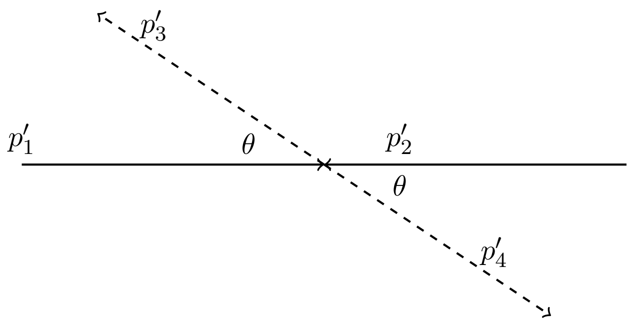
Calculations for arXiv:1002.0274v2
INTRODUCTION TO COLLIDER PHYSICS

1 Mandlestam variables

$(p_1,\, p_2) \rightarrow (p_3,\, p_4)$ scattering s.t. masses of particles are $m_1,\, m_2,\, m_3,\, m_4$ respectively.



Let the collision occur in the y-z plane s.t. m_1 moves along the z- direction and m_2 moves along the (-ve) z-direction. 4-momentum conservation: $p_1 + p_2 = p_3 + p_4$, where the respective 4-momenta are $p_1 = (E_1, 0, 0, p'_1)$, $p_2 = (E_2, 0, 0, -p'_1)$, $p_3 = (E_3, 0, p'_3 \sin\theta, -p'_3 \cos\theta)$, $p_4 = (E_4, 0, -p'_3 \sin\theta, p'_3 \cos\theta)$. Also, in the COM frame $p'_1 + p'_2 = 0$ and $p'_3 + p'_4 = 0$.

The three mandlestam variables, then, are:

1. $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$
2. $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$
3. $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$

The relation between s, t and u is:

$$\begin{aligned} s + t + u &= (p_1^2 + p_2^2)^2 + (p_1^2 - p_3^2)^2 + (p_1^2 - p_4^2)^2 \\ &= p_1^2 + p_2^2 + 2p_1.p_2 + p_1^2 + p_3^2 - 2p_1.p_3 + p_1^2 p_4^2 - 2p_1.p_4 \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1.(p_2 - p_3 - p_4) \\ &= p_1^2 + p_2^2 + p_3^2 + p_4^2 \end{aligned}$$

$$\boxed{s + t + u = p_1^2 + p_2^2 + p_3^2 + p_4^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2} \tag{1}$$

The incoming 3-momentum p'_1 in terms of the mandlestam variable and invariant masses is:

$$\begin{aligned} s &= (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1.p_2 \\ &= m_1^2 + m_2^2 + 2(E_1E_2 + p_1'^2) \end{aligned} \tag{2}$$

But $\sqrt{s} = E_1 + E_2 \implies E_2 = \sqrt{s} - E_1$
Therefore equation (2) becomes:

$$\begin{aligned} s - m_1^2 - m_2^2 &= 2(E_1\sqrt{s} - E_1^2 + p_1'^2) \\ &= 2E_1\sqrt{s} \\ &= 2\sqrt{p_1'^2 + m_1^2}\sqrt{s} \end{aligned}$$

$$\boxed{p_1'^2 = \frac{(s + m_1^2 - m_2^2)^2 - 4m_1^2 s}{4s}} \tag{3}$$

Similarly, the outgoing 3-momentum p'_3 in terms of the mandlestam variable and invariant masses is:

$$\boxed{p_3'^2 = \frac{(s + m_3^2 - m_4^2)^2 - 4m_3^2 s}{4s}} \tag{4}$$

The scattering angle:

$$\begin{aligned} t = (p_1 - p_3)^2 &= [(E_1, 0, 0, p'_1) - (E_3, 0, p'_3 \sin\theta), -p'_3\cos\theta]^2 \\ &= (E_1 - E_3)^2 - (p'_3\sin\theta)^2 - (p'_1 - p'_3\cos\theta)^2 \\ &= E_1^2 + E_3^2 - 2E_1E_3 - p_1'^2 - p_3'^2 + 2p'_1p'_3\cos\theta \\ &= m_1^2 + m_3^2 - 2\sqrt{p_1'^2 + m_1^2}\sqrt{p_3'^2 + m_3^2} + 2p'_1p'_3\cos\theta \end{aligned}$$

$$\implies \boxed{\cos\theta = \frac{t - m_1^2 + m_3^2 - 2\sqrt{p_1'^2 + m_1^2}\sqrt{p_3'^2 + m_3^2}}{2p'_1p'_3}} \tag{5}$$

If $m_1 = m_2$ and $m_3 = m_4$:

- Equations (3) and (4) become:

$$p'_{1,3} = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m_{1,3}^2}{s}}$$

- Equation (5) becomes:

$$\begin{aligned} \cos\theta &= \frac{t - m_1^2 + m_3^2 - 2\sqrt{p_1'^2 + m_1^2}\sqrt{p_3'^2 + m_3^2}}{2p'_1p'_3} \\ &= \frac{t - m_1^2 - m_3^2 - 2\sqrt{\frac{s}{4}(1 - \frac{4m_1^2}{s}) + m_1^2}\sqrt{\frac{s}{4}(1 - \frac{4m_3^2}{s}) + m_3^2}}{\frac{2s}{4}\sqrt{1 - \frac{4m_1^2}{s}}\sqrt{1 - \frac{4m_3^2}{s}}} \\ &= \frac{1 + \frac{2(t - m_1^2 - m_3^2)}{s}}{\sqrt{1 - \frac{4m_1^2}{s}}\sqrt{1 - \frac{4m_3^2}{s}}} \end{aligned}$$

Now, if the masses are identically zero:

$$\boxed{t = \frac{s}{2}(\cos\theta - 1)} \tag{6}$$