Lemma 1. *Let* $k' \geq \log n$ *and* $d' = \frac{\log n}{3(\log 2k' - \log\log n + 2)}$, *then* $\binom{k'+d'}{d'} < n$.[1]

Proof. First note that the denominator is at least 2 and so $d' \leq k'$. *Recall that* $\binom{r}{s} \leq 2^{H(s/r)r}$ *where* H *stands for the binary entropy function. In addition for* $x \leq 0.5$ *it holds that* $H(x) \leq 2x \log \frac{1}{x}$. *If we combine all these, we obtain:*

$$\log \binom{k'+d'}{d'} \leq H\left(\frac{d'}{k'+d'}\right)(k'+d') \leq 2d' \log \frac{k'+d'}{d'} \leq 2d'(\log 2k' - \log d') = \ldots$$

Let's make things simpler by substituting $a := \frac{\log n}{d'} = 3\left(\log 2k' - \log\log n + 2\right)$.

$$\ldots = 2\frac{\log n}{a}\left(\log 2k' - \log\log n + \log a\right) \leq 2\log n \cdot \left(\frac{a/3 - 2 + \log a}{a}\right) \leq \ldots$$

By using calculus we see that the fraction is maximal for $a = 4\mathrm{e}$.

$$\ldots \leq 2\log n \cdot \left(\frac{1}{3} + \frac{-2+2+\log \mathrm{e}}{4\mathrm{e}}\right) < \log n \cdot \left(\frac{2}{3} + \frac{\log \mathrm{e}}{2\mathrm{e}}\right) < \log n$$

□

[1] The proof can handle a little better constants: it works with $k' \geq \log n$ and $d' = \frac{\log n}{(2+\delta)(\log 2k' - \log\log n + c)}$ where $\delta > 0$ and $c \geq 1 + \log \frac{(2+\delta)\log \mathrm{e}}{\delta \mathrm{e}}$.