Time-Dependent Flow seen through Approximate Observer Killing Fields
SUPPLEMENTARY MATERIAL

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B THE FLOW OF A VECTOR FIELD

In this appendix, we briefly summarize the standard concepts of the *flow* of a vector field, as well as the corresponding linear map called the *differential* or *push-forward*, as they are typically defined in differential geometry. For details, we refer to the books by Lee [11], and Marsden and Hughes [12]. We follow the notation of Marsden and Hughes [12].

The flow of a *time-independent* vector field $\mathbf{u}$ on a manifold M is a map $\phi : J \times M \to M$ for a suitable interval $J \subseteq \mathbb{R}$, such that $t \mapsto \phi(t,x)$ is the unique maximal integral curve of $\mathbf{u}$ through $x \in M$ [11, Th. 9.12]. That is, ϕ maps a point x to its image along the integral curve of $\mathbf{u}$ after time t, which we also denote by $\phi_t(x)$. Important properties of ϕ are:

- The map $\phi_t : M \to M$ is a (local) diffeomorphism for all $t \in J$.
- For all $t_1, t_2 \in J$, $x \in M$, $\phi_{t_2}(\phi_{t_1}(x)) = \phi_{t_1+t_2}(x)$, $\phi_0(x) = x$. The inverse of ϕ_t is ϕ_{-t}, i.e., $\phi_t^{-1}(\phi_t(x)) = \phi_{-t}(\phi_t(x)) = x$. ϕ is an *action* of the additive group $\mathbb{R}$ on M, ϕ_t is a one-parameter group.
- The *linear* map $\mathrm{d}\phi_t : T_xM \to T_{\phi_t(x)}M$, called the differential of ϕ_t, or the (pointwise) *push-forward*, is an isomorphism between the two tangent spaces at each $x \in M$ and $\phi_t(x) \in M$, for each $t \in J$. $\mathrm{d}\phi_t$ maps tangent vectors to all possible curves through a point $x \in M$ to the corresponding tangent vectors of the images of these curves under the diffeomorphism ϕ_t, through the point $\phi_t(x) \in M$.

When the vector field $\mathbf{u}$ is *time-dependent*, the corresponding time-dependent flow $\psi : J \times J \times M \to M$ maps a point $x \in M$ to its image along the integral curve from time s to time t [11, Th. 9.48], which we denote by $\psi_{t,s}(x)$. The map ψ has similar properties to the map ϕ:

- The map $\psi_{t,s} : M \to M$ is a (local) diffeomorphism for all $s,t \in J$.
- For all $s, t_1, t_2 \in J$, $x \in M$, $\psi_{t_2,t_1}(\psi_{t_1,s}(x)) = \psi_{t_2,s}(x)$, $\psi_{s,s}(x) = x$. The inverse of $\psi_{t,s}$ is $\psi_{s,t}$, i.e., $\psi_{t,s}^{-1}(\psi_{t,s}(x)) = \psi_{s,t}(\psi_{t,s}(x)) = x$.
- The *linear* map $\mathrm{d}\psi_{t,s} : T_xM \to T_{\psi_{t,s}(x)}M$, called the differential (the *push-forward*) of $\psi_{t,s}$, is an isomorphism between the tangent spaces at each $x \in M$ and $\psi_{t,s}(x) \in M$, for each $s,t \in J$. $\mathrm{d}\psi_{t,s}$ maps tangent vectors to all possible curves through a point $x \in M$ to the corresponding tangent vectors of the images of these curves under the diffeomorphism $\psi_{t,s}$, through the point $\psi_{t,s}(x) \in M$.

We note that the notation $\psi_{t,s}(x)$ can of course also be consistently used for the case of time-independent flow. In that case, $\psi_{t,s}(x) = \phi_{t-s}(x)$.

C KILLING VECTOR FIELDS

In this appendix, we briefly summarize the standard concept of a *Killing vector field* (often also simply called a *Killing field*, or even simply a *Killing vector*), as it is typically defined in differential geometry. For details, we refer to the books by McInerney [14] and Petersen [15].

