

# **Renert School: Integration Bee 2025–2026**

**Problem 1.**

$$\int_1^{mower} \frac{1}{u} du$$

**Solution:**

$$\int_1^{mower} \frac{1}{u} du = \ln |u| \Big|_1^{mower} = \ln |mower|$$

Problems 2-17 are great review.

**Problem 2.**

$$\int_{-8}^{-1} \left( \frac{2}{x^{1/3}} + 4x \right) dx$$

**Solution:**

$$\begin{aligned} & \int_{-8}^{-1} \left( \frac{2}{x^{1/3}} + 4x \right) dx \\ &= \left( 3x^{2/3} + 2x^2 \right) \Big|_{-8}^{-1} \\ &= 3((-1)^{2/3} - (-8)^{2/3}) + 2((-1)^2 - (-8)^2) \\ &= 3(1 - 4) + 2(1 - 64) \\ &= -9 - 126 = -135 \end{aligned}$$

**Problem 3.**

$$\int \frac{\sqrt{r} - 5}{\sqrt{r}} dr$$

**Solution:** Divide to get

$$\int \frac{\sqrt{r} - 5}{\sqrt{r}} dr = \int (1 - 5r^{-1/2}) dr = r - 10\sqrt{r} + C$$

**Problem 4.**

$$\int s(\sqrt{s} + 1) ds$$

**Solution:** Expand to get

$$\int s(\sqrt{s} + 1) ds = \int (s^{3/2} + s) ds = \frac{2}{5}s^{5/2} + \frac{1}{2}s^2 + C$$

**Problem 5.**

$$\int_{\pi/6}^{\pi/3} \sin(2x) dx$$

**Solution:**

Method 1: Using  $u = 2x$ ,  $du = 2 dx$ ,  $\pi/6 \mapsto \pi/3$ ,  $\pi/3 \mapsto 2\pi/3$ ,

$$\begin{aligned} \int_{\pi/6}^{\pi/3} \sin(2x) dx &= \int_{\pi/3}^{2\pi/3} \frac{1}{2} \sin u du \\ &= -\frac{1}{2} \cos u \Big|_{\pi/3}^{2\pi/3} = -\frac{1}{2} \left( -\frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2} \end{aligned}$$

Method 2: Integrating directly,

$$\int_{\pi/6}^{\pi/3} \sin(2x) dx = -\frac{1}{2} \cos(2x) \Big|_{\pi/6}^{\pi/3} = -\frac{1}{2} \left( -\frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2}$$

**Problem 6.**

$$\int x^3 e^{x^4} dx$$

**Solution:** Using  $u = x^4$ ,

$$\int x^3 e^{x^4} dx = \int \frac{1}{4} e^u du = \frac{1}{4} e^{x^4} + C$$

**Problem 7.**

$$\int (2x - 1) \sin(4x) dx$$

**Solution:** Use  $f = 2x - 1$ ,  $g' = \sin(4x)$ , so  $f' = 2$ ,  $g = -\frac{1}{4} \cos(4x)$ . By IBP,

$$\begin{aligned} \int (2x - 1) \sin(4x) dx &= -\frac{2x - 1}{4} \cos(4x) + \frac{1}{2} \int \cos(4x) dx \\ &= -\frac{2x - 1}{4} \cos(4x) + \frac{1}{8} \sin(4x) + C \end{aligned}$$

**Problem 8.**

$$\int \frac{\sec \theta \tan \theta}{\sqrt{\sec \theta + 1}} d\theta$$

**Solution:** Using  $u = \sec \theta + 1$ ,  $du = \sec \theta \tan \theta d\theta$ , so

$$\int \frac{\sec \theta \tan \theta}{\sqrt{\sec \theta + 1}} d\theta = \int \frac{du}{\sqrt{u}} = 2u^{1/2} + C = 2\sqrt{\sec \theta + 1} + C$$

**Problem 9.**

$$\int \ln x^3 dx$$

**Solution:**

Method 1:

Use  $f = \ln x^3$ ,  $g' = 1$ , so  $f' = \frac{3x^2}{x^3} = \frac{3}{x}$ ,  $g = x$ . By IBP,

$$\int \ln x^3 dx = x \ln x^3 - \int 3 dx = x \ln x^3 - 3x + C$$

Method 2:

