Investigation of Dust-Ion Acoustic Solitary Structures in a Weakly Relativistic Plasma

Chirag Verma

27 August 2020

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Chapter 1

BASIC FORMULATION

In order to analytically study the propagation of solitary waves in an unmagnetized three component weak relativistically degenrate plasma we start with a set of interpenetrating fluids (three, here) characterized by :

1. The equations of continuity and motion of
   - ions,
   - electrons, and
   - dust particles
2. The Poisson's equation
3. The Pressure Law

1.1 The Governing Equations

- **For Dust particles**:

$$\frac{\partial n_d}{\partial t} + \frac{\partial (n_d u_d)}{\partial x} = 0 \tag{1.1}$$

$$\frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} = \frac{Q_d}{m_d} \frac{\partial \phi}{\partial x} \tag{1.2}$$

- **For Ions**:

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i u_i)}{\partial x} = 0 \tag{1.3}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{Q_i}{m_i} \frac{\partial \phi}{\partial x} - \frac{1}{m_i n_i} \frac{\partial p_i}{\partial x} \tag{1.4}$$

- **For Electrons**:

$$\frac{\partial n_e}{\partial t} + \frac{\partial (n_e u_e)}{\partial x} = 0 \tag{1.5}$$

$$\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} = -\frac{Q_e}{m_e}\frac{\partial \phi}{\partial x} - \frac{1}{m_e n_e}\frac{\partial p_e}{\partial x} + \frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial x}(\frac{1}{\sqrt{n_e}}\frac{\partial^2 \sqrt{n_e}}{\partial x^2}) \tag{1.6}$$

- **Poisson's Equation:**

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi(Q_e n_e - Q_i n_i + Q_d n_d) \tag{1.7}$$

1.2 Pressure Law

Following from Chandrasekhar (1939), The electron degenracy pressure in fully degenerate and relativistic form can be expressed as :

$$p_j = (\pi m_e^4 c^5/3h^3)[R_j(2R_j^2 - 3)\sqrt{1+R_j^2} + 3\sinh^{-1} R_j] \tag{1.8}$$

$$R_j = p_{Fj}/m_e c = [3h^3 n_j/8\pi m_e^3 c^3]^{1/3} = R_{j0} n_j^{1/3} \tag{1.9}$$

$$R_{j0} = (n_{j0}/n_0)^{1/3} = 5.9 \cdot 10^{29} cm^{-1}$$

For weak-relativistic degeneracy : $R_j \to 0$, and, hence

$$p_j = \frac{1}{20}(\frac{3}{\pi})^{\frac{2}{3}}\frac{h^2}{m_e}n_j^{\frac{5}{3}}$$

where the suffices d, e, i stand for dust particle, electron, and ion respectively; u_j and p_j are respectively the fluid velocity and degeneracy pressure of the j^{th} species, i.e. electron or dust particle or ion; ϕ is the electrostatic wave potential; and Q_j is the charge on the j^{th} species.

1.3 Normalization

Under the following normalized scheme: $n_j \to n_j/n_{i0}, t \to t\omega_{pd}, x \to x/\lambda_{De}$, $u_j \to u_j/C_d, \phi \to K_bT_e/m_d$

where $\omega_{pd} = (m_d/4\pi n_{i0}e^2)^{1/2}$ is the characteristic dust plasma frequency, $\lambda_{De} = (K_bT_e/4\pi n_{i0}e^2)^{1/2}$ is the Debye length, $C_d = (K_bT_e/m_d)^{1/2}$ is the dust acoustic speed; the above equations are then normalized to:

$$\frac{\partial n_d}{\partial t} + \frac{\partial(n_d u_d)}{\partial x} = 0 \tag{1.10}$$

$$\frac{\partial u_d}{\partial t} + u_d\frac{\partial u_d}{\partial x} = -Z_d\frac{\partial \phi}{\partial x} \tag{1.11}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial(n_i u_i)}{\partial x} = 0 \tag{1.12}$$

$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} = -\frac{1}{Q}(\frac{\partial \phi}{\partial x} + \frac{\alpha}{n_i}\frac{\partial p_i}{\partial x}) \tag{1.13}$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial(n_e u_e)}{\partial x} = 0 \tag{1.14}$$

$$\frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} = -\frac{1}{P}(\frac{\partial \phi}{\partial x} + \frac{1}{n_e}\frac{\partial p_e}{\partial x}) + \frac{1}{P^2}\frac{H^2}{2}\frac{\partial}{\partial x}(\frac{1}{\sqrt{n_e}}\frac{\partial^2\sqrt{n_e}}{\partial x^2}) \tag{1.15}$$

$$\frac{\partial^2 \phi}{\partial x^2} = (n_e - n_i + Z_d n_d) \tag{1.16}$$

where $Q = \frac{m_i}{m_d}, P = \frac{m_e}{m_d}, H = \frac{\hbar w_{pd}}{K_bT_e}, \alpha = \frac{T_i}{T_e}$

Chapter 2

ANALYTICAL STUDY

2.1 Perturbation Expansion

In order to investigate the nonlinear behaviour of electron-acoustic waves we make the following perturbation expansion, about their equilibrium quantities, for the aforementioned field quantities:

$$\begin{bmatrix} n_d \\ n_i \\ n_e \\ u_d \\ u_e \\ u_i \\ \phi \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ u_{0d} \\ u_{0e} \\ u_{0i} \\ \phi_0 \end{bmatrix} + \epsilon \begin{bmatrix} n_d^{(1)} \\ n_e^{(1)} \\ n_i^{(1)} \\ u_d^{(1)} \\ u_e^{(1)} \\ u_i^{(1)} \\ \phi^{(1)} \end{bmatrix} + \epsilon^2 \begin{bmatrix} n_d^{(2)} \\ n_e^{(2)} \\ n_i^{(2)} \\ u_d^{(2)} \\ u_e^{(2)} \\ u_i^{(2)} \\ \phi^{(2)} \end{bmatrix} + \cdots \tag{2.1}$$

