Master 2 - LP   Année 2020-2021

TP n° 1 de "Analyse statique et typage"

Lattices, theory ad practice

Here is a selection of definitions and exercises from [2] and [1], plus one last exercice to put the theory into practice.

We denote the set $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$. Write $m \preceq n$ if and only if there exsts a $k \in \mathbb{N}_0$ such that $km = n$. We have that $\langle \mathbb{N}_0, \preceq \rangle$ is an ordered set.

A *lattice* is a poset that has finite meets[1] and joins[2]. A *complete lattice* is a poset with arbitrary meets and joins.

A *sublattice* A of a lattice X is a subset $A \subseteq X$ for which given any finite subset $F \subseteq A$ we have $\bigvee_X F \in A$ and $\bigwedge_X F \in A$. This definition is tantamount to saying that if A is regarded as a poset by restricting the order from X, then $\bigvee_X F = \bigvee_A F$ and $\bigwedge_X F = \bigwedge_A F$. A subset A of a lattice X may be a lattice in its own right without being a sublattice of X.

Let $\langle X, \leq \rangle$ be a poset. If A is a subset of X we let

$$\begin{aligned} a^\ell &\stackrel{def}{=} \{x \in X \mid x \leq a\} \\ A^\ell &\stackrel{def}{=} \{x \in X \mid \forall a \in A.\, x \leq a\} \end{aligned}$$

The set A^ℓ contains the lower bounds of the set A.

The notation $X \longrightarrow Y$ denotes the set of all the functions with domain X and codomain Y. If $\langle Y, \sqsubseteq \rangle$ is a poset, we can order the functions in $X \longrightarrow Y$ via the *point-wise order*: for every $f, g \in X \longrightarrow Y$ we let $f \leq g$ if $\forall x \in X\,.f(x) \sqsubseteq g(x)$.

Exercice 1 [Warm-up]

1. Figure (1) shows the diagram of the subset $P = \{1, 2, 3, 4, 5, 6, 7\}$ of $\langle \mathbb{N}_0, \preceq \rangle$. Find the join and the meet, where they exist, of each of the following subsets of P. Either specify the join or the meet or indicate why it fails to exist. (i) $\{3\}$ (ii) $\{4, 6\}$ (iii) $\{2, 3\}$ (iv) $\{2, 3, 6\}$ (v) $\{1, 5\}$

2. Consider the subset $Q = \{1, 2, 4, 5, 6, 12, 20, 30, 60\}$ ordered by $\preceq$. Is it a lattice ?

3. Is Q from the previous exercice a sublattice of $\langle \mathbb{N}_0, \preceq \rangle$?
(*hint:* Can you find elements $a, b, c, d \in Q$ such that $a \vee_{\mathbb{N}_0} b$ and $c \wedge_{\mathbb{N}_0} d$ do not exist in Q ?)

4. Let $\langle \mathbb{N}^\top, \sqsubseteq \rangle$ be the poset of the natural numbers augmented with a top element $\top$, where for every $n, m \in \mathbb{N}^\top$ we have $n \sqsubseteq m$ if $m = \top$ or if $n \leq m$. Prove that $\langle \mathbb{N}^\top, \sqsubseteq \rangle$ is a complete lattice.

5. Let $\mathcal{P}(X) \stackrel{def}{=} \{Y \mid Y \subseteq X\}$. Prove that $\langle \mathcal{P}(X), \subseteq \rangle$ is a lattice.

[1] glbs
[2] lubs

Exercice 2 [Bounds]

Let $\langle X, \leq\rangle$ be an ordered set.

1. Prove that if $A \subseteq X$ and $\bigwedge A$ exists in X, then
$$\bigcap\{a^\ell \mid a \in A\} = (\bigwedge A)^\ell$$
2. Formulate and prove the dual result.
3. Show that if $A \subseteq B$ then $B^\ell \subseteq A^\ell$, and define two C, D such that $C \subseteq D$ and $C^\ell \not\subseteq D^\ell$.
4. Prove that for every two subsets A, B of a poset $\langle X, \leq\rangle$ we have
   (a) $(A \cup B)^\ell \subseteq A^\ell \cap B^\ell$
   (b) $A^\ell \cap B^\ell \subseteq (A \cup B)^\ell$
   (c) $A^\ell \cup B^\ell \not\subseteq (A \cup B)^\ell$
   (d) $(A \cap B)^\ell \not\subseteq A^\ell \cap B^\ell$
5. Prove that the set $\{Y^\ell \mid Y \subseteq X\}$ is a complete lattice via set intersection and set union.

Exercice 3 [Functions]

1. Find a counterexample to the following statement. A monotone function $f : X \longrightarrow Y$ between posets X and Y which is a bijection is necessarily an isomorphism.
2. Let X be a poset ad define a relation on the set X by letting $x \prec y$ just in case $x < y$ and there is no $z \in X$ for which $x < z < y$. Now let X be any set and Y a poset. Let $X \Rightarrow Y$ be the poset of functions $X \longrightarrow Y$ ordered pointwise. Show that for every $f, g \in X \Rightarrow Y$ we have $f \prec g$ if and only if (iff) there exists $\hat{x} \in X$ such that
   (a) $f(\hat{x}) \prec g(\hat{x})$ in Y, and
   (b) $f(x) = g(x)$ for each $x \in X \setminus \{\hat{x}\}$.
3. Now let X be a *finite* poset, and $X \Rightarrow Y$ the poset of *monotone* functions $X \longrightarrow Y$. Show that $f \prec g$ iff (a) and (b) above remain true, with the new definition of $X \Rightarrow Y$.
4. Let S be any set and X a poset. The set of functions $S \Rightarrow X$ with domain S and codomain (the underlying set of) X is a (complete) lattice whenever X is a (complete) lattice, with $S \Rightarrow X$ ordered point-wise.

Exercice 4 [Code] Implement a lattice of *finite* sets of elements of a generic type `t` in `ocaml`. The lattice must provide three functions: `leqthan, lub, glb`. Recall the existence in the standard library of the functor `Set.Make`.

References

[1] Roy L. Crole. *Categories for Types.* Cambridge mathematical textbooks. Cambridge University Press, 1993.

[2] Brian A. Davey and Hilary A. Priestley. *Introduction to lattices and order.* Cambridge University Press, Cambridge, 1990.

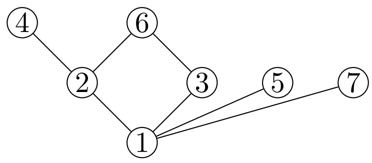

Figure 1: The poset for Exercice 1.