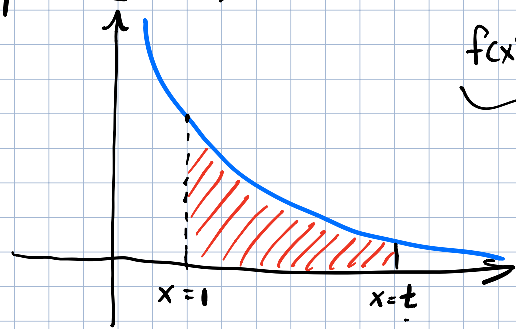
March 23

Improper Integrals.

Type I: Infinite Intervals.

$f(x) = \frac{1}{x^2}$



- let S be the infinite region, below $f(x)$, and to the right of $x=1$.

$$A(x) = \int_1^t \frac{1}{x^2}\,dx$$

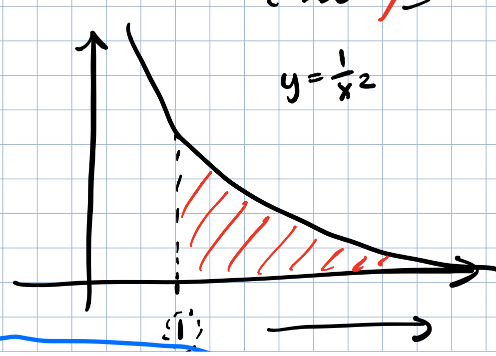
What happens as $t \to \infty$?

To Evaluate we take:

$$\lim_{t\to\infty} \int_1^t f(x)\,dx = \lim_{t\to\infty} \int_1^t x^{-2}\,dx$$

$$= \lim_{t\to\infty} \left[-x^{-1}\right]_1^t$$

$$\lim_{t\to\infty} -t^{-1} - (-1^{-1}) = \lim_{t\to\infty} \cancel{\left(-\frac{1}{t}\right)}^{0} + 1 = 1$$



Definition of Improper Integral type I

a) if $\int_a^t f(x)\,dx$ exists for all $t \geq a$

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(provided this limit exists)

b) if $\int_t^b f(x)\,dx$ exists. for all $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

(provided this limit exists).

- if in either case the limit exists we call the integral **Convergent**,
- if it does not exist, or is infinite, we call the integral **divergent**

c) if both $\int_a^\infty f(x)\,dx$, and $\int_{-\infty}^a f(x)\,dx$ are both convergent, we say:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

*Note: a may be any real number.

$$\left\{ \cancel{\int_{-\infty}^\infty x\,dx = 0} \right.$$ divergent

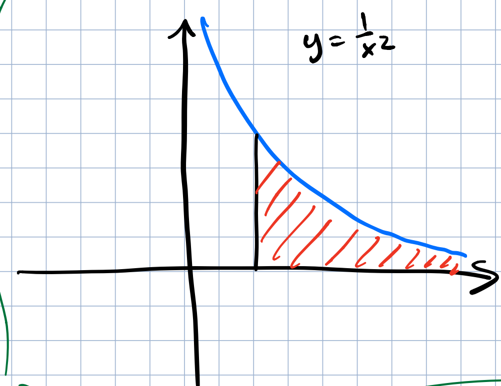
Ex

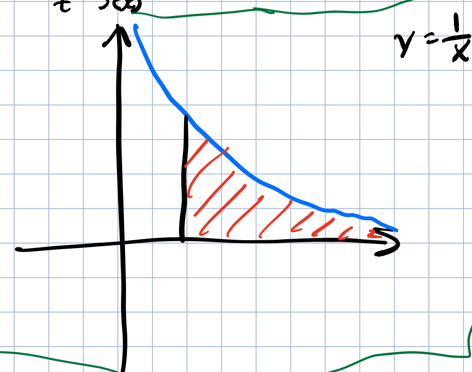
- Determine whether $\int_1^{\infty} \frac{1}{x}\,dx$ is convergent or divergent:

Simply take: $\lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(x)\Big|_1^{t}$

$= \lim_{t\to\infty} \ln(t) - \cancel{\ln(1)}^{\,0}$

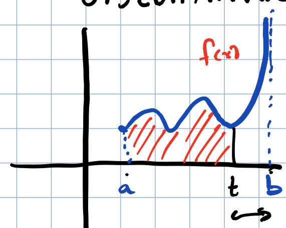
$= \lim_{t\to\infty} \ln(t) = \infty$





Type 2 : Discontinuous Integrands.

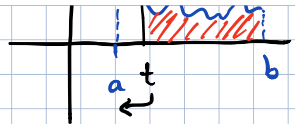
a) If $f(x)$ is continuous on $[a,b)$ and discontinuous at b, then.