2.2 Dispersion Characteristics

Substituting the expansion (3.1) in equations (2.10)-(2.16), and then linearizing and assuming that all field quantities vary as $e^{i(kx-\omega t)}$,we get for normalized wave frequency ω and wave number k, the following linear dispersion relation:

$$-1 = \frac{1}{P\omega^2 - \frac{5}{3}\alpha F_e k^2 - \frac{H^2k^4}{4P}} - \frac{1}{Q\omega^2 - \frac{5}{3}\alpha F_i k^2} + \frac{Z_d^2}{\omega^2} \tag{2.2}$$

where $F_i = F_e = \frac{1}{20}(\frac{3}{\pi})^{2/3}\frac{h^2}{m_e}$

In the long wavelength range, we ignore k^4 and equation (3.2) becomes quadratic in ω^2 as follows:

$$0 = PQ\omega^4 + (Q - \frac{5}{3}P\alpha F_i k^2 - \frac{5}{3}QF_e k^2 - P - Z_d^2 PQ)\omega^2 + (\frac{5}{3}F_e k^2 - \frac{5}{3}\alpha F_i k^2 + \frac{5}{3}Z_d^2 \alpha P F_i k^2 + \frac{5}{3}Z_d^2 Q F_e k^2) \tag{2.3}$$

The solutions of equation (3.3) are :

$$\omega_{1,2}^2 = -b \pm \sqrt{b^2 - 4c}$$

where

$$b = \frac{1}{P} - \frac{5}{3}\frac{1}{Q}\alpha F_i k^2 - \frac{5}{3}\frac{1}{P}F_e k^2 - P - Z_d^2$$
$$c = \frac{5}{3}\frac{1}{PQ}F_e k^2 - \frac{5}{3}\frac{1}{PQ}\alpha F_i k^2 + \frac{5}{3}\frac{1}{Q}Z_d^2 \alpha F_i k^2 + \frac{5}{3}\frac{1}{P}Z_d^2 F_e k^2$$

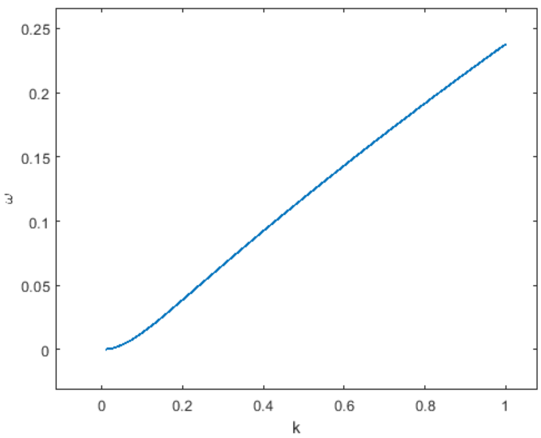


Figure 2.1: ω vs k: $F_i = F_e = 1.517, \alpha = 0.01, H = 2$

2.3 Derivation of KdV Equation

We would be using standard reductive perturation technique, which is:
$\xi = \varepsilon^{1/2}(x - V_0 t)$ and $\tau = \varepsilon^{3/2} t$
where V_0 is the normalized linear long wave phase velocity given by equation (), and ε is the smallness parameter measuring the dispersion and nonlinear effects.

Now writing the equations (2.10)-(2.16) in the terms of these stretched coordinates ξ and τ, and applying the perturbation expansion (3.1), and then

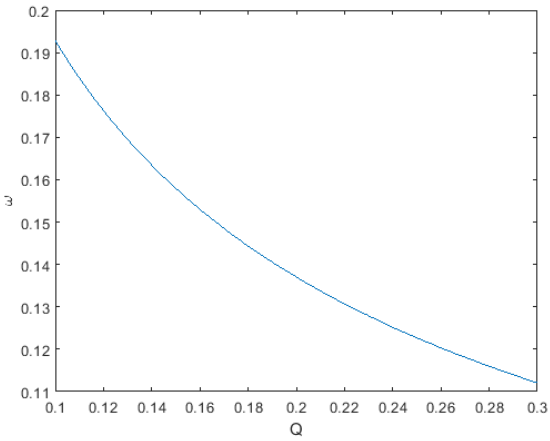


Figure 2.2: ω vs Q: $F_i = 1.517, \alpha = 0.01, Z_d = 100, H = 2, k = 0.5$

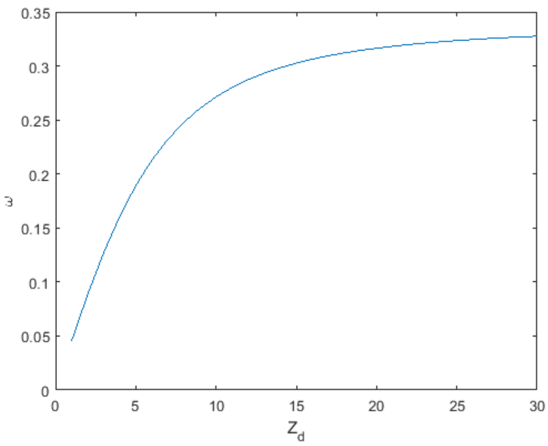


Figure 2.3: ω vs Z_d: $F_i = 1.517, \alpha = 0.01, Q = 0.1, H = 2, k = 0.5$

solving for the lowest order equation with boundary conditions that $n_j^{(1)}, u_j^{(1)}$, $\phi^{(1)} \to 0$ as $|\xi| \to 0$, the following solutions are obtained :

$$
\begin{aligned}
u_d^{(1)} &= \frac{Z_d}{M_d}\phi^{(1)}, n_d^{(1)} = \frac{Z_d}{M_d^2}\phi^{(1)} \\
u_i^{(1)} &= \frac{M_i}{A}\phi^{(1)}, n_i^{(1)} = \frac{1}{A}\phi^{(1)} \\
u_e^{(1)} &= \frac{M_e}{B}\phi^{(1)}, n_e^{(1)} = \frac{1}{B}\phi^{(1)}
\end{aligned}
\tag{2.4}
$$

where

$$M_d = V_0 - u_{0d} \tag{2.5}$$
$$M_i = V_0 - u_{0i} \tag{2.6}$$
$$M_e = V_0 - u_{0e} \tag{2.7}$$
$$A = QM_i^2 - \frac{5}{3}\alpha F_i \tag{2.8}$$
$$B = PM_e^2 - \frac{5}{3}F_e \tag{2.9}$$

Going for the next higher order terms in ε and following the usual method the desired Korteweg-de-Vries equation is obtained :

$$\frac{\partial \phi}{\partial \tau} + N\phi\frac{\partial \phi}{\partial \xi} + D\frac{\partial^3 \phi}{\partial \xi^3} \tag{2.10}$$

where

$$N = \frac{\frac{3Z_d^3}{M_d^4} + \frac{3PM_e^2}{B^3} - \frac{3QM_i^2}{A^3} + \frac{5\alpha F_i}{9A^3} - \frac{5F_i}{9B^3}}{\frac{2Z_d^2}{M_d^3} + \frac{2PM_e}{B^2} - \frac{2QM_i}{A^2}}$$
$$D = \frac{1 + \frac{H^@}{2PB^2}}{\frac{2Z_d^2}{M_d^3} + \frac{2PM_e}{B^2} - \frac{2QM_i}{A^2}}$$

2.4 Solution of KdV Equation

Consider the new variable
$\eta = \xi - M\tau$
where M is the normalized constant speed of the wave frame.