Simplify first, then

$$\int \ln x^3 dx = \int 3 \ln x dx = 3(x \ln x - x) + C$$

**Problem 10.**

$$\int \frac{1}{t^2 - 6t + 10} dt$$

**Solution:**

$$\int \frac{1}{t^2 - 6t + 10} dt = \int \frac{1}{(t - 3)^2 + 1} dt = \arctan(t - 3) + C$$

**Problem 11.**

$$\int_{-1}^1 \frac{1}{x^3} dx$$

**Solution:** This is an improper integral of type II - you cannot use FTC II or that this is an odd function over a symmetric interval. Instead, the discontinuity is at 0, so

$$\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx$$

Considering the second integral,

$$\begin{aligned} \int_0^1 \frac{1}{x^3} dx &= \lim_{c \rightarrow 0^+} \int_c^1 \frac{1}{x^3} dx = \lim_{c \rightarrow 0^+} -\frac{1}{2x^2} \Big|_c^1 \\ &= \lim_{c \rightarrow 0^+} \left( \frac{1}{2c^2} - \frac{1}{2} \right) = \infty \end{aligned}$$

Therefore, the integral diverges.

**Problem 12.**

$$\int \frac{8x^2 - 2x + 3}{2x - 1} dx$$

**Solution:**

$$\begin{array}{r} 4x + 1 \\ 2x - 1 \overline{) 8x^2 - 2x + 3} \\ \underline{- 8x^2 + 4x} \phantom{3} \\ 2x + 3 \\ \underline{- 2x + 1} \\ 4 \end{array}$$

$$\begin{aligned} \int \frac{8x^2 - 2x + 3}{2x - 1} dx &= \int \left( 4x + 1 + \frac{4}{2x - 1} \right) dx \\ &= 2x^2 + x + 2 \ln |2x - 1| + C \end{aligned}$$

**Problem 13.**

$$\int x^2 e^{2x} dx$$

**Solution:** Use  $f = x^2, g' = e^{2x}$ , so  $f' = 2x, g = \frac{1}{2}e^{2x}$ . By IBP,

$$\int x^2 e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \int x e^{2x} dx$$

Use  $f = x, g' = e^{2x}$ , so  $f' = 1, g = \frac{1}{2}e^{2x}$ . By IBP again,

$$\begin{aligned} \int x^2 e^{2x} dx &= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{2} \int e^{2x} dx \\ &= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C \end{aligned}$$

**Problem 14.**

$$\int_e^{e^2} \frac{1}{x \ln x} dx$$

**Solution:** Method 1: Using  $u = \ln x$ ,  $du = \frac{1}{x} dx$ ,  $e \mapsto 1$ ,  $e^2 \mapsto 2$ , so

$$\int_e^{e^2} \frac{1}{x \ln x} dx = \int_1^2 \frac{1}{u} du = \ln |u| \Big|_1^2 = \ln 2$$

Method 2: Integrating directly,

$$\int_e^{e^2} \frac{1}{x \ln x} dx = \ln \ln x \Big|_e^{e^2} = \ln \ln e^2 - \ln \ln e = \ln 2$$

**Problem 15.**

$$\int_0^4 \sqrt{3x+4} \, dx$$

**Solution:**

Method 1: Using  $u = 3x + 4$ ,  $du = 3 \, dx$ ,  $0 \mapsto 4$ ,  $4 \mapsto 16$ , so

$$\begin{aligned} \int_0^4 \sqrt{3x+4} \, dx &= \int_4^{16} \frac{1}{3} \sqrt{u} \, du = \frac{2}{9} u^{3/2} \Big|_4^{16} \\ &= \frac{2}{9} (16^{3/2} - 4^{3/2}) = \frac{2}{9} (4^3 - 2^3) = \frac{112}{9} \end{aligned}$$

Method 2: Integrating directly,

$$\begin{aligned} \int_0^4 \sqrt{3x+4} \, dx &= \frac{2}{9} \sqrt{3x+4} \Big|_0^4 = \frac{2}{9} (16^{3/2} - 4^{3/2}) \\ &= \frac{2}{9} (4^3 - 2^3) = \frac{112}{9} \end{aligned}$$

