Constructive Temporal Logic, Categorically

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Preface

This paper is dedicated to the memory of Professor Grigori Mints, to whom the first author owes a huge debt. Not only an intellectual debt (most people working with proof theory nowadays owe this one), not only a friendship one (there are plenty of us who owe this), but also a mentoring (and a personal help when I needed it) debt. Grisha would not be gratuitously conversational, you could say that his style was 'tough love': work hard, then he would talk to you. But when you needed him, he was there for you. This work might not be, yet, at the stage that he would approve of it, especially given all his work on Dynamic Topological Logic with Kremer and others [14], which might be related to what we describe here. However, this is the best that we can do in the time we have, so it will have to do.

The second author does owe Grigori an intellectual dept. While I never got to meet him in person I do remember fondly reading his "A Short Introduction to Intuitionistic Logic" [17]. His book gave me several realizations about intuitionistic logic that I had previously lacked. It was his book that turned on the light, and I thank him for that.

1 Motivation

Generally speaking, Temporal Logic is any system of rules and symbolism for representing, and reasoning about propositions qualified in terms of time. Temporal logic is also one of the most traditional kinds of modal logic, introduced by Arthur Prior in the late 1950s, but it is also one of the most controversial kinds of modal logic, as people have different intuitions about time, how to represent it, and how to reason about it.

There has been a huge amount of work in Modal Logic in the last sixty years, mainly in classical modal logic. We are mostly interested in constructive systems, not classical ones. In particular we are interested in a constructive version of temporal logic that satisfies some well-known and desirable proof-theoretical properties, but that is also algebraically and category-theoretically *well-behaved*.

$$\begin{array}{ccc} \Diamond(A \vee B) & \rightarrow & \Diamond A \vee \Diamond B \\ \Diamond\bot & \rightarrow & \bot \end{array}$$

Figure 1: Distributivity rules

Prior's 'Time and Modality' [21] introduced a propositional modal logic with two temporal connectives (modal operators), F and P, corresponding to "sometime in the Future" and "sometime in the Past". This has been called **tense logic** to distinguish it from other temporal systems.

Ewald [7] produced a first version of an intuitionistically based temporal logic system. The intuitive reading of the operators is very reasonable:

- P "It has at some time been the case that"
- F "It will at some time be the case that"
- H "It has always been the case that"
- G "It will always be the case that"

Ewald and most of the researchers that followed his path of constructivization of tense logic, did so assuming a symmetry between past and future. This symmetry, as well as the symmetry between universal and existential quantifiers, both in the past and in the future, are somewhat at odds with intuitionistic reasoning. In particular while an axiom like $A \rightarrow GPA$ "What is, will always have been" makes sense in a constructive way of thinking, the dual one $A \rightarrow HFA$ paraphrased in the SEP as "What is, has always been going to be" feels very classical.

Constructivizing a classical system is alwys prone to proliferation of the system, as is evident when considering the several versions on intuitionistic set theory, for example. In particular the basic constructive modal logic S4 (in Lewis original naming convention) has two main variants.

The first version of an intuitionsitic S4, originally presented by Dag Prawitz in his Natural Deduction book [20] does not satisfy the distributivity of the possibility operator $\Diamond$ over the logical disjunction. Prawitz's system satisfies neither the binary distribution nor its nullary form, as given in Figure 1. We call this system CS4. This system was investigated from a proof theoretical and categorical perspective in [5].

The second main version of an intuitionistic modal S4 does enforce these distributivities and it was thoroughly investigated by A. Simpson in [22]. This system is part of a framework for constructive modal logics, based on incorporating, as part of the syntax, the intended semantics of modal logics, as possible worlds. We call this system IS4.

Ewald's tense logic system consists of a pair of Simpson-style S4 operators [22], representing past and future over intuituionistic propositional logic. This is historically inaccurate, as Simpson based his systems in Ewald's, but it will serve to make some of our main points clearer below. The system we describe in this note is the tense logic system obtained by joining together two pairs of Prawitz-style S4 operators. So it satisfies some of the rules of Ewald's, but not all.

Simpson remarks that intuitionistic or constructive modal logic is full of interesting questions. As he says:

> Although much work has been done in the field, there is as yet no consensus on the correct viewpoint for considering intuitionistic modal logic. In particular, there is no single semantic framework rivalling that of possible world semantics for classical modal logic. Indeed, there is not even any general agreement on what the **intuitionistic analogue** of the basic modal logic, K, is.

