Markus Hadwiger, Thomas Theußl, Peter Rautek

Riemannian Geometry for Scientific Visualization

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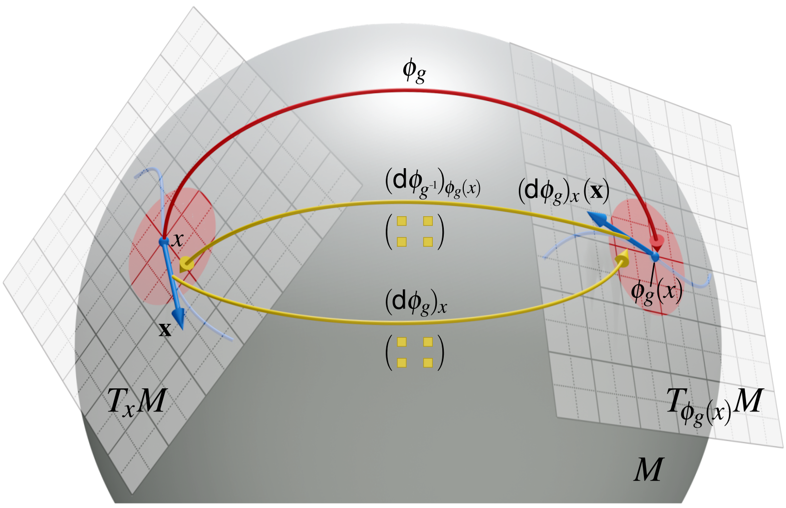

Markus Hadwiger
Visual Computing Center,
King Abdullah University of Science and Technology (KAUST)
23955-6900 Thuwal, Saudi Arabia

Thomas Theußl
Visualization Core Lab,
King Abdullah University of Science and Technology (KAUST)
23955-6900 Thuwal, Saudi Arabia

Peter Rautek
Visual Computing Center,
King Abdullah University of Science and Technology (KAUST)
23955-6900 Thuwal, Saudi Arabia

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Introduction

This tutorial introduces the most important basics of *Riemannian geometry* and related concepts with a specific focus on applications in *scientific visualization*. The main concept in Riemannian geometry is the presence of a *Riemannian metric* on a differentiable manifold, comprising a second-order tensor field that defines an inner product in each tangent space that varies smoothly from point to point. Technically, the metric is what allows defining and computing distances and angles in a coordinate-independent manner. However, even more importantly, it in a sense is really the major structure (on top of topological considerations) that *defines the space* where scientific data, such as scalar, vector, and tensor fields live.

However, the concept of a metric, and crucial related concepts such as *connections* and *covariant derivatives*, are not often used explicitly in visualization. In contrast to concepts of differential topology, which have been used extensively in visualization, for example in scalar and vector field topology, we believe that concepts from Riemannian geometry have been underrepresented in the visualization literature. One reason for this might be that most visualization techniques are developed for scalar, vector, or tensor fields given in Euclidean space $\mathbb{R}^2$ or $\mathbb{R}^3$, and data given on curved surfaces are usually treated explicitly through their embedding in $\mathbb{R}^3$. However, the presence of a Riemannian metric on a manifold has very important implications even for data given in Euclidean space, for example regarding the physical meaning of visualizations as well as for the use of non-Cartesian coordinates. Therefore, considering the metric tensor field explicitly provides several important benefits.

In this tutorial, we try to particularly highlight the additional *insight* that can be gained from employing concepts from Riemannian geometry in scientific visualization. However, although we believe that insight is the most important benefit to be gained from using these concepts, we also discuss *computational advantages*. In addition to Riemannian metrics, we also introduce the most important related concepts from modern, coordinate-free differential geometry, in particular general (non-Cartesian) tensor fields and differential forms, smooth mappings between manifolds, Lie derivatives, and Lie groups and Lie algebras. Throughout the tutorial, we use several examples from the scientific visualization literature, dealing with scalar, vector, or tensor fields, respectively, and highlight their implicit or explicit connections to Riemannian geometry.

Objectives

The main objective of this tutorial is to introduce the most important concepts of Riemannian geometry, and related concepts from modern differential geometry, with a strong focus on applications in scientific visualization. We describe these applications not only in the form of general concepts, but we also discuss individual research papers and representative techniques from the visualization literature to highlight how Riemannian geometry can provide additional insight on these techniques. Moreover, we discuss how Riemannian geometry can be seen as providing a *unifying framework* that facilitates comparing and discussing visualization techniques that in individual papers often look different and hard to compare.

Motivation and Audience

While there exist a lot of general mathematical textbooks and courses on differential geometry and Riemannian geometry, we are not aware of any course that specifically targets visualization researchers and practitioners. Furthermore, the concepts—and in particular the emphasis—most relevant and important for visualization techniques are hard to extract from standard geometry texts, which often cover a large amount of advanced material. At the same time, *time-dependent data*, such as unsteady vector fields, are not treated in sufficient detail in most geometry texts. This tutorial aims to start filling this gap for researchers and practitioners in visualization, on an *intermediate level*.

Outline and Sessions

The tutorial is structured in six sessions of about 30 minutes each, plus a short introduction in an initial session 0. The total time is planned for three hours, with one break in between.

We note that much of each session content expands on and builds on the content of the sessions coming before. Likewise, some sessions already provide previews of later sessions that will cover concepts in more detail. We therefore cover some concepts in multiple sessions, with the goal of highlighting different aspects and details as well as different connections to visualization in each session.

1 Overview

We first give an overview of the content and the main concepts to be introduced in this tutorial, highlighting example applications dealing with scalar fields, vector fields, and tensor fields. We introduce Riemannian geometry[1] in general, with important concepts that require a Riemannian metric vs. concepts that do not. In this chapter, we briefly highlight the major topics that we will cover: Riemannian manifolds and Riemannian metrics, and the most important related concepts of modern differential geometry, and why they are important for visualization. We motivate ways in which the presence of a metric tensor field on a manifold is crucial for insight into the meaning of visualization techniques.

[1] This is the part of *manifold theory*, the theory of manifolds (in particular, differential manifolds), that focuses on concepts requiring a Riemannian metric to be defined. However, many concepts that are important in this context are defined for any differential manifold, not just for those with Riemannian metrics.

1.1 Main Concepts

The main concepts that we will introduce in this tutorial are

- *Riemannian manifolds*. Starting with topological and then differential manifolds (topological manifolds with a differential structure), we will endow a differential manifold M with a Riemannian metric $\mathbf{g}$, thus obtaining a Riemannian manifold $(M, \mathbf{g})$.[2]
- *Riemannian metrics:* Riemannian metric tensor fields $\mathbf{g}$ on differential manifolds define an inner product $\langle\cdot,\cdot\rangle := \mathbf{g}(\cdot,\cdot)$ in each tangent space of the manifold, varying smoothly from point to point. Although technically, metric tensor fields just enable defining and computing inner products, they are in fact an important notion defining *what a space is*,[3] e.g., as the domain of scalar, vector, and tensor fields. This perspective enables important insights even in Euclidean space, regarding the *behavior* of fields. For example, fundamentally it is the metric that determines many types[4] of derivatives, such as $\nabla\mathbf{v}$, of vector and tensor fields.
- *Tensors* and *tensor fields*. Importantly, we do not constrain ourselves to Cartesian tensors (mathematical objects that only behave as tensors when Cartesian coordinates are used, but otherwise are non-tensorial), but we cover general tensors: *Contravariant* tensors, *covariant* tensors, and tensors of *mixed type*.[5] We will cover "classical" index computations, but in particular also emphasize the more modern purely geometric *coordinate-free* perspective without coordinates.[6] The general concept of tensor fields also includes scalar fields (0-order tensor fields) and vector fields (first-order contravariant tensor fields).
- *Differential forms:* Anti-symmetric[7] covariant tensors are particularly important for integration on manifolds, but provide more general insight and techniques even beyond. The gradient of a scalar function $f\colon M \to \mathbb{R}$, given on a manifold M, is naturally a *differential 1-form* df (a covariant vector, or covector),[8] and not a (contravariant) gradient vector ∇f. (Given a metric, the gradient vector can be computed via the *natural pairing* $\nabla f \mapsto \langle\nabla f,\cdot\rangle = df(\cdot)$, equivalent to *raising/lowering* indices in classical notation, where $(\nabla f)^i = g^{ij}\,(df)_j$, and $(df)_i = g_{ij}\,(\nabla f)^j$.)

[2] Despite the metric being the most important notion in our context, some fundamental concepts that we will cover do not require a metric to be defined. For example, the *gradient 1-form* and *Lie derivatives* are defined on any differential manifold. However, the metric can often provide additional insight or techniques. For example, "raising" the gradient *1-form* df to become the gradient *vector* ∇f. In coordinate-free notation, this can be denoted via the *musical isomorphisms*, where $\nabla f = (df)^\sharp$ and $df = (\nabla f)^\flat$, or via the *natural pairing* $\nabla f \mapsto \langle\nabla f,\cdot\rangle = df(\cdot)$.

[3] The most famous (pseudo-Riemannian) example is probably *general relativity*, where the four-dimensional spacetime metric *defines* what spacetime is, including its curvature (giving the effect of gravity) and how particles move along "straight" lines (geodesics) in curved spacetime.

[4] Not all derivatives depend on the metric, e.g., *covariant* derivatives (using a metric connection) do, *Lie* derivatives do not.

[5] Even linear maps between vector spaces are mixed tensors. However, the metric is an example of a covariant tensor.

[6] A vector $\mathbf{v}$ is a geometric vector in coordinate-free notation. A vector v^i given by (contravariant) components, however, depends on the particular choice of coordinates. Computing $\mathbf{v} = v^i\,\mathbf{e}_i$, with basis $\{\mathbf{e}_i\}$, links the two perspectives.

[7] "Anti-symmetry" for tensors of order less than two is trivially fulfilled. Therefore, every covector is a 1-form.

[8] For integration, the natural notion for *line integrals* along curves C is $\int_C df$, or, more generally, $\int_C \omega$, instead of $\int_C \nabla f \cdot d\mathbf{r}$ or $\int_C \mathbf{v}\cdot d\mathbf{r}$, respectively, integrating arbitrary 1-forms ω or *exact* 1-forms df instead of vector fields $\mathbf{v}$ or gradient vector fields ∇f.

- *Connections*.[9] We will focus on metric connections, i.e., connections that are compatible with a given metric $\mathbf{g}$, in particular the *Levi-Civita connection* given by *Christoffel symbols* $\Gamma^i{}_{jk}$. Connections are inherently related to the concept of *parallel transport* and *covariant derivatives*.
- *Covariant derivatives.* Given a connection on a manifold, we can compute *covariant* (here in the sense of *invariant* with respect to the choice of coordinate system, *not in the sense of covariant tensors!*[10]) derivatives of tensor fields. One hugely important example in flow visualization is the *velocity gradient tensor* $\nabla\mathbf{v}$. Contrary to widely-used definitions, in non-Cartesian coordinates this tensor cannot be defined as a Jacobian matrix of partial derivatives with respect to the coordinates.
- *Lie derivatives* and *Lie brackets*. This important type of derivative is independent of the metric, and computes a derivative of a tensor field with respect to the *flow* of an arbitrary given vector field. It is a very natural geometric concept that has important applications, such as in observer-relative computations in flow visualization (computing relative to the flow describing reference frame motion). Moreover, the Lie derivative of a vector field is the same as the *Lie bracket* between vector fields, which is an important concept in Lie theory (turning a Lie algebra from a vector space into an algebra, with the multiplication operation given by the Lie bracket). This leads to important applications in surface computation from vector fields in higher-dimensional ambient spaces, for example computing streamsurfaces from diffusion tensor fields[11].
- *Smooth maps between manifolds*, and their corresponding *pushforwards* and *pullbacks*. These are hugely important fundamental concepts that are required to determine how tensor fields (including vector fields) are mapped through diffeomorphisms, for example for active transformations or for observer-relative computations and the definition of *objectivity* in flow visualization and continuum mechanics.
- *Isometries* and *infinitesimal isometries*. Being able to define isometries precisely is very important to quantifying whether two spaces (Riemannian manifolds) are "the same" (including the same manifold before and after a transformation). This is crucial for observer (reference frame) transformations in flow visualization and continuum mechanics.
- *Lie groups* and *Lie algebras*. Lie theory enables understanding transformations as *symmetries*[12] within a very powerful framework. For example, the Lie groups $SO(2)$ and $SO(3)$, for two- and three-dimensional rotations, respectively, as well as the Euclidean groups (the transformation groups of all Euclidean isometries). The *exponential map* connects infinitesimal transformations (e.g., infinitesimal isometries) to the corresponding "finite" (i.e., integrated) transformations (e.g., isometries).[13]

1.2 *Motivating Examples*

Gradients of scalar fields

The meaning of the *gradient* of a scalar field $f\colon M \to \mathbb{R}$, on a manifold M, is completely independent of any underlying choice of coordinate system,

[9] "Connecting" two different, but infinitesimally close tangent spaces on a manifold. This is sometimes also called an *affine connection*. A connection is the major way in which tangent vectors can be transported from one tangent space to a neighboring tangent space. The *covariant derivative* measures the corresponding rate of change. When the covariant derivative of a vector field along a curve vanishes, the vector is *parallel-transported* along the curve.

[10] In fact, covariant derivatives are defined for any tensor field, i.e., covariant derivatives are defined for any contravariant, covariant, or mixed tensor field. The covariant derivative of a particular type of tensor in a given direction (given by a vector) is again a tensor of the same type. However, the covariant derivative as a map is one *covariant* order higher. For example, the covariant derivative of a vector field $\mathbf{v}$ is a second-order tensor of mixed contravariant (because of the contravariant field $\mathbf{v}$) and covariant type. The latter is because a tensor that accepts a direction vector as an argument is covariant in that argument.

[11] The integration of two vector fields in 3D may "fit together" to form a 2D surface, or not fit together, forming no surface. This "integrability" is determined by the *Frobenius theorem*, using the Lie bracket. Even without considering an embedding, the Lie bracket is needed to determine whether two vector fields *commute* or not: For example, the integration of a set of *basis vector fields* only forms a *coordinate system* if all pairs of basis vector fields commute (meaning the Lie bracket is zero). Otherwise, the vector fields can only define *non-coordinate* bases (or *frames*). This is important, because most *orthonormal* bases do not form coordinate bases, but *non-coordinate frames*. (The Cartesian basis, however, is a coordinate basis.)

[12] Lie groups are continuous *symmetry groups*, defining precisely what being symmetric (as a specific way of being "the same") means. For example, defining that all rotations of a circle or a sphere are still "the same circle" or "the same sphere."

[13] The *exponential map* maps an element $\mathfrak{g}$ of a Lie algebra, such as an infinitesimal rotation, e.g., given by an anti-symmetric matrix X, to the corresponding Lie group element $g \in G$, for a Lie group G, e.g., given by the corresponding "integrated" orthogonal rotation matrix $g(t) = \exp(tX)$, with $G = SO(n)$. The Lie group is also a (differential) manifold, and the path (parameterized by the parameter t, with $t = 0$ mapping to the identity) that connects the identity element of the Lie group (e.g., the identity matrix) with the element $g(t)$ is a geodesic in this manifold (e.g., all rotations $g(t)$ starting at the identity $g(0)$).

which is chosen purely for computational purposes. The most common way to compute the gradient of a scalar field f is to compute the *gradient vector* ∇f, whose *geometric meaning*[14] is that it points in the direction of fastest change (largest directional derivative), with its magnitude $\|\nabla f\|$ (requiring a metric to be defined) giving the rate of change in that direction.

The gradient vector allows computing the directional derivative $D_{\mathbf{u}}f$, of the scalar field f in the direction given by an arbitrary vector $\mathbf{u}$, as

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle \nabla f, \mathbf{u} \rangle. \tag{1.1}$$

This requires an inner product $\langle \cdot, \cdot \rangle$ to be defined in each tangent space,[15] given by a *metric tensor* field on M. The inner product, which is a geometric concept (see below), then enables determining the directional derivative, i.e., the rate of change[16] of the scalar field f in the direction given by $\mathbf{u}$.

Computationally, it is an important restriction of the definition of gradients as gradient vectors that *each choice of coordinate system requires a different formula for the gradient vector computation*, and also requires care (and again a different formula) with respect to which (coordinate) basis or (non-coordinate) frame is used.[17] The latter can often lead to misunderstandings, and thus wrong results, when the basis that corresponds to the gradient formula is not given explicitly or not fully understood.

In fact, instead of the gradient vector ∇f, the more "primary" notion for the concept of a gradient, allowing to compute directional derivatives without requiring an inner product to be defined, is given by the *differential 1-form* df of the scalar field f, also simply called the *differential* df. The 1-form df determines the directional derivative in the direction $\mathbf{u}$ as[18]

$$D_{\mathbf{u}}f = df(\mathbf{u}). \tag{1.2}$$

If a metric, defining the inner product, is given, the gradient vector ∇f is then *defined to be the unique vector such that* the following is true:

$$D_{\mathbf{u}}f = df(\mathbf{u}) = \langle \nabla f, \mathbf{u} \rangle. \tag{1.3}$$

This, in fact, as we will see in detail later on, is usually done by taking the 1-form df, which is a *covariant* first-order tensor (a *covector*),[19] and converting it into its associated (via the metric) *contravariant* vector[20] ∇f.

In contrast to these complications, the differential 1-form df can always, i.e., for *any* coordinate system, be obtained in components via

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{1.4}$$

Here, $\{dx^i\}$ is the *dual basis*, i.e., the covector basis of coordinate differentials corresponding to the coordinate basis $\{\mathbf{e}_i\}$. The dual basis is defined such that $dx^i(\mathbf{e}_j) = \delta^i_j$, where the latter is the *Kronecker delta* that is 1 when $i = j$, and 0 otherwise.[21] This implies that dx^i *reads off* the i'th component of a vector: Every vector $\mathbf{v}$ can be expanded as $\mathbf{v} = dx^i(\mathbf{v})\,\mathbf{e}_i$. This is the same as $\mathbf{v} = v^i \mathbf{e}_i$, with the components v^i obtained as $v^i = dx^i(\mathbf{v})$.

The coordinate basis vectors $\{\mathbf{e}_i\}$ themselves are *tangent* to the *coordinate lines* of the coordinate system[22] $\{x^i\}$, i.e., the curves along which *all other coordinates* are constant. They also comprise the partial derivative

[14] A geometric (tangent) vector in tensor calculus is a *contravariant* vector. Thus, on a manifold M, the gradient vector field is a *section* of the *tangent bundle* TM. Each gradient vector, at a point $x \in M$, is a member of T_xM, the tangent space at the point x. However, it is crucial to note that if a vector $\mathbf{n}$ is intended to be used in an inner product (like ∇f), for *active transformations* the desired geometric behavior is that of a *covariant* vector (a *1-form*). Gradient vectors (and surface normals) must then be treated as 1-forms, obtained via the *natural pairing* $\mathbf{n} \mapsto \langle \mathbf{n}, \cdot \rangle$, that are mapped through diffeomorphisms via *pullbacks* instead of *pushforwards*.

[15] This does not mean that the directional derivative can only be computed if a metric is defined! Quite to the contrary (Eq. 1.2).

[16] The rate is measured with $\mathbf{u}$ as the "unit," i.e., if $\|\mathbf{u}\|$ increases/decreases, the rate increases/decreases (see Eq. 1.5). If one wants to exclude this effect, $\mathbf{u}$ must be chosen as a unit vector, or the rate of change be divided by $\|\mathbf{u}\|$.

[17] For example, $\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y$ in 2D Cartesian coordinates, with the Cartesian coordinate basis $(\mathbf{e}_x, \mathbf{e}_y) = (\boldsymbol{\partial}_x, \boldsymbol{\partial}_y)$. However, in polar coordinates we instead get $\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$, with the corresponding coordinate basis $(\mathbf{e}_r, \mathbf{e}_\theta) = (\boldsymbol{\partial}_r, \boldsymbol{\partial}_\theta)$. Even more confusingly, if an orthonormal basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta\}$ is used instead, we get $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta$. (The coordinate basis in this case is not orthonormal; the orthonormal basis is thus a *non-coordinate frame*. For the former, $[\mathbf{e}_r, \mathbf{e}_\theta] = 0$, whereas for the latter $[\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta] \neq 0$, with $[\cdot, \cdot]$ the *Lie bracket*.)

[18] If this formulation is used, the gradient vector is not required *at all* in order to compute directional derivatives. Even in computer graphics, the normal vector can be seen as determining the directional derivative of the signed distance to a surface. The corresponding differential df is the fundamental reason why normal vectors in computer graphics do not transform as regular (geometric) vectors. They transform as differential 1-forms.

[19] A covariant first-order tensor, also called a *covector* or *1-form*, represents a scalar-valued linear function (or *functional*) mapping a (tangent) vector to a scalar.

[20] In "classical" tensor calculus using indices, the *"downstairs"* index of the covariant vector $v_i = (df)_i$ is *raised* to produce the associated contravariant vector with an *"upstairs"* index $v^i = (\nabla f)^i$. As we will see later, the (inverse) metric tensor is used to raise the index via $v^i = g^{ij} v_j$.

[21] In "matrix notation" the Kronecker delta is simply the identity matrix. In this sense, $\{dx^i\}$ and $\{\mathbf{e}_i\}$ are "inverses" of each other.

[22] With coordinate functions $\{x^i\}$, where each $x^i : M \supset U \to \mathbb{R}$.

operators $\mathbf{e}_i = \frac{\partial}{\partial x^i}$ (sometimes written in abbreviated notation as $\boldsymbol{\partial}_i$), because they are tangent to the direction in which only one coordinate is varied.[23]

The multitude of different formulas for ∇f, if even needed at all, then arise naturally[24] by raising the covector df to its dual vector ∇f.

Riemannian metrics

A Riemannian metric is a second-order tensor field $\mathbf{g}$ on a manifold M that defines an inner product $\langle \cdot, \cdot \rangle$ between two (tangent) vectors $\mathbf{v}$ and $\mathbf{w}$ as

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{g}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta, \tag{1.5}$$

where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$. This metric is required to be positive-definite,[25] i.e., $\mathbf{g}(\mathbf{v}, \mathbf{v}) > 0$ for all vectors $\mathbf{v} \neq 0$.

In our context, it is crucial to realize that the equation above corresponds to the *geometric meaning* of the inner product between two vectors, *without requiring coordinates*. It is therefore a *coordinate-free* definition.

That is, one should not right away think of the standard definition of the Euclidean inner product (the dot product) in Cartesian coordinates, where $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_i v_i w_i$ is often seen as the definition of the inner product.[26] However, in the more general context of Riemannian geometry and tensor calculus, this computation is merely a way for computing the scalar value given by the inner product of two vectors *in Euclidean space* given *by Cartesian components*.[27] This equation is neither valid[28] for non-Cartesian coordinates (even in Euclidean space), nor is it possible to use this definition in (intrinsically) curved spaces, e.g., on the surface of a sphere seen as a two-dimensional manifold. The metric tensor $\mathbf{g}$ is a purely geometric, i.e., *coordinate-free*, concept. However, for computations in coordinates, we can use the corresponding components g_{ij}, referred to a basis $\{\mathbf{e}_i\}$, given by

$$g_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle. \tag{1.6}$$

Here, the right-hand side is seen as being *evaluated geometrically*, i.e., without "circularly" using the definition of the (unknown) components g_{ij}. This can be done using the geometric definition given by Eq. 1.5,[29] or via some other geometric or physical means.[30] Of course, if the metric components g_{ij} are already known in some other coordinate system, these components can simply be *transformed* to any other coordinate system.[31]

Another important way to compute the metric components g_{ij} is determining the metric *induced* by a surrounding ambient space for which the metric is already known. For example, for curved surfaces embedded in $\mathbb{R}^3$, with the latter having the Euclidean metric, the 2D metric on the surface is then trivially induced by the 3D ambient metric. The right-hand side of Eq. 1.6 can then be computed via the inner product already known in $\mathbb{R}^3$.

Given the components g_{ij}, with respect to a given basis $\{\mathbf{e}_i\}$, we can evaluate the metric for two arbitrary vectors $\mathbf{v}$ and $\mathbf{w}$, given in components v^i and w^i with $\mathbf{v} = v^i \mathbf{e}_i$ and $\mathbf{w} = w^i \mathbf{e}_i$, respectively, as[32]

$$\begin{aligned} \mathbf{g}(\mathbf{v}, \mathbf{w}) &= \mathbf{g}\big(v^i \mathbf{e}_i, w^j \mathbf{e}_j\big) \\ &= v^i w^j \mathbf{g}\big(\mathbf{e}_i, \mathbf{e}_j\big) \\ &= g_{ij} v^i w^j. \end{aligned} \tag{1.7}$$

[23] This corresponds to the fact that vectors can be interpreted as directional derivative operators. For example, we always have $\boldsymbol{\partial}_i x^i = dx^i(\boldsymbol{\partial}_i) = 1$, and if a metric is given $\boldsymbol{\partial}_i x^i = \langle \nabla x^i, \boldsymbol{\partial}_i \rangle$. However, we also note that the basis vectors themselves are often also computed via partial differentiation.

[24] This is also called the *natural pairing* determined by a given metric $\mathbf{g}$, written as $\nabla f \mapsto \mathbf{g}(\nabla f, \cdot)$. That is, df is determined by $df(\cdot) = \langle \nabla f, \cdot \rangle$. This is sometimes also written as $df = \nabla f \cdot d\mathbf{r}$, where $d\mathbf{r}$ denotes a *vector-valued* 1-form.

[25] A generalized version are indefinite, *pseudo-Riemannian* metrics. The most famous example is the four-dimensional spacetime metric used in special and general relativity.

[26] In our context, we would write this as $\langle \mathbf{v}, \mathbf{w} \rangle = v_i w^i = v^i w_i$, because in tensor calculus indices are only allowed to be *contracted* in contravariant (upstairs) and covariant (downstairs) index pairs.

[27] In fact, the standard equation for the dot product results from the special case where the metric $\mathbf{g}$ in components is given by $g_{ij} = \delta_{ij}$ (i.e., by an identity matrix), because for the full expression (Eq. 1.7) we get $g_{ij} v^i w^j = \delta_{ij} v^i w^j = \sum_i v^i w^i = \sum_i v_i w_i$.

[28] We are referring to the fact that $\sum_i v_i w_i$ is non-tensorial, giving different results in different coordinate systems. However, both the expressions $v_i w^i$ and $v^i w_i$ are tensorial: They give the same result in all coordinate systems. (The full expression is $g_{ij} v^i w^j$, and $g_{ij} v^i = v_j$ and $g_{ij} w^j = w_i$.)

[29] We note that when the metric is already known in some other coordinate system, such as the "trivial" Euclidean metric in "background Cartesian coordinates," we can "cheat" and simply compute the geometric inner product from the known Cartesian components of the vectors $\mathbf{e}_i$.

[30] A famous example are the *Einstein field equations*, which determine the metric of curved spacetime from the energy-momentum tensor field corresponding to the distribution of energy and mass.

[31] Using the tensor transformation rules for a covariant second-order tensor.

[32] The derivation results directly from the bi-linearity of the metric $\mathbf{g}$.

Here, we are using the Einstein summation convention. See Chapter 2.1. Alternatively, we can arrange components in a symmetric matrix.[33] In 2D,

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \tag{1.8}$$

from which we can then compute the inner product above (Eqs. 1.5, 1.7) as

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{g}(\mathbf{v}, \mathbf{w}) = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}. \tag{1.9}$$

We note, however, that, as we will explain in detail later on, in order to fully write the metric tensor $\mathbf{g}$ by itself, in components referred to a basis $\{\mathbf{e}_i\}$, we need to use the corresponding basis for covariant second-order tensors[34] $\{\omega^i \otimes \omega^j\}$, with $\otimes$ the *tensor product*, giving the metric tensor as

$$\mathbf{g} = g_{ij}\, \omega^i \otimes \omega^j. \tag{1.10}$$

Velocity gradients

In flow visualization, many important properties of vector fields can be derived from a *first-order* Taylor expansion of a given vector field $\mathbf{v}$ around each point, which is determined by the *velocity gradient tensor* $\nabla\mathbf{v}$.[35]

The first-order directional derivative of the vector field $\mathbf{v}$ in a direction $\mathbf{w}$ is then obtained as $\nabla\mathbf{v}(\mathbf{w})$,[36] i.e., the second-order tensor $\nabla\mathbf{v}$ at a point is applied as a linear map to the vector $\mathbf{w}$ at that point, mapping it to the vector $\nabla_{\mathbf{w}}\mathbf{v} = \nabla\mathbf{v}(\mathbf{w})$ that comprises the directional derivative at that point.

In the flow visualization literature, $\nabla\mathbf{v}$ is often defined as a matrix of partial derivatives: The *Jacobian* (matrix) $\mathbf{J}$, giving (not writing the basis[37])

$$\nabla\mathbf{v} := \mathbf{J} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial v^i}{\partial x^j} = \partial_j v^i. \tag{1.11}$$

On the right-hand side, the vector components v^i are the components obtained by referring the vector $\mathbf{v}$ to a specific basis[38], and $(\partial/\partial x^j) =: \partial_j$ denotes taking the partial derivative with respect to the coordinate x^j.

For example, in 2D (again not writing the basis)

$$\nabla\mathbf{v} := \mathbf{J} = \begin{bmatrix} \partial_1 v^1 & \partial_2 v^1 \\ \partial_1 v^2 & \partial_2 v^2 \end{bmatrix}. \tag{1.12}$$

However, despite its wide-spread usage, *this definition is only valid in Cartesian (affine) coordinates*, but it is *not valid (not tensorial) in general*. For example, even in flat, two-dimensional Euclidean space with polar coordinates $(r, \theta) := (x^1, x^2) = (x^r, x^\theta)$, we have (not writing the basis)

$$\nabla\mathbf{v} \neq \mathbf{J} = \begin{bmatrix} \partial_r v^r & \partial_\theta v^r \\ \partial_r v^\theta & \partial_\theta v^\theta \end{bmatrix}. \tag{1.13}$$

Of course, one could simply go ahead and *define* $\nabla\mathbf{v}$ as above, for any arbitrary coordinate system. However, then this expression *does not define a tensor*, and it *will not give the correct geometric behavior*.[39] This implies that for each different coordinate system the corresponding (non-tensorial) definition differs from the behavior in another coordinate system. For

[33] In Euclidean space with Cartesian coordinates, this is the identity matrix. In Euclidean space with polar coordinates, $g_{11} = 1, g_{22} = r^2$, and $g_{12} = g_{21} = 0$.

[34] These bases comprise tensor products of 1-forms ω^i reading off coordinates of vector arguments, giving tensor bases $\{\omega^i \otimes \omega^j\}$. With arguments, $\mathbf{g}(\mathbf{v}, \mathbf{w}) = g_{ij}\, \omega^i(\mathbf{v})\, \omega^j(\mathbf{w}) = g_{ij} v^i w^j$. For dual coordinate bases, where $\omega^i = dx^i$, we can write the tensor bases as $\{dx^i \otimes dx^j\}$.

[35] At each point $x \in M$, this is a linear map $(\nabla\mathbf{v})_x \colon T_xM \to T_xM$. For the whole vector field $\mathbf{v}$, the map can be written as $\nabla\mathbf{v} \colon TM \to TM$, mapping a vector field of directions to the corresponding vector field of directional derivatives.

[36] Sometimes, e.g., in the Navier-Stokes equations, this is written as $\mathbf{w} \cdot \nabla\mathbf{v}$. However, we are using the more general notation here that emphasizes that $\nabla\mathbf{v}$ is a *function*.

[37] The tensor $\nabla\mathbf{v}$ is a second-order tensor of mixed *type*, with one contravariant and one covariant index. The corresponding basis is therefore $\mathbf{e}_i \otimes \omega^j$. Contrast this with the basis $\omega^i \otimes \omega^j$ of the covariant tensor $\mathbf{g}$, where both indices are covariant.

[38] The *components* must always be combined with the correct *basis*: For Eq. 1.12, this means the expansion $\mathbf{v} = v^x \mathbf{e}_x + v^y \mathbf{e}_y$, with components $(v^x, v^y) = (v^1, v^2)$, whereas for Eq. 1.13 this means the expansion $\mathbf{v} = v^r \mathbf{e}_r + v^\theta \mathbf{e}_\theta$, with components $(v^r, v^\theta) = (v^1, v^2)$. (In both, $\mathbf{v}$ is the same vector, just referred to different bases.)

[39] For $\nabla\mathbf{v}$, the directional derivatives of the vector field would be wrong (different for each coordinate system). The directional derivative of a vector field is a vector: The derivative of $\mathbf{v}$ in the direction $\mathbf{w}$ is the vector $\nabla_{\mathbf{w}}\mathbf{v} = \nabla\mathbf{v}(\mathbf{w})$. With a non-tensorial definition of $\nabla\mathbf{v}$, this vector will be different depending on the coordinate system: Its geometric meaning is lost.

physical computations, for example, this would imply that physical laws depend on the coordinate system, which is clearly not desirable.

The maybe simplest example to illustrate why this is a problem is the basic property that *a tensor that is zero in one coordinate system must be zero in all other coordinate systems*. Comparing Eq. 1.12 with Eq. 1.13 for the simple example of a constant vector field in the Euclidean plane,[40] once using Cartesian coordinates for the computation, and once using polar coordinates for "the same" computation, shows that this clearly is not the case when the definition of $\nabla\mathbf{v}$ given by Eq. 1.11 is used: The derivatives of a constant vector field in all directions should be zero (i.e., the zero vector). However, this is not the case when polar coordinates and Eq. 1.13 are used.

[40] We define a constant vector field $\mathbf{v}$, which in Cartesian coordinates has components $(v^x, v^y) = (U, V)$, for some constants U, V. However, in polar coordinates the components are $(v^r, v^\theta) = (U\cos\theta + V\sin\theta, -\frac{1}{r}U\sin\theta + \frac{1}{r}V\cos\theta)$, giving the derivatives in Eq. 1.13 as

$$\mathbf{J} = \begin{bmatrix} 0 & r v^\theta \\ -\frac{1}{r}v^\theta & -\frac{1}{r}v^r \end{bmatrix}.$$

Thus, $\mathbf{J}$ is non-zero for non-vanishing $\mathbf{v}$. Comparison with Eq. 1.17 shows why the correct expression yields $\nabla\mathbf{v} = 0$.

In these notes, we will explain in detail why, in the often very important general setting,[41] *a tensor is not just any array of numbers*. That is, for example, not every matrix gives a second-order tensor. A simple explanation is that it can only be a tensor if it *behaves like a tensor*, which is often determined via tensor *transformation rules* between coordinate systems, but this is essentially a geometric, and thus coordinate-free, concept.

[41] In tensor calculus, manifold theory, differential geometry in non-Cartesian coordinates, as well as in any physical setting, from Newtonian mechanics in non-Cartesian coordinates to special and general relativity.

So what is the correct general definition of the velocity gradient tensor? For this, we need the concept of *covariant derivative*, briefly introduced below, corresponding to a *connection* given on the underlying manifold M. In our context, this connection will be a *metric connection*, i.e., a connection that is compatible[42] with the Riemannian metric on M.

[42] This means that $\nabla\mathbf{g} = 0$, i.e., the covariant derivative of the metric tensor field is identically zero on M. (In components, this is written as $\nabla_k g_{ij} = 0$.) Equivalently, $\nabla_{\mathbf{u}}\langle\mathbf{v},\mathbf{w}\rangle = \langle\nabla_{\mathbf{u}}\mathbf{v},\mathbf{w}\rangle + \langle\mathbf{v},\nabla_{\mathbf{u}}\mathbf{w}\rangle$, for any vector $\mathbf{u}$ and any vector fields $\mathbf{v},\mathbf{w}$.

Connections and covariant derivatives

Instead of using partial derivatives ∂_j in the definition of $\nabla\mathbf{v}$, i.e., incorrectly defining $\nabla\mathbf{v} := \partial_j v^i\,\mathbf{e}_i \otimes \omega^j$, the general definition of the velocity gradient tensor must use *covariant derivatives* ∇_j, i.e., we define

$$\nabla\mathbf{v} := \nabla_j v^i\,\mathbf{e}_i \otimes \omega^j. \tag{1.14}$$

For example, in components in 2D (not writing the tensor basis $\mathbf{e}_i \otimes \omega^j$),

$$\nabla\mathbf{v} := \begin{bmatrix} \nabla_1 v^1 & \nabla_2 v^1 \\ \nabla_1 v^2 & \nabla_2 v^2 \end{bmatrix}. \tag{1.15}$$

The above is the correct (general) definition of the velocity gradient tensor as a covariant derivative, corresponding to a specific *connection*.[43]

[43] We will use the unique *Levi-Civita connection*, which is both metric (compatible with a given $\mathbf{g}$) and torsion-free.

The connection can be given in components via *Christoffel symbols* $\Gamma^i{}_{jk}$, with three indices[44] i, j, and k, each ranging from 1 to n. We then get for $\nabla\mathbf{v}$ in components (referred to the tensor basis $\mathbf{e}_i \otimes \omega^j$),

[44] Despite the slightly misleading notation, $\Gamma^i{}_{jk}$ is *not* a third-order tensor (and it correspondingly does not transform as one). The Christoffel symbols are an array of numbers indexed with three indices, but they do not obey the tensor transformation law for third-order tensors. In fact, if they would, they could not "make" $\nabla_j v^i$ behave correctly as a second-order tensor.

$$\nabla_j v^i := \partial_j v^i + \Gamma^i{}_{jk} v^k. \tag{1.16}$$

In (intrinsically) flat space with Cartesian or affine coordinates, all Christoffel symbols vanish ($\Gamma^i{}_{jk} \equiv 0$), but *only* in this special case is $\nabla_j v^i = \partial_j v^i$. Only in this case is Eq. 1.11 equivalent to Eq. 1.14 and therefore correct.

For example, the non-zero Christoffel symbols for polar coordinates are $\Gamma^1{}_{22} = -r$, and $\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r}$. For this case, we therefore get for Eq. 1.15

$$\nabla\mathbf{v} = \begin{bmatrix} \partial_1 v^1 & \partial_2 v^1 - r v^2 \\ \partial_1 v^2 + \frac{1}{r}v^2 & \partial_2 v^2 + \frac{1}{r}v^1 \end{bmatrix}. \tag{1.17}$$

The connection and the corresponding covariant derivative are inherently related to the concept of *parallel transport*. In fact, a vector field $\mathbf{v}$ is said to be parallel-transported along a curve $t \mapsto \gamma(t)$ with tangent vector field[45] $\mathbf{w}$ (with $\mathbf{v}$ given at least at each point along the curve[46]), when

$$\nabla\mathbf{v}(\mathbf{w}) = 0, \tag{1.18}$$

at each point along the curve. A special case are curves that are *geodesics*, for which the tangent vectors themselves are parallel-transported, i.e.,

$$\nabla\mathbf{w}(\mathbf{w}) = 0. \tag{1.19}$$

For (intrinsically) flat manifolds, parallel transport between two points is independent of the path (the curve) taken between the two points. For (intrinsically) curved manifolds, parallel transport is path-dependent.[47]

[45] If the curve is written as $\gamma(t)$, the tangent vectors $\mathbf{w}(t)$ are defined as $\mathbf{w}(t) := \gamma'(t)$.

[46] For $\nabla\mathbf{v}(\mathbf{w})$ to be defined, it is sufficient that the vector field $\mathbf{v}$ is defined at least in a neighborhood around each point $\gamma(t)$ along a curve with tangent parallel to $\mathbf{w}$.

[47] In fact, this is one way in which curvature can be *defined*, corresponding to the concept of *holonomy*. The (fourth-order) Riemann curvature tensor R^i_{jkl} captures the complete behavior of parallel transport of vectors around infinitesimal loops at a point. If parallel transport is path-independent, then the Riemann curvature tensor is zero, corresponding to the manifold being intrinsically flat. In the simple special case of a two-dimensional surface, this corresponds to the Gaussian curvature.