**Problem 16.**

$$\int \frac{x+1}{x(2x+1)} \, dx$$

**Solution:** If  $\frac{x+1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$ , the Heaviside cover-up method gives  $A = \frac{0+1}{2(0)+1} = 1$  and  $B = \frac{-\frac{1}{2}+1}{-\frac{1}{2}} = -1$ , so

$$\begin{aligned} \int \frac{x+1}{x(2x+1)} \, dx &= \int \left( \frac{1}{x} + \frac{-1}{2x+1} \right) \, dx \\ &= \ln |x| - \frac{1}{2} \ln |2x+1| + C \end{aligned}$$

**Problem 17.**

$$\int_0^2 \frac{1}{\sqrt{2-x}} \, dx$$

**Solution:** The upper limit is a discontinuity, so

$$\begin{aligned} \int_0^2 \frac{1}{\sqrt{2-x}} \, dx &= \lim_{b \rightarrow 2^-} \int_0^b \frac{1}{\sqrt{2-x}} \, dx = \lim_{b \rightarrow 2^-} (-2\sqrt{2-x}) \Big|_0^b \\ &= \lim_{b \rightarrow 2^-} (-2\sqrt{2-b} + 2\sqrt{2-0}) = 2\sqrt{2} \end{aligned}$$

Problems 18-30 are still good for review

**Problem 18.**

$$\int_0^3 \sqrt{9 - r^2} \, dr$$

**Solution:** The function is a quarter-circle of radius 3, so

$$\int_0^3 \sqrt{9 - r^2} \, dr = \frac{9\pi}{4}$$

**Problem 19.**

$$\int_{-2}^2 (x^2 + 1)(x - \sin x) \, dx$$

**Solution:** The integrand is odd and the interval is symmetric, so

$$\int_{-2}^2 (x^2 + 1)(x - \sin x) \, dx = 0$$

**Problem 20.**

$$\int_1^4 \frac{2 + 1/\sqrt{x}}{x + \sqrt{x}} \, dx$$

**Solution:** This is logarithmic differentiation:

$$\begin{aligned} \int_1^4 \frac{2 + 1/\sqrt{x}}{x + \sqrt{x}} \, dx &= 2 \ln |x + \sqrt{x}| \Big|_1^4 \\ &= 2 \ln |4 + 2| - 2 \ln |1 + 1| \\ &= 2 \ln 3 = \ln 9 \end{aligned}$$

**Problem 21.**

$$\int x(1 - x)^{2025} \, dx$$

**Solution:** Let  $u = 1 - x$ , so that  $du = -dx$ .

$$\begin{aligned} \int x(1 - x)^{2025} \, dx &= - \int (1 - u)u^{2025} \, du \\ &= \int (u^{2026} - u^{2025}) \, du \\ &= \frac{u^{2027}}{2027} - \frac{u^{2026}}{2026} + C \\ &= \frac{(1 - x)^{2027}}{2027} - \frac{(1 - x)^{2026}}{2026} + C \end{aligned}$$

Note: Wolfram Alpha cannot do this integral.

**Problem 22.**

$$\int_0^1 \frac{x}{1+x^4} dx$$

**Solution:**

$$\begin{aligned}\int_0^1 \frac{x}{1+x^4} dx &= \frac{1}{2} \arctan(x^2) \Big|_0^1 \\ &= \frac{1}{2} (\arctan 1 - \arctan 0) = \frac{\pi}{8}\end{aligned}$$

**Problem 23.**

$$\int \frac{1}{2025^x} dx$$

**Solution:**

$$\int \frac{1}{2025^x} dx = \int 2025^{-x} dx = -\frac{2025^{-x}}{\ln 2025} + C$$

**Problem 24.**

$$\int_1^\infty \frac{1}{x(x^2+1)} dx$$

**Solution:** Using partial fraction decomposition or inspection,

$$\begin{aligned}\int \frac{1}{x(x^2+1)} dx &= \int \left( \frac{1}{x} - \frac{x}{x^2+1} \right) dx \\ &= \ln|x| - \frac{1}{2} \ln|x^2+1| + C \\ &= \ln \left| \frac{x}{\sqrt{x^2+1}} \right| + C\end{aligned}$$