In an intuitionistic logic we do not expect perfect duality between quantifiers, ($\forall x.P(x)$ is not the same as $\neg\exists x.\neg P(x)$) or even between conjunction and disjunction (De Morgan laws do not hold for intuitionistic propositional logic). So one should not expect a perfect duality between possibility and necessity either. But just considerations from first principles do not seem to clearly indicate whether distributivity rules as the ones in Figure 1 should hold or not. Hence it seems sensible to develop different kinds of systems in parallel, proving equivalences, whenever possible. In this paper we develop the idea of tense logic in the Prawitz style. We recall some deductive systems for this tense logic and provide categorical semantics for it.

Much has been done recently in the proof theory of constructive modal logics using more informative sequent systems (e.g. hypersequents, labelled sequents, nested sequents, tree-style sequents, etc..) In particular nested sequents have been used to produce 'modal cubes' for the two variants of constructive modal logics described above. See the pictures below from [2, 23].

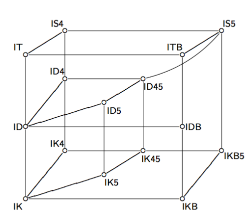

Fig. 2. The intuitionistic "modal cube"

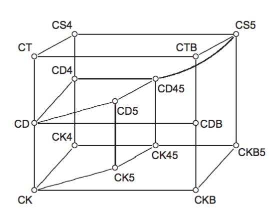

FIGURE 1. The constructive "modal cube"

Sequent calculi by themselves are not enough to provide us with Curry-Howard correspondences and/or term assignments for these systems. However, using the Prawitz S4 version of these modal systems we can easily produce a Curry-Howard correspondence and a categorical model for the Prawitz-style intuitionistic tense logic, our goal in this paper.

We start by recalling the system using axioms, plain sequent calculus and plain natural deduction. In the next section we described a term assignment

$$\frac{}{\Delta, A \vdash A}\ \mathrm{Id} \qquad \frac{\Gamma \vdash B \quad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \mathrm{Cut} \qquad \frac{}{\Gamma, \bot \vdash A}\ \bot_{\mathcal{L}}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee_{\mathcal{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{\mathcal{R}_1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{\mathcal{R}_2}$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\mathcal{L}_1} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\mathcal{L}_2} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_{\mathcal{R}}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ \to_{\mathcal{L}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to_{\mathcal{R}}$$

Figure 2: Intuitionistic Propositional Calculus (LJ)

based on the dual calculus described in [9] and show some of its syntactic prooperties. The next section introduces the categorical model (a cartesian closed category with two intertwinned adjunctions) and show the usual soundness and completeness results. Finally we discuss potential applications and limitations of our constructive tense logic.

2 Tense Logic CS4-style

We build up to the constructive tense logic we are interested in TCS4 in progressive steps. We start with the intuitionistic basis LJ, add the modalities to get the constructive S4 system, CS4, provide the dual context modification (to help with the reuse of libraries, amongst other things), obtaining dual CS4, DCS4 and then finally consider the two adjunctions to obtain the tense constructive system TCS4.

2.1 Intuitionistic Sequent Calculus

We start by recalling the basic sequent calculus for intuitionistic propositional logic, Gentzen's intuitionistic sequent calculus LJ. The syntax of formulas for LJ is defined by the following grammar:

$$A \quad ::= \quad p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to B$$

The formula p is taken from a set of countably many propositional atoms. The constant $\top$ could be added, but it is the negation of the the falsum constant $\bot$. The initial inference rules, which just model propositional intuitionistic logic, are as in Figure 2.

Sequents denoted $\Gamma \vdash C$ consist of a multiset of formulas, (written as either Γ, Δ, or a numbered version of either), and a formula C. The intuitive meaning is that the conjunction of the formulas in Γ entails the formula C. So far this is our intuitionistic basis.

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\ \Box_{\mathcal{L}} \qquad\qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma, \Delta \vdash \Box A}\ \Box_{\mathcal{R}}$$

$$\frac{\Box\Gamma, A \vdash \Diamond B}{\Delta, \Box\Gamma, \Diamond A \vdash \Diamond B}\ \Diamond_{\mathcal{L}} \qquad\qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{R}}$$

Figure 3: Constructive S4 modal rules (CS4)

2.2 Constructive modal S4

Next we recall the sequent calculus formalization of system CS4.