Flow maps

The flow map of a time-dependent vector field $\mathbf{v}(x,t)$ is given by

$$\begin{aligned} F_{t_0}^t \colon M &\to M, \\ x_0 &\mapsto F_{t_0}^t(x_0) =: x(t;x_0). \end{aligned} \tag{1.20}$$

This simply denotes the *diffeomorphism* $F_{t_0}^t(x_0) \colon x_0 \mapsto x(t;x_0)$, mapping from the manifold M to itself, mapping points x_0 at time t_0 to points x at time t, defined by for each point x_0 following the *path line* through x_0, at time t_0, until time t, at which time the path line passes through the point x.

From a flow map $F_{t_0}^t$, for a vector field given in Euclidean space, the corresponding spatial gradient $\nabla F_{t_0}^t$, also called the *deformation gradient*, is often computed.[48]. However, more generally, also including non-Euclidean manifolds, this deformation gradient in fact requires the more general concept of the *pushforward* or *differential* of the diffeomorphism $F_{t_0}^t$.[49] For a general diffeomorphism ϕ, the corresponding pushforward is a map between tangent bundles that is often written as the map ϕ_*. If the pushforward is called the differential of ϕ, it is also often written simply as $d\phi$. We will denote this map for the diffeomorphism $F_{t_0}^t$ by $dF_{t_0}^t$:

$$\begin{aligned} dF_{t_0}^t \colon TM &\to TM, \\ \mathbf{v} &\mapsto dF_{t_0}^t(\mathbf{v}). \end{aligned} \tag{1.21}$$

The pushforward defined on the entire tangent bundle is in general not a linear map. Considered point-wise, however, i.e., at each point x_0, it is a linear map between tangent spaces. We can write this linear map as

$$\begin{aligned} \left(dF_{t_0}^t\right)_{x_0} \colon T_{x_0}M &\to T_xM, \\ \mathbf{v}_{x_0} &\mapsto dF_{t_0}^t(\mathbf{v}_{x_0}). \end{aligned} \tag{1.22}$$

In the last row of the latter equation, the notation $\mathbf{v}_{x_0}$ refers to a single vector (not a vector field) in the tangent space $T_{x_0}M$ located at the point x_0. However, for brevity we will often use the simple notation $\mathbf{v}$ for both *vector fields* (on a tangent bundle) and individual *vectors* (in a specific tangent space), because the meaning is usually easy to infer from the context.

[48] For example, in FTLE computations, and in the definition and computation of many other Lagrangian concepts.

[49] We note that in general, even the name deformation gradient is misleading, since the pushforward of a diffeomorphism between two general manifolds (or a general manifold to itself) cannot be computed as a gradient. We note that in continuum mechanics, the pushforward at a given point on the manifold is also called a *two-point tensor*, because it maps between *two different* tangent spaces (which often also have, or must have (if there is no trivial parallel transport), two *different bases*).

2 Manifolds, coordinate charts, vector fields

In this chapter, we introduce coordinate systems and generalize them to differential manifolds via an atlas of coordinate charts. We introduce the fundamental notions of *coordinate bases* as well as *non-coordinate frames*. In the former, all basis vectors are derivatives of the coordinate functions $\{x^i\}$ of a given coordinate chart. In the latter, they are not, which for example is often the case for orthonormal frames. The *Lie bracket* is a very important concept that clarifies the difference. Vector fields on manifolds are defined on the *tangent bundle*.

Vector and tensor components referred to arbitrary coordinate systems; manifolds that cannot be covered by a single chart (e.g., the sphere); vector fields on curved manifolds, e.g., geophysical flows [1]; computing gradients in non-Euclidean coordinate systems, e.g., polar coordinates or curvilinear grids; preview of later sessions: velocity gradient tensors.

[1] P. Rautek, M. Mlejnek, J. Beyer, J. Troidl, H. Pfister, T. Theußl, and M. Hadwiger. Objective observer-relative flow visualization in curved spaces for unsteady 2d geophysical flows. *IEEE Transactions on Visualization and Computer Graphics*, 27 (2):283–293, 2021

The most important concepts covered in this chapter are:

- The Einstein summation convention.
- Manifolds; in particular *differentiable manifolds*.
- Coordinate systems. These are not sets of vectors (bases), but real-valued *coordinate functions*. Vector bases result as partial derivatives of these coordinate functions.
- Coordinate bases vs. non-coordinate frames.
- Change of basis.
- Dual bases and dual frames.
- Coordinate charts and atlases. Many manifolds cannot be covered by a single coordinate system (in this context called a coordinate *chart*), but must be covered by an *atlas* of multiple coordinate charts (with overlapping regions).
- Tensor transformations.
- Tangent spaces and tangent bundles.
- Vector fields.
- Lie brackets.

2.1 The Einstein Summation Convention

We first briefly summarize an important notational convention that we will use in many computations using coordinates. (In addition, in the tutorial we will also cover and use the more “modern” perspective of *coordinate-free* differential geometry, i.e., where geometric objects are used without coordinates, often enabling more geometric insight.)

We employ the Einstein summation convention (Frankel, 2011, p.59)[2], implying summation over indices occurring twice (once “upstairs” and “downstairs” each). For example, to represent a vector **v** referred to a basis $\{\mathbf{e}_i\}$, we pair[3] contravariant components v^i with the basis vectors via

$$\begin{aligned} \mathbf{v} = v^i\,\mathbf{e}_i &:= \sum_i v^i\,\mathbf{e}_i, \\ &= v^1\,\mathbf{e}_1 + v^2\,\mathbf{e}_2 + \ldots + v^n\,\mathbf{e}_n. \end{aligned} \tag{2.1}$$

Carrying out the implied summation is called a *contraction*. For an n-dimensional manifold, every paired (“dummy”) index represents a sum over all n dimensions (with n fixed and assumed known). Thus, an expression like $v^i\,\mathbf{e}_i$ above represents a sum over n terms, with $i \in \{1, \ldots, n\}$.

The actual letter (here, index i) for the index has no intrinsic meaning[4]. Indices occuring twice can be renamed freely, as long as the corresponding pairing in the summation convention stays the same. Indices that occur only once[5] can also be renamed freely; however, they also need to be renamed consistently everywhere, in particular also on both sides of an equation.

Likewise, a 1-form ω referred to a dual[6] 1-form basis $\{\omega^i\}$ expands as

$$\begin{aligned} \omega = v_i\,\omega^i &:= \sum_i v_i\,\omega^i, \\ &= v_1\,\omega^1 + v_2\,\omega^2 + \ldots + v_n\,\omega^n. \end{aligned} \tag{2.2}$$

Another example of a *tensor contraction* is applying a 1-form ω, with components v_i, to a vector **w**, with components w^i, to obtain the scalar[7]

$$\begin{aligned} \omega(\mathbf{w}) = v_i\,w^i &:= \sum_i v_i\,w^i, \\ &= v_1\,w^1 + v_2\,w^2 + \ldots + v_n\,w^n. \end{aligned} \tag{2.3}$$

In coordinate-free notation, the above contraction is simply $\omega(\mathbf{w})$ by itself, where a 1-form ω is contracted with a vector **w**, both without components.

In components, where the 1-form is expanded to $\omega = v_i\,\omega^i$, and the vector is expanded to $\mathbf{w} = w^i\,\mathbf{e}_i$, the above is the correct expression for the contraction, because[8] $\omega^i(\mathbf{e}_j) = \delta^i_j$. From this, we can derive the above as[9]

$$\begin{aligned} \omega(\mathbf{w}) &= (v_i\,\omega^i)(w^j\,\mathbf{e}_j), \\ &= v_i\,w^j\,\omega^i(\mathbf{e}_j), \\ &= v_i\,w^j\,\delta^i_j, \\ &= v_i\,w^i. \end{aligned} \tag{2.4}$$

In the last step, we have exploited that δ^i_j is 1 when $i = j$, and 0 otherwise:

$$\begin{aligned} v_i\,w^j\,\delta^i_j &:= \sum_{ij} v_i\,w^j\,\delta^i_j, \\ &= 1\cdot v_1\,w^1 + 0\cdot v_1\,w^2 + \ldots + 1\cdot v_n\,w^n, \\ &= v_1\,w^1 + v_2\,w^2 + \ldots + v_n\,w^n, \\ &= \sum_i v_i\,w^i = v_i\,w^i. \end{aligned} \tag{2.5}$$

[2] Theodore Frankel. *The Geometry of Physics: An Introduction*. Cambridge University Press, 3rd edition, 2011

[3] Paired indices are often called *dummy indices*. When a *contraction* (the summation over paired indices) is carried out, these indices “fall out” of the resulting expression. Here, the result is the geometric vector **v**, without any remaining index. However, see also the example below, where $w^j\,\delta^i_j$ is contracted to w^i: The dummy index j has fallen out; the free index i remains.

[4] This can be a great source of confusion when translating tensor expressions into matrix expressions and vice versa.

[5] These indices are often called *free indices*. There is no implied summation over free indices; e.g., the second-order tensor g_{ij}.

[6] See later for an explanation of dual bases.

[7] A 1-form is a linear function that, when applied to a vector, yields a scalar.

[8] This is the definition of a dual basis, i.e., a basis $\{\omega^i\}$ that is dual to the basis $\{\mathbf{e}_i\}$ is defined to be the $\{\omega^i\}$ that fulfills $\omega^i(\mathbf{e}_j) = \delta^i_j$, with δ^i_j the Kronecker delta that is 1 when $i = j$, and 0 otherwise.

[9] All expansions like this one here exploit the fact that tensors are *multi-linear* functions of their arguments, i.e., they are linear when all arguments except one are held fixed. For vectors and 1-forms this simply means that they are *linear* functions. This directly allows “pulling” coefficients in front of expressions, as is done here. More explicitly, this gives

$$\begin{aligned} (v_i\,\omega^i)(w^j\,\mathbf{e}_j) &= \Big(\sum_i v_i\,\omega^i\Big)\Big(\sum_j w^j\,\mathbf{e}_j\Big), \\ &= \sum_{i,j} \big((v_i\,\omega^i)(w^j\,\mathbf{e}_j)\big), \\ &= \sum_{i,j} v_i\,w^j\,\omega^i(\mathbf{e}_j) = \sum_i v_i w^i. \end{aligned}$$

That is, the summation convention works the same for multiple pairs of repeated indices. Computing the inner product between two vectors is then

$$\begin{aligned}\mathbf{g}(\mathbf{v},\mathbf{w}) = g_{ij}\,v^i\,w^j &:= \sum_{i,j} g_{ij}\,v^i\,w^j,\\ &= g_{11}\,v^1\,w^1 + g_{12}\,v^1\,w^2 + \ldots + g_{nn}\,v^n\,w^n.\end{aligned} \tag{2.6}$$

For an n-dimensional manifold, such an expression therefore automatically represents n^2 terms: Contracting k pairs of indices gives n^k terms.

Another important example is applying a linear map $\mathbf{T}$ to a vector $\mathbf{v}$, yielding another vector $\mathbf{w}$. This is equivalent to standard matrix-vector multiplication[10]. In tensor notation, this is written as

$$\begin{aligned}\mathbf{w} &= \mathbf{T}(\mathbf{v}),\\ &= \left(T^i_{\ j}\,v^j\right)\mathbf{e}_i := \sum_i \left(\sum_j T^i_{\ j}\,v^j\right)\mathbf{e}_i = w^i\,\mathbf{e}_i.\end{aligned} \tag{2.7}$$

As a last important example, the equivalent to matrix-matrix multiplication followed by matrix-vector multiplication, corresponding to the application of the composition of two linear maps to a vector, is[11]

$$\begin{aligned}\mathbf{w} &= (\mathbf{S}\circ\mathbf{T})(\mathbf{v}) = (\mathbf{S}(\mathbf{T}(\mathbf{v})),\\ &= \left(S^i_{\ k}\,T^k_{\ j}\,v^j\right)\mathbf{e}_i := \sum_i \left(\sum_{j,k} S^i_{\ k}\,T^k_{\ j}\,v^j\right)\mathbf{e}_i = w^i\,\mathbf{e}_i.\end{aligned} \tag{2.8}$$

Expressions like this one usually can be read in multiple ways. For example, here involving two simultaneous contractions in $w^i = S^i_{\ k}\,T^k_{\ j}\,v^j$, or first computing an intermediate vector $u^k = T^k_{\ j}\,v^j$ followed by $w^i = S^i_{\ k}\,u^k$. And, in fact, the geometric output vector $\mathbf{w}$ results from a third contraction ($w^i\,\mathbf{e}_i$).

2.2 Manifolds

Topological manifolds

A topological manifold M is a mathematical structure describing an n-dimensional[12] space[13], with the essential characteristic that the neighborhood[14] of every point $x \in M$, maps to a corresponding neighborhood of a point in $\mathbb{R}^n$ with a *homeomorphism*, i.e., a bijective map (one-to-one and onto) that is continuous and whose inverse is also continuous[15].

Naturally, this is possible for $\mathbb{R}^n$ itself, or any open set that is a (proper) subset of $\mathbb{R}^n$. However, more generally a manifold M is not a subset of $\mathbb{R}^n$ and is covered with multiple open subsets $U_\alpha \subset M$, where α is an index from some index set, and where the union of all subsets is the entire manifold M. Each open[16] set U_α is individually mapped to a corresponding open subset of $\mathbb{R}^n$ by the corresponding homeomorphism, which we denote by a map

$$\begin{aligned}\varphi_\alpha : M \supset U_\alpha &\to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n,\\ x &\mapsto \varphi_\alpha(x).\end{aligned} \tag{2.9}$$

An important technical property of this construction is that each point $x \in M$ mapping to $\varphi_\alpha(x) \subset \mathbb{R}^n$ in this way obtains a corresponding tuple of n-dimensional *coordinates*[17] for the point x. Roughly speaking, we can therefore say that a manifold is a topological space that has dimensionality

[10] Note, however, how the indices i, j in the tensor expression do not have any inherent meaning, corresponding to the fact that there are no "rows" or "columns" in tensor notation. Instead of the index names, for the tensor $T^i_{\ j}$ one can think about the *first* index (just here i) as rows, and the *second* index (just here j) as columns, respectively. (However, this is still partially just a (common) convention; writing $T_i^{\ j}\,v^i$ instead would be just as valid, although be slightly harder to map to matrix notation.)

[11] Again note that the indices do not have inherent meaning. Individually, the tensor components of the linear maps can be written as $T^i_{\ j}$ and $S^i_{\ j}$, respectively. However, for function composition, i.e., "matrix multiplication," some indices must be renamed and the corresponding expression is $S^i_{\ k}\,T^k_{\ j}$. (In contrast, $S^i_{\ j}\,T^i_{\ j}$ has the meaning of "component-wise matrix multiplication," the *Hadamard product*.)

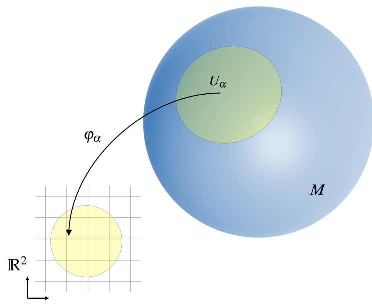


Figure 2.1: Coordinate chart on the sphere.

[12] It is important to note that the dimension n must be fixed and be the same everywhere on a given manifold.

[13] The space $\mathbb{R}^n$ is already such a manifold, but often we are interested in more complicated manifolds, such as a sphere, a torus, or a more general n-D manifold not embedded in some $\mathbb{R}^m$ with $m > n$.

[14] For completeness, we note that we only consider spaces that are *Hausdorff* ("any two distinct points have disjoint neighborhoods") and *second-countable* ("there exists a countable base for the topology"), excluding "pathological" cases.

[15] Homeomorphisms ("topological isomorphisms") are the most important functions in topology; they preserve all topological properties. In our context, this implies that the neighborhoods of $x \in M$ and of $\varphi(x) \in \mathbb{R}^n$ are *topologically the same*.

[16] The entire set $\mathbb{R}^n$, with the natural Euclidean topology (the topology induced by the Euclidean metric), is itself an *open set*. (In fact, in $\mathbb{R}^n$ it is *also* closed.)

[17] Later, we will call the pair $(U_\alpha, \varphi_\alpha)$ a coordinate chart, or simply a chart.

n everywhere and where it is possible to assign n-dimensional coordinates (a tuple of n coordinates, corresponding to a point in $\mathbb{R}^n$) in a continuous and continuously invertible way. However, this can only be done with a single map φ for very simple manifolds, and therefore, in the important general case, but even on relatively simple manifolds such as a sphere, multiple overlapping open sets U_α with corresponding maps φ_α are required[18].

A manifold with this structure is a *topological manifold*. Topological manifolds by themselves have already many important applications[19]. However, for our purposes we require calculus to be possible on manifolds. For calculus on manifolds, we additionally need a *differential structure* to be given on a topological manifold, turning it into a *differential manifold*.

[18] Because the sets U_α are open sets and we have to cover the entire manifold M, this naturally also means that there have to be overlapping regions in which individual points x will have multiple coordinates (each one coming from a different φ_α).

[19] We refer to the detailed treatment given by (Lee, 2011) :

John M. Lee. *Introduction to Topological Manifolds*. Springer-Verlag, 2nd edition, 2011

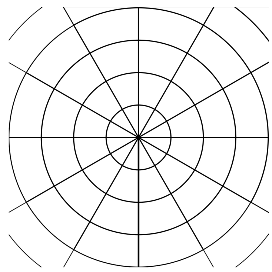


Figure 2.2: Polar coordinates are not defined everywhere in $\mathbb{R}^2$.

Differential manifolds

A topological manifold with an additional *differential structure* is a *differential manifold*, often also called a *smooth manifold*. Roughly speaking, the differential structure consists of a maximal set of *charts* $\{(U_\alpha, \varphi_\alpha)\}$ (called an *atlas*), whose union covers the entire manifold, with (partial) overlaps between charts, and where all coordinate transition functions between charts are *differentiable* in regions where multiple charts overlap.

Roughly speaking, we can also just say that a smooth manifold has a well-defined n-dimensional tangent space "attached" to every point $x \in M$ on the manifold M, where that tangent space is typically denoted by T_xM.

The full technical definition[20] is more complicated, with deep consequences in differential topology[21]. However, for our purposes this simple definition will suffice. For typical applications in our context, the differential structure simply means that the techniques of calculus are available to us on the manifold, by simply transferring them from $\mathbb{R}^n$ to M. This mainly means that we can use standard $\mathbb{R}^n$ calculus[22] "on" M, by performing computations in each coordinate chart (via simple computations in $\mathbb{R}^n$).

The differentiability of the transition functions in each chart overlap guarantees the consistency of calculus computations between charts, thus becoming intrinsic to the manifold, invariant with respect to—i.e., independent of—the subset of $\mathbb{R}^n$ that is being used.

In order to make the latter possible from a calculation perspective, calculus on manifolds usually makes use of tensor techniques, i.e., we build on methods from *tensor calculus*. These techniques have the crucial property that they result in *coordinate-independent* results, i.e., all computations agree and are independent of any particular choice of coordinate chart $(U_\alpha, \varphi_\alpha)$. We can think about this property even for the almost trivial[23] example of the manifold $M = \mathbb{R}^2$, but with *two different coordinate systems*, such as two different sets of 2D Cartesian coordinates (e.g., where one coordinate system is rotated with respect to the other one). A more illustrative example is using 2D Cartesian coordinates and polar coordinates, respectively. Naturally, we want all geometric meaning to be independent of the particular choice of coordinate system in $\mathbb{R}^2$. This is ensured in a well-defined way by the framework of tensor calculus techniques.

[20] For more complicated manifolds, such as the 7-dimensional sphere $\mathbb{S}^7$, there in fact exist multiple *different* differential structures.

[21] We refer to the detailed treatment given by (Lee, 2012) :

John M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, 2nd edition, 2012

[22] We highly recommend the concise book by (Spivak, 1965) :

Michael Spivak. *Calculus on Manifolds*. Benjamin Cummings, 1965

[23] We note, however, that things do not stay trivial for long. Even for $M = \mathbb{R}^2$, for example, one polar coordinate system is only defined for $U_\alpha = \varphi_\alpha^{-1}(\{r, \theta\})$ for all $0 < r < \infty$ and $0 < \theta < 2\pi$, with $(0,0)$ denoting the pole that is the origin of the coordinate system: At the pole, all angles θ would map to the same point, and thus there is no unique coordinate (r, θ); moreover, the angle θ has a discontinuity where $\theta = 0$ and $\theta = 2\pi$ denote the same line extending rightward from the origin. For simple calculations without calculus it sometimes does not matter much that a single point has "multiple coordinates," but for a differential manifold this is not allowed: There is no corresponding map φ_α that is a homeomorphism, because it would not be bijective. Therefore, in order to be able to include the pole and the angle discontinuity "half-line," a second overlapping chart with different pole and angle assignment is required (or another chart that does not have these constraints).

2.3 *Coordinate Systems*

In this section, we will start with the simplest case[24], where a single coordinate system maps the entire n-dimensional manifold M to $\mathbb{R}^n$.

In the general notion of a coordinate system, accommodating arbitrary coordinate systems, such as curvilinear coordinates, as well as non-Euclidean manifolds, such as a sphere, a coordinate system does not consist of n coordinate vectors[25]. Instead, for an n-dimensional manifold it consists of n *coordinate functions* $\{x^i\}$, with each

$$\begin{aligned} x^i \colon M &\to \mathbb{R}, \\ x &\mapsto x^i(x). \end{aligned} \tag{2.10}$$

[24] Below, we will extend this notion to that of *coordinate charts*, each of which maps an open subset $U \subset M$ of an n-dimensional manifold M to a subset of $\mathbb{R}^n$. Multiple (overlapping) coordinate charts then comprise a *coordinate atlas* that jointly covers the entire manifold M.

[25] In contrast, basis *vectors* live in each tangent space T_xM, at $x \in M$, and they are often obtained as *derivatives* of the coordinate functions.

This approach is crucial for non-affine (including Cartesian) coordinate systems: Coordinate vectors are only possible in linear spaces[26], but coordinate functions (as $\mathbb{R}$-valued functions x^i on a manifold M) can be defined on any manifold.

[26] Where the whole manifold M is a vector space with the origin at 0, or an affine space with an arbitrarily-chosen origin.

For example, for $n = 2$, the two coordinate functions x^1, x^2,

$$\begin{aligned} x^1 \colon M &\to \mathbb{R}, \\ x^2 \colon M &\to \mathbb{R}. \end{aligned} \tag{2.11}$$

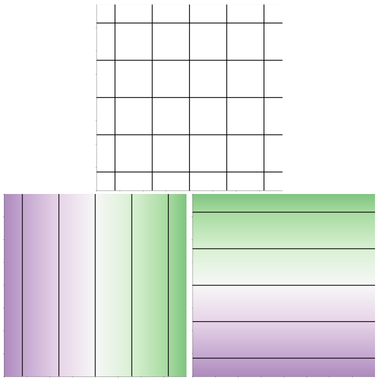


Figure 2.3: Cartesian coordinates in $\mathbb{R}^2$.

Sometimes, the coordinate functions are labeled differently to highlight the particular coordinate system used. For example, for 2D polar coordinates $(r, \theta) = \big(r(x), \theta(x)\big)$, we can define $(x^1, x^2) := (x^r, x^\theta)$, writing[27]

$$\begin{aligned} x^r \colon M &\to \mathbb{R}, \\ x^\theta \colon M &\to \mathbb{R}. \end{aligned} \tag{2.12}$$

Altogether, the n-tuple of n coordinates of any point $x \in M$, where M is n-dimensional, can be obtained via a *coordinate map*

$$\begin{aligned} \varphi \colon M &\to \mathbb{R}^n, \\ x &\mapsto \varphi(x) = \big(x^1(x), x^2(x), \ldots, x^n(x)\big). \end{aligned} \tag{2.13}$$

If desired, we can again make the particular coordinate system more explicit, for example denoting the polar coordinate map from above as

$$\begin{aligned} \varphi \colon M &\to \mathbb{R}^2, \\ x &\mapsto \varphi(x) = \big(x^r(x), x^\theta(x)\big) = \big(r(x), \theta(x)\big). \end{aligned} \tag{2.14}$$

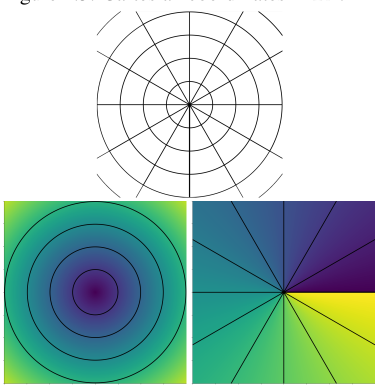


Figure 2.4: Polar coordinates in $\mathbb{R}^2$.

[27] We also could denote x^r simply by a function r, and x^θ by a function θ. Usually, the context makes clear whether a value r or a function r (as $r(x)$ with $r \colon M \to \mathbb{R}, x \mapsto r(x)$) is meant.

2.4 *Coordinate Curves*

Coordinate curves on a manifold, of a coordinate system or of the coordinate map of a given coordinate chart,[28] are simply the 1-manifold pre-images of the coordinate map, when varying one chosen coordinate while keeping all other coordinates constant. When a coordinate map $x \mapsto \varphi(x)$ is given, we can choose one coordinate x^i to vary, and define the curve

[28] For coordinate charts see Sec. 2.8.

$$t \mapsto x(t) := \varphi^{-1}\left(x^1, x^2, \ldots, t, \ldots, x^n\right) \in M. \tag{2.15}$$

Here, the curve parameter t replaces the i'th coordinate, and all other coordinates x^j, $j \neq i$, are held fixed.[29] See Fig. 2.5 for examples of coordinate curves on manifolds, i.e., examples for the 1D pre-images just defined.

[29] Which coordinate curve we get depends on the choice of constants for $\{x^j\}$, $j \neq i$.

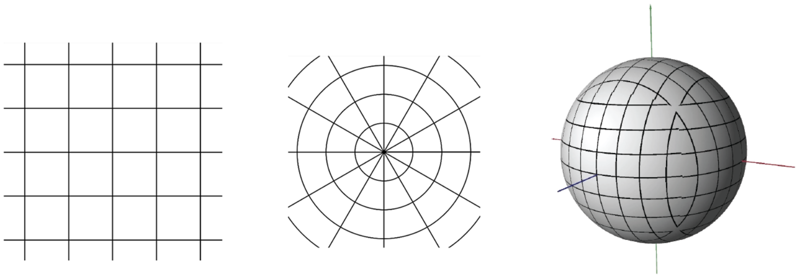


Figure 2.5: Coordinate curves of different coordinate maps, on different manifolds: (Left) Cartesian coordinate curves on $M = \mathbb{R}^2$; (Center) Polar coordinate curves on $M = \mathbb{R}^2$; (Right) Coordinate curves of multiple coordinate charts on the sphere, $M = S^2$. For charts, see Sec. 2.8.

2.5 Coordinate Bases vs. Non-Coordinate Frames

Vectors are geometric objects independent of any chosen basis. In order to refer a vector $\mathbf{v}$ to a basis, we expand it in components as $\mathbf{v} = v^i\,\mathbf{e}_i$, where $\{\mathbf{e}_i\}$ is a basis for T_xM, and, in general[30], the T_xM at different $x \in M$ have different bases. It is crucial to understand that the vectors $\mathbf{e}_i$ are themselves also seen as *geometric* vectors (e.g., as the *tangent* vectors of coordinate curves) that exist without having to refer them to another basis[31].

[30] In principle, they are *always* different bases. However, when all tangent spaces are identical copies of each other, with the manifold M having trivial parallel transport, one can say in this context all bases are "the same." (E.g., Cartesian/affine coordinates on (intrinsically) flat manifolds.)

[31] Otherwise, this would seem to be a "circular" definition.

(That said, however, *if* we know two different (geometric) bases, e.g., the tangent vectors to two different coordinate systems, we can refer one of them to the other one[32], obtaining the corresponding expansions $\tilde{\mathbf{e}}_j = \tilde{e}^i{}_j\,\mathbf{e}_i$).

[32] This is exactly what we have to do to perform a *change of basis*, see below.

A *coordinate basis* for the coordinate system $\{x^i\}$ is denoted, in each tangent space T_xM at the point $x \in M$, by

$$\mathbf{e}_i := \frac{\partial}{\partial x^i} =: \boldsymbol{\partial}_i. \tag{2.16}$$

From the coordinate functions $\{x^i\}$, we obtain *coordinate basis vectors* as

$$\boldsymbol{\partial}_i := \frac{\partial}{\partial x^i} := \frac{\partial}{\partial x^i}\varphi^{-1}\left(x^1, x^2, \ldots, x^n\right). \tag{2.17}$$

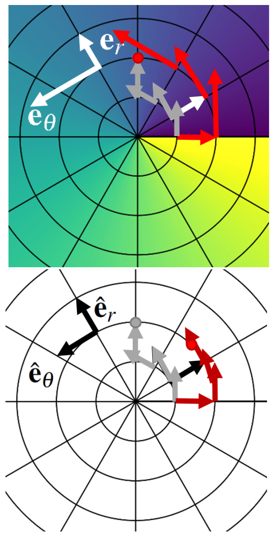


Figure 2.6: (Non-)commutativity of basis vector fields.

This definition evaluated at a single point $x \in M$ gives basis vectors $\boldsymbol{\partial}_i$ in the corresponding tangent space T_xM. Evaluating for *all* points $x \in M$ gives vectors $\boldsymbol{\partial}_i$ *in each* tangent space T_xM, thus for each i giving a corresponding *basis vector field* (see below), which we also denote by $\boldsymbol{\partial}_i$.

Each coordinate basis vector $\boldsymbol{\partial}_i$ is the *tangent vector*, at the point $x \in M$ of evaluation, to the corresponding coordinate curve on M, through the point x, along which all coordinates x^j with $j \neq i$ are *constant*. It is crucial to understand that the $\boldsymbol{\partial}_i$ are *geometric vectors* in each tangent space T_xM[33].

Coordinate basis vector fields[34] always *commute*, stated explicitly by

$$[\boldsymbol{\partial}_i, \boldsymbol{\partial}_j] = 0. \tag{2.18}$$

[33] The corresponding vectors in components $\mathbb{R}^n$, with respect to the same coordinate system, are always $\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1)\}$.

The operator $[\cdot,\cdot]$ is the *Lie bracket* (of vector fields), which is described below. It essentially measures the non-commutativity of vector fields.

In contrast, in a *non-coordinate basis* $\{\mathbf{e}_i\}$, where the $\mathbf{e}_i$ in each tangent space T_xM are arbitrary linearly-independent vectors, with $\mathbf{e}_i \neq \boldsymbol{\partial}_i$, the corresponding basis (or frame) vector fields *do not commute*. That is,

[34] See below. This commutativity, and the corresponding Lie bracket, only make sense for *vector fields*, not for individual vectors in a single tangent space.

$$[\mathbf{e}_i, \mathbf{e}_j] \neq 0. \tag{2.19}$$

Basis vector fields

It is often very important to consider that the notation $\{\mathbf{e}_i\}$ does not only correspond to a basis of n linearly-independent vectors in one tangent space T_xM, but that there is one such basis in *every* tangent space T_xM[35]. Therefore, the notation $\{\mathbf{e}_i\}$ also refers to a set of n tangent vector fields, the *basis vector fields*, where the corresponding vectors in each T_xM at each point $x \in M$ must be linearly-independent in each vector space T_xM. In each T_xM, the set of vectors $\{\mathbf{e}_i\}$ forms a vector space basis for T_xM.

[35] See below for the description of vector fields as *sections* of the *tangent bundle* TM.

Frame fields

We refer to the basis vector fields of non-coordinate bases, or non-coordinate frames, as *frame fields*. However, in general one needs to be careful what is meant by these terms and make sure that it is clear whether a basis or frame is a coordinate or a non-coordinate basis (or frame).

2.6 *Change of Basis*

Given two different bases $\{\tilde{\mathbf{e}}_i\}$ and $\{\mathbf{e}_i\}$, we can refer one of the two bases (here, $\{\tilde{\mathbf{e}}_i\}$) to the other one (here, $\{\mathbf{e}_i\}$), by expanding[36]

$$\tilde{\mathbf{e}}_j = \tilde{e}^i{}_j \, \mathbf{e}_i. \tag{2.20}$$

The $n \times n$ components $\tilde{e}^i{}_j$ are simply the n components $\tilde{e}^i{}_j$, with the second index j held fixed, for each basis vector $\tilde{\mathbf{e}}_j$, for all n basis vectors $\{\tilde{\mathbf{e}}_j\}$, with j not held fixed. We can write this schematically in "matrix form" as[37]

$$\begin{bmatrix} \tilde{\mathbf{e}}_1 \\ \tilde{\mathbf{e}}_2 \\ \vdots \\ \tilde{\mathbf{e}}_n \end{bmatrix} = \begin{bmatrix} \tilde{e}^1{}_1 & \tilde{e}^2{}_1 & \dots & \tilde{e}^n{}_1 \\ \tilde{e}^1{}_2 & \tilde{e}^2{}_2 & \dots & \tilde{e}^n{}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{e}^1{}_n & \tilde{e}^2{}_n & \dots & \tilde{e}^n{}_n \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{bmatrix}. \tag{2.21}$$

Likewise, we can expand in the other direction, as

$$\mathbf{e}_j = e^i{}_j \, \tilde{\mathbf{e}}_i. \tag{2.22}$$

Again in schematic matrix form, we can write this as

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} e^1{}_1 & e^2{}_1 & \dots & e^n{}_1 \\ e^1{}_2 & e^2{}_2 & \dots & e^n{}_2 \\ \vdots & \vdots & \ddots & \vdots \\ e^1{}_n & e^2{}_n & \dots & e^n{}_n \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{e}}_1 \\ \tilde{\mathbf{e}}_2 \\ \vdots \\ \tilde{\mathbf{e}}_n \end{bmatrix}. \tag{2.23}$$

The two matrices of components are matrix inverses[38] of each other, i.e.,

$$\left[\, e^i{}_j \,\right] = \left[\, \tilde{e}^i{}_j \,\right]^{-1}. \tag{2.24}$$

We know that we can refer an arbitrary geometric vector $\mathbf{v}$ to either basis as

$$\mathbf{v} = \tilde{v}^i \, \tilde{\mathbf{e}}_i = v^i \, \mathbf{e}_i, \tag{2.25}$$

and therefore, if we know the components v^i, inserting Eq. 2.20 into the above[39], we obtain the components $\tilde{v}^i$, with $e^i{}_j$ given by Eq. 2.24, as

[36] Note that the index j is shown offset to the right, indicating that it is the second, and not the first, index. (The first index here is i.) This will make mapping the tensor transformation laws of *components* (as opposed to the basis vectors here) to standard matrix notation easier. See below.

[37] This expression must be evaluated using the same multiplications and additions of elements as in standard matrix-vector multiplication, keeping each geometric vector $\mathbf{e}_i, \tilde{\mathbf{e}}_i$ as a single mathematical object.
Note, however, that due to the arrangement using standard matrix notation, in the use of $\tilde{e}^i{}_j$ here, the first index (i) is the column index, and the second index (j) the row index, respectively: One can think of this matrix as having been "*transposed*." However, in tensor notation, as in Eq. 2.20, this is irrelevant: "Transposition" merely results from using matrix notation. (Alternatively, the basis vectors could have been arranged as row instead of column vectors.)

[38] These are standard matrices of coefficients, i.e., $[e^i{}_j] \cdot [\tilde{e}^i{}_j] = [\tilde{e}^i{}_j] \cdot [e^i{}_j] = \mathbf{I}$, giving the identity. (It does not matter here whether all matrices are transposed or not.) In tensor notation, the equivalent expression is denoted by $e^i{}_k \, \tilde{e}^k{}_j = \tilde{e}^i{}_k \, e^k{}_j = \delta^i{}_j$.

[39] We have $\tilde{v}^i \, \tilde{\mathbf{e}}_i = \tilde{v}^j \, \tilde{e}^i{}_j \, \mathbf{e}_i = v^i \, \mathbf{e}_i$, and therefore $v^i = \tilde{e}^i{}_j \, \tilde{v}^j$, giving $\tilde{v}^i = e^i{}_j \, v^j$, because $e^i{}_k \, \tilde{e}^k{}_j = \delta^i{}_j$. (The naming of i and j has no meaning by itself: Only the matching of indices for contraction must be correct, and therefore as long as this stays the same, indices can freely be renamed.)

$$\tilde{v}^i = e^i{}_j v^j. \tag{2.26}$$

This gives the *change of basis*, from the basis $\{\mathbf{e}_i\}$ to the basis $\{\tilde{\mathbf{e}}_i\}$. We call the matrix $[e^i{}_j] = [\tilde{e}^i{}_j]^{-1}$ the *change of basis matrix*, from basis $\{\mathbf{e}_i\}$ to basis $\{\tilde{\mathbf{e}}_i\}$. In matrix form, using the known components $\tilde{e}^i{}_j$, this is[40]

$$\begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \\ \vdots \\ \tilde{v}^n \end{bmatrix} = \begin{bmatrix} \tilde{e}^1{}_1 & \tilde{e}^1{}_2 & \dots & \tilde{e}^1{}_n \\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2 & \dots & \tilde{e}^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{e}^n{}_1 & \tilde{e}^n{}_2 & \dots & \tilde{e}^n{}_n \end{bmatrix}^{-1} \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}. \tag{2.27}$$

Comparing Eqs. 2.21 and 2.27 we now see that the components v^i transform *contravariantly*, i.e., inversely to the transformation of the basis[41].

[40] Now, in contrast to above, the first index (i) is in fact the row index, and the second index (j) is the column index.

[41] Computing the analogous transformation for 1-forms (covectors), in components referred to a 1-form basis (the *dual basis*, see below), we see that they transform directly (not inversely) as the basis transforms, i.e., 1-forms transform *covariantly*. See below.

2.7 Dual Bases and Dual Frames

We will also need the concept of a *dual basis* $\{\omega^i\}$, of 1-forms ω^i, where

$$\omega^i(\mathbf{e}_j) = \delta^i{}_j. \tag{2.28}$$

The Kronecker delta $\delta^i{}_j = 1$ for $i = j$, and zero otherwise.

Each ω^i is a *covector*, or *1-form*, which is a linear function mapping a vector to a scalar[42]. The dual basis $\{\omega^i\}$ *reads off* the components v^i of a contravariant vector $\mathbf{v}$ referred to the basis $\{\mathbf{e}_i\}$, such that[43]

$$\mathbf{v} = \omega^i(\mathbf{v})\, \mathbf{e}_i = v^i\, \mathbf{e}_i. \tag{2.29}$$

[42] Such a map is also called a *functional*.

[43] This results from linearity and Eq. 2.28:

$\omega^i(\mathbf{v}) = \omega^i(v^j \mathbf{e}_j) = v^j \omega^i(\mathbf{e}_j) = v^j \delta^i{}_j = v^i$.

