Fast discrete arctangent computation

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1 Introducion

Discrete arctangent in that article is a two argument function $\mathcal{F}(x, y)$ parametrized by integral parameter N. Current library supports only $N = 8k, k \in Z, N > 8$. It returns zero-based "sector number" in which lies beam from coordinate origin to point (x, y). Sectors are counted from 0 to N-1 counter clock-wise, first sector contains beams for angles $[0, \frac{2\pi}{N})$.

More formally speaking, assume α is an angle between x axis and point (x, y) value of $\mathcal{F}(x, y) \in Z$ so that

$$\mathcal{F}(x, y)\frac{2\pi}{N} \leq \alpha < (\mathcal{F}(x, y) + 1)\frac{2\pi}{N} \tag{1}$$

or, equivalently

$$\mathcal{F}(x, y) = \lfloor N\frac{\alpha}{2\pi}\rfloor$$

In this document we'll use either two-argument form $\mathcal{F}(x, y)$ when calclating arcatngent for point (x, y) or single argument form $\mathcal{F}(\alpha)$ for simplicity - here α is an angle between x axis and (x, y) point, as written above.

2 Math background

2.1 Reduction problem to angles only in $[0, \frac{\pi}{4})$

If $\alpha \geq \pi$ then we can compute

$$\mathcal{F}(\alpha - \pi) = \lfloor\frac{(\alpha - \pi)N}{2\pi}\rfloor = \lfloor\frac{\alpha N}{2\pi} - \frac{N}{2}\rfloor = \lfloor\frac{\alpha N}{2\pi}\rfloor - \frac{N}{2} = \mathcal{F}(\alpha) - \frac{N}{2} \tag{2}$$

Here last expression is valid since $\frac{N}{2} \in Z$ by definition of N suported by the library. Re-ordering last inequality gives us

$$\mathcal{F}(\alpha) = \mathcal{F}(\alpha - \pi) + \frac{N}{2} \tag{3}$$

Calculating point (x', y') for angle $\alpha - \pi$ from point (x, y) is straighforward. We can re-write original point coords as

$$x = R\cos(\alpha); y = R\sin(\alpha) \tag{4}$$

where R is a distance from coordinate origin to point (x, y). So

$$x' = R\cos(\alpha - \pi) = R(\cos(\alpha)\cos(\pi) + \sin(\alpha)\sin(\pi)) = -R\cos(\alpha) = -x$$
$$y' = R\sin(\alpha - \pi) = R(\sin(\alpha)\cos(\pi) - \cos(\alpha)\sin(\pi)) = -R\sin(\alpha) = -y \quad (5)$$

If $\pi > \alpha \geq \frac{\pi}{2}$ then same as in (3) we get

$$\mathcal{F}(\alpha) = \mathcal{F}(\alpha - \frac{\pi}{2}) + \frac{N}{4} \quad (6)$$

And new coords (x', y')

$$x' = R\cos(\alpha - \frac{\pi}{2}) = R(\cos(\alpha)\cos(\frac{\pi}{2}) + \sin(\alpha)\sin(\frac{\pi}{2})) = R\sin(\alpha) = y$$
$$y' = R\sin(\alpha - \frac{\pi}{2}) = R(\sin(\alpha)\cos(\frac{\pi}{2}) - \cos(\alpha)\sin(\frac{\pi}{2})) = -R\cos(\alpha) = -x \quad (7)$$

And finally if $\frac{\pi}{2} > \alpha \geq \frac{\pi}{4}$ then same as in (3) we get

$$\mathcal{F}(\alpha) = \mathcal{F}(\alpha - \frac{\pi}{4}) + \frac{N}{8} \quad (8)$$

And new coords (x', y')

$$x' = R\cos(\alpha - \frac{\pi}{4}) = R(\cos(\alpha)\cos(\frac{\pi}{4}) + \sin(\alpha)\sin(\frac{\pi}{4})) = R\frac{\sqrt{2}}{2}(\cos(\alpha) + \sin(\alpha)) = \frac{\sqrt{2}}{2}(x+y)$$
$$y' = R\sin(\alpha - \frac{\pi}{4}) = R(\sin(\alpha)\cos(\frac{\pi}{4}) - \cos(\alpha)\sin(\frac{\pi}{4})) = R\frac{\sqrt{2}}{2}(\sin(\alpha) - \cos(\alpha)) = \frac{\sqrt{2}}{2}(y-x) \quad (9)$$

Since multiplying both coords to the same scalar does not change angle α we're interested in, we can use in our calculations.

$$x' = (x + y)$$
$$y' = (y - x) \quad (10)$$

2.2 Computing function for angles in $[0, \frac{\pi}{4})$

Assume $f = \mathcal{F}(\alpha)$ for $\alpha \in [0, \frac{\pi}{4})$. On that interval range for f, assuming N is multiple by 8.

$$0 \leq f < \frac{N}{8} \quad (11)$$

Or, since $f \in Z$

$$0 \leq f \leq \frac{N}{8} - 1 \quad (12)$$

Since tan function monotonically increases on that interval we can write

$$\tan(f\frac{2\pi}{N}) \leq \tan(\alpha) < \tan((f + 1)\frac{2\pi}{N}) \quad (13)$$

Let's introduce table T:

$$T[k] = \tan(\frac{2\pi k}{N}), k \in Z, k \in [0, \frac{N}{8}] \tag{14}$$

Note that $T[k+1] > T[k] \forall k$ since tan is monotonically increasing.

$$T[f] \leq \tan(\alpha) < T[f+1] \tag{15}$$

and our task is quickly find elemend f, satisfying this inequality.

