# PDE I Final HW

Victor Churchill

I chose the second problem.

**II**

1. To obtain an equation with $||u||_{L^2}^2$, we multiply (10) by $u$ and strategically simplify to get

$$\Big(\frac{1}{2}u^2\Big)_t + |\nabla u|^2 - \nabla \cdot (u\nabla u) + uf(u) = 0 \tag{1}$$

Integrating over $\Omega$ and using the divergence theorem and our boundary condition yields:

$$\frac{1}{2}\frac{d}{dt}||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + \int_\Omega uf(u)dx = 0 \tag{2}$$

This is our equation with $||u||_{L^2}^2$.

Now we must show $E(u)$ is decreasing. To show $E(u)$ is decreasing, we need to show $\partial_t\big[E(u)\big] \leq 0$. Start with:

$$E(u) = \frac{1}{2}\int_\Omega \nabla u \nabla u dx + \int_\Omega F(u)dx \tag{3}$$

$$= -\frac{1}{2}\int_\Omega u\Delta u dx + \int_\Omega F(u)dx \tag{4}$$

Taking the derivative of that equation with respect to time, we have:

$$\frac{d}{dt}E(u) = -\frac{1}{2}\int_\Omega [u_t\Delta u + u\Delta u_t]dx + \int_\Omega u_t F'(u)dx \tag{5}$$

$$= \int_\Omega \big[ - u_t\Delta u + u_t f(u)\big]dx \tag{6}$$

Now we need to use the given info. Multiplying (10) by $u_t$ and integrating over $\Omega$, we have:

$$\int_\Omega \left[u_t^2 - u_t\Delta u + u_t f(u)\right] dx = 0 \tag{7}$$

This means that:

$$\frac{d}{dt}E(u) = -\int_\Omega u_t^2 dx \implies \frac{d}{dt}E(u) \leq 0 \tag{8}$$

as desired. Therefore $E(u)$ is decreasing.

2. We assume $g \in L^2(\Omega)$. Since $|f(u)| \leq C(1+|u|)$, then $|uf(u)|$ is less than or equal to a polynomial of degree 2 with positive leading coefficient. So there exist constants $\alpha > 0$, $\beta \geq 0$ such that

$$\alpha||u||_{L^2}^2 \leq \int_\Omega uf(u)dx + \beta \tag{9}$$

Therefore we have

$$\frac{1}{2}\sup_{[0,T]}||u||_{L^2}^2 + \int_0^T ||\nabla u||_{L^2}^2 dt + \alpha\int_0^T ||u||_{L^2}^2 dt \leq \beta T + \frac{1}{2}||g||_{L^2}^2 \tag{10}$$

Note if $||u||_{L^2} \leq \infty$ then $||f(u)||_{L^2} \leq \infty$ since

$$\int_\Omega |f(u)|^2 dx \leq \alpha\int_\Omega |u|^2 dx + \beta \leq \alpha||u||_{L^2} + \beta \tag{11}$$

So in forming a weak solution we use test functions: $v \in H_0^1(\Omega) \cap L^2(\Omega)$ such that $(\nabla u, \nabla v)_{L^2}$ and $(f(u), v)_{L^2}$ are well-defined.

The Galerkin approximations $\{u_m\}$ take values in a finite dimensional subspace $E_m \subset H_0^1(\Omega) \cap L^2(\Omega)$ and satisfy the following with $P_m$ the projection of $E_m$ onto $L^2(\Omega)$

$$(u_m)_t = \Delta u_m + P_m f(u_m) \tag{12}$$

Similar to the solution $u$, these approximations satisfy

$$\frac{1}{2}\sup_{[0,T]}||u_m||_{L^2}^2 + \int_0^T ||\nabla u_m||_{L^2}^2 dt + \alpha \int_0^T ||u_m||_{L^2}^2 dt \leq \beta T + \frac{1}{2}||g||_{L^2}^2 \tag{13}$$

The Galerkin ODEs have a unique local solution since the nonlinear terms are Lipschitz continuous functions of $u_m$. Furthermore, the local solutions are bounded, and so they exist globally for all times $t \geq 0$.

Since our estimates hold uniformly for $m$, we can pick a subsequence that weakly converges $u_m \to u$ to a limiting function

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1) \cap L^2(0,T;L^2) \tag{14}$$

And furthermore we can say that $u_t \in L^2(0,T;H^{-1}) + L^2(0,T;L^2)$.

