

CS 247 – Scientific Visualization

Lecture 12: Scalar Fields, Pt.8 [preview]

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Reading Assignment #6 (until Mar 7)



Read (required):

- Real-Time Volume Graphics, Chapter 2
(*GPU Programming*)
- Real-Time Volume Graphics, Chapters 5.5 and 5.6 (you already had to read - 5.4)
(*Local Volume Illumination*)
- Refresh your memory on eigenvectors and eigenvalues:
https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

Look at (optional):

- Riemannian Geometry for Scientific Visualization (notes and videos [part 1])
<https://vccvisualization.org/RiemannianGeometryTutorial/>

Gradient and Directional Derivative



Gradient $\nabla f(x, y, z)$ of scalar function $f(x, y, z)$: (in Cartesian coordinates)

$$\nabla f(x, y, z) = \left(\frac{\partial f(x, y, z)}{\partial x}, \frac{\partial f(x, y, z)}{\partial y}, \frac{\partial f(x, y, z)}{\partial z} \right)^T$$

Directional derivative in direction \mathbf{u} :

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

And therefore also:

$$D_{\mathbf{u}}f(x, y, z) = \|\nabla f\| \|\mathbf{u}\| \cos \theta$$

Gradient and Directional Derivative



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(Cartesian vector components; basis vectors not shown)

But: always need **basis vectors**! With Cartesian basis:

$$\nabla f(x, y, z) = \frac{\partial f(x, y, z)}{\partial x} \mathbf{i} + \frac{\partial f(x, y, z)}{\partial y} \mathbf{j} + \frac{\partial f(x, y, z)}{\partial z} \mathbf{k}$$

What about the Basis?



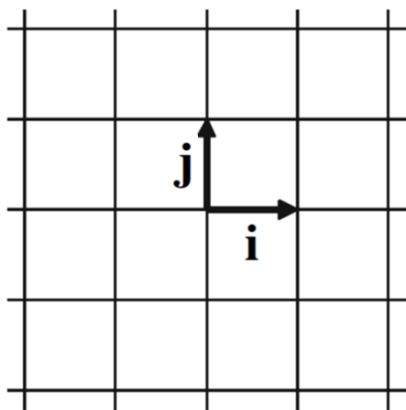
On the previous slide, this actually meant

$$\nabla f(x, y, z) = \frac{\partial f(x, y, z)}{\partial x} \mathbf{i}(x, y, z) + \frac{\partial f(x, y, z)}{\partial y} \mathbf{j}(x, y, z) + \frac{\partial f(x, y, z)}{\partial z} \mathbf{k}(x, y, z)$$

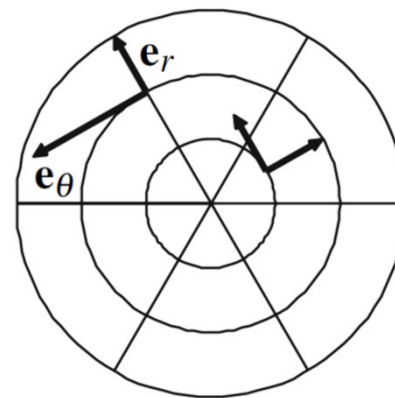
It's just that the Cartesian basis vectors are the same everywhere...

But this is not true for many other coordinate systems:

Cartesian
coordinates



polar
coordinates



What about the Basis?



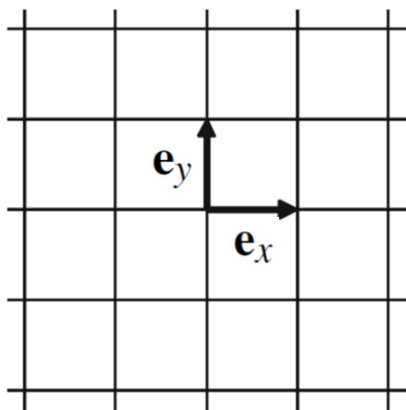
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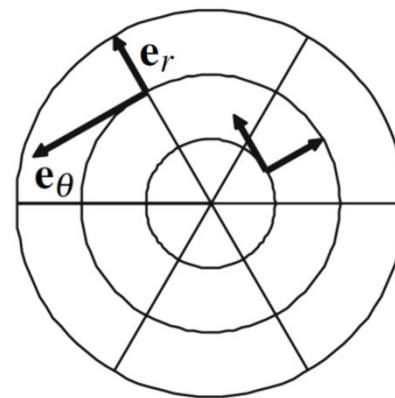
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The Gradient as a Differential Form



The gradient as a *differential* (differential 1-form) is the “primary” concept (also “total differential” or “total derivative”)

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

A differential 1-form is a scalar-valued linear function that takes a (direction) vector as input, and gives a scalar as output

Each of the 1-forms df, dx, dy, dz takes direction vector as input, gives scalar output

In the expression of the gradient df above, all 1-forms on the right-hand side get the same vector as input

df is simply a linear combination of the coordinate differentials dx, dy, dz

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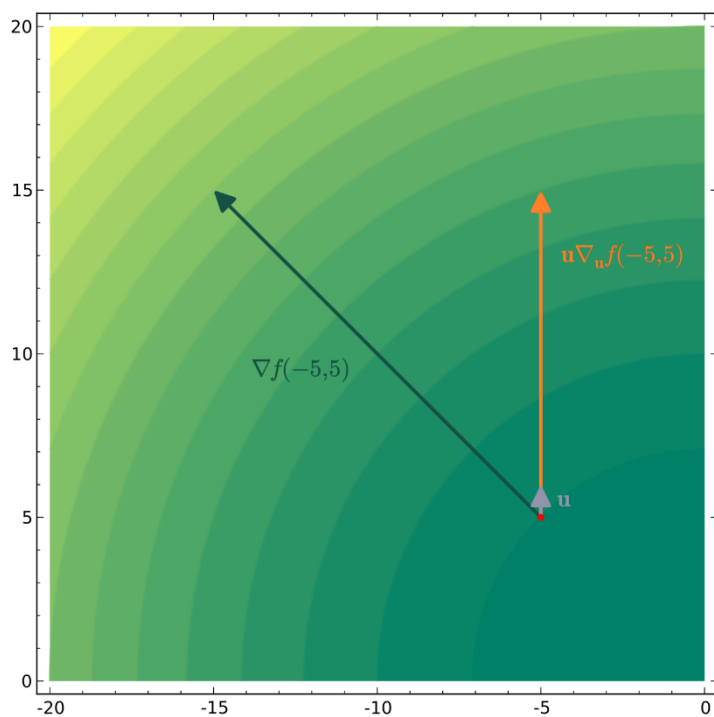
The directional derivative and the gradient vector

$$\begin{aligned} D_{\mathbf{u}}f &= df(\mathbf{u}) \\ df(\mathbf{u}) &= \nabla f \cdot \mathbf{u} \end{aligned}$$

The gradient vector is then *defined*, such that:

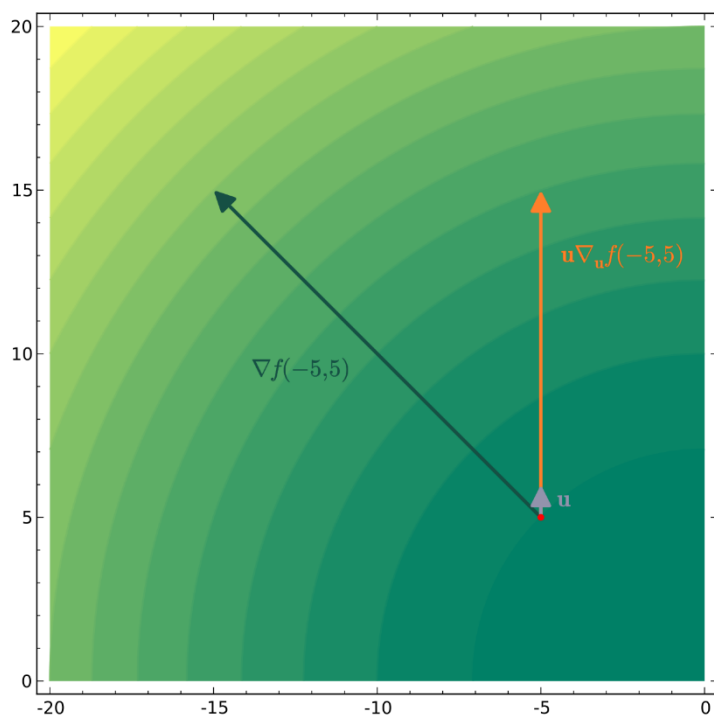
$$\nabla f \cdot \mathbf{u} := df(\mathbf{u})$$

Gradient Vectors and Differential 1-Forms



from Wikipedia (for \mathbf{u} a unit vector),
the function here is $f(x, y) = x^2 + y^2$
 $\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$

Gradient Vectors and Differential 1-Forms



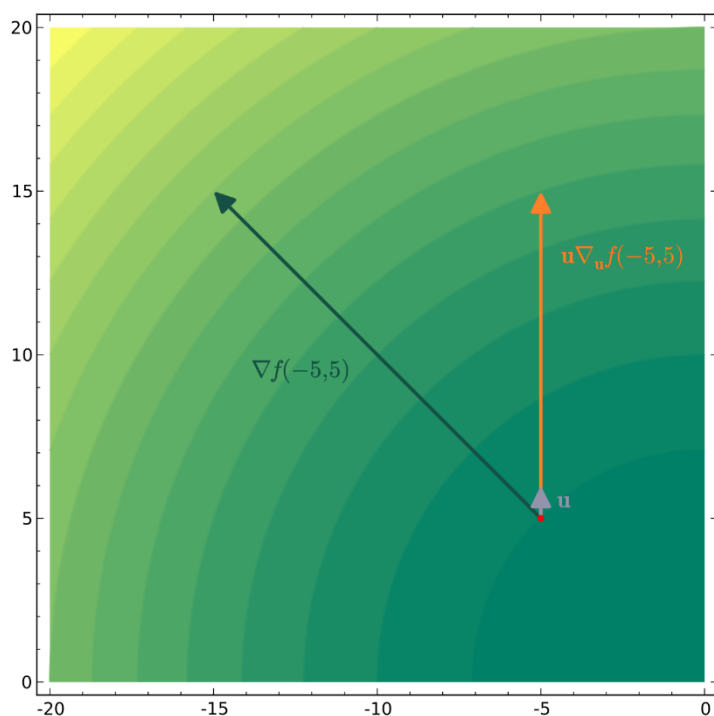
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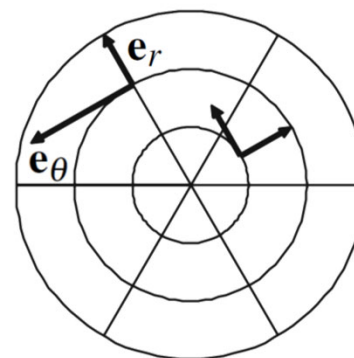
$$\nabla f(x, y) = 2x\mathbf{e}_x + 2y\mathbf{e}_y$$

$$df(x, y) = 2x dx + 2y dy$$

Gradient Vectors and Differential 1-Forms



how about in polar coordinates?



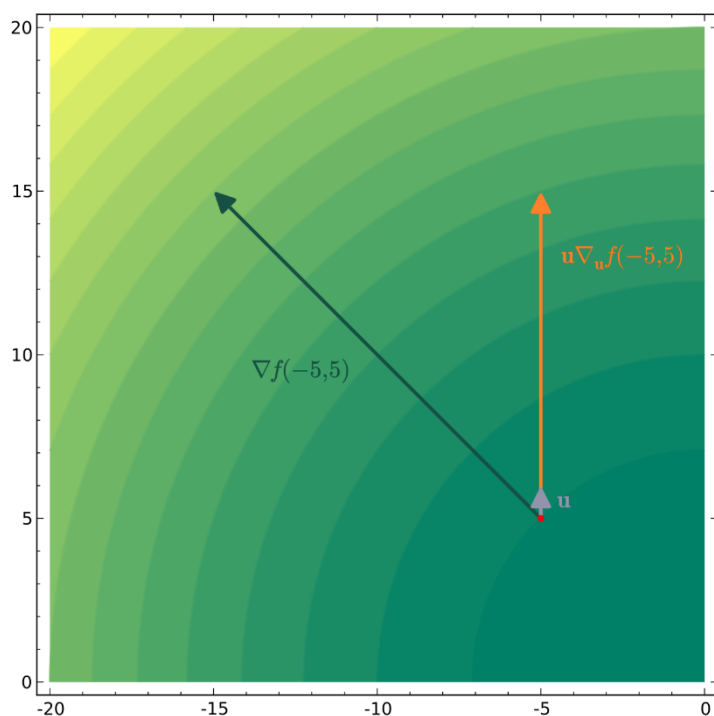
from Wikipedia (for \mathbf{u} a unit vector),

the function here is $f(r, \theta) = r^2$

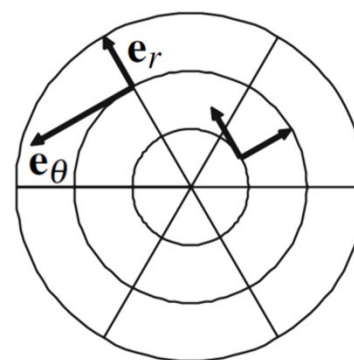
$$\nabla f(r, \theta) = 2r \mathbf{e}_r + 0 \frac{1}{r^2} \mathbf{e}_\theta = 2r \mathbf{e}_r$$

$$df(r, \theta) = 2r dr + 0 d\theta = 2r dr$$

Gradient Vectors and Differential 1-Forms



how about in polar coordinates?



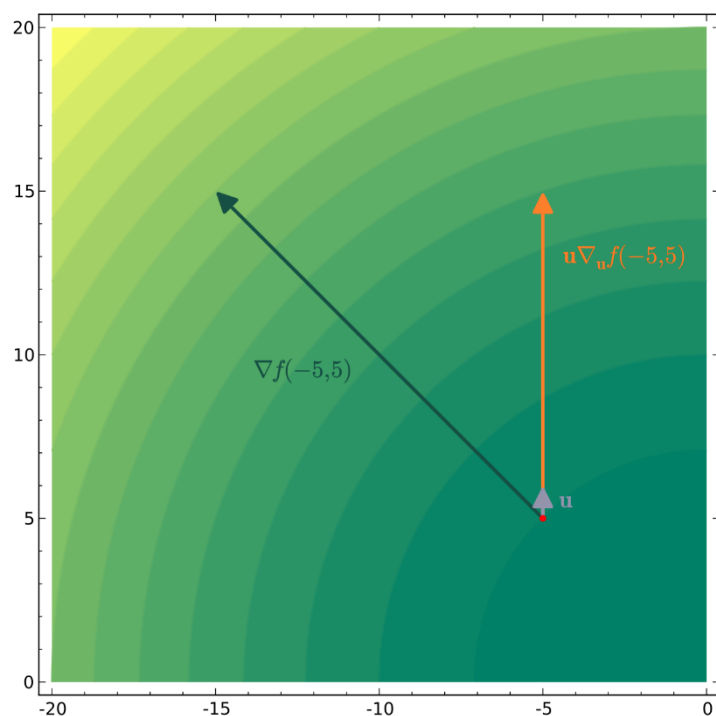
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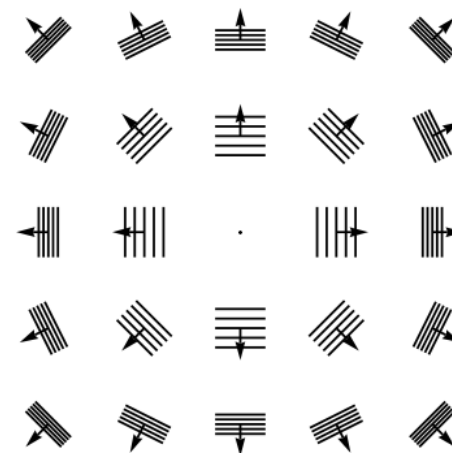
Gradient Vectors and Differential 1-Forms



different 1-forms
evaluated in some direction



1-form (field) df



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$$\nabla f(r, \theta) = 2r \mathbf{e}_r + 0 \frac{1}{r^2} \mathbf{e}_\theta = 2r \mathbf{e}_r$$

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Einstein Summation Convention (1)



Implicit summation over paired indices

- Pairs of “upstairs” and “downstairs” indices

$$\mathbf{v} = v^i \mathbf{e}_i := \sum_i v^i \mathbf{e}_i$$

$$= v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + \dots + v^n \mathbf{e}_n$$

Einstein Summation Convention (2)



Implicit summation over paired indices

- Pairs of “upstairs” and “downstairs” indices

$$\mathbf{g}(\mathbf{v}, \mathbf{w}) = g_{ij} v^i w^j := \sum_{i,j} g_{ij} v^i w^j$$

$$= g_{11} v^1 w^1 + g_{12} v^1 w^2 + \dots + g_{nn} v^n w^n$$

Interlude: Tensor Calculus



In tensor calculus, first-order tensors can be

- Contravariant
- Covariant

$$\mathbf{v} = v^i \mathbf{e}_i$$

$$\boldsymbol{\omega} = v_i \boldsymbol{\omega}^i$$

The gradient vector is a contravariant vector

$$\mathbf{v} = v^i \partial_i$$

The gradient 1-form is a covariant vector (a covector)

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Very powerful; necessary for non-Cartesian coordinate systems

On (intrinsically) curved manifolds (sphere, ...):

Cartesian coordinates not even possible

Interlude: Tensor Calculus



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This is also the fundamental reason why in graphics a normal vector transforms differently: as a covector, not as a vector!

(typical graphics rule: \mathbf{n} transforms with transpose of inverse matrix)

Inner Products and Metric Tensor (Field)



Symmetric, covariant second-order tensor field:
defines inner product on manifold (in each tangent space)

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{g}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

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$$\begin{array}{l|l} \langle \mathbf{v}, \mathbf{w} \rangle & \|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle \\ = \mathbf{g}(\mathbf{v}, \mathbf{w}) & = \mathbf{g}(\mathbf{v}, \mathbf{v}) \\ = g_{ij} v^i w^j & = g_{ij} v^i v^j \\ = \mathbf{v}^T \mathbf{g} \mathbf{w} & = \mathbf{v}^T \mathbf{g} \mathbf{v} \end{array}$$

Inner Products and Metric Tensor (Field)



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$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{g}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \quad (2D)$$

$$\begin{array}{l|l} \langle \mathbf{v}, \mathbf{w} \rangle & \|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle \\ = \mathbf{g}(\mathbf{v}, \mathbf{w}) & = \mathbf{g}(\mathbf{v}, \mathbf{v}) \\ = g_{ij} v^i w^j & = g_{ij} v^i v^j \\ = \mathbf{v}^T \mathbf{g} \mathbf{w} & = \mathbf{v}^T \mathbf{g} \mathbf{v} \end{array} \quad \|\mathbf{v}\|^2 = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

Inner Products and Metric Tensor (Field)



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Inner Products and Metric Tensor (Field)



Components of metric referred to coordinates

$$g_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle$$

A second-order tensor field is bi-linear, i.e.,
linear in each (vector/covector) argument separately

From bi-linearity we immediately get:

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle &= \mathbf{g}(v^i \mathbf{e}_i, w^j \mathbf{e}_j) \\ &= v^i w^j \langle \mathbf{e}_i, \mathbf{e}_j \rangle \\ &= g_{ij} v^i w^j \end{aligned}$$

Gradient Vector from Differential 1-Form



The metric (and inverse metric) *lower* or *raise* indices
(i.e., convert between covariant and contravariant tensors)

$$v^i = g^{ij} v_j$$

$$v_i = g_{ij} v^j$$

$$v^i \mathbf{e}_i = g^{ij} v_j \mathbf{e}_i$$

$$v_i \boldsymbol{\omega}^i = g_{ij} v^j \boldsymbol{\omega}^i$$

Inverse metric (contravariant)

$$[g^{ij}] = [g_{ij}]^{-1}$$

$$g^{ik} g_{kj} = \delta_j^i$$

Kronecker delta behaves
like identity matrix

Gradient Vector from Differential 1-Form



So the gradient vector is

$$\nabla f = \left(g^{ij} \frac{\partial f}{\partial x^j} \right) \mathbf{e}_i$$

Vector-valued 1-form

$$d\mathbf{r} = dx^i \mathbf{e}_i$$

$$d\mathbf{r}(\cdot) = dx^i(\cdot) \mathbf{e}_i$$

Directional derivative via inner product:

$$\begin{aligned} \langle \nabla f, \cdot \rangle &= g_{kj} g^{ik} \frac{\partial f}{\partial x^i} dx^j(\cdot) \\ &= \delta^i_j \frac{\partial f}{\partial x^i} dx^j(\cdot) \\ &= \frac{\partial f}{\partial x^i} dx^i(\cdot) \end{aligned}$$

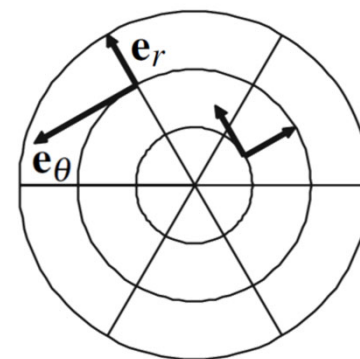
$$\begin{aligned} \nabla f \cdot d\mathbf{r} &= g_{kj} g^{ik} \frac{\partial f}{\partial x^i} dx^j \\ &= \delta^i_j \frac{\partial f}{\partial x^i} dx^j \\ &= \frac{\partial f}{\partial x^i} dx^i \end{aligned}$$

Example: Polar Coordinates



Metric tensor and inverse metric for polar coordinates

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}$$



Gradient vector from 1-form: raise index with inverse metric

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{bmatrix} = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta$$

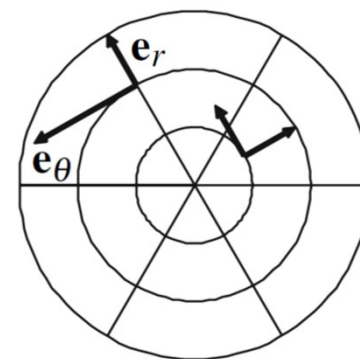
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Gradient vector from 1-form: raise index with inverse metric

$$\nabla f(r, \theta) = \frac{\partial f(r, \theta)}{\partial r} \mathbf{e}_r(r, \theta) + \frac{1}{r^2} \frac{\partial f(r, \theta)}{\partial \theta} \mathbf{e}_\theta(r, \theta)$$

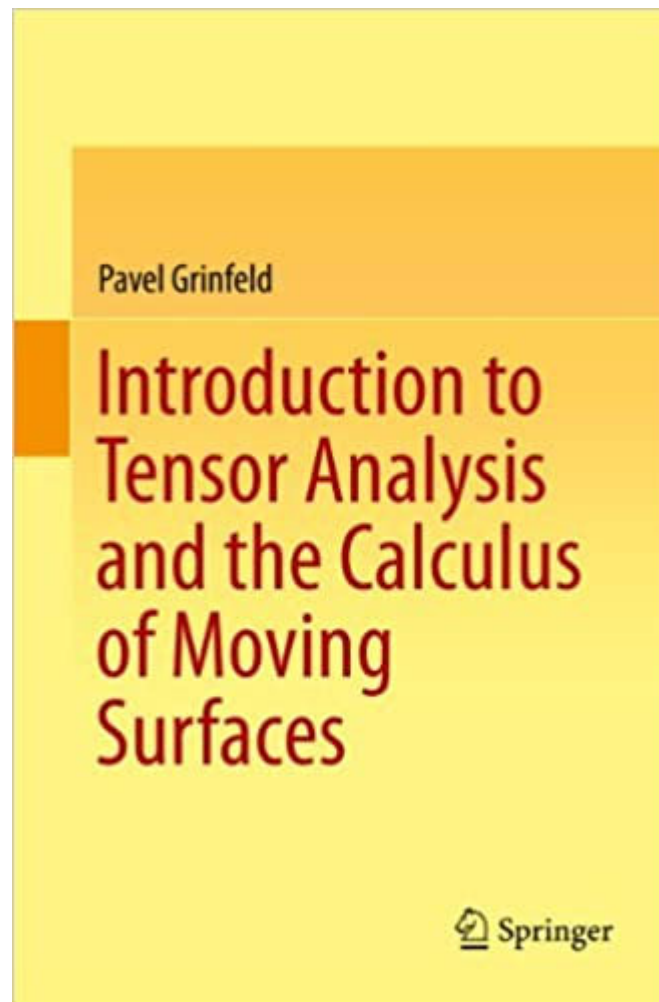
don't forget that all of this is position-dependent!

Tensor Calculus



Highly recommended:

Very nice book,
complete lecture on Youtube!



Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama