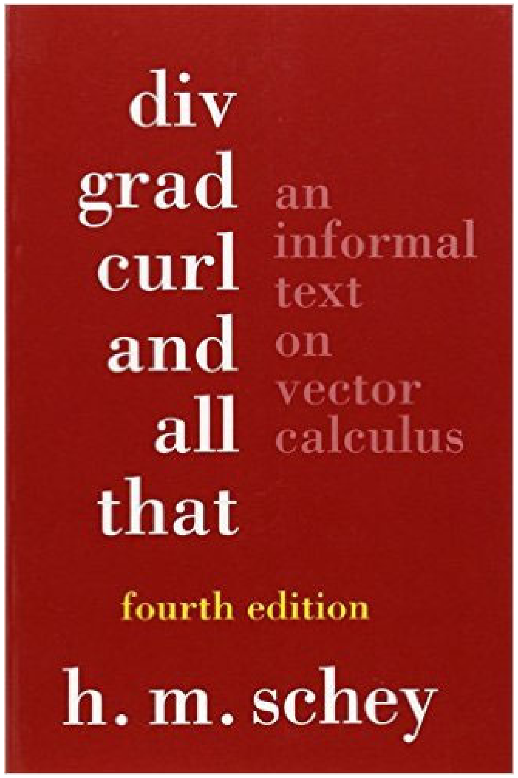
Recommended Books (1)

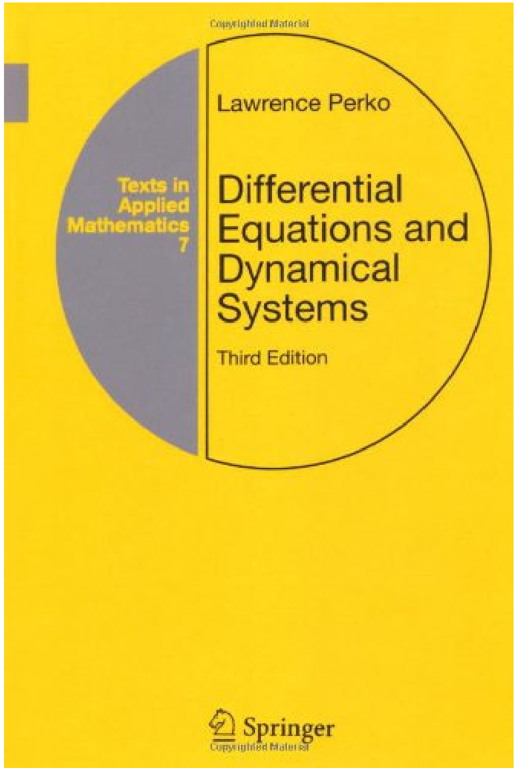


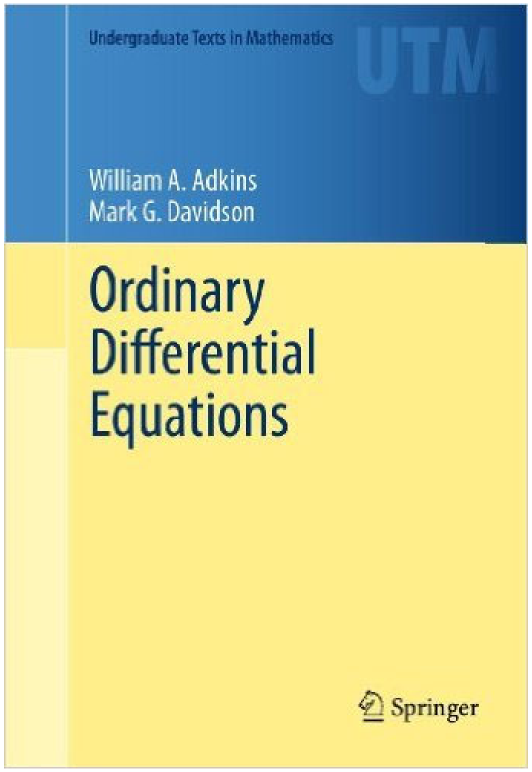




Recommended Books (2)



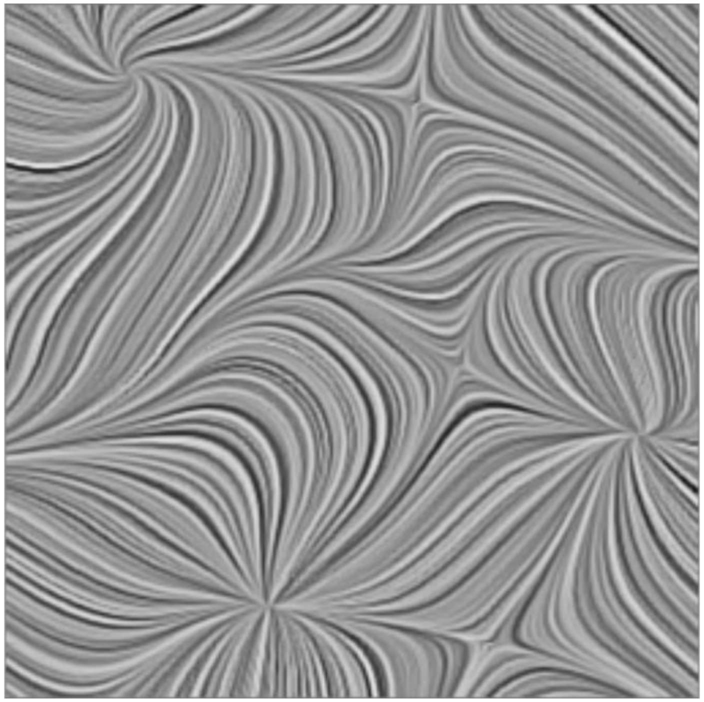




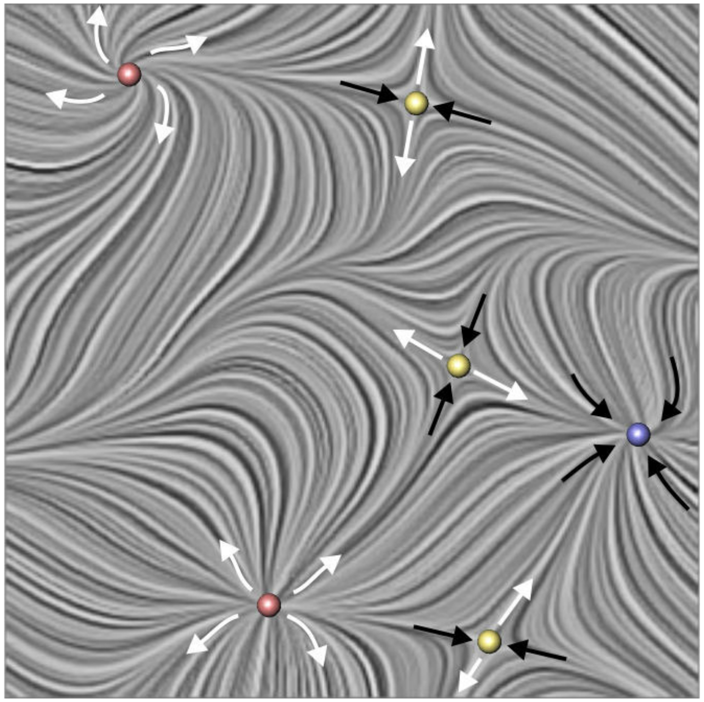
Critical Point Analysis

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point



stream lines (LIC)



critical points (**v** = 0)