We recap the modality rules in Figure 3. These, in addition to the initial set of inference rules, define the sequent calculus for CS4. In Figure 3, we write $\Box\Gamma$ for the sequence of boxed formulas $\Box G_1, \Box G_2, \ldots, \Box G_k$ where Γ is the set $G_1, G_2, \ldots, G_k$.

Note that we do have right rules and left rules for introducing the new modal operators $\Box$ (necessity) and $\Diamond$ (posssibility), but these rules are not as symmetric as the propositional ones. Most importantly, we have a local restriciton on the rule that introduces the $\Box$ operator: We can only introduce $\Box$ in the conclusion, if all the assumptions are already boxed. Also the rules for the $\Diamond$ operator presuppose that you have already $\Box$ operators. This system is indeed constructive, $\Box$ and $\Diamond$ are independent logical operators and $\Box A$ is not logically equivalent to $\neg\Diamond\neg A$, nor is $\Diamond A$ logically equivalent to $\neg\Box\neg A$.

This system has a nice proof theory. Bierman and de Paiva [5] show that it has a Hilbert-style presentation, a Natural Deduction presentation, as well as a sequent calculus presentation and these presentations are proved equivalent, that is, they prove the same theorems. The sequent calculus satisfies cut-elimination, an old result from Ohnishi and Matsumoto [18], as well as the subformula property. The Natural Deduction formulation has a colourful history: one of its distinct features is that it was described in Prawitz' seminal book in Natural Deduction [20], hence it is sometimes called Prawitz S4 intuitionistic modal logic. Most interestingly the system has both Kripke and categorical semantics, described respectively in [1] and [5] as well as an independent mathematical semantics in terms of simplicial sets, described by Goubault-Larrecq [11].

2.3 The dual context modal S4 calculus

An equivalent (in terms of provability) but more type-theoretic system can be produced for the modal logic CS4. This is not so well-known, but this system can be given a presentation in terms of a categorical adjunction, between two cartesian closed categories, as we will describe in the next section. This categorical presentation has been described both in [5] and in [9], in the former, this is called the **multi-context** formulation of CS4 and the rules are given in

$$\frac{\Gamma; \emptyset \vdash A}{\Gamma; \Delta \vdash \Box A}\ \Box_{\mathcal{I}} \qquad \frac{\Gamma; \Delta \vdash \Box A \quad \Gamma, A; \Delta \vdash B}{\Gamma; \Delta \vdash B}\ \Box_{\mathcal{E}}$$

$$\frac{\Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash \Diamond A}\ \Diamond_{\mathcal{I}} \qquad \frac{\Gamma; \Delta \vdash \Diamond A \quad \Gamma; A \vdash \Diamond B}{\Gamma; \Delta \vdash \Diamond B}\ \Diamond_{\mathcal{E}}$$

Figure 4: The dual context modal calculus (DCS4)

Figure 4. (We prefer to call it the dual context sequent calculus.) Note that the rules are Natural Deduction rules, as it should be clear from the fact that they are introduction and elimination rules.

The main difference in the dual context formulation of CS4 is the fact that the context now has modal formulas and non-necessarily modal ones, separated by a semi-colon as in $\Gamma; \Delta$. The difficult rule of $\Box$ introduction now insists that to introduce a necessity operator $\Box$ on a conclusion, we need to have an empty context of non-modal assumptions. This corresponds to the traditional idea that to prove something is necessarily the case, all its assumptions have to be also necessary (or it has no assumptions whatsoever).

These rules have been shown by Benton [4] and Barber [3] to correspond to an adjunction of the categories, in the case where the basis is Linear Logic and the modalities correspond to the exponentials. Instead of Linear Logic, we deal with constructive modal logic and the adjunction is between functors corresponding to operators $\Diamond \vdash \Box$.

2.4 The tense CS4 calculus

Finally to get to the tense logic which is the main aim of this note, we need two such adjunctions, but intertwined. This follows the pattern explained by Ewald [7]. Thus $\Diamond$ is left-adjoint to $\blacksquare$ and $\blacklozenge$ is left-adjoint to $\Box$, where we are writing $\blacksquare$ for the operator we called past universal H before and $\Box$ for the future necessity operator G. The past existential P is $\Diamond$ and the future existential is $\blacklozenge$ or F.

