Assignment 4

Numerical Methods, 2024 Spring

111550057 資工15 莊婷馨

1.

First calculate the divided difference table in Matlab.

```
x = 0.3:0.2:1.5;
y = [0.3985, 0.6598, 0.9147, 1.1611, 1.3971, 1.6212, 1.8325];
div = divided_diff(x,y);

% calculate the divided difference table
function table = divided_diff(x,y)
    n = length(x);
    table = zeros(n,n);
    table(:,1) = y';   % first column is f(x)
    for j = 2:n
        for i = j:n
            % calculate the value based on the previous column
            table(i,j) = (table(i,j-1) - table(i-1,j-1)) / (x(i) - x(i-j+1));
        end
    end
end
```

(a)

Choose starting i = 1

$$f(x) \approx f[x_1, x_2] + f[x_1, x_2, x_3](x - x_1 + x - x_2) +$$
$$f[x_1, x_2, x_3, x_4][(x - x_3)(x - x_1) + (x - x_3)(x - x_2) + (x - x_1)(x - x_2)]$$

The result :

```
(a)
    1.2505
```

(b)

Choose starting i = 4

$$f(x) \approx f[x_4, x_5] + f[x_4, x_5, x_6](x - x_4 + x - x_5)$$

The result :

```
(b)
    1.0789
```

(c)

Choose staring i = 0

$$f(x) \approx f[x_0, x_1] + f[x_0, x_1, x_2](x - x_0 + x - x_1) +$$
$$f[x_0, x_1, x_2, x_3][(x - x_0)(x - x_1) + (x - x_0)(x - x_2) + (x - x_1)(x - x_2)]$$
$$f[x_0, x_1, x_2, x_3, x_4][(x - x_0)(x - x_1)(x - x_2) + (x - x_0)(x - x_2)(x - x_3)$$
$$+(x - x_1)(x - x_2)(x - x_3) + (x - x_0)(x - x_1)(x - x_3)]$$

The result :

```
(c)
    1.2923
```

2.

$f_n = f(x_n)$

$p(u) = au^4 + bu^3 + cu^2 + du + e$

$p(-2h) = f_{-2}$, $p(-h) = f_{-1}$, $p(0) = f_0$, $p(h) = f_1$,

$p(2h) = f_2$

$f''(x)$

$f''(x_0) = C_{-2} f_{-2} + C_{-1} f_{-1} + C_0 f_0 + C_1 f_1 + C_2 f_2$

Set $P(u) = 1$

$f_{-2} = f_{-1} = f_0 = f_1 = f_2 = 1$

$f''(x_0) = C_{-2} + C_{-1} + C_0 + C_1 + C_2$

$P''(0) = 0 = f''(x_0) = C_{-2} + C_{-1} + C_0 + C_1 + C_2$

$C_{-2} + C_{-1} + C_0 + C_1 + C_2 = 0$

Set $P(u) = u$

$f_{-2} = -2h,\ f_{-1} = -h,\ f_0 = 0,\ f_1 = h,\ f_2 = 2h$

$f''(x_0) = -2hC_{-2} + -hC_{-1} + 0 + hC_1 + 2hC_1$

$P''(0) = 0 = -2hC_{-2} - hC_{-1} + hC_1 + 2hC_1$

Set $P(u) = u^2$

$f_{-2} = 4h^2,\ f_{-1} = h^2,\ f_0 = 0,\ f_1 = h^2,\ f_2 = 4h^2$

$f''(x_0) = 4h^2C_{-2} + h^2C_{-1} + h^2C_1 + 4h^2C_2 = P''(0) = 2$

Set $P(u) = u^3$

$f_{-2} = -8h^3,\ f_{-1} = -h^3,\ f_0 = 0,\ f_1 = h^3,\ f_2 = 8h^3$

$f''(x_0) = -8h^3C_{-2} - h^3C_{-1} + h^3C_1 + 8h^3C_2 = P''(0) = 0$

Set $P(u) = u^4$

$f_{-2} = 16h^4,\ f_{-1} = h^4,\ f_0 = 0,\ f_1 = h^4,\ f_2 = 8h^3$

$f''(x_0) = 16h^4C_{-2} + h^4C_{-1} + h^4C_1 + 16h^4C_2 = P''(0) = 0$

Solve

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8 & -1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2c_{-2}+c_{-1}=c_1+2c_2 \\ 8c_{-2}+c_{-1}=c_1+8c_2 \end{cases} \Rightarrow c_1=c_{-1},\ c_2=c_{-2}$$

$$\begin{cases} 2c_1+2c_2+c_0=0 \\ 2h^2c_1+8h^2c_2=2 \\ 2c_1+32c_2=0 \end{cases} \Rightarrow c_1=-16c_2$$

