PDE I Final HW

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II

1. We can write the smooth solution $u = \sum_k c_k e^{-\lambda_k t} w_k$ for $\lambda_k \leq \lambda_{k+1}$ eigenvalues of the $-\Delta$ operator with zero Dirichlet boundary conditions, w_k are the corresponding eigenfuctions, and $c_k = \int_\Omega g(x) w_k(x) dx$ are the Fourier coefficients with respect to the orthonormal basis $(w_k)_k$.

   Multiplying (10) by u, we have:

   $$u\partial_t u - u\Delta u + uf(u) = 0$$

   Plugging in, we have:

   $$\begin{gathered}\sum_k c_k e^{-\lambda_k t} w_k(x)[\partial_t \sum_k c_k e^{-\lambda_k t} w_k(x)] - \sum_k c_k e^{-\lambda_k t} w_k(x)[\Delta \sum_k c_k e^{-\lambda_k t} w_k(x)] + \\ \sum_k c_k e^{-\lambda_k t} w_k(x) f(\sum_k c_k e^{-\lambda_k t} w_k(x)) = 0 \\ -\sum_k \lambda_k c_k^2 e^{-2\lambda_k t} w_k(x) - \sum_k c_k e^{-\lambda_k t} w_k(x)[\Delta \sum_k c_k e^{-\lambda_k t} w_k(x)] + \\ \sum_k c_k e^{-\lambda_k t} w_k(x) f(\sum_k c_k e^{-\lambda_k t} w_k(x)) = 0 \\ \implies \beta ||u||_{L^2}^2 + B[u,u] + uf(u) = 0\end{gathered}$$

   for some constant β.

   It makes sense intuitively that the function decreases. Multiplying (10) by $\partial_t u$, we have:

   $$(\partial_t u)^2 - \partial_t u \Delta u + \partial_t u f(u) = 0$$

   The decreasing nature of $E(u)$ is furthermore clear when we consider the Poincare inequality:

   $$\int_\Omega |u|^2 dx \leq C \int_\Omega |\nabla u|^2 dx$$

   We need to show that the gradient is less than 0.

2.

3. (a)

   First, let w_k be the complete set of appropriately normalized eigenfunctions for $-\Delta$, which is an orthonormal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$.

Write the smooth solution $u = \sum_k c_k w_k$. Note here that (,) denotes the L^2 inner product. The u satisfies:

$$(\partial_t u, w_k) + B[u, w_k; t] + f(u) = 0$$