A version of a sequent calculus system for this constructive tense logic is given by the rules in Figure 5. This can be transformed into Natural Deduction in the style of [5] as shown in Figure 6. The problem is that the last two adjunction rules in Figure 5 (that relate the two sets of modalities) are extremely badly-behaved proof-theoretically (no cut-elimination and no subformula property even for cut-free proofs), as discussed in page 35 of the full report by Benton [4]. In fact they are the reason for moving to a dual context calculus, as explained in that paper and also in Barber's work [3].

The dual context systems, as described in Barber and Benton's work, are proved equivalent to the system with a single modality operator, either '!' or '$\Box$'. This is because in Intuitionistic Linear Logic one is not usually interested in either *why not?* '?' or '$\Diamond$'. (In Classical Linear Logic the possibility modality

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\ \Box_{\mathcal{L}} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma, \Delta \vdash \Box A}\ \Box_{\mathcal{R}}$$

$$\frac{\Box\Gamma, A \vdash \blacklozenge B}{\Delta, \Box\Gamma, \blacklozenge A \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{R}}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \blacksquare A \vdash B}\ \blacksquare_{\mathcal{L}} \qquad \frac{\blacksquare\Gamma \vdash A}{\blacksquare\Gamma, \Delta \vdash \blacksquare A}\ \blacksquare_{\mathcal{R}}$$

$$\frac{\blacksquare\Gamma, A \vdash \Diamond B}{\Delta, \blacksquare\Gamma, \Diamond A \vdash \Diamond B}\ \Diamond_{\mathcal{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{R}}$$

$$\frac{A \vdash \Box B}{\blacklozenge A \vdash B}\ adj1_{\mathcal{L}} \qquad \frac{\blacklozenge A \vdash B}{A \vdash \Box B}\ adj1_{\mathcal{R}}$$

$$\frac{A \vdash \blacksquare B}{\Diamond A \vdash B}\ adj2_{\mathcal{L}} \qquad \frac{\Diamond A \vdash B}{A \vdash \blacksquare B}\ adj2_{\mathcal{R}}$$

Figure 5: Tense S4 sequent rules (biCS4)

is defined by negation of the necessity modality, so this extension is easier to make [19].) Given that our main goal is to discuss categorical semantics, which we can do easily for the necessity modalities, in this note we consider only two necessity-like modalities $\Box$ and $\blacksquare$, to begin with.

We would like to have a natural deduction version of the tense calculus in dual context style. A dual context-style presentation of a single necessity modality has been presented in Figure 4. Now we need to add another necessity-like modality and discuss their interaction. A preliminary attempt at such calculus is given in Figure 7.

This corresponds to an intuitionistic tense logic obtained by extending IPL with two pairs of adjoint modalities $(\blacklozenge, \Box)$ and $(\Diamond, \blacksquare)$, with no explicit relationship between the modalities of the same colour, namely, $(\blacklozenge, \blacksquare)$ and $(\Diamond, \Box)$.

2.5 Axioms

Axiom sets for the system TCS4 are easier to provide. We need a set for the basic system intuitionistic logic LJ, any traditional set would do, plus the axioms for modalities, as well as the rules **modus ponens** and **necessitation** for the two necessity operators:

We have similar axioms to Ewald's [7], except that the duality between necessity and possibility is not strict (Ewald's original axioms (7) and (7') in page 171 of [7] are not valid) and that the possibility modalities we deal with, do not distribute over disjunction (Ewald's axioms (4) and (4') are not valid). Also note we do have introspection and reflexivity valid, which correspond to

$$\frac{\Gamma \vdash \Box B}{\Gamma \vdash B}\ \Box_{\mathcal{E}} \qquad \frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k \quad \Box A_1, \ldots \Box A_k \vdash B}{\Gamma \vdash \Box B}\ \Box_{\mathcal{I}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{I}}$$

$$\frac{\Gamma \vdash \blacksquare A}{\Gamma \vdash A}\ \blacksquare_{\mathcal{E}} \qquad \frac{\Gamma \vdash \blacksquare A_1, \ldots, \Gamma \vdash \blacksquare A_k \quad \blacksquare A_1, \ldots \blacksquare A_k \vdash B}{\Gamma \vdash \blacksquare B}\ \blacksquare_{\mathcal{I}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{I}}$$