If the basis is a coordinate basis, i.e., $\mathbf{e}_i := \boldsymbol{\partial}_i$, we can use the corresponding coordinate 1-forms $\omega^i := dx^i$, and write

$$\begin{aligned} \mathbf{v} &= dx^i(\mathbf{v})\, \mathbf{e}_i = v^i\, \mathbf{e}_i, \\ &= dx^i(\mathbf{v})\, \boldsymbol{\partial}_i = v^i\, \boldsymbol{\partial}_i. \end{aligned} \tag{2.30}$$

An arbitrary 1-form ω can be referred to a 1-form basis $\{\omega^i\}$, giving covariant components $\{v_i\}$, such that we have

$$\omega = v_i\, \omega^i. \tag{2.31}$$

Analogous, but inverse, to Eq. 2.26, for a change of basis, and thus a corresponding change of dual basis, the covariant components $\{v_i\}$ transform to covariant components $\{\tilde{v}_i\}$ according to[44]

$$\tilde{v}_j = \tilde{e}^i{}_j v_i. \tag{2.32}$$

In matrix notation, again analogous but inverse to Eq. 2.27, this gives[45]

$$\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_n \end{bmatrix} = \begin{bmatrix} \tilde{e}^1{}_1 & \tilde{e}^2{}_1 & \dots & \tilde{e}^n{}_1 \\ \tilde{e}^1{}_2 & \tilde{e}^2{}_2 & \dots & \tilde{e}^n{}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{e}^1{}_n & \tilde{e}^2{}_n & \dots & \tilde{e}^n{}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \tag{2.33}$$

Comparing Eqs. 2.21 and 2.33, we see that the components v_i transform *covariantly*, i.e., exactly the same way as the basis $\{\mathbf{e}_i\}$ transforms.

[44] Again, an advantage of tensor index notation is that the matrix transposes sometimes required in the equivalent matrix notation are not necessary.

[45] However, again note the matrix "transposition" compared to the matrix in Eq. 2.27.

We note that in matrix notation, covectors are often written as row vectors instead of as column vectors, which then gives the equivalent[46]

$$\begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \dots & \tilde{v}_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} \tilde{e}^1_1 & \tilde{e}^1_2 & \dots & \tilde{e}^1_n \\ \tilde{e}^2_1 & \tilde{e}^2_2 & \dots & \tilde{e}^2_n \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{e}^n_1 & \tilde{e}^n_2 & \dots & \tilde{e}^n_n \end{bmatrix}. \tag{2.34}$$

[46] Note that now the matrix is *not transposed*, like the matrix in Eq. 2.27.

2.8 Coordinate Charts and Atlases

Coordinate charts

A *coordinate chart* (or, simply, a *chart*) is a pair (U, φ), where $U \subset M$ is an open subset of an n-dimensional manifold M. The *chart map* φ maps the region U to a subset of $\mathbb{R}^n$,

$$\begin{aligned} \varphi\colon M \supset U &\to \varphi(U) \subset \mathbb{R}^n, \\ x &\mapsto \varphi(x). \end{aligned} \tag{2.35}$$

To specify the chart map φ, we use n coordinate functions $\{x^i\}$,

$$\begin{aligned} x^i\colon U &\to \mathbb{R}, \\ x &\mapsto x^i(x). \end{aligned} \tag{2.36}$$

The chart map again maps each point $x \subset U$ to a tuple of n coordinates,

$$\begin{aligned} \varphi\colon U &\to \varphi(U) \subset \mathbb{R}^n, \\ x &\mapsto \varphi(x) = \left(x^1(x), x^2(x), \dots, x^n(x)\right). \end{aligned} \tag{2.37}$$

Coordinate atlases

If an atlas comprising multiple charts is used—or has to be used, as is the case even for a sphere, for example—to cover the manifold M, the individual charts described above can be labeled

$$(U_\alpha, \varphi_\alpha), \tag{2.38}$$

where the index α comes from an index set. The union of all open sets U_α is required to cover all of M. We note that, naturally, the coordinate functions $\{x^i\}$ for each chart $(U_\alpha, \varphi_\alpha)$ are usually different for each chart.

As above[47], each chart map φ_α is a map

$$\varphi_\alpha\colon M \supset U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n, \tag{2.39}$$

$$x \mapsto \varphi_\alpha(x). \tag{2.40}$$

[47] Also, what we called a *coordinate system* in our context is essentially a single chart. For example, $U_\alpha = M = \mathbb{R}^n$, with only a single possible index α. (Also note, for consistency, that $\mathbb{R}^n$ is an open set.)

See, e.g., Lee (Lee, 2012)[48].

[48] John M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, 2nd edition, 2012

Chart transitions

In the overlapping region $U_\alpha \cap U_\beta$ of two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), with $U_\alpha \cap U_\beta$ non-empty, we can define the *chart transition map* $\varphi_\beta \circ \varphi_\alpha^{-1}$, mapping the overlapping region from the chart $(U_\alpha, \varphi_\alpha)$ to the chart (U_β, φ_β), as a map

$$\begin{aligned}\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) &\to \varphi_\beta(U_\alpha \cap U_\beta), \\ \varphi_\alpha(x) &\mapsto (\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(x)) = \varphi_\beta(x).\end{aligned} \tag{2.41}$$

Written with coordinate n-tuples[49] in $\mathbb{R}^n$, this gives

$$\begin{aligned}\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) &\to \varphi_\beta(U_\alpha \cap U_\beta), \\ \left(x^1, x^2, \ldots, x^n\right) &\mapsto \left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)\left(x^1, x^2, \ldots, x^n\right) = \left(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n\right).\end{aligned} \tag{2.42}$$

Since φ_β consists of the individual coordinate functions $\{\tilde{x}^i\}$, this means[50]

$$\begin{aligned}\left(x^1, x^2, \ldots, x^n\right) &\mapsto \tilde{x}^1 = \left(\tilde{x}^1 \circ \varphi_\alpha^{-1}\right)\left(x^1, x^2, \ldots, x^n\right), \\ \left(x^1, x^2, \ldots, x^n\right) &\mapsto \tilde{x}^2 = \left(\tilde{x}^2 \circ \varphi_\alpha^{-1}\right)\left(x^1, x^2, \ldots, x^n\right), \\ &\vdots \\ \left(x^1, x^2, \ldots, x^n\right) &\mapsto \tilde{x}^n = \left(\tilde{x}^n \circ \varphi_\alpha^{-1}\right)\left(x^1, x^2, \ldots, x^n\right).\end{aligned} \tag{2.43}$$

We note that here we have used the notation $\tilde{x}^i$ both for a specific coordinate (value) $\tilde{x}^i$ as well as for the *function* $\tilde{x}^i \colon U_\beta \to \varphi_\beta(U_\beta), x \mapsto \tilde{x}^i(x)$. This simplifies the notation, and which exact meaning is meant is usually easy to see from the context. Going even further, sometimes it is convenient (and common) to also just write $\tilde{x}^i$ for the function $\tilde{x}^i \circ \varphi_\alpha^{-1}$. Then we can write[51]

$$\begin{aligned}\left(x^1, x^2, \ldots, x^n\right) &\mapsto \tilde{x}^1 = \tilde{x}^1\left(x^1, x^2, \ldots, x^n\right), \\ \left(x^1, x^2, \ldots, x^n\right) &\mapsto \tilde{x}^2 = \tilde{x}^2\left(x^1, x^2, \ldots, x^n\right), \\ &\vdots \\ \left(x^1, x^2, \ldots, x^n\right) &\mapsto \tilde{x}^n = \tilde{x}^n\left(x^1, x^2, \ldots, x^n\right).\end{aligned} \tag{2.44}$$

This means that in this context, $\tilde{x}^i$ also denotes a function

$$\tilde{x}^i \colon \mathbb{R}^n \supset \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n. \tag{2.45}$$

We can now define the $n \times n$ *Jacobian matrix*[52] of the coordinate transition

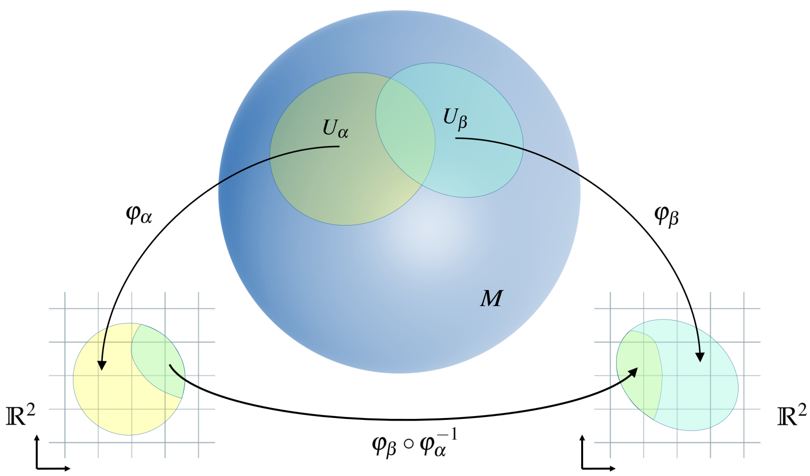


Figure 2.7: Two charts and transition map.

[49] We denote the coordinate functions comprising the map φ_α by $\{x^i\}$ and those comprising the map φ_β by $\{\tilde{x}^i\}$.

[50] For example, mapping from polar coordinates in $U_\alpha = \mathbb{R}^2 - \{(0,0)\}$ to 2D Cartesian coordinates in $U_\beta = U_\alpha$, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is the map $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, comprising the individual functions $(r,\theta) \mapsto r\cos\theta, (r,\theta) \mapsto r\sin\theta$.

[51] This is really quite a misuse of notation, but it is very convenient and usually not too ambiguous. In particular, it makes denoting and computing derivatives very convenient.

[52] As a matrix $[J^i{}_j]$; here with row index i and column index j. However, that the actual letter for the index does not have any intrinsic meaning: Indices can be renamed freely in any expression, as long as the pairing of indices stays the same. Even more importantly, in tensor notation there is no concept of "rows" and "columns": Indices are paired automatically through the Einstein summation convention. However, to make these expressions less error-prone to map to matrix equivalents, we can use the convention that the *first* index will correspond to the row index, and the *second* index will correspond to the column index. For mixed tensors, we indicate the index order by writing $J^i{}_j$ instead of J^i_j, where the second index is visibly offset to the right (and so forth, for more indices).

$$J^i{}_j := \frac{\partial \tilde{x}^i}{\partial x^j} := \frac{\partial(\tilde{x}^i \circ \varphi_\alpha^{-1})}{\partial x^j}\left(x^1, x^2, \ldots, x^n\right). \tag{2.46}$$

The inverse (see below) Jacobian matrix maps in the opposite direction,

$$\tilde{J}^i{}_j := \frac{\partial x^i}{\partial \tilde{x}^j} := \frac{\partial(x^i \circ \varphi_\beta^{-1})}{\partial \tilde{x}^j}\left(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n\right). \tag{2.47}$$

These two Jacobian matrices are in fact *matrix inverses* of one another, as[53]

$$J^i{}_k \tilde{J}^k{}_j = \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \tilde{x}^j} = \delta^i_j, \quad \text{and therefore,} \quad \left[J^i{}_j\right] = \left[\tilde{J}^i{}_j\right]^{-1}. \tag{2.48}$$

[53] This is easy to see even without computation: The coordinates are *defined* such that along each fixed coordinate curve, all other coordinates are constant.

2.9 Tensor Transformations

Given a chart transition as described above, and the corresponding Jacobian matrices for the coordinate transformations for contravariant and covariant vectors[54], respectively, we obtain the following tensor transformation rules, corresponding to the *change of coordinates* induced by the chart transition.

[54] The complete transformation laws for tensors of arbitrary order and variance will be described later. However, we note that they are the straightforward extension of the transformation laws given here, using the exact same Jacobian matrices.

Contravariant vectors

A contravariant vector, given in components v^i with respect to a coordinate basis $\{\boldsymbol{\partial}_i := \frac{\partial}{\partial x^i}\}$, transforms to components $\tilde{v}^i$ with respect to a coordinate basis $\{\tilde{\boldsymbol{\partial}}_i := \frac{\partial}{\partial \tilde{x}^i}\}$, according to the *contravariant* transformation rule[55]

$$\tilde{v}^i = J^i{}_j v^j = \frac{\partial \tilde{x}^i}{\partial x^j} v^j. \tag{2.49}$$

[55] This results from the fact that $\{\boldsymbol{\partial}_i\}$ transforms with $\tilde{J}^i{}_j$, i.e., $\tilde{\boldsymbol{\partial}}_j = \tilde{J}^i{}_j \boldsymbol{\partial}_i$, and because the vector $\mathbf{v}$ should be kept the same, the components v^i must transform with the inverse matrix, which is $J^i{}_j$. I.e.,

$$\begin{aligned}\mathbf{v} = v^i \boldsymbol{\partial}_i &= v^j \delta^i_j \boldsymbol{\partial}_i, \\ &= v^j (\tilde{J}^i{}_k J^k{}_j) \boldsymbol{\partial}_i, \\ &= (v^j J^i{}_j)(\tilde{J}^k_i \boldsymbol{\partial}_k) = \tilde{v}^i \tilde{\boldsymbol{\partial}}_i.\end{aligned}$$

In fact, the Jacobian matrix $J^i{}_j$ used here is the same as the (more general) inverse matrix $[\tilde{e}^i{}_j]^{-1}$ from Eq. 2.27, if the basis $\{\mathbf{e}_i\} = \{\boldsymbol{\partial}_i\}$. That is, $J^i{}_j = e^i{}_j$, and $[e^i{}_j] = [\tilde{e}^i{}_j]^{-1}$. However, the definition of the Jacobian matrix $J^i{}_j$, and the corresponding tensor transformation, is only valid for the transformation from one *coordinate basis* to another *coordinate basis*.

For *non-coordinate bases*, the tensor transformation law for a contravariant vector is the general transformation[56] given by Eqs. 2.26 and 2.27.

[56] Naturally, the more general approach is of course also correct for coordinate bases.

Covariant vectors

A covariant vector, given in components v_i, with respect to a dual coordinate basis $\{dx^i\}$, transforms to components $\tilde{v}_i$, with respect to a dual coordinate basis $\{d\tilde{x}^i\}$, according to the *covariant* transformation rule[57]

$$\tilde{v}_j = \tilde{J}^i{}_j v_i = \frac{\partial x^i}{\partial \tilde{x}^j} v_i. \tag{2.50}$$

[57] Analogously, but inversely, to above,

$$\begin{aligned}\omega = v_i\, dx^i &= v_i (\tilde{J}^i{}_j J^j{}_k)\, dx^k, \\ &= (v_i \tilde{J}^i{}_j)(J^j{}_k\, dx^k) = \tilde{v}_i\, d\tilde{x}^i.\end{aligned}$$

Here, we have used that the dual basis transforms as $d\tilde{x}^i = J^i{}_j\, dx^j$, resulting from

$$\begin{aligned}d\tilde{x}^i(\tilde{\boldsymbol{\partial}}_j) &= (J^i{}_k\, dx^k)(\tilde{\boldsymbol{\partial}}_j), \\ &= (J^i{}_k\, dx^k)(\tilde{J}^l{}_j \boldsymbol{\partial}_l), \\ &= J^i{}_k \tilde{J}^l{}_j\, dx^k(\boldsymbol{\partial}_l), \\ &= J^i{}_k \tilde{J}^l{}_j \delta^k_l = J^i{}_k \tilde{J}^k_j = \delta^i_j.\end{aligned}$$

Analogously (but inversely) to the contravariant transformation rule given above, the Jacobian matrix $\tilde{J}^i{}_j = \tilde{e}^i{}_j$, with $\tilde{e}^i{}_j$ as in Eq. 2.32, if the basis $\{\omega^i\} = \{dx^i\}$. Here, matrix notation must be used with care: The matrix in Eq. 2.34 is indeed $[\tilde{J}^i{}_j]$. However, the matrix in Eq. 2.33 is $[\tilde{J}^i{}_j]^T$. Again, our definition of the Jacobian matrix $\tilde{J}^i{}_j$ of partial derivatives, and the corresponding tensor transformation, is only defined for the transformation from one *dual coordinate basis* to another *dual coordinate basis*.

For *non-coordinate dual bases*, the tensor transformation law for a covariant vector is the transformation[58] given by Eqs. 2.32, 2.33, and 2.34.

[58] Again, the more general approach is of course also correct for coordinate bases.

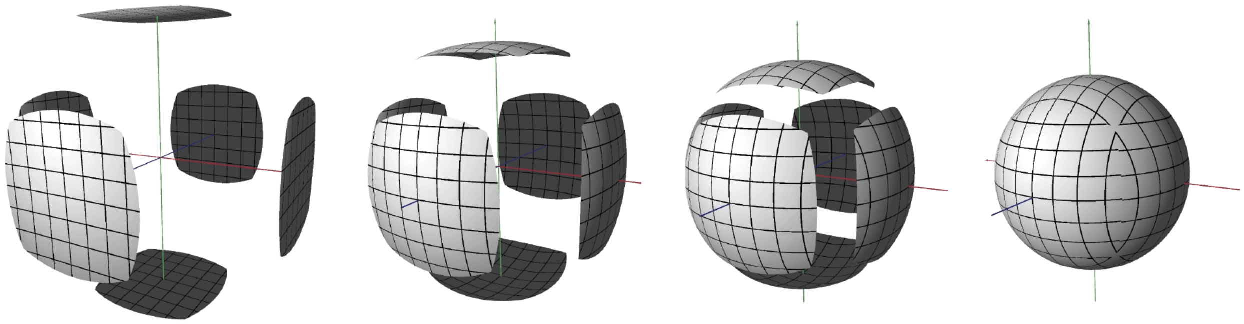


Figure 2.9: Charts projected to the sphere, with coordinate curves (coordinate isolines) shown. We note that the discontinuities in the coordinate curves between charts, which are visible here, are in fact removed smoothly by the coordinate transition maps in chart overlap regions (not shown here).

2.10 *Example: Charts on the Sphere*

As a full example, in this section we will derive six charts covering a sphere via simple orthogonal projection for each chart, i.e., we project a subset of the "planar" coordinate space $\mathbb{R}^2$ onto the sphere to obtain coordinates on the sphere. To simplify the discussion, we will denote the region of $\mathbb{R}^2$ that corresponds to the coordinates used in each chart by $\bar{U}$, defining

$$\bar{U} := \varphi(U) \subset \mathbb{R}^2. \tag{2.51}$$

That is, $\bar{U}$ is the region in "coordinate space" for the chart (U,φ). Furthermore, we will denote the two corresponding coordinate functions x^1, x^2 by u and v instead, i.e., we will use $u := x^1$ and $v := x^2$, with

$$\begin{aligned} u &\colon U \to \mathbb{R}, \\ v &\colon U \to \mathbb{R}. \end{aligned} \tag{2.52}$$

In a typical implementation, a (u,v) coordinate can then be taken to correspond directly to 2D texture coordinates, which can be used, for example, for LIC (Line Integral Convolution) computations.

Figure 2.8: Six charts on the sphere.

We now explicitly derive the coordinate bases and their derivatives of charts (U,φ) on a sphere mapping to the coordinate region $\bar{U} \subset \mathbb{R}^2$, corresponding to orthogonal projection of the region $\bar{U}$ onto a hemisphere of some arbitrary radius r. See Figs. 2.8 and 2.9. From these, we can derive the metric tensor and the Christoffel symbols referred to the chart, given analytically for any position referenced by (u,v) coordinates.

Furthermore, although we use six charts to cover the sphere, below we mainly derive a single chart, because all other charts are completely analogous. Even more simple, the metric as well as the Christoffel symbols that we derive for one chart are *identical* for all other charts, due to symmetry.

We emphasize that, although below we perform some derivations in the ambient space $\mathbb{R}^3$, the resulting metric (Eq. 2.71) and the Christoffel symbols (Eq. 2.77) are completely *intrinsic*, i.e., independent of the immersion in $\mathbb{R}^3$, and correspondingly are 2D quantities. Using the Christoffel symbols, we can compute the covariant derivative of any vector field $\mathbf{v}$ (Eq. 2.81, Eq. 6.8) in a completely intrinsic manner.

Due to projection, each region $\bar{U} \subset \mathbb{R}^2$ is limited to an (open) disk of radius r in the (u,v) plane, i.e.,

$$\bar{U} = \{(u,v) \colon u^2 + v^2 < r^2\}. \tag{2.53}$$

We describe the map from $\bar{U} \subset \mathbb{R}^2$ (intrinsic view; corresponding to U on the manifold M) to ambient $\mathbb{R}^3$ (extrinsic view) via the *inclusion map*[59]

$$\begin{aligned} \iota^{x\perp y} \colon \bar{U} \subset \mathbb{R}^2 &\hookrightarrow \mathbb{R}^3, \\ (u,v) &\mapsto (u,v,\bar{w}). \end{aligned} \tag{2.54}$$

[59] We map the disk in $\mathbb{R}^2$ to the corresponding hemisphere in $\mathbb{R}^3$.

We define the third component $\bar{w}$ to be[60]

$$\bar{w} := \sqrt{r^2 - u^2 - v^2}. \tag{2.55}$$

[60] This is simply the third component of a vector pointing to a point on a hemisphere of radius r, in Cartesian 3D coordinates $(u,v,\bar{w})$. (Located in the 3D center of the hemisphere.)

This chart is defined via projection onto the hemisphere on the x,y plane, denoted by $x \perp y$. The entire sphere is covered by six analogous charts. In total, we define the six orthogonally projected charts

$$\begin{aligned} \iota^{x\perp y} &\colon (u,v) \mapsto (u,v,\bar{w}), & \iota^{-x\perp y} &\colon (u,v) \mapsto (-u,v,-\bar{w}), \\ \iota^{z\perp y} &\colon (u,v) \mapsto (-\bar{w},v,u), & \iota^{-z\perp y} &\colon (u,v) \mapsto (\bar{w},v,-u), \\ \iota^{x\perp -z} &\colon (u,v) \mapsto (u,\bar{w},-v), & \iota^{x\perp z} &\colon (u,v) \mapsto (u,-\bar{w},v). \end{aligned} \tag{2.56}$$

To avoid too severe distortions, apart from overlaps to facilitate transitions between neighboring charts, each chart is only used where (see Fig. 2.10)

$$\begin{aligned} u^2 &\le \bar{w}^2, \quad \textit{and} \\ v^2 &\le \bar{w}^2. \end{aligned} \tag{2.57}$$

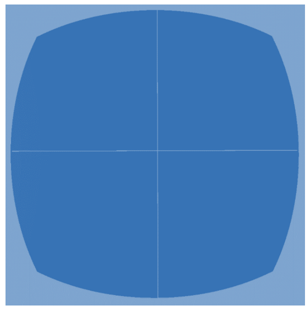


Figure 2.10: Used region (dark blue) in orthogonally projected chart on the sphere.

Outside this region, another chart will be used.

We now consider the basis vectors

$$\mathbf{e}_i = \boldsymbol{\partial}_i = \frac{\partial}{\partial x^i}, \quad i \in \{1,2\}, \tag{2.58}$$

denoting coordinate functions $x^1, x^2 := u, v$. In the chart, $\mathbf{e}_1, \mathbf{e}_2$ are by definition given by components $(1,0)$, $(0,1)$, respectively.

In ambient space $\mathbb{R}^3$, however, for the chart $x \perp y$, they map to the partial derivatives of Eq. 2.54, i.e.,

$$\tilde{\mathbf{e}}_1\Big|_{(u,v)} = \begin{pmatrix} 1 \\ 0 \\ -u/\bar{w} \end{pmatrix}, \quad \tilde{\mathbf{e}}_2\Big|_{(u,v)} = \begin{pmatrix} 0 \\ 1 \\ -v/\bar{w} \end{pmatrix}. \tag{2.59}$$

These components are referred to Cartesian coordinates in $\mathbb{R}^3$. We will now also use the shorthand notations

$$\begin{aligned} a^2 &:= r^2 - u^2, \\ b^2 &:= r^2 - v^2. \end{aligned} \tag{2.60}$$

The dual basis ω^i, with $\omega^i(\mathbf{e}_j) = \delta^i_{\ j}$, mapped to ambient space $\mathbb{R}^3$, is

$$\tilde{\omega}^1\Big|_{(u,v)} = \frac{1}{r^2}\begin{pmatrix} a^2 \\ -u\,v \\ -u\,\bar{w} \end{pmatrix}, \quad \tilde{\omega}^2\Big|_{(u,v)} = \frac{1}{r^2}\begin{pmatrix} -u\,v \\ b^2 \\ -v\,\bar{w} \end{pmatrix}. \tag{2.61}$$

In order to be able to directly use Eq. 2.76 below, these two dual basis vectors $\tilde{\omega}^1$ and $\tilde{\omega}^2$ were computed such that they correspond to *orthogonal projection* from the ambient space $\mathbb{R}^3$ into the tangent plane of the immersion of M into $\mathbb{R}^3$. An easy way to do this is to compute an orthogonal third extrinsic basis vector $\tilde{\mathbf{e}}_3 := \tilde{\mathbf{e}}_1 \times \tilde{\mathbf{e}}_2$, and compute the extrinsic dual basis by inverting the 3×3 matrix with columns $\{\tilde{\mathbf{e}}_i\}$ to get $\{\tilde{\omega}^i\}$.

The basis vector $\tilde{\mathbf{e}}_3$, and its corresponding dual $\tilde{\omega}^3$, are

$$\tilde{\mathbf{e}}_3\Big|_{(u,v)} = \begin{pmatrix} u/\bar{w} \\ v/\bar{w} \\ 1 \end{pmatrix}, \quad \tilde{\omega}^3\Big|_{(u,v)} = \frac{1}{r^2}\begin{pmatrix} u\bar{w} \\ v\bar{w} \\ \bar{w}^2 \end{pmatrix}. \tag{2.62}$$

2.11 *Tangent Spaces and Tangent Bundles*

The tangent bundle TM, over a base manifold M, is the disjoint union of all tangent spaces T_xM for all points $x \in M$. See Fig. 2.11.

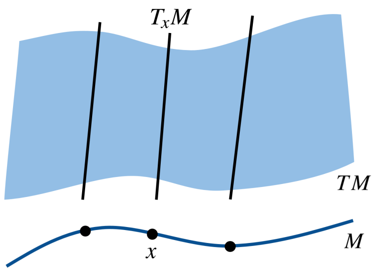


Figure 2.11: Tangent bundle over base manifold M.

Formally, a tangent bundle is often denoted by

$$\pi\colon TM \to M. \tag{2.63}$$

Here, the manifold M is called the *base manifold*, and the map π is the *bundle projection map* that takes a point (vector) on the bundle to the corresponding point on the base manifold, such that every point (vector) in the tangent space T_xM (which is a subset of the bundle manifold) is taken to the corresponding point $x \in M$ on the base manifold. Given the definition of a vector field as a section of the tangent bundle (see below), for any vector field $\mathbf{v}\colon M \to TM$ we correspondingly have

$$\pi(\mathbf{v}(x)) = x, \quad \text{at all points } x. \tag{2.64}$$

The tangent bundle construction is very important to be able to see (and work with) vector fields as geometric entities independent of coordinates. For example, in a region $U \subset M$, there is always a *local trivialization*

$$TU \cong U \times \mathbb{R}^n. \tag{2.65}$$

Such a local trivialization is a local isomorphism that can be determined by choosing a set of *basis vector fields* in the corresponding region U,[61] or, more precisely, choosing n linearly independent basis vector fields

[61] And, of course, we also choose a coordinate chart for the region U.

$$\mathbf{e}_i\colon M \supset U \to TU \subset TM. \tag{2.66}$$

Because every vector field $\mathbf{v}$ can then locally be given in n components $v^i(x)$ at every point $x \in U$, altogether giving an n-tuple of coordinates in $\mathbb{R}^n$ for each vector at each point $x \in U$, referred to the corresponding basis in the corresponding tangent space T_xM with $x \in U$, i.e., where we have

$$\mathbf{v}(x) = v^i(x)\,\mathbf{e}_i(x), \tag{2.67}$$

the choice of basis fields determines a specific isomorphism[62] between a region of the tangent bundle and the Cartesian product of U and $\mathbb{R}^n$.

[62] Again, we of course also have a coordinate chart. To specify the whole isomorphism from a point in TU to the corresponding point in $U \times \mathbb{R}^n$, we therefore need $2n$ numbers. This also shows that the manifold TU (and TM) is $2n$-dimensional.

However, it is a crucial fact that for many manifolds (such as a sphere) giving such a trivialization is not possible globally, and therefore a vector field cannot simply be defined as a section of $U \times \mathbb{R}^n$. If a global trivialization is indeed possible, the manifold M is called *parallelizable*. On the sphere, for example, where one knows that no global set of basis vector fields exists, this immediately shows that the sphere is not parallelizable.

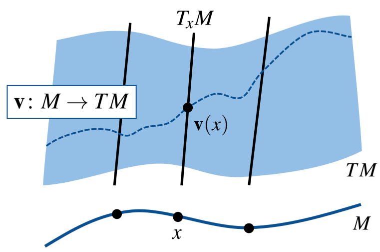


Figure 2.12: Vector fields are sections of the tangent bundle.

This leads to the fact that a vector field as a "set of components" per point is, in general, not a global construction. However, a vector field as a section of the tangent bundle (see below) is a global (coordinate-independent) construction. The use of multiple overlapping coordinate charts, and the corresponding local basis vector fields, corresponds to using multiple different local trivializations, exactly corresponding to the (local) expansion of vector fields referred to the local set of basis vector fields.

Tensor bundles

The same construction as for the tangent bundle can be used for tensor fields of arbitrary order and type (variance), for example the *tensor bundle* for $\binom{0}{2}$ tensor fields, where a specific section is the metric on the manifold M. Another example is the bundle for $\binom{0}{1}$ tensor fields, which is usually called the *cotangent bundle*.

2.12 Vector Fields

A smooth vector field $\mathbf{v}$ on a manifold M is a smooth function giving a vector $\mathbf{v}(x)$ at every point $x \in M$ on the manifold. We write this as

$$\begin{aligned} \mathbf{v}\colon M &\to TM, \\ x &\mapsto \mathbf{v}(x). \end{aligned} \tag{2.68}$$

TM refers to the *tangent bundle* of M, the manifold of all tangent spaces of M, and a vector field is also referred to as a *section* of TM. That is, a vector field, as a section of the tangent bundle, is a smooth assignment of one element $\mathbf{v}(x)$ of the tangent bundle per base point $x \in M$, such that we have $\pi(\mathbf{v}(x)) = x$, meaning that the vector $\mathbf{v}(x)$ is a member of the tangent space T_xM.

Where no confusion arises (between vector fields as functions $\mathbf{v}$ and individual vectors $\mathbf{v}(x)$, an individual vector $\mathbf{v}(x)$ is often also simply denoted by $\mathbf{v}$. However, one needs to keep in mind whether the current context is referring to whole vector fields or just to individual vectors.

2.13 Lie Brackets

The Lie bracket (of vector fields) is a map

$$\begin{aligned} [\cdot,\cdot]\colon \mathscr{X}(M) \times \mathscr{X}(M) &\to \mathscr{X}(M), \\ (\mathbf{v},\mathbf{w}) &\mapsto [\mathbf{v},\mathbf{w}]. \end{aligned} \tag{2.69}$$

Here, $\mathscr{X}(M)$ denotes the space of smooth vector fields on the manifold M. That is, the Lie bracket maps a pair of vector fields on M to another vector field M.

In Lie theory, the space $\mathscr{X}(M)$ constitutes a Lie algebra comprised of the vector space $\mathscr{X}(M)$ together with a multiplication operation given by the Lie bracket.

2.14 Example: Metric on the Sphere

For each of the charts on the sphere given in Sec. 2.10, the components of the metric tensor $\mathbf{g}$ can be referred to components given by

$$g_{ij} = \tilde{\mathbf{e}}_i \cdot \tilde{\mathbf{e}}_j. \tag{2.70}$$

Here, $\tilde{\mathbf{e}}_i$ denotes the basis vectors $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$, because $n = 2$. However, here, as indicated by the tilde, we are referring to 3D vectors in 3D tangent spaces $T_x\mathbb{R}^3$ of the *ambient space* $\mathbb{R}^3$, instead of to 2D vectors in the intrinsic tangent spaces $T_x\mathbb{S}^2$ of the sphere as a two-manifold. Correspondingly, the $\cdot$ in Eq. 2.70 denotes the usual Euclidean dot product in $\mathbb{R}^3$, or, more specifically, the standard dot product in each tangent space $T_x\mathbb{R}^3$.

For our charts (Sec. 2.10), the metric $\mathbf{g}$ (in components g_{ij}) at a point x identified by coordinates $(u,v) \in \bar{U} \subset \mathbb{R}^2$ in the chart, and its inverse $\mathbf{g}^{-1}$ (in components g^{ij}), are then given by[63]

$$\begin{aligned} g_{ij}\Big|_{(u,v)} &= \frac{1}{\bar{w}^2}\begin{bmatrix} b^2 & uv \\ uv & a^2 \end{bmatrix}, \\ g^{ij}\Big|_{(u,v)} &= \frac{1}{r^2}\begin{bmatrix} a^2 & -uv \\ -uv & b^2 \end{bmatrix}. \end{aligned} \tag{2.71}$$

[63] As usual, this means $[g^{ij}] = [g_{ij}]^{-1}$ as matrix inverses, and in tensor notation we have $g^{ik}g_{kj} = \delta^i{}_j$, and $g_{ik}g^{kj} = \delta_i{}^j$.

Here, we have again used the shorthand notations[64]

$$\begin{aligned} a^2 &:= r^2 - u^2, \\ b^2 &:= r^2 - v^2, \\ \bar{w}^2 &:= r^2 - u^2 - v^2. \end{aligned} \tag{2.72}$$

[64] By r we denote the radius of the sphere, and (u,v) are the coordinates of a point x in the chart, with $u^2 + v^2 < r^2$.

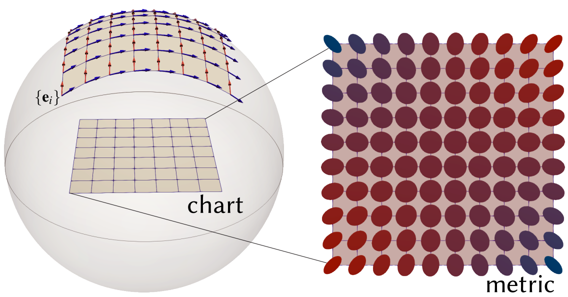


Figure 2.15: Metric in a chart on the sphere. We describe everything intrinsically in 2D coordinate charts. At each coordinate (u,v) in a region $\bar{U} \subset \mathbb{R}^2$, corresponding to the open set U on the sphere, with $\bar{U} = \varphi(U)$, we know the corresponding metric tensor (glyph visualization on the right) in components g_{ij}.

Transformation rule

Since the metric $\mathbf{g}$ is a covariant second-order tensor, its components g_{ij} transform according to

$$\tilde{g}_{ij} = \tilde{J}^k{}_i \tilde{J}^l{}_j g_{kl} = \frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j} g_{kl}. \tag{2.73}$$

In case of a non-coordinate basis, the more general rule is[65]

$$\tilde{g}_{ij} = \tilde{e}^k{}_i \tilde{e}^l{}_j g_{kl}. \tag{2.74}$$

[65] In matrix notation, the equivalent expression is $[\tilde{g}_{ij}] = [\tilde{e}^i{}_j]^T [g_{ij}] [\tilde{e}^i{}_j]$. We note that this is *not* the same transformation rule as for a linear map between vectors, i.e., it does not apply the transformation $[\tilde{e}^i{}_j]^{-1}$. This is a crucial difference, except when $[\tilde{e}^i{}_j]$ is an orthogonal matrix. See below.

However, due to the covariant nature of the metric tensor, and the corresponding covariant transformation rule, *this transformation does not* *correspond to the definition of similar matrices*. Two matrices $\mathbf{A}$ and $\mathbf{B}$ are similar if there exists an invertible matrix $\mathbf{P}$, such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}$. Among other properties, similar matrices always have the same eigenvalues. However, in general, this property does not hold for the transformation given above[66]. For this reason, *the metric tensor in general, as a matrix, does not have the same eigenvalues in different coordinate systems*[67].

[66] In matrix notation, the transformation of a covariant second-order tensor corresponds to $\mathbf{B} = \mathbf{P}^T\mathbf{A}\,\mathbf{P}$ instead of $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}$. The two are only equal for *orthogonal* change-of-basis matrices $\mathbf{P}$.

[67] An exception is when the transformation is simply rotating the basis. Only then is $\mathbf{P}$ orthogonal, and therefore $\mathbf{P}^{-1} = \mathbf{P}^T$. Then, the two matrices are in fact similar and therefore do have the same eigenvalues.

2.15 Example: Christoffel Symbols on the Sphere

One simple way to derive the Christoffel symbols $\Gamma^i{}_{jk}$, in a specific chart such as that described in Sec. 2.10, is to make use of the immersion[68] of the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. To do this, we first compute the partial derivatives of the basis vectors in ambient $\mathbb{R}^3$, in the directions $x^1, x^2 := u, v$, evaluated at the coordinate $(u,v) \in \bar{U} \subset \mathbb{R}^2$, as

$$\begin{aligned} \partial_1 \tilde{\mathbf{e}}_1\Big|_{(u,v)} &= -\frac{1}{\bar{w}^3}\begin{pmatrix}0\\0\\b^2\end{pmatrix}, & \partial_1 \tilde{\mathbf{e}}_2\Big|_{(u,v)} &= -\frac{1}{\bar{w}^3}\begin{pmatrix}0\\0\\uv\end{pmatrix}, \\ \partial_2 \tilde{\mathbf{e}}_1\Big|_{(u,v)} &= -\frac{1}{\bar{w}^3}\begin{pmatrix}0\\0\\uv\end{pmatrix}, & \partial_2 \tilde{\mathbf{e}}_2\Big|_{(u,v)} &= -\frac{1}{\bar{w}^3}\begin{pmatrix}0\\0\\a^2\end{pmatrix}. \end{aligned} \tag{2.75}$$

[68] Of course, *if* such an immersion (or an embedding) is given. If none is given or known, the more general, fully intrinsic approach must be used. (We note that for this computation, it does not matter if an embedding or "only" an immersion (with potential self-intersections in the ambient space) is known, since the whole computation is done locally.)

Now, from these basis vector field partial derivatives $\partial_j \tilde{\mathbf{e}}_i$, reading off components in the tangent plane with the dual basis gives

$$\Gamma^i{}_{jk} = \tilde{\omega}^i\big(\partial_j \tilde{\mathbf{e}}_k\big), \qquad \text{for} \quad i,j,k \in \{1,2\}. \tag{2.76}$$

Due to the way in which we have computed the dual basis $\{\tilde{\omega}^1, \tilde{\omega}^2\}$, this is equivalent to a completely intrinsic computation from the metric using Eq. 6.10, but easier to compute. We emphasize that using this extrinsic "shortcut" computation does not in any way change the fact that afterwards we can perform all computations requiring Christoffel symbols, i.e., covariant derivatives, in a fully intrinsic manner.