On applying the boundary conditions as $\eta \to \pm\infty; \phi, \frac{\partial \phi}{\partial \eta}, \frac{\partial^2 \phi}{\partial \eta^2} \to 0$, the possible stationary solution of equation (3.8) is :

$$\phi = \phi_m \operatorname{sech}^2\left(\frac{\eta}{\Delta}\right) \tag{2.11}$$

where the amplitude ϕ_m and the width Δ of the soliton are given by :

$\phi_m = 3\frac{M}{N}$ and $\Delta = \sqrt{\frac{4D}{M}}$

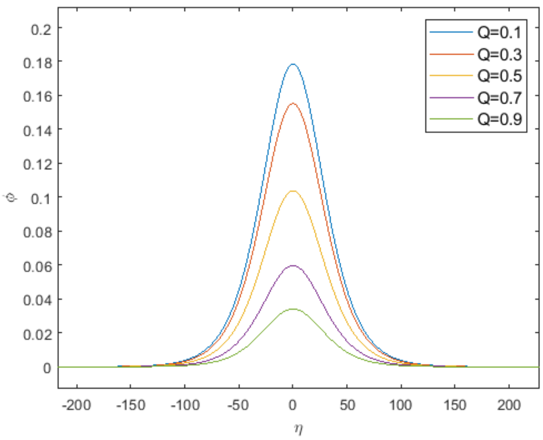
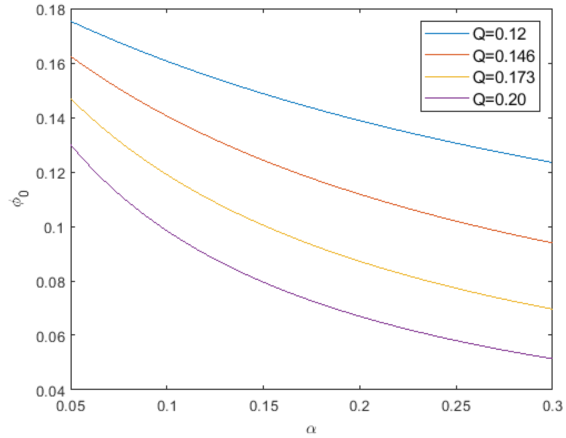
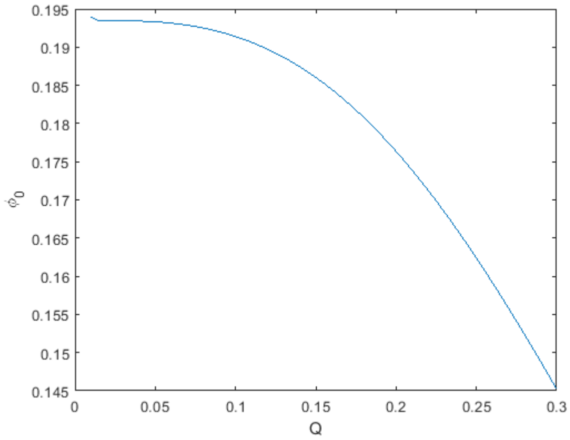


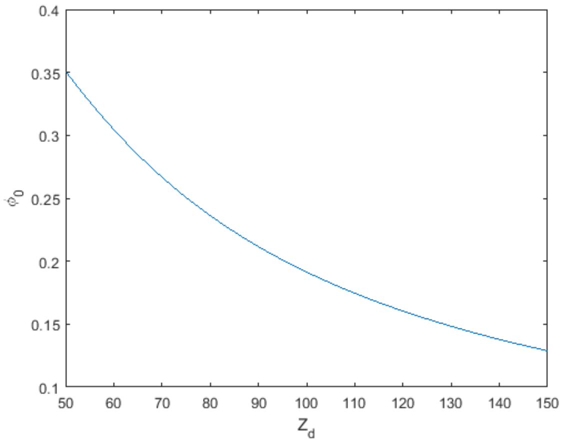
Figure 2.4: ϕ vs η: $k = 0.8, u_0 = 1, \omega = 0.1892, V = 10, M = V - u_0, M_n = 1$



(a) ϕ_0 vs α: $Z = 100, P = 1/9000, Q = 1/10, F = 1.517, H = 3, k = 0.4, u_0 = 0.3, V = 10, M = V - u_0, M_n = 1$



(b) ϕ_0 vs Q :$Z = 100, P = 1/9000, F_i = 1.517, a = 0.01, H = 3, k = 0.4, u_0 = 0.3, V = 10, M = V - u_0, M_n = 1$



(c) ϕ_0 vs Z_d:$P = 1/9000, Q = 1/10, F_i = 1.517, a = 0.01, H = 3, k = 0.4, u_0 = 0.3, V = 10, M = V - u_0, M_n = 1$

Figure 2.5: Variation of wave's amplitude with different quantities

Chapter 3

RESULTS AND DISCUSSIONS

Using the model described above and standard perturbative techniques, a linear dispersion relation was found taking into account quantum effects of electrons and relativistic effects.

Figure 3.1 shows variation of ω with wave number k. ω initially has a higher degree variation with respect to wave number k but shows linear variation at larger values of $k (k > 0.1)$ for fixed values of other parameters. Initially the k^4 term is significant, but its influence disappears as k increases, validating our assumption to simplify the linear dispersion relation earlier.

Figure 3.2 shows variation of omega with ion-dust mass ratio Q. Omega decreases with increase in Q, and the rate of change reduces as Q increases. This is due to dependence of constant A on Q, which is quite large. Figure 3.3 shows variation of omega with dust charge Z_d. There is a sharp rise in the value of omega at smaller values of Z_d but flattens out to a nearly constant value at higher values of Z_d for fixed values of other parameters.

The KdV equation was derived to study nonlinear characteristics of the waves, where the coefficients are found to have been modified by the quantum and relativistic effects.

Figure 3.4 shows the soliton profiles obtained for different values of Q. The amplitude of the soliton decreases with increase in Q.

Figure 3.5(a) shows variation of amplitude with temperature ratio for different values of Q. The amplitude decreases with increase in alpha, and as observed earlier there is an overall reduction in amplitude with increase in Q.