Then

$$\begin{aligned}\int_1^\infty \frac{1}{x(x^2+1)} dx &= \lim_{b \rightarrow \infty} \ln \left| \frac{x}{\sqrt{x^2+1}} \right| \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \left( \ln \left| \frac{1}{\sqrt{1+\frac{1}{b^2}}} \right| - \ln \left| \frac{1}{\sqrt{2}} \right| \right) \\ &= -\ln 2^{-1/2} = \frac{\ln 2}{2}\end{aligned}$$



**Problem 25.**

$$\int \frac{2^x + 3^x}{5^x} dx$$

**Solution:** Simplify first,

$$\begin{aligned} \int \frac{2^x + 3^x}{5^x} dx &= \int \left( \left( \frac{2}{5} \right)^x + \left( \frac{3}{5} \right)^x \right) dx \\ &= \frac{(2/5)^x}{\ln(2/5)} + \frac{(3/5)^x}{\ln(3/5)} + C \end{aligned}$$

**Problem 26.**

$$\int \frac{1}{(x^2 - 1)^2} dx$$

**Solution:** By partial fractions,

$$\begin{aligned} &\int \frac{1}{(x^2 - 1)^2} dx \\ &= \int \frac{1}{(x + 1)^2(x - 1)^2} dx \\ &= \int \left( \frac{1/4}{x + 1} + \frac{1/4}{(x + 1)^2} + \frac{-1/4}{x - 1} + \frac{1/4}{(x - 1)^2} \right) dx \\ &= \frac{1}{4} \left( \ln|x + 1| - \frac{1}{x + 1} - \ln|x - 1| - \frac{1}{x - 1} \right) + C \\ &= \frac{1}{4} \left( -\frac{2x}{x^2 - 1} + \ln|x + 1| - \ln|x - 1| \right) + C \\ &= \frac{1}{4} \left( -\frac{2x}{x^2 - 1} + \ln \left| \frac{x + 1}{x - 1} \right| \right) + C \end{aligned}$$

Any of the last three or their distributed forms are acceptable.

**Problem 27.**

$$\int_1^e \ln(x^5) dx$$

**Solution:** Using integration by parts,

$$\begin{aligned} \int_1^e \ln(x^5) dx &= 5 \int_1^e \ln x dx \\ &= 5(x \ln x - x) \Big|_1^e \\ &= 5(e - e - 0 + 1) = 5 \end{aligned}$$

**Problem 28.**

$$\int_{-4}^0 \sqrt{-4x - x^2} dx$$

**Solution:** Completing the square,  $-4x - x^2 = 4 - (x + 2)^2$ . Then the integral is the area under the circle of radius 2 centred at  $(-2, 0)$  and above the  $x$ -axis, so

$$\int_{-4}^0 \sqrt{-4x - x^2} dx = \frac{2^2\pi}{2} = 2\pi$$

Alternatively, upon completing the square use  $x + 2 = 2 \sin \theta$ ,  $-4 \mapsto -\pi/2$ ,  $4 \mapsto \pi/2$ , so

$$\begin{aligned} \int_{-4}^0 \sqrt{-4x - x^2} dx &= \int_{-\pi/2}^{\pi/2} \sqrt{4 - 4 \sin^2 \theta} \cdot 2 \cos \theta d\theta \\ &= \int_{-\pi/2}^{\pi/2} 4 \cos^2 \theta d\theta \\ &= 2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= (2\theta + \sin 2\theta) \Big|_{-\pi/2}^{\pi/2} = 2\pi \end{aligned}$$

**Problem 29.**

$$\int \frac{1}{1 + e^{-x}} dx$$

**Solution:** This is a logarithmic integral:

$$\int \frac{1}{1 + e^{-x}} dx = \int \frac{e^x}{e^x + 1} dx = \ln |e^x + 1| + C$$

**Problem 30.**

$$\int_{-2}^2 (x + 3)\sqrt{4 - x^2} dx$$

**Solution:** Breaking this integral into two parts,

$$\begin{aligned} &\int_{-2}^2 (x + 3)\sqrt{4 - x^2} dx \\ &= \int_{-2}^2 x\sqrt{4 - x^2} dx + 3 \int_{-2}^2 \sqrt{4 - x^2} dx \end{aligned}$$

The first integral is of an odd integrand over a symmetric interval, so gives 0. The second integral is a semi-circle of radius 2. Then

$$\int_{-2}^2 (x + 3)\sqrt{4 - x^2} dx = 3 \cdot \frac{1}{2}\pi(2^2) = 6\pi$$