The Christoffel symbols that we need, given with respect to $(u,v) \in \bar{U} \subset \mathbb{R}^2$, are (only six are unique, because $\Gamma^1_{12} = \Gamma^1_{21}$, $\Gamma^2_{12} = \Gamma^2_{21}$),

$$\begin{aligned} \Gamma^1_{11}\Big|_{(u,v)} &= c\,u\,b^2, & \Gamma^1_{21}\Big|_{(u,v)} &= c\,u^2\,v, \\ \Gamma^1_{12}\Big|_{(u,v)} &= c\,u^2\,v, & \Gamma^1_{22}\Big|_{(u,v)} &= c\,u\,a^2, \\ \Gamma^2_{11}\Big|_{(u,v)} &= c\,v\,b^2, & \Gamma^2_{21}\Big|_{(u,v)} &= c\,u\,v^2, \\ \Gamma^2_{12}\Big|_{(u,v)} &= c\,u\,v^2, & \Gamma^2_{22}\Big|_{(u,v)} &= c\,v\,a^2. \end{aligned} \tag{2.77}$$

Here, we have used the shorthand notations

$$\begin{aligned} a^2 &:= r^2 - u^2, \\ b^2 &:= r^2 - v^2, \\ \bar{w}^2 &:= r^2 - u^2 - v^2, \\ c &:= \frac{1}{r^2\bar{w}^2}. \end{aligned} \tag{2.78}$$

One can verify that with these Christoffel symbols we now have, extrinsically in $\mathbb{R}^3$,

$$\nabla_{\tilde{\mathbf{e}}_j} \tilde{\mathbf{e}}_k = \Gamma^i{}_{jk} \tilde{\mathbf{e}}_i, \quad \text{for} \quad i,j,k \in \{1,2\}, \tag{2.79}$$

where $\nabla_{\tilde{\mathbf{e}}_j} \tilde{\mathbf{e}}_k$ always lies in the tangent plane at the point corresponding to (u,v). However, most importantly, we now never need to refer to the ambient space $\mathbb{R}^3$ again, and can compute everything intrinsically in the chart, with the same values for the Christoffel symbols $\Gamma^i{}_{jk}$, giving

$$\nabla_{\mathbf{e}_j} \mathbf{e}_k = \Gamma^i{}_{jk} \mathbf{e}_i, \quad \text{for} \quad i,j,k \in \{1,2\}. \tag{2.80}$$

Because the covariant derivative is linear in each of its arguments, Eq. 2.80 determines Eq. 6.8 for the covariant derivative $\nabla \mathbf{v}$ of any vector field $\mathbf{v}$. In a 2D chart, we can thus expand Eq. 6.8 as the matrix

$$\begin{bmatrix} \nabla_1 v^1 & \nabla_2 v^1 \\ \nabla_1 v^2 & \nabla_2 v^2 \end{bmatrix} = \begin{bmatrix} \partial_1 v^1 + \Gamma^1{}_{11} v^1 + \Gamma^1{}_{12} v^2 & \partial_2 v^1 + \Gamma^1{}_{21} v^1 + \Gamma^1{}_{22} v^2 \\ \partial_1 v^2 + \Gamma^2{}_{11} v^1 + \Gamma^2{}_{12} v^2 & \partial_2 v^2 + \Gamma^2{}_{21} v^1 + \Gamma^2{}_{22} v^2 \end{bmatrix}. \tag{2.81}$$

Evaluating $\nabla_{\mathbf{x}} \mathbf{v} = \nabla \mathbf{v}(\mathbf{x})$ (Eq. 6.9) in the chart thus becomes a matrix-vector multiply of the matrix $\nabla_j v^i$, times the vector components x^j:

$$\begin{bmatrix} (\nabla_{\mathbf{x}} \mathbf{v})^1 \\ (\nabla_{\mathbf{x}} \mathbf{v})^2 \end{bmatrix} = \begin{bmatrix} \nabla_1 v^1 & \nabla_2 v^1 \\ \nabla_1 v^2 & \nabla_2 v^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}. \tag{2.82}$$

Writing the basis $\{\mathbf{e}_i\}$ explicitly, this, as usual, means

$$\nabla_{\mathbf{x}} \mathbf{v} = [\mathbf{e}_1 \; \mathbf{e}_2] \begin{bmatrix} (\nabla_{\mathbf{x}} \mathbf{v})^1 \\ (\nabla_{\mathbf{x}} \mathbf{v})^2 \end{bmatrix} = (\nabla_{\mathbf{x}} \mathbf{v})^1 \mathbf{e}_1 + (\nabla_{\mathbf{x}} \mathbf{v})^2 \mathbf{e}_2. \tag{2.83}$$

All six charts for the sphere

Due to the symmetry of all charts, the metric components (Eq. 2.71) and the Christoffel symbols (Eq. 2.77) are *the same* in all charts, although above we have derived them only for the chart $x \perp y$.

2.16 *Numerical Computation of Partial Derivatives in Charts*

We again work with coordinates in $\bar{U} \subset \mathbb{R}^2$. Each $\bar{U}$ is triangulated, with mesh vertices $\{x_k\}$ at 2D coordinates $\big(u(x_k), v(x_k)\big) = (u_k, v_k) \in \mathbb{R}^2$. To compute the partial derivatives $\partial_1 v^i$ and $\partial_2 v^i$ of an $\mathbb{R}$-valued function $v^i(x)$ given at the vertices, we consider the 1-form dv^i, with basis $\{\omega^i\}$,

$$dv^i = (\partial_1 v^i)\, \omega^1 + (\partial_2 v^i)\, \omega^2. \tag{2.84}$$

Approximation in a single triangle

To compute dv^i for a single triangle comprising the vertices x_0, x_1, x_2, with coordinates $(u_0, v_0), (u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, and known function values $v^i(x_0), v^i(x_1), v^i(x_2) \in \mathbb{R}$, we can solve the 2×2 linear system

$$\begin{bmatrix} (u_1 - u_0) & (v_1 - v_0) \\ (u_2 - u_0) & (v_2 - v_0) \end{bmatrix} \begin{bmatrix} \partial_1 v^i \\ \partial_2 v^i \end{bmatrix} = \begin{bmatrix} v^i(x_1) - v^i(x_0) \\ v^i(x_2) - v^i(x_0) \end{bmatrix}, \tag{2.85}$$

in order to obtain $\partial_1 v^i$ and $\partial_2 v^i$.

Approximation in a vertex 1-ring

For a 1-ring around a given vertex x_0 (see Fig. 2.16), labeling its vertices as $x_0, x_1, x_2, x_3, \ldots, x_{n-1}$, we can solve, in the least-squares sense, the over-determined $(n-1)\times 2$ system

$$\begin{bmatrix} (u_1-u_0) & (v_1-v_0) \\ (u_2-u_0) & (v_2-v_0) \\ \vdots & \vdots \\ (u_{n-1}-u_0) & (v_{n-1}-v_0) \end{bmatrix} \begin{bmatrix} \partial_1 v^i \\ \partial_2 v^i \end{bmatrix} = \begin{bmatrix} v^i(x_1)-v^i(x_0) \\ v^i(x_2)-v^i(x_0) \\ \vdots \\ v^i(x_{n-1})-v^i(x_0) \end{bmatrix}. \tag{2.86}$$

If we write the system above in the abbreviated form

$$\mathbf{A}\,\mathbf{d} = \mathbf{v}, \tag{2.87}$$

we can solve the 2×2 square system

$$\mathbf{A}^T\mathbf{A}\,\mathbf{d} = \mathbf{A}^T\mathbf{v}. \tag{2.88}$$

That is, we obtain

$$\mathbf{d} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{v}, \tag{2.89}$$

corresponding to the normal equations of the least-squares problem.

We can simplify the structure of this computation by computing weights $\{(w_j^1, w_j^2)\}_{j=0}^{n-1}$ for each vertex x_j in the 1-ring of vertex x_0.

These weights form an n-tap filter stencil for computing a weighted average of the 1-ring neighborhood of vertex x_0. From them, we can compute the components $\partial_1 v^i, \partial_2 v^i$ of the 1-form dv^i at vertex x_0 as

$$\begin{aligned} \partial_1 v^i\Big|_{(u_0,v_0)} &= w_0^1\, v^i(x_0) + w_1^1\, v^i(x_1) + \ldots + w_{n-1}^1\, v^i(x_{n-1}), \\ \partial_2 v^i\Big|_{(u_0,v_0)} &= w_0^2\, v^i(x_0) + w_1^2\, v^i(x_1) + \ldots + w_{n-1}^2\, v^i(x_{n-1}). \end{aligned} \tag{2.90}$$

In order to compute all weights $\{(w_j^1, w_j^2)\}_{j=0}^{n-1}$ in the stencil, we introduce the $2\times(n-1)$ matrix

$$\mathbf{W} := (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T, \tag{2.91}$$

with components $\mathbf{W}_{ij}$, with i the row and j the column index, respectively.

Considering the structure of the $(n-1)\times 1$ right-hand side above, we directly obtain

$$\begin{aligned} w_0^i &= -\sum_{j=1}^{n-1} \mathbf{W}_{ij}, \qquad i \in \{1,2\}, \\ w_j^i &= \mathbf{W}_{ij}, \qquad i \in \{1,2\}; 1 \le j \le (n-1). \end{aligned} \tag{2.92}$$

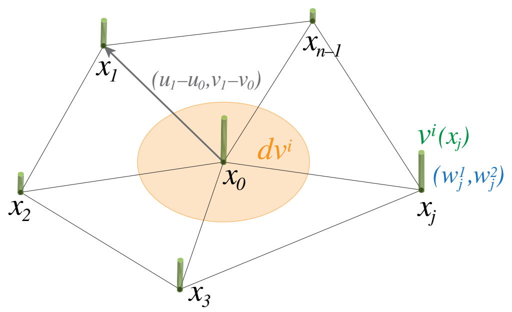


Figure 2.16: **1-ring neighborhood** of a triangle vertex x_0 for approximating 1-forms $dv^i = (\partial_1 v^i)\,\omega^1 + (\partial_2 v^i)\,\omega^2$ of $\mathbb{R}$-valued functions v^i on M.

We pre-compute the $2n$ weights of each 1-ring neighborhood, with n vertices, storing them with the corresponding center vertex (above: x_0).

We note that all filter stencils depend solely on the geometry (the vertex positions) of the triangle mesh, but not on any specific function $v^i(x)$. We can therefore associate the filter weights with each triangle vertex, and then use them to compute the partial derivatives of arbitrary functions, e.g., the v^1 and v^2 of the previous section, see Eq. 2.81.

We also emphasize that these partial derivatives are the only numerically approximated quantities. The metric components g_{ij} (Eq. 2.71) and the Christoffel symbols $\Gamma^i{}_{jk}$ (Eq. 2.77) are accurately computed analytically.

3 Tensor fields and differential forms

We start by looking at tensors defined in a linear space[1] and then extend this notion to tensor fields on a manifold[2]. In the following, V is a finite-dimensional linear space and V^* its dual space[3]. There are essentially three approaches to defining tensors:

[1] In our case this will always be the tangent space at a point of a manifold.

[2] Where components vary smoothly in each chart.

[3] See Section 3.1.

T1 A tensor of type $\binom{r}{s}$ is an element of an abstract space, denoted by

$$\underbrace{V \otimes \cdots \otimes V}_{r \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ copies}}, \tag{3.1}$$

and defined as the quotient space of the free linear space on the Cartesian product

$$\underbrace{V \times \cdots \times V}_{r \text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{s \text{ copies}} \tag{3.2}$$

by a suitable equivalence relation [4]. This approach is very elegant and generic, but also very abstract. It is often used by more mathematically oriented texts like (Lee, 2012, Chapter 12)[5] or (Tu, 2017, §18)[6].

[4] So that the tensor product has the desired properties like bilinearity.

[5] John M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, 2nd edition, 2012

[6] Loring W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Springer-Verlag, 2017

T2 A tensor of type $\binom{r}{s}$ is a multi-linear[7] map from the Cartesian product[8]

$$\underbrace{V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \cdots \times V}_{s \text{ copies}} \tag{3.3}$$

to the scalar field $\mathbb{R}$. The linear space of all such multi-linear functions is denoted by

$$L(\underbrace{V^*, \ldots, V^*}_{r \text{ copies}}, \underbrace{V, \ldots, V}_{s \text{ copies}}; \mathbb{R}). \tag{3.4}$$

[7] That is, linear in each entry when the others are held fixed.

[8] Note the reversal of the order of the spaces V and V^* as compared to Equation 3.2. This will be explained below.

This approach is more concrete and is favored by modern Physics texts like (Frankel, 2011, Chapter 2.4)[9] but also older ones like (Bishop and Goldberg, 1980, Chapter 2.10)[10].

[9] Theodore Frankel. *The Geometry of Physics: An Introduction*. Cambridge University Press, 3rd edition, 2011

[10] Richard L. Bishop and Samuel I. Goldberg. *Tensor analysis on manifolds*. Dover, 1980

T3 A tensor of type $\binom{r}{s}$ is a family of numbers (depending on r, s and the dimension of the space) that transform in a specific way. This approach corresponds to how tensors were originally thought of and is taken in classical texts like (Aris, 1990, Chapter 2)[11] or (Dubrovin et al., 1984, §17)[12].

[11] Rutherford Aris. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Publications, Inc., 1990

[12] B.A Dubrovin, A.T. Fomenko, and S.P. Novikov. *Modern Geometry - Methods and Applications: Part I. The Geometry of Surfaces, Transformation Groups, and Fields*. Graduate Texts in Mathematics. Springer-Verlag New York, 1984

We adopt the modern view **T2** of a tensor as a multi-linear *coordinate-independent* map in these notes. The transformation laws of **T3** then follow naturally from the transformation laws of vectors and covectors[13] by multi-linearity. The link between the approaches **T1** and **T2**, on the other hand, is given by the fact that there is a *canonical* isomorphism between $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ and $L(V^*, \ldots, V^*, V, \ldots, V; \mathbb{R})$[14]. This means that whenever we see a symbol like $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$, we can think of it as the space of multi-linear functions $L(V^*, \ldots, V^*, V, \ldots, V; \mathbb{R})$.

The most important concepts covered in this chapter are:

[13] For covectors see Def. 1.

[14] (Lee, 2012, Proposition 12.10) calls this "Abstract vs. Concrete Tensor Products".
John M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, 2nd edition, 2012

- Scalar and vector fields as tensor fields
- Contravariant and covariant tensors and transformation laws
- Differential forms
- Cotangent space and cotangent bundle
- Preview of later sessions: metric tensor fields.

3.1 The Dual Space

Both approaches **T1** and **T2** make heavy use of the dual space V^* of a linear space V, so we look at this first. We start with a result from linear algebra which looks unimpressive but is fundamental.

Theorem 1. *If V is a linear space of dimension n, then any linear map to any other linear space W is uniquely specified by prescribing n values at a basis. That is, if $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V, any linear map $f : V \to W$ is uniquely specified by letting*

$$\begin{aligned} f(\mathbf{b}_1) &= \mathbf{w}_1 \\ &\vdots \\ f(\mathbf{b}_n) &= \mathbf{w}_n \end{aligned} \tag{3.5}$$

for $\mathbf{w}_1, \ldots \mathbf{w}_n \in W$. [15]

[15] This is often expressed by saying that there *exists* a *unique* linear map $f : V \to W$ with $f(\mathbf{b}_i) = \mathbf{w}_i$, $i = 1 \ldots n$.

To see this we note that if $\mathbf{v}$ is in V and f is supposed to be linear we necessarily have

$$f(\mathbf{v}) = f\left(v^i \mathbf{b}_i\right) = v^i f(\mathbf{b}_i) = v^i \mathbf{w}_i \in W. \tag{3.6}$$

so that we can *define* f in this way and indeed get a map from V to W with $f(\mathbf{b}_i) = \mathbf{w}_i$, $i = 1 \ldots n$ because

$$f(\mathbf{b}_i) = f(\delta_i^j \mathbf{b}_j) = \delta_i^j f(\mathbf{b}_j) = \delta_i^j \mathbf{w}_j = \mathbf{w}_i, \quad i = 1 \ldots n. \tag{3.7}$$

This map is linear because for $\mathbf{u}, \mathbf{v}$ in V we have

$$\begin{aligned} f(\mathbf{u} + \mathbf{v}) &= f\left(u^i \mathbf{b}_i + v^i \mathbf{b}_i\right) = f\left((u^i + v^i)\mathbf{b}_i\right) = (u^i + v^i) f(\mathbf{b}_i) \\ &= u^i f(\mathbf{b}_i) + v^i f(\mathbf{b}_i) = u^i \mathbf{w}_i + v^i \mathbf{w}_i = f(\mathbf{u}) + f(\mathbf{v}). \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} f(\alpha \mathbf{v}) &= f(\alpha(v^i \mathbf{b}_i)) = f((\alpha v^i)\mathbf{b}_i) = (\alpha v^i) f(\mathbf{b}_i) \\ &= \alpha(v^i f(\mathbf{b}_i)) = \alpha f(v^i \mathbf{b}_i) = \alpha f(\mathbf{v}) \end{aligned} \tag{3.9}$$

for $\alpha \in \mathbb{R}$ and $\mathbf{v} \in V$.

Furthermore, if g is another linear map with $g(\mathbf{b}_i) = \mathbf{w}_i$, $i = 1 \ldots n$ we have

$$g(\mathbf{v}) = g(v^i \mathbf{b}_i) = v^i g(\mathbf{b}_i) = v^i \mathbf{w}_i = v^i f(\mathbf{b}_i) = f(v^i \mathbf{b}_i) = f(\mathbf{v}) \tag{3.10}$$

for all $\mathbf{v} \in V$ so that g is actually equal to f and this map is indeed unique.

Example 1. *An important example is when the target linear space $W = \mathbb{R}$, that is the scalar field of V viewed as a linear space over itself.*[16] *Let's start with the simple case of a two-dimensional linear space V with basis $\{\mathbf{b}_1, \mathbf{b}_2\}$, to make it even more concrete let's pick*

$$V = \mathbb{R}^2 \text{ with } \{\mathbf{b}_1, \mathbf{b}_2\} = \{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0)^\intercal, (0,1)^\intercal\}. \tag{3.11}$$

We can define two linear functionals ω^1 and ω^2 in the following way:

$$\begin{aligned} \omega^1((1,0)^\intercal) &= 1, \quad \omega^2((1,0)^\intercal) = 0, \\ \omega^1((0,1)^\intercal) &= 0 \quad \omega^2((0,1)^\intercal) = 1 \end{aligned} \tag{3.12}$$

Extending these linear functionals as in Eq. 3.6 we see that they act on arbitrary vectors $(x,y)^\intercal \in \mathbb{R}^2$ by

$$\begin{aligned} \omega^1((x,y)^\intercal) &= \omega^1(x(1,0)^\intercal + y(0,1)^\intercal) \\ &= x \cdot \omega^1((1,0)^\intercal) + y \cdot \omega^1((0,1)^\intercal) \\ &= x \cdot 1 + y \cdot 0 = x \\ \omega^2((x,y)^\intercal) &= \omega^2(x(1,0)^\intercal + y(0,1)^\intercal) \\ &= x \cdot \omega^2((1,0)^\intercal) + y \cdot \omega^2((0,1)^\intercal) \\ &= x \cdot 0 + y \cdot 1 = y, \end{aligned} \tag{3.13}$$

that is, they pick the first and second component of the vector $(x,y)^\intercal$ respectively.

This last example works in arbitrary dimensions n and we note that, using the Kronecker delta symbol, we can write the equations defining the linear functionals for any basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ (as in Eq. 3.12) concisely as

$$\omega^i(\mathbf{b}_j) = \delta^i_j, \quad i = 1 \ldots n. \tag{3.14}$$

If a vector $\mathbf{v} \in V$ is expanded in the basis $(\mathbf{b}_1, \ldots \mathbf{b}_n)$, that is, $\mathbf{v} = v^i \mathbf{b}_i$ we get

$$\omega^i(\mathbf{v}) = \omega^i(v^j \mathbf{b}_j) = v^j \omega^i(\mathbf{b}_j) = v^j \delta^i_j = v^i, \quad i = 1 \ldots n. \tag{3.15}$$

That is, ω^i picks the i^{th} component of $\mathbf{v}$ with respect to the basis $(\mathbf{b}_1, \ldots \mathbf{b}_n)$.

If we define addition and scalar multiplication of linear functionals[17] by

$$\begin{aligned} f + g &:= f(\mathbf{v}) + g(\mathbf{v}), \qquad && \text{for all } \mathbf{v} \in V, \\ \alpha f &:= \alpha f(\mathbf{v}). && \text{for all } \alpha \in \mathbb{R}, \mathbf{v} \in V. \end{aligned} \tag{3.16}$$

we get a linear space.

Definition 1 (*Dual space*). Let V^* denote the linear functionals on a linear space V[18] and define addition and scalar multiplication for V^* by Eq. 3.16. Then, V^* is itself a linear space called the *dual space* of V and its elements are called linear functionals or *covectors*.

It may not be clear at this point what elements of V^* look like or how they can be obtained. However, if we have a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ then we immediately get n such elements.

---

[16] The linear functions from a linear space V to its scalar field make up a linear space themselves, as we will see below. This space is usually denoted by V^* and its elements are called linear functionals.

[17] in the usual pointwise way

[18] That is, linear functions from V to the scalar field $\mathbb{R}$.

Definition 2 (*Dual basis*)**.** Let V be a linear space with basis $(\mathbf{b}_1,\dots,\mathbf{b}_n)$. The n covectors obtained via Eq. 3.14[19], which we denote by $(\omega^1,\dots,\omega^n)$, are called the *dual basis* of $(\mathbf{b}_1,\dots,\mathbf{b}_n)$.

[19] and extended linearly via Eq. 3.6

The name suggests that all linear functionals of V can be obtained as linear combinations[20] of the dual basis. This is justified by the following theorem.

[20] in a unique way

Theorem 2. *If V is a linear space of dimension n with basis $(\mathbf{b}_1,\dots\mathbf{b}_n)$, then the dual basis $(\omega^1,\dots\omega^n)$[21] is a basis of the dual space V^*[22]. Therefore, V^* has dimension n as well.*

[21] as given by Definition 2

[22] as given by Definition 1

A basis has the properties that it is linearly independent and spans the whole space, that is, any element can be written as a linear combination of the basis elements. Let's first look at linear independence[23]. So assume that there are $\alpha_i,\dots,\alpha_n$ such that $\alpha_i\omega^i = 0$[24]. If this is the case, we especially get zero when applying $\alpha_i\omega^i$ to every $\mathbf{b}_i \in V$. It follows that

$$\begin{aligned} 0 &= \alpha_i\omega^i(\mathbf{b}_1) = \alpha_i\delta^i_1 = \alpha_1 \\ &\vdots \\ 0 &= \alpha_i\omega^i(\mathbf{b}_n) = \alpha_i\delta^i_n = \alpha_n \end{aligned} \tag{3.17}$$

and the ω^i are linearly independent.

[23] Elements $\mathbf{v}_1,\dots,\mathbf{v}_k$ of a linear space are linear independent, if from $\alpha^1\mathbf{v}_1 + \dots\alpha^k\mathbf{v}_k = 0$ follows that necessarily $\alpha^1 = \dots = \alpha^k = 0$.

[24] 0 here being the zero covector, that is, $\alpha_i\omega^i(\mathbf{v}) = 0$ (now the real number $0 \in \mathbb{R}$) for all $\mathbf{v} \in V$.

To see that the ω^i span V^* let ω be an arbitrary element of V^*. We have to show that ω can be written as a linear combination of the ω^i assuming only the linearity of ω. To do this, set $v_i := \omega(\mathbf{b}_i)$[25] and note that, since $v^i = \omega^i(\mathbf{v})$[26], we have

$$\omega(\mathbf{v}) = \omega(v^i\mathbf{b}_i) = v^i\omega(\mathbf{b}_i) = v_i\omega^i(\mathbf{v}), \tag{3.18}$$

that is $\omega = v_i\omega^i$. In conclusion, we have shown that the ω^i are linearly independent and they span V^*, which means they are a basis of V^*[27].

[25] This is a fixed number for each $i = 1,\dots,n$ since the basis has been chosen.

[26] Equation 3.15, ω^i picks the i^{th} component

[27] Which means in particular that every $\omega \in V^*$ can be written as a *unique* linear combination of the ω^i.

We note that, if we choose a basis $(\mathbf{b}_1,\dots\mathbf{b}_n)$ for a linear space V, we can define a map

$$\begin{aligned} \varphi : V &\to V^* \\ \mathbf{v} = v^i\mathbf{b}_i &\mapsto \sum_{i=1}^{n} v^i\omega^i, \end{aligned} \tag{3.19}$$

which is then an isomorphism[28] from V to V^*[29], but this isomorphism is not canonical[30].[31].

[28] Use Theorem 1 to show that φ is linear and Theorem 2 that it is injective.

[29] with dual basis $(\omega^1,\dots\omega^n)$

[30] In the sense that a different basis will result in a different isomorphism, that is, the isomorphism depends on the choice of the basis.

[31] The fact that we had to use a sum sign in the definition of this isomorphism is a hint.

3.2 Tensors as Multi-Linear Maps, and Their Bases

Now that we have the notion of a dual space at our disposal, we can go on and define tensors. According to **T2**, a tensor of *type* $\binom{r}{s}$, and correspondingly of *order* $(r+s)$, on a linear space V with dual space V^* is a multi-linear function

$$\mathbf{T} : \underbrace{V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \cdots \times V}_{s \text{ copies}} \to \mathbb{R}. \tag{3.20}$$

That is, it acts on r covector arguments and s vector arguments and is linear in each slot when the other variables are held fixed:[32]

$$\begin{aligned} &\mathbf{T}(\sigma^1, \ldots, \alpha\sigma^i + \beta\tilde{\sigma}^i, \ldots, \sigma^r, \mathbf{v}_1, \ldots, \mathbf{v}_s) = \\ &\qquad \alpha\mathbf{T}(\sigma^1, \ldots, \sigma^i, \ldots, \sigma^r, \mathbf{v}_1, \ldots, \mathbf{v}_s) + \\ &\qquad \beta\mathbf{T}(\sigma^1, \ldots, \tilde{\sigma}^i, \ldots, \sigma^r, \mathbf{v}_1, \ldots, \mathbf{v}_s), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\mathbf{T}(\sigma^1, \ldots, \sigma^r, \mathbf{v}_1, \ldots, \alpha\mathbf{v}_i + \beta\tilde{\mathbf{v}}_i, \ldots, \mathbf{v}_s) = \\ &\qquad \alpha\mathbf{T}(\sigma^1, \ldots, \sigma^r, \mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_s) + \\ &\qquad \beta\mathbf{T}(\sigma^1, \ldots, \sigma^r, \mathbf{v}_1, \ldots, \tilde{\mathbf{v}}_i, \ldots, \mathbf{v}_s) \end{aligned} \tag{3.22}$$

for $\alpha, \beta \in \mathbb{R}$.

[32] scalar multiplication and addition for covectors is defined in Eq. 3.16

Definition 3. The space of all multi-linear functions as in Eq. 3.20 is denoted by

$$L(\underbrace{V^*, \ldots, V^*}_{r \text{ copies}}, \underbrace{V, \ldots, V}_{s \text{ copies}}; \mathbb{R}) \tag{3.23}$$

or short[33]

$$\mathbf{T}^r_s(V). \tag{3.24}$$

[33] ⚠ Notation differs here, sometimes this is written as $\mathbf{T}^{(r,s)}(V)$ or even $\mathbf{T}^s_r(V)$ or some variation of this.

For $\sigma^i = v^i_j \omega^j$ and $\mathbf{v}_j = v^i_j \mathbf{e}_i$[34] we get because of multi-linearity

$$\begin{aligned} \mathbf{T}\left(\sigma^1, \ldots, \sigma^r, \mathbf{v}_1, \ldots \mathbf{v}_s\right) &= \mathbf{T}\left(v^1_{i_1}\omega^{i_1}, \ldots, v^r_{i_r}\omega^{i_r}, v^{j_1}_1 \mathbf{e}_{j_1}, \ldots v^{j_s}_s \mathbf{e}_{j_s}\right) \\ &= v^1_{i_1} \cdots v^r_{i_r} v^{j_1}_1 \cdots v^{j_s}_s \cdot \mathbf{T}\left(\omega^{i_1}, \ldots, \omega^{i_r}, \mathbf{e}_{j_1}, \ldots \mathbf{e}_{j_s}\right) \end{aligned} \tag{3.25}$$

[34] where $(\omega^1, \ldots \omega^n)$ is the dual basis to $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$

If V is of dimension n[35], we define n^{r+s} functions[36]

$$\begin{aligned} &\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}(\omega^{k_1}, \ldots, \omega^{k_r}, \mathbf{e}_{l_1}, \ldots, \mathbf{e}_{l_s}) \\ &\qquad = \delta^{(k_1, \ldots, k_r, j_1, \ldots, j_s)}_{(i_1, \ldots, i_r, l_1, \ldots, l_s)}, \quad i_m, j_m, k_m, l_m = 1, \ldots, n \end{aligned} \tag{3.26}$$

and extend them multi-linearily to elements of $L(V^*, \ldots, V^*, V, \ldots, V; \mathbb{R})$. We see that

$$\begin{aligned} &\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}(\sigma^1, \ldots, \sigma^r, \mathbf{v}_1, \ldots \mathbf{v}_s) \\ &\qquad = v^1_{i_1} \cdots v^r_{i_r} v^{j_1}_1 \cdots v^{j_s}_s. \end{aligned} \tag{3.27}$$

[35] and therefore V^* as well

[36] The Kronecker symbol works for tuples the same way as for single numbers, i.e., it is 1 if $(i_1, \ldots, i_r, l_1, \ldots, l_s) = (k_1, \ldots, k_r, j_1, \ldots, j_s)$ and 0 otherwise. In detail, this means that it is 1 if $i_1 = k_1, \ldots, i_r = k_r$ and $j_1 = l_1, \ldots, j_s = l_s$ and 0 otherwise, since two tuples are the same if their entries are the same .

If we set

$$T^{i_1, \ldots, i_r}_{j_1, \ldots, j_s} := \mathbf{T}\left(\omega^{i_1}, \ldots, \omega^{i_r}, \mathbf{e}_{j_1}, \ldots \mathbf{e}_{j_s}\right) \tag{3.28}$$

we can write[37]

$$\mathbf{T} = T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \cdot \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}. \tag{3.29}$$

Pfuh. This still seems to be hopelessly abstract, so lets take a look at an example.

[37] The $T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}$ are called the *components* of $\mathbf{T}$ with respect to the *basis* $\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}$. We will prove below that this terminology is justified. In particular, this means that the space of $\binom{r}{s}$ tensors on an n-dimensional linear space has dimension n^{r+s}.

Example 2. *A $\binom{1}{1}$ tensor takes the form*

$$\mathbf{T} : V^* \times V \to \mathbb{R} \tag{3.30}$$

since it is a member of the space $L(V^*, V; \mathbb{R})$. If V is a two-dimensional linear space, we fix a basis $(\mathbf{e}_1, \mathbf{e}_2)$[38] and define four multi-linear functions, lets call them $h^1_1, h^1_2, h^2_1, h^2_2$ for now[39], by

[38] which defines the dual basis (ω^1, ω^2)

[39] we will see below that these are just the $n^{r+s} = 2^{1+1} = 4$ functions defined in Eq. 3.26

$$\begin{aligned} h^1_1(\omega^1, \mathbf{e}_1) &= 1 \quad & h^1_1(\omega^1, \mathbf{e}_2) &= 0 \\ h^1_1(\omega^2, \mathbf{e}_1) &= 0 \quad & h^1_1(\omega^2, \mathbf{e}_2) &= 0 \end{aligned} \tag{3.31}$$

$$\begin{aligned} h^1_2(\omega^1, \mathbf{e}_1) &= 0 \quad & h^1_2(\omega^1, \mathbf{e}_2) &= 1 \\ h^1_2(\omega^2, \mathbf{e}_1) &= 0 \quad & h^1_2(\omega^2, \mathbf{e}_2) &= 0 \end{aligned} \tag{3.32}$$

$$\begin{aligned} h^2_1(\omega^1, \mathbf{e}_1) &= 0 \quad & h^2_1(\omega^1, \mathbf{e}_2) &= 0 \\ h^2_1(\omega^2, \mathbf{e}_1) &= 1 \quad & h^2_1(\omega^2, \mathbf{e}_2) &= 0 \end{aligned} \tag{3.33}$$

$$\begin{aligned} h^2_2(\omega^1, \mathbf{e}_1) &= 0 \quad & h^2_2(\omega^1, \mathbf{e}_2) &= 0 \\ h^2_2(\omega^2, \mathbf{e}_1) &= 0 \quad & h^2_2(\omega^2, \mathbf{e}_2) &= 1 \end{aligned} \tag{3.34}$$

extend them multi-linearly to $\sigma = v_j \omega^j$ and $\mathbf{v} = v^i \mathbf{e}_i$[40] and compute

[40] so $h^i_j \in L(V^*, V; \mathbb{R})$, $i, j = 1, 2$

$$\begin{aligned} h^1_1(\sigma, \mathbf{v}) &= h^1_1\left(v_i \omega^i, v^j \mathbf{e}_j\right) = v^i \cdot v_j \cdot h^1_1\left(\omega^i, \mathbf{e}_j\right) = v^1 \cdot v_1 \\ h^1_2(\sigma, \mathbf{v}) &= h^1_2\left(v_i \omega^i, v^j \mathbf{e}_j\right) = v^i \cdot v_j \cdot h^1_2\left(\omega^i, \mathbf{e}_j\right) = v^1 \cdot v_2 \\ h^2_1(\sigma, \mathbf{v}) &= h^2_1\left(v_i \omega^i, v^j \mathbf{e}_j\right) = v^i \cdot v_j \cdot h^2_1\left(\omega^i, \mathbf{e}_j\right) = v^2 \cdot v_1 \\ h^2_2(\sigma, \mathbf{v}) &= h^2_2\left(v_i \omega^i, v^j \mathbf{e}_j\right) = v^i \cdot v_j \cdot h^2_2\left(\omega^i, \mathbf{e}_j\right) = v^2 \cdot v_2 \end{aligned} \tag{3.35}$$

We see that h^i_j picks the product of the i^{th} component of σ and the j^{th} component of $\mathbf{v}$. We set

$$\mathbf{e}_i \otimes \omega^j := h^i_j \tag{3.36}$$

and write the defining Eqs. 3.31 – 3.34 concisely as[41]

[41] compare Eq. 3.26

$$\mathbf{e}_i \otimes \omega^j(\omega^k, \mathbf{e}_l) = \delta^{(k,l)}_{(i,j)}. \tag{3.37}$$

We then have

$$\mathbf{e}_i \otimes \omega^j(\sigma, \mathbf{v}) = v_i \cdot v^j. \tag{3.38}$$

Setting $T^i_j := \mathbf{T}(\omega^i, \mathbf{e}_j)$ this lets us write[42]

[42] We can also write this in matrix form as

$$\begin{aligned} v^i \cdot v_j \cdot T^i_j &= \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} v_1 \cdot T^1_1 + v_2 \cdot T^1_2 \\ v_1 \cdot T^2_1 + v_2 \cdot T^2_2 \end{bmatrix} \\ &= \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} T^1_1 & T^1_2 \\ T^2_1 & T^2_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \end{aligned}$$

$$\mathbf{T}(\sigma, \mathbf{v}) = T^i_j \cdot \mathbf{e}_i \otimes \omega^j(\sigma, \mathbf{v}). \tag{3.39}$$

or short

$$\mathbf{T} = T^i_j \cdot \mathbf{e}_i \otimes \omega^j. \tag{3.40}$$

Theorem 3. *If V is a linear space of dimension n with basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$, then*

$$\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}, \quad i_m, j_m, k_m, l_m = 1, \ldots, n \tag{3.41}$$

is a basis of the space of $\binom{r}{s}$ tensors

$$\mathbf{T}^r_s(V) = L(\underbrace{V^*, \ldots, V^*}_{r\ copies}, \underbrace{V, \ldots, V}_{s\ copies}; \mathbb{R}). \tag{3.42}$$

In particular this space has dimension n^{r+s}.

It is important to realize that, although the notation $\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}$ might look rather abstract, it simply denotes a multi-linear function taking vectors and covectors as arguments and producing a real number.

Transformation Rules

To derive the transformation rules for tensors, we start with a linear space V[43] of dimension n and fix a basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$. This determines the dual basis $(\omega^1, \ldots, \omega^n)$ as well as a basis for each space of $\binom{r}{s}$ tensors on V[44]. We have seen in Section 2.6 that, if we change basis $\tilde{\mathbf{e}}_j = \tilde{e}^i{}_j \mathbf{e}_i$ the components of a vector $\mathbf{v} = v^i \mathbf{e}_i$ change by $\tilde{v}^i = e^i{}_j v^j$, where $\left[e^i{}_j \right] = \left[\tilde{e}^i{}_j \right]^{-1}$. Correspondingly, if we use the dual basis, covectors acting on the transformed basis vectors are defined by

$$\tilde{\omega}^i (\tilde{\mathbf{e}}_j) = \delta^i{}_j. \tag{3.43}$$

But we also have that

$$\delta^i{}_j = \delta^j{}_i = \omega^j (\mathbf{e}_i) = \omega^j \left(e^i{}_j \tilde{\mathbf{e}}_i \right) = e^i{}_j \, \omega^j (\tilde{\mathbf{e}}_i) \tag{3.44}$$

so that we get

$$\tilde{\omega}^i (\tilde{\mathbf{e}}_j) = e^i{}_j \, \omega^j (\tilde{\mathbf{e}}_i). \tag{3.45}$$

Extending by linearity we see that $\tilde{\omega}^i = e^i{}_j \, \omega^j$. Finally, we have also seen in Section 2.6 that the components of covectors with respect to the dual basis transform by $\tilde{v}_j = \tilde{e}^i{}_j v_i$. We summarize this discussion in Table 3.1.
Then, if we change the basis in this way, the components of a $\binom{r}{s}$ tensor change in the following way:

$$\begin{aligned}
\tilde{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} &= \mathbf{T}\left(\tilde{\omega}^{i_1}, \ldots, \tilde{\omega}^{i_r}, \tilde{\mathbf{e}}_{j_1}, \ldots \tilde{\mathbf{e}}_{j_s}\right) \\
&= \mathbf{T}\left(e^{i_1}_{k_1} \omega^{k_1}, \ldots, e^{i_r}_{k_r} \omega^{k_r}, \tilde{e}^{l_1}_{j_1} \mathbf{e}_{l_1}, \ldots \tilde{e}^{l_s}_{j_s} \mathbf{e}_{l_s}\right) \\
&= e^{i_1}_{k_1} \cdots e^{i_r}_{k_r} \cdot \tilde{e}^{l_1}_{j_1} \cdots \tilde{e}^{l_s}_{j_s} \cdot \mathbf{T}\left(\omega^{k_1}, \ldots, \omega^{k_r}, \mathbf{e}_{l_1}, \ldots \mathbf{e}_{l_s}\right) \\
&= e^{i_1}_{k_1} \cdots e^{i_r}_{k_r} \cdot \tilde{e}^{l_1}_{j_1} \cdots \tilde{e}^{l_s}_{j_s} \cdot T^{k_1,\ldots,k_r}_{l_1,\ldots,l_s}.
\end{aligned} \tag{3.46}$$

In this generic form this still looks a bit complicated, so let's look at a few specific examples.

[43] again, think of the tangent space at a point of a manifold

[44] as in Theorem 3

basis vectors	basis covectors
$\tilde{\mathbf{e}}_j = \tilde{e}^i_{\ j}\,\mathbf{e}_i$	$\tilde{\omega}^i = e^i_{\ j}\,\omega^j$
vector components	covector components
$\tilde{v}^i = e^i_{\ j}\,v^j$	$\tilde{v}_j = \tilde{e}^i_{\ j}\,v_i.$

Table 3.1: Summary of transformation rules for vectors and covectors, as well as vector and covector components.

$\binom{0}{1}$ *Tensors: 1-Forms or Covectors*

For a $\binom{0}{1}$ tensor, Eq. 3.20 takes the form

$$\mathbf{T} : V \to \mathbb{R}. \tag{3.47}$$

That is, $\binom{0}{1}$ tensors are just linear functionals, 1-forms, or covectors as defined in Def. 1. We usually denote a 1-form by σ instead of $\mathbf{T}$ and expand it in the dual basis as $\sigma = v_i\,\omega^i$, where we have set $v_i := T_i$. If we change basis according to Table 3.1, so that the components of σ change by $\tilde{v}_j = \tilde{e}^i_{\ j}\,v_i$, this is consistent with $\tilde{T}_{j_1} = \tilde{e}^{l_1}_{\ j_1}\,T_{l_1}$ according to Eq. 3.46.

$\binom{1}{0}$ *Tensors: Vectors*

For a $\binom{1}{0}$ tensor, Eq. 3.20 takes the form

$$\mathbf{T} : V^* \to \mathbb{R}. \tag{3.48}$$

We note that, defined this way, $\mathbf{T}$ is an element of V^{**}[45], the dual space of the dual space of V. It might be confusing why we call it a vector, which is just an element of V.