Assume we can build index table J of size S, so that

$$T[J[k]-1] < \frac{k}{S} \leq T[J[k]], J[k] \geq 0 \forall k \tag{16}$$

We can satisfy this inequality only if $S > Smin$. This inequality becomes possible when $\frac{1}{S} < \min_i(T[i+1] - T[i])$. That is evident - since values $\frac{k}{S}$ placed uniformly, in that case between any $T[i]$ and $T[i+1]$ will be at least one $\frac{k}{S}$ value. So

$$\begin{aligned} \frac{1}{S} &< \min_i(\tan(\frac{2\pi(i+1)}{N}) - \tan(\frac{2\pi i}{N})) \\ &= \min_i(\frac{\sin(\frac{2\pi(i+1)}{N} - \frac{2\pi i}{N})}{\cos(\frac{2\pi(i+1)}{N})\cos(\frac{2\pi i}{N})}) = \min_i(\frac{\sin(\frac{2\pi}{N})}{\cos(\frac{2\pi(i+1)}{N})\cos(\frac{2\pi i}{N})}) \end{aligned} \tag{17}$$

Since cos funcion is monotonically decreasing on $[0, \frac{\pi}{2})$, maximum value in denominator achieved when $i = 0$. So we get

$$\frac{1}{S} < \frac{\sin(\frac{2\pi}{N})}{\cos(\frac{2\pi}{N})} = \tan(\frac{2\pi}{N}), S > Smin = \frac{1}{\tan(\frac{2\pi}{N})} \tag{18}$$

Having table J we can quickly find values of f with it we can prove that

$$T[J[\lfloor \tan(\alpha)S \rfloor - 1]] \leq T[f] \leq T[J[\lfloor \tan(\alpha)S \rfloor]] \tag{19}$$

in that case we need make only one comparison after computing $\tan(\alpha) = \frac{y}{x}$.

First prove left part of (19). From (16) we get

$$T[J[\lfloor \tan(\alpha)S \rfloor - 1]] < \frac{\lfloor \tan(\alpha)S \rfloor}{S} \leq \frac{\tan(\alpha)S}{S} = \tan\alpha \tag{20}$$

From this and from (15) we get

$$T[J[\lfloor \tan(\alpha)S \rfloor - 1]] < T[f+1] \tag{21}$$

Because values in T monotonically increase, and $f \in Z$ from (21) we have

$$T[J[\lfloor \tan(\alpha)S \rfloor - 1]] \leq T[f] \tag{22}$$

that is the left part of (19), Q.E.D.

Proof of the right part of (19). Assume that it is wrong, that is

$$T[f] > T[J[\lfloor \tan(\alpha)S \rfloor]] \tag{23}$$

From (14) and (1)

$$T[f] = \tan(\frac{2\pi f}{N}) = \tan(\frac{2\pi \lfloor N\frac{\alpha}{2\pi} \rfloor}{N})$$

$$\tan(\frac{2\pi \lfloor N\frac{\alpha}{2\pi} \rfloor}{N}) < \tan(\frac{2\pi(\frac{N\alpha}{2\pi} - 1)}{N}) = \tan(\alpha - \frac{2\pi}{N}) \tag{24}$$

Substititute this and (16) to (23) and get

$$\tan(\alpha - \frac{2\pi}{N}) > T[f] > T[J[\lfloor \tan(\alpha)S \rfloor]] \geq \frac{\lfloor \tan(\alpha)S \rfloor}{S}$$

$$\tan(\alpha - \frac{2\pi}{N}) > \frac{\lfloor \tan(\alpha)S \rfloor}{S} > \frac{\tan(\alpha)S - 1}{S} = \tan(\alpha) - \frac{1}{S}$$

$$\frac{1}{S} > \tan(\alpha) - \tan(\alpha - \frac{2\pi}{N}) = \frac{\sin(\frac{2\pi}{N})}{\cos(\alpha)\cos(\frac{2\pi}{N})} =$$

$$\tan(\frac{2\pi}{N})\frac{1}{\cos(\alpha)} \geq \tan(\frac{2\pi}{N}) \tag{25}$$

But before in (18) we took $\frac{1}{S} < \tan(\frac{2\pi}{N})$. We get to contradiction, so right part of (19) is proven $\forall S > Smin$.

3 Algorighm

1. Build table T as described in (14).
2. Take $S = \lceil \frac{1}{\tan(\frac{2\pi}{N})} \rceil$.
3. Build table J as described in (16).
4. For each input point:
   (a) For input point (x, y) get corresponding point (x', y'), arctangent of which lies in $[0, \frac{\pi}{4})$. Also compute required offset for result. Use for that consequently equations (3), (5), (6), (7), (8), (10).
   (b) Calculate $\tan(\alpha) = \frac{x'}{y'}$.
   (c) Make initial guess $r = J[\lfloor \tan(\alpha)S \rfloor] - 1$. From (19) we know that either $f = r$ or $f = r + 1$.
   (d) Compare $\tan(\alpha)$ with $T[r + 1]$. Accorgind to (15) must be $\tan(\alpha) < T[f + 1]$ if this inequality does not hold for $f = r$, we must use $f = r + 1$.
   (e) Get final function result $\mathcal{F}(x, y)$ having computed value of f for (x', y') and offset for it.

For sake of better CPU cache usage we can put both value of $J[r]$ and $T[r + 1]$ in the same table item.