Figure 3.5(b) shows variation of amplitude with Q. The initial rate of change is small but gets steeper as value of Q increases.

Figure 3.5(c) shows variation of amplitude with dust charge Z_d. The amplitude decreases with increase in Z_d for fixed values of other parameters.

Chapter 4

CONCLUSION

The properties and characteristics of dust-ion acoustic waves in weakly relativistic plasma were investigated. The properties of the solitons under consideration depends mainly on the ion-dust mass ratio, the dust charge and temperature ratio. The investigation of these waves may prove helpful in astrophysical studies of interstellar media, astronomical objects such as neutron stars and laser interaction experiments where the additional quantum and relativistic effects may become influential.

Chapter 5

APPENDICES

5.1 Detailed Calculations

The symbols used are defined in Chapter 2.
Also $Q_i = e, Q_d = -Z_d e, Q_e = -e$.

5.1.1 Normalization

We introduce the dummy variables as follows:
$\bar{x} = x/\lambda_D e, \bar{n}_j = n_j/n_{i0}, \bar{u}_j = u_j/C_d,$
$\bar{t} = t/\omega pd, \bar{\phi} = \frac{\phi e}{K_b T_e}$

Equation (2.1) becomes:

$$\frac{\partial(\bar{n}_d n_{i0})}{\partial\bar{t}}\frac{\partial\bar{t}}{\partial t} + \frac{\partial(\bar{n}_d n_{i0}\bar{u}_d C_d)}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x} = 0$$
$$\implies \frac{\partial\bar{n}_d}{\partial\bar{t}} + \frac{\partial(\bar{n}_d\bar{u}_d)}{\partial\bar{x}} = 0$$

OR, on renaming the variables:

$$\frac{\partial n_d}{\partial t} + \frac{\partial(n_d u_d)}{\partial x} = 0 \tag{5.1}$$

Similarly equation (2.3) and (2.5) become:

$$\frac{\partial n_i}{\partial t} + \frac{\partial(n_i u_i)}{\partial x} = 0 \tag{5.2}$$
$$\frac{\partial n_e}{\partial t} + \frac{\partial(n_e u_e)}{\partial x} = 0 \tag{5.3}$$

Equation(2.2) becomes:

$$\frac{\partial(C_d\bar{u}_d)}{\partial\bar{t}}\frac{\partial\bar{t}}{\partial t}+(C_d\bar{u}_d)\frac{\partial(C_d\bar{u}_d)}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x}=\frac{Q_d}{m_d}\frac{\partial(K_bT_e\bar{\phi}/e)}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x}$$
$$\implies C_d^2\frac{\partial\bar{u}_d}{\partial\bar{t}}+C_d^2u_d\frac{\partial\bar{u}_d}{\partial\bar{x}}=\frac{Q_d}{m_d}\frac{K_bT_e}{e}\frac{\partial\bar{\phi}}{\partial\bar{x}}$$

OR, on renaming the variables:

$$\frac{\partial u_d}{\partial t}+u_d\frac{\partial u_d}{\partial x}=-Z_d\frac{\partial\phi}{\partial x} \tag{5.4}$$

Equation (2.4) becomes:

$$\frac{\partial(\bar{u}_dC_d)}{\partial\bar{t}}\frac{\partial\bar{t}}{\partial t}+C_d\bar{u}_i\frac{\partial(\bar{u}_iC_d)}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x}$$
$$=\frac{-1}{m_i}[Q_i\frac{\partial(K_bT_e\bar{\phi}/e)}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x}+\frac{1}{20\bar{n}_in_{i0}}(\frac{3}{\pi})^{\frac{2}{3}}\frac{h^2}{m_e}\frac{\partial(n_{i0}\bar{n}_i)^{\frac{5}{3}}}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x}]$$

On simplification and renaming, the above equation reduces to:

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x}=-\frac{1}{Q}(\frac{\partial\phi}{\partial x}+\frac{5}{3}\frac{F_i\alpha}{n_i^{\frac{2}{3}}}\frac{\partial n_i}{\partial x} \tag{5.5}$$

where $F_i=\frac{1}{20}(\frac{3}{\pi})^{2/3}\frac{h^2}{m_e}$

Similarly, equation (2.6) reduces to

$$\frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} = -\frac{1}{P}(\frac{\partial \phi}{\partial x} + \frac{5}{3}\frac{\alpha F_e}{n_e^{\frac{1}{3}}}\frac{\partial n_e}{\partial x}) + \frac{1}{P^2}\frac{H^2}{2}\frac{\partial}{\partial x}(\frac{1}{\sqrt{n_e}}\frac{\partial^2\sqrt{n_e}}{\partial x^2}) \tag{5.6}$$

where the last term on right hand side was reduced as follows:

$$\frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial x}(\frac{1}{\sqrt{n_e}}\frac{\partial^2\sqrt{n_e}}{\partial x^2}) = \frac{\hbar^2}{2m_e^2}(\frac{\partial \bar{x}}{\partial x})^3\frac{\partial}{\partial \bar{x}}(\frac{1}{\sqrt{\bar{n}_e}}\frac{\partial^2\sqrt{\bar{n}_e}}{\partial \bar{x}^2})$$

Lastly, equation (2.7) becomes:

$$\frac{\partial^2(K_bT_e\phi/e)}{\partial x^2}(\frac{\partial \bar{x}}{\partial x})^2 = Q_e\bar{n}_en_{i0} + Q_d\bar{n}_dn_{i0} + Q_i\bar{n}_in_{i0}$$

OR, on simplifying and renaming:

$$\frac{\partial^2\phi}{\partial x^2} = (n_e - n_i + Z_dn_d) \tag{5.7}$$

5.1.2 Dispersion Characteristics

The eigenvalues are: $\frac{\partial}{\partial x} = ik$, $\frac{\partial}{\partial t} = -\iota\omega$

From equations (2.10),(2.12), and (2.14), we get:

$$n_d^{(1)} = \frac{-ku_d^{(1)}}{ku_{d0} - \omega} \tag{5.8}$$

$$n_i^{(1)} = \frac{-ku_i^{(1)}}{ku_{i0} - \omega} \tag{5.9}$$

$$n_e^{(1)} = \frac{-ku_e^{(1)}}{ku_{e0} - \omega} \tag{5.10}$$

From equation(2.11), (2.13), and (2.15) we get:

$$u_d^{(1)}(ku_{d0} - \omega) = -Z_dk\phi^{(1)} \tag{5.11}$$

$$(ku_{i0} - \omega)u_i^{(1)} = \frac{1}{Q}[k\phi^{(1)} + \frac{5}{3}\alpha F_in_i^{(1)}] \tag{5.12}$$

$$(ku_{e0} - \omega)u_e^{(1)} = \frac{1}{P}[-k\phi^{(1)} - \frac{5}{3}F_ikn_e^{(1)}] - \frac{H^2k^3}{2P^2}\frac{n_e^{(1)}}{2} \tag{5.13}$$