(Non-Linear) Dynamical Systems

Start with system of linear ODEs (with constant coefficients)

- Non-linear systems can be linearized around critical points
- Use linearization for characterization

$\dot{\mathbf{x}} = A\mathbf{x}$ $\quad$ A is an $n \times n$ matrix $\quad \longrightarrow \quad \boxed{\begin{aligned} \mathbf{v} &= A\mathbf{x}, \\ \nabla\mathbf{v} &= A. \end{aligned}}$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

solution: $\mathbf{x}(t) = e^{At}\mathbf{x}_0$

characterize behavior
through eigenvalues of A

A Few Facts about Eigenvalues and –vectors

The matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ has eigenvalues $\lambda_1 = c + s\mathbf{i} \quad \lambda_2 = c - s\mathbf{i}$

with eigenvectors $u_1 = \begin{bmatrix} 1 \\ -\mathbf{i} \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ +\mathbf{i} \end{bmatrix}$

If c = 0, this is a skew-symmetric matrix

Skew-symmetric matrices: “infinitesimal rotations” (infinitesimal generators of rot.)

For $c = \cos\theta$ and $s = \sin\theta$: 2x2 rotation matrix with

$$\begin{aligned} \lambda_1 = e^{\mathbf{i}\theta} &= \cos\theta + \mathbf{i}\sin\theta \\ \lambda_2 = e^{-\mathbf{i}\theta} &= \cos\theta - \mathbf{i}\sin\theta \end{aligned}$$

Eigenvalues

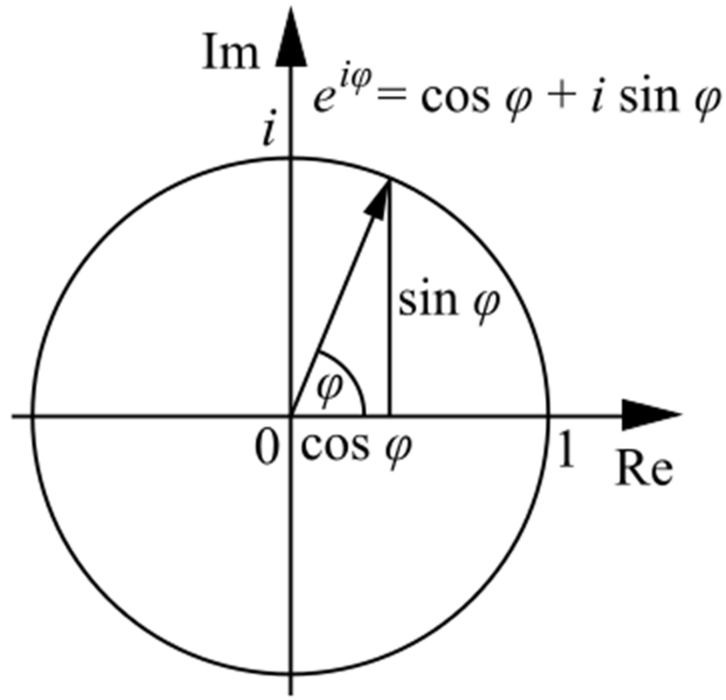
- Symmetric matrix: all eigenvalues are *real*
- Skew-symmetric matrix: all eigenvalues are *pure imaginary*

Euler's Formula

Can be derived from the infinite power series for exp(), cos(), sin()

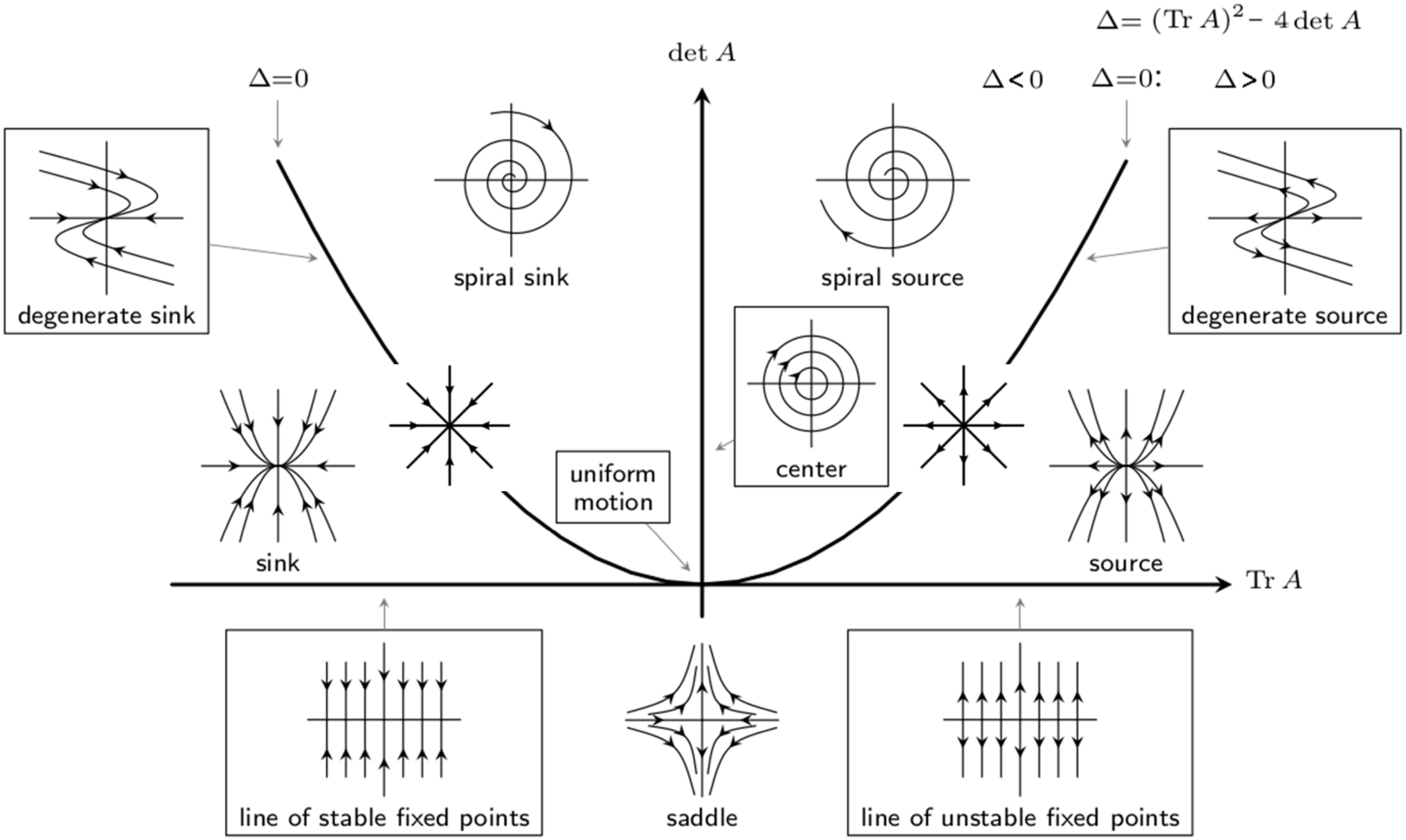
$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} + 1 = 0$$



Critical Points (Steady Flow!)

Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \mathrm{Tr}\, A)$-plane



Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

Easy if *X* is diagonalizable

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Exponentials of anti-symmetric matrices are rotation matrices

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

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Complex eigenvalues lead to rotation

$$X = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix} \qquad \exp(tX) = e^{at} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$$

$$\lambda_{1,2} = a \pm \mathbf{i}\omega$$

Classification of Critical Points

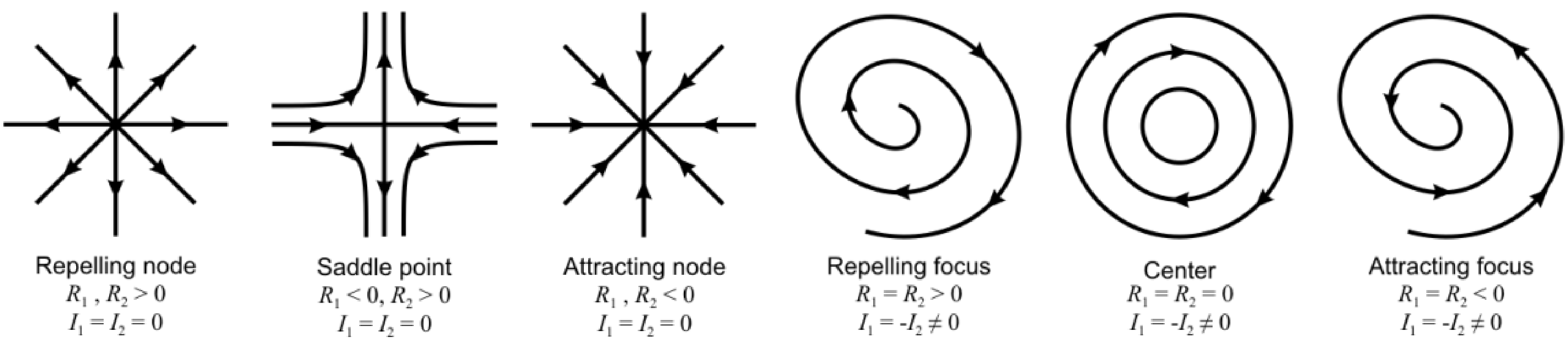
(Isolated) critical point (equilibrium point)

- Velocity vanishes (all components zero)

$$\mathbf{v}(\mathbf{x}_\mathrm{c}) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_\mathrm{c} \pm \boldsymbol{\epsilon}) \neq \mathbf{0} \qquad\qquad \det(\, \nabla\mathbf{v}(\mathbf{x}_\mathrm{c})\,) \neq 0$$

Characterize using velocity gradient $\nabla\mathbf{v}$ at critical point $\mathbf{x}_\mathrm{c}$

- Look at eigenvalues (and eigenvectors) of $\nabla\mathbf{v}$

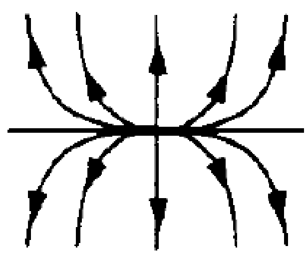


the first three phase portraits are special cases, see later slides!

A Few Details (1)

Repelling/attracting nodes

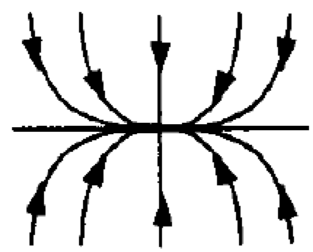
- Do not necessarily imply that streamlines are straight lines
  (do not confuse with the linear system of ODEs!)
- They are only straight lines when both eigenvalues are real and have the same sign, *and are also equal* (as in the phase portraits before)
- If they are not equal:



Repelling Node

R1, R2 > 0

I1, I2 = 0



Attracting Node

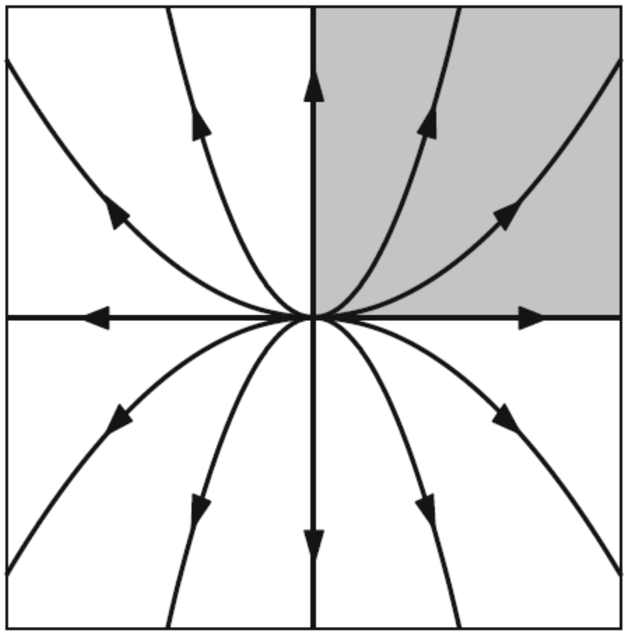
R1, R2 < 0

I1, I2 = 0

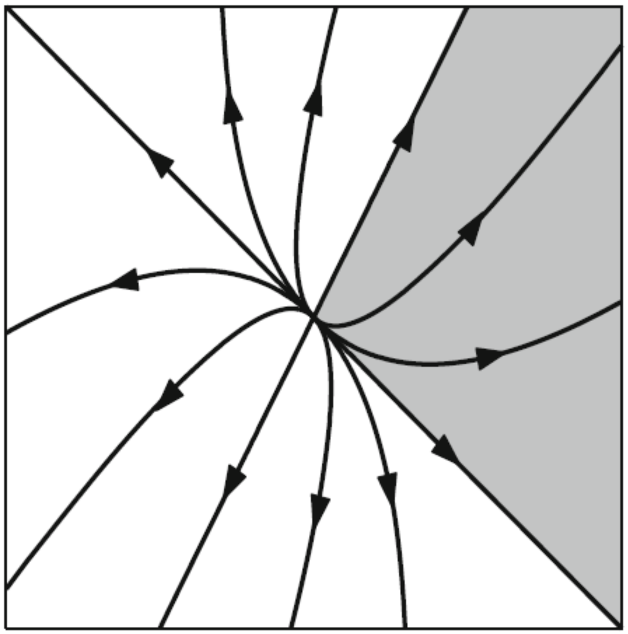
A Few Details (2)

What about skew axes?