Now we need to show that this $u$ is actually a solution of the original PDE. To do this, we must show $f(u_m) \to f(u)$. We find some subsequence of $u_m$ such that $u_m \to u$ strongly in $L^2(0,T;L^2)$, which is the same as a strong $L^2$ convergence for $\Omega \times (0,T)$. By the Riesz-Fisher theorem, we can then extract a subsequence such that $u_m(x,t) \to u(x,t)$ pointwise on $\Omega \times (0,T)$. Then we use the dominated convergence theorem and the uniform bounds of $u_m$ to find that for every test function $v \in H_0^1(\Omega) \cap L^2(\Omega)$

$$(f(u_m(t)), v)_{L^2} \to (f(u(t)), v)_{L^2} \tag{15}$$

pointwise on $[0,T]$.

So here we've shown that the PDE has a global unique weak solution with the desired properties.

3. (a) Now we have $g \in H_0^1(\Omega)$ and $uf(u)$ is a monomial of degree $p+1$ with $1 < p < 5$ and leading coefficient $\lambda$.

   There exist constants $\alpha > 0$, $\beta \geq 0$ such that

$$\alpha||u||_{L^{p+1}}^2 \leq \int_\Omega uf(u)dx + \beta \tag{16}$$

   Therefore we have

$$\frac{1}{2}\sup_{[0,T^*)}||u||_{L^2}^2+\int_0^{T^*}||\nabla u||_{L^2}^2dt+\alpha\int_0^{T^*}||u||_{L^{p+1}}^2dt\leq\beta T^*+\frac{1}{2}||g||_{H_0^1}^2 \tag{17}$$

Note if $||u||_{L^{p+1}}\leq\infty$ then $||f(u)||_{L^{\frac{p+1}{p}}}\leq\infty$ since

$$\int_\Omega|f(u)|^{\frac{p+1}{p}}dx\leq\alpha\int_\Omega|u|^{p+1}dx+\beta\leq\alpha||u||_{L^{p+1}}+\beta \tag{18}$$

Note that we use $\frac{p+1}{p}$ in this case since it is the Holder conjugate of $p+1$. So in forming a weak solution we use test functions: $v\in H_0^1(\Omega)\cap L^{p+1}(\Omega)$ such that $(\nabla u,\nabla v)_{L^2}$ and $(f(u),v)_{L^2}$ are well-defined.

The Galerkin approximations $\{u_m\}$ take values in a finite dimensional subspace $E_m\subset H_0^1(\Omega)\cap L^{p+1}(\Omega)$ and satisfy the following with $P_m$ the projection of $E_m$ onto $L^2(\Omega)$

$$(u_m)_t=\Delta u_m+P_mf(u_m) \tag{19}$$

Similar to the solution $u$, these approximations satisfy

$$\frac{1}{2}\sup_{[0,T^*)}||u_m||_{L^2}^2+\int_0^{T^*}||\nabla u_m||_{L^2}^2dt+\alpha\int_0^{T^*}||u_m||_{L^{p+1}}^2dt\leq\beta T^*+\frac{1}{2}||g||_{H_0^1}^2 \tag{20}$$

The Galerkin ODEs have a unique local solution since the nonlinear terms are Lipschitz continuous functions of $u_m$. Furthermore, the local solutions are bounded, and so they exist globally for all times $t\geq0$.

Since our estimates hold uniformly for $m$, we can pick a subsequence that weakly converges $u_m\to u$ to a limiting function

$$u\in L^\infty([0,T^*);H_0^1\cap L^{p+1}(\Omega))\cap L^2([0,T^*);H_0^1)\cap L^{p+1}([0,T^*)\times\Omega) \tag{21}$$

Now we need to show that this $u$ is actually a solution of the original PDE. To do this, we must show $f(u_m)\to f(u)$. We find some subsequence of $u_m$ such that $u_m\to u$ strongly in $L^{p+1}([0,T^*)\times\Omega)$. By the Riesz-Fisher theorem, we can then extract a subsequence such that $u_m(x,t)\to u(x,t)$ pointwise on $[0,T^*)\times\Omega$. Then we use the dominated convergence theorem and the uniform bounds of $u_m$ to find that for every test function $v \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$

$$(f(u_m(t)), v)_{L^2} \to (f(u(t)), v)_{L^2} \tag{22}$$

pointwise on $[0, T^*)$.

So here we've shown that the PDE has a global unique weak solution with the desired properties. In particular, $T^* > 0$ exists.

(b) So part $(a)$ shows that there exists a unique solution defined on some maximal time interval $[0, T^*)$, $u \in L^\infty([0, T^*); H_0^1 \cap L^{p+1}(\Omega))$ for all $T < T^*$. For $\lambda \geq 0$,this means there is a potential blowup: either $T^* = +\infty$ or $T^* < \infty$ and $||u||_{L^\infty} \to \infty$ as $t \to T^*$. However $||u||_{L^\infty}$ is bounded, so $T^* = +\infty$.

(c) i. First, $-4E(g) = -2||\nabla g||^2_{L^2(\Omega)} - 4\int_\Omega F(g)dx > 0$. Also $y^{\frac{p+1}{2}} = ||u||^{p+1}_{L^2(\Omega)}$. Lastly, $\dot{y} = 2u_t||u||_{L^2(\Omega)}$.