From equation (2.16):

$$-k^2\phi^{(1)} = n_e^{(1)} + Z_d n_d^{(1)} - n_i^{(1)} \tag{5.14}$$

From equation (6.1) and equation (6.4):

$$n_d^{(1)}(ku_{d0} - \omega)^2 = Z_d k\phi^{(1)} \tag{5.15}$$

From equation (6.2) and equation (6.5), we get:

$$[Q(ku_{i0} - \omega)^2 - \frac{5}{3}\alpha F_i k^2]n_i^{(1)} = -k^2\phi^{(1)} \tag{5.16}$$

And lastly, from equation (6.3) and equation (6.6), we get:

$$[P(ku_{e0} - \omega^2) - \frac{5}{3}\alpha F_i k^2 - \frac{H^2k^4}{4P}]n_e^{(1)} = k^2\phi^{(1)} \tag{5.17}$$

Putting equations (6.8)-(6.10) in equation (6.7):

$$\begin{aligned} -k^2\phi^{(1)} = \frac{k^2\phi^{(1)}}{P(ku_{e0} - \omega)^2 - \frac{5}{3}\alpha F_e k^2 - \frac{H^2k^4}{4P}} - \frac{k^2\phi^{(1)}}{Q((ku_{i0} - \omega)^2 - \frac{5}{3}\alpha F_i k^2} \\ + \frac{Z_d^2 k^2\phi^{(1)}}{(ku_{d0} - \omega)^2} \end{aligned} \tag{5.18}$$

OR , on putting $u_{j_0} = 0$, we get:

$$-1 = \frac{1}{P\omega^2 - \frac{5}{3}\alpha F_e k^2 - \frac{H^2k^4}{4P}} - \frac{1}{Q\omega^2 - \frac{5}{3}\alpha F_i k^2} + \frac{Z_d^2}{\omega^2} \tag{5.19}$$

On simplifying the above expression we get:

$$\begin{aligned} 0 = PQ\omega^6 + [-\frac{5}{3}P\alpha F_i k^2 - \frac{5}{3}QF_e k^2 - P + Q - Z_d^2 PQ]\omega^4 + [\frac{25}{9}\alpha F_i F_e k^4 \\ + \frac{5}{3}F_e k^2 + \frac{H^2k^4}{4} - \frac{5}{3}\alpha F_i k^2 - \frac{QH^2k^4}{4} + \frac{5}{3}\frac{H^2k^4\alpha F_i k^2}{4} + \frac{5}{3}Z_d^2\alpha F_i k^2 \\ + \frac{5}{3}QZ_d^2 F_e k^2]\omega^4 - \frac{25}{9}Z_d^2\alpha F_i F_e k^4 \end{aligned}$$

Ignoring k^4 terms, and, hence, in the long wavelength limit:

$$\begin{aligned} 0 =& PQ\omega^4 + (Q - \frac{5}{3}P\alpha F_i k^2 - \frac{5}{3}QF_e k^2 - P - Z_d^2 PQ)\omega^2 \\ &+ (\frac{5}{3}F_e k^2 - \frac{5}{3}\alpha F_i k^2 + \frac{5}{3}Z_d^2\alpha PF_i k^2 + \frac{5}{3}Z_d^2 QF_e k^2) \end{aligned} \tag{5.20}$$

5.1.3 KdV Equation

The stretching of coordinates used is : $\xi = \varepsilon^{1/2}(x - V_0 t)$ and $\tau = \varepsilon^{3/2} t$, which gives:

$\frac{\partial}{\partial x} = \epsilon^{1/2}\frac{\partial}{\partial xi}, \frac{\partial}{\partial x} = \epsilon^{3/2}\frac{\partial}{\partial \tau} - V_0\epsilon^{1/2}\frac{\partial}{\partial \xi}$

Using the Pertubation Expansion given in Chapter 3, we transform the earlier derived normalized equations as follows:

$$
\begin{aligned}
&[\epsilon^{3/2}\frac{\partial}{\partial \tau} - V_0\epsilon^{1/2}\frac{\partial}{\partial \xi}][1 + \epsilon n_j^{(1)} + \epsilon^2 n_j^{(2)}] \\
&+ \epsilon^{1/2}\frac{\partial}{\partial \xi}[(1 + \epsilon n_j^{(1)} + \epsilon^2 n_j^{(2)} + \cdots)(u_{0j} + \epsilon u_j^{(1)} + \epsilon^2 u_j^{(2)} + \cdots)] = 0
\end{aligned}
\tag{5.21}
$$

where j represents the j^{th} species.

Now, on collecting the $\epsilon^{3/2}$ and $\epsilon^{5/2}$ terms, we get:

$\epsilon^{3/2}$ terms:

$$
-V_0\frac{\partial n_j^{(1)}}{\partial \xi} + \frac{\partial (u_j^{(1)} + u_{0j}^{(1)} n_j^{(1)})}{\partial \xi} = 0
$$

OR

$$
(u_{0j}^{(1)} - V_0)\frac{\partial n_j}{\partial \xi} + \frac{\partial u_j^{(1)}}{\partial \xi} = 0 \tag{5.22}
$$

Similarly, $\epsilon^{5/2}$ terms:

$$
\frac{\partial n_j^{(1)}}{\partial \tau} + (u_{0j} - V_0)\frac{\partial n_j^{(2)}}{\partial \xi} + \frac{\partial (n_j^{(1)} u_j^{(1)} + u_j^{(2)})}{\partial \xi} = 0 \tag{5.23}
$$