- Both of the systems below have eigenvalues 3 and 6
- Jordan normal form (Jordan canonical form) gives details



$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$



$$\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$$

Jordan Normal Form (2x2 Matrix)

For every real 2x2 matrix A there is an invertible P such that

$P^{-1}AP$ is one of the following Jordan matrices (all entries are real):

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$ (defective matrix)

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Each of these has its corresponding rule for constructing P

- Example on prev. slide (the two eigenvectors are not orthogonal):

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

See also *algebraic* and *geometric multiplicity* of eigenvalues

Jordan Normal Form (2x2 Matrix)

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same eigenvalues,
trace, determinant!

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Each of these has its corresponding rule for constructing P

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See also *algebraic* and *geometric multiplicity* of eigenvalues

Another Example

$P^{-1}AP$ has form J_1

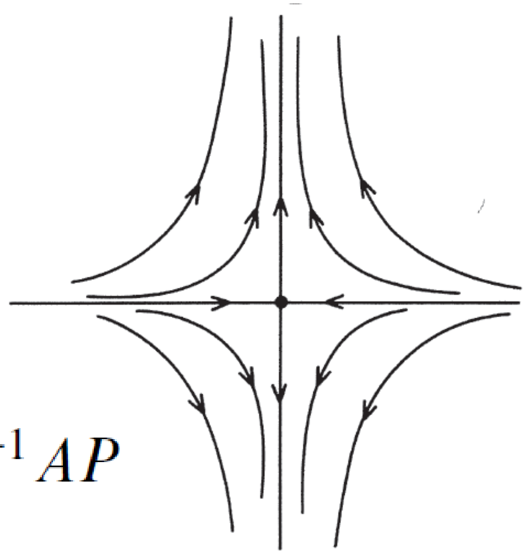
$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

Eigenvalues:

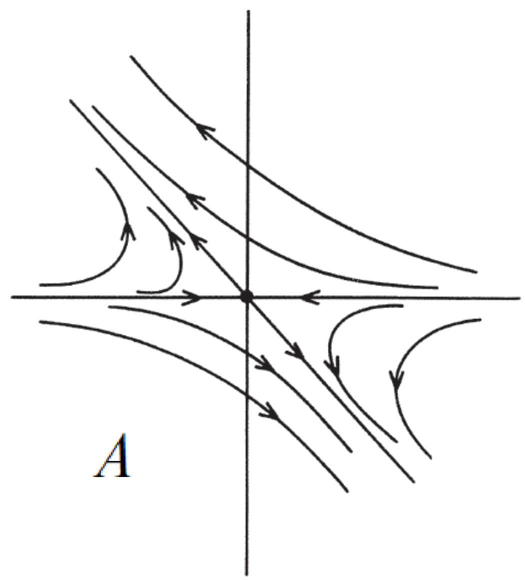
$\lambda_1 = -1$

$\lambda_2 = 2$

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



$P^{-1}AP$

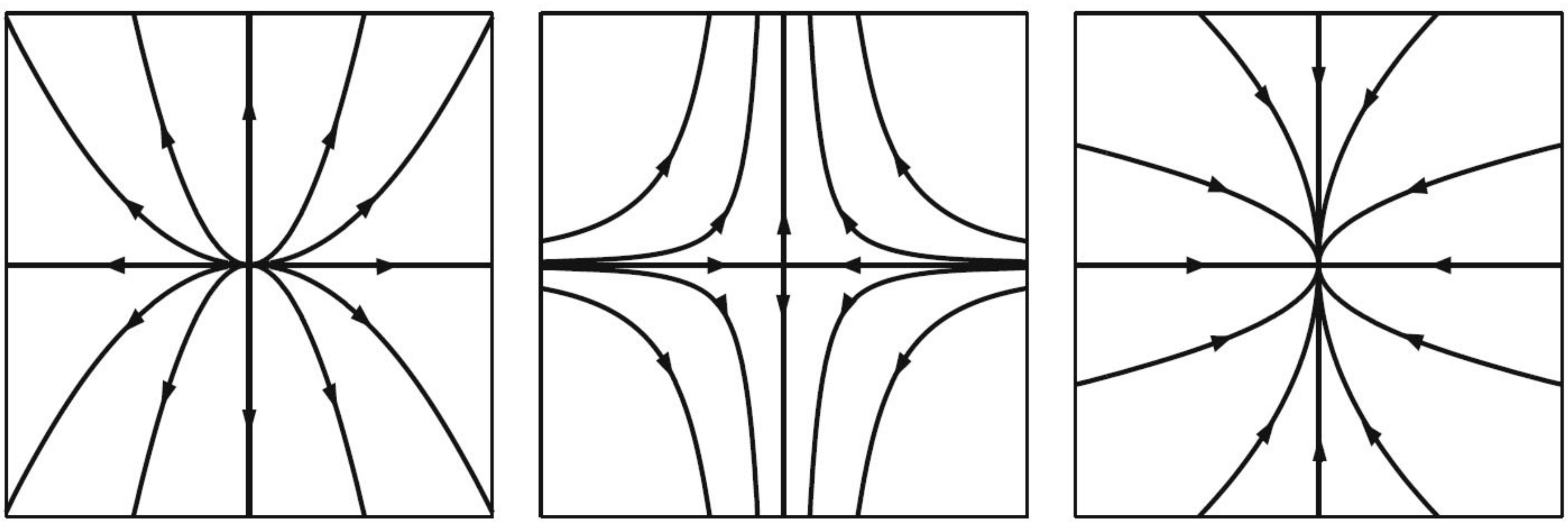


A

Jordan Form Characterization (1)

Phase portraits corresponding to Jordan matrix

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



$0 < \lambda_1 < \lambda_2$
unstable node

$\lambda_1 < 0 < \lambda_2$
saddle

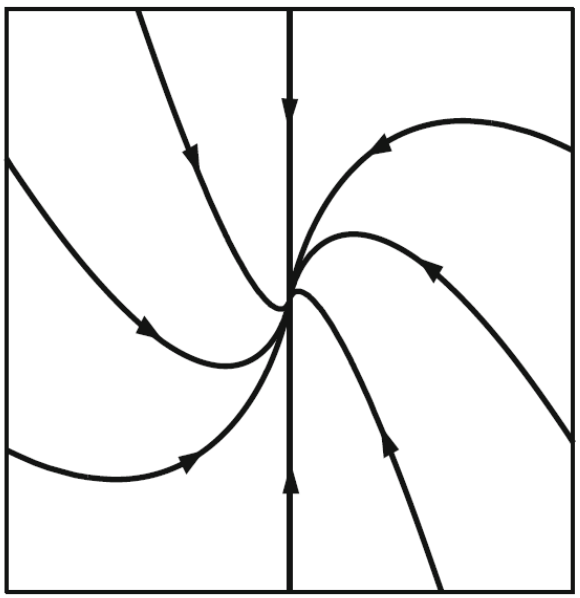
$\lambda_1 < \lambda_2 < 0$
stable node

Jordan Form Characterization (2)