$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

b) If $f(x)$ is continuous on $(a,b]$, and discontinuous at a, then.

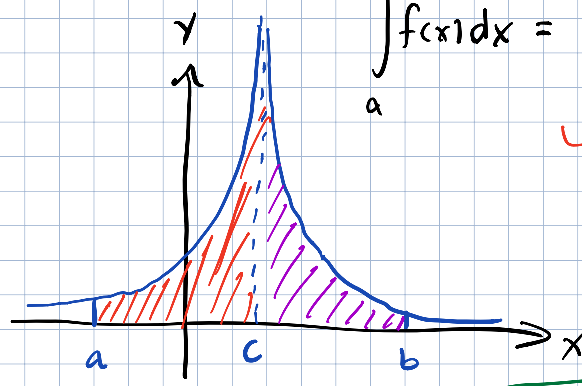
$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$



· Similar to before if the limit <u>exists</u> we call the Integral <u>Convergent</u>.

c) If $f(x)$ has discontinuity at c, where $a \le c \le b$, <u>and</u> both $\int_a^c f(x)\,dx$ <u>and</u> $\int_c^b f(x)\,dx$ are <u>convergent</u>, then:

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx} + \underbrace{\int_c^b f(x)\,dx}$$



<u>Ex</u>: Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

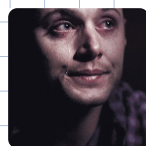
$$= \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big|_t^5$$

$$= \lim_{t\to 2^+} 2\sqrt{3} - 2\sqrt{t-2}$$

$$t \to 2$$

$$= 2\sqrt{3} - \lim_{t\to 2^+} \cancel{2\sqrt{t-2}}^{\,0}$$

$$= \boxed{2\sqrt{3}}$$

For  (tomorrow) :

For what values of p is the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ convergent?

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$p = 2 \implies$ Convergent

$p = 1 \implies$ divergent.

$p > 1$ $\qquad\qquad$ $p < 1$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^{t} x^{-p}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{-p+1} x^{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty} \frac{1}{-p+1} t^{-p+1} - \frac{1}{-p+1} 1^{-p+1} \qquad - \frac{1}{-p+1}$$

$$= \lim_{t\to\infty} \frac{1}{-p+1} t^{-p+1} + \frac{1}{p-1} \qquad + \frac{1}{p-1}$$

$p > 1$ $\qquad$ $p < 1$ $\qquad$ $p = 1 \Rightarrow$ diverges.

Convergent $\qquad$ divergent

when $p \leq 1$ $\quad \displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is divergent

when $p > 1$ $\quad \displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent

$$\int_{-\infty}^{-2} \frac{e^{2x}}{1+e^{2x}}\,dx$$

$u = 1+e^{2x}$

$du = 2e^{2x}dx$

$\frac{1}{2}du = e^{2x}dx$

$$I = \lim_{t \to -\infty} \int_t^{-2} \frac{e^{2x}}{1+e^{2x}}\,dx$$

$$= \lim_{t \to -\infty} \int_{x=t}^{x=-2} \frac{1}{2u}\,du$$

$$\lim_{t \to -\infty} \frac{1}{2}\ln|1+e^{2x}| \Big|_t^{-2} = \lim_{t \to -\infty} \frac{1}{2}\ln|1+e^{-4}| - \frac{1}{2}\ln|1+e^{2t}|$$

$$= \frac{1}{2}\ln|1+e^{-4}| - \lim_{t \to -\infty} \frac{1}{2}\ln|1+\cancel{e^{2t}}|$$

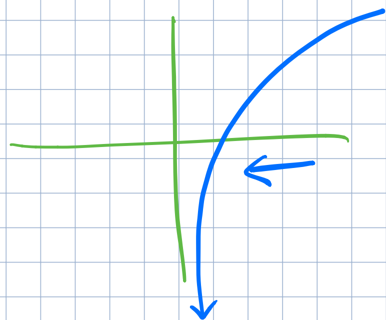
0

$$\boxed{= \frac{1}{2}\ln|1+e^{-4}|}$$

1 $\qquad$ 2

Ex:

$$\int_{-2}^{2}\frac{1}{v}\,dv = \lim_{t\to 0^-}\int_{-2}^{t}\frac{1}{v}\,dv + \lim_{t\to 0^+}\int_{t}^{2}\frac{1}{v}\,dv$$



$$= \lim_{t\to 0^-}\ln|v|\Big|_{-2}^{t} + \lim_{t\to 0^+}\ln|v|\Big|_{t}^{2}$$

$$= \lim_{t\to 0^-}\left(\ln|t| - \ln|-2|\right) + \lim_{t\to 0^+}\left(\ln|2| - \ln|t|\right)$$

$$= \underbrace{\left(\lim_{t\to 0^-}\ln|t|\right)}_{} - \ln|-2| + \ln|2| - \underbrace{\left(\lim_{t\to 0^+}\ln|t|\right)}_{-\infty}$$