$$-32h^2c_2+8h^2c_2=2$$

$$c_2=\frac{1}{-12h^2},\quad c_1=\frac{16}{12h^2}$$

$$c_0=\frac{2}{12h^2}+\frac{-32}{12h^2}=\frac{-30}{12h^2}$$

$$f''(x_0)\approx\frac{-f_{-2}+16f_{-1}-30f_0+16f_1-f_2}{12h^2}$$

#

$f'''(x)$

$f'''(x_0)=c_{-2}f_{-2}+c_{-1}f_{-1}+c_0f_0+c_1f_1+c_2f_2$

Same process as $f''(x)$. Solve

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix}\begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16 & 1 & 0 & 1 & 16 \end{bmatrix}\begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

$$\begin{cases} 4c_{-2}+c_{-1}=-c_1-4c_2 \\ 16c_{-2}+c_{-1}=-c_1-16c_2 \end{cases} \Rightarrow \begin{matrix} c_{-2}=-c_2 \\ c_{-1}=-c_1 \end{matrix}$$

$$c_0=0$$

$$\begin{cases} 4c_2+2c_1=0 \\ 16h^3c_2+2h^3c_1=6 \end{cases}$$

$$\begin{cases} 4h^3c_2+2h^3c_1=0 \\ 16h^3c_2+2h^3c_1=6 \end{cases} \Rightarrow \begin{matrix} 12h^3c_2=6 \\ c_2=\frac{1}{2h^3},\quad c_1=\frac{-1}{h^3} \end{matrix}$$

$$f'''(x_0)\approx\frac{-f_{-2}+2f_{-1}-2f_1+f_2}{2h^3}$$

#

$f^{(4)}(x)$

$$f^{(4)}(x_0) = c_{-2}f_{-2} + c_{-1}f_{-1} + c_0f_0 + c_1f_1 + c_2f_2$$

Same process as $f''(x)$. Solve

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{bmatrix}$$

$$\begin{cases} 2c_{-2} + c_{-1} = c_1 + 2c_2 \\ 8c_{-2} + c_{-1} = c_1 + 8c_2 \end{cases} \Rightarrow c_2 = c_{-2},\ c_1 = c_{-1}$$

$$8c_2 + 2c_1 = 0 \quad \Rightarrow \quad c_1 = -4c_2$$

$$32h^4c_2 + 2h^4c_1$$

$$= 24h^4c_2 = 24 \quad \Rightarrow \quad c_2 = \frac{1}{h^4},\ c_1 = \frac{-4}{h^4}$$

$$\frac{2}{h^4} + \frac{-8}{h^4} + c_0 = 0 \quad \Rightarrow \quad c_0 = \frac{6}{h^4}$$

$$f^{(4)}(x_0) \approx \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4}$$

3.

Suppose $f(x)$ is a cubic function between $x=a$ and $x=b$. $p(x)$ is a parabola that matches $f(x)$ at $x=a$, $x=b$ and $x=\frac{a+b}{2}$.

Then $f(x)-p(x)$ has three roots at $x=a$, $x=b$, $x=\frac{a+b}{2}$. $f(x)-p(x)$ can be expressed as:

$$f(x)-p(x) = c\cdot(x-a)(x-b)(x-\frac{a+b}{2})$$

where c is a constant

Shift $f(x)-p(x)$ horizontally by $\frac{a+b}{2}$, then we get

$$c\cdot(x-\frac{b-a}{2})(x-\frac{-(b-a)}{2})(x) \quad , \text{ let } k=\frac{b-a}{2}$$

$$= c\cdot(x-k)(x+k)x$$

$$= c(x^3-k^2x)$$

which is an odd function.

$$\therefore \int_{-k}^{k} c(x^3-k^2x)\,dx = 0$$

Shift $f(x)-p(x)$ back, then we get

$$\int_a^b [f(x)-p(x)]\,dx = 0$$

which means

$$\int_a^b f(x)\,dx = \int_a^b p(x)\,dx \quad ※$$

4.

First, calculate the true answer by the function "integral" in Matlab.

```
f = @(x) sin(x)./x;
q = vpa(integral(f,0,1));
disp('True answer:')
disp(q)
```

Then, use formula of Simpson's 1/3 rule:

$$\int_{x_0}^{x_2} f(x) \approx \frac{h}{3}(f_0+4f_1+f_2)$$

```
% Simpson's 1/3 rule, h = 0.5
h = 0.5;
x0 = 0;
x1 = 0.5;
x2 = 1;
approx1 = vpa((h/3) * ( 1 + 4*f(x1) + f(x2) )) ;
disp('Simpson, h = 0.5:')
disp(approx1)

% Simpson's 1/3 rule, h = 0.25
h = 0.25;
x0 = 0;
x1 = 0.25;
x2 = 0.5;
x3 = 0.75;
x4 = 1;
approx2 = vpa((h/3) * ( 1 + 4*f(x1) + 2*f(x2) + 4*f(x3) + f(x4) ));
disp('Simpson, h = 0.25:')
disp(approx2)
```

Since the error term of Simpson's 1/3 rule is $\dfrac{-1}{90}h^5 f^{(4)}(\xi)$, we can do extrapolation by

$$\text{Extrapolated result} = [g(\frac{h}{2}) - g(h) \times \frac{1}{2^5 - 1}] \times \frac{2^5}{2^5 - 1}$$

```
% Extrapolation (reach 5th order)
ex = vpa((approx2 - (approx1/32)) * (32/31));
disp('Extrapolation:')
disp(ex)
```

After the extrapolation, the order of error is 5, since the 4th order error is removed.

The result :

```
True answer:
0.94608307

Simpson, h = 0.5:
0.94614588

Simpson, h = 0.25:
0.946086934

Extrapolation:
0.94608503
```

The answer after extrapolation has the best accuracy compared to the Simpson's method. Using h=0.25 has better accuracy than h=0.5.

5.

(a) trapezoidal rule

Using the formula

$$\int_{x_0}^{x_n} f(x) \approx \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n)$$

Calculate by loops in Matlab

```
% trapezoidal rule
ix = zeros(17,1);
for i = 0:16   % x
    xcur = xmin + h*i;
    % y
    temp = f(xcur , ymin);
    for j = 1:21
        temp = temp + 2 * f(xcur, ymin+h*j);
    end
    temp = temp + f(xcur, ymax);
    ix(i+1) = temp * (h/2) ;
end

tra = ix(1) + ix(17);
for i = 2:16
    tra = tra + 2*ix(i);
end
tra = tra * (h/2);
disp('By trapezoidal rule:')
disp(vpa(tra))
```

The result :

```
By trapezoidal rule:
0.3683399551
```

(b) Simpson's 1/3 rule

Using the formula

$$\int_{x_0}^{x_n} f(x) \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 4f_{n-1} + f_n)$$

Calculate by loops in Matlab

```
% Simpson 1/3
ix = zeros(17,1);
for i = 0:16   % x
    xcur = xmin + h*i;
    % y
    temp = f(xcur , ymin);
    for j = 1:21
        if rem(j,2)==1
            fac = 4;
        else
            fac = 2;
        end
        temp = temp + fac * f(xcur, ymin+h*j);
    end
    temp = temp + f(xcur, ymax);
    ix(i+1) = temp * (h/3) ;
end
```

The result :

```
By Simpson 1/3:
0.3692685195
```

(c) 3-term Gaussian quadrature

Transform the integral interval of x and y to [-1,1]

$$x = \frac{1.4-0.2}{2} + \frac{1.4+0.2}{2}u = 0.6 + 0.8u$$

$$y = \frac{2.6+0.4}{2} + \frac{2.6-0.4}{2}v = 1.5 + 1.1v$$

Then we can calculate the following formula to get the approximation.

$$0.88 \times \int_{-1}^{1}\int_{-1}^{1} e^{(0.6+0.8u)} \sin(3 + 2.2v)dv\,du$$

$$\approx 0.88 \sum_{i=1}^{3}\sum_{i=1}^{3} w_i e^{(0.6+0.8t_i)} w_j \sin(3 + 2.2t_i)$$

Using values and weights of 3-term Gaussian quadrature

Number of terms	Values of t	Weighting factor	Valid up to degree
2	−0.57735027 0.57735027	1.0 1.0	3
3	−0.77459667 0.0 0.77459667	0.55555555 0.88888889 0.55555555	5

Calculate in Matlab

```
% 3-term Gaussian
t = [-0.77459667, 0, 0.77459667];
w = [0.55555555, 0.88888889, 0.55555555];

f21 = @(u) exp(1).^(0.6+0.8.*u);
f22 = @(v) sin(3+2.2.*v);

sum21 = w(1)*f21(t(1)) + w(2)*f21(t(2)) + w(3)*f21(t(3));
sum22 = w(1)*f22(t(1)) + w(2)*f22(t(2)) + w(3)*f22(t(3));
gau = 0.88 * sum21 *sum22;
disp('3-term Gaussian formula:')
disp(vpa(gau))
```

The result :

```
3-term Gaussian formula:
0.3723777165
```

- Comparison with analytical solution
  Obtain the analytical result by "integral2" function in Matlab.

```
Analytical solution:
0.3692650166

By trapezoidal rule:
0.3683399551

By Simpson 1/3:
0.3692685195

3-term Gaussian formula:
0.3723777165
```

Comparing with the analytical result, Simpson's 1/3 rule has the best accuracy on this problem. The 3-term Gaussian quadrature has the worst accuracy on this problem.

6.

Get random points from the interval $R = [-2,3] \times [-1,2]$

```
N = 100;    % number of random points
x_rand = x_min + (x_max - x_min) * rand(N, 1);
y_rand = y_min + (y_max - y_min) * rand(N, 1);
```

Get the approximation by Monte Carlo with sample mean method

$$\int\int_R f(x,y) \approx (3+2) \times (2+1) \times \frac{1}{\mathrm{N}} \sum_{i=1}^{N} f(x_i, y_i)$$

The result of different N values:

```
N = 100
   37.0392

N = 10000
   36.1498

N = 1000000
   35.9635
```

Comparison of results with analytical result :

N = 100	N = 10000	N = 1000000	Analytical
37.0392	36.1498	35.9635	35.9375
38.3521	36.0812	36.0035	35.9375
38.9896	36.0421	35.9379	35.9375
33.2476	35.4727	35.9564	35.9375

Overall, the more random points, the better accuracy.