Phase portraits corresponding to Jordan matrix

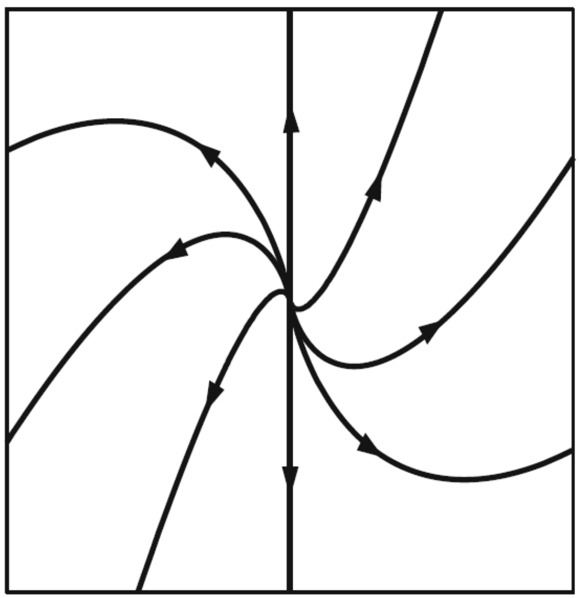
(matrix is defective: eigenspaces collapse, geometric multiplicity less than algebraic multiplicity)

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$



$\lambda < 0$

stable improper node



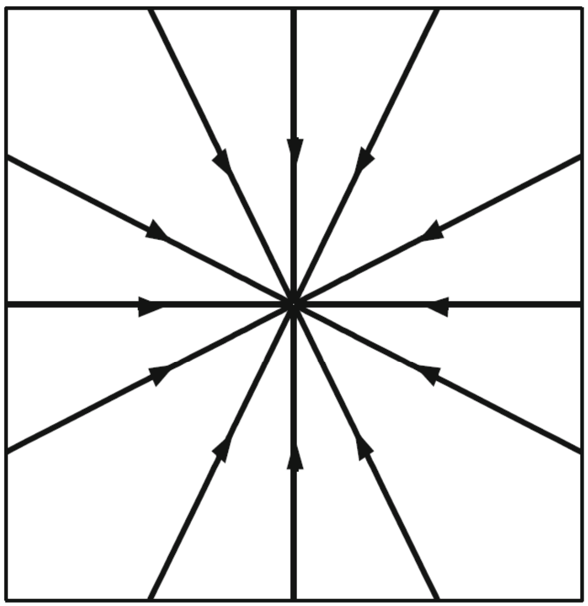
$\lambda > 0$

unstable improper node

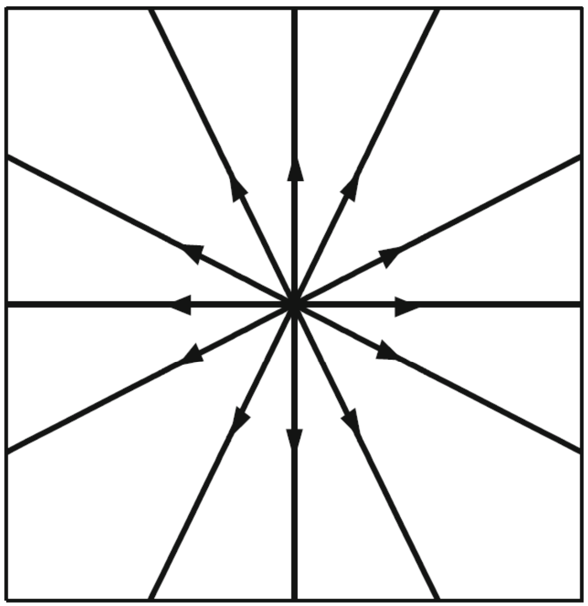
Jordan Form Characterization (3)

Phase portraits corresponding to Jordan matrix

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$



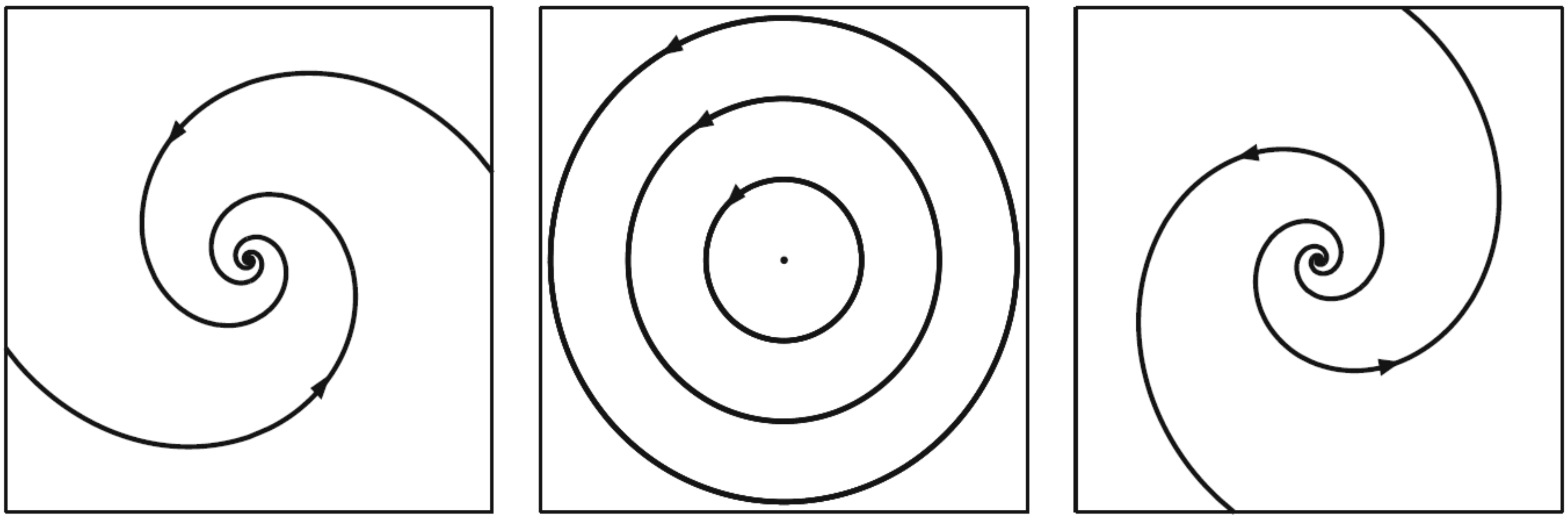
$\lambda < 0$
stable star node



$\lambda > 0$
unstable star node

Jordan Form Characterization (4)

Phase portraits corresponding to Jordan matrix $$J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