This can be explained in the following way. For a vector $\mathbf{v}$ define an element $\iota(\mathbf{v}) \in V^{**}$ by $\iota(\mathbf{v})(\sigma) := \sigma(\mathbf{v})$. We see that $\iota(\mathbf{v})$ is indeed a mapping[46] $V^* \to \mathbb{R}$, that is, an element of V^{**}. We therefore get another mapping

$$\begin{aligned} \iota : V &\to V^{**} \\ \mathbf{v} &\mapsto \iota(\mathbf{v}) \end{aligned} \tag{3.49}$$

which is linear[47] and injective[48]. Since V^{**}, being the dual space of V^* has the same dimension as V^* which in turn has the same dimension as V, this mapping is also surjective and therefore an *isomorphism*. Since it further only depends on the vector space structure of V[49], ι is called *canonical* and is used to identify V with V^{**}.

One immediate consequence is, that we can now also call $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ the dual basis of $(\omega^1, \dots, \omega^n)$ since $\iota(\mathbf{e}_i)\left(\omega^j\right) = \omega^j(\mathbf{e}_i) = \delta^j_i$. That is, $(\iota(\mathbf{e}_1), \dots, \iota(\mathbf{e}_n))$ is actually the dual basis of $(\omega^1, \dots, \omega^n)$, but we can identify it with $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ via the canonical isomorphism ι.

This means that we can think of a vector $\mathbf{v}$ alternatively as a real-valued linear map that takes a 1-form as input, defined by $\mathbf{v}(\sigma) := \sigma(\mathbf{v})$.

[45] actually $(V^*)^*$, but since there is only one way to interpret this, it is usually simply denoted by V^{**}

[46] when σ, ω are elements of V^*, which is a linear space by Definition 1, we have

$$\begin{aligned} \iota(\mathbf{v})(a\sigma + b\omega) &= (a\sigma + b\omega)(\mathbf{v}) \\ &= a\sigma(\mathbf{v}) + b\omega(\mathbf{v}) \\ &= a\iota(\mathbf{v})(\sigma) + b\iota(\mathbf{v})(\omega). \end{aligned}$$

[47] For $\sigma \in V^*$ we get $\iota(a\mathbf{v} + b\mathbf{w})(\sigma) = \sigma(a\mathbf{v} + b\mathbf{w}) = a\sigma(\mathbf{v}) + b\sigma(\mathbf{w}) = a\iota(\mathbf{v})(\sigma) + b\iota(\mathbf{w})(\sigma)$. Since this holds for all $\sigma \in V^*$ this means that $\iota(a\mathbf{v} + b\mathbf{w}) = a\iota(\mathbf{v}) + b\iota(\mathbf{w})$.

[48] if $\iota(\mathbf{v}) = 0$ then $\sigma(\mathbf{v}) = 0$ for all $\sigma \in V^*$, this is only possible if $\mathbf{v} = 0$

[49] no choices, like picking a basis, have to be made

$\binom{1}{1}$ *Tensors*

For a $\binom{1}{1}$ tensor, which is a bi-linear map of one covector and one vector argument to a scalar, Eq. 3.20 takes the form

$$\mathbf{T} : V^* \times V \to \mathbb{R}. \tag{3.50}$$

According to Theorem 3

$$(\mathbf{e}_1 \otimes \omega^1, \mathbf{e}_1 \otimes \omega^2, \ldots, \mathbf{e}_{n-1} \otimes \omega^n, \mathbf{e}_n \otimes \omega^n) \tag{3.51}$$

is a basis of the space of all $\binom{1}{1}$ tensors. That is, we can expand an arbitrary $\binom{1}{1}$ tensor as

$$\mathbf{T} = T^i{}_j \mathbf{e}_i \otimes \omega^j \tag{3.52}$$

and under a change of basis these components transform as

$$\tilde{T}^i{}_j = e^i_k \tilde{e}^l_j T^k_l \tag{3.53}$$

according to Eq. 3.46.

An important property of $\binom{1}{1}$ tensors is that we can interpret them as a linear map of vectors, and vice versa[50] , that is, a linear map can be interpreted as a $\binom{1}{1}$ tensor.

[50] Mathematically, this means that there is a *canonical* isomorphism between $L(V)$, the space of linear maps on V, and $\mathbf{T}^1_1(V) = L(V^*, V; \mathbb{R})$, the space of $\binom{1}{1}$ tensors on V.

We show this in detail. First, if we are given a linear map $A : V \to V$, we can define

$$\begin{aligned} \mathbf{T}_A : V^* \times V &\to \mathbb{R} \\ (\sigma, \mathbf{v}) &\mapsto \sigma(A\mathbf{v}) \end{aligned} \tag{3.54}$$

which is bi-linear and real-valued, that is, a $\binom{1}{1}$ tensor.

On the other hand, if we are given a $\binom{1}{1}$ tensor $\mathbf{T}$, we define

$$\begin{aligned} A_{\mathbf{T}}\mathbf{v} : V^* &\to \mathbb{R} \\ \sigma &\mapsto \mathbf{T}(\sigma, \mathbf{v}), \end{aligned} \tag{3.55}$$

which is a linear functional on V^*, that is, an element of V^{**}. Using the (inverse of) the canonical isomorphism defined in Eq. 3.49 we obtain a linear map by $\mathbf{w} := A\mathbf{v} := \iota^{-1}(A_{\mathbf{T}}\mathbf{v})$. To actually compute this map, we expand $A_{\mathbf{T}}\mathbf{v}$ in the basis $(\iota(\mathbf{e}_1), \ldots, \iota(\mathbf{e}_n))$ of V^{**}, that is, $A_{\mathbf{T}}\mathbf{v} = a^j \iota(\mathbf{e}_j)$ for some coefficients a^j and note that

$$A_{\mathbf{T}}\mathbf{v}(\omega^i) = a^j \iota(\mathbf{e}_j)(\omega^i) = a^j \omega^i(\mathbf{e}_j) = a^j \delta^i{}_j = a^i. \tag{3.56}$$

But by definition we also have that $A_{\mathbf{T}}\mathbf{v}(\omega^i) = \mathbf{T}(\omega^i, \mathbf{v})$ and we can compute

$$\mathbf{w} = A\mathbf{v} = \iota^{-1}(A_{\mathbf{T}}\mathbf{v}) = \iota^{-1}(a^i \iota(\mathbf{e}_i)) = a^i \iota^{-1}(\iota(\mathbf{e}_i)) = \mathbf{T}(\omega^i, \mathbf{v})\mathbf{e}_i. \tag{3.57}$$

We see that we in this way get a linear map[51]

[51] Note, that $A_{\mathbf{T}}\mathbf{v}$ denotes the map defined by Eq. 3.55 and $A_{\mathbf{T}}$ denotes a *new* map.

$$\begin{aligned} A_{\mathbf{T}} : V &\to V \\ \mathbf{v} &\mapsto \mathbf{w} = \iota^{-1}(A_{\mathbf{T}}\mathbf{v}), \end{aligned} \tag{3.58}$$

Further, the components of $\mathbf{w} = w^i \mathbf{e}_i$ are just given by

$$w^i = \mathbf{T}(\omega^i, \mathbf{v}) = v^j \mathbf{T}(\omega^i, \mathbf{e}_j) = T^i{}_j v^j \tag{3.59}$$

for $\mathbf{v} = v^j \mathbf{e}_j$. This means, that the linear map $A_\mathbf{T}$ has the same components as the $\binom{1}{1}$ tensor $\mathbf{T}$.

Further[52],

$$\begin{aligned}
\mathbf{T}_{(A_\mathbf{T})}(\sigma, \mathbf{v}) &= \sigma((A_\mathbf{T})\,\mathbf{v}) = \sigma\left(\iota^{-1}\left(A_\mathbf{T}\mathbf{v}\right)\right) = \sigma\left(\mathbf{T}(\omega^i, \mathbf{v})\mathbf{e}_i)\right) \\
&= \mathbf{T}(\omega^i, \mathbf{v})\sigma(\mathbf{e}_i) = \mathbf{T}(\omega^i, \mathbf{v}) v_j \omega^j(\mathbf{e}_i) \\
&= \mathbf{T}(\omega^i, \mathbf{v}) v_j \delta^j{}_i = \mathbf{T}(\omega^i, \mathbf{v}) v_i = \mathbf{T}(v_i \omega^i, \mathbf{v}) \\
&= \mathbf{T}(\sigma, \mathbf{v}).
\end{aligned} \tag{3.60}$$

That is, these identifications of $\binom{1}{1}$ tensors with linear maps are indeed inverses of each other.

Remark 1. We note that an expression such as $T^i{}_j\, \mathbf{e}_i \otimes \omega^j$ is often used simply as a linear map $\mathbf{T}$, acting on a vector $\mathbf{v}$, and giving a result vector $\mathbf{T}(\mathbf{v})$, by writing[53]

$$\begin{aligned}
\mathbf{T}(\mathbf{v}) &= \left(T^i{}_j\, \mathbf{e}_i \otimes \omega^j\right)(\mathbf{v}), \\
&= T^i{}_j\, \mathbf{e}_i\, \omega^j(\mathbf{v}) = T^i{}_j\, \omega^j(\mathbf{v})\, \mathbf{e}_i, \\
&= T^i{}_j v^j\, \mathbf{e}_i.
\end{aligned} \tag{3.61}$$

We note that when the 1-form ω^j is applied to the vector argument $\mathbf{v}$, the tensor product $\otimes$ simply turns into a regular product. This behavior is part of the definition of the tensor product. It corresponds to the fact that the contraction $\omega^j(\mathbf{v})$ has turned the $\binom{1}{1}$ tensor $\mathbf{T}$ into a $\binom{1}{0}$ tensor $\mathbf{T}(\mathbf{v})$, i.e., a *vector*. Correspondingly, the remaining basis is solely the basis $\{\mathbf{e}_i\}$ for vectors. We note that the first-order tensor $\mathbf{T}(\mathbf{v})$ can be interpreted directly as a vector, or still be interpreted as a scalar-valued *function* acting on the argument of a *covector* (as one definition of a vector in tensor analysis). For example, we get the *scalar* that is the i'th component of the vector $\mathbf{T}(\mathbf{v})$ referred to the basis $\{\mathbf{e}_i\}$, by computing

$$\mathbf{T}(\mathbf{v})\left(\omega^i\right) := \omega^i\left(\mathbf{T}(\mathbf{v})\right) = \omega^i\left(T^i{}_j v^j\, \mathbf{e}_i\right) = T^i{}_j v^j \omega^i\left(\mathbf{e}_i\right) = T^i{}_j v^j. \tag{3.62}$$

When "executed" for all "rows" i, the final expression $T^i{}_j v^j$ is a matrix-vector multiplication of components. However, in the entire derivation above, the notation has helped us avoid mixing components of different variance and the corresponding bases. Overall, tensor notation is a powerful way of using basis vectors and 1-forms, and tensors referred to components, in a general context, simplifying the use of arguments of different variance (covariant, contravariant) and higher-order tensors.

A concrete example of a $\binom{1}{1}$ tensor is the definition of the covariant derivative $\nabla\mathbf{v}$ of a vector field $\mathbf{v}$ in Eqs. 6.8 and 6.9 below. However, we note that in that context, "covariant" refers to "general covariance," not to covariant arguments[54].

$\binom{0}{2}$ *Tensors*

For a $\binom{0}{2}$ tensor[55], which is a bi-linear map of two vector arguments to a

[52] To avoid confusion with the map $A_\mathbf{T}\mathbf{v}$ we write $(A_\mathbf{T})\,\mathbf{v}$ for the map $A_\mathbf{T}$ applied to the vector $\mathbf{v}$.

[53] the problem here is that the expression $(T^i{}_j\, \mathbf{e}_i \otimes \omega^j)\,(\mathbf{v})$ is imprecise because $\mathbf{e}_i \otimes \omega^j$ actually takes a covector and a vector as arguments, to make this precise we have to use the argument above

[54] See for example the discussion in (Frankel, 2011, p. 430) or (Lee, 2018, p. 89) .

Theodore Frankel. *The Geometry of Physics: An Introduction*. Cambridge University Press, 3rd edition, 2011; and John M. Lee. *Introduction to Riemannian Manifolds*. Springer-Verlag, 2nd edition, 2018

[55] or a covariant second-order tensor

scalar, Eq. 3.20 takes the form

$$\mathbf{T} : V \times V \to \mathbb{R}. \tag{3.63}$$

According to Theorem 3

$$(\omega^1 \otimes \omega^1, \omega^1 \otimes \omega^2, \ldots, \omega^{n-1} \otimes \omega^n, \omega^n \otimes \omega^n) \tag{3.64}$$

is a basis of the space of all $\binom{0}{2}$ tensors. That is, we can expand an arbitrary $\binom{0}{2}$ tensor as

$$\mathbf{T} = T_{ij} \omega^i \otimes \omega^j \tag{3.65}$$

and under a change of basis these components transform as

$$\tilde{T}_{ij} = \tilde{e}_i^l \tilde{e}_j^k T_{lk} \tag{3.66}$$

according to Eq. 3.46. An important example of a covariant second-order tensor[56] is the metric tensor $\mathbf{g}$, referred to a basis $\{\omega^i \otimes \omega^j\}$ it is usually expanded as $g_{ij}\, \omega^i \otimes \omega^j$.

[56] a $\binom{0}{2}$ tensor

$\binom{2}{0}$ *Tensors*

For a $\binom{2}{0}$ tensor[57], which is a bi-linear map of two covector arguments to a scalar, Eq. 3.20 takes the form

[57] or a contravariant second-order tensor

$$\mathbf{T} : V^* \times V^* \to \mathbb{R}. \tag{3.67}$$

According to Theorem 3

$$(\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \ldots, \mathbf{e}_{n-1} \otimes \mathbf{e}_n, \mathbf{e}_n \otimes \mathbf{e}_n) \tag{3.68}$$

is a basis of the space of all $\binom{2}{0}$ tensors. That is, we can expand an arbitrary $\binom{2}{0}$ tensor as

$$\mathbf{T} = T^{ij} \mathbf{e}_i \otimes \mathbf{e}_j \tag{3.69}$$

and under a change of basis these components transform as

$$\tilde{T}^{ij} = e_l^i e_k^j T^{lk} \tag{3.70}$$

according to Eq. 3.46. An important contravariant second-order tensor[58] is the inverse metric $\mathbf{g}^{-1}$, which is usually expanded in a basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ as $g^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$.

[58] a $\binom{2}{0}$ tensor

0-Tensors

For a 0-tensor, or a $\binom{0}{0}$ tensor, it seems unclear how to interpret Eq. 3.20:

$$\mathbf{T} : ? \to \mathbb{R} \tag{3.71}$$

However, being strict, we can say that a 0-tensor takes zero vectors and zero covectors and produces a number. So it is just a number depending on no input. This seems to be a useless concept, but for tensor fields we will see that 0-tensor fields just correspond to smooth functions on the manifold.

3.3 Tensor Fields on Manifolds

Now that we know how tensors are defined in one linear space, e.g., in the tensor space at one individual point $x \in M$ of the manifold M, we can extend this construction to a whole tensor field on the manifold M. As a coordinate-free geometric construct, each of these tensor fields is a *section* of the corresponding *tensor bundle*, without referring to a particular basis.

However, analogously to vector fields on the manifold M, we can choose tensor basis fields in local neighborhoods $U \subset M$ that then allow us to describe tensor fields in U in components referred to that basis. For a tensor field, these component functions are required to vary smoothly from point to point on the manifold.

That is, where above we have used a single basis, for example at one specific point $x \in M$, such as a basis $\mathbf{e}_i$ for vectors, i.e., $\binom{1}{0}$ tensors, or a basis $\mathbf{e}_i \otimes \omega^j$ for $\binom{1}{1}$ tensors, etc., we now simply extend this approach to smoothly varying *tensor basis fields* on M, where the basis tensors vary (smoothly) from point to point of the manifold, and we refer tensors to these basis fields via smoothly varying component functions.

$\binom{1}{0}$ *Tensor fields: Vector fields*

As above, as coordinate-free geometric constructs, vector fields are in fact sections of the tangent bundle TM, i.e., a vector field $\mathbf{v}$ is a map

$$\mathbf{v} \colon M \to TM. \tag{3.72}$$

However, to refer vector fields (i.e., sections of the tangent bundle) to coordinates, we choose n linearly-independent basis vector fields $\mathbf{e}_i$ in a neighborhood U. At every point $x \in U \subset M$, i.e., in the corresponding linear space T_xM, this gives us a linearly-independent basis $\mathbf{e}_i(x)$. A specific vector field can then be given in components referred to these basis vector fields, which at any point x means

$$\mathbf{v}(x) = v^i(x)\,\mathbf{e}_i(x). \tag{3.73}$$

$\binom{0}{1}$ *Tensor fields: Differential 1-forms*

As above, as coordinate-free geometric constructs, 1-form (covector) fields are in fact sections of the cotangent bundle T^*M, i.e., a 1-form (covector) field ω is a map

$$\omega \colon M \to T^*M. \tag{3.74}$$

However, to refer 1-form (covector) fields (i.e., sections of the cotangent bundle) to coordinates, we choose n linearly-independent basis 1-form (covector) fields ω^i. At every point $x \in M$, i.e., in the corresponding linear space T_x^*M, this gives us a linearly-independent basis $\omega^i(x)$. A specific 1-form (covector) field can then be given in components referred to these basis 1-form (covector) fields, which at any point x means

$$\omega(x) = v_i(x)\,\omega^i(x). \tag{3.75}$$

We can visualize 1-form (covector) fields using a "stack" visualization in each cotangent space, as depicted in Fig. 3.1 for two example cotangent spaces at two different points on the manifold $M = \mathbb{S}^2$.

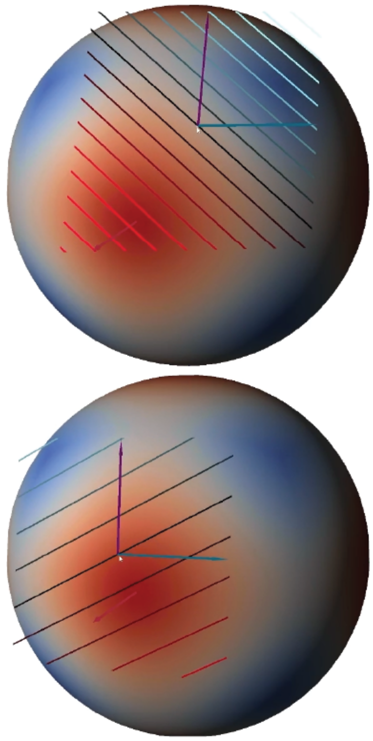


Figure 3.1: Gradient 1-forms df on a manifold M, here visualized with a "stack" (isolines of the linearized function f) in each (co)tangent space, for the color-coded scalar function $f : M \to \mathbb{R}$ on the manifold $M = \mathbb{S}^2$ shown here.

$\binom{r}{s}$ *Tensor fields*

For higher-order tensors, of different type (variance), we simply extend the above approach. Each tensor field is a section of the corresponding tensor bundle. In coordinates, we obtain higher-order tensor basis fields using the tensor product. For general $\binom{r}{s}$ tensor fields that means that[59]

[59] compare Equation 3.29

$$\mathbf{T}(x) = T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}(x) \cdot \left(\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}\right)(x). \tag{3.76}$$

where the component functions $T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}(x)$ vary smoothly from point to point.

For example, for a $\binom{0}{2}$ tensor field we need to choose basis tensor fields $\omega^i \otimes \omega^j$, for a $\binom{1}{1}$ tensor field we need to choose basis tensor fields $\mathbf{e}_i \otimes \omega^j$, and so forth. As a specific example the $\binom{0}{2}$ covariant metric tensor field $\mathbf{g}$ is then given at a point $x \in U \subset M$ by

$$\mathbf{g}(x) = g_{ij}(x)\,(\omega^i \otimes \omega^j)(x). \tag{3.77}$$

0-*Tensor fields*

We can now also understand the interpretation of 0-tensor fields as smooth functions. Such a 0-tensor field is simply

$$\mathbf{T}(x) = T(x), \tag{3.78}$$

since it takes no inputs[60]. It is just a smooth function on the manifold.

[60] and therefore requires no indices

3.4 *Differential Forms*

Differential k-forms are $\binom{0}{k}$ tensor fields that are completely anti-symmetric, i.e., on exchange of the order of any two vector arguments, the sign of the resulting scalar changes.

Differential forms are the most important construct in *exterior calculus*, and are the natural mathematical entities that constitute *integrands*, i.e., mathematical objects to be integrated.[61]

[61] In a future version of these notes, we are planning to extend this part significantly. For the time being, however, this is left as future work.

Further Reading

For a detailed description, including the general concept of tensor bundles over a manifold M, we refer to the books by Spivak [62] and Frankel [63].

[62] Michael Spivak. *A Comprehensive Introduction to Differential Geometry (5 volumes)*. Publish or Perish Press, 3rd edition, 1999

[63] Theodore Frankel. *The Geometry of Physics: An Introduction*. Cambridge University Press, 3rd edition, 2011

4 Riemannian metrics and connections

In this chapter, we introduce (Riemannian) *metric tensor fields*, and the corresponding concepts of *connections* and *parallel transport*. As an important example we discuss the *Levi-Civita connection*, i.e., the on a Riemannian manifold uniquely-defined connection that is both compatible with the metric and torsion-free. While connections and covariant derivatives are very related, and are sometimes seen as a single larger concept, in this session we will emphasize the concept of connection first, and covariant derivatives will be covered in Covariant derivatives and Lie derivatives.

Important concepts that we will cover are Riemannian manifolds and Riemannian metrics, connections and parallel transport, the Levi-Civita connection, torsion and compatibility of a connection with the metric.

4.1 Riemannian Metrics

So far, we have looked at manifolds that have a differential structure, which allows us to do calculus on these manifolds. We cannot, however, define geometric notions[1] yet, what is missing is a special second-order tensor field[2], to turn the smooth manifold into a Riemannian manifold.

[1] such as lengths of vectors or curves or angles between vectors

[2] the metric tensor field

Coordinate-free definition. A (Riemannian) metric $\mathbf{g}$ on a manifold M defines an inner product on each tangent space T_xM. This is usually written as $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{g}(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in T_xM$. Specifically, $\mathbf{g}$ is

1. *symmetric*, that is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle \tag{4.1}$$

   for all $\mathbf{x}, \mathbf{y} \in T_xM$,

2. *bilinear*, that is

$$\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a\langle \mathbf{x}, \mathbf{z} \rangle + b\langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{z}, a\mathbf{x} + b\mathbf{y} \rangle \tag{4.2}$$

   for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in T_xM$, and all $a, b \in \mathbb{R}$, and

3. *positive definite*, that is

$$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0 \tag{4.3}$$

   for all $\mathbf{x} \in T_xM$ with $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = 0$.

Furthermore, $\mathbf{g}$ is required to be smooth in the sense that in all charts the coordinate functions are smooth. Consequently, $\mathbf{g}$ is a covariant second-order tensor field.

Example 3. *1. The length of a vector* $\mathbf{v}$ *is defined by*

$$\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}. \tag{4.4}$$

2. *The angle between two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *is defined by*

$$\cos\theta := \frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|} \tag{4.5}$$

3. *The length* ℓ *of a curve* $c : [t_1, t_2] \to M$ *is defined by*

$$\ell := \int_{t_1}^{t_2} \|c'(t)\|\, dt. \tag{4.6}$$

The metric tensor on a Riemannian manifold, however, has another important consequence. It allows us to identify a tangent space with its dual space.

Identification with the dual space. On a generic manifold, although a tangent space and its dual are isomorphic[3], there is no isomorphism that is in some sense singled out[4]. If we have a metric tensor[5], this tensor provides such a specific isomorphism,

For a vector $\mathbf{x}$ define a covector $\omega_{\mathbf{x}}$ by $\omega_{\mathbf{x}}(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y}\rangle$, one writes[6] $\omega_{\mathbf{x}} = \mathbf{x}_\flat$.

On the other hand, for a covector ω there is a unique vector $\mathbf{x}_\omega$ such $\omega(\mathbf{y}) = \langle \mathbf{x}_\omega, \mathbf{y}\rangle$, one writes $\mathbf{x}_\omega = \omega^\sharp$.

That is, the operators $\flat$ and $\sharp$ provide an isomorphic identification of each tangent space of a Riemannian manifold with its dual space.

Computation in a chart. If we define $g_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j\rangle$, we get

$$\begin{aligned} \mathbf{g}(\mathbf{x},\mathbf{y}) &= \langle \mathbf{x},\mathbf{y}\rangle = \langle x^i\mathbf{e}_i, y^j\mathbf{e}_j\rangle = x^i y^j \langle \mathbf{e}_i, \mathbf{e}_j\rangle = x^i y^j g_{ij} \\ &= \omega^i(\mathbf{x})\omega^j(\mathbf{y}) g_{ij} = g_{ij}(\omega^i \otimes \omega^j)(\mathbf{x},\mathbf{y}), \end{aligned} \tag{4.7}$$

since $\{\omega^i\}$ is dual[7] to $\{\mathbf{e}_i\}$, i.e., $\omega^i(\mathbf{e}_j) = \delta^i_{\ j}$, and the tensor product of two covectors is simply their product. That is, the g_{ij} are the components of the metric tensor $\mathbf{g}$ with respect to the basis $\omega^i \otimes \omega^j$, $\mathbf{g} = g_{ij}\omega^i \otimes \omega^j$.

We can write $x^i y^j g_{ij}$ in matrix notation as

$$\mathbf{x}^T \mathbf{g}\mathbf{y}. \tag{4.8}$$

Inverse metric. Given a vector $\mathbf{x} = x^i\mathbf{e}_i$, the map $\mathbf{y} \mapsto \mathbf{g}(\mathbf{x},\mathbf{y})$ defines a covector (or 1-form). From Eq. 4.7, we get

$$\begin{aligned} \mathbf{g}(\mathbf{x},\mathbf{y}) &= g_{ij} x^i \omega^j(\mathbf{y}), \quad \text{or} \\ (\mathbf{y} \mapsto \mathbf{g}(\mathbf{x},\mathbf{y})) &= g_{ij} x^i \omega^j. \end{aligned} \tag{4.9}$$

This means that the components of the covector are simply $g_{ij}x^i$. If we set $x_j := g_{ij}x^i$, we can write the covector[8] as $x_j\omega^j$. Thus, using the metric to convert a vector into a covector, we have effectively *lowered* the index of the components.

The matrix $\mathbf{g}$ with components g_{ij} is invertible with inverse $\mathbf{g}^{-1}$, whose components are denoted by g^{ij}. This means that

$$g_{jk}g^{ki} = g^{ik}g_{kj} = \delta^i_{\ j}, \tag{4.10}$$

or in matrix notation

$$\mathbf{g}\mathbf{g}^{-1} = \mathbf{g}^{-1}\mathbf{g} = I, \tag{4.11}$$

[3] since they have the same dimension

[4] compare Equation 3.19 and the discussion thereafter

[5] that is, a Riemannian manifold

[6] pronounced x flat, the motivation for the musical notation comes from the fact, that in a chart the indices of components are effectively lowered, see below

[7] this also means that $\omega^i(\mathbf{x}) = x^i$ if $\mathbf{x} = x^i\mathbf{e}_i$, that is, ω^i reads off the i^{th} component of $\mathbf{x}$ in the basis $\mathbf{e}_i$

[8] that is, $\omega_{\mathbf{x}} = x_j\omega^j$, or $\omega_{\mathbf{x}} = \mathbf{x}_\flat$

with I the identity matrix. Given the components x_j of a covector[9] ω, we now obtain the components of the corresponding vector by *raising* the index[10]: $x^i = g^{ij}x_j$. See Lee (Lee, 2018, p. 26)[11] for more details.

[9] $\omega = x_j\omega^j$

[10] that is $\mathbf{x}_\omega = x^i\mathbf{e}_i$ or $\mathbf{x}_\omega = \omega^\sharp$

[11] John M. Lee. *Introduction to Riemannian Manifolds*. Springer-Verlag, 2nd edition, 2018

Example 4. *As an example we look at the gradient of scalar fields on the plane again. Let $f : M \to \mathbb{R}$ with $M = \mathbb{R}^2$. On a generic smooth manifold*[12] *there is no gradient vector, all we have is the differential df of f, and its representation in components*[13]

[12] without a Riemannian metric

[13] using the dual basis of covectors

$$df = \frac{\partial f}{\partial x^i}dx^i = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{4.12}$$

If we use Cartesian coordinates we use the chart[14] $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $\varphi = id_{\mathbb{R}^2}$, *that is,* $\varphi((x,y)) = (x,y)$. *Then,* $\varphi^{-1} = id_{\mathbb{R}^2}$ *as well and we get the coordinate basis vectors*[15]

[14] note, that the first $\mathbb{R}^2$ is seen as a manifold, where the chart provides the differential structure, and the second $\mathbb{R}^2$ as the parameter space where we can use multi-variable calculus, especially partial derivatives

[15] compare Equation **??**

$$\begin{aligned}\mathbf{e}_1 = \boldsymbol{\partial}_1 = \frac{\partial}{\partial x^1} = \frac{\partial}{\partial x} = \frac{\partial}{\partial x}\varphi^{-1}(x,y) = (1,0),\\ \mathbf{e}_2 = \boldsymbol{\partial}_2 = \frac{\partial}{\partial x^2} = \frac{\partial}{\partial y} = \frac{\partial}{\partial y}\varphi^{-1}(x,y) = (0,1).\end{aligned} \tag{4.13}$$

We compute the components of the metric tensor and its inverse as

$$g_{ij} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad g^{ij} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{4.14}$$

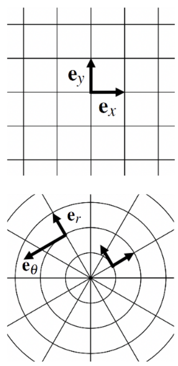


Figure 4.1: Coordinate basis vectors in Cartesian and polar coordinates.

To convert the differential of f[16] *to the gradient vector, we have to use the inverse metric, which is just the identity in this case, to raise the indices and get the components for the gradient vector as*

[16] which is a covector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_2. \tag{4.15}$$

That is, in this case the components of the gradient vector are just the partial derivatives as well.

If we use polar coordinates[17]*, however, things change. First, we use the chart*[18] $\varphi((x,y)) = (\sqrt{x^2+y^2}, \arctan\frac{y}{x})$ *and its inverse*[19] $\varphi^{-1}((r,\theta)) = (r\cos\theta, r\sin\theta)$. *We get for the coordinate basis vectors*

[17] it is important to note, that we are still on the same flat manifold, the plane, we are just using different coordinates

[18] on a suitable domain

[19] note that we now also clearly distinguish between points on the manifold, denoted by x and y, and the parameter domain, denoted by r and θ

$$\begin{aligned}\mathbf{e}_1 = \mathbf{e}_r = \boldsymbol{\partial}_1 = \frac{\partial}{\partial x^1} = \frac{\partial}{\partial r} = \frac{\partial}{\partial r}\varphi^{-1}(r,\theta) = (\cos\theta, \sin\theta),\\ \mathbf{e}_2 = \mathbf{e}_\theta = \boldsymbol{\partial}_2 = \frac{\partial}{\partial x^2} = \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta}\varphi^{-1}(r,\theta) = (-r\sin\theta, r\cos\theta).\end{aligned} \tag{4.16}$$

Finally, the components of the metric tensor and its inverse in this chart are given[20] *by*

[20] $\cos^2\theta + \sin^2\theta = 1!$

$$g_{ij} = \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}, \qquad g^{ij} = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{r^2}\end{pmatrix}. \tag{4.17}$$

That is the gradient vector in polar coordinates is given by[21]

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta. \tag{4.18}$$

[21] to be precise, we should write this as

$$\nabla f|_p = \left.\frac{\partial f}{\partial r}\right|_p \mathbf{e}_r + \frac{1}{r^2}\left.\frac{\partial f}{\partial \theta}\right|_p \mathbf{e}_\theta.$$

for $p = (x,y)$, since the coordinate basis vectors now are different for different points, see Figure 4.1

4.2 *The Directional Derivative*

Although the metric tensor provides a way to define geometric notions, it does not allow us yet to take the directional derivative of a vector field[22], for example. The notion that is missing is called a *connection*, and it is a structure that has to be provided. On a Riemannian manifolds it will turn out that there is a unique connection[23], the Levi-Civita connection. The motivation for this connection is that is generalizes directional derivatives in Euclidean space. So to understand connections better we first take a closer look at the directional derivative in Euclidean space, using multi-variable calculus, and then see how to generalize it to curved manifolds.

[22] or higher order tensor fields

[23] with certain properties, see below

Let $a \in \mathbb{R}^n$, that is, $a = (a^1, \ldots, a^n)$. Then[24], $X_p = a^i \partial / \partial x^i\big|_p$ is a differential operator D_{x_p}, the directional derivative in direction a at p. Then, setting $x^i = p^i + ta^i$, we can differentiate a scalar field in the following way:

[24] Einstein summation convention!

$$\begin{aligned} D_{X_p} f &= \lim_{t \to 0} \frac{f(p+ta) - f(p)}{t} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(p+ta) \\ &= \left.\frac{\partial f}{\partial x^i}\right|_p \cdot \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\partial f}{\partial x^i}\right|_p \cdot a^i \\ &= \left(a^i \left.\frac{\partial}{\partial x^i}\right|_p \right) f = X_p f. \end{aligned} \tag{4.19}$$

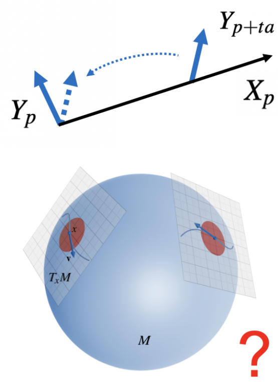


Figure 4.2: A vector field in the plane can be differentiated by moving all vectors to one point and differentiating them there (top). On the sphere it is not clear how to do this (bottom).

We can also differentiate a vector field $Y = b^i \partial / \partial x^i$ in Euclidean space, by simply moving all vectors to the point p, which just means that we can differentiate the component functions:

$$D_{X_p} Y = \left(X_p b^i \right) \left.\frac{\partial}{\partial x^i}\right|_p \tag{4.20}$$

However, on curved manifolds it is not all clear how to do this, since we do not know how to move vectors from one tangent space to another (see Figure 4.2).

To generalize the directional derivative to arbitrary manifolds, we note that we can extend Equation 4.20 to whole vector fields by defining it pointwise:

$$(D_X Y)_p = D_{X_p} Y \tag{4.21}$$

This defines an operator[25]

[25] $\mathfrak{X}(\mathbb{R}^n)$ denotes all vector fields on $\mathbb{R}^n$

$$\begin{aligned} D : \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) &\to \mathfrak{X}(\mathbb{R}^n), \\ (X, Y) &\mapsto D(X, Y) = D_X Y \end{aligned} \tag{4.22}$$

where we write $D_X Y$ instead of $D(X,Y)$.[26]. We now apply a common strategy in mathematics to generalize something. We look for properties of the notion we want to generalize and see if we can find something more general with the same properties. If this more general notion also fits the original one, we consider it a generalization.

[26] the reason is that it is not completely symmetric, see below, but also to make the meaning of the arguments clear, $D_X Y$ is the directional derivative of Y in direction X

Properties of the directional derivative

Define multiplication of a vector field by a smooth function pointwise: $(fX)_p := f(p)X_p$. The directional derivative has the following properties:

1. It is $\mathbb{R}$-linear in X and Y: For all $a,b \in \mathbb{R}$ and $X,Y,Z \in \mathfrak{X}(\mathbb{R}^n)$

$$\begin{aligned} D_{aX+bY}Z &= aD_XZ + bD_YZ \\ D_Z(aX+bY) &= aD_ZX + bD_ZY \end{aligned} \tag{4.23}$$

2. It is $\mathfrak{F}$-linear in X: For all[27] $f \in C^\infty(\mathbb{R}^n)$ and $X,Y \in \mathfrak{X}(\mathbb{R}^n)$

$$D_{fX}Y = fD_XY \tag{4.24}$$

3. The Leibniz rule holds in Y: For all $f \in C^\infty(\mathbb{R}^n)$ and $X,Y \in \mathfrak{X}(\mathbb{R}^n)$

$$D_X(fY) = (Xf)Y + fD_XY \tag{4.25}$$

[27] $C^\infty(\mathbb{R}^n)$ denotes all smooth functions on $\mathbb{R}^n$

4.3 Connections

This way of looking at the directional derivative lets us now define connections[28]. That is, a (linear or affine)[29] connection on a manifold M is an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \tag{4.26}$$

with the properties

1. It is $\mathfrak{F}$-linear in X: For all $f \in C^\infty(\mathbb{R}^n)$ and $X,Y \in \mathfrak{X}(\mathbb{R}^n)$

$$\nabla_{fX}Y = f\nabla_XY \tag{4.27}$$

2. The Leibniz rule holds in Y: For all $f \in C^\infty(\mathbb{R}^n)$ and $X,Y \in \mathfrak{X}(\mathbb{R}^n)$

$$\nabla_X(fY) = (Xf)Y + f\nabla_XY \tag{4.28}$$

[28] that is, generalizations of the directional derivative to arbitrary manifolds

[29] although there are subtle mathematical differences between these two notions, they need not concern us here

4.4 Parallel Transport

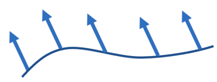


Figure 4.3: Parallel transport in the plane.

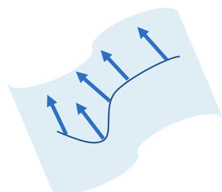


Figure 4.4: Parallel transport on a curved manifold.

Let $\gamma : I \to M$ be a curve in the manifold. A connection determines a parallel transport operator

$$P^\gamma_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M \tag{4.29}$$

We could also go in the other direction and define a parallel transport operator which then in turn defines a connection:

$$\nabla_X Y\big|_p = \lim_{h\to 0} \frac{P^\gamma_{h0} Y_{\gamma(h)} - Y_p}{h} \tag{4.30}$$

where $\gamma(0) = p$. See Lee (Lee, 2018, p. 108)[30] for details. This parallel transport operator now allows us to move vectors to one tangent space, so we can differentiate a vector field. However, we still have the problem that there exist many different connections on a manifold[31]. We therefore apply the same strategy as before and look at more properties of the directional derivative and see if they narrow down our choice for a connection.

[30] John M. Lee. *Introduction to Riemannian Manifolds*. Springer-Verlag, 2nd edition, 2018

[31] and therefore different notions for directional derivatives, parallel transport, or geodesics

Other properties of the directional derivative

1. The first property we look at is called torsion, it measure how much the directional derivative commutes, that is, what is the difference between $D_X Y$ and $D_Y X$. It turns out that in Euclidean space this is always given by the Lie bracket $[X,Y]$. So one defines the torsion T of a connection as

$$T(X,Y) := D_X Y - D_Y X - [X,Y] = 0 \tag{4.31}$$

and says that the Euclidean connection has zero torsion.

2. We can also define curvature using only the connection, and it is always zero as well:

$$R(X,Y) := [D_X, D_Y] - D_{[X,Y]} = D_X D_Y - D_Y D_X - D_{[X,Y]} = 0 \tag{4.32}$$

that is

$$R(X,Y)Z := 0 \tag{4.33}$$

3. To motivate the last property, we note that so far the two notions of metric and connection are completely independent of each other. We certainly want to have some kind of compatibility of the two. It turns out that the directional derivative is compatible with the metric in Euclidean space in the sense that it obeys a kind of product rule:

$$D_Z\langle X,Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle \tag{4.34}$$

4.5 The Levi-Civita Connection

If we now want to use these properties on curved manifolds, we certainly have to drop the property of zero curvature. It turns out, however, that, when we require a connection to have zero torsion which is compatible with the metric, there is a unique such connection. This connection is called the Levi-Civita[32] connection.

So to summarize, on a Riemannian manifold M there is a unique connection, this is now written as ∇, with the properties

[32] or sometimes Riemannian connection

1. Zero torsion: For all[33] $X,Y \in \mathfrak{X}(M)$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0 \tag{4.35}$$

[33] $\mathfrak{X}(M)$ now denotes all vector fields on the manifold M

2. Compatibility with the metric: For all $X,Y,Z \in \mathfrak{X}(M)$

$$\nabla_Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \tag{4.36}$$

Computation

In a chart with coordinate basis vector fields $\mathbf{e}_i = \boldsymbol{\partial}_i = \frac{\partial}{\partial x^i}$ we can look at all combinations

$$\nabla_{\mathbf{e}_j}\mathbf{e_k}. \tag{4.37}$$

These are vectors which can be expressed as linear combinations of the coordinate basis vectors again:

$$\nabla_{\mathbf{e}_j}\mathbf{e}_k = \Gamma^i{}_{jk}\mathbf{e}_i. \tag{4.38}$$

The symbols $\Gamma^i{}_{jk}$ corresponding to the unique Levi-Civita connection for a given metric $\mathbf{g}$ on M are called the *Christoffel symbols*. We now can write in a chart[34] for vector fields $X = a^j\mathbf{e}_j$ and $Y = b^k\mathbf{e}_k$

$$\nabla_X Y = \left(X(b^k) + a^j b^k \Gamma^i{}_{jk}\right)\mathbf{e}_i. \tag{4.39}$$

The Christoffel symbols can also be derived intrinsically from the components g_{ij} of $\mathbf{g}$, referred to the same basis (and its dual)[35], via

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im}\left(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk}\right). \tag{4.40}$$

See, e.g., the book by do Carmo [36]. The components g_{ij} give the metric $\mathbf{g}$ referred to the basis $\{\omega^i \otimes \omega^j\}$, and g^{ij} is its inverse $\mathbf{g}^{-1}$, i.e., $g^{ik}g_{kj} = \delta^i_j$, referred to the basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$.

[34] if we use Cartesian coordinates, the Christoffel symbols all all zero, so this expression results in the usual differentiation of the coordinate functions

[35] This equation is specified for the use of a coordinate basis $\mathbf{e}_i = \boldsymbol{\partial}_i$, with $[\mathbf{e}_i, \mathbf{e}_j] = 0$, and its dual coordinate basis $\omega^i = dx^i$.

[36] Manfredo Perdigao do Carmo. *Riemannian Geometry*. Birkhäuser, 1992

5 Smooth maps between manifolds, isometries

Smooth maps (including *diffeomorphisms*[1]) between manifolds, denoted

$$\phi : M \to N, \tag{5.1}$$

including the special case $N = M$, are crucial for many important applications in flow visualization and continuum mechanics, in particular for characterizing deformations of fluids and solids, as well as for defining reference frame transformations and the corresponding concept of *objectivity*. Moreover, a smooth map between two manifolds induces the corresponding *pushforward* as well as the corresponding *pullback* operations.

For each $x \in M$, the *pushforward* $\mathrm{d}\phi$ (or ϕ_*) is a *linear* map, also called the *differential*, from the tangent space at $x \in M$, i.e., T_xM, to the corresponding tangent space at $\phi(x) \in N$, i.e., $T_{\phi(x)}N$. This map enables "pushing forward" individual tangent vectors to other points, where the "source" point and the "destination" point are connected by the map ϕ, and in the case of diffeomorphisms enables pushing forward entire vector fields.

We further introduce the corresponding *pullback* ϕ^*, which can always "pull back" covariant tensor fields, including differential forms. This includes the important application of *pullback metrics*, i.e., the pullback of a second-order covariant metric tensor field, which enables a general characterization of *isometries* and *infinitesimal isometries*.

In contrast to smooth maps that are not diffeomorphisms, if ϕ is indeed a diffeomorphism[2], we can push forward (and pull back) whole vector fields from the manifold M to the manifold N, and vice versa. Moreover, for diffeomorphisms, pushforwards and pullbacks enable mapping tensor fields of mixed type (mixed variance) between manifolds, by applying the pushforward and the pullback operations, respectively, to each tensor argument (vector or 1-form, i.e., covector) individually.

The most important concepts covered in this chapter are:

- Diffeomorphisms.
- Pushforwards and pullbacks.
- Pullback metrics.
- Active deformations.
- Isometries and infinitesimal isometries.
- Flows and flow maps of vector fields.

[1] Other important smooth maps are *embeddings*, *immersions*, and *submersions*, which, unlike diffeomorphisms, are not bijective. We touch on these maps, but mainly focus on diffeomorphisms.

[2] A smooth map ϕ is called a *diffeomorphism* if it is (1) bijective, i.e., it is both one-to-one and onto, and (2) if it also has a smooth inverse ϕ^{-1}.

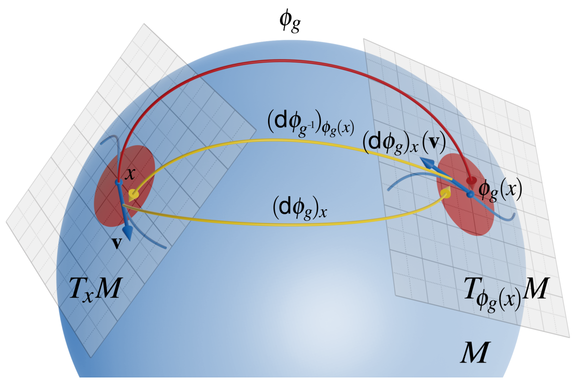


Figure 5.1: **Pushforward and pullback** of a diffeomorphism ϕ_g are linear maps between the tangent spaces T_xM and $T_{\phi_g(x)}M$, and cotangent spaces $T^*_{\phi_g(x)}M$ and T^*_xM, respectively. The pushforward $(\mathrm{d}\phi_g)_x$ maps a tangent vector $\mathbf{x} \in T_xM$ to the vector $(\mathrm{d}\phi_g)_x(\mathbf{x}) \in T_{\phi_g(x)}M$. The pullback ϕ_g^* maps a covector (1-form) $\omega \in T^*_{\phi_g(x)}M$ to the covector (1-form) $\phi_g^*\omega \in T^*_xM$.

5.1 *Notation*

There are various notations for pushforwards and pullbacks. A very common notation used in mathematics is to write

$$\text{smooth map: } \phi, \quad \text{pushforward: } \phi_*, \quad \text{pullback: } \phi^*, \tag{5.2}$$

for the smooth map (including diffeomorphisms), the corresponding pushforward, and the corresponding pullback, respectively. Because the pushforward is the differential of a smooth map, these three entities are often also written as

$$\text{smooth map: } \phi, \quad \text{pushforward: } \mathrm{d}\phi, \quad \text{pullback: } \phi^*, \tag{5.3}$$

again for the smooth map (including diffeomorphisms), the corresponding pushforward, and the corresponding pullback, respectively.

5.2 *Diffeomorphisms*

A smooth map $\phi : M \to N$ that is smooth and bijective, i.e., one-to-one and onto, and that has an inverse map $\phi^{-1} : N \to M$ that is also smooth, is called a diffeomorphism between the manifolds M and N.[3] The special case where $M = N$ is also included in this definition, and is particularly important for active deformations of a manifold M and for reference frame transformations, i.e., where we have $\phi : M \to M$.

[3] If such a map exists, the manifolds M and N are called *diffeomorphic*.

Sometimes we denote diffeomorphisms by maps ϕ_g, where $g \in G$ is an element of a transformation group (a Lie group) G, for example all rotations of three-dimensional Euclidean space, where $G = SO(3)$, and each g is a 3×3 rotation matrix. The map ϕ_g is then the diffeomorphism corresponding to the transformation described by g, i.e., we have an isomorphism between the group G and the group of diffeomorphisms ϕ_g, for example all maps

$$\phi_g : M \to M. \tag{5.4}$$

The inverse diffeomorphism then corresponds to the inverse group element $g^{-1} \in G$, where we then have the inverse maps

$$(\phi_g)^{-1} = \phi_{g^{-1}}. \tag{5.5}$$

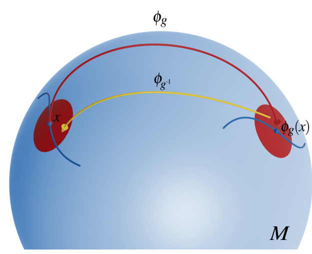


Figure 5.2: A diffeomorphism ϕ_g always has a smooth inverse $\phi_{g^{-1}}$.

Local diffeomorphisms

We can also define diffeomorphisms only locally, in order to restrict the region of definition, for example to the neighborhood of a given point. Local diffeomorphisms are particularly important for the flows of vector fields. If a vector field is not complete, i.e., when it cannot be integrated for all time (where stream or path lines enter and exit the domain boundary, for example), then the *flow* of the vector field only gives local diffeomorphisms, instead of diffeomorphisms of the entire manifold.

5.3 Pushforwards and Pullbacks

Pushforwards

A smooth map $\phi : M \to N$ induces for each $x \in M$ a *linear* map

$$\begin{aligned}(\mathrm{d}\phi)_x \colon T_xM &\to T_{\phi(x)}N, \\ \mathbf{x} &\mapsto (\mathrm{d}\phi)_x(\mathbf{x}),\end{aligned} \tag{5.6}$$

called the *differential* or *pushforward*, from the tangent space at $x \in M$, i.e., T_xM, to the tangent space at $\phi(x) \in N$, i.e., $T_{\phi(x)}N$. While the map ϕ does not have to be a diffeomorphism, i.e., it does not have to be one-to-one or onto (for example for embeddings, immersions, or submersions), diffeomorphisms and the special case where $M = N$ are of course included in this definition. This is illustrated geometrically in Fig. 5.4 for a diffeomorphism $\phi \colon M \to M$: Choosing a smooth curve through the point $x \in M$ defines a tangent vector $\mathbf{v} \in T_xM$. The map ϕ maps this smooth curve to another smooth curve through the point $\phi(x) \in M$, defining the corresponding tangent vector $(\mathrm{d}\phi)_x(\mathbf{v}) \in T_{\phi(x)}M$. Now, the pushforward of a smooth map corresponds exactly to this mapping of curves, but it allows pushing forward individual tangent vectors to other points on the manifold without having to explicitly consider the curves to which they are tangent.

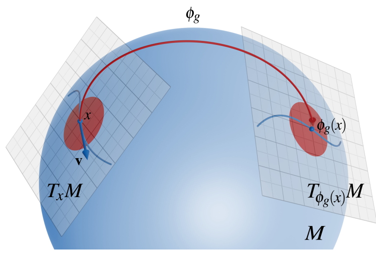


Figure 5.3: Diffeomorphisms on smooth manifolds link tangent spaces.

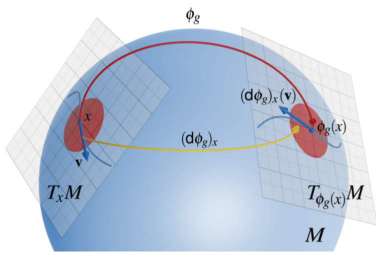


Figure 5.4: Pushforward of a tangent vector, induced by the diffeomorphism.

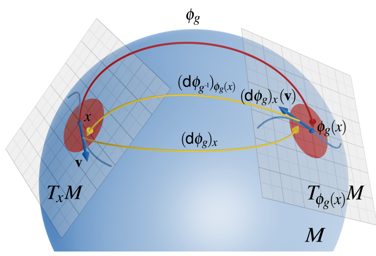


Figure 5.5: Due to the inverse, we can pushforward vectors in both directions.

Pushforwards of diffeomorphisms

If a smooth map ϕ happens to be both one-to-one and onto, with smooth inverse ϕ^{-1}, i.e., when the smooth map ϕ is a diffeomorphism, then it uniquely defines another vector *field* with the tangent vectors being the pointwise pushforwards. This is not possible with an arbitrary smooth map ϕ, as this fails to define tangent vectors at points not hit by ϕ (when ϕ is not onto), or might define tangent vectors ambiguously at points hit several times (when ϕ is not one-to-one). That is, we can use a map ϕ to pushforward whole vector fields precisely when ϕ is a diffeomorphism.

Pullbacks

The analogous concept to the pushforward of a *vector* field is the *pullback* of a *covector* (1-form) field. The pullback ϕ^* of a 1-form (covector) field is

$$\begin{aligned}(\phi^*)_x : T^*_{\phi(x)}M &\to T^*_xM, \\ \omega &\mapsto (\phi^*)_x(\omega).\end{aligned} \tag{5.7}$$

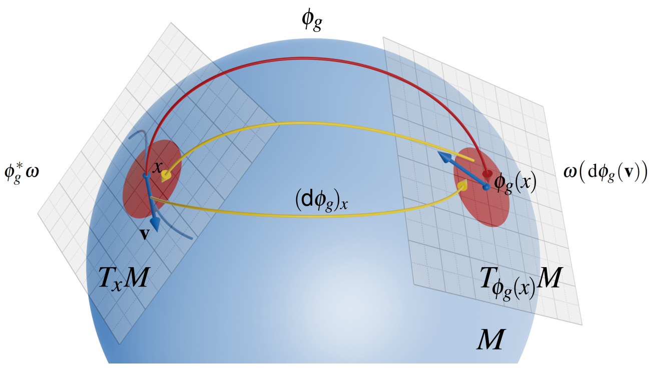


Figure 5.6: Pullback of a 1-form ω.

Since $(\phi^*)_x(\omega)$ is a 1-form (covector), in order to define it we have to specify how it acts on a vector $\mathbf{x} \in T_xM$. We simply use the pushforward and the fact that ω is a 1-form (covector) in the cotangent space at $\phi(x)$:

$$(\phi^*)_x(\omega)(\mathbf{x}) := \omega((\mathrm{d}\phi)_x(\mathbf{x})). \tag{5.8}$$

In contrast to vector fields, 1-forms (covector fields) always pull back to 1-forms (covector fields), even when the map ϕ is not a diffeomorphism.

5.4 Pullback Metrics

Given a smooth map ϕ from a differential manifold M to a Riemannian manifold $(N,\mathbf{h})$, with a metric tensor field $\mathbf{h}$ given on N, the pullback metric $\mathbf{g} := \phi^*\mathbf{h}$ defines a metric on the manifold M, i.e., via the pullback metric we obtain the Riemannian manifold

$$(M,\mathbf{g}) := (M,\phi^*\mathbf{h}). \tag{5.9}$$

The pullback metric is defined via the pushforward of both argument vectors, i.e., we define

$$\begin{aligned}\phi^*\mathbf{h}\colon\ & T_xM \times T_xM \to \mathbb{R}, \\ & (\mathbf{v},\mathbf{w}) \mapsto \phi^*\mathbf{h}(\mathbf{v},\mathbf{w}) := \mathbf{h}\left((\mathrm{d}\phi)_x(\mathbf{v}),(\mathrm{d}\phi)_x(\mathbf{w})\right).\end{aligned} \tag{5.10}$$

For brevity, we can then define a metric $\mathbf{g}$ on M as

$$\mathbf{g}(\mathbf{v},\mathbf{w}) := \phi^*\mathbf{h}(\mathbf{v},\mathbf{w}). \tag{5.11}$$

Here, $\mathbf{h}$ is a metric on the manifold N, and $\mathbf{g} := \phi^*\mathbf{h}$ is a metric on the manifold M, defined as the pullback metric of $\mathbf{h}$, pulled back from N to M.

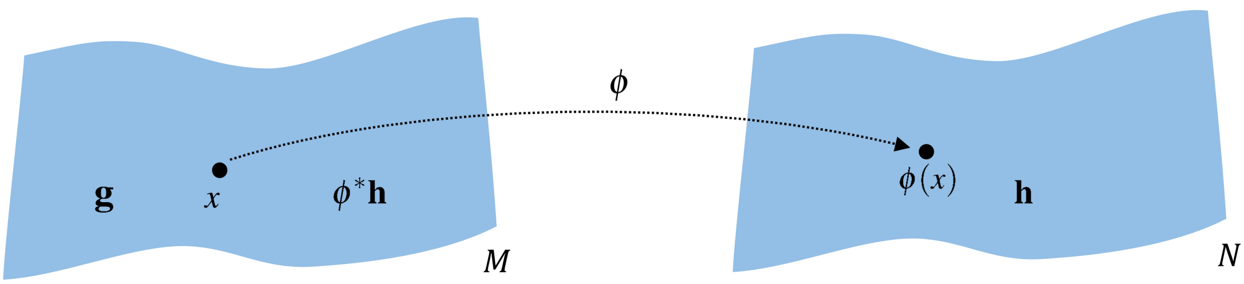


Figure 5.7: Pullback metric and isometries.

5.5 Active Deformations

We can model active deformations as diffeomorphisms ϕ, with

$$\begin{aligned}\phi : M &\to M, \\ x &\mapsto \phi(x),\end{aligned} \tag{5.12}$$

where the metric $\mathbf{g}$ on M is fixed. A general active deformation is not an isometry (see below). However, it can be the special case of an isometry defined via an active transformation, however without deformation.[4]

A simple instructive example is a scaling of "space" (in the form of a manifold $M = \mathbb{R}^n$), and considering the transformation properties of the gradient 1-form df, and comparing with the (wrong) transformation of the gradient vector ∇f (if it is transformed like a regular vector).

[4] An active transformation that is an isometry, for example, would be a rotation of $\mathbb{R}^n$, or any other isometry (translations and rotations) of Euclidean space.

5.6 Isometries and Infinitesimal Isometries

A smooth map $\phi : M \to N$, where both $(M, \mathbf{g})$ and $(N, \mathbf{h})$ are given Riemannian manifolds, is an *isometry*, if we have[5]

$$\mathbf{g} = \phi^* \mathbf{h}. \tag{5.13}$$

[5] It is crucial to note that, here, we *compare* an existing metric $\mathbf{g}$ to the pullback metric $\phi^*\mathbf{h}$. (Instead of defining the metric $\mathbf{g}$ to be identical to the pullback metric.)

Alternatively, we can also write this directly with inner products on M and N, respectively, where for all pairs of vectors $\mathbf{v}$ and $\mathbf{w}$ we must have

$$\langle \mathbf{v}, \mathbf{w} \rangle_x = \langle (\mathrm{d}\phi)_x(\mathbf{v}), (\mathrm{d}\phi)_x(\mathbf{w}) \rangle_{\phi(x)}. \tag{5.14}$$

As a direct consequence, if ϕ is an isometry we also have

$$\|\mathbf{v}\|_x = \|(\mathrm{d}\phi)_x(\mathbf{v})\|_{\phi(x)}. \tag{5.15}$$

We note that the map ϕ can be an *embedding* or an *immersion*, or a *diffeomorphism*, for example. If the map ϕ is a diffeomorphism, we would say that if it is also an isometry, then the manifolds M and N are, with respect to the metric, "the same" Riemannian manifold.

Infinitesimal isometries

If we have a one-parameter group of diffeomorphisms ϕ_t, i.e., we have $t \mapsto \phi_t$, we can take the derivative with respect to the parameter t, and in this way obtain a vector field on the manifold N. If the diffeomorphisms ϕ_t are isometries, than the corresponding vector fields are called *infinitesimal isometries*, because they are the derivatives of isometries. This particular kind of vector field is also called a *Killing vector field*.

5.7 Mapping Tensor Fields between Manifolds

Because all tensors are multi-linear maps taking an ordered list of vector and 1-form arguments, tensors of arbitrary order and variance can be mapped between manifolds by simply using the constructions of the push-forward (for vector arguments) and the pullback (for 1-form arguments) for each argument individually. However, one has to be careful that several of the following constructions can only be defined for diffeomorphisms, and not for general smooth maps, between manifolds, because they require the map to be smoothly invertible.

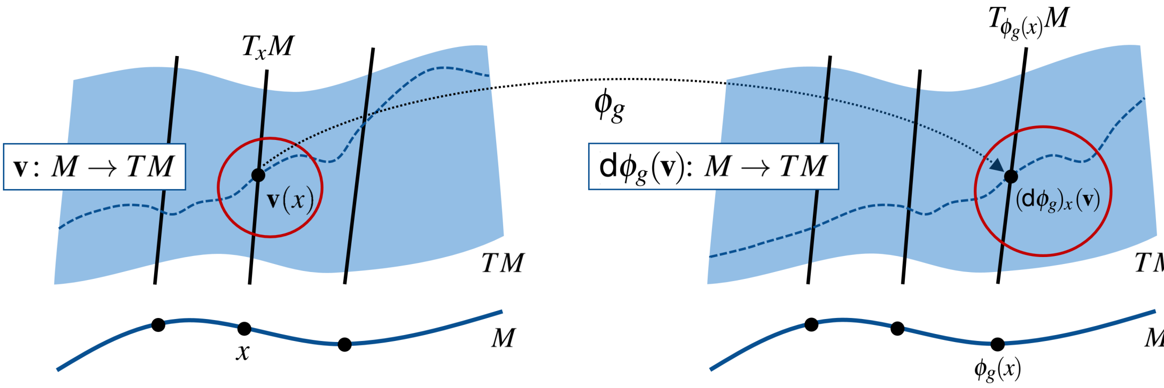


Figure 5.8: Pushforward of vector field on tangent bundle (for diffeomorphisms).

Pullbacks of 1-forms and k-forms (through any smooth map)

As above, the pullback $\phi_g^* \omega$ of a 1-form (field) ω is

$$(\phi_g^* \omega)(\mathbf{v}) := \omega\big(\mathrm{d}\phi_g(\mathbf{v})\big). \tag{5.16}$$

This definition can always pull back entire 1-form fields, and not just an individual 1-form at a specific point, even for smooth maps that are not diffeomorphisms.

For k-forms, i.e., differential forms acting on k vector arguments, we simply use an analogous definition to the above, where each vector argument is separately pushed forward by the same map $\mathrm{d}\phi_g$. That is, we get the pullback as

$$(\phi_g^* \omega)(\mathbf{v}_1, \ldots, \mathbf{v}_k) := \omega\big(\mathrm{d}\phi_g(\mathbf{v}_1), \ldots, \mathrm{d}\phi_g(\mathbf{v}_k)\big). \tag{5.17}$$

Pushforwards of vector fields (through diffeomorphisms)

In contrast to smooth maps that are not diffeomorphisms, where we can only pushforward individual vectors from one tangent space to another tangent space, for a diffeomorphism ϕ_t we can pushforward an entire vector field as

$$\mathrm{d}\phi_g\colon TM \to TN, \tag{5.18}$$

$$\mathbf{v} \mapsto \mathrm{d}\phi_g(\mathbf{v}). \tag{5.19}$$

This means that we have defined a vector feld via the point-wise mapping

$$\big(\mathrm{d}\phi_g(\mathbf{v})\big)_{\phi_g(x)} := \big(\mathrm{d}\phi_g\big)_x(\mathbf{v}). \tag{5.20}$$

In this way, the diffeomorphism in fact maps a vector field as a section of one tangent bundle, i.e., $\mathbf{v}\colon M \to TM$, to the corresponding vector field as a section of another tangent bundle, i.e., $\mathrm{d}\phi_g(\mathbf{v})\colon N \to TN$.

Pullbacks of vector fields (through diffeomorphisms)

For diffeomorphisms ϕ_t, instead of using the pushforwards of the corresponding inverse diffeomorphisms ϕ_t^{-1}, we can also choose to use the equivalent pullback ϕ_t^*, in order to "map back" vectors from points $\phi_t(x)$ to points x by defining

$$(\phi_t^* \mathbf{v})_x := \big(\mathrm{d}\phi_t^{-1}\big)_{\phi_t(x)}(\mathbf{v}). \tag{5.21}$$

Mapping mixed tensors (through diffeomorphisms)

For diffeomorphisms ϕ_g, the pushforward and pullback directly generalize to general maps for $\binom{r}{s}$ tensor fields $\mathbf{T}\left(\omega^1,\ldots,\omega^r,\mathbf{v}_1,\ldots,\mathbf{v}_s\right)$, with r 1-form (covector) arguments ω^j, and s vector arguments $\mathbf{v}_i$, because any tensor $\mathbf{T}$ is a multi-linear function of these arguments[6]. We can thus apply the definition for vectors argument-wise. We can write this pullback as

$$\phi_g^* \mathbf{T}. \tag{5.22}$$

[6] For brevity, we list the arguments in this order, but the analogous applies for an argument list of contravariant and covariant arguments in any order.

Writing this out fully with the diffeomorphism ϕ_g, corresponding to a transformation $g \in G$, and its corresponding inverse transformation $\phi_{g^{-1}}$, we get the pullback of an arbitrary $\binom{r}{s}$ tensor field $\mathbf{T}$ as

$$\begin{aligned}\phi_g^* \mathbf{T}\big(\omega^1,\ldots,\omega^r,\mathbf{v}_1,\ldots,\mathbf{v}_s\big) &= \\ &\mathbf{T}\big(\phi_{g^{-1}}^*\omega^1,\ldots,\phi_{g^{-1}}^*\omega^r,\mathrm{d}\phi_g(\mathbf{v}_1),\ldots,\mathrm{d}\phi_g(\mathbf{v}_s)\big).\end{aligned} \tag{5.23}$$

Using the action $g\cdot$ of the transformation group element $g \in G$, and the corresponding inverse action $g^{-1}\cdot$, we can also write this more succinctly as

$$\begin{aligned}\phi_g^* \mathbf{T}\big(\omega^1,\ldots,\omega^r,\mathbf{v}_1,\ldots,\mathbf{v}_s\big) &= \\ &\mathbf{T}\big(g^{-1}\cdot\omega^1,\ldots,g^{-1}\cdot\omega^r,g\cdot\mathbf{v}_1,\ldots,g\cdot\mathbf{v}_s\big).\end{aligned} \tag{5.24}$$

We emphasize, however, that evaluating this pullback is only possible for diffeomorphisms, because we need the inverse $\phi_{g^{-1}}$. For a diffeomorphism this inverse is guaranteed to exist and it is smooth.

For non-invertible maps, the pullback of tensors of mixed variance is not defined.

Mapping linear maps (through diffeomorphisms)

The special case of the pullback of a $\binom{1}{1}$ tensor field $\mathbf{T}$, i.e., a bi-linear map that maps one 1-form (covector) and one vector to a scalar, which can also be seen as a linear map from one input vector to one corresponding output vector, is given by

$$\begin{aligned}\phi_g^* \mathbf{T}\big(\omega,\mathbf{v}\big) &= \mathbf{T}\big(\phi_{g^{-1}}^*\omega,\, \mathrm{d}\phi_g(\mathbf{v})\big), \\ &= \mathbf{T}\big(g^{-1}\cdot\omega,\, g\cdot\mathbf{v}\big).\end{aligned} \tag{5.25}$$

For a diffeomorphism ϕ_g, we can also write the pullback ϕ_g^* of $\mathbf{T}$, interpreted as a linear map $\mathbf{v}\mapsto\mathbf{T}(\mathbf{v})$ from vector to vector, as

$$\begin{aligned}(\phi_g^*\mathbf{T})_x\colon T_xM &\to T_xM, \\ \mathbf{v} &\mapsto (\phi_g^*\mathbf{T})_x(\mathbf{v}) := \mathrm{d}\phi_g^{-1}\big(\mathbf{T}\big(\mathrm{d}\phi_g(\mathbf{v})\big)\big).\end{aligned} \tag{5.26}$$

Again, while pullbacks in general are defined for smooth maps that need not be diffeomorphisms, the above definitions require the map ϕ_g to be a diffeomorphism (guaranteed to have an inverse) to allow mapping back vectors in the inverse direction ϕ_g^{-1}.

Example: Pullback of vector fields and linear maps on the sphere

For an isometry ϕ_t, the pullback ϕ_t^* of a vector field $\mathbf{v}$ on the sphere is

$$\phi_t^*\,\mathbf{v} = (\mathbf{B}^*)'\,\mathbf{R}^T(t)\,\mathbf{B}\,\mathbf{v}. \tag{5.27}$$

The pullback ϕ_t^* of a second-order tensor field $\mathbf{T}$ on the sphere is

$$\phi_t^* \mathbf{T} = (\mathbf{B}^*)' \mathbf{R}^T(t) \mathbf{B} \ \mathbf{T} \ \mathbf{B}' \mathbf{R}(t) \mathbf{B}^*. \tag{5.28}$$

The $\mathbf{R}(t)$ are 3D rotation matrices determining the corresponding isometry ϕ_t of the sphere. The 3×2 matrices $\mathbf{B}$, $\mathbf{B}^*$ map vectors $\mathbf{v}$, at $\phi_t(x)$, and $\mathbf{v}^*$, at x, respectively, from two components referred to tangent space bases embedded in $\mathbb{R}^3$, $\{\mathbf{b}_1, \mathbf{b}_2\}$, at $\phi_t(x)$, and $\{\mathbf{b}_1^*, \mathbf{b}_2^*\}$, at x, respectively, to their embedding in $\mathbb{R}^3$. The corresponding 2×3 matrices $\mathbf{B}'$, $(\mathbf{B}^*)'$ perform the "inverse" operation, mapping 3D vectors, tangent to the sphere, back from three to two components, again referred to the bases $\{\mathbf{b}_1, \mathbf{b}_2\}$, $\{\mathbf{b}_1^*, \mathbf{b}_2^*\}$, respectively[7].

[7] In case the vectors are given embedded in $\mathbb{R}^3$, instead of as intrinsic 2D vectors, and the tensors $\mathbf{T}$ are likewise given embedded in 3D, the above matrices $\mathbf{B}$, $\mathbf{B}^*$, $\mathbf{B}'$, $(\mathbf{B}^*)'$ simply become identity matrices.

5.8 *Example: Rotation-Invariant Vortices*

One nice example where using the concept of the pullback metric allows one to gain additional insight into an existing visualization method are the *rotation-invariant vortices for flow visualization* by Günther et al. [8].

The idea of this approach is to "transform rotations to translations" by setting up local diffeomorphisms for local neighborhoods that "rectify" a polar coordinate system in $\mathbb{R}^2$, without the point $(0,0)$, to a Cartesian coordinate system in $\mathbb{R}^2$. The original approach does not employ the concept of a pullback metric. However, it can be reformulated as follows:

[8] Tobias Günther, Maik Schulze, and Holger Theisel. Rotation invariant vortices for flow visualization. *IEEE Transactions on Visualization and Computer Graphics*, 22(1):817–826, 2016

- Set up an active deformation with local diffeomorphisms ϕ that map points in $\mathbb{R}^2 - \{(0,0)\}$ referred to polar coordinates to corresponding points in $\mathbb{R}^2$ referred to Cartesian coordinates[9]. See Fig. 5.9.
- Define the metric in the original space as the corresponding pullback metric $\phi^*\mathbf{h}$, where $\mathbf{h}$ is the standard Euclidean metric for $\mathbb{R}^2$.
- Compute the metric connection that is compatible with the pullback metric $\phi^*\mathbf{h}$, the *pullback connection*, on $\mathbb{R}^2$. This will not be the standard Euclidean connection, but will be compatible with the Riemannian manifold $(\mathbb{R}^2, \phi^*\mathbf{h})$, i.e., it will be a non-Euclidean "polar connection."

[9] We can simply do this by defining $\phi(r,\theta) := (x(r,\theta), y(r,\theta))$, with the functions $x(r,\theta) = r$, and $y(r,\theta) = \theta$. Conceptually, the map ϕ must be a *local* diffeomorphism to avoid the discontinuity in the angle function θ: We can define this map only locally. We do this by simply choosing the polar axis, where the angle $\theta = 0$ with the corresponding discontinuity, to be outside the local neighborhood where we define ϕ. *However, we note that we will not even have to do this explicitly, because we will not use the map ϕ directly, and in the connection that we derive below, the discontinuity in θ does not occur.*

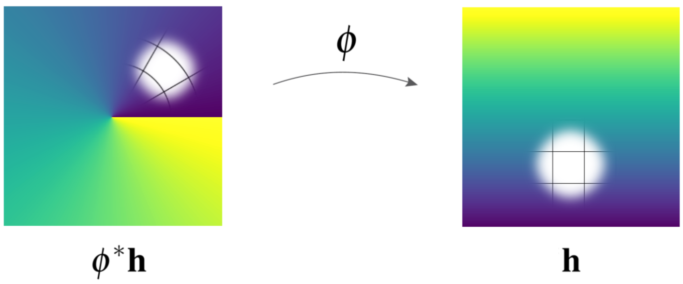


Figure 5.9: Active deformation ϕ between polar and locally-defined "Cartesian" (r,θ) coordinates, "rectifying" the polar coordinate system. Note that the local diffeomorphism ϕ must be defined such that the discontinuity where the angle function $\theta = 0$ is not included in the domain of definition. However, for any other neighborhood we simply choose the polar axis with $\theta = 0$ somewhere else.

The entire computation above can be "condensed" into simply modifying the computation of the velocity gradient tensor $\nabla\mathbf{v}$, by defining the Christoffel symbols $\Gamma^i{}_{jk}$, for the *non-Euclidean* "polar connection" $(\nabla)_{polar}$, to be *zero in polar coordinates*. If we want to work directly in Cartesian coordinates, this non-Euclidean connection can be evaluated directly by simply *subtracting* the contribution of the Christoffel symbols $\Gamma^i{}_{jk}$ of the *Euclidean connection* in polar coordinates, i.e., with the standard $\Gamma^1{}_{22} = -r$, and $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$, with application of the corresponding change of basis given by a coordinate Jacobian $\mathbf{J}$, from the partial derivatives. This gives[10]

$$(\nabla \mathbf{v})_{polar} = \begin{bmatrix} \nabla_x v^x & \nabla_y v^x \\ \nabla_x v^y & \nabla_y v^y \end{bmatrix} = \begin{bmatrix} \partial_x v^x & \partial_y v^x \\ \partial_x v^y & \partial_y v^y \end{bmatrix} - \mathbf{J}^{-1} \begin{bmatrix} 0 & -r\, v^\theta \\ \frac{1}{r}\, v^\theta & \frac{1}{r}\, v^r \end{bmatrix} \mathbf{J}. \tag{5.29}$$

[10] Without writing the basis $\mathbf{e}_i \otimes \omega^j$

Here, $\mathbf{J}$ is the Jacobian matrix of the coordinate change from Cartesian to polar coordinates, i.e., at a position $r = \sqrt{x^2 + y^2}$, $\theta = atan2(y, x)$, we have

$$\mathbf{J}(r, \theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}, \text{ and } \begin{bmatrix} v^r \\ v^\theta \end{bmatrix} = \mathbf{J} \begin{bmatrix} v^x \\ v^y \end{bmatrix}. \tag{5.30}$$

This approach allows one to directly use the components (v^x, v^y) and the partial derivatives $\partial_x v^x, \partial_y v^x, \partial_x v^y, \partial_y v^y$ from a Cartesian coordinate system.

This reformulation now allows one to gain additional insight:

- Because the metric (and, correspondingly, the metric connection) is modified, $(\mathbb{R}^2, \phi^*\mathbf{h})$ *is not a Euclidean space*, although it is a flat space. (The latter can be seen by directly computing the curvature of the metric connection defined above, or by observing that parallel transport in the whole domain of definition, i.e., $\mathbb{R}^2 - \{(0,0)\}$, is path-independent.)
- Concepts determined by the connection, such as parallel transport of vectors and geodesics, are accordingly modified, and do not agree with the corresponding behavior in Euclidean $\mathbb{R}^2$. See Fig. 5.10.

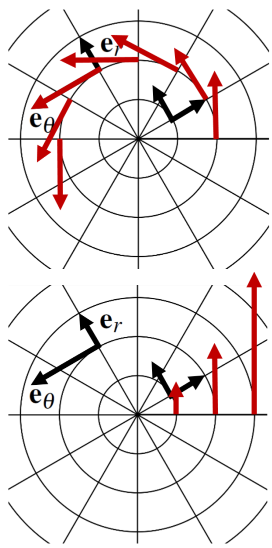


Figure 5.10: Parallel transport compatible with pullback metric of active deformation.

Example. We can now look at the example of the vector field $\mathbf{v} = a\,\mathbf{e}_\theta = -ay\,\mathbf{e}_x + ax\,\mathbf{e}_y$, describing a rotation with angular velocity a in the counter-clockwise direction. See the red vectors in Fig. 5.10. For this field, we have for the derivatives $\partial_x v^x = \partial_y v^y = 0$, and $\partial_y v^x = -a, \partial_x v^y = a$. For the "polar" velocity gradient in Cartesian coordinates, we therefore get, at any position (x, y), (r, θ),

$$\begin{aligned} (\nabla \mathbf{v})_{polar} &= \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} - \mathbf{J}^{-1} \begin{bmatrix} 0 & -ra \\ \frac{1}{r}a & 0 \end{bmatrix} \mathbf{J}, \\ &= \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = 0. \end{aligned} \tag{5.31}$$

Thus, this field is *parallel-transported* with respect to the pullback metric $\phi^*\mathbf{h}$, because its velocity gradient is identically zero. See Fig. 5.10.

5.9 *Flows of Vector Fields*

In this section, we briefly summarize the standard concepts of the *flow* of a vector field, as well as the corresponding linear map called the *differential* or *push-forward*, as they are typically defined in differential geometry. For details, we refer to the books by Lee [11], and Marsden and Hughes [12]. We follow the notation of Marsden and Hughes.

The flow of a *time-independent* vector field $\mathbf{u}$ on a manifold M is a map

$$\phi : J \times M \to M, \tag{5.32}$$

[11] John M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, 2nd edition, 2012

[12] Jerrold E. Marsden and Thomas J.R. Hughes. *Mathematical Foundations of Elasticity*. Dover Publications, Inc., 1994

for a suitable interval $J \subseteq \mathbb{R}$, such that $t \mapsto \phi(t,x)$ is the unique maximal integral curve of $\mathbf{u}$ through $x \in M$. That is, ϕ maps a point x to its image along the integral curve of $\mathbf{u}$ after time t, which we also denote by $\phi_t(x)$.

Important properties of ϕ are:

- The map $\phi_t \colon M \to M$ is a (local) diffeomorphism for all $t \in J$.
- For all $t_1, t_2 \in J$, $x \in M$, $\phi_{t_2}(\phi_{t_1}(x)) = \phi_{t_1+t_2}(x)$, $\phi_0(x) = x$. The inverse of ϕ_t is ϕ_{-t}, i.e., $\phi_t^{-1}(\phi_t(x)) = \phi_{-t}(\phi_t(x)) = x$. ϕ is an *action* of the additive group $\mathbb{R}$ on M, ϕ_t is a one-parameter group.
- The *linear* map $\mathrm{d}\phi_t \colon T_xM \to T_{\phi_t(x)}M$, called the differential of ϕ_t, or the (pointwise) *push-forward*, is an isomorphism between the two tangent spaces at each $x \in M$ and $\phi_t(x) \in M$, for each $t \in J$. $\mathrm{d}\phi_t$ maps tangent vectors to all possible curves through a point $x \in M$ to the corresponding tangent vectors of the images of these curves under the diffeomorphism ϕ_t, through the point $\phi_t(x) \in M$.

When the vector field $\mathbf{u}$ is *time-dependent*, the corresponding time-dependent flow

$$\psi \colon J \times J \times M \to M, \tag{5.33}$$

maps a point $x \in M$ to its image along the integral curve from time s to time t [13], which we denote by $\psi_{t,s}(x)$.

The map ψ has similar properties to the map ϕ:

- The map $\psi_{t,s} \colon M \to M$ is a (local) diffeomorphism for all $s, t \in J$.
- For all $s, t_1, t_2 \in J$, $x \in M$, $\psi_{t_2,t_1}(\psi_{t_1,s}(x)) = \psi_{t_2,s}(x)$, $\psi_{s,s}(x) = x$. The inverse of $\psi_{t,s}$ is $\psi_{s,t}$, i.e., $\psi_{t,s}^{-1}(\psi_{t,s}(x)) = \psi_{s,t}(\psi_{t,s}(x)) = x$.
- The *linear* map $\mathrm{d}\psi_{t,s} \colon T_xM \to T_{\psi_{t,s}(x)}M$, called the differential (the *push-forward*) of $\psi_{t,s}$, is an isomorphism between the tangent spaces at each $x \in M$ and $\psi_{t,s}(x) \in M$, for each $s, t \in J$. $\mathrm{d}\psi_{t,s}$ maps tangent vectors to all possible curves through a point $x \in M$ to the corresponding tangent vectors of the images of these curves under the diffeomorphism $\psi_{t,s}$, through the point $\psi_{t,s}(x) \in M$.

We note that the notation $\psi_{t,s}(x)$ can of course also be consistently used for the case of time-independent flow. In that case, $\psi_{t,s}(x) = \phi_{t-s}(x)$.

[13] John M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, 2nd edition, 2012

5.10 Example: Flow Maps

The flow map of a time-dependent vector field $\mathbf{v}(x,t)$ is given by[14]

$$\begin{aligned} F_{t_0}^t \colon M &\to M, \\ x_0 &\mapsto F_{t_0}^t(x_0) =: x(t; x_0). \end{aligned} \tag{5.34}$$

This simply denotes the *diffeomorphism* $F_{t_0}^t(x_0) \colon x_0 \mapsto x(t;x_0)$, mapping from the manifold M to itself, mapping points x_0 at time t_0 to points x at time t, defined by for each point x_0 following the *path line* through x_0, at time t_0, until time t, at which time the path line passes through the point x.

[14] We note that, with the notation used in the previous section, we simply have $F_{t_0}^t = \psi_{t,t_0}$. In fact, there exist many different (but equivalent) notations for the concept of a flow map (or, likewise, the flow of a vector field), for both steady and unsteady vector fields.

Flow map gradients

From a flow map $F_{t_0}^t$, as above, for a vector field given in Euclidean space, the corresponding spatial gradient is often computed[15], and often specifically denoted in the case of Euclidean space by

$$\nabla F_{t_0}^t. \tag{5.35}$$

This gradient is often called the flow map gradient, or sometimes—especially in elasticity—the *deformation gradient* of a diffeomorphism[16].

However, more generally, also including non-Euclidean manifolds, this deformation gradient in fact requires the more general concept of the *pushforward* or *differential* of the diffeomorphism $F_{t_0}^t$, as described above[17]. As above, for any diffeomorphism ϕ, the corresponding pushforward is a map between tangent bundles that is usually written as the map $\mathrm{d}\phi$, or ϕ_*. For the specific diffeomorphism $F_{t_0}^t$, we will denote this map by $dF_{t_0}^t$:

$$\begin{aligned} dF_{t_0}^t : TM &\to TM, \\ \mathbf{v} &\mapsto dF_{t_0}^t(\mathbf{v}). \end{aligned} \tag{5.36}$$

The pushforward defined on the entire tangent bundle is in general not a linear map. Considered point-wise, however, i.e., at each point x_0, it is a linear map between tangent spaces. We can write this linear map as

$$\begin{aligned} \left(dF_{t_0}^t\right)_{x_0} : T_{x_0}M &\to T_xM, \\ \mathbf{v}_{x_0} &\mapsto dF_{t_0}^t(\mathbf{v}_{x_0}). \end{aligned} \tag{5.37}$$

In the last row of the latter equation, the notation $\mathbf{v}_{x_0}$ refers to a single vector (not a vector field) in the tangent space $T_{x_0}M$ located at the point x_0.

Now, having defined the flow map gradient as the pushforward of a specific diffeomorphism, because pushforwards are defined for any smooth manifold we in this way have obtained a general map $dF_{t_0}^t$ that can be used on any manifold for the same computations as those using the flow map gradient $\nabla F_{t_0}^t$ in Euclidean space.

[15] For example, in FTLE computations, and in the definition and computation of many other Lagrangian concepts.

[16] We note that, here, both the time t_0 as well as the time t are held fixed, in order to obtain a specific diffeomorphism of which the flow map gradient is computed.

[17] In contrast to covariant derivatives, and the corresponding connection, using the ∇ operator in this context is not appropriate for diffeomorphisms on general manifolds, in particular on curved manifolds. See the book by Marsden and Hughes (Marsden and Hughes, 1994) for a discussion of why, despite wide-spread usage of the term, the deformation gradient is in fact (quote) "not a gradient at all, but is simply the derivative of the deformation."

Jerrold E. Marsden and Thomas J.R. Hughes. *Mathematical Foundations of Elasticity*. Dover Publications, Inc., 1994

6 Covariant derivatives and Lie derivatives

In this chapter, we discuss how the *covariant derivative* generalizes the directional derivative of tensor fields in Euclidean space with Cartesian coordinates to arbitrary manifolds with arbitrary coordinates. We introduce the (intrinsic) velocity gradient tensor $\nabla \mathbf{v}$ as the covariant derivative of a vector field $\mathbf{v}$ on a given manifold M, directly corresponding to the metric on M (if a metric connection is used). See Sec. 6.1.

In contrast, the *Lie derivative* is independent of the metric on M. It measures the rate of change of a tensor field on a manifold M with respect to the *flow* generated by a given vector field on M. An important application that combines both derivatives is that isometries can be quantified by computing the Lie derivative of the metric. We also introduce how the Lie derivative can be computed from the Levi-Civita connection. See Sec. 6.3.

Example applications are computing derivatives of vector fields in non-Cartesian coordinate systems, e.g., polar coordinates, or in intrinsically curved spaces, extrinsic vs. intrinsic computations, determining vortices from velocity gradients in non-Cartesian coordinates, Killing vector fields as infinitesimal isometries, Killing vector fields and covariant derivatives on the sphere, and observed time derivatives and objectivity [1].

[1] Markus Hadwiger, Matej Mlejnek, Thomas Theußl, and Peter Rautek. Time-dependent flow seen through approximate observer Killing fields. *IEEE Transactions on Visualization and Computer Graphics*, 25(1):1257–1266, 2019

6.1 Covariant Derivatives

The *covariant derivative* (also called an *affine connection*) generalizes the directional derivative of tensor fields in Euclidean space to arbitrary manifolds (Tu, Chapter 6) [2].

[2] Loring W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Springer-Verlag, 2017

Coordinate-free definition

We define the (intrinsic) covariant derivative $\nabla \mathbf{v}$, of a vector field $\mathbf{v}$, on a given manifold M with the following properties:

1. The map $(\mathbf{v}, \mathbf{w}) \mapsto \nabla_{\mathbf{w}} \mathbf{v}$ is $\mathbb{R}$-bilinear, that is

$$\begin{aligned} \nabla_{a\mathbf{w}_1 + b\mathbf{w}_2} \mathbf{v} &= a\nabla_{\mathbf{w}_1} \mathbf{v} + b\nabla_{\mathbf{w}_2} \mathbf{v}, \quad \text{and} \\ \nabla_{\mathbf{w}} (a\mathbf{v}_1 + b\mathbf{v}_2) &= a\nabla_{\mathbf{w}} \mathbf{v_1} + b\nabla_{\mathbf{w}} \mathbf{v_2} \end{aligned} \tag{6.1}$$

   for all $a, b \in \mathbb{R}$.

2. The map $\mathbf{w} \mapsto \nabla_{\mathbf{w}} \mathbf{v}$ (or $\nabla \mathbf{v}(\mathbf{w})$) is linear with respect to smooth functions, that is

$$\nabla_{\mathbf{w}} (f\mathbf{v}_1 + g\mathbf{v}_2) = f\nabla_{\mathbf{w}} \mathbf{v}_1 + g\nabla_{\mathbf{w}} \mathbf{v}_2 \tag{6.2}$$

   for all smooth functions f, g on M.

3. The map $\mathbf{v} \mapsto \nabla_{\mathbf{w}} \mathbf{v}$ is a *derivation*, i.e., it satisfies the *Leibniz rule*

$$\nabla_{\mathbf{w}} (f\mathbf{v}) = (\mathbf{w}f)\mathbf{v} + f\nabla_{\mathbf{w}} \mathbf{v} \tag{6.3}$$

for all smooth functions f on M.

If we write Eq. 6.1 as

$$\nabla\mathbf{v}(a\mathbf{w}_1 + b\mathbf{w}_2) = a\nabla\mathbf{v}(\mathbf{w}_1) + b\nabla\mathbf{v}(\mathbf{w}_2), \tag{6.4}$$

and define

$$\begin{aligned} \nabla\mathbf{v}\colon T_x^*M \times T_xM &\to \mathbb{R}, \\ (\omega, \mathbf{w}) &\mapsto \omega\big(\nabla\mathbf{v}(\mathbf{w})\big), \end{aligned} \tag{6.5}$$

it follows that $\nabla\mathbf{v}$ is a multi-linear map, and thus a $\binom{1}{1}$ tensor (field) (see Sec. 3.2). In addition, on a (Riemannian) manifold with a metric $\mathbf{g}$, there is a *unique* covariant derivative (Theorem 6.6) [3] that is

1. compatible with the metric, that is[4]

$$\nabla\mathbf{g} = 0, \qquad \text{and} \tag{6.6}$$

2. torsion-free, that is

$$\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{v} - [\mathbf{v}, \mathbf{w}] = 0. \tag{6.7}$$

The notation $[\mathbf{v}, \mathbf{w}]$ gives the Lie bracket of the vector fields $\mathbf{v}$ and $\mathbf{w}$. This unique connection is called the *Levi-Civita connection*.

[3] Loring W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Springer-Verlag, 2017

[4] This is often written in the equivalent, but less intuitive, form $\nabla_{\mathbf{u}}\langle\mathbf{v}, \mathbf{w}\rangle = \langle\nabla_{\mathbf{u}}\mathbf{v}, \mathbf{w}\rangle + \langle\mathbf{v}, \nabla_{\mathbf{u}}\mathbf{w}\rangle$ (Lee, Proposition 5.5) .

John M. Lee. *Introduction to Riemannian Manifolds*. Springer-Verlag, 2nd edition, 2018

Computation in a chart

Referred to a coordinate basis $\{\mathbf{e}_i \otimes \omega^j\}$, the (intrinsic) velocity gradient $\nabla\mathbf{v}$ as a *covariant derivative* is given by

$$\nabla\mathbf{v} = \left(\nabla_j v^i\right)\mathbf{e}_i \otimes \omega^j := \left(\partial_j v^i + \Gamma^i{}_{jk} v^k\right)\mathbf{e}_i \otimes \omega^j. \tag{6.8}$$

The tensor $\nabla\mathbf{v}$ evaluated in direction $\mathbf{x}$ is the vector (see Sec. 3.2),

$$\nabla\mathbf{v}(\mathbf{x}) = \nabla_{\mathbf{x}}\mathbf{v} = \left(\partial_j v^i + \Gamma^i{}_{jk} v^k\right)\omega^j(\mathbf{x})\,\mathbf{e}_i. \tag{6.9}$$

The Christoffel symbols $\Gamma^i{}_{jk}$, corresponding to the (unique) Levi-Civita connection for a metric $\mathbf{g}$ on M, can be derived intrinsically from the components g_{ij} of $\mathbf{g}$, referred to the same basis (and its dual), via

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im}\left(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk}\right). \tag{6.10}$$

See, e.g., the book by do Carmo [5]. g_{ij} is the metric $\mathbf{g}$ referred to the basis $\{\omega^i \otimes \omega^j\}$, and g^{ij} is its inverse $\mathbf{g}^{-1}$, i.e., $g^{ik}g_{kj} = \delta^i{}_j$, referred to the basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$.

[5] Manfredo Perdigao do Carmo. *Riemannian Geometry*. Birkhäuser, 1992

Relation to Cartesian tensors

The tensor $\nabla\mathbf{v}$ *only* consists solely of partial derivatives when (1) affine or Cartesian coordinates are used; and thus (2) the manifold is intrinsically flat, such as $M = \mathbb{R}^n$ with the standard Euclidean metric. Only then do the Christoffel symbols on M vanish. The above intrinsic formulation can be used on abstract manifolds M, without any known immersion into a Euclidean ambient space. However, even when an immersion of M into a higher-dimensional ambient space $\mathbb{R}^m$ is known, such as for a two-manifold embedded as a curved surface in $\mathbb{R}^3$, it is extremely useful for intrinsic (lower-dimensional) computations.

6.2 Example: Killing Fields in the Plane and on the Sphere

Killing vector fields on manifolds correspond to the *infinitesimal isometries* of a manifold, i.e., to the derivatives of isometries. Any isometry[6] can thus be obtained by *integrating* a (time-dependent) Killing vector field.

Each Killing field is characterized by the fact that the covariant derivative $\nabla \mathbf{w}$ of any Killing field $\mathbf{w}$ is anti-symmetric.[7] We can state this as $\langle \nabla \mathbf{w}(\mathbf{x}), \mathbf{x} \rangle = 0$, for all tangent vectors $\mathbf{x}$. If we consider a basis $\{\mathbf{e}_i\}_{i=1}^k$ of Killing fields, where for each basis element we have $\langle \nabla \mathbf{e}_i(\mathbf{x}), \mathbf{x} \rangle = 0$, the linearity of the covariant derivative immediately gives that each linear combination of these basis vector fields will again be Killing. For example, if we have $\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, we immediately know that $\langle \nabla \mathbf{w}(\mathbf{x}), \mathbf{x} \rangle = 0$.

Fig. 6.1 depicts four example Killing vector fields in the plane $\mathbb{R}^2$.

[6] The meaning of an isometry of a manifold is determined by the metric $\mathbf{g}$ defined on the manifold. A vector field $\mathbf{w}$ is an *infinitesimal isometry*, if we have $\mathscr{L}_{\mathbf{w}}\mathbf{g} = 0$. The differential operator $\mathscr{L}$ denotes the Lie derivative defined in Sec. 6.3.

[7] The general statement is given as *Killing's equation* $\nabla_i w_j + \nabla_j w_i = 0$.

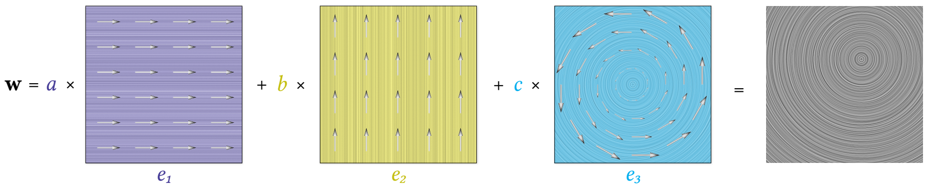


Figure 6.1: The set of all Killing vector fields in the plane forms a three-dimensional *vector space*. (Together with the *Lie bracket* operation, they form a *Lie algebra*; see Sec. 7.4.) As for any three-dimensional vector space, any vector in the space can be referred to a basis comprising three basis vectors. Here, we are referring to a *vector space of vector fields in the plane*, and therefore the three basis vectors are three *basis vector fields in the plane*. Here, we show one particular example basis. Each Killing field corresponds to the main property that its velocity gradient, as determined by the *covariant derivative* $\nabla \mathbf{w}$, is anti-symmetric, i.e., $\langle \nabla \mathbf{w}(\mathbf{x}), \mathbf{x} \rangle = 0$.

The set of all Killing fields in $\mathbb{R}^2$ forms a three-dimensional vector space.[8] Thus, because the vector fields $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent, they form a *basis* for *all* Killing fields in $\mathbb{R}^2$. Therefore, we can write any Killing field $\mathbf{w}$ in $\mathbb{R}^2$ as $\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, with coefficients $a, b, c \in \mathbb{R}$.

As another example, Fig. 6.2 depicts four different Killing vector fields on the sphere. Because the set of all Killing fields on the sphere again forms a three-dimensional vector space,[9] and because the three vector fields $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ shown in the figure are again linearly independent, they again form a *basis* for all Killing vector fields, this time on the sphere, such as the example Killing field $\mathbf{w}$ shown in the figure.

Corresponding to infinitesimal isometries also means that in the plane there exists an isomorphism between all Killing fields and all derivatives of Euclidean isometries in the plane, i.e., the derivatives of all 2D translations and rotations, which can bee seen to correspond to linear and angular velocities, respectively.

Likewise, there exists an isomorphism between all Killing vector fields on the sphere and all derivatives of isometries of the sphere, which are the infinitesimal rotations of the sphere. There is also an isomorphism between all Killing fields on the sphere and all anti-symmetric 3×3 matrices (bottom row of Fig. 6.2). This is explained fully by Lie theory, and the relationship to matrix Lie groups, as described in Sec. 7.4.

[8] In $\mathbb{R}^3$, the set of all Killing fields forms a six-dimensional vector space. (Corresponding to three degrees of freedom for translation, plus three degrees of freedom for rotation, respectively.)

[9] Corresponding to three degrees of freedom of rotations of the sphere.

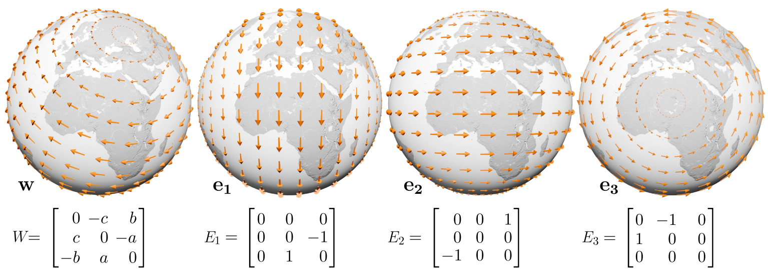


Figure 6.2: The set of all Killing vector fields on the sphere forms a three-dimensional *vector space*. (Together with the *Lie bracket* operation, they form a *Lie algebra*; see Sec. 7.4.) As for any three-dimensional vector space, any vector in the space can be referred to a basis comprising three basis vectors. Here, we are referring to a *vector space of vector fields on the sphere*, and therefore the three basis vectors are three *basis vector fields on the sphere*. Here, we show one particular example basis. Each Killing field corresponds to the main property that its velocity gradient, as determined by the *covariant derivative* $\nabla \mathbf{w}$, is anti-symmetric, i.e., $\langle \nabla \mathbf{w}(\mathbf{x}), \mathbf{x} \rangle = 0$.

Covariant derivatives of basis Killing fields in the plane

For the example in Fig. 6.1, we construct the following three linearly-independent basis Killing fields in $\mathbb{R}^2$, where the vectors at any point $x = (\hat{x}, \hat{y}) \in \mathbb{R}^2$, with respect to a Cartesian coordinate system in $\mathbb{R}^2$, are

$$\mathbf{e}_1(\hat{x},\hat{y}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_2(\hat{x},\hat{y}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{e}_3(\hat{x},\hat{y}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} - \hat{x}_0 \\ \hat{y} - \hat{y}_0 \end{bmatrix}. \tag{6.11}$$

For the field $\mathbf{e}_3$, we have to choose a “center point” $(\hat{x}_0, \hat{y}_0)$. Corresponding to vector fields given on a rectangular domain $D = [x_a, x_b] \times [y_a, y_b] \subset \mathbb{R}^2$, we define $(\hat{x}_0, \hat{y}_0) := \frac{1}{2}(x_a + x_b, y_a + y_b)$.[10]

Each basis element must be a Killing field. To confirm, we compute

$$\nabla \mathbf{e}_1 = 0, \quad \nabla \mathbf{e}_2 = 0, \quad \nabla \mathbf{e}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{6.12}$$

Therefore, we have, for all $i \in \{1,2,3\}$,

$$\langle \nabla \mathbf{e}_i(\mathbf{x}), \mathbf{x} \rangle = 0, \tag{6.13}$$

for all vectors $\mathbf{x} \in T_x M$, at all points $x \in M = \mathbb{R}^2$.

Using this basis, we can therefore write any Killing field $\mathbf{w}$ on $M = \mathbb{R}^2$ as $\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, with three coefficients (a,b,c). We note that, due to this particular choice of basis, (a,b) have the meaning of a linear velocity vector (given with two Cartesian components), and the third coefficient c has the meaning of angular velocity.[11] In fact, the linear velocity is the same constant vector at all points $x \in \mathbb{R}^2$.

[10] We note that this basis is an *orthogonal basis*, which can be confirmed by computing inner products between the basis vector fields (an inner product between functions).

[11] As for any vector space, other bases are of course also possible. Here then, however, no single coefficient would be the angular velocity. See Fig. 7.8.

Covariant derivatives of basis Killing fields on the sphere

For the example in Fig. 6.2, we construct the following three basis Killing fields for the sphere $\mathbb{S}^2 := \{(\hat{x}, \hat{y}, \hat{z}) | \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 1\}$ embedded in $\mathbb{R}^3$, where the vectors at any point $x = (\hat{x}, \hat{y}, \hat{z})$, as elements of the tangent space embedded in $\mathbb{R}^3$ at that point, are given by

$$\begin{aligned} \mathbf{e}_1(\hat{x},\hat{y},\hat{z}) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}, \\ \mathbf{e}_2(\hat{x},\hat{y},\hat{z}) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}, \\ \mathbf{e}_3(\hat{x},\hat{y},\hat{z}) &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}. \end{aligned} \tag{6.14}$$

Using this basis, we can write any Killing field $\mathbf{w}$ on $M = \mathbb{S}^2$ as $\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, with three coefficients (a,b,c). We note that, due to the above choice of basis, the coefficients (a,b,c) determine a 3D angular velocity vector, and $\omega^2 := a^2 + b^2 + c^2$ is the corresponding (squared) angular velocity (magnitude). Since $\langle \nabla \mathbf{e}_i(\mathbf{x}), \mathbf{x} \rangle = 0$ (equivalent to Killing's equation) is intrinsically defined in each tangent space, to see that the 2D tensors $\nabla \mathbf{e}_i$

are anti-symmetric, we must compute the covariant derivatives $\nabla \mathbf{e}_i$ at all $x \in \mathbb{S}^2$. Using a right-handed orthonormal basis in each tangent plane to the sphere, they are

$$(\nabla \mathbf{e}_i)_x = \begin{bmatrix} 0 & -\cos\varphi_i(x) \\ \cos\varphi_i(x) & 0 \end{bmatrix}. \tag{6.15}$$

Here, the angle $\varphi_i(x) \in [0,\pi]$ is the colatitude[12] of the point x away from the "north pole" of the respective $\mathbf{e}_i$, i.e., $\hat{x} = 1$ for $\mathbf{e}_1$, $\hat{y} = 1$ for $\mathbf{e}_2$, $\hat{z} = 1$ for $\mathbf{e}_3$. Therefore, we again have $\langle \nabla \mathbf{e}_i(\mathbf{x}), \mathbf{x} \rangle = 0$, for all $i \in \{1,2,3\}$, and for all vectors $\mathbf{x} \in T_x M$, at all points $x \in M = \mathbb{S}^2$.

[12] Where zero degrees is at the "north pole" instead of at the "equator."

6.3 Lie Derivatives

The *Lie derivative* measures the rate of change of a tensor field on a manifold M with respect to the *flow* (Sec. 5.9) generated by a vector field on M. For a time-independent tensor field $\mathbf{t}$, the Lie derivative $\mathscr{L}_{\mathbf{u}}\,\mathbf{t}$ with respect to a vector field $\mathbf{u}$ with flow ϕ_t, is defined, at $x \in M$, as

$$\left(\mathscr{L}_{\mathbf{u}}\,\mathbf{t}\right)_x := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{d}\phi_{-t}\left(\mathbf{t}_{\phi_t(x)}\right). \tag{6.16}$$

Here, $\mathrm{d}\phi_t$ is the differential of the flow ϕ_t, and $\phi_{-t} = \phi_t^{-1}$. When $\mathbf{t}$ is a vector field $\mathbf{v}$, the Lie derivative $\mathscr{L}_{\mathbf{u}}\mathbf{v}$ is the same as the Lie bracket. See Frankel (Ch. 4) [13] between the two vector fields, i.e., $\mathscr{L}_{\mathbf{u}}\mathbf{v} = [\mathbf{u},\mathbf{v}]$. See Fig. 6.3.

For any given torsion-free connection on a manifold M, such as the Levi-Civita connection corresponding to a given metric, the Lie bracket, and thus the Lie derivative, is then (cf. Eq. 6.7)

$$\mathscr{L}_{\mathbf{u}}\,\mathbf{v} = \nabla\mathbf{v}\,(\mathbf{u}) - \nabla\mathbf{u}\,(\mathbf{v})\,. \tag{6.17}$$

[13] Theodore Frankel. *The Geometry of Physics: An Introduction*. Cambridge University Press, 3rd edition, 2011

If the field $\mathbf{t}$ is time-dependent, the definition of the Lie derivative must be extended to the time-dependent Lie derivative [14], which is

$$\left(L_{\mathbf{u}}\,\mathbf{t}\right)_x := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} \psi^*_{t,s}\left(\mathbf{t}_{\psi_{t,s}(x)}\right) = \left(\frac{\partial\mathbf{t}}{\partial t} + \mathscr{L}_{\mathbf{u}}\,\mathbf{t}\right)_x, \tag{6.18}$$

at the point $x \in M$, at time s. The pullback $\psi^*_{t,s}$ is given by $\psi^*_{t,s} = \mathrm{d}\psi_{s,t}$.

[14] Jerrold E. Marsden and Thomas J.R. Hughes. *Mathematical Foundations of Elasticity*. Dover Publications, Inc., 1994

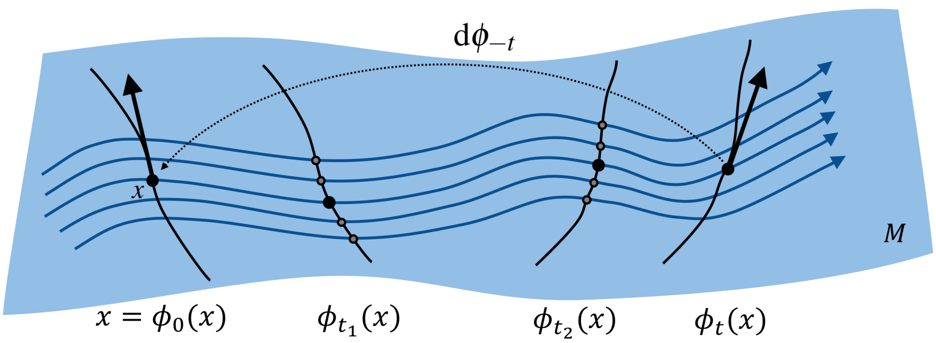


Figure 6.3: The Lie derivative measures the rate of change of a tensor field (shown here: a vector field, i.e., a first-order tensor field, of tangent vectors of curves depicted in black) with respect to the flow of a given vector field (shown here: the vector field whose integral curves are depicted in blue).

6.4 Example: Observed Time Derivative

An example for the use of the Lie derivative in visualization is the definition of the *observed time derivative*. Originally given by Hadwiger et al. [15] for flat (Euclidean) spaces (in 2D and 3D), because it is a Lie derivative it is also well-defined for curved spaces [16]. On any smooth manifold (with or without a metric, flat or curved), it is given by the differential operator

$$\frac{\mathscr{D}}{\mathscr{D}t} := \frac{\partial}{\partial t} + \mathscr{L}_{\mathbf{u}}. \tag{6.19}$$

This operator gives the derivative of a time-dependent tensor field[17] with respect to the flow of a given vector field, here denoted by **u**. A crucial point is that the semantic meaning of the field **u** is that of an observer velocity field describing (infinitesimal) reference frame motion.

Fig. 6.4 shows an unsteady (i.e., time-dependent) 2D velocity vector field relative to two different observers (i.e., reference frames). On the left, the reference frame motion is given by an *observer velocity field* relative to which the observed time derivative, relative to this observer motion, is minimized. On the right, the same velocity field is shown relative to the input reference frame, relative to which the vortex structures developing behind a circular obstacle are hard to discern.

[15] Markus Hadwiger, Matej Mlejnek, Thomas Theußl, and Peter Rautek. Time-dependent flow seen through approximate observer Killing fields. *IEEE Transactions on Visualization and Computer Graphics*, 25(1):1257–1266, 2019

[16] P. Rautek, M. Mlejnek, J. Beyer, J. Troidl, H. Pfister, T. Theußl, and M. Hadwiger. Objective observer-relative flow visualization in curved spaces for unsteady 2d geophysical flows. *IEEE Transactions on Visualization and Computer Graphics*, 27 (2):283–293, 2021

[17] This naturally includes scalar fields, in which case the observed time derivative coincides with the *material time derivative* $\partial/\partial t + \mathbf{u} \cdot \nabla$. For higher-order tensor fields (including vector fields), however, the observed time derivative and the material time derivative are not the same.

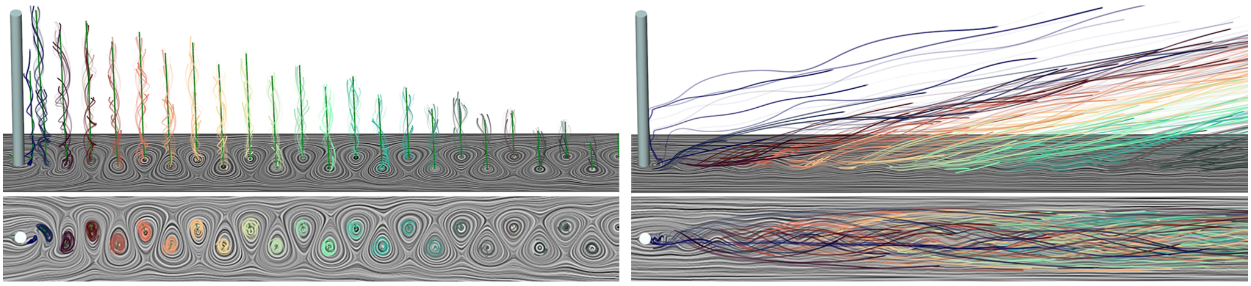


Figure 6.4: The *observed time derivative* is a specific time-dependent Lie derivative, with respect to the flow of an *observer velocity field*. The latter property gives this derivative its special meaning: Measuring the rate of change of a vector field, with respect to time, with respect to a specific observer motion: The motion determined by the observer velocity field. Here, each top image visualizes the 2D flow depicted in the bottom image in 3D space-time (the vertical axes map to time), respectively.

We can obtain the time derivative of any input flow field **v**, as observed relative to the reference frame motion determined by an observer velocity field **u**, by applying the operator above to the relative velocity field $(\mathbf{v} - \mathbf{u})$,

$$\frac{\mathscr{D}}{\mathscr{D}t}(\mathbf{v} - \mathbf{u}) = \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{v}(\mathbf{u}) - \nabla \mathbf{u}(\mathbf{v}). \tag{6.20}$$

On the right-hand side, we have simply expanded the (autonomous) Lie derivative, which for vector fields is identical to the differential-geometric Lie bracket, as given by Eq. 6.17.

As we will see below, since the definition of the Lie derivative is independent of the metric, the observed time derivative as just defined can also be used directly in curved spaces without changing the definition (Sec. 6.6).

6.5 *Lie Derivatives in Curved Spaces*

Lie derivatives are independent of the metric $\mathbf{g}$ defined on the manifold M. For a vector field $\mathbf{v}$, this can be seen by expanding

$$\begin{aligned}\mathscr{L}_{\mathbf{u}}\,\mathbf{v} &= \nabla\mathbf{v}\,(\mathbf{u}) - \nabla\mathbf{u}\,(\mathbf{v})\,,\\ &= \left(\nabla_j v^i u^j - \nabla_j u^i v^j\right)\mathbf{e}_i,\\ &= \left(\left(\partial_j v^i + \Gamma^i{}_{jk}v^k\right)u^j - \left(\partial_j u^i + \Gamma^i{}_{jk}u^k\right)v^j\right)\mathbf{e}_i,\\ &= \left(\partial_j v^i u^j + \Gamma^i{}_{jk}v^k u^j - \partial_j u^i v^j - \Gamma^i{}_{jk}u^k v^j\right)\mathbf{e}_i,\\ &= \left(\partial_j v^i u^j - \partial_j u^i v^j\right)\mathbf{e}_i.\end{aligned} \tag{6.21}$$

That is, all terms with Christoffel symbols $\Gamma^i{}_{jk}$ cancel out. This property always holds, given that the connection is torsion-free, which means that the symmetry $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ holds (for $\{\mathbf{e}_i\}$ a coordinate basis). This applies in our framework, because we are using the *Levi-Civita connection*, which, by definition, is both metric-compatible and torsion-free.

To make parsing the tensor expressions above easier, we note that an expression like $\partial_j v^i$ can be seen as a matrix of partial derivatives, with row index i and column index j, and $\partial_j v^i u^j$ is equivalent to matrix-vector multiplication with a column vector u^j with row index j. We also give the explicit expansion and summations for the 2D case:

$$\begin{aligned}\mathscr{L}_{\mathbf{u}}\,\mathbf{v} =& \nabla\mathbf{v}\,(\mathbf{u}) - \nabla\mathbf{u}\,(\mathbf{v})\,,\\ =& \left(\sum_{j=1,2}\left(\left(\partial_j v^1\right)u^j - \left(\partial_j u^1\right)v^j\right)\right)\mathbf{e}_1 +\\ & \left(\sum_{j=1,2}\left(\left(\partial_j v^2\right)u^j - \left(\partial_j u^2\right)v^j\right)\right)\mathbf{e}_2.\end{aligned} \tag{6.22}$$

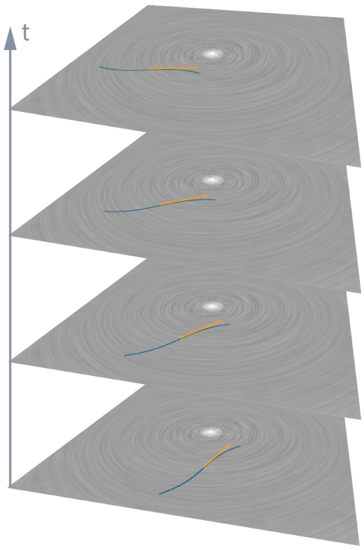


Figure 6.5: Lie dragging ("advecting") a curve with the flow of a vector field on the manifold determines how individual tangent vectors are Lie-dragged. These are exactly the tangent vector fields whose Lie derivative, relative to this flow, is zero.

6.6 *Example: Observed Time Derivative in Curved Spaces*

As we have derived in the previous section, Lie derivatives are independent of the metric defined on a given manifold. Therefore, the definition of the observed time derivative given in Eq. 6.19 is also independent of the metric, and is therefore defined for any manifold on which an observer velocity field **u** is given.

Analogously to the example for unsteady 2D flow in the plane given above, we can therefore also minimize the observed time derivative on curved surfaces. This enables following a flow feature, such as a hurricane, over time by minimizing the observed time derivative on the sphere representing the Earth's surface, as depicted in Fig. 6.6.

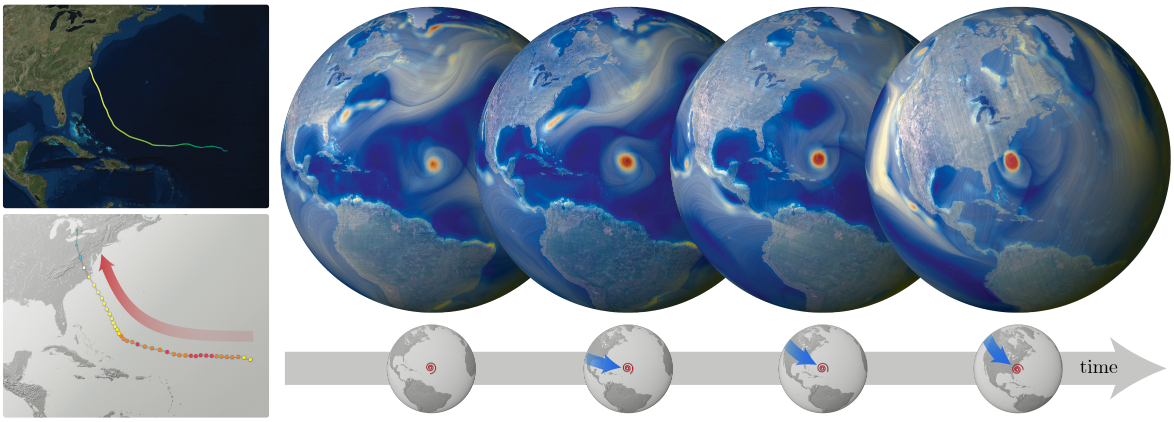


Figure 6.6: The observed time derivative in curved spaces is the same time-dependent Lie derivative as in flat space, because its definition is independent of the metric. Minimizing it therefore also makes an arbitrary unsteady input vector field, such as here on the sphere, as steady as possible: Features such as the hurricane shown in the center here appear as evolving in place instead of moving over the Earth's surface.

The visualization in Fig. 6.7 contrasts the two perspectives: In the upper two images, a hurricane is moving across the Earth while the reference frame fixed to the Earth is shown as not moving. In contrast, in the bottom images, the reference frame is co-moving with the hurricane, and therefore the Earth is moving underneath the hurricane in that reference frame.

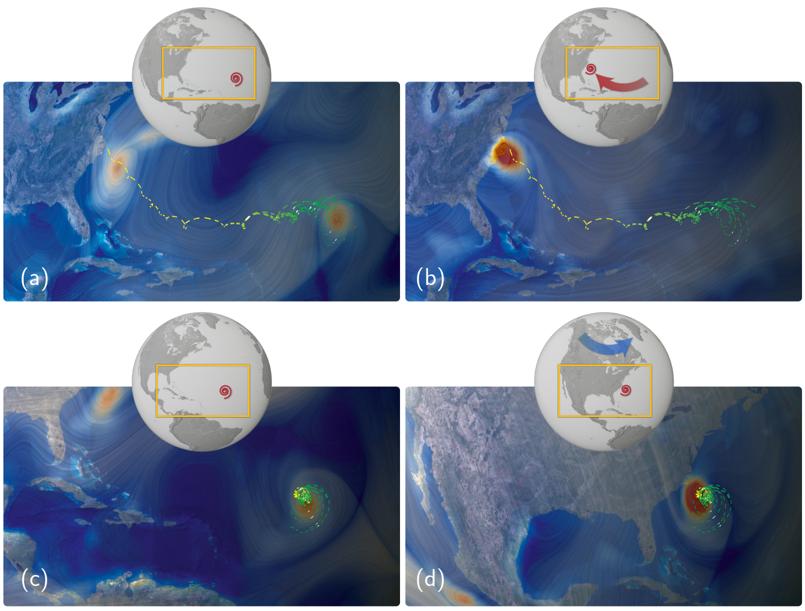


Figure 6.7: Feature-centric visualization. (a,b) A flow feature such as the hurricane shown here moves relative to the Earth's surface, holding the Earth's reference frame in place. (c,d) The reference frame is now co-moving with the hurricane: The hurricane seems to evolve in place, whereas now the Earth seems to be moving underneath, in the opposite direction as the hurricane's motion depicted in (a,b).

7 Lie groups and Lie algebras

This chapter is an introduction to Lie theory from the point of view of continuous symmetry or transformation groups. For computational purposes we emphasize matrix Lie groups and their derivatives, which comprise the Lie algebra of the Lie group. We further cover the exponential map which maps elements from the Lie algebra, called infinitesimal generators, to the Lie group. Finally, we introduce the important concept of a group action and demonstrate how Lie theory, with Lie groups or their Lie algebras acting on a manifold, can be used in visualization algorithms, for example in reference frame optimization.

Important concepts that we will cover are Lie groups and Lie algebras, the exponential map, group actions, and reference frame transformations and objectivity.

Example applications are how to use Lie theory to extend the definition of objectivity to non-Euclidean spaces with non-trivial continuous isometries, e.g., the sphere [1], and how to find good reference frames, e.g., to make a vector field as-steady-as-possible, in this context.

