Contemporary Mathematics
Volume 92, 1989

The Dialectica categories

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Abstract. This paper is a resumé of my work on a categorical version of Godel's "Dialectica interpretation" of higher-order arithmetic. The idea is to analyse the Dialectica interpretation using a category DC where objects are relations on objects of a basic category C and maps are pairs of maps of C satisfying a certain pullback condition. If C has finite limits then DC exists and has a symmetric monoidal structure. If C is cartesian closed, DC is monoidal closed; if C has stable coproducts, DC has products and weak- coproducts. Moreover, if C has free monoids then DC has cofree comonoids and we define an endofunctor ! on DC which is a comonad. Using the structure above, DC is a categorical model for intuitionistic linear logic. The category of !-coalgebras is isomorphic to the category of comonoids in DC and the!-Kleisli category corresponds to the Diller-Nahm variant of the Dialectica interpretation.

Introduction

These notes contain a resumé of work I have been doing on a categorical version of the "Dialectica Interpretation" of higher order arithmetic.They form, more or less, an extended abstract of the first few sections of the Ph.D. thesis I am currently preparing under the supervision of Martin Hyland.

The original idea, as suggested to me by Hyland, was to consider the interpretation in a way now familiar from the "propositions as types" school of categorical proof-theory. As always the objects of the category are well-determined, in our case they represent essentially the Φ^D, where Φ is a formula in higher-order arithmetic and $(\ )^D$ is the Dialectica translation, see [T]. The maps however are more problematic - looked at from the proof-theoretic point of view they should represent normalisation classes of proofs, but more abstractly a map from Φ^D to Ψ^D can be taken to be some kind of realisation of the formula "$\Phi^D \rightarrow \Psi^D$". Hyland's observation was that in the case of the Dialectica Interpretation this realisation could be given very abstractly, leading to the notion of a Dialectica category DC for an arbitrary category C with limits. The objects of the category are relations in the base category C, which we write as

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$U \xleftarrow{\alpha} X$ and the maps from an object $U \xleftarrow{\alpha} X$ to another $V \xleftarrow{\beta} Y$ are pairs of maps $f{:}U \to V$ and $F{:}U \times Y \to X$ in C, satisfying a certain condition. The motivation behind such an odd definition of maps can be found in the Dialectica translation of implication. Implication, by far the most interesting rule in the Dialectica translation, is described by Troelstra [p.231] as:

$$\begin{aligned}
(A \Rightarrow B)^D &\equiv (\exists u \forall x A_D \Rightarrow \exists v \forall y B_D)^D \\
&\equiv [\forall u (\forall x A_D \Rightarrow \exists v \forall y B_D)]^D \\
&\equiv [\forall u\, \exists v\, (\forall x A_D \Rightarrow \forall y B_D)]^D \\
&\equiv [\forall u\, \exists v \forall y\, (\forall x A_D \Rightarrow B_D)]^D \\
&\equiv [\forall u\, \exists v \forall y\, \exists x (A_D \Rightarrow B_D)]^D \\
&\equiv \exists \mathcal{U} \mathcal{H} \forall uy\, (A_D(u, \mathcal{H}(u,y)) \Rightarrow B_D(\mathcal{U}u, y))
\end{aligned}$$

So to translate implication we need the functionals $\mathcal{U}{:}U \to V$ and $\mathcal{H}{:}U \times Y \to X$, which correspond to (f,F) in our definition.

A Dialectica category, however, differs from conventional proof-theoretic categories in that it is not cartesian closed. In fact usually in a Dialectica category, we have two constructions that seem to correspond to the interpretation of conjunction and it is the construction which is not a product, just a tensor, that provides us with a "good" categorical structure - the interpretation of implication gives us a monoidal closed category.

A new input came when we received accounts of Girard's work on linear logic (cf.[G]), and realised that there were many aspects of his work that seemed close to the categorical behaviour of the Dialectica categories. Indeed it became clear that we had a categorical version of the intuitionistic fragment of linear logic (cf.[G-L]). The main remaining problem was the interpretation of the operator ! (pronounced "of course"). That was solved by looking at co-free comonoid structures in DC and as a spin-off from this categorical setting we got another category $DC_!$, the !-Kleisli category, which corresponds to the variant of the Dialectica Interpretation described by Diller and Nahm. These notes are divided into two sections, the first constructs the Dialectica category DC and establishes its categorical properties, the second describes the connector "of course".

Section 1 The Dialectica category DC

1.1 Basic definitions

In this section we describe the general Dialectica category DC associated with a basic category C with finite limits. Martin Hyland's idea was to build a category on relations between objects of the basic category. So if U and X are objects of C, a typical object of DC would be $\alpha{:}A \rightarrowtail U \times X$, a subobject of the product $U \times X$. A map between two such objects $\alpha{:}A \rightarrowtail U \times X$ and $\beta{:}B \rightarrowtail V \times Y$ would be a pair of maps of C, (f,F) $f{:}U \to V$, $F{:}U \times Y \to X$ such that a non-trivial condition is satisfied, namely pulling back $A \rightarrowtail U \times X$ and $B \rightarrowtail V \times Y$ as the diagram shows, the first subobject A' is smaller than the second B', i.e there is a map making the triangle commute:

$$\begin{tikzcd}
& A' \arrow[r] \arrow[d] \arrow[dl, dash] & A \arrow[d] \\
B' \arrow[r, tail] \arrow[d] & U\times Y \arrow[r, "\langle p,F\rangle"'] \arrow[d, "f\times Y"] & U\times X \\
B \arrow[r, tail] & Y\times Y &
\end{tikzcd}$$

Given this presentation it is not obvious that DC is indeed a category. To simplify notation we write $(U \xleftarrow{\alpha} X)$ for $\alpha: A \rightarrowtail U\times X$. Then a map in DC can be represented as the pair (f,F) in the diagram below

$$\begin{tikzcd}
U \arrow[d, "f"'] & X \arrow[l, "\alpha"'] \\
V & Y \arrow[l, "\beta"] \arrow[ul, "F"']
\end{tikzcd}
\qquad \text{where } f:U\to V,\ \ F:U\times Y \to X \text{ satisfy } \langle p,F\rangle^{-1}(\alpha) \le (f\times Y)^{-1}(\beta) \quad [*]$$

If, intuitevely, one thinks of α and β as usual set-theoretic relations, condition [*] says that if $u\alpha F(u,y)$ then $f(u)\beta y$. Since [*] is not a straightforward categorical condition we will show that DC is a category. Given two maps $(f,F):\alpha\to\beta$ and $(g,G):\beta\to\gamma$ their composition $(g,G)\circ(f,F)$ is $gf:U\to W$ in the first coordinate and $G\circ F: U\times Z\to X$ given by:

$$U\times Z \xrightarrow{\delta\times Z} U\times U\times Z \xrightarrow{U\times f\times Z} U\times V\times Z \xrightarrow{U\times G} U\times Y \xrightarrow{F} X$$

in the second. To check that the new map $(gf, G\circ F):\alpha\to\gamma$ satisfies condition [*] we use pullback patching. It is easy to see that composition is associative and that identities are (id_U, π_1) where $id_U: U\to U$ is the identity and $\pi_1: U\times X\to X$ is the canonical second projection in C.

1.2 The monoidal structure in DC

DC has a natural symmetric monoidal structure. For objects $U \xleftarrow{\alpha} X$ and $V \xleftarrow{\beta} Y$, we take their tensor product $\alpha\otimes\beta$ to be $U\times V \xleftarrow{\alpha\times\beta} X\times Y$. Note that the tensor product above does not define a product, since we in general do not have projections. The operation $\otimes$ is a bifunctor and one can easily check that it defines a symmetric monoidal structure with $I=(1\leftarrow 1)$ as a unit.

1.3 The category DC is monoidal closed

Assuming C is finitely complete, we defined DC and verified that it has a monoidal structure. Now if C is also a cartesian closed category we want to show that the monoidal structure on DC is closed. To show that we have to define internal homs in DC (cf. [K]) and check the adjunction $(-)\otimes\beta \dashv [\beta\to(-)]$. Define the internal homs in DC by recalling that intuitively $[\beta\to\gamma]$ should represent "the set of pairs of maps in C, $f:V\to W$, $F:V\times Z\to Y$

satisfying the [*] condition". So it is fairly obvious that $[\beta \to \gamma]$ should be a subobject of $W^V \times Y^{V\times Z} \times V \times Z$. Consider the following diagram:

$$\begin{tikzcd}
 & B' \arrow[r] \arrow[d] \arrow[dl, dotted] & B \arrow[d] \\
C' \arrow[r] \arrow[d] & W^V \times Y^{V\times Z} \times V \times Z \arrow[r, "\langle p, eval\rangle"'] \arrow[d, "eval \times Z"] & V \times Y \\
C \arrow[r] & W \times Z &
\end{tikzcd}$$

where B' and C' are defined by the pullbacks.

Finally we define $[\beta \to \gamma]$ to be the greatest subobject $A \rightarrowtail W^V \times Y^{V\times Z} \times V \times Z$ such that $A \wedge B' \leq C'$ ($\wedge$ here means pullback over $W^V \times Y^{V\times Z} \times V \times Z$). Note that this is the usual categorical translation of Heyting implication.To garantee the existence of the greatest subobject, we ask for C locally cartesian closed as well as cartesian closed. Having defined $[\beta \to \gamma]$ we check that $DC(\alpha \otimes \beta, \gamma)$ is naturally isomorphic to $DC(\alpha, [\beta \to \gamma])$. To see that $DC(\alpha \otimes \beta, \gamma)$ is in bijective correspondence with $DC(\alpha, [\beta \to \gamma])$ take (f,F) in $DC(\alpha \otimes \beta, \gamma)$, it is of the form $(f, \langle F_1, F_2\rangle)$ where $f: U \times V \to W$, $F_1: U \times V \times Z \to X$ and $F_2: U \times V \times Z \to Y$. The map f is bijectively associated (by exponential transpose in C) to $f: U \to W^V$ and analogously F_2 is associated to $F_2: U \to Y^{V\times Z}$. So the mapping $(f, \langle F_1, F_2\rangle) \to (\langle f, F_1\rangle, F_2)$ has appropriate domain and codomain and is clearly bijective. Also a long and tedious arrow-chasing proves that $(f, \langle F_1, F_2\rangle)$ is a map in DC if and only if so is $(\langle f, F_1\rangle, F_2)$.

1.4 Products and weak-coproducts in DC

In this subsection we consider C a cartesian closed category with stable and disjoint coproducts. Then we shall define categorical products and weak-coproducts in DC and show a weak form of distributivity of product over weak-coproducts.

The product $U \times V \xleftarrow{\alpha \& \beta} X+Y$ of two objects $U \xleftarrow{\alpha} X$ and $V \xleftarrow{\beta} Y$ of DC is obtained by considering the subobjects $U \times B \xrightarrow{1 \times \beta} U \times V \times Y$ and $A \times V \xrightarrow{\alpha \times 1} U \times X \times V$ and adding them up. Thus $\alpha \& \beta = A \times V + U \times B \xrightarrow{1\times\beta + \alpha\times 1} U \times X \times V + U \times V \times Y \cong U \times V \times (X+Y)$. To check that '$\alpha \& \beta$' gives a categorical product, note that:

1. There are canonical projections given by $(\pi_0, i\pi_1)$ and $(\pi_1, j\pi_2)$, i and j canonical injections in C.
2. The object $\alpha \& \beta$ has the universal property, i.e given maps $(f,F): \delta \to \alpha$ and $(g,G): \delta \to \beta$

there is a unique map in DC, namely $(\langle f,g\rangle, F+G)$, making the diagram

$$\begin{tikzcd}
 & \delta \arrow[dl, "{(f,F)}"'] \arrow[d] \arrow[dr, "{(g,G)}"] & \\
\alpha & \alpha\&\beta \arrow[l] \arrow[r] & \beta
\end{tikzcd}$$

commute. Again some pullback patching is needed to show $(\langle f,g\rangle, F+G)$ is a map in DC.

We do not seem to have all coproducts in DC, but we do have some special ones, e.g for $U \xleftarrow{\alpha} X$ and $V \xleftarrow{\beta} X$ the object $U+V \xleftarrow{\alpha+\beta} X$ is a coproduct, where the relation '$\alpha+\beta$' is defined in the obvious way.

More importantly we always have weak-coproducts, i.e there is an operation $\oplus$, not a bifunctor, which satisfies:

1. There are canonical injections i_1, i_2.
2. If there are maps $(f,F):\alpha \to \delta$ and $(g,G):\beta \to \delta$, then there is a map $\alpha\oplus\beta \to \delta$, but it is not necessarily unique. To define $\oplus$ first take the pullback of $A \rightarrowtail^{\alpha} U\times X$ along $U\times X^U \xrightarrow{<p,ev>} U\times X$, multiply the new α' by Y^V, and add it to the correspondent new β.So that $\alpha\oplus\beta = U+V \xleftarrow{\alpha+\beta} X^U\times Y^V$. For the canonical injections we just use canonical injections in C and evaluation. For the map $\alpha\oplus\beta \to \delta$ we use the natural map $U+V \to W$ and any of the possible maps in the second coordinate.

Another point to mention is that we do not have distributivity of tensor product over weak-coproduct, nor of categorical product over weak-coproduct. But we do have a map (i,I) going from $\alpha\otimes(\beta\oplus\gamma)$ to $(\alpha\otimes\beta)\oplus(\alpha\otimes\gamma)$ and a map (j,J) coming back and these maps (i,I),(j,J) form a retraction. The map (i,I) will be necessary in **1.5.** It consists of the usual iso $i:U\times(V+W) \to U\times V+U\times W$ in the first coordinate ; $I:U\times(V+W)\times(X\times Y)^{U\times V}\times(X\times Z)^{U\times W} \to X\times Y^V\times Z^W$ in the second coordinate can be decomposed as (H,M,N) where $H=H_1+H_2$, all of them consisting of evaluations and projections.

Recapitulating:

	If C	then DC
has	products, 1, pbs	exists with $\otimes$
is	cartesian closed	is monoidal closed
has	stable, disjoint coproducts	has products, weak coproducts weak-distributivity

1.5 Linear Categories

Linear categories as described in [G-L] are categories with enough structure to model at least part of Intuitionistic Linear Logic, which means that they are symmetric monoidal closed categories with categorical products and coproducts. That implies the existence of units for product (t), for tensor (I) and for coproduct (0).

The aim of this section is to show that DC, despite not being a linear category, (not all coproducts) can be considered as a model for linear logic. That is, the constructions in DC, even the weak-coproduct, satisfy the rules of the Gentzen style system for propositional intuitionistic linear logic. We recall those rules:

Structural Rules:

1. $\dfrac{}{A \vdash A}$ 2. $\dfrac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$ 3. $\dfrac{\Gamma, \Delta, A, B \vdash C}{\Gamma, \Delta, B, A \vdash C}$

Logical Rules:

4. $\dfrac{}{\vdash 1}$ 5. $\dfrac{\Gamma \vdash A}{\Gamma, 1 \vdash A}$ 6. $\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$

7. $\dfrac{}{\Gamma \vdash t}$ 8. $\dfrac{}{\Gamma, 0 \vdash A}$ 9. $\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$

10. $\dfrac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C}$ 11. $\dfrac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}$ 12. $\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}$

13. $\dfrac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}$ 14. $\dfrac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$ 15. $\dfrac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, (A \oplus B) \vdash C}$

16. $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$ 17. $\dfrac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$

where $\Gamma = \langle G_1, \ldots, G_n \rangle$ and Δ are strings of formulae and Γ, Δ is juxtaposition.

We define an interpretation of (the propositional part of) intuitionistic linear logic into a category D, as a map $|-|_0$ which associates to each atomic formula A an object of the category **D**. That interpretation can be extended to all the formulae of propositional intuitionistic linear logic, via $|\text{—}|$:Form i.L.L$\to$**D**, if we can associate constructions in the category **D** to the logical connectives in linear logic. We say that **D** is a categorical model for (propositional) intuitionistic linear logic, if using the interpretation $|\text{—}|$ we can define an appropriate categorical notion of $\vdash_D$ such that the following holds: $\Gamma\vdash_{Lin} A \Rightarrow |\Gamma|\vdash_D |A|$

In our case we want to read $\vdash_D$ as "there exists a map in **D** from $|\Gamma|$ to $|A|$". So to show now that DC is a model of propositional intuitionistic linear logic we suppose we are given an interpretation of atomic formulae as objects of DC, $|A| = U\xleftarrow{\alpha}X$, we extend that interpretation to the sets of formulae by setting $|\Gamma| = |G_1,\ldots,G_n| = |G_n|\otimes\ldots\otimes|G_1|$ (note the reverse order) and by interpreting the connectives $\otimes$, &, $\oplus$, $-\!\!\circ$ as the corresponding constructions in DC. Then it is straightforward to check that the structures defined for DC satisfy the rules above. Rule **1** is ensured by the existence of identities and rule **2** is obtained by tensoring and composing the maps. The symmetric monoidal structure in DC ensures **3** to 6 and **9**. Rules **7** and **10** to **12** are obtained by interpreting |A&B| as the categorical product of objects |A| and |B|, |A|&|B|. In addition, there are logical rules corresponding to the weak-coproduct **8** , **13** and **14**. Rule 15, since we are reading the rules only downwards, corresponds to the weak-distributivity, i.e to the existence of the map $(i,I):\alpha\otimes(\beta\oplus\gamma)\to(\alpha\otimes\beta)\oplus(\alpha\otimes\gamma)$ mentioned before. Finally, rules **16** and **17** reflect the monoidal closed structure.

As a last remark in this section we note that we are amalgamating two steps, i.e we are considering "propositions as types" and "types as objects of a category", which gives us "propositions as objects of a (suitable) category". Suitable, at this level, only means that linear entailment corresponds to existence of a map in the category.

Section 2. The linear connective !

The logical idea behind the connective ! in linear logic is that it should give you the possibility of using the same hypothesis as many times as you wish. So, even if there is no diagonal map in DC we would like to have a natural map $!A\to!A\otimes!A$. From that to develop the idea that ! should be, not only an endofunctor, but a comonad in DC is not, perhaps the most natural thought, but it seems to work.

In this section we recall some basic facts about comonads and state a folklore proposition that we shall need later. Then assuming DC with all the structure described in section 1, we discuss the comonad !, its basic properties and some logical consequences .

2.1 Basic facts about comonads

We recall some results from chapter 6 in [CWM], using comonads instead of monads. They shall be stated as facts, since their proofs are just dualizations of MacLane's.

FACT 1: Every adjunction $<F,G,\eta,\varepsilon>:\mathbf{D}\to\mathbf{C}$ gives rise to a monad in the category **D** and a comonad in C. The comonad is given by the endofunctor FG, the co-unit of the comonad by the co-unit of the adjunction $\varepsilon:FGU\to U$ and the unit of the adjunction $\eta:I\to GF$ yields by composition a natural transformation μ, where $\mu=F\eta G:FGU\to FGFGU$.

FACT 2: Every comonad $T:C\to C$ gives rise to two categories, the category C^T of T-coalgebras (or Eilenberg-Moore category) and the T-Kleisli category, C_T. The category C^T has as objects T-coalgebras, that is pairs $<U,h:U\to TU>$, where U is an object of C and h is a map, called the structure map of the algebra, which makes both diagrams commute:

$$\begin{tikzcd} TTU & TU \arrow[l, "Th"'] & & U \arrow[r, "\eta"] \arrow[d, "i_U"'] & TU \arrow[d, "h"] \\ TU \arrow[u, "\mu_U"] & U \arrow[l, "h"] \arrow[u, "h"'] & & U \arrow[r, phantom, "="] & U \end{tikzcd}$$

A morphism of T-algebras is an arrow $f:U\to U'$ of C which renders commutative the diagram:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "f"'] & TU \arrow[d, "Tf"] \\ U' \arrow[r, "h'"'] & TU' \end{tikzcd}$$

The category C_T, the T-Kleisli category has the same objects as C, but $HomC_T(X,Y)$ corresponds to $HomC(TX,Y)$. Composition of $f:TX\to Y$ and $g:TY\to Z$ is given by:

$$TX \xrightarrow{\mu} TTX \xrightarrow{Tf} TY \xrightarrow{g} Z$$

FACT 3: Let $<F,G,\eta,\varepsilon>:D\to C$ be an adjunction, $T=<FG,\varepsilon,\mu>$ the comonad it defines in C. Then there are unique functors $K:D\to C^T$ and $L:C_T\to D$ making the following diagram commute:

$$\begin{tikzcd} C_T \arrow[r, dashed, "L"] \arrow[d, shift right, "F_T"'] & D \arrow[r, dashed, "K"] \arrow[d, shift right, "F"'] & C^T \arrow[d, shift right, "F^T"'] \\ C \arrow[u, shift right, "G_T"'] \arrow[r, phantom, "="] & C \arrow[u, shift right, "G"'] \arrow[r, phantom, "="] & C \arrow[u, shift right, "G^T"'] \end{tikzcd}$$

Under certain hypothesis the unique functor $K:D\to C^T$ can be an equivalence as we shall discuss.

2.2

Recall that if D is any symmetric monoidal category we can consider the category **Mon D**, consisting of monoid objects in D, i.e triples $(M,\mu_M:M\otimes M\to M,\eta_M:I\to M)$, where M is an object of D, μ is its monoid multiplication and η its unit and these maps make the following diagrams commute:

$$\begin{tikzcd} M\otimes M\otimes M \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & M\otimes M \arrow[d, "\mu"] \\ M\otimes M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} I\otimes M \arrow[r, "\eta\times 1"] \arrow[d, "\lambda"] & M\otimes M \arrow[d, "\mu"] & M\otimes I \arrow[l, "1\times\eta"'] \arrow[d, "\rho"] \\ M \arrow[r, phantom, "="] & M \arrow[r, phantom, "="] & M \end{tikzcd}$$

Maps $f:(M,\mu,\eta)\to(M',\mu',\eta')$ are maps $f:M\to M'$, which preserve the structure. Notice that as a monoidal structure is self-dual, one can dually define the category **Comon D** of comonoids on D, where objects are triples $(C,\gamma_C:C\to C\otimes C,\varepsilon_C:C\to I)$. In this subsection we want to show the following proposition, which is just an application of Beck's Theorem, as well as being part of the folklore of monoidal closed categories.

Proposition: If D is any monoidal category and U:**Comon D**$\to$D has a right-adjoint R, then U is comonadic, i.e **Comon D**$\cong$T-coalgebras, where T is the comonad defined in D by the adjunction $U\dashv R$.

Proof: To show the proposition we quote Beck's theorem in an appropriate form and verify that the forgetful functor U satisfies the condition required.

Theorem: Let $\langle U,R,\eta,\varepsilon\rangle$: **Comon D**$\to$D be an adjunction, $T=\langle UR,\varepsilon,\mu\rangle$ the comonad it determines in D, D^T the category of coalgebras for this comonad. Then the following are equivalent:

1.The unique comparison functor K: **Comon D**$\to D^T$ is an equivalence.

2.The functor U:**Comon D**$\to$D creates equalizers for those parallel pairs f,g in D for which Uf,Ug has an absolute equalizer in **Comon D.**

So to use the theorem we have to show condition 2 for U: **Comon D**$\to$D above.

If the following diagram is an absolute equalizer in D we want it to be an equalizer in **Comon D.**

$$\begin{tikzcd} E \arrow[r, tail, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

But since the equalizer is an absolute equalizer we have that

$$\begin{tikzcd} E\times E \arrow[r, tail, "e\times e"] & UA\times UA \arrow[r, shift left, "Uf\times Uf"] \arrow[r, shift right, "Ug\times Ug"'] & UB\times UB \end{tikzcd}$$

is an equalizer too. Using that we define a comultiplication in E, induced by the comultiplication in A. Similarly we can define a co-unit for E, $E \to I$. Then E with the induced structure is a comonoid object, the map $E \rightarrowtail^{e} A$ is a comonoid homomorphism and it is easy to see that (E,e) is an equalizer in **Comon D**.

2.3

Now we want to apply the theory of sections **2.1** and **2.2** to the Dialectica categories DC. For that we consider C cartesian closed, with stable and disjoint coproducts and with free monoid structures. So there exists a functor $*: C \to$ **Mon** C, which is left-adjoint to the forgetful functor U:**Mon** $C \to C$.

To define the endofunctor $!: DC \to DC$, which is the promised comonad, we need first to define special maps $C_{(-,-)}$ for pairs of objects V,Y of C. Define $C_{V,Y}$ using the following transformations:

$$\begin{array}{ll}
\underline{V \times Y \to (V \times Y)^*} & \eta \text{ co-unit of adjunction} \\
\underline{Y \longrightarrow (V \times Y)^{*V}} & * \dashv U \\
\underline{Y^* \to (V \times Y)^{*V}} & \\
V \times Y^* \xrightarrow[C_{V,Y}]{} (V \times Y)^* &
\end{array}$$

Using the maps above, to define the endofunctor ! we recall that objects in DC are monos in C, i.e arrows of the form $A \rightarrowtail^{\alpha} U \times X$, so $U \xleftarrow{\alpha} X$ or $\alpha: A \rightarrowtail U \times X$ is taken by ! to $U \xleftarrow{!\alpha} X^*$, where the relation $!\alpha$ is given by the pullback:

$$\begin{tikzcd}
!A \arrow[r] \arrow[d, tail, "!\alpha"'] & A^* \arrow[d, tail, "\alpha^*"] \\
U \times X^* \arrow[r, "C_{U,X}"'] & (U \times X)^*
\end{tikzcd}$$

Intuitively the relation "$u\alpha x$" is transformed into "$u(!\alpha)[x_1 \ldots x_k]$ iff $u\alpha x_1$ and $u\alpha x_2$ and...and $u\alpha x_k$". A map in DC $(f,F): \alpha \to \beta$ goes to $!(f,F)=(f,!F): !\alpha \to !\beta$, where $!F: U \times Y^* \to X^*$ is the composite

$$U \times Y^* \xrightarrow{C_{U,X}} (U \times Y)^* \xrightarrow{F^*} X^*$$

It is easy to check that $(f,!F)$ is a map in DC and that ! does define an endofunctor. Now to show that ! is a comonad we have to exhibit two natural transformations, $\mu: ! \to !!$ and $\varepsilon: ! \to Id$ which make the usual diagrams commute. Specifically the natural transformation $\mu_\alpha: !\alpha \to !!\alpha$

is given by the identity in the first coordinate and "forgetting parenthesis" $P: U\times X^{**}\to X^*$ in the second coordinate, while $\varepsilon_\alpha : !\alpha \to \alpha$ is given by identity and "repetition of the argument" on the second coordinate, $R: U\times X \to X^*$. An easy manipulation shows that the two natural transformations satisfy the comonad equations:

1. $!\mu_\alpha \circ \mu_\alpha = \mu_{!\alpha} \circ \mu_\alpha$
2. $\varepsilon_{!\alpha} \circ \mu_\alpha = id_{!\alpha} = !\varepsilon \circ \mu_\alpha$

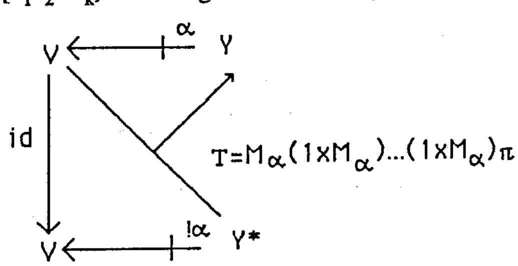
The next step is to show that ! provides us with good categorical structure, so:

Proposition: There exists a functor $F: DC \to$ **Comon** DC such that $U \dashv F$ and $UF \cong !$.

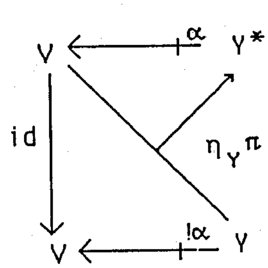
Proof: To define F we just check that objects $!\alpha$ admit a natural comonoid structure. Indeed, there is $\mu_{!\alpha} : !\alpha \to !\alpha \otimes !\alpha$, given by diagonal in C, $\Delta: U \to U\mathrm{x}U$ and concatenation of sequences $C: U\times X^* \times X^* \to X^*$.

Also $\varepsilon_{!\alpha} : !\alpha \to I$ is easily seen as the canonical unique map to terminal object, $t: U \to 1$ and canonical injection into coproduct, $U\times 1 \to X^*$. So $!\alpha$ has a natural comonoid structure and we call F the functor from DC to **Comon** DC which takes α to $<!\alpha, \mu_{!\alpha} : !\alpha \to !\alpha \otimes !\alpha, \varepsilon_{!\alpha} : !\alpha \to 1>$.

To show $U \dashv F$, that is Hom DC $[U\alpha, \beta] \cong$ Hom ComonDC $[\alpha, F\beta]$, we have to describe another natural transformation $\tau_\alpha : \alpha \to !\alpha$, which exists only for α a comonoid. The natural transformation $\tau_\alpha : \alpha \to !\alpha$, is given by identity in the first coordinate and $T: U\times X^* \to X$ in the second, and T is obtained using the comonoid multiplication as many times as necessary to transform the sequence $[x_1 x_2 \ldots x_k]$ into a single element of X, as in:



So given $(f,F): U\alpha \to \beta$ we get $(g,G): \alpha \to !\beta$ via composition, $(g,G) = !(f,F).\tau_\alpha$. Conversely given $(t,T): \alpha \to F\beta$ to get $(s,S): U\alpha \to \beta$ we simply compose (t,T) with the natural map $!\beta \to \beta$, given by identity and co-unit of the adjunction on C,



Obviously $UF \cong !$.

2.4

We apply the general theory in section **2.1** to the comonad ! in DC.

As we have shown $U \dashv F$:**Comon** DC$\to$DC, **FACT 1** only says we have a comonad in DC, namely !. FACT **2** says that the comonad $<!,\mu_!,\varepsilon_!>$ in DC gives rise to two categories, respectively, $DC^!$ the !-coalgebras and $DC_!$, the !-Kleisli category.

The objects of $DC^!$ are pairs $<\alpha, h:\alpha\to !\alpha>$, where h satisfies the !-coalgebra diagrams and morphisms are maps in DC, which preserve the coalgebra structure. $DC_!$, on the other hand, has the same objects as DC but $DC_![\alpha,\beta]=DC\,[!\alpha,\beta]$, that is a map in $DC_!$ from α to β corresponds to a map in DC from $!\alpha$ to β. It is easy to see that $DC_!$ inherits the cartesian products from DC, since $DC_!(\gamma,\alpha\&\beta)=DC(!\gamma,\alpha\&\beta)=DC(!\gamma,\alpha)\times DC(!\gamma,\beta)$ $=DC_!(\gamma,\alpha)\times DC_!(\gamma,\beta)$.

FACT **3** says we can draw the following diagram:

$$\begin{tikzcd}
DC_! \arrow[r, dashed, "L"] \arrow[d, shift right, "U_!"'] & \text{Comon } DC \arrow[r, dashed, "K", "\cong"'] \arrow[d, shift right, "U"'] & DC^! \arrow[d, shift right, "U^!"'] \\
DC \arrow[u, shift right, "F_!"'] \arrow[r, phantom, "="] & DC \arrow[u, shift right, "F"'] \arrow[r, phantom, "="] & DC \arrow[u, shift right, "F^!"']
\end{tikzcd}$$

and proposition **2.2** tells us that K is an equivalence of categories, i.e that $DC^!\cong$**ComonDC.** We can also easily verify that the tensor product of two comonoids is a comonoid and has natural projections.

One could be tempted to say that the tensor product in DC becomes a cartesian product in ComonDC, but for that we need the additional hypothesis of commutativity of the comonoid structure (see [F]).If we assume commutativity of the comonoid structures then we can have ComonDC is cartesian, but not necessarily closed, since the internal hom of two comonoids $[\alpha\to\beta]$ need not be a comonoid. More importantly we can have the following isomorphism $!(\alpha\&\beta)\cong !\alpha\otimes !\beta$, which leads directly to the nice result that $DC_!$ is cartesian closed.

2.5

In this subsection we consider the functor ***:Mon C$\to$C as giving us free** commutative monoids in C. Then we can, as before, define the comonad !:DC$\to$DC and it naturally defines a functor F:DC$\to$$\text{Comon}_c$DC, where the subscript c only serves to remind us that we are considering now commutative comonoids. Everything goes as before, with $\text{Comon}_c DC\cong DC^!$ and all the previous adjunctions. We also have that Comon_cDC is cartesian, its product being $\alpha\otimes\beta$ and the mentioned **Lemma:** In Comon_cDC the isomorphism $!(\alpha\&\beta)\cong !\alpha\otimes !\beta$ holds.

Proof: Since F is a left-adjoint it preserves products, $(\alpha\&\beta)$ is the product in DC, so $F(\alpha\&\beta)=(!(\alpha\&\beta),\mu,\varepsilon)$ is isomorphic to the product of comonoids $!\alpha\otimes !\beta$.

That takes us to the last result in this short subsection, which has a one-line proof.

Proposition: The Kleisli category $DC_!$ associated with the comonad ! is cartesian closed.

Proof: To check that the internal hom $[\beta,\gamma]_{DC!}$ is $[!\beta,\gamma]_{DC}$ we look at:
$DC_!(\alpha\&\beta,\gamma)=DC(!(\alpha\&\beta),\gamma)\cong DC(!\alpha\otimes!\beta,\gamma)\cong DC(!\alpha,[!\beta,\gamma]_{DC}=DC_!(\alpha,[!\beta,\gamma]_{DC})=$
$DC_!(\alpha,[\beta,\gamma]_{DC!})$.

2.6

The aim of this subsection is to tie up the linear logic aspects with the category theory presented in the last subsections. So we shall show that (DC,!) corresponds to a model of Intuitionistic Linear logic with modality and that $DC_!$, the Kleisli category of DC, which is cartesian closed, can be related to DC in a interesting way.

We start by reading off the rules for the modality from Girard's paper.These are:

$$\text{I. } \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \text{II. } \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}$$

$$\text{III. } \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \qquad \text{IV. } \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

Then it is again very easy (cf **1.4**) to check that the category DC with the comonad $<!,\mu_!,\varepsilon_!>$ described in **2.3** is a model of intuitionistic linear logic with modality. To have rule I it is enough to have a map $!A\to A$ which we have since ! is a comonad.To have rules II and III it is enough to have maps $!A\to I$ and $!A\to !A\otimes !A$ which we have since $!A$ is a comonoid. Finally, to have rule IV it is enough to have half of the adjunction $U\dashv F$ or

$DC(U!B,A)\cong DC_!(!B,!A)$, since we only need that given a map $!B\to A$ we can get a map $!B\to !A$ in a natural way.

The modality was introduced by Girard to recover the strenght of intuitionistic logic, by means of the following translation:

$$A^t = A \quad \text{for } A \text{ atomic}$$
$$(A\wedge B)^t = A^t \,\&\, B^t$$
$$(A\vee B)^t = !A^t \oplus !B^t$$
$$(A\to B)^t = !A^t \multimap B^t$$
$$(\neg A)^t = !A^t \multimap 0$$

So using that translation we want to show the following proposition, which is slightly stronger than the corresponding one in [G-L].

Proposition: $\Gamma \vdash_{Int} A$ iff $!\Gamma^t \ \vdash_{Lin} A^t$

Proof: We show the direct implication by structural induction on the deduction $\Gamma \vdash_{Int} A$, i.e we look at the last application of any of the rules of I.L and we check that if the premises have been translated then using L.L+modality rules, we can get a translation of the consequence. For instance, the CUT-rule i.e

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

becomes the following deduction:

$$\frac{\dfrac{!\Gamma^t \vdash_{Lin} A^t}{!\Gamma^t \vdash_{Lin} !A^t} \qquad !\Delta^t, !A^t \vdash_{Lin} B^t}{!\Gamma^t, \Delta^t \vdash_{Lin} B^t}$$

All the other structural rules are straightforward applications of the modality rules, but for the exchange rule which is simply linear exchange. For the logical rules we have to add some steps, in general very easy ones as in the introduction of conjuntion which becomes

$$\frac{\dfrac{\dfrac{!\Gamma^t, !A^t \vdash_{Lin} C^t}{!\Gamma^t, !A^t, !B^t \vdash_{Lin} C^t}}{!\Gamma^t, !A^t \otimes !B^t \vdash_{Lin} C^t}}{!\Gamma^t, !(A\&B)^t \vdash_{Lin} C^t}$$

where we use the isomorphism $!(A\&B) \cong !A \otimes !B$. The only slightly complicated case is the introduction of $\rightarrow$ on the left, which requires the lemma $!(!A \multimap B) \ \vdash_{Lin} !A \multimap !B$. The lemma is easily given by:

$$\frac{\dfrac{\dfrac{!A \multimap B, !A \vdash_{Lin} B}{!(!A \multimap B), !A \vdash_{Lin} B}}{!(!A \multimap B), !A \vdash_{Lin} !B}}{!(!A \multimap B) \vdash_{Lin} !A \multimap !B}$$

and all the other rules are similar to the ones above.

To show the converse we follow the suggestion in [G-L] again and look at the translation which takes linear logic into intuitionistic logic via:

$|!A| = A$ for A atomic
$|A\&B| = |A| \wedge |B|$
$|A \otimes B| = |A| \wedge |B|$
$|A \oplus B| = |A| \vee |B|$
$|A \multimap B| = |A| \rightarrow |B|$

Then it is trivial to check that for A an intuitionistic formula $|A^t| = A$. Moreover, if $\Delta \vdash_{Lin} B$ then $|\Delta| \vdash_{Int} |B|$, so applying this to $!\Gamma^t \vdash_{Lin} A^t$ we have $|!\Gamma^t| \vdash_{Int} |A^t|$ which implies $\Gamma \vdash_{Int} A$.

Conceptually the proposition above reflects the fact that in the same way as DC is a model for intuitionistic linear logic, its Kleisli category $DC_!$ is a model for Intuitionistic logic. It is interesting to note that the Kleisli category corresponds exactly to the Diller-Nahm variant of the Dialectica Interpretation. As a remark we recall that the original Dialectica assumed decidability of the atomic formulae. Decidability was essential to prove the consistency of $A \rightarrow A \wedge A$ and the soundness of the whole system depended upon it, cf. Troelstra's comments, page 230. The categorical model gives us a glimpse of why that happens. There we do not have, in general, maps $A \rightarrow A \otimes A$, but if A is decidable we can use the following trick to get a map in DC from a to $\alpha \otimes \alpha$:

$$
\begin{tikzcd}
U \arrow[d, "\Delta"'] & X \arrow[l, "\alpha"'] \\
U \times U & X \times X \arrow[l, "\alpha \wedge \alpha"] \arrow[u, "D"']
\end{tikzcd}
$$

$$
\text{where } D(u,x,x') = \begin{cases} x' & \text{if } u\alpha x \\ x & \text{otherwise} \end{cases}
$$

makes (Δ, D) a map of the category.

2.7 Conclusions and acknowledgements

In this paper we have had space to discuss just one of the relations between the Dialectica Interpretation and Linear Logic using categorical models. There is another which I am working on which is the result of a suggestion of Girard. It provides a way to model Classical Linear Logic via certain *-autonomous categories.

I would like to thank R.A.G Seely for the opportunity of presenting this paper, as well as the encouragement to do so. At a different, more personal level I would like to thank Edmund Robinson and Pino Rosolini for unending explanations and discussions. Also the whole "team" in Cambridge, including Peter Johnstone, Glynn Winskel, Thierry Coquand and Thomas Forster, provided a lively and estimulating atmosphere in which work was a pleasure.

But to conclude, I would like one day, perhaps, to be able to express the immense debt I owe my supervisor Martin Hyland. Any one who has ever worked with him, knows exactly what I mean, not only in terms of his fascinating mathematical insights, but also the feelings of friendship and cooperation that he communicates to his colleagues and students.

References

[F] T.FOX, *Coalgebras and cartesian categories*, Comm. Alg. (7) 4 (1976), 665-667.

[G] J.-Y.GIRARD, *Linear Logic*, Theo. Comp. S. 46, (1986), 1-102.

[G-L] J.-Y.GIRARD and Y.LAFONT, *Linear Logic and Lazy Computation*, preprint, 1986

[K] G.M. KELLY, *Basic Concepts of Enriched Category Theory*, Cambridge University Press,1982.

[CWM] S.MACLANE, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[T] A.S.TROELSTRA, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Springer-Verlag, 1973.

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