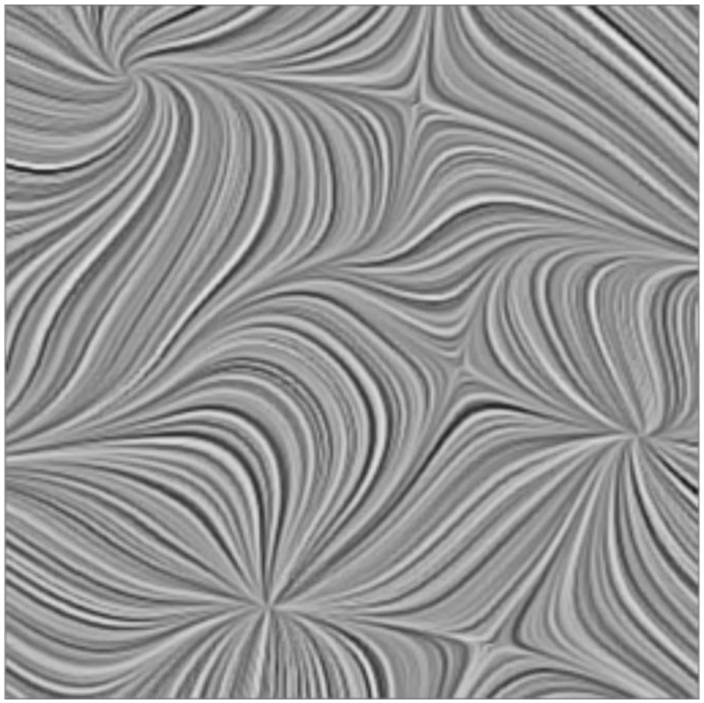
$a < 0$ — stable spiral node

$a = 0$ — center

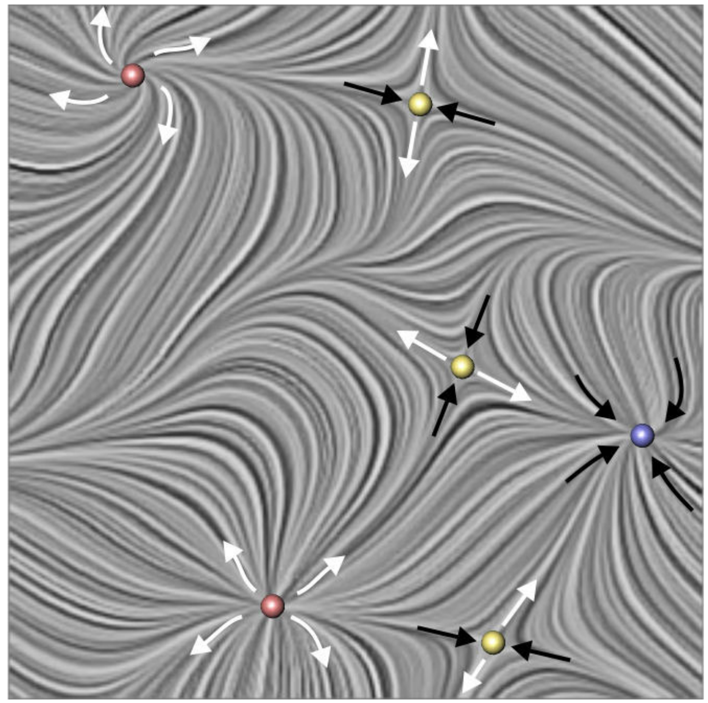
$a > 0$ — unstable spiral node

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point



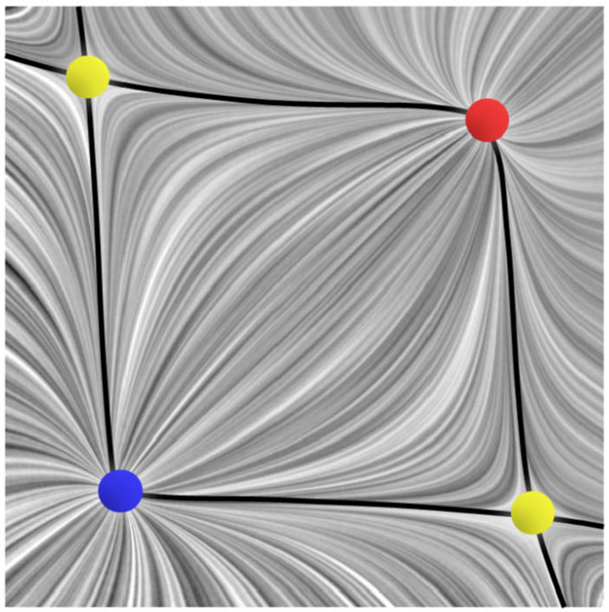
stream lines (LIC)

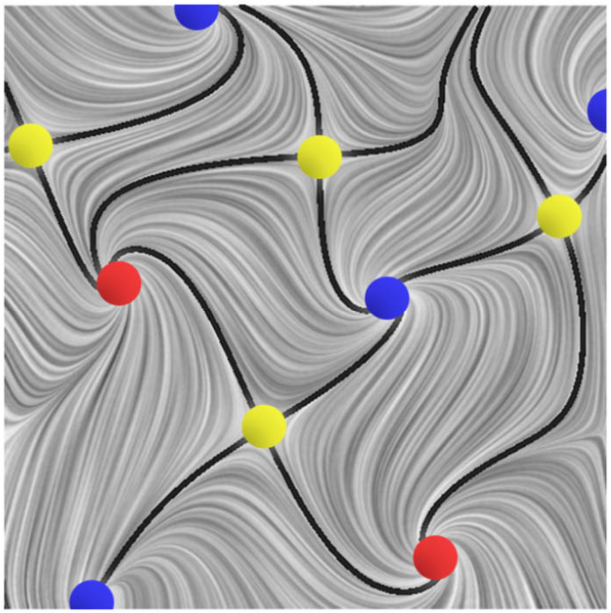


critical points (**v** = 0)

Vector Field Topology: Topological Skeleton

Connect critical points by *separatrices*

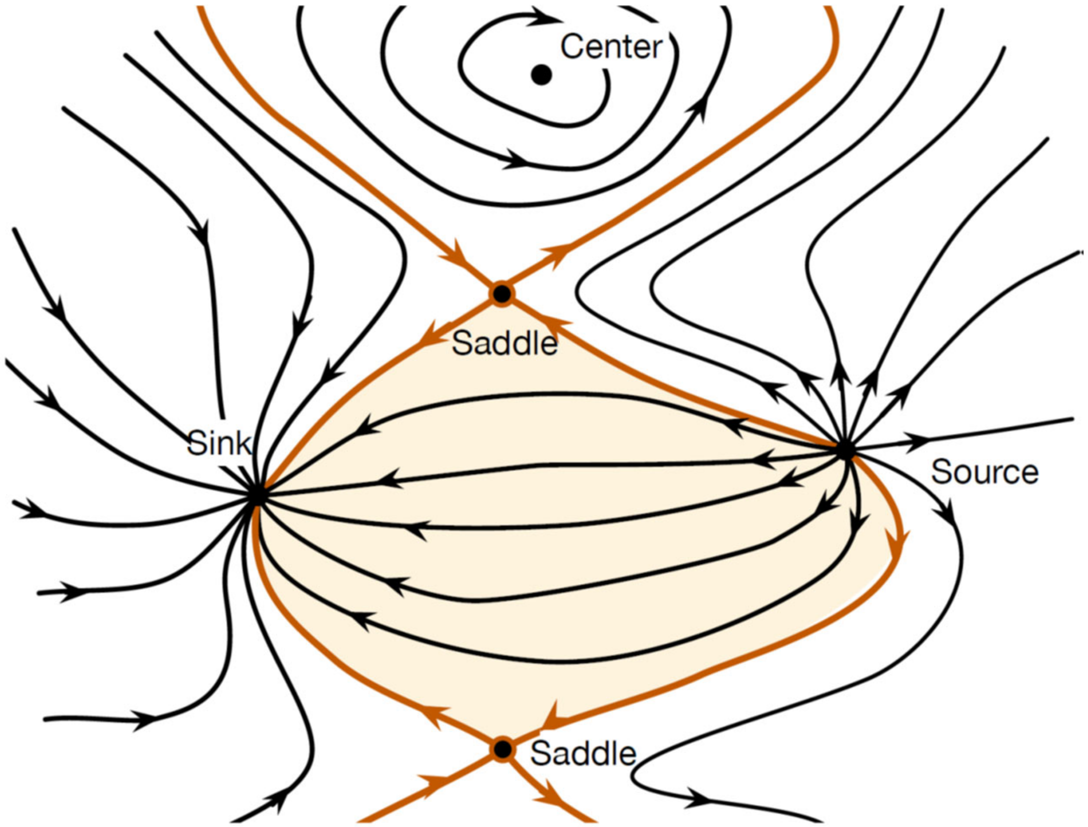




Sources (red), sinks (blue), saddles (yellow)