In order to be able to describe the deformations a vector field generates, in particular to be able to talk about *isometries*, we equip the manifold M with a *Riemannian metric* [4, 15], which is a smoothly varying symmetric positive-definite bilinear form, i.e., an inner product, g, or g_x, on the tangent space T_xM of each $x \in M$. It is customary to suppress the suffix x if no confusion arises. For two arbitrary vectors $\mathbf{x}, \mathbf{y} \in T_xM$, this inner product is often denoted by any of the following:

$$g(\mathbf{x},\mathbf{y}) = g_x(\mathbf{x},\mathbf{y}) = \langle \mathbf{x},\mathbf{y} \rangle = \langle \mathbf{x},\mathbf{y} \rangle_x . \tag{C.1}$$

The flow ϕ, or $\phi_t(x)$, of a *time-independent* vector field $\mathbf{u}$ is said to generate an *isometry* if it preserves the metric structure, i.e., if

$$\langle \mathbf{x},\mathbf{y} \rangle_x = \left\langle \mathrm{d}\phi_t(\mathbf{x}), \mathrm{d}\phi_t(\mathbf{y}) \right\rangle_{\phi_t(x)}, \tag{C.2}$$

for all $x \in M$, tangent vectors $\mathbf{x}, \mathbf{y} \in T_xM$, and $t \in J \subseteq \mathbb{R}$. That is, the map ϕ_t is a Riemannian isometry for all $t \in J$. Eq. C.2 for all $x, t, \mathbf{x}, \mathbf{y}$ is equivalent to the flow ϕ generating an *infinitesimal isometry*. That is,

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \mathrm{d}\phi_t(\mathbf{x}), \mathrm{d}\phi_t(\mathbf{y}) \right\rangle_{\phi_t(x)} \\ &:= \lim_{t \to 0} \frac{\left\langle \mathrm{d}\phi_t(\mathbf{x}), \mathrm{d}\phi_t(\mathbf{y}) \right\rangle_{\phi_t(x)} - \left\langle \mathbf{x},\mathbf{y} \right\rangle_x}{t} = 0, \end{aligned} \tag{C.3}$$

under the same conditions. Another equivalent condition is that the metric g vanishes under the *Lie derivative* [15, Prop. 8.1.1]:

$$(\mathscr{L}_{\mathbf{u}}\, g)_x = 0, \qquad \text{for all } x \in M. \tag{C.4}$$

In detail, this means that for all $x \in M$, and all $\mathbf{x}, \mathbf{y} \in T_xM$,

$$(\mathscr{L}_{\mathbf{u}}\, g)_x(\mathbf{x},\mathbf{y}) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \mathrm{d}\phi_t(\mathbf{x}), \mathrm{d}\phi_t(\mathbf{y}) \right\rangle_{\phi_t(x)} = 0. \tag{C.5}$$

This is usually compactly written as the condition $\mathscr{L}_{\mathbf{u}}\, g = 0$.

The *time-dependent* case is very similar. Here, a flow $\psi_{t,s}$ of a time-dependent vector field $\mathbf{u}$ is said to generate an isometry if it preserves the metric structure for each fixed start time $s_0 \in J$, i.e., if

$$\langle \mathbf{x},\mathbf{y} \rangle_x = \left\langle \mathrm{d}\psi_{t,s_0}(\mathbf{x}), \mathrm{d}\psi_{t,s_0}(\mathbf{y}) \right\rangle_{\psi_{t,s_0}(x)}, \tag{C.6}$$

again for all points $x \in M$, tangent vectors $\mathbf{x}, \mathbf{y} \in T_xM$, and $t \in J \subseteq \mathbb{R}$ [12, Def. 6.15]. That is, ψ_{t,s_0} is a Riemannian isometry for all start times $s_0 \in J$ and end times $t \in J$. This is again equivalent to the flow ψ generating an infinitesimal isometry, which similar to above means

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s_0} \left\langle \mathrm{d}\psi_{t,s_0}(\mathbf{x}), \mathrm{d}\psi_{t,s_0}(\mathbf{y}) \right\rangle_{\psi_{t,s_0}(x)} \\ &:= \lim_{t \to s_0} \frac{\left\langle \mathrm{d}\psi_{t,s_0}(\mathbf{x}), \mathrm{d}\psi_{t,s_0}(\mathbf{y}) \right\rangle_{\psi_{t,s_0}(x)} - \left\langle \mathbf{x},\mathbf{y} \right\rangle_x}{t - s_0} = 0, \end{aligned} \tag{C.7}$$

under the same conditions. Equivalently, the metric g vanishes under the, now however time-dependent, Lie derivative [12, Prop. 6.16]:

$$(L_{\mathbf{u}}\, g)_x = 0, \qquad \text{for all } x \in M. \tag{C.8}$$

This now means that for all $x \in M$, all $\mathbf{x}, \mathbf{y} \in T_xM$, and all $s_0 \in J$,

$$(L_{\mathbf{u}}\, g)_x(\mathbf{x},\mathbf{y}) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s_0} \left\langle \mathrm{d}\psi_{t,s_0}(\mathbf{x}), \mathrm{d}\psi_{t,s_0}(\mathbf{y}) \right\rangle_{\psi_{t,s_0}(x)} = 0. \tag{C.9}$$

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This is again usually compactly written as the condition $L_{\mathbf{u}}\, g = 0$.

If one of the equivalent Eqs. C.2, C.3, or C.4 (time-independent flow) holds for all points $x \in M$, the vector field $\mathbf{u}$ generating the flow is called a *Killing vector field* [15, Ch. 8], named after the mathematician Wilhelm Killing [10]. If one of the equivalent Eqs. C.6, C.7, or C.8 (time-dependent flow) holds for all points $x \in M$, the vector field $\mathbf{u}$ generating the flow is called a *time-dependent Killing vector field*.

Another equivalent condition is that a vector field $\mathbf{u}$ (time-dependent or time-independent) is a Killing field, if and only if the $\binom{1}{1}$ tensor $\nabla\mathbf{u}$, denoting the once-contravariant, once-covariant second-order tensor called the (spatial) velocity gradient tensor, is *skew-symmetric* at all points $x \in M$. We note that this condition is independent of coordinates. See Petersen [15, Prop. 8.1.2] for the time-independent case, and Marsden and Hughes [12, p.99] for the general formulation using $L_{\mathbf{u}}\, g$.

We can also state this condition by saying that $\mathbf{u}$ is a Killing field if and only if, at all points $x \in M$ on the manifold M, we have [1, 17],

$$\Big\langle \nabla\mathbf{u}\,(\mathbf{v})\,,\mathbf{v} \Big\rangle_x = 0, \tag{C.10}$$

where $\nabla\mathbf{u}$, the velocity gradient tensor of $\mathbf{u}$ at the point x, is evaluated for the direction $\mathbf{v}$ given at the same point x, i.e., $\mathbf{v} \in T_xM$. This equation must hold for all directions $\mathbf{v} \in T_xM$ and all points $x \in M$ [1, 17].

D Time Derivatives Relative to Rigid Observer Motion (Rotating and Translating Reference Frames)

To be able to compare two observers $\mathbb{O}_0$ and $\mathbb{O}_1$, we will use two different bases. For vectors seen by $\mathbb{O}_0$, we will use a basis $\{\mathbf{e}_i\}$ that is not spinning, and for $\mathbb{O}_1$ we will use a basis $\{\tilde{\mathbf{e}}_i\}$ that is spinning relative to the basis $\{\mathbf{e}_i\}$, i.e., relative to $\mathbb{O}_0$, as given by the spin $\mathbf{\Omega}$. That is, for $\mathbb{O}_0$ the time derivative of each basis vector $\tilde{\mathbf{e}}_i$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{\mathbf{e}}_i = \mathbf{\Omega}\,\tilde{\mathbf{e}}_i. \tag{D.1}$$

We now use the Einstein summation convention for implied summation over repeated indexes [5, p.59], and refer geometric vectors $\mathbf{v}$ to a basis $\{\tilde{\mathbf{e}}_i\}$ by writing $\mathbf{v} = \tilde{v}^i\,\tilde{\mathbf{e}}_i$ with components $\tilde{v}^i$. We can then expand

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{v}^i\,\tilde{\mathbf{e}}_i\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{v}^i\right)\tilde{\mathbf{e}}_i + \tilde{v}^i\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{\mathbf{e}}_i\right). \tag{D.2}$$

Here, $\mathrm{d}/\mathrm{d}t$ denotes the derivative in the direction $\mathbf{u}$ for the observer $\mathbb{O}_0$. The second term on the right-hand side results from the product rule applied to both the components $\tilde{v}^i$ *and* the basis $\{\tilde{\mathbf{e}}_i\}$. For $\mathbb{O}_0$, $\{\tilde{\mathbf{e}}_i\}$ is spinning according to Eq. D.1, i.e., this term will be $\mathbf{\Omega}\,\mathbf{v}$. However, we can also write this for $\mathbb{O}_0$ with respect to the basis $\{\mathbf{e}_i\}$, where we get

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(v^i\,\mathbf{e}_i\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,v^i\right)\mathbf{e}_i = \frac{\partial\mathbf{v}}{\partial t} + \nabla\mathbf{v}\,(\mathbf{u})\,. \tag{D.3}$$

In contrast, however, for $\mathbb{O}_1$ the right-hand term in Eq. D.2 will be zero for the basis $\{\tilde{\mathbf{e}}_i\}$, because for $\mathbb{O}_1$ the basis $\{\tilde{\mathbf{e}}_i\}$ is not spinning. Therefore, comparing the two observers gives the observed time derivative

$$\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{v} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{v}^i\right)\tilde{\mathbf{e}}_i = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \mathbf{\Omega}\,\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + \nabla\mathbf{v}\,(\mathbf{u}) - \mathbf{\Omega}\,\mathbf{v}. \tag{D.4}$$

Likewise, since the vector field $\mathbf{u}$ describes rigid motion, where $\nabla\mathbf{u} = \mathbf{\Omega}$, for the observed time derivative of $\mathbf{u}$ we obtain

$$\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{u} = \frac{\partial\mathbf{u}}{\partial t} + \mathbf{\Omega}\,\mathbf{u} - \mathbf{\Omega}\,\mathbf{u} = \frac{\partial\mathbf{u}}{\partial t}. \tag{D.5}$$

Putting both results together, we finally obtain the full expression

$$\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{v}_{\mathbf{u}} = \frac{\partial\mathbf{v}}{\partial t} - \frac{\partial\mathbf{u}}{\partial t} + \nabla\mathbf{v}\,(\mathbf{u}) - \mathbf{\Omega}\,\mathbf{v}. \tag{D.6}$$

This agrees with the general derivation using Lie derivatives, giving Eq. 16 in the paper, when the field $\mathbf{u}$ is an exact Killing field ($\nabla\mathbf{u} = \mathbf{\Omega}$).

E The Derivative (Tangent) of Observed Path Lines

We can compute the derivative of an observed path line $t \mapsto p_{\mathbf{u}}(t,r)$ with r fixed (Eq. 44 in the paper) as follows. To identify the path line, we denote some fixed position on the path line at some time s by $p_s := p(s) \in M$. We denote another position on the same path line at some other time τ by $p_\tau := \psi_{\tau,s}(p_s) \in M$ (using ψ, corresponding to $\mathbf{v}$, not $\psi^{\mathbf{u}}$). See Eq. 35 in the paper. In the derivation below, it is important to note that we use τ as a *fixed* parameter, and p_τ as a *fixed* spatial position, at which we want to evaluate a specific derivative.

Starting from Eq. 36 in the paper, using the chain rule we can derive

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=\tau} t \mapsto p_{\mathbf{u}}(t,r) &= \frac{\partial}{\partial t}\bigg|_{t=\tau} \psi^{\mathbf{u}}_{r,t}\big(\psi_{t,s}(p_s)\big),\\
&= \frac{\partial}{\partial t}\bigg|_{t=\tau} \psi^{\mathbf{u}}_{r,t}(p_\tau) + \frac{\partial}{\partial x}\bigg|_{x=p_\tau} \psi^{\mathbf{u}}_{r,\tau}(x)\,\cdot\,\frac{\partial}{\partial t}\bigg|_{t=\tau} \psi_{t,s}(p_s),\\
&= \mathrm{d}\psi^{\mathbf{u}}_{r,\tau}(p_\tau)\Big(-\mathbf{u}(p_\tau,\tau)\Big) + \mathrm{d}\psi^{\mathbf{u}}_{r,\tau}(p_\tau)\Big(\mathbf{v}(p_\tau,\tau)\Big),\\
&= \mathrm{d}\psi^{\mathbf{u}}_{r,\tau}(p_\tau)\Big(\mathbf{v}_{\mathbf{u}}(p_\tau,\tau)\Big).
\end{aligned} \tag{E.1}$$

The differential $\mathrm{d}\psi^{\mathbf{u}}_{t,s}(x)$ is the *push-forward* of $\psi^{\mathbf{u}}_{t,s}(x)$, which linearly maps vectors from the tangent space T_xM at x to vectors in the tangent space $T_{\psi^{\mathbf{u}}_{t,s}(x)}M$ at $\psi^{\mathbf{u}}_{t,s}(x)$. In the derivation above, we have used that

$$\frac{\partial}{\partial t}\bigg|_{t=\tau} \psi^{\mathbf{u}}_{\tau,t}(p_\tau) = -\frac{\partial}{\partial t}\bigg|_{t=\tau} \psi^{\mathbf{u}}_{t,\tau}(p_\tau) = -\mathbf{u}\,(p_\tau,\tau)\,. \tag{E.2}$$

Using this relation, we can compute the first derivative term above as

$$\frac{\partial}{\partial t}\bigg|_{t=\tau} \psi^{\mathbf{u}}_{r,t}(p_\tau) = \mathrm{d}\psi^{\mathbf{u}}_{r,\tau}(p_\tau)\Big(-\mathbf{u}(p_\tau,\tau)\Big). \tag{E.3}$$

F Objectivity of Relative Velocities

We briefly prove that the relative velocity field $\mathbf{v}_{\mathbf{u}} = \mathbf{v} - \mathbf{u}$ is objective. In our case, $\mathbf{v}$ denotes the input field, $\mathbf{u}$ denotes the observer field, and $\mathbf{v}_{\mathbf{u}}$ denotes the observed field. However, the property of a relative velocity field being objective is independent of these semantics.

We consider the observer transformation of the relative velocity field

$$\mathbf{v}_{\mathbf{u}}(x,t) = \mathbf{v}(x,t) - \mathbf{u}(x,t), \tag{F.1}$$

which, following Truesdell and Noll [18, p.43], with slightly adapted notation, transformed from an observer $\mathbb{O}_0$ to another observer $\bar{\mathbb{O}}_0$, denoting a different reference frame by $\bar{(\cdot)}$ instead of by $(\cdot)^*$, is

$$\begin{aligned}
(\bar{\mathbf{v}} - \bar{\mathbf{u}})(x,t) = &\Big(\mathbf{Q}(t)\,\mathbf{v}(x,t) + \dot{c}(t) + \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T\big(x - c(t)\big)\Big) - \\
&\Big(\mathbf{Q}(t)\,\mathbf{u}(x,t) + \dot{c}(t) + \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T\big(x - c(t)\big)\Big),
\end{aligned} \tag{F.2}$$

where the point $c(t)$ is an arbitrary origin. See also Holzapfel [8, p.184]. It is crucial to note that here $x, c(t) \in M$ denote specific points on the manifold M, independent of any coordinate system, and not relative position vectors. We can further clarify by defining the vector-valued function $\mathbf{w}(t) := \dot{c}(t)$, and the point $o(t) := c(t)$, to write this as

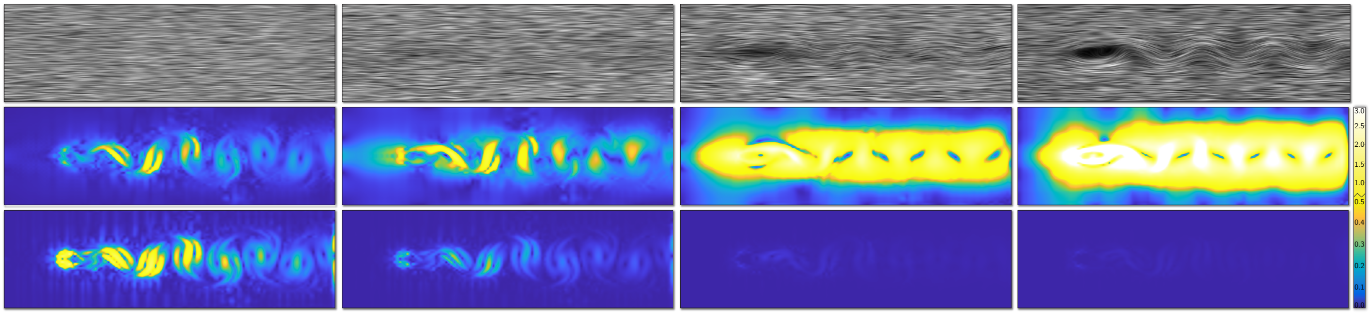
$$\begin{aligned}
(\bar{\mathbf{v}} - \bar{\mathbf{u}})(x,t) = &\Big(\mathbf{Q}(t)\,\mathbf{v}(x,t) + \mathbf{w}(t) + \mathbf{\Omega}(t)\big(x - o(t)\big)\Big) - \\
&\Big(\mathbf{Q}(t)\,\mathbf{u}(x,t) + \mathbf{w}(t) + \mathbf{\Omega}(t)\big(x - o(t)\big)\Big),
\end{aligned} \tag{F.3}$$

with $\mathbf{\Omega} := \dot{\mathbf{Q}}\mathbf{Q}^T$ the skew-symmetric tensor giving the spin of observer $\mathbb{O}_0$ relative to $\bar{\mathbb{O}}_0$, and $\mathbf{w}(t)$ the relative velocity of the point $o(t)$. We can now see that, in fact, $\mathbf{w}(t)$ and $\mathbf{\Omega}(t)$ describe a Killing field (Eq. 2).

All terms except the relative active rotation $\mathbf{Q}$, applied pointwise to velocity vectors in each tangent space T_xM, cancel out, and so we get

$$\bar{\mathbf{v}}_{\bar{\mathbf{u}}}(x,t) = \mathbf{Q}(t)\,\mathbf{v}_{\mathbf{u}}(x,t). \tag{F.4}$$

Therefore, according to the definition and transformation properties of an objective vector field [18, p.42], the vector field $\mathbf{v}_{\mathbf{u}}$ is objective.



(a) λ=0.5 μ=0.5 (b) λ=1 μ=2 (c) λ=10 μ=6 (d) λ=16 μ=18

Fig. 11. **Optimizing observer fields** for the 2D VORTEX STREET with different parameters λ, μ, increasing μ from left to right leading to successively higher Killing energy: (a) $\lambda = 0.5, \mu = 0.5$; (b) $\lambda = 1.0, \mu = 2.0$; (c) $\lambda = 10.0, \mu = 6.0$; (d) $\lambda = 16.0, \mu = 18.0$. (Top row) LIC of each observer field $\mathbf{u}$ with velocity masking; (Middle row) Killing energy $\|K\mathbf{u}\|_F$; (Bottom row) Observed time derivative $\|\mathscr{D}/\mathscr{D}t\ \mathbf{v}_\mathbf{u}\|_2$.

G Parameter Exploration

Fig. 11 shows results for different parameter settings λ and μ in addition to Fig. 4 in the main paper. Here we can see that increasing μ starts to develop swirly motion in the observer field, due to matching the input flow more closely. This reduces the time derivative but results in significantly higher Killing energy corresponding to more deformation.

H Discussion of Differences from *Topology-Inspired Galilean Invariant Vector Field Analysis* [2]

The Galilean-invariant approach by Bujack et al. [2] is similar to ours in spirit, as their result is also not, quoting, “one well-suited frame of reference, but the simultaneous visualization of the dominating frames of reference in the different areas of the flow field” [2, abstract].