$$\frac{\Gamma \vdash \Box A_1 \ldots \Gamma \vdash \Box A_k \ \Gamma \vdash \blacklozenge B \quad \Box A_1 \ldots \Box A_k, B \vdash \blacklozenge C}{\Gamma \vdash \blacklozenge C}\ \blacklozenge_{\mathcal{E}}$$

$$\frac{\Gamma \vdash \blacksquare A_1 \ldots \Gamma \vdash \blacksquare A_k \ \Gamma \vdash \Diamond B \quad \blacksquare A_1 \ldots \blacksquare A_k, B \vdash \Diamond C}{\Gamma \vdash \Diamond C}\ \Diamond_{\mathcal{E}}$$

Figure 6: Tense S4 rules ND first version (NDCS4)

$$\frac{\Gamma_1; \emptyset \vdash A}{\Gamma_1; \Delta \vdash \Box A}\ \Box_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \Box A \quad \Gamma_1; A; \Delta \vdash B}{\Gamma_1; \Delta \vdash B}\ \Box_{\mathcal{E}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \blacklozenge A \quad \Gamma_1; A \vdash \blacklozenge B}{\Gamma_1; \Delta \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{E}}$$

$$\frac{\Gamma; \emptyset \vdash A}{\Gamma; \emptyset \vdash \blacksquare A}\ \blacksquare_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \blacksquare A \quad \Gamma_1; A; \Delta \vdash B}{\Gamma_1; \Delta \vdash B}\ \blacksquare_{\mathcal{E}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \Diamond A \quad \Gamma_1; A \vdash \Diamond B}{\Gamma_1; \Delta \vdash \Diamond B}\ \Diamond_{\mathcal{E}}$$

Figure 7: biS4 rules, dual context version (ND2CS4)

$$\frac{}{\Gamma_1; \Gamma_2, A; \Delta \vdash A}\ \Box\mathrm{Id} \qquad \frac{}{\Gamma_1, A; \Gamma_2; \Delta \vdash A}\ \blacksquare\mathrm{Id}$$

$$\frac{\Gamma_1; \Gamma_2; \emptyset \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \Box A}\ \Box_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \Box A \quad \Gamma_1; \Gamma_2, A; \Delta \vdash B}{\Gamma_1; \Gamma_2; \Delta \vdash B}\ \Box_{\mathcal{E}}$$

$$\frac{\Gamma_1; \Gamma_2; \Delta \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge A \quad \Gamma_1; \Gamma_2; A \vdash \blacklozenge B}{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{E}}$$

$$\frac{\Gamma_1; \Gamma_2; \emptyset \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \blacksquare A}\ \blacksquare_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \blacksquare A \quad \Gamma_1, A; \Gamma_2; \Delta \vdash B}{\Gamma_1; \Gamma_2; \Delta \vdash B}\ \blacksquare_{\mathcal{E}}$$

$$\frac{\Gamma_1; \Gamma_2; \Delta \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \Diamond A}\ \Diamond_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge A \quad \Gamma_1; \Gamma_2; A \vdash \blacklozenge B}{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{E}}$$

Figure 8: Dual context 2CS4 calculus (TCS4)

$$\frac{A}{\Box A}\ \Box_{Nec} \qquad \frac{A}{\blacksquare A}\ \blacksquare_{Nec} \quad \frac{A \to B \quad A}{B}\ \mathcal{MP}$$

Figure 9: Axiomatic Rules

Ewald's axioms (12) and (12'), as well as (13) and (13').

We are interested in the term assignment system and its properties, as our aim is to use these as type systems for innovative programming languages. So we needed to provide the systematic work that shows basic properties of the type system TCS4 we are interested in, this is what we do in the next section.

3 Term Assignment

In this section we provide a term assignment to constructive tense logic with only $\Box$ and $\blacksquare$. We leave term assignments to the other varieties of tense logic with $\Diamond$ and $\blacklozenge$ for future work.

The typing rules can be found in Figure 11 with the typed equality rules in Figure 12. Here we can see that types are tense S4 formulas. The sequents have the form $\Gamma \vdash t : A$ and $\Gamma \vdash s \approx t : A$ where Γ is a multiset of free variables and their types denoted $x : A$, and s and t are terms with the following syntax:

$$\begin{array}{lcl} t & := & x \mid \lambda x : A.t \mid s\ t \mid \mathsf{let}_{\Box}\ x_1 : \Box A_1, \ldots, x_k : \Box A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t \mid \\ & & \mathsf{let}_{\blacksquare}\ x_1 : \blacksquare A_1, \ldots, x_k : \blacksquare A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t \mid \mathsf{unbox}_{\Box}\ t \mid \mathsf{unbox}_{\blacksquare}\ t \end{array}$$

Equality is straightforward where it is apparent that the let-expressions model explicit substitutions. These substitutions are triggered when they are applied to an unbox-expression.

propositional basic intuitionistic axioms

$$
\begin{array}{ccc}
\Box(A \to B) & \to & (\Box A \to \Box B) \\
\Box(A \to B) & \to & (\blacklozenge A \to \blacklozenge B) \\
(\Box A \to A) & \wedge & (A \to \blacklozenge A) \\
(\Box A \to \Box\,\Box\, A) & \wedge & (\blacklozenge\,\blacklozenge\, A \to \blacklozenge A) \\
\blacksquare(A \to B) & \to & (\blacksquare A \to \blacksquare B) \\
\blacksquare(A \to B) & \to & (\Diamond A \to \Diamond B) \\
(\blacksquare A \to A) & \wedge & (A \to \Diamond A) \\
(\blacksquare A \to \blacksquare\blacksquare A) & \wedge & (\Diamond\Diamond A \to \Diamond A) \\
\Diamond\blacksquare A \to A & \wedge & A \to \blacksquare\Diamond A \\
\blacklozenge\,\Box\, A \to A & \wedge & A \to \Box\,\blacklozenge\, A
\end{array}
$$

Figure 10: Axioms for $\mathsf{TCS4}$

$$
\frac{}{\Gamma, x : A \vdash x : A}\mathrm{Id} \qquad \frac{}{\Gamma, x : \perp \vdash \mathsf{contra} : A}\perp_{\mathcal{E}} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.t : A \to B}\to_{\mathcal{I}}
$$

$$
\frac{\Gamma \vdash t_1 : A \to B \quad \Gamma \vdash t_2 : A}{\Gamma \vdash t_1\, t_2 : B}\to_{\mathcal{E}} \qquad \frac{\Gamma \vdash t : \Box\, B}{\Gamma \vdash \mathsf{unbox}_{\Box}\ t : B}\Box_{\mathcal{E}}
$$

$$
\frac{\Gamma \vdash t_1 : \Box\, A_1, \ldots, \Gamma \vdash t_k : \Box\, A_k \quad x_1 : \Box\, A_1, \ldots, x_k : \Box\, A_k \vdash t : B}{\Gamma \vdash \mathsf{let}_{\Box}\, x_1 : \Box\, A_1, \ldots, x_k : \Box\, A_k \;\mathsf{be}\; t_1, \ldots, t_k \;\mathsf{in}\; t : \Box\, B}\Box_{\mathcal{I}}
$$

$$
\frac{\Gamma \vdash t : \blacksquare\, B}{\Gamma \vdash \mathsf{unbox}_{\blacksquare}\ t : B}\blacksquare_{\mathcal{E}}
$$

$$
\frac{\Gamma \vdash t_1 : \blacksquare\, A_1, \ldots, \Gamma \vdash t_k : \blacksquare\, A_k \quad x_1 : \blacksquare\, A_1, \ldots, x_k : \blacksquare\, A_k \vdash t : B}{\Gamma \vdash \mathsf{let}_{\blacksquare}\, x_1 : \blacksquare\, A_1, \ldots, x_k : \blacksquare\, A_k \;\mathsf{be}\; t_1, \ldots, t_k \;\mathsf{in}\; t : \blacksquare\, B}\blacksquare_{\mathcal{I}}
$$

Figure 11: TCS4 Typing Rules

$$\frac{\Gamma, x : A \vdash t_2 \approx s_2 : B \quad \Gamma \vdash t_1 \approx s_1 : A}{\Gamma \vdash (\lambda x : A.t_2)\, t_1 \approx [s_1/x]s_2 : B}\beta$$

$$\frac{\Gamma \vdash t_1 \approx s_1 : \Box A_1, \ldots, \Gamma \vdash t_k \approx s_k : \Box A_k \quad x_1 : \Box A_1, \ldots, x_k : \Box A_k \vdash t \approx s : B}{\Gamma \vdash \mathsf{unbox}_\Box\ (\mathsf{let}_\Box\, x_1 : \Box A_1, \ldots, x_k : \Box A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t) \approx [s_1/x_1] \ldots [s_k/x_k]s : B}\Box$$

$$\frac{\Gamma \vdash t_1 \approx s_1 : \blacksquare A_1, \ldots, \Gamma \vdash t_k \approx s_k : \blacksquare A_k \quad x_1 : \blacksquare A_1, \ldots, x_k : \blacksquare A_k \vdash t \approx s : B}{\Gamma \vdash \mathsf{unbox}_\blacksquare\ (\mathsf{let}_\blacksquare\, x_1 : \blacksquare A_1, \ldots, x_k : \blacksquare A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t) \approx [s_1/x_1] \ldots [s_k/x_k]s : B}\blacksquare$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \approx t : A}\mathrm{refl} \qquad \frac{\Gamma \vdash t_2 \approx t_1 : A}{\Gamma \vdash t_1 \approx t_2 : A}\mathrm{sym}$$

$$\frac{\Gamma \vdash t_1 \approx t_2 : A \quad \Gamma \vdash t_2 \approx t_3 : A}{\Gamma \vdash t_1 \approx t_3 : A}\mathrm{trans}$$

Figure 12: TCS4 Equality Rules

We have the following basic properties of this term assignment.

Lemma 1 (Substitution for Typing)**.** *If* $\Gamma \vdash t_1 : A$*, and* $\Gamma, x : A \vdash t_2 : B$*, then* $\Gamma \vdash [t_1/x]t_2 : B$*.*

Proof. This proof hold by straightforward induction on the form of the assumed typing derivation. □

Lemma 2 (Weakening)**.** *If* $\Gamma \vdash t : B$*, then* $\Gamma, x : A \vdash t : B$*.*

Proof. This proof hold by straightforward induction on the form of the assumed typing derivation. □

4 The Categorical Model

There is not much essentially new in what we discuss here about the tense logic based on CS4. Similar ideas were discussed by Ghilardi and Meloni [10], Makkai and Reyes [15] and more recently in by Dzik et al [6] and Menni and Smith [16].

The upshot of our discussion is that the categorical model we advance is a cartesian closed category endowed with two adjunctions, corresponding to the (limited) universal and existential quantifications relative to the past and to the future that correspond to the two sets of necessity and possibility operators.

This setting is though different enough from the precursors we know about, to justify this note. First, as discussed elsewhere [5], we see no reason for the monads/comonads emerging from this setting to be **idempotent** operators, as they are in [10] or [15]. Secondly we see no reason to take our models as part of toposes, as we are not interested in extra structure provided by toposes. However, we also see no reason to confine ourselves to algebraic models such as

Heyting algebras with operators, as degenerate posetal categories, as both [6] and [16] do. Different proofs of the same theorem are important to us as they correspond to different morphisms in the category between the same objects. Thus we are interested in **proof relevant** semantics, not simply provability.

We build our main definition in stages. A categorical model of propositional intuitionistic logic is a **cartesian closed** category with coproducts $\mathcal{C}$.

Definition 3 (adjoint model)**.** *An adjoint categorical model of dual context modal logic* $\mathsf{DCS4}$ *consists of the following data:*

1. *A cartesian closed category with coproducts* $(\mathcal{C}, 1, 0, \times, +, \to)$*;*
2. *A monoidal adjunction* $F \dashv G$*, where* (F, m) *and* $(G, n) \colon \mathcal{C} \longrightarrow \mathcal{C}$ *are monoidal functors such that their composition* GF *is a monoidal comonad, written as* $\Box$*;*
3. *The monad* $(\Diamond, \eta, \mu, \mathsf{st}_{A,B})$*, induced by the adjunction* $F \dashv G$*, is* $\Box$*-strong.*

Definition 4 (tense calculus model)**.** *A model of* $\mathsf{TCDS4}$ *is a cartesian closed category* $\mathcal{C}$ *as above, together with two intertwined adjunctions* $(\blacklozenge \dashv \Box, \Diamond \dashv \blacksquare)$*. The adjunctions* $(\blacklozenge \dashv \Box)$ *and* $(\Diamond \dashv \blacksquare)$ *on* $\mathcal{C}$ *are connected by Fisher-Servi axioms.*

Categorical soundness is proved, as usual, checking the natural deduction rules preserve validity of the constructions used, i.e function spaces, products, coproducts and the two adjunctions.

Define an interpretation $[\![\_]\!] : \mathsf{TCS4} \to \mathcal{C}$ which takes the types and sequents of $\mathsf{TCS4}$ (over a basic set of types) to a model $\mathcal{C}$ as follows:

$$\begin{aligned}
[\![p]\!] &= I(p) \text{ for } p \text{ a base type} \\
[\![\top]\!] &= \top \\
[A \times B] &= [A] \times [B] \\
[\![A \to B]\!] &= [\![A]\!] \to [\![B]\!] \\
[\![\Box A]\!] &= FG([\![A]\!]) \\
[\![\blacksquare A]\!] &= F'G'([\![A]\!])
\end{aligned}$$

We extend this interpretation to to lists of types by saying that for a list $A_1, ..., A_n$ of types, the interpretation of the list is the list of the interpretations $[\![A_1, ...A_n]\!] = ([\![A_1]\!], ...[\![A_n]\!])$ contexts $(A_1, ...A_n |\ B_1, ...B_n)$ by adding (two kinds of) boxes to the modal assumptions and defining $[\![A_1, \ldots, A_n |\ B_1, \ldots, B_n]\!] = \Box[\![A_1]\!] \times \cdots \times \Box[\![A_n]\!] \times\ [\![B_1]\!] \times \cdots \times [\![B_n]\!]$. The interpretation will take a sequent $\Gamma|\Delta \vdash t : A$ to an arrow $[\![\Gamma|\Delta \vdash t : A]\!] : [\![\Gamma|\Delta]\!] \to [\![A]\!]$ in the tense modal category.

Theorem 5. *The type theory* $\mathsf{TCS4}$ *has sound models provided by the structures* $\mathcal{C}$ *defined above. In other words, given an tense adjoint modal category* $\mathcal{C}$*, using the above interpretation, the following hold:*

- *Assume* $\Gamma|\Delta \vdash t : A$ *in* $\mathsf{TCS4}$*. Then* $[\![\Gamma|\Delta \vdash t\colon A]\!]$ *is a morphism with domain* $[\![\Gamma|\Delta]\!]$ *and codomain* $[\![A]\!]$*;*

- *Assume* $\Gamma \vdash t = s\colon A$*. Then* $[\![\Gamma \vdash t\colon A]\!] = [\![\Gamma \vdash s\colon A]\!]$.

We also have completeness of the tense modal categories.

Theorem 6. *The adjoint modal models are complete in the appropriate sense for the type theory* $\mathsf{TCS4}$*. This is to say, if we have equality of the interpretations* $[\![\Gamma \vdash t\colon A]\!] = [\![\Gamma \vdash s\colon A]\!]$ *(where* $[\![\ ]\!]$ *is the interpretation defined above) in the tense modal category* $\mathcal{C}$ *for any derived sequents* $\Gamma \vdash t\colon A$ *and* $\Gamma \vdash s\colon A$ *then we can derive the equation in the type theory* $\mathsf{TCS4}$ $\quad \Gamma \vdash t = s\colon A$*.*

Categorical completeness requires providing an equivalence relation in the Lindenbaum algebra of the formulae, as usual in algebraic semantics. The basic calculations, for traditional algebraic semantics in Heyting algebras were provided, for instance, by Figallo et al in [8]. *Mutatis mutantis* these calculations will apply for our version of the system (no distribution of diamonds over disjunctions, no definibility of diamonds in terms of negated boxes).

5 Conclusions

We have described a tense version of constructive temporal logic, conceived as a basic category of propositions, together with two adjunctions, corresponding to two kinds of necessity modalities, in the future and in the past. This system is based on traditional work of Ewald in [7], where we simply do the modifications required to account for the categorical model desired. This work is somewhat inspired by recent work on Functional Reactive Programming (FRP) by Jeltsch [13] and Jeffrey [12], independently. Both of these works consider Curry-Howard correspondents to temporal logic, but they tend to concentrate on the *next* temporal operator, originally considered in LTL (Linear Temporal Logic) as suggested by Davies. The temporal operators we consider are more abstract and one can hope that they may shed some light on the issues of FRP. But this is future work.

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