Vector Field Topology: Topological Skeleton

Connect critical points by *separatrices*



Index of Critical Points / Vector Fields

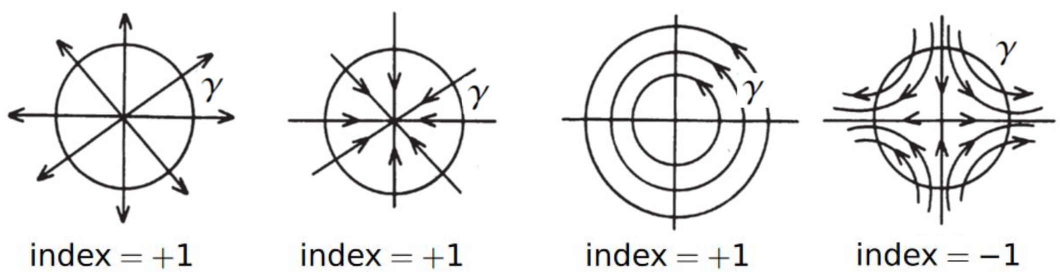
Poincaré index (in scalar field topology we had the *Morse index*)

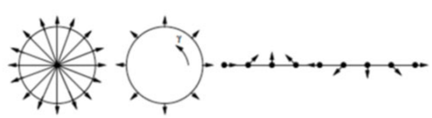
- Can compute index (winding number) for each critical point
- Index of a region is the sum of the critical point indexes inside
- Sum of all indexes over a manifold is its Euler characteristic

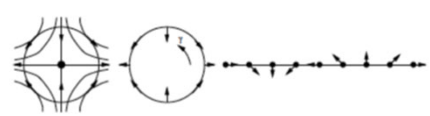
Do a loop (Jordan curve) around each critical point: the index is its (Brouwer) degree: integer how often the vector field along the loop turns around (determined by angle 1-form integrated over oriented 1-manifold)

$$\mathrm{index}_\gamma = \frac{1}{2\pi} \oint_\gamma \mathrm{d}\alpha$$

$$\alpha = \arctan \frac{v}{u}$$





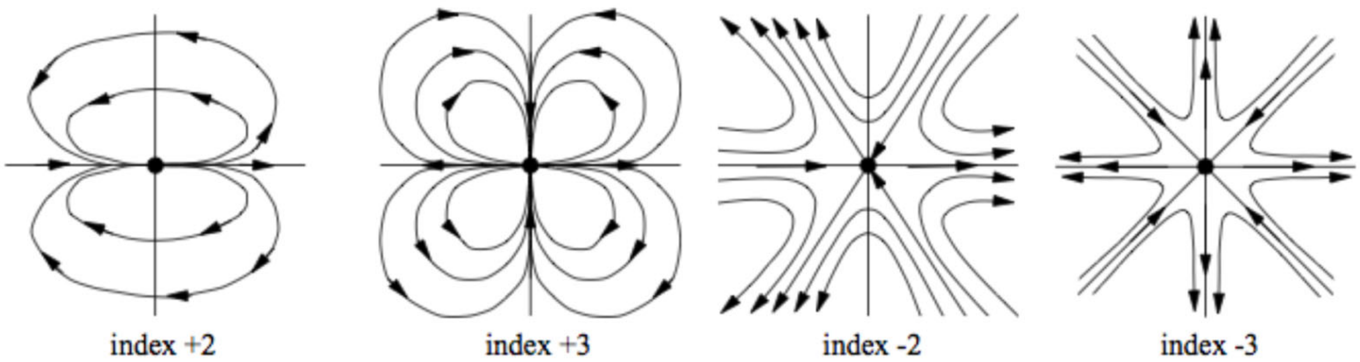


Higher-Order Critical Points

Higher than first-order

- Sectors can by elliptic, parabolic, hyperbolic
- For index sum over number of elliptic and hyperbolic sectors

$$\text{index}_{cp} = 1 + \frac{n_e - n_h}{2}$$



(dipole) (see monkey saddle)

Example: Differential Topology

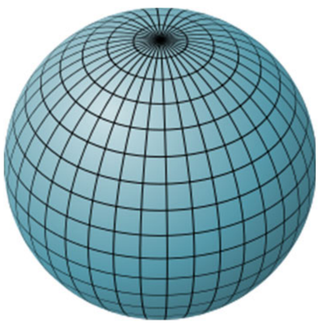
Topological information from vector fields on manifold

- Independent of actual vector field! Poincaré-Hopf theorem
- Useful constraints: vector field editing, simplification, sphere always has critical point, …

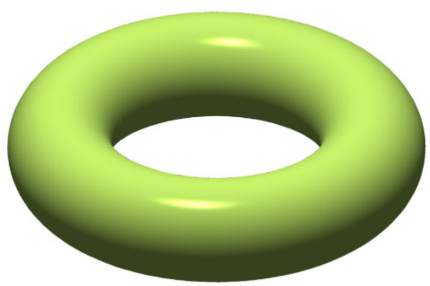
Topological invariant: Euler characteristic $\chi(M)$ of manifold M

(for 2-manifold mesh: $\chi(M) = V - E + F$)

$\chi = 2 - 2g$ (orientable)