Following the above procedure of applying the perturbation expansion, to the normalized equations, and collecting the similar terms together, we get the following set of equations:

$$
(u_{0d} - V_0)\frac{\partial u_d^{(1)}}{\partial \xi} = -Z_d\frac{\partial \phi^{(1)}}{\partial \xi} \tag{5.24}
$$

$$
\frac{\partial u_d^{(1)}}{\partial \tau} + (u_{0d} - V_0)\frac{\partial u_d^{(2)}}{\partial \xi} + u_d^{(1)}\frac{\partial u_d^{(1)}}{\partial \xi} = -Z_d\frac{\partial \phi^{(2)}}{\partial \xi} \tag{5.25}
$$

$$(u_{0i}-V_0)\frac{\partial u_i^{(1)}}{\partial \xi}=\frac{-1}{Q}[\frac{\partial \phi^{(1)}}{\partial \xi}+\frac{5}{3}\alpha F_i\frac{\partial n_i^{(1)}}{\partial \xi}] \tag{5.26}$$

$$\frac{\partial u_i^{(1)}}{\partial \tau}+(u_{0i}-V_0)\frac{\partial u_i^{(2)}}{\partial \xi}+u_i^{(1)}\frac{\partial u_i^{(1)}}{\partial \xi}=\frac{-1}{Q}[\frac{\partial \phi^{(2)}}{\partial \xi}+\frac{5}{3}\alpha F_i(\frac{\partial n_i^{(2)}}{\partial \xi}-\frac{n_i^{(1)}}{3}\frac{\partial n_i^{(1)}}{\partial \xi})] \tag{5.27}$$

$$(u_{0e}-V_0)\frac{\partial u_e^{(1)}}{\partial \xi}=\frac{-1}{P}[\frac{\partial \phi^{(1)}}{\partial \xi}+\frac{5}{3}\alpha F_e\frac{\partial n_e^{(1)}}{\partial \xi}] \tag{5.28}$$

$$\frac{\partial u_e^{(1)}}{\partial \tau}+(u_{0e}-V_0)\frac{\partial u_e^{(2)}}{\partial \xi}+u_e^{(1)}\frac{\partial u_e^{(1)}}{\partial \xi}=\frac{-1}{P}[\frac{\partial \phi^{(2)}}{\partial \xi}+\frac{5}{3}\alpha F_e(\frac{\partial n_e^{(2)}}{\partial \xi}-\frac{n_e^{(1)}}{3}\frac{\partial n_e^{(1)}}{\partial \xi})] +\frac{H^2}{2P^2}\frac{\partial^3 n_e^{(1)}}{\partial \xi^3} \tag{5.29}$$

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2}=n_e^{(2)}+Z_d n_d^{(2)}-n_i^{(2)} \tag{5.30}$$

$$\frac{\partial^3 \phi^{(1)}}{\partial \xi^3}=\frac{\partial n_e^{(2)}}{\partial \xi}+Z_d\frac{\partial n_d^{(2)}}{\partial \xi}-\frac{\partial n_i^{(2)}}{\partial \xi} \tag{5.31}$$

From equations (6.22) and(6.24) for j = d, we get:

$$u_d^{(1)}=(V_0-u_{0d})n_d^{(1)}$$

and

$$(V_0-u_{0d})u_d^{(1)}=Z_d\phi^{(1)}$$

Also, let

$$M_j=V_0-u_{0j}$$

Therefore

$$n_d^{(1)}=\frac{Z_d}{M_d^2}\phi^{(1)} \tag{5.32}$$

$$u_d^{(1)}=\frac{Z_d}{M_d}\phi^{(1)} \tag{5.33}$$

Following the same steps, and writing other field quantities in terms of $\phi^{(1)}$, we get the following set of equations:

$$u_i^{(1)} = \frac{M_i}{A}\phi^{(1)} \tag{5.34}$$

$$n_i^{(1)} = \frac{1}{A}\phi^{(1)} \tag{5.35}$$

$$u_e^{(1)} = \frac{M_e}{B}\phi^{(1)} \tag{5.36}$$

$$n_e^{(1)} = \frac{1}{B}\phi^{(1)} \tag{5.37}$$

where

$$A = QM_i^2 - \frac{5}{3}\alpha F_i$$

$$B = PM_e^2 - \frac{5}{3}F_e$$

Now, on substituting equations (6.32)-(6.37) in equation (6.23) for j = i,e,d respectively, we get:

$$\frac{Z_d}{M_d^2}[\frac{\partial\phi^{(1)}}{\partial\tau} + 2\frac{Z_d}{M_d}\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}] = M_d\frac{\partial n_d^{(2)}}{\partial\xi} - \frac{\partial u_d^{(2)}}{\partial\xi} \tag{5.38}$$

$$\frac{1}{A}[\frac{\partial\phi^{(1)}}{\partial\tau} + 2\frac{M_i}{A}\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}] = M_i\frac{\partial n_i^{(2)}}{\partial\xi} - \frac{\partial u_i^{(2)}}{\partial\xi} \tag{5.39}$$

$$\frac{1}{B}[\frac{\partial\phi^{(1)}}{\partial\tau} + 2\frac{M_e}{B}\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}] = M_e\frac{\partial n_e^{(2)}}{\partial\xi} - \frac{\partial u_e^{(2)}}{\partial\xi} \tag{5.40}$$

Similarly, on substituting equations (6.32)-(6.37) in equations (6.25), (6.27), and (6.29), we get:

$$\frac{Z_d}{M_d}\frac{\partial\phi^{(1)}}{\partial\tau} + \frac{Z_d^2}{M_d^2}\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi} = -Z_d\frac{\partial\phi^{(2)}}{\partial\xi} + M_d\frac{\partial u_d^{(2)}}{\partial\xi} \tag{5.41}$$

$$-Q[\frac{M_i}{A}\frac{\partial\phi^{(1)}}{\partial\tau} + \frac{M_i^2}{A^2}\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}] + \frac{5}{9}\alpha F_i\frac{\phi^{(1)}}{A^2}\frac{\partial\phi^{(1)}}{\partial\xi} = \frac{\partial\phi^{(2)}}{\partial\xi} + \frac{5}{3}\alpha F_i\frac{\partial n_i^{(2)}}{\partial\xi} - QM_i\frac{\partial u_i^{(2)}}{\partial\xi} \tag{5.42}$$

$$-P[\frac{M_e}{B}\frac{\partial\phi^{(1)}}{\partial\tau} + \frac{M_e^2}{B^2}\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}] + \frac{H^2}{2PB}\frac{\partial^3\phi^{(1)}}{\partial\xi^3} + \frac{5}{9}\alpha F_e\frac{\phi^{(1)}}{B^2}\frac{\partial\phi^{(1)}}{\partial\xi} = \frac{\partial\phi^{(2)}}{\partial\xi} + \frac{5}{3}F_e\frac{\partial n_e^{(2)}}{\partial\xi} - PM_e\frac{\partial u_e^{(2)}}{\partial\xi} \tag{5.43}$$