They achieve this by observing that the sign of the determinant of the velocity gradient tensor (called the Jacobian in [2]) is Galilean-invariant. Based on this, they define Galilean-invariant critical points as the critical points of the determinant of the velocity gradient tensor.

They then use contour tree pruning to keep only the most prominent critical points, and cluster the vector field domain according to these critical points. Finally, they subtract a weighted sum of the velocities at the critical points (in our terminology an observer velocity field) to construct a Galilean-invariant vector field.

This differs from our approach in the following ways:

- Their approach is Galilean-invariant (although, in fact, it is slightly more since the translation can be time-dependent) but not objective, whereas our approach is objective.
- They compute a finite number of observers, whereas we compute a continuous field of observers.
- Their approach works on one time step at a time without considering time derivatives, i.e., each time step is treated as a separate time-independent vector field. Achieving smooth results over time is left for future work. In contrast, our approach treats time inherently, optimizing over all time steps simultaneously.
- Their approach was designed for 2D vector fields. An extension to 3D is left as future work. In contrast, our approach was designed to seamlessly work in both 2D and 3D.
- They require several parameters for topological simplification, clustering, and constructing the Galilean-invariant vector field, whose influence on the final visualization is not clear. In contrast, our approach has essentially one parameter (λ) that chooses a smooth trade-off between the as-rigid-as-possible (approximate Killing field) property and the as-steady-as-possible (small observed time derivative) property of the observer velocity field.

The paper [2] also briefly discusses issues of possible ambiguities and misinterpretations of the resulting visualizations, as the resulting images show different regions of the flow for different observers. Also, the authors argue that interactive methods to switch between different observers might be helpful in understanding the visualizations. We leave this as future work.

I Flow Field Data Sets

We briefly give more information on the data sets that we have used.

I.1 Four Centers

The four centers vector field given by Günther et al. [6, Sec. 6.2] is computed from the originally steady 2D vector field in the spatial domain $[-2,2] \times [-2,2]$, and the temporal domain $[0, 2\pi]$, defined by

$$\mathbf{v}(x,y) := \begin{pmatrix} -x(2y^2-1)e^{-(x^2+y^2)} \\ y(2x^2-1)e^{-(x^2+y^2)} \end{pmatrix}, \tag{I.1}$$

with the observer transformation (which makes the final field unsteady)

$$\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{I.2}$$

For this paper, the field was sampled on a $64 \times 64 \times 64$ Cartesian grid.

I.2 2D Vortex Street

This data set has been simulated by Tino Weinkauf [20] using the Free Software *Gerris Flow Solver* [16].

Data downloaded from
`http://www.csc.kth.se/~weinkauf/notes/cylinder2d.html`

I.3 3D Vortex Street

This is a direct numerical Navier Stokes simulation by Simone Camarri and Maria-Vittoria Salvetti (University of Pisa), Marcelo Buffoni (Politecnico of Torino), and Angelo Iollo (University of Bordeaux I) [3] which is publicly available [9].

We use a uniformly resampled version which has been provided by Tino Weinkauf and has been used in von Funck et al. [19].

Data downloaded from
`http://www.csc.kth.se/~weinkauf/notes/squarecylinder.html`

I.4 Ocean

This is a three-dimensional unsteady flow field obtained by Haller et al. [7] from the Southern Ocean state estimation model of Mazloff et al. [13]. The spatial domain is bounded by longitudes $[11°E, 16°E]$, latitudes $[37°S, 33°S]$, and depth $[7, 2000]$ m.

In this paper we use a time-dependent 2D slice corresponding to the ocean surface given on a 390×210 regular grid with 14 time steps.

Data downloaded from
`https://github.com/LCSETH/Lagrangian-Averaged-Vorticity-Deviation-LAVD/tree/master/3D/data`

J Source Code

MATLAB source code for the major computations described in this paper can be found online in the source code repository listed at
`http://vccvisualization.org/research/killingobservers/`

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