genus $g = 0$
Euler characteristic $\chi = 2$

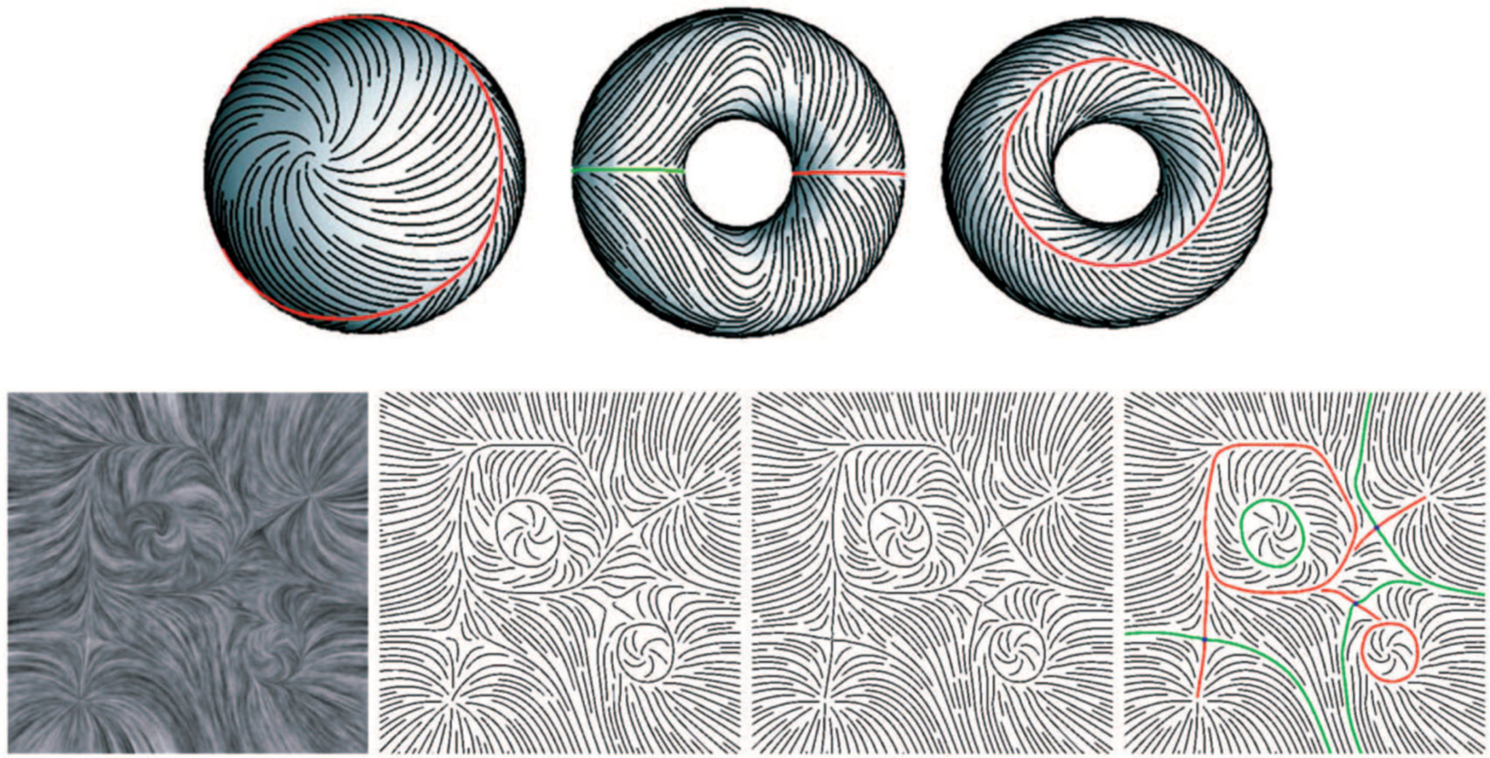


genus $g = 1$
Euler characteristic $\chi = 0$

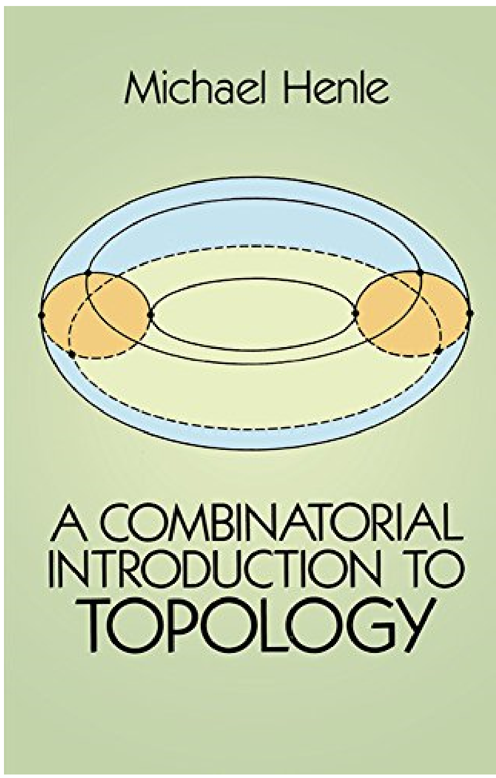
genus $g = 2$
Euler characteristic $\chi = -2$

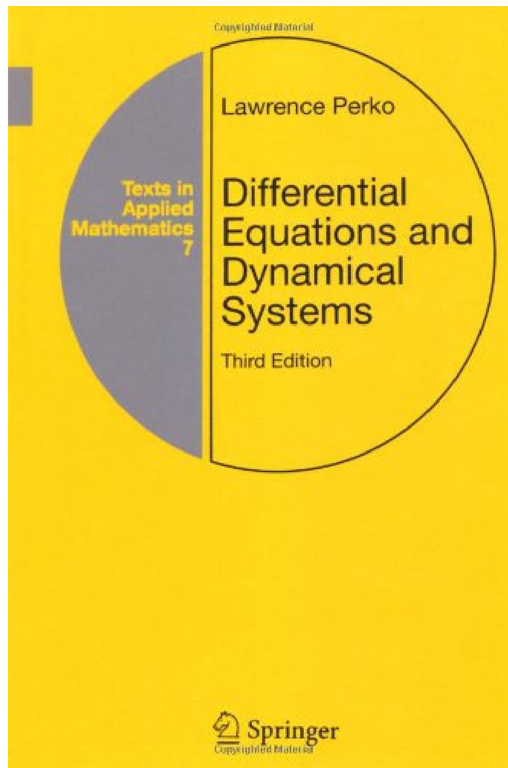
Example: Vector Field Editing

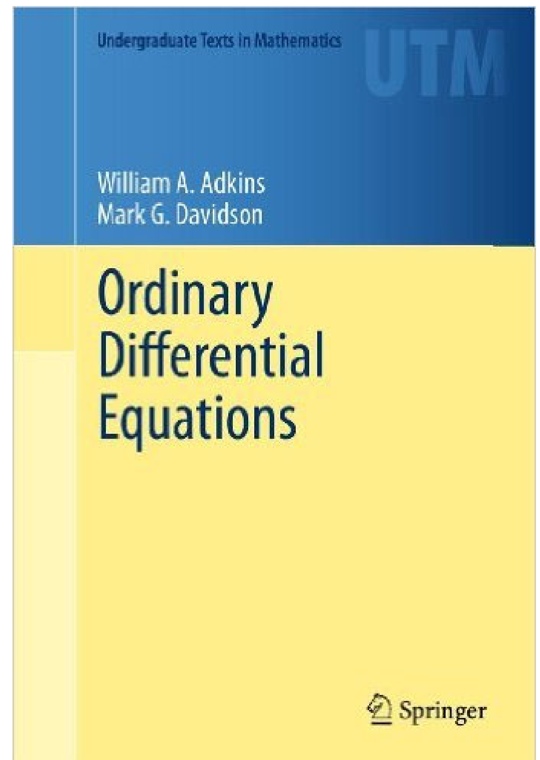
Guoning Chen et al., Vector Field Editing and Periodic Orbit Extraction Using Morse Decomposition, IEEE TVCG, 2007



Recommended Books







Thank you.

Thanks for material

- Helwig Hauser
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- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama