Intuitionistic Propositional Logic with Galois Connections

Wojciech Dzik* Jouni Järvinen† Michiro Kondo‡

Abstract

In this work, an intuitionistic propositional modal logic IntGC is introduced. In addition to the intuitionistic logic axioms and inference rule of modus ponens, the logic contains only two rules of inference mimicking the performance of Galois connections. Both Kripke-style and algebraic semantics are presented for IntGC, and IntGC is proved to be complete with respect to both of these semantics. We show that IntGC has the finite model property and is decidable, but so-called Glivenko's Theorem does not hold. Duality between algebraic and Kripke semantics is presented, and a representation theorem for Heyting algebras with Galois connections is proved. In addition, an application to rough L-valued sets is presented.

Keywords: Galois connection, Intuitionistic logic, Algebraic semantics, Kripke-semantics, Representation

1 Introduction

In [8], Information Logic of Galois Connections (ILGC) suited for approximate reasoning about knowledge is introduced. ILGC is just classical propositional logic with two unary connectives $\blacktriangle$ and $\triangledown$ mimicking the performance of Galois connection maps.

The set of connectives of ILGC consists of logical symbols $\to$, $\neg$, $\blacktriangle$, and $\triangledown$. In addition to the three classical propositional logic axioms and the inference rule of modus ponens, ILGC contains only two rules of inference:

$$\text{(GC1)} \quad \frac{A \to \triangledown B}{\blacktriangle A \to B} \qquad\qquad \text{(GC2)} \quad \frac{\blacktriangle A \to B}{A \to \triangledown B}$$

*Institute of Mathematics, University of Silesia, ul. Bankowa 12, 40-007 Katowice, Poland (dzikw@silesia.top.pl)

†Department of Information Technology, University of Turku, FI-20014 Turku, Finland (Jouni.Jarvinen@utu.fi)

‡School of Information Environment, Tokyo Denki University, Inzai, 270-1382, Japan (kondo@sie.dendai.ac.jp)

It is proved that ILGC is with respect to provability equivalent to the minimal tense logic K_t. This means that ILGC can be viewed as a simple formulation of K_t.

In this paper, intuitionistic logic with Galois connections (IntGC) is defined by extending intuitionistic propositional logic with rules (GC1) and (GC2) and connectives $\blacktriangle$ and $\triangledown$ which appear to be modal.

IntGC is weaker than ILGC, since IntGC has the same inference rules, but the base logic is weaker than the one for ILGC. Galois connections are studied in an environment of intuitionistic logic in which none of the connectives is definable by the other connectives. Completeness of IntGC with respect to both algebraic semantics and Kripke semantics as well as the finite model property and decidability are proved. Duality between the two kinds of semantics leads to the representation theorem. The Kripke semantics provide applications to rough L-sets with upper and lower approximations in which the Kripke-frame accessibility relation coincides with the indiscernibility relation for rough sets.

The paper is presented as follows: The second section introduces the logic IntGC. We define its language and syntax, and present some essential provable formulas and rules. In Section 3, we introduce HGC-algebras, which are Heyting algebras equipped with an order-preserving Galois connection. We define an algebraic semantics for IntGC in terms of HGC-algebras and prove the completeness of IntGC with respect to the semantics in Section 4. We also show that IntGC is conservative over intuitionistic logic, in other words, the formulas of intuitionistic logic are provable in IntGC if and only if they are provable in intuitionistic logic. We also show that the so-called Glivenko's Theorem does not hold between ILGC and IntGC – this is done by providing such a formula A that it is provable in ILGC, but $\neg\neg A$ is not provable in IntGC. Section 5 is devoted to Kripke-frames. We introduce a Kripke-style semantics for our logic and prove its Kripke-completeness. We also use the method known as unravelling to show that IntGC is complete with respect to irreflexive frames. In the other words, IntGC does not distinguish irreflexive relations from all relations. In Section 6, we prove that IntGC has the so-called finite model property. This implies that every IntGC-formula is provable if and only if it is valid in every finite IntGC-model. Because IntGC is finitely axiomatized and it has the finite model property, IntGC is decidable. A couple of representation theorems is given in Section 7 stating that every HGC-algebra is embeddable into the complex algebra of the canonical frame, and that every Kripke-frame can be embedded into the complex algebra of its canonical frame. In addition, in this section an application to rough L-valued sets is given. Finally, in Section 8 some conditions related to finiteness are presented. In particular, we prove that an IntGC-formula A is provable if and only if A is valid in any finite distributive lattice with an additive and normal operator, and this is equivalent to the validity of A in any finite Kripke-model.

2 Logic IntGC

In this section, we define the language and the syntax of the logic IntGC. IntGC is suitable for dealing with, for instance, rough L-valued sets. Note that IntGC is a generalization of ILGC [8], because IntGC has the same inference rules (GC1) and (GC2) for modal connectives, but the base logic is weaker than the one of ILGC.

Let P be an enumerable set, whose elements are called *propositional variables*. The set of *connectives* consists of logical symbols $\neg$, $\rightarrow$, $\vee$, $\wedge$, $\blacktriangle$, and $\triangledown$. *Formulas* of IntGC are defined inductively as follows:

(i) Every propositional variable is a formula.

(ii) If A and B are formulas, then so are $\neg A$, $A \rightarrow B$, $A \vee B$, $A \wedge B$, $\blacktriangle A$, and $\triangledown A$.

We agree that in formulas implication has weaker precedence than conjunction and disjunction, which in turn have weaker precedence than $\neg$ and the modal connectives $\blacktriangle$ and $\triangledown$.

Let us denote by Φ the set of all IntGC-formulas. The logical system IntGC has the following 11 axioms of the intuitionistic logic (see [13, p. 379], for instance).

(Ax1) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

(Ax2) $A \rightarrow A \vee B$

(Ax3) $B \rightarrow A \vee B$

(Ax4) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$

(Ax5) $A \wedge B \rightarrow A$

(Ax6) $A \wedge B \rightarrow B$

(Ax7) $(C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow A \wedge B))$

(Ax8) $(A \rightarrow (B \rightarrow C)) \rightarrow (A \wedge B \rightarrow C)$

(Ax9) $(A \wedge B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$

(Ax10) $A \wedge \neg A \rightarrow B$

(Ax11) $(A \rightarrow A \wedge \neg A) \rightarrow \neg A$

As the intuitionistic logic, IntGC has modus ponens as its inference rule:

(MP) $$\frac{A \qquad A \rightarrow B}{B}$$

In addition, IntGC has two rules

(GC1) $$\frac{A \rightarrow \triangledown B}{\blacktriangle A \rightarrow B}$$ (GC2) $$\frac{\blacktriangle A \rightarrow B}{A \rightarrow \triangledown B}$$

mimicking the definition of Galois connections.

An IntGC-formula A is said to be *provable* in IntGC, if there is a finite sequence $A_1, A_2, \ldots, A_n$ of IntGC-formulas such that $A = A_n$ and for every $1 \leq i \leq n$:

(i) either A_i is an axiom of IntGC

(ii) or A_i is the conclusion of some inference rules, whose premises are in the set $\{A_1, \ldots, A_{i-1}\}$.

That A is provable in IntGC is denoted by $\vdash A$.

Several provable formulas of intuitionistic logic can be found in [13, p. 388] – these formulas are naturally provable also in IntGC. In the next proposition, we give some essential provable formulas and inference rules of IntGC involving the modal connectives $\blacktriangle$ and $\triangledown$. Let p be some fixed propositional variable. Then, we may define the constant *true* by setting

$$\top := p \to p$$

for some fixed propositional variable $p \in P$. The constant *false* is defined by

$$\bot := \neg\top.$$

It is obvious that $A \to \top$ and $\bot \to A$ for all $A \in \Phi$.

Proposition 2.1. *For all* IntGC-*formulas* $A, B \in \Phi$, *the following assertions hold.*

(i) $\dfrac{A \to B}{\triangledown A \to \triangledown B}$ *and* $\dfrac{A \to B}{\blacktriangle A \to \blacktriangle B}$.

(ii) $\vdash A \to \triangledown\blacktriangle A$ *and* $\vdash \blacktriangle\triangledown A \to A$.

(iii) $\vdash \blacktriangle\triangledown\blacktriangle A \to \blacktriangle A$ *and* $\vdash \blacktriangle A \to \blacktriangle\triangledown\blacktriangle A$.

(iv) $\vdash \triangledown\blacktriangle\triangledown A \to \triangledown A$ *and* $\vdash \triangledown A \to \triangledown\blacktriangle\triangledown A$.

(v) $\vdash \top \to \triangledown\top$ *and* $\blacktriangle\bot \to \bot$.

(vi) $\dfrac{A}{\triangledown A}$.

(vii) $\vdash \triangledown(A \wedge B) \to \triangledown A \wedge \triangledown B$ *and* $\vdash \triangledown A \wedge \triangledown B \to \triangledown(A \wedge B)$.

(viii) $\vdash \blacktriangle(A \vee B) \to \blacktriangle A \vee \blacktriangle B$ *and* $\vdash \blacktriangle A \vee \blacktriangle B \to \blacktriangle(A \vee B)$.

(ix) $\vdash \triangledown(A \to B) \to (\triangledown A \to \triangledown B)$.

Proof. (i) Suppose $\vdash A \to B$. Since $\vdash \triangledown A \to \triangledown A$ holds trivially, we have $\vdash \blacktriangle\triangledown A \to A$ by (GC1). Hence, by (Ax1) and (MP), $\vdash \blacktriangle\triangledown A \to B$. This gives $\vdash \triangledown A \to \triangledown B$ by (GC2). For the other part, suppose that $\vdash A \to B$. Since $\vdash \blacktriangle B \to \blacktriangle B$, we have $\vdash B \to \triangledown\blacktriangle B$ by (GC2). Thus, $\vdash A \to \triangledown\blacktriangle B$. This implies $\vdash \blacktriangle A \to \blacktriangle B$ by (GC1).

(ii) Because $\vdash \blacktriangle A \to \blacktriangle A$, we have $\vdash A \to \triangledown\blacktriangle A$ by (GC2). Similarly, $\vdash \triangledown A \to \triangledown A$ gives $\vdash \blacktriangle\triangledown A \to A$ by (GC1).

(iii) From (ii), $\vdash \blacktriangle\triangledown\blacktriangle A \to \blacktriangle A$ follows directly. On the other hand, $\vdash A \to \triangledown\blacktriangle A$ implies $\vdash \blacktriangle A \to \blacktriangle\triangledown\blacktriangle A$. Claim (iv) can be proved in a similar manner.

(v) It is clear that $\vdash \blacktriangle\top \to \top$. This implies $\vdash \top \to \triangledown\top$ by (GC2). Analogously, $\vdash \bot \to \triangledown\bot$ yields $\vdash \blacktriangle\bot \to \bot$ by (GC1).

(vi) Assume $\vdash A$. This means $\vdash \top \to A$ and we get $\vdash \triangledown\top \to \triangledown A$ by (i). By (v), we have $\vdash \top \to \triangledown\top$. Thus, $\vdash \top \to \triangledown A$ and $\vdash \triangledown A$.

(vii) Since $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, we have $\vdash \triangledown(A \wedge B) \to \triangledown A$ and $\vdash \triangledown(A \wedge B) \to \triangledown B$. Hence, $\vdash \triangledown(A \wedge B) \to \triangledown A \wedge \triangledown B$ by (Ax7) and (MP). On the other hand, $\vdash \triangledown A \wedge \triangledown B \to \triangledown A$ yields $\vdash \blacktriangle(\triangledown A \wedge \triangledown B) \to A$ by (GC1). Similarly, we may show $\vdash \blacktriangle(\triangledown A \wedge \triangledown B) \to B$. These give $\vdash \blacktriangle(\triangledown A \wedge \triangledown B) \to A \wedge B$. Therefore, $\vdash \triangledown A \wedge \triangledown B \to \triangledown(A \wedge B)$ by (GC2).

(viii) Because $\vdash A \to A \vee B$ and $\vdash B \to A \vee B$, we have $\vdash \blacktriangle A \to \blacktriangle(A \vee B)$ and $\vdash \blacktriangle B \to \blacktriangle(A \vee B)$. These give $\vdash \blacktriangle A \vee \blacktriangle B \to \blacktriangle(A \vee B)$ by (Ax4) and (MP). Furthermore, $\vdash \blacktriangle A \to \blacktriangle A \vee \blacktriangle B$ and $\vdash \blacktriangle B \to \blacktriangle A \vee \blacktriangle B$ imply $\vdash A \to \triangledown(\blacktriangle A \vee \blacktriangle B)$ and $\vdash B \to \triangledown(\blacktriangle A \vee \blacktriangle B)$. We get $\vdash A \vee B \to \triangledown(\blacktriangle A \vee \blacktriangle B)$, from which $\vdash \blacktriangle(A \vee B) \to \blacktriangle A \vee \blacktriangle B$ follows.

(ix) Since $\vdash A \wedge (A \to B) \to B$, we have $\vdash \triangledown(A \wedge (A \to B)) \to \triangledown B$. Furthermore, by (vii), we obtain $\vdash \triangledown A \wedge \triangledown(A \to B) \to \triangledown(A \wedge (A \to B))$. Thus, $\vdash \triangledown A \wedge \triangledown(A \to B) \to \triangledown B$, which is equivalent to $\vdash \triangledown(A \to B) \to (\triangledown A \to \triangledown B)$. □

Note that by (vii) and (viii), $\triangledown$ and $\blacktriangle$ are *modal connectives* of necessity and possibility, respectively, but the duality between them does not hold; moreover (vi) says that the necessitation rule is admissible in IntGC.

3 HGC-algebras

In this section, we introduce HGC-algebras, which will be used to define an algebraic semantics for IntGC. An HGC-*algebra* $(L, \wedge, \vee, \to, f, g, 0, 1)$ is an algebra of type $(2, 2, 2, 1, 1, 0, 0)$ such that

(i) $(L, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra, that is, L is a relatively pseudocomplemented lattice with the least element 0. In particular, for all $x, y, z \in L$,

$$x \leq y \to z \text{ if and only if } x \wedge y \leq z.$$

The greatest element of L is $1 = x \to x$.

(ii) For all $x, y \in L$, $f(x) \leq y$ if and only if $x \leq g(y)$.

In other words, HGC-algebras are Heyting algebras equipped with an order-preserving Galois connection. Notice that in [13] Heyting algebras are considered under the name *pseudo-Boolean algebras*.

In any Heyting algebra each element x has a pseudocomplement

$$\neg x = x \to 0.$$

Therefore, an HGC-algebra can be considered also as an algebra

$$(L, \wedge, \vee, \to, \neg, f, g, 0, 1).$$

Note also that $f(x) \to y = 1$ if and only if $x \to g(y) = 1$.

Next we present some properties of HGC-algebras. First we recall the following well-known lemma (see [5], for example).

Lemma 3.1. *Let $f \colon P \to Q$ and $g \colon Q \to P$ be maps between ordered sets P and Q. The pair (f, g) is a Galois connection if and only if*

(i) *$p \leq gf(p)$ for all $p \in P$ and $fg(q) \leq q$ for all $q \in Q$;*

(ii) *the maps f and g are order-preserving.*

We show that the class of HGC-algebras is equational. Equational axioms for Heyting algebras can be found in [1, 3]. Let us introduce the following equations:

(HGC1) $f(x \vee y) = f(x) \vee f(y)$, that is, f is *additive*;

(HGC2) $g(x \wedge y) = g(x) \wedge g(y)$, that is, g is *multiplicative*;

(HGC3) $x \leq gf(x)$;

(HGC4) $fg(x) \leq x$.

Note that (HGC3) and (HGC4) have the equivalent forms $x \to gf(x) = 1$ and $fg(x) \to x = 1$, respectively. Now we can write the following lemma.

Lemma 3.2. *All* HGC-*algebras form an equational class, that is, an algebra* $(L, \wedge, \vee, \to, f, g, 0, 1)$ *is an* HGC-*algebra if and only if it satisfies the axioms for Heyting algebras and* (HGC1)–(HGC4).

Proof. Let us suppose that $(L, \wedge, \vee, \to, f, g, 0, 1)$ is an HGC-algebra. We will show that (HGC1)–(HGC4) hold. By Lemma 3.1(i), (HGC3) and (HGC4) are satisfied. By Lemma 3.1(ii), f and g are order-preserving. This implies that $f(x) \vee f(y) \leq f(x \vee y)$. In addition, if z is an upper bound of $f(x)$ and $f(y)$, then $x \leq gf(x) \leq g(z)$ and $y \leq gf(y) \leq g(z)$. These imply $x \vee y \leq g(z)$

and $f(x \vee y) \leq z$. Thus, $f(x \vee y) = f(x) \vee f(y)$ and (HGC1) holds. In an analogous manner, we can show that also (HGC2) is satisfied.

Conversely, suppose that (HGC1)–(HGC4) hold. Due to Lemma 3.1, it is enough to show that f and g are order-preserving. Assume $x \leq y$. Then $f(x) \vee f(y) = f(x \vee y) = f(y)$, that is, $f(x) \leq f(y)$. Similarly, $g(x) \wedge g(y) = g(x \wedge y) = g(x)$, which is equivalent to $g(x) \leq g(y)$. $\square$

Because all HGC-algebras form an equational class, by the famous theorem of Birkhoff, this class is a variety (see e.g. [3] for further details).

It is also easy to see that in any HGC-algebra, $f(0) = 0$ and $g(1) = 1$, that is, f is *normal* and g is *co-normal*. This is because $0 \leq g(0)$ implies $f(0) \leq 0$, and $f(1) \leq 1$ implies $1 \leq g(1)$. Therefore, in an HGC-algebra $(L, \wedge, \vee, \rightarrow, f, g, 0, 1)$, the map f can be viewed as an *operator* in the sense of [10]. However, in Heyting algebras the duality that holds in Boolean algebras does not hold. More precisely, if we define the maps f^d and g^d by setting

$$f^d(x) = \neg f(\neg x) \quad \text{and} \quad g^d(x) = \neg g(\neg x), \tag{Dual}$$

then the maps f^d and g^d do not form a Galois connection, as is the case of Boolean lattices. However, the following lemma gives some partial results. Let us denote

$$S(L) = \{\neg x \mid x \in L\}.$$

Lemma 3.3. *Let (f, g) be a Galois connection on an* HGC*-algebra L, and let the maps f^d and g^d be defined as in* (Dual)*. Then for all $x, y \in L$ and $a, b \in S(L)$,*

(i) $x \leq y$ *implies* $f^d(x) \leq f^d(y)$.

(ii) $x \leq y$ *implies* $g^d(x) \leq g^d(y)$.

(iii) $g^d(f^d(x)) \leq \neg\neg x$.

(iv) $a \leq f^d(b)$ *implies* $g^d(a) \leq b$.

(v) $g^d(f^d(a)) \leq a$.

Proof. (i) If $x \leq y$, then $\neg x \geq \neg y$. This implies that $f(\neg x) \geq f(\neg y)$ and $f^d(x) = \neg f(\neg x) \leq \neg f(\neg y) = f^d(y)$. Case (ii) can be proved analogously.

(iii) $g^d(f^d(x)) = \neg g(\neg\neg f(\neg x)) \leq \neg g(f(\neg x)) \leq \neg\neg x$.

(iv) Let $a = \neg x$ and $b = \neg y$, and assume that $\neg x \leq f^d(\neg y) = \neg f(\neg\neg y)$. Then, $f(\neg\neg y) \leq \neg\neg f(\neg\neg y) \leq \neg\neg x$, which implies $\neg\neg y \leq g(\neg\neg x)$ and $g^d(\neg x) = \neg g(\neg\neg x) \leq \neg y$.

Claim (v) follows directly from (iii). $\square$

4 Algebraic Semantics and Completeness

Let $(L, \wedge, \vee, \rightarrow, \neg, f, g, 0, 1)$ be an HGC-algebra. Recall that we denote by P the set of all propositional variables. Let v be a function $v \colon P \rightarrow L$ assigning to each propositional variable p in P an element $v(p)$ of the lattice L. Such functions are called *valuations*. The valuation v can be extended uniquely to the set Φ of all IntGC-formulas inductively by the following way:

$$\begin{aligned} v(A \wedge B) &= v(A) \wedge v(B) & \qquad v(A \vee B) &= v(A) \vee v(B) \\ v(A \rightarrow B) &= v(A) \rightarrow v(B) & v(\neg A) &= \neg v(A) \\ v(\blacktriangle A) &= f(v(A)) & v(\triangledown A) &= g(v(A)) \end{aligned}$$

We say that an IntGC-formula $A \in \Phi$ is *valid* if $v(A) = 1$ for any valuation $v \colon \Phi \rightarrow L$ on any HGC-algebra $(L, \wedge, \vee, \rightarrow, \neg, f, g, 0, 1)$.

Theorem 4.1 (**Soundness I**)**.** *Every provable* IntGC*-formula is valid.*

Proof. The proof concerning the axioms of intuitionistic logic is standard (see e.g. [13]). We have to only show that the rules (GC1) and (GC2) preserve validity. Let v be a valuation from the set of all formulas Φ to some HGC-algebra L. Then, for all $A, B \in \Phi$,

$$\begin{aligned} v(A \rightarrow \triangledown B) = 1 &\iff v(A) \rightarrow v(\triangledown B) = 1 \iff v(A) \leq f(v(B)) \\ &\iff g(v(A)) \leq v(B) \iff v(\blacktriangle A) \rightarrow v(B) = 1 \\ &\iff v(\blacktriangle A \rightarrow B) = 1. \end{aligned}$$

□

To obtain completeness, we show that the valid formulas are provable by applying so-called Lindenbaum–Tarski algebras (see [13], for instance). First, we define an equivalence relation $\equiv$ on the set Φ:

$$A \equiv B \iff \vdash A \rightarrow B \text{ and } \vdash B \rightarrow A.$$

Lemma 4.2. *The equivalence relation $\equiv$ is a congruence relation on the formula algebra.*

Proof. It suffices to consider only the connectives $\blacktriangle$ and $\triangledown$. Let $A \equiv B$. Then $\vdash A \rightarrow B$ and $\vdash B \rightarrow A$, which give $\vdash \blacktriangle A \rightarrow \blacktriangle B$ and $\vdash \blacktriangle B \rightarrow \blacktriangle A$ by Proposition 2.1(i). Hence, $\blacktriangle A \equiv \blacktriangle B$. Analogously, we can show $\triangledown A \equiv \triangledown B$. □

For an IntGC-formula $A \in \Phi$, we denote by $[A]$ the equivalence class of A, that is,

$$[A] = \{B \in \Phi \mid A \equiv B\}.$$

The set of all Φ-classes $\{\,[A] \mid A \in \Phi\}$ is denoted by $\Phi/{\equiv}$.

Because the equivalence $\equiv$ is a congruence on the set Φ, we may now define its *quotient algebra* (cf. [3]) by introducing the following operations on the quotient set $\Phi/{\equiv}$. For $A, B \in \Phi$, we set:

$$\begin{array}{rcl} [A] \vee [B] = [A \vee B] & \qquad & [A] \wedge [B] = [A \wedge B] \\ [A] \to [B] = [A \to B] & & \neg[A] = [\neg A] \\ f([A]) = [\blacktriangle A] & & g([A]) = [\triangledown A] \\ \mathbf{0} = [\bot] & & \mathbf{1} = [\top]. \end{array}$$

Since the class of HGC-algebras is equational by Proposition 3.2 and the relation $\equiv$ is a congruence by Lemma 4.2, the quotient algebra is an HGC-algebra (see e.g. [2] for details) as stated in the following proposition.

Proposition 4.3. *The quotient algebra* $(\Phi/\equiv, \vee, \wedge, \to, \neg, \mathbf{0}, \mathbf{1}, f, g)$ *is an* HGC-*algebra.*

The HGC-algebra $(\Phi/{\equiv}, \vee, \wedge, \to, \neg, \mathbf{0}, \mathbf{1}, f, g)$ is referred to the *Lindenbaum–Tarski HGC-algebra.* We may now define a valuation $v^* \colon P \to \Phi/{\equiv}$ by setting

$$v^*(p) = [p].$$

It can be easily verified by a straightforward formula induction that we have

$$v^*(A) = [A]$$

for all IntGC-formulas $A \in \Phi$.

Lemma 4.4. *For any* IntGC-*formula* $A \in \Phi$*,*

$$\vdash A \iff v^*(A) = \mathbf{1}.$$

Proof. If $\vdash A$, then $\vdash \top \to A$. The inverse, $\vdash A \to \top$, holds always. Thus, $A \equiv \top$ and $v^*(A) = [A] = [\top] = \mathbf{1}$. Conversely, if $v^*(A) = [A] = \mathbf{1}$, then $\vdash \top \to A$. Hence, $\vdash A$. □

The next theorem shows the algebraic completeness of IntGC.

Theorem 4.5 (Completeness I). *An* IntGC-*formula is provable if and only if it is valid.*

Proof. Suppose that A is valid. Then $v(A) = 1$ for every valuation v on any HGC-algebra. In particular, we have $v^*(A) = \mathbf{1}$ in the Lindenbaum–Tarski HGC-algebra. From Lemma 4.4 we obtain that A must be provable. The other direction is already proved (Theorem 4.1). □

We end this section by proving that IntGC is *conservative* over intuitionistic logic Int, and by showing that so-called *Glivenko's Theorem* does not hold between ILGC and IntGC. The theorem states that an arbitrary propositional formula A is classically provable if and only if $\neg\neg A$ is intuitionistically provable.

We start by recalling ILGC [8] which is an extension of classical propositional logic by the rules of (GC1) and (GC2). Of course, the language of the logic has $\triangledown$ and $\blacktriangle$ as additional logical symbols.

Let Φ^* denote the set of all Int-formulas, that is, the set Φ^* is the subset of IntGC-formulas Φ not containing the symbols $\blacktriangle$ and $\triangledown$. Obviously, Φ^* is the set of all formulas of intuitionistic propositional logic Int. For any Int-formula $A \in \Phi^*$, $\vdash_{\mathrm{Int}} A$ denotes that A is provable in Int. This means that there is a proof of A that uses the axioms (Ax1)–(Ax11) and the modus ponens only.

Proposition 4.6 (Conservativeness I). IntGC *is conservative over intuitionistic logic, that is, for any* Int-*formula* $A \in \Phi^*$,

$$\vdash A \quad \textit{if and only if} \quad \vdash_{\mathrm{Int}} A.$$

Proof. Let $A \in \Phi^*$ and suppose that $\nvdash_{\mathrm{Int}} A$. Since intuitionistic logic Int is known to be complete with respect to Heyting algebra semantics, there exists a Heyting algebra $(L, \wedge, \vee, \to, 0, 1)$ and a valuation $v^* \colon \Phi^* \to L$ such that $v^*(A) \neq 1$. By setting $f(x) = x$ and $g(x) = x$ for any $x \in L$, we may expand this algebra into a HGC-algebra $(L, \wedge, \vee, \to, 0, 1, f, g)$. Also the valuation v^* can be extended to a valuation $v \colon \Phi \to L$ by setting $v(\triangledown p) = v^*(p)$ and $v(\blacktriangle p) = v^*(p)$ for all $p \in P$. We have $v(A) \neq 1$, which means that $\nvdash A$ by Theorem 4.5.

The converse is trivial. $\square$

By a similar technique we can prove the following result.

Corollary 4.7 (Conservativeness II). ILGC *is conservative over the classical propositional logic.*

As a corollary of Proposition 4.6 and Corollary 4.7, we obtain that the classical propositional logic with Galois Connections (ILGC) is a particular axiomatic extension of IntGC. Namely, it is shown in [13, p. 410] that the classical propositional logic can be obtained from intuitionistic logic by adding to axioms (Ax1)–(Ax11) the axiom schema $A \vee \neg A$. Similar results can be proved also for the schemata $\neg\neg A \to A$ and $((A \to B) \to A) \to A$.

Corollary 4.8. ILGC *can be obtained by inserting to the set of* IntGC-*axioms any of the axioms:*

(i) $A \vee \neg A$

(ii) $\neg\neg A \to A$

(iii) $((A \to B) \to A) \to A$

We conclude this section by showing that so-called *Glivenko's Theorem* does not hold between ILGC and IntGC. Let A be a formula of ILGC of the form $\triangledown(p \vee \neg p)$, where p a propositional variable. It is clear that A is provable in ILGC, because $p \vee \neg p$ is provable in the classical propositional logic. On the other hand, the formula $\neg\neg A$ is not IntGC-provable, because we can construct a counter model for it.

Let $(\{0, a, 1\}, \wedge, \vee, \neg, 0, 1)$ be the 3-element Heyting algebra with the order $0 < a < 1$. Then, the joins and the meets are given by the maximums and by the minimums. In addition, $\neg 0 = 1$, $\neg 1 = 0$ and $\neg a = 0$. If we define two maps f and g by setting

$$f(0) = 0, f(a) = 1, f(1) = 1, g(0) = 0, g(a) = 0, g(1) = 1,$$

then this algebra is an HGC-algebra.

If we define a valuation v on the algebra such that $v(p) = a$, then the value of $\neg\neg A = \neg\neg\triangledown(p \vee \neg p)$ is

$$\begin{aligned} v(\neg\neg A) &= v(\neg\neg\triangledown(p \vee \neg p)) = \neg\neg v(\triangledown(p \vee \neg p)) = \neg\neg g(v(p \vee \neg p)) \\ &= \neg\neg g(v(p) \vee v(\neg p)) = \neg\neg g(v(p) \vee \neg v(p)) = \neg\neg g(a \vee \neg a) \\ &= \neg\neg g(a) = \neg\neg 0 = 0 \neq 1. \end{aligned}$$

This means that the formula $\neg\neg A$ is not provable in IntGC by the completeness theorem of IntGC. Therefore, there exists an IntGC-formula A which is provable in ILGC, but $\neg\neg A$ is not provable in IntGC and so the Glivenko's theorem does not hold.

5 Kripke-Semantics and Completeness

At first, we define Kripke-frames for IntGC. A structure $(X, \leq, R)$ is called a *Kripke-frame*, if X is a non-empty set, $\leq$ is a preorder on X, and R is a relation on X such that

$$x \leq x', x\, R\, y \text{ and } y' \leq y \text{ imply } x'\, R\, y' \tag{CR}$$

In details, a Kripke-frame can be viewed as a structure $(X, \leq, R, R^{-1})$, where $(X, \leq)$ is a frame for intuitionistic logic, R is the accessibility relation corresponding to $\blacktriangle$, R^{-1} is the accessibility relation of $\triangledown$, and condition (CR) holds. In this paper, we use $(X, \leq, R)$ for brevity.

A map $\xi\colon P \times X \to \{0,1\}$ is called an *assignment* on the Kripke-frame $(X, \leq, R)$, where P is the set of all propositional variables. It can be extended to the map $\xi^*\colon \Phi \times X \to \{0,1\}$ as follows:

$$
\begin{aligned}
\xi^*(A \wedge B, x) = 1 &\iff \xi^*(A, x) = \xi^*(B, x) = 1 \\
\xi^*(A \vee B, x) = 1 &\iff \xi^*(A, x) = 1 \ \text{ or } \ \xi^*(B, x) = 1 \\
\xi^*(A \to B, x) = 1 &\iff (\forall y \in X)\ x \leq y \ \text{ implies } \ \xi^*(A, y) \leq \xi^*(B, y) \\
\xi^*(\neg A, x) = 1 &\iff (\forall y \in X)\ x \leq y \ \ \xi^*(A, y) = 0 \\
\xi^*(\blacktriangle A, x) = 1 &\iff (\exists y \in X)\ x\, R\, y \ \text{ and } \ \xi^*(A, y) = 1 \\
\xi^*(\triangledown A, x) = 1 &\iff (\forall y \in X)\ y\, R\, x \ \text{ implies } \ \xi^*(A, y) = 1
\end{aligned}
$$

In the sequel, we will use the symbol ξ also for the extended map ξ^* for the sake of simplicity. For a Kripke-frame $(X, \leq, R)$ and an assignment ξ on, the structure $(X, \leq, R, \xi)$ is called a *Kripke-model*.

An IntGC-formula A is *valid in a Kripke-model* $(X, \leq, R, \xi)$, if $\xi(A, x) = 1$ for all $x \in X$. The formula A is said to be *valid in a Kripke-frame* $(X, \leq, R)$ if A is valid in every model based on this frame and A is *Kripke-valid* if A is valid in every Kripke-frame.

To prove the soundness of IntGC by Kripke-models is straightforward, so we may present the following theorem.

Theorem 5.1 (**Soundness II**). *Every provable* IntGC*-formula is Kripke-valid.*

To prove the completeness theorem by Kripke-models, we use the completeness theorem of IntGC with respect to HGC-algebras and canonical frames. In Section 4 we showed that an IntGC-formula A is provable in IntGC if and only if $v(A) = 1$ for any valuation v on any HGC-algebra.

Given an HGC-algebra $(L, \wedge, \vee, \to, 0, 1, f, g)$, the *canonical frame* of L is a Kripke-frame $(X(L), \leq, R)$ defined as follows:

(Kr1) $X(L)$ is the set of all prime filters of L; recall that a *prime filter* is a proper filter F such that $a \vee b \in F$ implies $a \in F$ or $b \in F$.

(Kr2) For all $x, y \in X(L)$, $x \leq y$ if and only if $x \subseteq y$.

(Kr3) For all $x, y \in X(L)$, the relation R is defined by

$$x\, R\, y \iff (\forall a \in L)\ a \in y \ \text{ implies } \ f(a) \in x.$$

Suppose that A is not provable. It is sufficient to construct a Kripke-model in which A is not Kripke-valid. Because A is not provable, there exists an HGC-algebra $(L, \wedge, \vee, \to, 0, 1, f, g)$ and a valuation v on it such that $v(A) \neq 1$. For the HGC-algebra L and the valuation v, we construct a Kripke-model $(X(L), \leq, R, \xi)$ by defining the assignment ξ on the canonical frame $(X(L), \leq, R)$ of L as follows:

(Kr4) $\xi(A,x)=1$ if and only if $v(A)\in x$.

We first proof the following simple lemmas.

Lemma 5.2. *For all $x,y\in X(L)$,*

$$y\,R\,x \iff (\forall a\in L)\ g(a)\in x \ \ \textit{implies}\ \ a\in y.$$

Proof. Suppose that $y\,R\,x$. By definition, this means that $a\in x$ implies $f(a)\in y$ for all $a\in L$. If $g(a)\in x$, then $fg(a)\in y$. Because $fg(a)\le a$ and y is a filter, we get $a\in y$.

Conversely, assume that for all $a\in L$, $g(a)\in x$ implies $a\in y$. Suppose $a\in x$. Now $a\le gf(a)$ gives $gf(a)\in x$ and $f(a)\in y$. Thus, $y\,R\,x$. □

Lemma 5.3. $(X(L),\le,R)$ *is a Kripke-frame.*

Proof. Because $\le$ is the same as $\subseteq$, $\le$ is a preorder. Suppose that $x\le x'$, $x\,R\,y$, and $y'\le y$. For all $a\in L$,

$$\begin{aligned} a\in y' &\Longrightarrow a\in y && \text{(by } y'\subseteq y)\\ &\Longrightarrow f(a)\in x && \text{(by } x\,R\,y)\\ &\Longrightarrow f(a)\in x' && \text{(by } x\subseteq x') \end{aligned}$$

This means that $x'\,R\,y'$ and also condition (CR) holds. □

Note that if x is a filter, then for all $a,b\in L$, $a\in x$ and $a\to b\in x$ imply $b\in x$. Namely, if $a\in x$ and $a\to b\in x$, then $a\wedge(a\to b)\in x$. Because L is a Heyting algebra, $a\wedge(a\to b)\le b$ and hence $b\in x$.

The following lemma plays an important role to prove the completeness theorem of Kripke-models.

Lemma 5.4 (Prime Filter Theorem). *Let u be a filter and $a\in L$. If $a\notin u$, then there exists a prime filter x such that $u\subseteq x$ and $a\notin x$.*

Proof. Let $\Gamma=\{v\mid v \text{ is a filter}, u\subseteq v, \text{ and } a\notin v\}$. Since $u\in\Gamma$, Γ is not empty. For every chain $\{v_i\}_{i\in I}$ in Γ, it is easy to show that $\bigcup_{i\in I}v_i\in\Gamma$. By Zorn's Lemma, there is a maximal element $x\in\Gamma$. Thus, x is a filter, $u\subseteq x$, and $a\notin x$. We claim that x is a prime filter.

Suppose that $b\vee c\in x$ but $b,c\notin x$. Let us denote by x_b and x_c the filters generated by $x\cup\{b\}$ and $x\cup\{c\}$ (see e.g. [13, p. 45] for the notion). Since x is the maximal element in Γ, $x\subset x\cup\{b\}\subseteq x_b$, and $x\subset x\cup\{c\}\subseteq x_c$, we have that $a\in x_b$ and $a\in x_c$. Then, there are elements $h,k\in x$ such that $h\wedge b\le a$ and $k\wedge c\le a$. Since $h\wedge k\in x$, $b\vee c\in x$, and the HGC-algebra L is a distributive lattice, we obtain $(h\wedge k)\wedge(b\vee c)\in x$ and

$$(h\wedge k)\wedge(b\vee c)=(h\wedge k\wedge b)\vee(h\wedge k\wedge c)\le(h\wedge b)\vee(k\wedge c)\le a\vee a=a.$$

Therefore, $a\in x$, contradiction. This means that $b\in x$ or $c\in x$, and x is a prime filter. □

Lemma 5.5. *Let x be a filter. If $a \to b \notin x$, then there exists a prime filter y such that $x \subseteq y$ and $a \in y$, but $b \notin y$.*

Proof. Let $\Gamma = \{c \in L \mid a \to c \in x\}$. Since $a \to a = 1 \in x$, we have $a \in \Gamma$. We show that Γ is a filter. If $c \in \Gamma$ and $d \in \Gamma$, then $a \to c \in x$ and $a \to d \in x$. Then, $a \to (c \wedge d) = (a \to c) \wedge (a \to d) \in x$, which implies $c \wedge d \in \Gamma$. On the other hand, if $c \in \Gamma$ and $c \leq d$, then $a \to c \in x$ and $a \to c \leq a \to d$ imply $a \to d \in x$ and $d \in \Gamma$. Thus, Γ is a filter and $b \notin \Gamma$.

By the Prime Filter Theorem, there exists a prime filter y such that $\Gamma \subseteq y$ and $b \notin y$. If $c \in x$, then $c \to (a \to c) = 1 \in x$ and thus $a \to c \in x$. This implies $c \in \Gamma$ and $x \subseteq \Gamma \subseteq y$. Note that $a \in \Gamma$ gives $a \in y$. $\square$

A non-empty set $u \subseteq L$ is called a *co-filter* if $a \vee b \in u$ implies $a \in u$ or $b \in u$ for all $a, b \in L$. Notice that the term "co-filter" originate in the paper [11] by Orłowska and Rewitzky, but our co-filters are slightly weaker than theirs. Also the following lemma is adapted from [11].

Lemma 5.6. *Let x be a filter and u a co-filter such that $x \subseteq u$. Then there exists a prime filter y such that $x \subseteq y \subseteq u$.*

Proof. Let $\Gamma = \{z \mid z \text{ is a filter and } x \subseteq z \subseteq u\}$. By Zorn's Lemma, there is a maximal element $y \in \Gamma$. Thus, $x \subseteq y \subseteq u$ and y is a filter. We show that y is a prime filter.

Assume that there are elements $a, b \in L$ such that $a \vee b \in y$ but $a \notin y$ and $b \notin y$. Let y_a and y_b be the filters generated by the sets $y \cup \{a\}$ and $y \cup \{b\}$. Because y is maximal, $y \subset y_a$, and $y \subset y_b$, we get that $y_a \not\subseteq u$ and $y_b \not\subseteq u$. Therefore, there exist $c, d \in L$ such that $c \in y_a$ and $d \in y_b$, but $c \notin u$ and $d \notin u$. Since u is a co-filter, we must have $c \vee d \notin u$. On the other hand, since $c \in y_a$ and $d \in y_b$, there are elements $h, k \in y$ such that $h \wedge a \leq c$ and $k \wedge b \leq d$. Now

$$(h \wedge k) \wedge (a \vee b) = (h \wedge k \wedge a) \vee (h \wedge k \wedge b) \leq (h \wedge a) \vee (k \wedge b) \leq c \vee d.$$

Because $h \wedge k \in y$ and $a \vee b \in y$, we have $c \vee d \in y \subseteq u$, a contradiction. Therefore, $a \in y$ or $b \in y$, and y is a prime filter. $\square$

Now we can show completeness of IntGC in terms of Kripke-models $(X(L), \leq, R, \xi)$. The following lemma is essential.

Lemma 5.7 (Key Lemma). *For all $x \in X(L)$ and $A \in \Phi$, we have $\xi(A, x) = 1$ if and only if $v(A) \in x$.*

Proof. We only show the cases (i) $A \to B$ and (ii) $\blacktriangle A$ by induction on the length of a formula. The rest may be be proved in an analogous manner.

(i) Let $x \in X(L)$ be a prime filter. If $v(A \to B) = v(A) \to v(B) \notin x$, then by Lemma 5.5, there exists a prime filter $y \in X(L)$ such that $x \subseteq y$ and

$v(A) \in y$, but $v(B) \notin y$. By the induction hypothesis, we have $\xi(A, y) = 1$ and $\xi(B, y) = 0$. Since $x \subseteq y$, $x \leq y$ holds implying $\xi(A \to B, x) = 0$.

Conversely, suppose that $\xi(A \to B, x) = 0$. Then, there exists a prime filter $y \in X(L)$ such that $x \leq y$ and $\xi(A, y) = 1$, but $\xi(B, y) = 0$. By the induction hypothesis, we have $v(A) \in y$ and $v(B) \notin y$ for $x \subseteq y$. This implies $v(A \to B) = v(A) \to v(B) \notin y$, because $v(A) \to v(B) \in y$ would imply $v(A) \wedge (v(A) \to v(B)) \in y$ and $v(A) \wedge (v(A) \to v(B)) \leq v(B)$, that is, $v(B) \in y$. Since $v(A \to B) \notin y$, we have $v(A \to B) \notin x$.

(ii) If $\xi(\blacktriangle A, x) = 1$, then there exists $y \in X(L)$ such that $x \, R \, y$ and $\xi(A, y) = 1$. By the induction hypothesis, we have $v(A) \in y$. Since $x \, R \, y$, we also have $f(v(A)) = v(\blacktriangle A) \in x$.

For the converse, let us assume $v(\blacktriangle A) \in x$. Clearly, the principal filter ${\uparrow}v(A) = \{a \in L \mid v(A) \leq a\}$ is a filter. Also the preimage $f^{-1}(x) = \{b \in L \mid f(b) \in x\}$ is a co-filter, since x is a prime filter and the function f is additive. It is easy to see that ${\uparrow}v(A) \subseteq f^{-1}(x)$, because if $v(A) \leq a$, then $f(v(A)) = v(\blacktriangle A) \in x$, and $f(v(A)) \leq f(a)$ implies $f(a) \in x$, that is, $a \in f^{-1}(x)$. Therefore, by Lemma 5.6, there exists a prime filter $y \in X(L)$ such that ${\uparrow}v(A) \subseteq y \subseteq f^{-1}(x)$. The fact that $v(A) \in {\uparrow}v(A)$ implies $v(A) \in y$ and therefore $\xi(A, y) = 1$ by the induction hypothesis. We also have $x \, R \, y$, because $y \subseteq f^{-1}(x)$, that is, $a \in y$ implies $f(a) \in x$. Hence, $\xi(\blacktriangle A, x) = 1$. □

Now we have everything to present the completeness theorem.

Theorem 5.8 (Completeness II). *Every* IntGC*-formula is provable if and only if it is Kripke-valid.*

Proof. We have already noted that every provable IntGC-formula is Kripke-valid. On the other hand, if an IntGC-formula A is not provable, then there exists an HGC-algebra L and a valuation v such that $v(A) \neq 1$. From the algebra L and the valuation v, we construct the canonical frame $(X(L), \leq, R)$ and the Kripke-model $(X(L), \leq, R, \xi)$ as above. By the Principal Filter Theorem, there exists a prime filter $x \in X(L)$ such that $v(A) \notin x$. By the Key Lemma, we conclude that A is not Kripke-valid. □

We end this section by showing that for the completeness of IntGC we can restrict to the class of Kripke-models and -frames with irreflexive accessibility relation R. Recall that a binary relation on X is *irreflexive* if $(x, x) \notin R$ for all $x \in X$. Frames and models containing an irreflexive R are called similarly *irreflexive*. For the next proposition, we apply the method called *unravelling* [2].

Proposition 5.9. *An* IntGC*-formula is provable if and only if it is valid in every irreflexive Kripke-model.*

Proof. For every Kripke-model $(X, \leq, R, \xi)$, we construct a Kripke-model $(X^*, \leq^*, R^*, \xi^*)$ as follows. The set X^* is such that for each element $x \in X$, there are two distinct elements x^l and x^r in X^*. The relation $\leq^*$ is defined in X^* so that if $x \leq y$ holds in X, then $x' \leq^* y'$ holds for all $x' \in \{x^l, x^r\}$ and $y' \in \{y^l, y^r\}$. The relation $\leq^*$ is clearly a preorder.

The relation R^* is defined so that $x \, R \, x$ implies $x^l \, R^* \, x^r$ and $x^r \, R^* \, x^l$, and if $(x, x) \notin R$, then $(x^l, x^r) \notin R$ and $(x^r, x^l) \notin R$. Thus, $(x^l, x^l) \notin R^*$ and $(x^r, x^r) \notin R^*$ for all $x \in X$. For two distinct elements $x \neq y$ in X such that $x \, R \, y$, we set $x' \, R^* \, y'$ for all $x' \in \{x^l, x^r\}$ and $y' \in \{y^l, y^r\}$. Clearly, the relation R^* is irreflexive.

It is also straightforward to show that condition

$$x \leq^* x', x \, R^* \, y \text{ and } y' \leq^* y \text{ imply } x' \, R^* \, y'$$

is satisfied since (CR) holds for the relations $\leq$ and R. Thus, $(X^*, \leq^*, R^*)$ is a Kripke-frame such that R^* is irreflexive.

For any valuation $\xi \colon \Phi \times X \to \{0, 1\}$, we define the valuation $\xi^* \colon \Phi \times X^* \to \{0, 1\}$ by setting $\xi^*(A, x^l) = \xi^*(A, x^r) = \xi(A, x)$ for any $A \in \Phi$ and $x \in X$.

Now, if an IntGC-formula A is provable, then it is valid in every Kripke-model. This implies trivially that A is valid in all irreflexive Kripke-models. Conversely, suppose that A is not provable. Then there exists a Kripke-model $(X, \leq, R, \xi)$ such that $\xi(A, x) = 0$ for some $x \in X$. By the above construction, we have a Kripke-model $(X^*, \leq^*, R^*, \xi^*)$ with an irreflexive R^* such that $\xi^*(A, x^l) = \xi^*(A, x^r) = \xi(A, x) = 0$. Thus, A is not valid in this irreflexive frame. $\square$

Observe that if a Kripke-model $(X, \leq, R, \xi)$ is finite, then corresponding irreflexive Kripke-model $(X^*, \leq^*, R^*, \xi^*)$ is finite as well.

6 Finite Model Property and Decidability

In this short section, we show that the logic IntGC has the so-called *finite model property*, that is, for every formula A of IntGC, if A is not provable in IntGC, then there exists a finite counter Kripke-model.

Let A be an IntGC-formula which is not provable. By Theorem 5.8, there exists a Kripke-model $\mathcal{M} = (X, \leq, R, \xi)$ in which A is not valid.

Let Φ^* be the subset of all subformulas of A (cf. [2], for example). It is easy to show that Φ^* is closed under subformulas, that is, if $B \in \Phi^*$ and C is a subformula of B, then $C \in \Phi^*$. We define a relation $\sim$ on X by setting:

$$x \sim y \quad \text{if and only if} \quad (\forall B \in \Phi^*) \, \xi(B, x) = \xi(B, y).$$

Obviously, the relation $\sim$ is an equivalence relation. Let us write $[x] = \{y \in X \mid x \sim y\}$ and $X^* = \{[x] \mid x \in X\}$. We define the relations $\leq^*$ and R^*, and the valuation $\xi^*\colon \Phi^* \times X^* \to \{0,1\}$ as follows:

$$[x] \leq^* [y] \text{ iff } (\forall B \in \Phi^*)\, \xi(B,x) \leq \xi(B,y)$$

$$[x]\, R^*\, [y] \text{ iff } a\, R\, b \text{ for some } a \in [x] \text{ and } b \in [y]$$

$$\xi^*(A,[x]) = \begin{cases} \xi(A,x) & \text{if } A \in \Phi^* \\ 1 & \text{otherwise} \end{cases}$$

The Kripke-model $(X^*, \leq^*, R^*, \xi^*)$ is called a Φ^*-*filtration.* It is clear that since Φ^* is finite, also X^* is finite. Therefore, $(X^*, \leq^*, R^*, \xi^*)$ is a finite Kripke-model such that $x\, R\, y$ implies $[x]\, R^*\, [y]$.

Lemma 6.1. *For all $A \in \Phi^*$ and $x \in X$,*

$$\xi^*(A,[x]) = \xi(A,x).$$

Proof. (By induction on formulas) We consider the case $\triangledown B \in \Phi^*$ only. The other cases can be proved analogously.

Suppose that $\xi^*(\triangledown B,[x]) = 1$ for some $x \in X$. We will prove that $\xi(\triangledown B, x) = 1$. Assume that $y\, R\, x$. Then $[y]\, R^*\, [x]$, which implies $\xi^*(B,[y]) = 1$. This gives $\xi(B,y) = 1$ by the induction hypothesis. Hence, $\xi(\triangledown B, x) = 1$.

Conversely, assume $\xi(\triangledown B, x) = 1$ for some $x \in X$. We show that $\xi^*(\triangledown B,[x]) = 1$. If $[y]\, R^*\, [x]$, then there exist $a \in [x]$ and $b \in [y]$ such that $b\, R\, a$. Because $a \sim x$, we have $\xi(\triangledown B, a) = 1$. Since $b\, R\, a$, we obtain $\xi(B,b) = 1$. The fact $b \sim y$ gives $\xi(B,y) = 1$. By the induction hypothesis, we get $\xi^*(B,[y]) = 1$ yielding $\xi^*(\triangledown B,[x]) = 1$. $\square$

By applying the previous lemma, we may easily prove the following theorem.

Theorem 6.2. IntGC *has the finite model property.*

Proof. Suppose that an IntGC-formula $A \in \Phi$ is not provable. Then there exists a Kripke-model $(X, \leq, R, \xi)$ and $x \in X$ such that $\xi(A,x) = 0$. Lemma 6.1 means that also for the finite Φ^*-filtration $(X^*, \leq^*, R^*, \xi^*)$, $\xi^*(A,[x]) = 0$. $\square$

Since we have proved that an IntGC-formula A formula is provable if and only if A is valid in any Kripke-model, and that the logic IntGC has a finite model property, we can write also the following "stronger completeness theorem".

Corollary 6.3 (Completeness III). *Every* IntGC*-formula is provable if and only if it is valid in any finite Kripke-model.*

It is a well-known fact that if a logic is finitely axiomatized with the finite model property, then the logic is decidable.

Corollary 6.4. IntGC *is decidable.*

7 Representation Theorem for HGC-algebras and an Application to Rough L-Valued Sets

In this section, we show that each HGC-algebra is embeddable into an HGC-algebra of sets. More exactly, each HGC-algebra is embeddable into the complex algebra of the canonical frame of the HGC-algebra. This is a part of the discrete duality in the spirit of Orłowska and Rewitzky [11].

An *Alexandrov topology* is a topology $\mathcal{T}$ that contains also all arbitrary intersections of its members. Every Alexandrov topology $\mathcal{T}$ has the property that each point $x \in X$ has a *smallest neighbourhood* $N_{\mathcal{T}}(x) = \bigcap\{Y \in \mathcal{T} \mid x \in Y\}$. This means that $N_{\mathcal{T}}(x)$ is the smallest set in the topology $\mathcal{T}$ containing the point x.

There is a close connection between preorders and Alexandrov topologies. Let $\leq$ be a preorder on a set X. We may now define an Alexandrov topology $\mathcal{T}_{\leq}$ on X consisting of all upward-closed subsets of X with respect to the relation $\leq$, that is,

$$\mathcal{T}_{\leq} = \{A \subseteq X \mid (\forall x, y \in X)\, x \in A \ \ \& \ \ x \leq y \Longrightarrow y \in A\}.$$

On the other hand, let us denote for any $x \in X$, the *principal up-set* of x (cf. [4]) by ${\uparrow}x = \{y \in X \mid x \leq y\}$. The set ${\uparrow}x$ is the smallest neighbourhood of the point x in the Alexandrov topology $\mathcal{T}_{\leq}$ and clearly $y \in {\uparrow}x$ if and only if $x \leq y$. This hints how we may also define preorders by means of Alexandrov topologies. If $\mathcal{T}$ is an Alexandrov topology on X, then we define a preorder $\leq_{\mathcal{T}}$ on X by setting

$$x \leq_{\mathcal{T}} y \iff y \in N_{\mathcal{T}}(x).$$

The correspondences $\leq \,\mapsto\, \mathcal{T}_{\leq}$ and $\mathcal{T} \,\mapsto\, \leq_{\mathcal{T}}$ are one-to-one (see [9] for details).

Next we show that similar correspondences can be found between Kripke-frames of IntGC and HGC-algebras. This is done by applying Alexandrov topologies, because it is well known that each Alexandrov space $(X, \mathcal{T})$ can be viewed as a (complete) Heyting algebra $(\mathcal{T}, \cup, \cap, \to, \emptyset, X)$ such that the relative pseudocomplement for any $A, B \in \mathcal{T}$ is

$$A \to B = \mathcal{I}_{\mathcal{T}}(-A \cup B),$$

where $\mathcal{I}_{\mathcal{T}}$ is the interior operator of $\mathcal{T}$, that is, $\mathcal{I}_{\mathcal{T}}(A) = \bigcup\{Y \in \mathcal{T} \mid Y \subseteq A\}$.

First we show how to construct HGC-algebras from Kripke-frames. Let $(X, \leq, R)$ be a Kripke-frame. We define the standard modal operators of possibility ${}^{\blacktriangle}\colon \wp(X) \to \wp(X)$ and necessity ${}^{\triangledown}\colon \wp(X) \to \wp(X)$ in terms of R and R^{-1}:

$$\begin{aligned} A^{\blacktriangle} &= \{x \in X \mid (\exists y \in X)\, x \,R\, y \ \text{ and } \ y \in A\}; \\ A^{\triangledown} &= \{x \in X \mid (\forall y \in X)\, y \,R\, x \ \text{ implies } \ y \in A\}. \end{aligned}$$

Then, the following lemma holds.

Lemma 7.1. *Let $(X, \leq, R)$ be a Kripke-frame. Then $A^\blacktriangle$ and $A^\triangledown$ belong to $\mathcal{T}_\leq$ for all $A \subseteq X$.*

Proof. Let $x \in A^\blacktriangle$ and $x \leq y$. There exists $z \in A$ such that $x\,R\,z$. We have

$$x \leq y, \ \ x\,R\,z, \ \ z \leq z$$

implying $y\,R\,z$ because $(X, \leq, R)$ is a Kripke-frame. Thus, $y \in A^\blacktriangle$ and therefore $A^\blacktriangle$ is an upward-closed set. So, $A^\blacktriangle \in \mathcal{T}_\leq$.

Let $x \in A^\triangledown$ and $x \leq y$. If $z\,R\,y$, then

$$z \leq z, \ \ z\,R\,y, \ \ x \leq y$$

imply $z\,R\,x$. Since $x \in A^\triangledown$, we get $z \in A$. Hence, $y \in A^\triangledown$ and $A^\triangledown \in \mathcal{T}_\leq$. $\square$

We have now showed how a Kripke-frame determines an HGC-algebra. For a Kripke-frame $(X, \leq, R)$, the algebra

$$(\mathcal{T}_\leq, \cup, \cap, \rightarrow, \emptyset, X, ^\blacktriangle, ^\triangledown)$$

is called the *complex algebra* of the frame $(X, \leq, R)$. Notice that we sometimes denote $\mathcal{T}_\leq$ by $L(X)$, which emphasizes that $L(X)$ is the carrier of an complex algebra determined by a frame on X.

Conversely, for an HGC-algebra $(L, \vee, \wedge, \rightarrow, 0, 1, f, g)$, we have defined in Section 5 page 12 the canonical frame $(X(L), \subseteq, R)$, where $X(L)$ consists of all prime filters of L and the relation R is defined on $X(L)$ by

$$\begin{aligned} F\,R\,G \ \text{ iff } \ & (\forall a \in L)\, a \in G \implies f(a) \in F \\ \text{ iff } \ & (\forall a \in L)\, g(a) \in G \implies y \in F. \end{aligned}$$

By Lemma 5.3, $(X(L), \subseteq, R)$ is a Kripke-frame.

Orłowska and Rewitzky [11] have proved the following correspondences for Heyting algebras and Heyting-frames. A *Heyting-frame* is a pair $(X, \leq)$, where $\leq$ is a preorder on X. Note that complex algebras of Heyting-frames and canonical frames of Heyting algebras are defined as above.

(i) Every Heyting algebra can be embedded into the complex algebra of its canonical frame. The mapping $h\colon L \to L(X(L))$, defined for any $x \in L$ by
$$h(x) = \{F \in X(L) \mid x \in F\},$$
is the required embedding.

(ii) Every Heyting-frame can be embedded into the canonical frame of its complex algebra. The mapping $k\colon X \to X(L(X))$, defined for any $x \in X$ by
$$k(x) = \{A \in L(X) \mid x \in A\},$$
is the required embedding, that is, for any $x, y \in X$, $x \leq y$ if and only if $k(x) \subseteq k(y)$.

Our next theorem extends the above result to HGC-algebras and Kripke-frames of IntGC.

Theorem 7.2. (a) *Every* HGC-*algebra is embeddable into the complex algebra of its canonical frame. In particular, if an* HGC-*algebra is finite, then it is isomorphic to the complex algebra of its canonical frame.*

(b) *Every Kripke-frame can be embedded into the complex algebra of its canonical frame.*

Proof. By above, the claim holds for Heyting algebras and frames.

(a) We show first that for all $x \in L$,

$$h(g(x)) = h(x)^{\triangledown}.$$

Let $F \in h(g(x))$, that is, $g(x) \in F \in X(L)$. Suppose $F \notin h(x)^{\triangledown}$. Then there exists $G \, R \, F$ such that $G \notin h(x)$. Now $g(x) \in F$ implies $x \in G$, that is, $G \in h(x)$, a contradiction. So, $F \in h(x)^{\triangledown}$.

Conversely, if $F \in h(x)^{\triangledown}$, then $G \, R \, F$ implies $G \in h(x)$, that is, $x \in G$. Suppose that $F \notin h(g(x))$, that is, $g(x) \notin F$. Because g is multiplicative and order-preserving, the preimage $g^{-1}(F) = \{a \in L \mid g(a) \in F\}$ is a filter since F is a filter. Clearly, $x \notin g^{-1}(F)$. Then, by the Prime Filter Theorem, there exists a prime filter G such that $g^{-1}(F) \subseteq G$ and $x \notin G$. Now $g^{-1}(F) \subseteq G$ means that $g(a) \in F$ implies $a \in G$ for all $a \in L$. By Lemma 5.2 this gives $G \, R \, F$. Therefore, $x \in G$, a contraction. So, $F \in h(g(x))$

Similarly, we show that

$$h(f(x)) = h(x)^{\blacktriangle}.$$

Assume that $F \in h(x)^{\blacktriangle}$. Then there exists $G \in h(x)$ such that $F \, R \, G$. Since $G \in h(x)$ is equivalent to $x \in G$, we have $f(x) \in F$, that is, $F \in h(f(x))$.

Conversely, suppose that $F \in h(f(x))$, that is, $f(x) \in F$. Clearly, $f^{-1}(F) = \{a \in L \mid f(a) \in L\}$ is a co-filter and we can easily show that ${\uparrow}x \subseteq f^{-1}(F)$. By Lemma 5.6, there is a prime filter H such that ${\uparrow}x \subseteq H \subseteq f^{-1}(F)$. This means that for all $a \in L$, $a \in H$ implies $f(a) \in F$, that is, $F \, R \, H$. Because $x \in H$, we have $H \in h(x)$ and $F \in h(x)^{\blacktriangle}$.

Finally, let L be a finite HGC-algebra and let $J(L)$ be the set of all join-irreducible elements of the lattice L; recall that an element $a \in L$ is *join-irreducible* if $a \neq 0$ and $a = b \vee c$ implies $a = b$ or $a = c$. Because L is a finite lattice, all filters of L are known to be principal. Furthermore, for each prime filter F, there exists a join-irreducible element $a \in J(L)$ such that $F = {\uparrow}a$ (see e.g. [1]). This then means that $X(L) = \{{\uparrow}a \mid a \in J(L)\}$.

We show that the map h is onto $L(X(L))$. Assume that $A \in L(X(L))$. This means that A is an upward-closed subset of $X(L)$. Let us set $x = \bigvee\{a \in J(L) \mid {\uparrow}a \in A\}$. Now $h(x) = \{{\uparrow}a \mid a \leq x \text{ and } a \in J(L)\}$.

If $\uparrow c \in h(x)$, then $c \in J(L)$ and $c \leq \bigvee\{a \in J(L) \mid \uparrow a \in A\}$. Because L is finite and c is join-irreducible, we have that $c \leq y$ for some $y \in \{a \in J(L) \mid \uparrow a \in A\}$. Now $c \leq y$ implies $\uparrow y \subseteq \uparrow c$. Since A is upward-closet set, we have $\uparrow c \in A$.

Conversely, if $\uparrow c \in A$, then $c \leq x$, which implies directly $\uparrow c \in h(x)$.

(b) It suffices to show that for any $x, y \in X$,

$$x \, R \, y \iff k(x) \, R^* \, k(y),$$

where R^* is the relation of the canonical frame of $L(X)$. By the definition of R^*,

$$\begin{aligned} k(x) \, R^* \, k(y) \ \text{ iff } \ & (\forall X \in \mathcal{T}_\leq) \, X \in k(y) \Longrightarrow X^\blacktriangle \in k(x) \\ \text{iff } \ & (\forall X \in \mathcal{T}_\leq) \, y \in X \Longrightarrow x \in X^\blacktriangle. \end{aligned}$$

Assume that $x \, R \, y$. Then, for all $X \in \mathcal{T}_\leq$, $y \in X$ implies trivially $x \in X^\blacktriangle$. So, $k(x) \, R^* \, k(y)$.

Conversely, suppose $k(x) \, R^* \, k(y)$. Then for all $X \in \mathcal{T}_\leq$, $y \in X$ implies $x \in X^\blacktriangle$. Then, in particular for $\uparrow y \in \mathcal{T}_\leq$, we have $y \in \uparrow y$. Therefore, $x \in (\uparrow y)^\blacktriangle$ and so there exists $z \in \uparrow y$ such that $x \, R \, z$. Now

$$x \leq x, \ x \, R \, z, \ y \leq z$$

imply $x \, R \, y$ by (CR). □

Remark 7.3. The embedding $h \colon L \to L(X(L))$ in Theorem 7.2 can not be replaced, in general, by an isomorphism, since both the complex algebra $L(X(L))$ of the canonical frame of an HGC-algebra L are complete lattices, while a HGC-algebra L may not be a complete lattice. In fact, differences between an HGC-algebra L and the algebra $L(X(L))$ are much deeper.

We end this section by considering L-valued sets. Goguen generalized fuzzy sets to L-fuzzy sets in [6]. An *L-fuzzy set* φ on X is a mapping $\varphi \colon X \to L$, where L is a "transitive partially ordered set". The term *L-valued set* is commonly used instead of an L-fuzzy set. In this work, L is assumed to be a preordered set. Typically, L may consist of linguistic membership values such as "good", "excellent", "poor", and "adequate", and the preorder relation $v_1 \leq v_2$ holds between two values v_1 and v_2, if v_2 is "stronger" than v_1. For instance, "poor" $\leq$ "excellent".

Any L-valued set φ on X determines naturally a preorder on X, as presented in [9]. The idea behind this definition is that in fact only the *order* of φ-values is important, not the values itself. A preorder $\leq_\varphi$ is defined by setting for all $x, y \in X$,

$$x \leq_\varphi y \iff \varphi(x) \leq \varphi(y).$$

Since there is one-to-one correspondence between preorders and Alexandrov topologies on X, each L-valued set induces also an Alexandrov topology $\mathcal{T}_\varphi$ consisting of upward-closed subsets of $\leq_\varphi$. Let us denote the principal up-set $\uparrow x$ of x with respect to the preorder $\leq_\varphi$ by $N_\varphi(x)$, that is, $N_\varphi(x) = \{y \mid \varphi(x) \leq \varphi(y)\}$. It is clear that

$$A \in \mathcal{T}_\varphi \iff A = \bigcup\{N_\varphi(x) \mid x \in A\}$$

and $N_\varphi(x)$ is the smallest neighbourhood of x in the Alexandrov topology $\mathcal{T}_\varphi$. Because $\mathcal{T}_\varphi$ is an Alexandrov space, the algebra

$$(\mathcal{T}_\varphi, \cup, \cap, \to, \emptyset, X)$$

is an Heyting algebra such that for all $A, B \in \mathcal{T}_\leq$,

$$A \to B = \mathcal{I}_\varphi(A^c \cup B),$$

where $\mathcal{I}_\varphi$ is the interior operator of the Alexandrov space $(X, \mathcal{T}_\varphi)$. The interior operator is defined by setting for all $A \subseteq X$,

$$\begin{aligned} \mathcal{I}_\varphi(A) &= \{x \in X \mid x \leq_\varphi y \text{ implies } y \in A\} \\ &= \{x \in X \mid \varphi(x) \leq \varphi(y) \text{ implies } y \in A\}. \end{aligned}$$

Note that since $\mathcal{T}_\varphi$ consists of fixed points of $\mathcal{I}_\varphi$, it can be viewed to consist of elements which are exact or definable with respect to the L-valued set φ.

As in rough set theory [12], we may define for any $A \in \mathcal{T}$ the rough approximations. Let R be a binary relation on U called *indiscernibility relation.* The *upper approximation* $A^\blacktriangle$ is defined by

$$x \in A^\blacktriangle \iff (\exists y \in U)\, x\, R\, y \text{ and } y \in A$$

and the *lower approximation* $A^\triangledown$ is specified by the condition:

$$x \in A^\triangledown \iff (\forall y \in U)\, x\, R\, y \text{ implies } y \in A.$$

Now $x \in A^\blacktriangle$ if there exists an object in A similar to x, and if $x \in A^\triangledown$, then all objects that are similar to x are in must be A. Therefore, $A^\blacktriangle$ and $A^\triangledown$ can be viewed as sets of elements *possibly* and *certainly* belonging to A.

It is now clear that if $\varphi \colon X \to L$ is an L-valued set and R is an indiscernibility relation on X, then $(X, \leq_\varphi, R)$ is a Kripke-frame for IntGC. Similarly, the algebra

$$(\mathcal{T}_\varphi, \cup, \cap, \to, ^\blacktriangle, ^\triangledown, \emptyset, X)$$

is an HGC-algebra. Therefore, we may define semantics for IntGC with respect to these structures.

8 Logic of Finite Distributive Lattices with Galois Connections

In this section, we consider issues related to finite structures. First we recall some definitions. A complete lattice L satisfies the *join-infinite distributive law* if for any $S \subseteq L$ and $x \in L$,

$$x \wedge \Big(\bigvee S\Big) = \bigvee \{x \wedge y \mid y \in S\}. \tag{JID}$$

It is well known that a complete lattice is a Heyting algebra if and only if it satisfies (JID) (see e.g. see [7]). For a complete lattice L satisfying (JID), the relative pseudocomplement is defined as

$$x \to y = \bigvee \big\{z \in L \mid z \wedge x \leq y\big\}.$$

Hence, every complete lattice L which satisfies (JID) and is equipped with a Galois connection (f, g) determines an HGC-algebra $(L, \wedge, \vee, \to, f, g, 0, 1)$. Additionally, if $f \colon L \to L$ is a complete join-morphism on a complete lattice L, that is, $f(\bigvee S) = \bigvee f(S)$ for all $S \subseteq L$, then the pair (f, g_f) is a Galois connection, where $g_f \colon L \to L$ is defined by

$$g_f(x) = \bigvee \{y \in L \mid f(y) \leq x\}. \tag{Adj1}$$

Similarly, if $g \colon L \to L$ is a complete meet-morphism on L, that is, $g(\bigwedge S) = \bigwedge g(S)$ for all $S \subseteq L$, then the pair (f_g, g) is a Galois connection, where $f_g \colon L \to L$ is defined by

$$f_g(x) = \bigwedge \{y \in L \mid x \leq g(y)\}. \tag{Adj2}$$

The above considerations can be summarized by the following proposition.

Proposition 8.1. *Let L be a complete lattice L which satisfies* (JID)*.*

(a) *Any complete join-morphism f determines an* HGC*-algebra*

$$(L, \wedge, \vee, \to, f, g_f, 0, 1).$$

(b) *Any complete meet-morphism g determines an* HGC*-algebra*

$$(L, \wedge, \vee, \to, f_g, g, 0, 1).$$

It is clear that in a finite lattice L, each additive and normal map is f is a complete join-morphism. Therefore, f induces a Galois connection (f, g_f), where g_f is defined as in (Adj1). Similarly, any multiplicative and co-normal function g determines a Galois connection (f_g, g), with $f_g \colon L \to L$ defined as in (Adj2). Since all finite distributive lattices satisfy (JID), they always are Heyting algebras. Hence, for finite lattices we can write also the following corollary.

Corollary 8.2. *Let L be a finite distributive lattice.*

(a) *Any additive and normal map on L induces an* HGC*-algebra.*

(b) *Any multiplicative and co-normal map on L induces an* HGC*-algebra.*

We may now summarize our considerations of this work by the following theorem.

Theorem 8.3. *For every* IntGC*-formula A, the following conditions are equivalent:*

(i) *A is provable.*

(ii) *A is valid in any finite* HGC*-algebra L, that is, L is a finite Heyting-algebra equipped with a Galois connection (f, g).*

(iii) *A is valid in any finite distributive lattice with an additive and normal operator f.*

(iv) *A is valid in any finite distributive lattice with a multiplicative and co-normal operator g.*

(v) *A is valid in any finite Kripke-model for* IntGC*.*

(vi) *A is valid in any finite Kripke-model for* IntGC *with irreflexive accessibility relation.*

We may also present the following corollary related to universal algebra.

Corollary 8.4. *The variety* **HGC** *of* HGC*-algebras is generated by its finite members.*

Proof. Let us denote by **HGC** and **f-HGC** the varieties generated by of all HGC-algebras and finite HGC-algebras, respectively. Since both of these varieties are equational classes, there exists two sets Σ and Γ of equations that characterize **HGC** and **f-HGC**, respectively, that is, $\mathbf{HGC} = M(\Sigma)$ and $\textbf{f-HGC} = M(\Gamma)$ by using the notation of [3].

It is trivial that $\textbf{f-HGC} \subseteq \mathbf{HGC}$. On the other hand, for any equation $\phi \approx \psi \in \Gamma$, the corresponding formulas $\phi \to \psi$ and $\psi \to \phi$ are valid on every finite HGC-algebra. This means that the formulas $\phi \to \psi$ and $\psi \to \phi$ are provable in IntGC. Therefore, the formulas $\phi \to \psi$ and $\psi \to \phi$ are valid on any HGC-algebra. This implies that any HGC-algebra satisfies the equation $\phi \approx \psi$. Therefore, we have $\Gamma \subseteq \Sigma$ and also $\mathbf{HGC} = M(\Sigma) \subseteq M(\Gamma) = \textbf{f-HGC}$. $\square$

A Concluding Remark

We end this work by briefly considering the *positive logic* of Galois Connections. This is achieved by removing the negation sign $\neg$ and the falsity $\bot$ from the language, and considering relatively pseudocomplemented lattices equipped with Galois Connections for $\blacktriangle$ and $\triangledown$. Recall that relatively pseudocomplemented lattices have the greatest element 1, but not necessarily the least element 0. It is known that for the connectives $\vee$, $\wedge$, and $\rightarrow$ satisfying (Ax1)–(Ax9), the resulting positive logic is sound and complete with respect to relatively pseudocomplemented lattices, see [13, pp. 460–466]. One may check that also for positive logic, most of the results and proofs – the ones that do not concern negation $\neg$ or falsity $\bot$ – presented in this work can be repeated.

References

[1] Raymond Balbes and Philip Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, Missouri, 1974.

[2] Patrick Blackburn, Maarten de Rijke, and Yde Venema, *Modal Logic*, Cambridge University Press, 2001.

[3] Stanley N. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics, vol. 78, Springer-Verlag, New York, Heidelberg, Berlin, 1981.

[4] Brian A. Davey and Hilary A. Priestley, *Introduction to Lattices and Order*, Second Edition, Cambridge University Press, 2002.

[5] M. Erné, J. Koslowski, A. Melton, and G. E. Strecker, *A primer on Galois connections*, Annals of the New York Academy of Sciences **704** (1993), 103–125.

[6] Joseph A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications **18** (1967), 145–174.

[7] George Grätzer, *General Lattice Theory*, Second Edition, Birkhäuser, Basel, 1998.

[8] Jouni Järvinen, Michiro Kondo, and Jari Kortelainen, *Logics from Galois connections*, International Journal of Approximate Reasoning **49** (2008), 595–606.

[9] Jouni Järvinen and Jari Kortelainen, *A unifying study between modal-like operators, topologies, and fuzzy sets*, Fuzzy Sets and Systems **158** (2007), 1217–1225.

[10] Bjarni Jónsson and Alfred Tarski, *Boolean algebras with operators. Part I*, American Journal of Mathematics **73** (1951), 891–939.

[11] Ewa Orłowska and Ingrid Rewitzky, *Discrete dualities for Heyting algebras with operators*, Fundamenta Informaticae **81** (2007), 275–295.

[12] Zdzisław Pawlak, *Rough sets*, International Journal of Computer and Information Sciences **11** (1982), 341–356.

[13] Helena Rasiowa and Roman Sikorski, *The Mathematics of Metamathematics*, Second Edition, PWN-Polish Scientific Publishers, Warsaw, 1968.