Now, we replace the Left Hand Side of equations (6.38)-(6.43) as arbritrary functions of $\phi^{(1)}$, i.e. $G_1(\phi^{(1)}), G_2(\phi^{(1)}), G_3(\phi^{(1)}), G_4(\phi^{(1)})$, $G_5(\phi^{(1)}), G_6(\phi^{(1)})$ respectively; where $G_i(\phi^{(1)})$ is a partial differential equation in $\phi^{(1)}$. Therefore, we get:

$$G_1(\phi^{(1)}) = M_d\frac{\partial n_d^{(2)}}{\partial\xi} - \frac{\partial u_d^{(2)}}{\partial\xi} \tag{5.44}$$

$$G_2(\phi^{(1)}) = M_i\frac{\partial n_i^{(2)}}{\partial\xi} - \frac{\partial u_i^{(2)}}{\partial\xi} \tag{5.45}$$

$$G_3(\phi^{(1)}) = M_e\frac{\partial n_e^{(2)}}{\partial\xi} - \frac{\partial u_e^{(2)}}{\partial\xi} \tag{5.46}$$

$$G_4(\phi^{(1)}) = -Z_d\frac{\partial\phi^{(2)}}{\partial\xi} + M_d\frac{\partial u_d^{(2)}}{\partial\xi} \tag{5.47}$$

$$G_5(\phi^{(1)}) = \frac{\partial\phi^{(2)}}{\partial\xi} + \frac{5}{3}\alpha F_i\frac{\partial n_i^{(2)}}{\partial\xi} - QM_i\frac{\partial u_i^{(2)}}{\partial\xi} \tag{5.48}$$

$$G_6(\phi^{(1)}) = \frac{\partial\phi^{(2)}}{\partial\xi} + \frac{5}{3}\alpha F_e\frac{\partial n_e^{(2)}}{\partial\xi} - PM_e\frac{\partial u_e^{(2)}}{\partial\xi} \tag{5.49}$$

From equations (6.44) and (6.47):

$$\frac{\partial n_d^{(2)}}{\partial\xi} = \frac{1}{M_d^2}[G_4 + Z_d\frac{\partial\phi^{(2)}}{\partial\xi} + M_dG_1] \tag{5.50}$$

From equations (6.45) and (6.48):

$$\frac{\partial n_i^{(2)}}{\partial\xi} = \frac{-1}{A}[G_5 - \frac{\partial\phi^{(2)}}{\partial\xi} - QM_iG_2] \tag{5.51}$$

From equations (6.46) and (6.49):

$$\frac{\partial n_e^{(2)}}{\partial\xi} = \frac{-1}{B}[G_6 - \frac{\partial\phi^{(2)}}{\partial\xi} - PM_eG_3] \tag{5.52}$$

$$\tag{5.53}$$

On putting equations (6.50)-(6.52) in equation (6.31); and letting $\phi^{(1)} \rightarrow \phi$, we get:

$$APM_eG_3 + A\frac{\partial \phi^{(2)}}{\partial \xi} - AG_6 + \frac{Z_d}{M_d^2}ABG_4 + \frac{Z_d^2}{M_d^2}AB\frac{\partial \phi^{(2)}}{\partial \xi} + \frac{Z_d}{M_d}ABG_1 + BG_5 \\ -B\frac{\partial \phi^{(2)}}{\partial \xi} - BQM_iG2 = AB\frac{\partial^3 \phi}{\partial \xi}$$

On applying boundary condition: $\phi^{(2)} \rightarrow 0$, and simplifying we get the final KdV equation:

$$\frac{\partial \phi}{\partial \tau} + N\phi\frac{\partial \phi}{\partial \xi} + D\frac{\partial^3 \phi}{\partial \xi^3} \tag{5.54}$$

where

$$N = \frac{\frac{3Z_d^3}{M_d^4} + \frac{3PM_e^2}{B^3} - \frac{3QM_i^2}{A^3} + \frac{5\alpha F_i}{9A^3} - \frac{5F_i}{9B^3}}{\frac{2Z_d^2}{M_d^3} + \frac{2PM_e}{B^2} - \frac{2QM_i}{A^2}} \tag{5.55}$$

$$D = \frac{1 + \frac{H^{@}}{2PB^2}}{\frac{2Z_d^2}{M_d^3} + \frac{2PM_e}{B^2} - \frac{2QM_i}{A^2}}$$

5.2 MATLAB Codes

5.2.1 Variation of ω vs Z_d, and ω vs Q

```
format compact
P = 1/(1840*Z);
Q = 1/Z;
F = 1.517;
a = 0.01;
H = 2;
k = 0.5;

omega = [];
Z_l = linspace(1, 30, 30);

for Z=1:30
    c1 = P*Q;
    c2 = -F*P*a*(k^2) - F*Q*(k^2) - P + Q - (Z^2)*P*Q;
    c3 = a*F*F*(k^4) + F*(k^2) + ((H^2)*(k^4))/4 - F*a
        *(k^2) ...
        - (Q*(H^2)*(k^4))/4 + (F*(H^2)&(k^6)*a)/4 ...
        + (Z^2)*F*a*P*(k^2) + (Z^2)*Q*F*(k^2);
```

```
    c4 = F*F*a*(Z^2)*(k^4);

    syms x
    eqn = c1*x^3 + c2*x^2 + c3*x - c4;
    sol = vpa(solve(eqn, x));
    omega(Z) = sqrt(sol(1));
end

figure(1)
plot(Z_l, omega)
xlabel("Z_d")
ylabel("\omega")

Q_new = linspace(0.1, 0.3, 100);
omega_q = [];

for j=1:100
     c1 = P*Q_new(j);
    c2 = -F*P*a*(k^2) - F*Q_new(j)*(k^2) - P + Q_new(j)
        - (Z^2)*P*Q_new(j);
    c3 = a*F*F*(k^4) + F*(k^2) + ((H^2)*(k^4))/4 - F*a
       *(k^2) ...
        - (Q_new(j)*(H^2)*(k^4))/4 + (F*(H^2)&(k^6)*a)
          /4 ...
        + (Z^2)*F*a*P*(k^2) + (Z^2)*Q_new(j)*F*(k^2);
    c4 = F*F*a*(Z^2)*(k^4);

    syms x
    eqn = c1*x^3 + c2*x^2 + c3*x - c4;
    sol = vpa(solve(eqn, x));
    omega_q(j) = sqrt(sol(1));
end

figure(2)
plot(Q_new, omega_q)
xlabel('Q')
ylabel('\omega')
```