[1] P. Rautek, M. Mlejnek, J. Beyer, J. Troidl, H. Pfister, T. Theußl, and M. Hadwiger. Objective observer-relative flow visualization in curved spaces for unsteady 2d geophysical flows. *IEEE Transactions on Visualization and Computer Graphics*, 27 (2):283–293, 2021

7.1 Groups and Lie Groups

A group G is a set with an operation[2] $\circ : G \times G \to G$ with the properties

[2] "composition", it is important that $\circ(g,h) = g \circ h$ is another element of the same group G

1. Associativity: For all $g, h, j \in G$

$$(g \circ h) \circ j = g \circ (h \circ j), \tag{7.1}$$

2. Existence of a neutral element: There is an $e \in G$ so that for all $g \in G$

$$g \circ e = e \circ g = g. \tag{7.2}$$

3. Existence of inverses: For all $g \in G$ there is another element denoted by $g^{-1} \in G$ so that

$$g \circ g^{-1} = g^{-1} \circ g = e. \tag{7.3}$$

If the composition has to be made explicit, we write $(G, \circ)$ for the group.

Example 5. *1. The real numbers with addition* $(\mathbb{R}, +)$*, but also the rational numbers* $(\mathbb{Q}, +)$ *or even the integers* $(\mathbb{Z}, +)$ *are all groups.*

2. Often groups are part of a larger mathematical structure. For example, any field (like $\mathbb{R}$ *or* $\mathbb{C}$*) with only addition as operation[3] is a group. So both* $(\mathbb{R}, +)$ *and* $(\mathbb{Q}, +)$*, seen as part of the fields* $(\mathbb{R}, +, \cdot)$ *and* $(\mathbb{Q}, +, \cdot)$*, are groups.*

[3] that is, we disregard multiplication

3. If we leave away zero[4] from a field and look only at multiplication, we also get a group. That is, both $(\mathbb{R} \setminus \{0\}, \cdot)$ *and* $(\mathbb{Q} \setminus \{0\}, \cdot)$ *are groups.*

[4] which has no inverse

4. *Any linear space V with only the vector addition, that is $(V,+)$, is a group.*
5. *The most important examples of groups for us are the invertible matrices with matrix multiplication*[5]*. These groups are usually written as $GL(n)$*[6] *for $n \times n$ matrices.*

[5] from Linear Algebra we know that matrix multiplication is associative, the neutral element is given by the identity matrix, and each matrix has by definition an inverse

[6] which stands for general linear group

A *Lie group* now is a group which is also a manifold, such that the group operation as a function on the manifolds is smooth. This definition, although very concise, is also very generic and abstract[7]. We will use a slightly less generic, but still extensive enough, concept, that of a matrix Lie group.

[7] it is favored by more mathematically inclined texts like for example (Lee, 2012, Chapter 7)

John M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, 2nd edition, 2012

Definition 4. A *Matrix Lie group* is a closed subgroup of the group of invertible matrices.

G being a subgroup of $GL(n)$[8] means that it is a group itself[9]. So one usually has to check that for any two matrices $A, B \in G$ their product AB is in G as well, and that the identity is in G[10].

The property of G being closed in $GL(n)$ is a generic topological property. In more concrete terms it means that if M_n is a sequence of matrices in G and it converges to some matrix M, then M is either in G or not invertible. Since convergence can be checked component-wise, this property is usually easily verified with concrete matrix Lie groups as in the following examples.

[8] the group of invertible matrices of size $n \times n$

[9] with the same group operation, that is, matrix multiplication

[10] every element of G will have an inverse, since it is also in $GL(n)$ by definition and therefore invertible

---

Example 6. *1. The translation group of the plane*[11] *can be represented by the following matrices*

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{7.4}$$

.

This shows that this matrix Lie group is just another plane.

[11] the plane being situated at $z = 1$

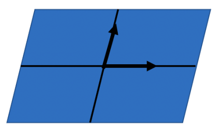


Figure 7.1: The translation group of the plane is just another plane.

2. *Similarly for the rotation group of the plane, which is just a circle:*

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \tag{7.5}$$

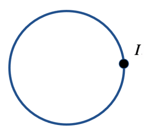


Figure 7.2: The rotation group of the plane $SO(2)$ is just a circle.

3. *These two matrix Lie groups can be combined to the affine group of the plane:*[12]

$$\begin{pmatrix} \cos t & \sin t & t_x \\ -\sin t & \cos t & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{7.6}$$

[12] the plane is again situated at $z = 1$

Example 7. *Lets work out in detail that the translation group of Example 6.1, lets denote it by G here, is actually a matrix Lie group. Firstly, the identity matrix is obviously*[13] *in G. Secondly, if we have a convergent sequence in G, say*

$$\begin{pmatrix} 1 & 0 & (t_x)_n \\ 0 & 1 & (t_y)_n \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.7}$$

then obviously $(t_x)_n$ converges to some t_x and $(t_y)_n$ to some t_y, so the limit is in G as well. Thus, G is a matrix Lie group.

[13] for $t_x = t_y = 0$

Note that, since a Lie group is also a manifold, it has a tangent space at each point[14]. A tangent space of a smooth manifold is just a linear space, however, since a Lie group is also a group, this tangent space inherits a very special structure from the group operation. It is even a Lie algebra.

[14] that notion has no meaning for a generic group!

7.2 *Lie Algebras*

In mathematics, an algebra usually denotes a linear space which has an additional operation[15]. A good example is the matrix algebra, where matrices can be added and linear combinations be taken, but matrices can also be multiplied. In this case, the multiplication is associative and the algebra called an associative algebra. A *Lie algebra* is a slightly different kind of algebra, especially the multiplication is in general not associative. The definition is as follows.

[15] usually called multiplication

Definition 5. A *Lie algebra* is a linear space (vector space) V with an additional operation[16]

$$\begin{aligned} [\cdot,\cdot] : V \times V &\to V \\ (X,Y) &\mapsto [X,Y], \end{aligned} \tag{7.8}$$

with the properties:

[16] in this case called the *Lie bracket*

1. Bi-linearity: For all $X,Y \in V$ and $\alpha, \beta \in \mathbb{R}$

$$[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z] \tag{7.9}$$

$$[Z, \alpha X + \beta Y] = \alpha[Z,X] + \beta[Z,Y] \tag{7.10}$$

2. Skew-symmetry: For all $X,Y \in V$[17]

$$[X,Y] = -[Y,X] \tag{7.11}$$

[17] this implies that $[X,X] = 0$ for all $X \in V$

3. Jacobi identity: For all $X,Y,Z \in V$

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \tag{7.12}$$

Example 8. *The group of rotations of $\mathbb{R}^3$, called*[18] $SO(3)$ *is a three dimensional manifold sitting in the space of* 3×3 *matrices*[19]*. This can probably not be visualized directly, but the situation is the same as for a surface in three dimensional space, as depicted in Figure 7.3.*

[18] which stands for special orthogonal group

[19] which is nine dimensional

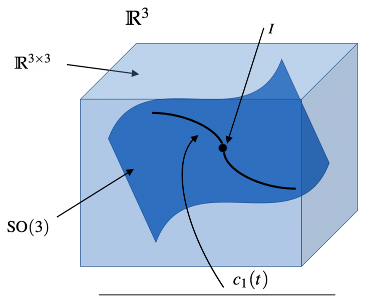


Figure 7.3: The rotation group $SO(3)$ as a submanifold of $\mathbb{R}^{3\times 3}$.

Obviously,

$$c_1(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.13}$$

is a smooth curve through the identity, that is $c_1(t) \in SO(3)$ *for all t and*

$$c_1(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I. \tag{7.14}$$

Its derivative is given by

$$c_1'(t) = \begin{pmatrix} -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.15}$$

and thus

$$X_1 := c_1'(0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.16}$$

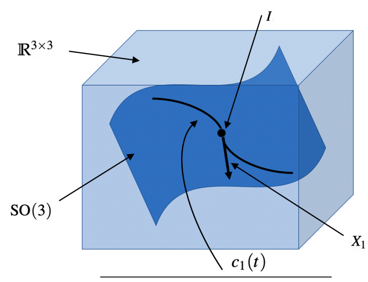


Figure 7.4: A tangent vector to $SO(3)$.

is a tangent vector at the identity I of $SO(3)$*, as depicted in Figure 7.4.*

With the analogous curves[20]

[20] the opposite signs in c_2 are convention

$$c_2(t) = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix} \text{ and } c_3(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix} \tag{7.17}$$

we get two more tangent vectors

$$X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{7.18}$$

These three tangent vectors are linearly independent, since from[21]

[21] 0 here denotes the zero matrix

$$0 = \alpha X_1 + \beta X_2 + \gamma X_3 = \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix} \tag{7.19}$$

it follows that $\alpha = \beta = \gamma = 0$[22]*. Thus, since* $SO(3)$ *is a three-dimensional manifold,* X_1, X_2, X_3 *constitute a basis of the tangent space of* $SO(3)$ *at the identity I and thus of the Lie algebra* $\mathfrak{so}(3)$*, as depicted in Figure 7.5.*

[22] this the real number zero

The Lie bracket is given by

$$[X,Y] = XY - YX \tag{7.20}$$

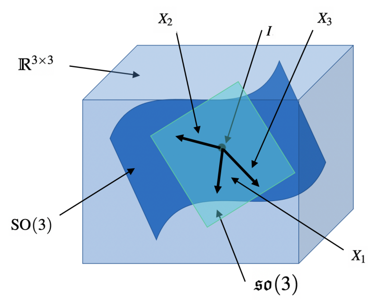


Figure 7.5: The tangent space of $SO(3)$, that is, the Lie algebra $\mathfrak{so}(3)$.

7.3 *The Exponential Map*

So we have seen how to move from the Lie group to its Lie algebra[23]. We will also need a way to move back to the Lie group. This is done with the exponential map,

[23] by finding curves through the identity and differentiating them

Abstractly and generically, the *exponential map* is a mapping from a Lie algebra $\mathfrak{g}$ to the corresponding Lie group G, i.e.,

$$\exp\colon \mathfrak{g} \to G, \tag{7.21}$$
$$X \mapsto \exp(X). \tag{7.22}$$

In case of a matrix Lie group it is defined for a matrix X in the Lie algebra, like the exponential map for real numbers, by the power series

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots, \tag{7.23}$$

where I is the identity matrix.

Example 9. *For example, for $G = SO(2)$, the rotation group of the plane*[24]*, and its Lie algebra*[25] *$\mathfrak{g} = \mathfrak{so}(2)$, if we choose the basis vector (matrix) $X \in \mathfrak{so}(2)$*

[24] see Example 6.2

[25] which are the skew-symmetric 2×2 matrices

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{7.24}$$

we can compute

$$\begin{aligned}
(tX)^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
(tX)^1 &= \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}, \\
(tX)^2 &= \begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix}, \\
(tX)^3 &= \begin{pmatrix} 0 & t^3 \\ -t^3 & 0 \end{pmatrix}, \\
(tX)^4 &= \begin{pmatrix} t^4 & 0 \\ 0 & t^4 \end{pmatrix} = t^4 (tX)^0,
\end{aligned} \tag{7.25}$$

and conclude that $(tX)^{k+4} = t^4 (tX)^k$ by induction. Thus,

$$\begin{aligned}
\exp(tX) &= e^{tX} \\
&= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{pmatrix} \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2).
\end{aligned} \tag{7.26}$$

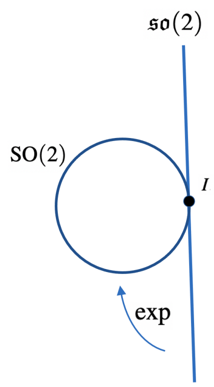


Figure 7.6: The exponential map maps the tangent space $\mathfrak{so}(2)$ at the identity of $SO(2)$ to $SO(2)$. It is only a local diffeomorphism, if you move away too far from the identity you will wrap around and the mapping will not be bijective anymore.

See Figure 7.6. For $G = SO(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$, with essentially the same calculations, we get for the basis vectors (matrices)

$X_i \in \mathfrak{so}(3)$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (7.27)$$

the corresponding exponentials

$$\begin{aligned} \exp(tX_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \in SO(3), \\ \exp(tX_2) &= \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix} \in SO(3), \quad (7.28) \\ \exp(tX_3) &= \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3). \end{aligned}$$

7.4 Example: The Lie Algebra of Observer Motions

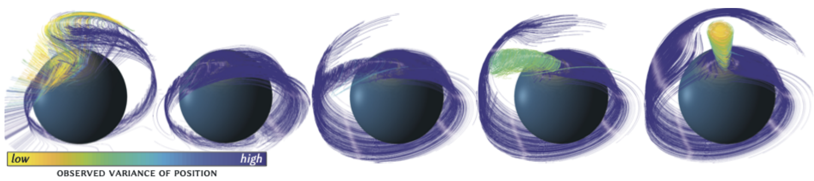
For an application of these abstract concepts of Lie groups and Lie algebras we want to show how to use them to explore 2D time-dependent flow with observers[26]. The core idea here is, since observers are transformations of space, we can exploit the linear structure of the Lie algebra of these transformation to, for example, interpolate between different observers.

In the following figure[27] we see a time-dependent flow on a sphere[28]. On the left, the flow seems to be very chaotic. On the right, an observer was chosen who centers on a vortex. In between, the two observer are interpolated which shows that the chaotic movement on the left can be seen as resulting from the movement of an observer.

[26] for details we refer to Xingdi Zhang, Markus Hadwiger, Thomas Theußl, and Peter Rautek. Interactive exploration of physically-observable objective vortices in unsteady 2d flow. *IEEE Transactions on Visualization and Computer Graphics (Proceedings IEEE VIS 2021)*, 28(2):1–1, 2022. DOI: 10.1109/TVCG.2021.3115565

[27] taken from the paper

[28] visualized with path lines, which are, for visualization purposes, moving away orthogonally from the sphere over time.



In flow visualization and continuum mechanics, we can model any possible observer (i.e., reference frame) motion as an element[29] of the corresponding Lie algebra. This Lie algebra therefore comprises all possible observer motions.[30] In the framework presented by Zhang et al., [31] each Lie algebra element is in fact a whole Killing vector field $\mathbf{w}$ on the underlying manifold M, where M constitutes the domain where the input vector field as well as any Killing field describing observer motion are given. For example, $M = \mathbb{R}^2$ or $M = \mathbb{S}^2$. See Fig. 7.7 for an illustration of the case $M = \mathbb{R}^2$.

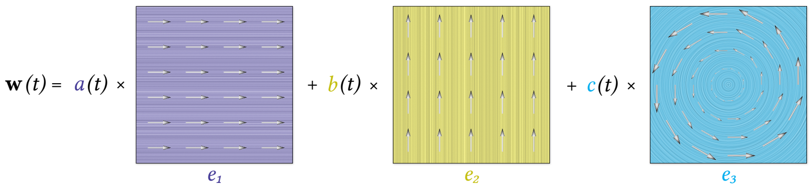
The vector space structure means that a Lie algebra has a basis, i.e., a spanning set of k linearly-independent basis vectors (basis vector fields) for

[29] In terms of the vector space structure, an abstract vector as a mathematical object. Here, these objects are *vector fields*.

[30] Usually, we want *physically-realizable* observer motions. Disregarding special and general relativity, they are rigid motions in Euclidean space, including rotations of a sphere embedded in the ambient space $\mathbb{R}^3$, e.g., modeling the surface of the Earth.

[31] Xingdi Zhang, Markus Hadwiger, Thomas Theußl, and Peter Rautek. Interactive exploration of physically-observable objective vortices in unsteady 2d flow. *IEEE Transactions on Visualization and Computer Graphics (Proceedings IEEE VIS 2021)*, 28(2):1–1, 2022. DOI: 10.1109/TVCG.2021.3115565

an k-dimensional Lie algebra. The Lie algebras of all rigid motions for the manifolds $M = \mathbb{R}^2$ and $M = \mathbb{S}^2$, respectively, are each three-dimensional as a vector space. (This makes these two cases nicely uniform for observer motions, despite otherwise many differences between planes and spheres.)



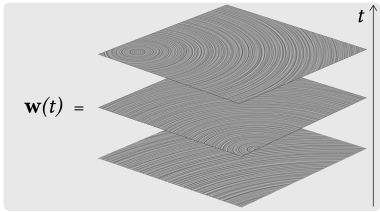


Figure 7.7: The set of all Killing vector fields in the plane forms a three-dimensional *vector space*, and, together with the Lie bracket operation, forms a three-dimensional *Lie algebra*. As for any three-dimensional vector space, any vector in the space can be referred to a basis comprising three basis vectors. Here, we are referring to a *vector space of vector fields in the plane*, and therefore the three basis vectors are three *basis vector fields in the plane*. Here, we show one particular example basis $\{\mathbf{e}_i\}$ on the manifold $M = \mathbb{R}^2$, from which we can obtain an arbitrary Killing field $\mathbf{w}$, by simply specifying three real coefficients (a, b, c), or an arbitrary *time-dependent* Killing field $\mathbf{w}(t)$, by simply specifying three real coefficients $(a(t), b(t), c(t))$ for every time t, respectively. In the latter case, this means we specify a reference frame motion via a function $t \mapsto (a(t), b(t), c(t))$.

This implies that any element of such a Lie algebra, i.e., any infinitesimal rigid motion, can be referred to three basis vectors (here, basis vector fields), giving three corresponding real coefficients (per time t, for time-dependent rigid motions, corresponding to time-dependent Killing fields):[32]

$$\mathbf{w}(x,t) = a(t)\,\mathbf{e}_1(x) + b(t)\,\mathbf{e}_2(x) + c(t)\,\mathbf{e}_3(x). \tag{7.29}$$

This expression is a linear combination of three basis vector fields with three real coefficients: Here, one triplet $(a(t), b(t), c(t))$ per time t.[33]

Scalar multiplication: Scalar times Lie algebra element

The scalar multiplication for the vector space structure, e.g., $a\,\mathbf{e}_1$, where $\mathbf{e}_1$ denotes a vector field on M, is defined pointwise in each tangent space via standard scalar multiplication in each tangent space:

$$(a\,\mathbf{e}_1)(x) := a\,\mathbf{e}_1(x). \tag{7.30}$$

That is, in each tangent space T_xM at a point $x \in M$, the basis vector in that tangent space, i.e., $\mathbf{e}_1(x)$, is multiplied by the same coefficient a.

Vector addition: Addition of two Lie algebra elements

Likewise, the vector addition for the vector space structure is also defined by pointwise addition of vectors in each tangent space:

$$(\mathbf{e}_1 + \mathbf{e}_2)(x) := \mathbf{e}_1(x) + \mathbf{e}_2(x). \tag{7.31}$$

Linear independence of the Lie algebra basis

It is important to realize that the linear independence of our Lie algebra basis needs to be verified as the linear independence of vector fields, not as that for individual vectors in some tangent space. (Otherwise, in a two-dimensional tangent space, for instance, more than two vectors would always be linearly dependent; but we have three linearly independent basis vector fields.) That is, linear independence of three basis fields $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$ is given if there are no coefficients $\lambda, \mu \in \mathbb{R}$ such that

$$\mathbf{e}_i = \lambda\,\mathbf{e}_j + \mu\,\mathbf{e}_k, \tag{7.32}$$

[32] A time-dependent Killing vector field $\mathbf{w}(t)$ is simply a Killing field $\mathbf{w}_T$ for each fixed time T: $\mathbf{w}_T := \mathbf{w}(T)$, and $\langle \nabla \mathbf{w}_T(\mathbf{x}), \mathbf{x} \rangle = 0$. Writing out points $x \in M$ explicitly, we can write the same as $\mathbf{w}_T(x) := \mathbf{w}(x,T)$.

[33] It is nice that the basis itself does not need to be time-dependent: All time-dependence is encoded in the coefficient function $t \mapsto (a(t), b(t), c(t))$.

for every (cyclic) permutation of $(i, j, k) = (1,2,3)$. This equation must be read such that for fixed $\lambda, \mu \in \mathbb{R}$, for all points $x \in M$, the vectors in each tangent space T_xM would have to be

$$\mathbf{e}_i(x) = \lambda \mathbf{e}_j(x) + \mu \mathbf{e}_k(x). \tag{7.33}$$

For $M = \mathbb{R}^2$ and the basis given by Eq. 6.11 (depicted in Fig. 7.7), this is trivial to see for $\mathbf{e}_i = \mathbf{e}_3$. However, it is also not hard to see in the other cases. For $M = \mathbb{S}^2$ and the basis given by Eq. 6.14 (depicted in Fig. 7.9), we can imagine choosing two basis fields to reproduce a non-zero vector of the third basis field at some point, and then considering the "pole" of the third field, where the third field has a critical point (i.e., the vector in the tangent space at that point is zero), but the linear combination that we just considered gives a non-zero vector. (That is, linear independence is given as long as the poles, i.e., the critical points, do not coincide.)

Time dependence

While above we have considered the individual scalar multiplications, vector (field) additions, and linear independence, neglecting the time-dependence of vector fields on M, i.e., we have mainly considered

$$\mathbf{w}(x) = a\,\mathbf{e}_1(x) + b\,\mathbf{e}_2(x) + c\,\mathbf{e}_3(x), \tag{7.34}$$

everything trivially extends to time-dependent Killing fields with time-dependent coefficients $t \mapsto (a(t), b(t), c(t))$, giving Eq. 7.29 above, i.e.,

$$\mathbf{w}(x,t) = a(t)\,\mathbf{e}_1(x) + b(t)\,\mathbf{e}_2(x) + c(t)\,\mathbf{e}_3(x).$$

All basis Killing fields $\mathbf{e}_i$ are time-independent, and therefore we can directly compute the partial (Eulerian) time derivative of any time-dependent infinitesimal observer motion as the time-dependent vector field[34]

$$\frac{\partial \mathbf{w}}{\partial t}(x,t) = \frac{da(t)}{dt}\mathbf{e}_1(x) + \frac{db(t)}{dt}\mathbf{e}_2(x) + \frac{dc(t)}{dt}\mathbf{e}_3(x). \tag{7.35}$$

From this expression, we can immediately see that this time derivative is again a Killing field, because it is a linear combination of the same basis.

Together with the fact that the Lie bracket of two Killing fields is also always a Killing field (because the Lie bracket is an operation in the Lie algebra), and the definitions of the observed time derivative operator (Eq. 6.19), and the observed time derivative of a relative velocity field (Eq. 6.20), respectively, we can furthermore see that these *observed time derivatives* again yield Killing vector fields, if both the observed motion as well as the reference frame motion are rigid motions, respectively.[35]

It is also useful to observe the following relationship, resulting directly from the linearity of the covariant derivative,

$$\nabla \mathbf{w}(x,t) = a(t)\,\nabla \mathbf{e}_1(x) + b(t)\,\nabla \mathbf{e}_2(x) + c(t)\,\nabla \mathbf{e}_3(x). \tag{7.36}$$

[34] It is important to note here that this means that the entire time derivative, which is a vector field and not a single vector, can nevertheless be computed by simply computing the time derivatives of the functions $a(t), b(t), c(t)$, respectively.

[35] This can come up, for example, when computing a *change of reference frame*.

Alternative basis

Although the basis depicted in Fig. 7.7 (cf. Eq. 6.11) is probably more intuitive (and it lends itself more intuitively to defining an isomorphism with the matrix Lie algebra $\mathfrak{se}(2)$ illustrated in Eq. 7.43 below), because any set of three linearly independent Killing vector fields *must* be a basis for the vector space of Killing fields in the plane, the three linearly independent Killing fields depicted in Fig. 7.8 also comprise a basis.

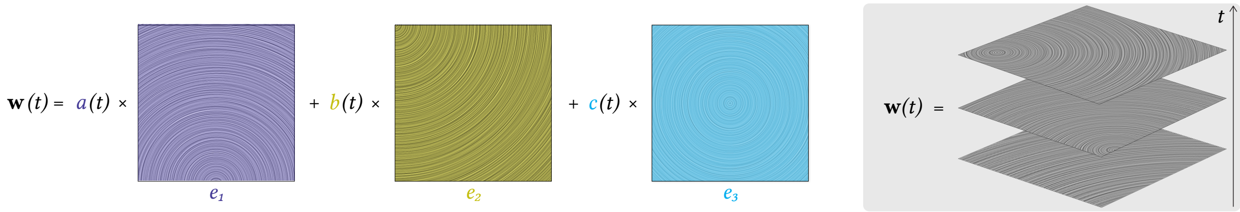


Figure 7.8: Alternative basis for all Killing vector fields (and, thus, all infinitesimal rigid motions) in the plane. As any other basis, this particular basis can represent all infinitesimal translations and rotations, although the "translation component" is not easily apparent from looking at these basis vector fields (they all have $\nabla \mathbf{e}_i \neq 0$).

We can give this basis $\{\mathbf{e}_i\}$ explicitly by defining the vectors at points $x = (\hat{x}, \hat{y}) \in \mathbb{R}^2$, with respect to a Cartesian coordinate system in $\mathbb{R}^2$, as

$$\mathbf{e}_i(\hat{x}, \hat{y}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} - \hat{x}_0(i) \\ \hat{y} - \hat{y}_0(i) \end{bmatrix}, \quad \begin{bmatrix} \hat{x}_0(i) \\ \hat{y}_0(i) \end{bmatrix} \in \mathbb{R}^2, \;\; i \in \{1, 2, 3\}. \tag{7.37}$$

Here, for linear independence, the three "center" points $(\hat{x}_0(i), \hat{y}_0(i))^T$ need to be chosen in general position, i.e., such that none of the points coincide, and no point lies on the line defined by the other two points.

While this alternative basis might at first seem to be "missing" an infinitesimal translation component, because it is a basis (for this three-dimensional Lie algebra) it is guaranteed to be able to represent all infinitesimal translations and rotations, just as the basis in Fig. 7.7 is, albeit of course with different coefficients (a, b, c) for a given Killing field $\mathbf{w}$.[36]

One easy way to compute the coefficients (a, b, c) with respect to this alternative basis is to compute the coefficients (a, b) by projecting the vector $\mathbf{w}(x)$, at the spatial point $x \in \mathbb{R}^2$ where the field $\mathbf{e}_3$ has its critical point, into the tangent space basis $\mathbf{e}_1(x)$ and $\mathbf{e}_2(x)$, of the tangent space T_xM ($M = \mathbb{R}^2$) at the same point x, and then computing the coefficient c such that the angular velocity of the Killing field $\mathbf{w}$ (which can be read off from $\nabla \mathbf{w}$) is $\omega = a + b + c$. That is, after we know (a, b), we compute $c = \omega - a - b$.[37]

[36] We note, however, that in contrast to the basis given in Eq. 6.11, this alternative basis is *not* an orthogonal basis, which can be confirmed by computing inner products between the basis vector fields (as the inner product between functions). Non-orthogonality also corresponds to the fact that when one basis vector field is changed, more than one coefficient changes.

[37] One illustrative example field $\mathbf{w}$ to consider is one with $\omega = 0$, i.e., $\nabla \mathbf{w} = 0$. Naturally, the approach given here results in coefficients (a, b, c), such that $\mathbf{w}(x) = a\,\mathbf{e}_1(x) + b\,\mathbf{e}_2(x) + c\,\mathbf{e}_3(x) = \mathbf{v}$, at any point x, for a constant velocity vector $\mathbf{v}$.

Observer motions on the sphere

We can directly use the same approach as above for observer motions on the sphere, with the important real-world example of fluid flow on the Earth's surface modeled as a sphere. Fig. 7.9 shows a basis $\{\mathbf{e}_i\}$ for the three-dimensional Lie algebra of observer motions on the sphere, i.e., all infinitesimal rotations,[38] comprising three basis Killing fields on the sphere.

Apart from the manifold $M = \mathbb{S}^2$ being the sphere, and all vector fields correspondingly being vector fields defined on this manifold, everything else works exactly the same as above, most importantly Eq. 7.29.

Here, we also illustrate that we can define an isomorphism between the Lie algebra of all Killing vector fields on $\mathbb{S}^2$ and the matrix Lie algebra of

[38] These are all infinitesimal isometries of the sphere. Through integration, we can therefore obtain all rotations of the sphere.

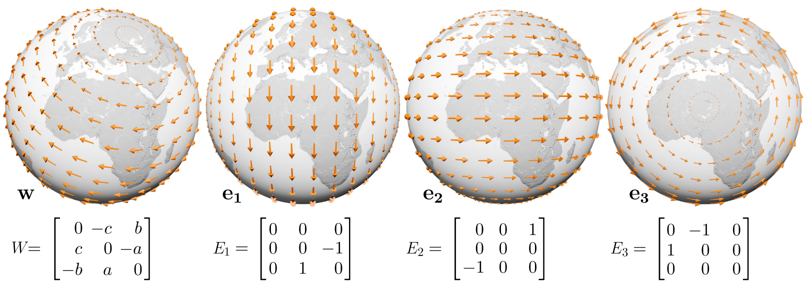


Figure 7.9: As in the plane, the set of all Killing vector fields on the sphere forms a three-dimensional Lie algebra. However, a *basis* of this vector space comprises three linearly independent *basis vector fields on the sphere*. The top row shows four elements of the Lie algebra of Killing fields on the sphere. The three vector fields $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent and therefore form a *basis* for the space of all Killing fields. Here, $\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$. Each Killing field is isomorphic to an anti-symmetric 3×3 matrix. If we *choose* the particular isomorphism shown here for E_1, E_2, E_3, the entire isomorphism is given, due to linearity: $W = aE_1 + bE_2 + cE_3$.

all anti-symmetric 3×3 matrices. Accordingly, for the vector space structure of the latter, we likewise have (see Fig. 7.9, bottom row)

$$W = aE_1 + bE_2 + cE_3. \tag{7.38}$$

Where for the Lie algebra of Killing vector fields, the Lie bracket is the *differential geometric Lie bracket*[39] between vector fields, i.e., $[\mathbf{v}, \mathbf{w}] = \mathscr{L}_{\mathbf{v}}\mathbf{w}$, the corresponding Lie bracket of the matrix Lie algebra of anti-symmetric 3×3 matrices is simply the *matrix commutator*[40]

$$[V, W] = VW - WV, \tag{7.39}$$

where the right-hand side is simply matrix multiplication and subtraction, and the left-hand side is the resulting anti-symmetric 3×3 matrix.

Finally, we note that the particular isomorphism is determined by (arbitrarily[41]) *choosing* an isomorphism between three basis vector fields and three linearly independent anti-symmetric matrices, respectively. That is, the isomorphism depicted in Fig. 7.9 is just one example of such an isomorphism.[42] After the isomorphism between the bases is chosen, the correspondence for any other vector field and matrix, respectively, follows directly from the linearity of the vector space structure.

[39] A differential operator that maps two vector fields to a vector field.

[40] Simply mapping two (anti-symmetric) matrices to a (anti-symmetric) matrix.

[41] As long as linear independence is preserved.

[42] Simply imagine different rotations of the sphere for choosing basis elements. However, any other choice is also valid, as long as the basis vector fields as well as the basis matrices are linearly independent, respectively.

From Lie algebra to Lie group

Given the description of observer motions via *derivatives*,[43] as elements of a Lie algebra, we can obtain the corresponding *integrated* observer motions, as elements of a Lie group. That is, we integrate Lie algebra elements to obtain the corresponding Lie group element. We often also say that we *integrate a path through the Lie algebra*, i.e., a function

$$t \mapsto X(t) \in \mathfrak{g}, \tag{7.40}$$

where each $X(t)$ is an element of a given Lie algebra $\mathfrak{g}$, and the corresponding integral is an element $g \in G$ of a Lie group G.

Analogously, we can also speak of a *path through a Lie group*, which is a function

$$t \mapsto g(t) \in G, \tag{7.41}$$

where each $g(t)$ is an element of a given Lie group G. We can start integrating a path $t \mapsto X(t)$ at the identity element $e \in G$, with $g(0) = e$, and the Lie group element $g(t)$ is obtained by integrating from $X(0)$ to $X(t)$. For the

[43] All these derivatives are evaluated at the identity element e of the corresponding Lie group.

case where $X(t) = X$, for a fixed Lie algebra element X, this integration can be obtained directly from the *exponential map* described in Sec. 7.3, with

$$g(t) := \exp(tX). \tag{7.42}$$

Where the Lie algebra of Killing vector fields in the plane is isomorphic to the Lie algebra of infinitesimal rigid motions in the plane, often denoted by $\mathfrak{se}(2)$,[44] the corresponding integrated motions form the Lie group $SE(2)$.[45] The Lie algebra $\mathfrak{se}(2)$ is isomorphic to all 3×3 matrices of the form[46]

$$\begin{bmatrix} 0 & -c & a \\ c & 0 & b \\ 0 & 0 & 0 \end{bmatrix}, \quad a,b,c \in \mathbb{R}, \tag{7.43}$$

and the Lie group $SE(2)$ is isomorphic to all 3×3 matrices of the form[47]

$$\begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}, \quad \phi \in [0, 2\pi); x,y \in \mathbb{R}. \tag{7.44}$$

For observer motions on the sphere, the Lie algebra of Killing vector fields on the sphere is isomorphic to the Lie algebra of infinitesimal rigid motions (rotations) of $\mathbb{S}^2$, which is $\mathfrak{so}(3)$,[48] with the corresponding integrated motions forming the Lie group $SO(3)$.[49] The Lie algebra $\mathfrak{so}(3)$ is isomorphic to all 3×3 anti-symmetric matrices, i.e., all matrices of the form[50]

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}, \quad a,b,c \in \mathbb{R}, \tag{7.45}$$

and the Lie group $SO(3)$ is isomorphic to all 3×3 rotation matrices, i.e., all orthogonal 3×3 matrices with determinant $+1$, also called the *proper orthogonal matrices*.[51]

Because of these isomorphisms, we call the Lie groups above *matrix Lie groups*, and the identity elements e in the matrix Lie groups $SE(2)$ and $SO(3)$, respectively, are simply the 3×3 identity matrices.

[44] The Lie algebra of the *special Euclidean group* in 2D, i.e., derivatives (evaluated at the identity group element e) of rigid motions without reflections in the plane.

[45] The Lie group of rigid motions without reflections: The *special Euclidean group*.

[46] The elements (a,b) in this matrix correspond to a linear velocity, and the element c corresponds to angular velocity.

[47] The arrangement in the matrix is identical to how rotations and translations are represented in computer graphics using homogeneous coordinates.

[48] The Lie algebra of the *special orthogonal group* in 3D.

[49] The *special orthogonal group* in 3D, i.e., all rotations without reflections in 3D, isomorphic to all invertible 3×3 matrices with unit determinant.

[50] The elements (a,b,c) in this matrix correspond to a 3D angular velocity vector.

[51] If reflections were also included (including orthogonal matrices with determinant -1), we would get the *orthogonal group* $O(3)$.

7.5 *Lie Group Actions*

Another important concept here is that of a Lie group action. Lie groups themselves are abstracts objects, but we want to actually use them in practice, usually together with some manifold[52]. A *group action* Φ, more specifically a smooth left action, of a Lie group G on a manifold M now does exactly this. It is a smooth map[53]

$$\begin{aligned} \Phi\colon G \times M &\to M, \\ (g,x) &\mapsto \Phi(g,x), \end{aligned} \tag{7.46}$$

such that

1. $\Phi(e,x) = x$, for all[54] $x \in M$, and
2. $\Phi(g, \Phi(h,x)) = \Phi(gh,x)$, for all $g,h \in G$ and $x \in M$,

[52] which represents the space under considerations, for example the plane, 3D space, or the sphere

[53] that is, it takes an element of a Lie group and an element of a manifold and produces another element of the manifold, one says that g acts on x

[54] where e is the identity of G, and one says that the identity acts the way that the identity is supposed to act

By setting $\phi_g(x) := \Phi(g,x)$ the properties of the group action can be written in the more concise form

$$\phi_g \phi_h = \phi_{gh} \quad \text{and} \quad \phi_e = \mathrm{id}_M. \tag{7.47}$$

It follows[55] that $\phi_{g^{-1}} = (\phi_g)^{-1}$ and the map

$$\begin{aligned} \phi_g : M &\to M, \\ x &\mapsto \phi_g(x) \end{aligned} \tag{7.48}$$

is a *diffeomorphism* for every $g \in G$[56].

[55] since $\phi_g \phi_{g^{-1}} = \phi_{gg^{-1}} = \phi_e = \mathrm{id}_M$

[56] since both ϕ_g and $\phi_{g^{-1}}$ are smooth by definition

---

Example 10. *For a specific example we can look at the situation where we use rotation matrices to rotate the plane, say. On the one hand we can multiply a matrix with a point:*

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7.49}$$

but we can also multiply the matrices, and we know from computer graphics that applying two rotation matrices to a point one after the other is the same as multiplying the matrices first and then apply them to the point, that is

$$\begin{aligned} &\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \left[\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right] \\ &\qquad = \left[\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \right] \cdot \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned} \tag{7.50}$$

And of course multiplying a point by the identity matrix results in just that point. So in this context

1. *Multiplying two matrices corresponds to the Lie group operation.*
2. *Multiplying a matrix with a point corresponds to the action of the Lie group on the manifold.*

7.6 Example: Objectivity

As an application we use this abstract concept of Lie group actions to shed some light on the concept of objectivity. Objectivity is usually defined to be invariance under rotations and translations of the reference frame [57]. A common criterion, for a vector field for example, to be objective is, that if one reference frame $\mathbf{x}^*$ is connected to another reference frame $\mathbf{x}$ by a time-dependent translation and rotation, that is

$$\mathbf{x}^* = c(t) + Q(t)\mathbf{x}, \tag{7.51}$$

then the vector field $\mathbf{v}^*$ in the transformed reference frame is linked to the vector field $\mathbf{v}$ via a time-dependent rotation in each tangent space, via[58]

[57] Clifford Truesdell and Walter Noll. *The Nonlinear Field Theories of Mechanics*. Springer-Verlag, 1965

[58] It is crucial, though, that the vector $\mathbf{v}^*$ is defined in the vector space at the point $\mathbf{x}^*$, whereas the vector $\mathbf{v}$ is defined in the vector space at the point $\mathbf{x}$.

$$\mathbf{v}^* = Q(t)\mathbf{v}. \tag{7.52}$$

We can reformulate this criterion using the Lie group action of the isometry group of the underlying space by writing[59]

$$\mathbf{v}^*_{\phi_g(x)} = (\mathrm{d}\phi_g)_x(\mathbf{v}). \tag{7.53}$$

That is, the change of reference frame is now encoded in the Lie group action of ϕ_g. This reformulation has two advantages:

1. Because we are only using generic notions like manifolds and Lie group actions, this definition works in any space with a non-trivial isometry group.[60]
2. We also gain some geometric insight: Objectivity is invariance under the action of the differential of the isometry group.[61]

For more details on this generalization, how to define objectivity in general spaces, like the sphere, and how to find good reference frames in general spaces, we refer to the papers by Rautek et al. [62] and by Zhang et al. [63].

[59] or short $\mathbf{v}^* = g\mathbf{v}$, that is g acting on $\mathbf{v}$, to make the connection to Equation 7.52 more clear, but it is important to note that the meaning of this shorthand notation is that given in Equation 7.53

[60] for example the sphere

[61] in the sense that tensor fields are simply pushed forward by that differential

[62] P. Rautek, M. Mlejnek, J. Beyer, J. Troidl, H. Pfister, T. Theußl, and M. Hadwiger. Objective observer-relative flow visualization in curved spaces for unsteady 2d geophysical flows. *IEEE Transactions on Visualization and Computer Graphics*, 27 (2):283–293, 2021

[63] Xingdi Zhang, Markus Hadwiger, Thomas Theußl, and Peter Rautek. Interactive exploration of physically-observable objective vortices in unsteady 2d flow. *IEEE Transactions on Visualization and Computer Graphics (Proceedings IEEE VIS 2021)*, 28(2):1–1, 2022. DOI: 10.1109/TVCG.2021.3115565

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