Theory Assignment II

Automata Theory Monsoon 2023, IIIT Hyderabad

September 18, 2023

Total Marks: 25 points

Due date: **16/09/23 11:59 pm**

General Instructions: All symbols have the usual meanings (example: $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of natural numbers, and so on). FSM stands for finite state machine. DFA stands for deterministic finite automata. NFA stands for non-deterministic finite automata. a^* is the Kleene Star operation.

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1. [6 points] Prove that the following languages are not context-free using the pumping lemma:

   (a) $\{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$.

   (b) $\{ww \mid w \in \{0,1\}^*\}$.

   (c) $\{a^{n!} \mid n \geq 0\}$.

   [CO-2,CO-3]

2. [3 points] We define the operation Interleave F on strings as follows:

   $$\mathrm{F}(a, b) = a_0 b_0 a_1 b_1 \dots a_n b_n,$$

   where a_i is the i^{th} character of the string (zero-indexed). Are recursively enumerable (RE) languages closed under this operation? If your answer is no, provide a counterexample of two RE languages L_1, L_2 such that $F(L_1, L_2)$ is not RE. If you answer yes, prove it. **[CO-4]**

3. [4 points] A push-down automaton (PDA) has a stack. The machine can push, pop or read from the stack. What is the computational power of a PDA that has a binary tree instead of the stack? We allow the following operations:

   - Read the current node
   - Create a left or a right node with a value
   - Go to any child node
   - Go to any parent node

   Suppose we do not allow the traversal back to the parent, but the machine can read any of its ancestral nodes. Does the power change? **[CO-3,CO-4]**

4. [2 points] Construct a Turing Machine that given a binary string $w \in \{0,1\}^n$, outputs the reverse of string. In other words, the Turing Machine you construct halts with w^R on its tape. **[CO-2,CO-3]**

5. [2 points] Prove that the set of all binary strings is *countably infinite*, but the set of all languages is *uncountably infinite*. **[CO-4]**

6. [2 points] Show that Recursively Enumerable Languages are closed under the star operation. **[CO-4]**

7. [6 points] We will define a Lilliputian Turing Machine (LTM) as one that has the following description: $\langle Q, \Sigma, \delta, \Gamma, q_{start}, q_{accept}, q_{reject} \rangle$. The transition function is as follows: $\delta : \{Q \times \Gamma^k\} \rightarrow \{Q \times \Gamma^k \times \mathbb{Z}\}$. Here $\mathbb{Z}$ is the set of all integers. Each cell is given an address, starting with index 0. So, the cell's addresses are 0, 1, 2, ... and so on.

   Each machine comes with a (well paid) Lilliputian, who helps the machine function. Note that k should be treated as a constant for an LTM. Initially, the tape head starts off at address 0.

   The working of an LTM for $k = 3$ has been described below. Say the tape head is on address $p = 0$. For a δ transition, $\delta(q_i, \{a_1, a_2, a_3\}) = (q_j, \{b_1, b_2, b_3\}, r)$, the Lilliputian inside the LTM M checks if the current state is q_i and if the symbols at addresses p, $p+1$, $p+2$ are equal to a_1, a_2, and a_3 respectively. If these conditions are met, it changes the machine's state to q_j and the symbols at those addresses to b_1, b_2, and b_3. After that, he moves the tape head to the address $p + r$, which is a valid address ($r \in \mathbb{Z}$). For any general k, if the tape head is at cell p, then in transition, the Lilliputian changes cells p, $p+1$, ... $p+k-1$. If the tape head moves out of bounds in the one way infinite tape, then it just goes to address 0.

   Essentially, in one δ transition, the LTM can modify k cells and then after that the tape head can jump to any other cell instantly (relative to the current position). Your task is to prove that the Lilliputian Turing Machine is as powerful as a standard Turing Machine in terms of the languages it can recognize.

   Hint: You need to prove that a normal TM can simulate an LTM, and that an LTM can simulate a normal TM. Your answer can be an informal algorithm that describes how these simulations can be done, but the steps should be laid out clearly. **[CO-1,CO-2,CO-3,CO-4]**