5.2.2 Variation of ω vs k

```
format compact
Z=5;
P = 1/(1840*Z);
Q = 1/Z;
F = 1.517;
```

```
a = 0.01;
H = 2;

omega = [];
k = linspace(0.01, 1, 100);

for i=1:100
    c1 = P*Q;
    c2 = -F*P*a*(k(i)^2) - F*Q*(k(i)^2) - P + Q - (Z^2)
       *P*Q;
    c3 = a*F*F*(k(i)^4) + F*(k(i)^2) + ((H^2)*(k(i)^4))
       /4 - F*a*(k(i)^2) ...
        - (Q*(H^2)*(k(i)^4))/4 + (F*(H^2)&(k(i)^6)*a)/4
            ...
        + (Z^2)*F*a*P*(k(i)^2) + (Z^2)*Q*F*(k(i)^2);
    c4 = F*F*a*(Z^2)*(k(i)^4);

    syms x
    eqn = c1*x^3 + c2*x^2 + c3*x - c4;
    sol = vpa(solve(eqn, x));
    omega(i) = sqrt(sol(1));
end

figure(1)
plot(k, omega, 'LineWidth', 1.5)
xlabel('k')
ylabel('\omega')
```

5.2.3 Variation of ϕ_0 vs Z_d, ϕ_0 vs Q, and ϕ_0 vs α

```
format compact
opengl software
Z = 100;
P = 1/9000;
Q = 1/10;
F = 1.517;
a = 0.01;
H = 3;
k = 0.4;
u_0 = 0.3;
omega = 0.1892;
V = 10;
M = V - u_0;
M_n = 1;
```

```
A = Q*M^2 - F*a;
B = P*M^2 - F;
deno = ((2*Z^2)/M^3) + ((2*P*M)/B^2) - ((2*Q*M)/A^2);
N_num = ((3*Z^3)/M^4) + ((3*P*M^2)/B^3) - ((3*Q*M^2)/A
    ^3) + ((F*a)/3*A^3) - (F/(3*B^3));
D_num = 1 + (H^2/(2*P*B^2));
N = N_num/deno;
D = D_num/deno;

eta = linspace(-500, 500, 200);
phi_0 = 3*M_n/N;
delta = sqrt(4*D/M_n);
phi = [];

for j=1:200
    phi(j) = phi_0*(sech(eta(j)/delta))^2;
end

figure(1)
plot(eta, phi)
xlabel('\eta')
ylabel('\phi')

a_new = linspace(0.05, 0.3, 100);
Q_new = linspace(0.12, 0.2, 4);
phi_0a = [];
for k=1:4
    for j=1:100
        A_new = Q_new(k)*M^2 - F*a_new(j);
        B_new = P*M^2 - F;
        deno_new = ((2*Z^2)/M^3) + ((2*P*M)/B_new^2) -
            ((2*Q_new(k)*M)/A_new^2);
        N_new_sub = ((3*Z^3)/M^4) + ((3*P*M^2)/B_new^3)
            - ((3*Q_new(k)*M^2)/A_new^3) + ((F*a_new(j)
            )/3*A_new^3) - (F/(3*B_new^3));
        N_new = N_new_sub/deno_new;
        phi_0a(j) = 3*M_n/N_new;
    end

    figure(2)
    plot(a_new, phi_0a)
    lgd1 = legend('Q=0.12','Q=0.146','Q=0.173','Q=0.20'
        );
    lgd1.FontSize = 11;
```

```
    xlabel('\alpha')
    ylabel('\phi_0')
    hold on
end

Z_new = linspace(50,150,100);
phi_0z = [];
for j=1:100
        A_n = Q*M^2 - F*a;
        B_n = P*M^2 - F;
        deno_new = ((2*Z_new(j)^2)/M^3) + ((2*P*M)/B_n
           ^2) - ((2*Q*M)/A_n^2);
        N_new_sub = ((3*Z_new(j)^3)/M^4) + ((3*P*M^2)/
           B_n^3) - ((3*Q*M^2)/A_n^3) + ((F*a)/3*A_n^3)
            - (F/(3*B_n^3));
        N_new = N_new_sub/deno_new;
        phi_0z(j) = 3*M_n/N_new;
end

figure(3)
plot(Z_new, phi_0z)
xlabel('Z_d')
ylabel('\phi_0')

Q_x = linspace(0.01, 0.3, 100);
phi_0q = [];
for j=1:100
    A_new = Q_x(j)*M^2 - F*a;
    B_new = P*M^2 - F;
    deno_new = ((2*Z^2)/M^3) + ((2*P*M)/B_new^2) - ((2*
       Q_x(j)*M)/A_new^2);
    N_new_sub = ((3*Z^3)/M^4) + ((3*P*M^2)/B_new^3) -
       ((3*Q_x(j)*M^2)/A_new^3) + ((F*a)/3*A_new^3) - (
       F/(3*B_new^3));
    N_new = N_new_sub/deno_new;
    phi_0q(j) = 3*M_n/N_new;
end

figure(4)
plot(Q_x, phi_0q)
xlabel('Q')
ylabel('\phi_0')
```

5.2.4 Variation of ϕ vs η, i.e Soliton Profile

```
format compact
opengl software
Z = 100;
P = 1/5000;
F = 1.517;
a = 0.01;
H = 3;
k = 0.8;
u_0 = 1;
omega = 0.1892;
V = 10;
M = V - u_0;
M_n = 1;
Q = linspace(0.1, 0.9, 5);

for k=1:5
    A = Q(k)*M^2 - F*a;
    B = P*M^2 - F;
    deno = ((2*Z^2)/M^3) + ((2*P*M)/B^2) - ((2*Q(k)*M)/
        A^2);
    N_num = ((3*Z^3)/M^4) + ((3*P*M^2)/B^3) - ((3*Q(k)*
        M^2)/A^3) + ((F*a)/3*A^3) - (F/(3*B^3));
    D_num = 1 + (H^2/(2*P*B^2));
    N = N_num/deno;
    D = D_num/deno;

    eta = linspace(-500, 500, 500);
    phi_0 = 3*M_n/N;
    delta = sqrt(4*D/M_n);
    phi = [];

    for j=1:500
        phi(j) = phi_0*(sech(eta(j)/delta))^2;
    end

    figure(1)
    plot(eta, phi)
    hold on
    xlabel('\eta')
    ylabel('\phi')
    lgd1 = legend('Q=0.1','Q=0.3','Q=0.5','Q=0.7','Q
        =0.9');
    lgd1.FontSize = 11;
end
```