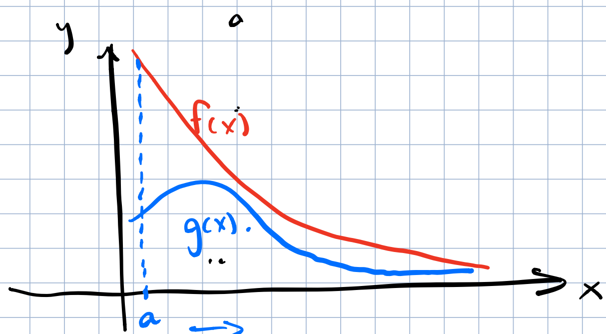
Divergent

{A Comparison Test for Improper Integrals}

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

2↳ If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent

2↳ if $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.



Ex: Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

for $x \geq 1$ we have $x^2 \geq x$, so $-x^2 \leq -x$

and then $\underbrace{e^{-x^2} \leq e^{-x}}$. We may write.

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{some constant}} + \underbrace{\int_1^\infty e^{-x^2}dx}_{\text{Convergent}}$$

Convergent

if $\underbrace{\int_1^\infty e^{-x}dx \text{ converges}}$, then $\int_1^\infty e^{-x^2}dx$ converges.

$$\underbrace{\int_1^\infty e^{-x}dx} = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t$$

$$= \lim_{t\to\infty} \cancel{-e^{-t}}^{\,0} + e^{-1}$$

$$= e^{-1}$$

Anika +1

Ex: Show $\int_1^\infty \frac{1+e^{-x}}{x}$ is divergent

using Comparison theorem with $f(x) = \frac{1}{x}$

we know

$$\underbrace{\frac{1+e^{-x}}{x}} > \underbrace{\frac{1}{x}}$$

Divergent

HW:

- $\displaystyle\int_0^3 \frac{dx}{x-1}$ Evaluate

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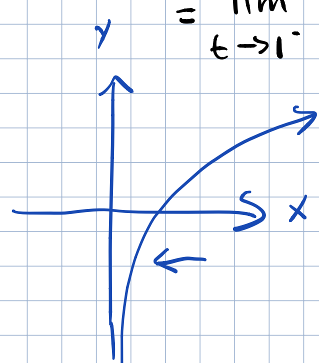
- $\displaystyle\int_0^1 \ln(x)\,dx$ Evaluate

$$\int_0^3 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx + \lim_{t\to 1^+} \int_t^3 \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \ln|x-1|\Big|_0^t + \lim_{t\to 1^+} \ln|x-1|\Big|_t^3$$

$$= \lim_{t\to 1^-} \ln|t-1| - \ln|0-1| + \lim_{t\to 1^+} \ln|2| - \ln|t-1|$$

$$\underbrace{\lim_{t\to 1^+} \ln|t-1|}_{-\infty}$$



Divergent

2. $\displaystyle\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+} \int_t^1 \ln(x)\,dx$

$$u = \ln(x) \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = x$$

$$I = \ln(x)\cdot x - \int \frac{1}{x}\cdot x\,dx$$

$$= x\ln(x) - \int dx = x\ln(x) - x$$

$$\lim_{t\to 0^+} \left(x\ln(x) - x\right)\Big|_t^1 = \lim_{t\to 0^+} \left(\overbrace{1\ln(1)}^{0} - 1\right) - \left(t\ln(t) - \overbrace{t}^{0}\right)$$

$$\lim_{t\to 0^+} -1 - t\ln(t) = -1 - \cancel{\lim_{t\to 0^+} t\ln(t)}$$

$\underbrace{\text{'}0\,(\infty)\text{'}}$ indeterminate?

$$\lim_{t\to 0^+} t\ln(t) = \lim_{t\to 0^+} \frac{\ln(t)}{\left(\frac{1}{t}\right)}$$

∞

$$= \lim_{t\to 0^+} \frac{\left(\frac{1}{t}\right)}{\left(-\frac{1}{t^2}\right)}$$

$$= \cancel{\lim_{t\to 0^+} -t} \to 0$$

Converges to $\boxed{-1}$

Quiz Problems

$$\int_{-3}^{0} \frac{4}{\sqrt{(x+3)^3}}\,dx = \lim_{t\to -3^+} \int_{t}^{0} \frac{4}{(x+3)^{3/2}}\,dx$$

$$\frac{8}{\sqrt{t+3}}$$

$$= \lim_{t\to 3^+} \int_{t}^{0} 4(x+3)^{-3/2}\,dx$$

$$= \lim_{t\to 3^+} -8(x+3)^{-1/2}\Big|_{t}^{0}$$

$$= \lim_{t\to 3^+} -8/\sqrt{3} + 8(t+3)^{-1/2} \to \infty$$

$$= -8/\sqrt{3} + \lim_{t\to 3^+} 8(t+3)^{-1/2}$$

$$= \infty \quad \therefore \text{Divergent.}$$

$$\int_{1}^{\infty} x e^{-4x}\,dx = \lim_{t\to\infty} \int_{1}^{t} x e^{-4x}\,dx$$

x	$\oplus$	e^{-4x}
1	$\ominus$	$-\frac{1}{4}e^{-4x}$
0		$\frac{1}{16}e^{-4x}$

$$= \lim_{t\to\infty} \left(-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x}\right)\Big|_{1}^{t}$$

$$= \lim_{t\to\infty} \left(-\frac{1}{4}te^{-4t} - \frac{1}{16}e^{-4t}\right) - \left(-\frac{1}{4}e^{-4} - \frac{1}{16}e^{-4}\right)$$

$$\boxed{= \frac{1}{4}e^{-4} + \frac{1}{16}e^{-4}}$$

3

$$\int_2^{\infty} \frac{4}{x^2+4x+3}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{4}{x^2+4x+3}\,dx$$

$$= \lim_{t\to\infty} \int_2^{t} \frac{4}{\underbrace{(x+1)(x+3)}}\,dx$$

$$\frac{A}{(x+1)} + \frac{B}{(x+3)} = \frac{4}{(x+1)(x+3)}$$

$A(x+3) + B(x+1) = 4$

$Ax + 3A + Bx + B = 4$

$Ax + Bx + 3A + B = 4$

$(A+B)x + 3A + B = 4$

$A + B = 0 \qquad A = -B$

$3A + B = 4$

$-3B + B = 4$

$-2B = 4$

$B = -2$

$A = 2$

$$= \lim_{t\to\infty} \int_2^{t} \left(\frac{2}{x+1}\right) - \left(\frac{2}{x+3}\right) dx$$

$$= \lim_{t\to\infty} \left(2\ln|x+1| - 2\ln|x+3|\right)\Big|_2^t$$

$$= \lim_{t\to\infty} \left(2\ln|t+1| - 2\ln|t+3|\right) - \left(2\ln|3| - 2\ln|5|\right)$$

$$= \lim_{t\to\infty} \underbrace{2\ln\left|\frac{t+1}{t+3}\right|}_{0} - \left(2\ln(3) - 2\ln(5)\right)$$

$$= \cancel{2\ln(1)} - 2\ln(3) + 2\ln(5)$$

$$= \boxed{-2\ln(3) + 2\ln(5)}$$

Evaluate $\displaystyle\int \frac{4}{(x+1)^2(x+2)}\,dx$

$$\frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)} = \frac{4}{(x+1)^2(x+2)}$$

$$A(x+1)(x+2) + B(x+2) + C(x+1)^2 = 4$$

$$A(x^2+3x+2) + B(x+2) + C(x^2+2x+1) = 4$$

$$(A+C)x^2 + (3A+B+2C)x + (2A+2B+C) = 4$$

$A + C = 0 \Rightarrow A = -C$

$3A + B + 2C = 0$ $\quad B - C = 0 \Rightarrow B = C$

$B = C = -A$

$2A + 2B + C = 4$

$-2C + C + 2C = 4$

$C = 4$

$B = 4$

$A = -4$

$\dfrac{4}{(x+1)^2}\,dx$

$$\int \frac{-4}{(x+1)} + \frac{4}{(x+1)^2} + \frac{4}{(x+2)}\,dx$$

$$\boxed{-4\ln|x+1| - \frac{4}{(x+1)} + 4\ln|x+2| + C}$$

$$\int \frac{1}{(x+1)(x+2)(x+3)}\,dx$$

$$\frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)} = \frac{1}{(x+1)(x+2)(x+3)}$$

$$A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) = 1$$

$$A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2) = 1$$

$$(A+B+C)x^2 + (5A+4B+3C) + (6A+3B+2C) = 1$$

$$\begin{cases} A+B+C = 0 \\ 5A+4B+3C = 0 \\ 6A+3B+2C = 1 \end{cases}$$