We also need to use Holder's inequality $||f||_p||g||_q \geq ||fg||_1$ is $\frac{1}{p} + \frac{1}{q} = 1$, and also $\int_\Omega |f(x)g(x)|dx \leq (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}} (\int_\Omega |g(x)|^q dx)^{\frac{1}{q}}$.

Maybe an exponential function?

Unfortunately not sure where to take it from here.

ii. For this to be true we would need $2\int_\Omega F(g)dx < -||\nabla g||^2_{L^2(\Omega)}$.

4. (a) We have $g \in L^2(\Omega)$ and $uf(u)$ is some $p+1$ degree monomial with $1 < p < 5$ and positive leading coefficient $\lambda$.

There exist constants $\alpha > 0$, $\beta \geq 0$ such that

$$\alpha||u||^2_{L^{p+1}} \leq \int_\Omega uf(u)dx + \beta \tag{23}$$

Therefore we have

$$\frac{1}{2}\sup_{[0,T]}||u||^2_{L^2} + \int_0^T ||\nabla u||^2_{L^2}dt + \alpha\int_0^T ||u||^2_{L^{p+1}}dt \leq \beta T + \frac{1}{2}||g||^2_{L^2} \tag{24}$$

Note if $||u||_{L^{p+1}} \leq \infty$ then $||f(u)||_{L^{\frac{p+1}{p}}} \leq \infty$ since

$$\int_{\Omega} |f(u)|^{\frac{p+1}{p}} dx \leq \alpha \int_{\Omega} |u|^{p+1} dx + \beta \leq \alpha ||u||_{L^{p+1}} + \beta \tag{25}$$

Note that we use $\frac{p+1}{p}$ in this case since it is the Holder conjugate of $p+1$. So in forming a weak solution we use test functions: $v \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$ such that $(\nabla u, \nabla v)_{L^2}$ and $(f(u), v)_{L^2}$ are well-defined.

The Galerkin approximations $\{u_m\}$ take values in a finite dimensional subspace $E_m \subset H_0^1(\Omega) \cap L^{p+1}(\Omega)$ and satisfy the following with $P_m$ the projection of $E_m$ onto $L^2(\Omega)$

$$(u_m)_t = \Delta u_m + P_m f(u_m) \tag{26}$$

Similar to the solution $u$, these approximations satisfy

$$\frac{1}{2} \sup_{[0,T]} ||u_m||_{L^2}^2 + \int_0^T ||\nabla u_m||_{L^2}^2 dt + \alpha \int_0^T ||u_m||_{L^{p+1}}^2 dt \leq \beta T + \frac{1}{2} ||g||_{L^2}^2 \tag{27}$$

The Galerkin ODEs have a unique local solution since the nonlinear terms are Lipschitz continuous functions of $u_m$. Furthermore, the local solutions are bounded, and so they exist globally for all times $t \geq 0$.

Since our estimates hold uniformly for $m$, we can pick a subsequence that weakly converges $u_m \to u$ to a limiting function

$$u \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1) \cap L^{p+1}((0, T) \times \Omega) \tag{28}$$

And furthermore we can say that $u_t \in L^2(0, T; H^{-1}) + L^{\frac{p+1}{p}}((0, T) \times \Omega)$.

Now we need to show that this $u$ is actually a solution of the original PDE. To do this, we must show $f(u_m) \to f(u)$. We find some subsequence of $u_m$ such that $u_m \to u$ strongly in $L^{p+1}((0, T) \times \Omega)$. By the Riesz-Fisher theorem, we can then extract a subsequence such that $u_m(x, t) \to u(x, t)$ pointwise on $(0, T) \times \Omega$. Then we use the dominated convergence theorem and the uniform bounds of $u_m$ to find that for every test function $v \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$

$$(f(u_m(t)), v)_{L^2} \to (f(u(t)), v)_{L^2} \tag{29}$$

pointwise on $[0, T]$.

So here we've shown that the PDE has a global unique weak solution with the desired properties.

(b) Following the same proof above for a $g \in H_0^1 \cap L^{p+1}$ and for every $T > 0$ we will get a unique solution $u \in L^\infty(0, T; H_0^1 \cap L^{p+1}(\Omega)) \cap L^2(0, T; H_0^1) \cap L^{p+1}((0, T) \times \Omega)$ which of course means $u \in L^\infty(0, T; H_0^1 \cap L^{p+1}(\Omega))$. Perhaps I missed the point of this question and there is some property about the $H_0^1$ space that makes the conclusion clear from the above result.

(c) For both $(a)$ and $b$, I believe yes uniqueness is possible, as in my proofs.

Thanks for a great semester! Happy New Year!