# Oregon Lectures

## Logics & Lambda Calculi 2

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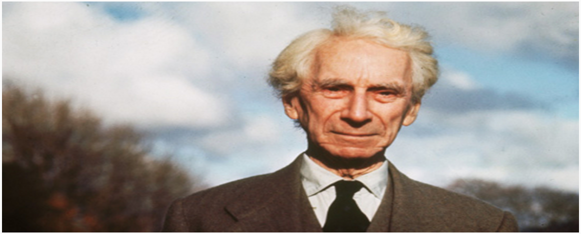
# Modal Logic



*...there is no one fundamental logical notion of necessity, nor consequently of possibility. If this conclusion is valid, the subject of* ***modality ought to be banished*** *from logic, since propositions are simply true or false...*

[Russell, 1905]

# Modal Logic

*One often hears that modal logic is pointless because it can be translated[...] There is no weight to the claim that the original system must therefore be replaced by the new one. What is essential is to* ***single out important concepts*** *and to investigate their properties. [Scott 1972]*

# Classical Modal Logic

- Modalities: the most successful logical framework in CS
- $\Box A$ = $A$ is necessarily the case, $A$ holds for all times, etc
  $\Diamond A$ = $A$ is possibly the case, $A$ holds at some time, etc
- Temporal logic, knowledge operators, BDI models, denotational semantics, effects, security modelling and verification, databases, etc..
- Logic representation of info and to reason about it. Usually only **classical** modalities...

# Classical Modalities

- Reasoning about [concurrent] programs Pnueli, 1977 ACM Turing Award, 1996.
- Reasoning about hardware; model-checking, Clarke, Emerson, 1981. ACM Kanellakis Award, 1999
- Knowledge Representation 2003 **Semantic Web**

They are also essential in AI, Cognitive Science, Philosophy, Decision Theory, etc...

Thanks Frank Pfenning!

# Constructive reasoning

- What: Reasoning principles that are safer
- if I ask you whether "is there an $x$ such that $P(x)$?"
- I'm happier with the answer "yes, $x_0$", than with an answer "yes, for all $x$ it is not the case that not $P(x)$"
- Why: want reasoning to be as precise and safe as possible
- How: constructive reasoning as much as possible, classical if need be, but tell me where and why...
- Today: constructive modalities!

# Intuitionistic Modal Logic

Basic idea: Modalities over an Intuitionistic propositional Basis:

$$\wedge, \vee, \rightarrow, \neg$$

- Which modalities?
- Which intuitionistic basis?
- **Why? How?**
- Why so many modal logics?
- How to choose?
- Can relate to others?
- Which are the important theorems?
- Which are the most useful applications?

# Constructive Modal Logic

- Constructive logic = a logical basis for programming via Curry-Howard correspondences
- Modalities useful in Computing
- Constructive modalities ought to be **twice as useful**
- But which constructive modalities?
- Usual phenomenon: classical facts can be ‘constructivized’ in many different ways. Hence constructive notions multiply...
- Too many choices

# Constructive Modal Logics

- Operators Box $\Box$, Diamond $\Diamond$ (like forall/exists) not interdefinable
- How do these two modalities interact?
- Depends on expected behavior and on tools you want to use
- The proof theory of modal logic is difficult, Handbook of Philosophical Logic – Segerberg 1984
- Solutions add to syntax: hypersequents, labelled deduction systems, (linear) nested sequents, tree-sequents, all add some semantics to syntax (many ways...)

# Intuitionistic Modal Logic and Applications (IMLA)

- a loose association of researchers, meetings and a certain amount of mathematical common ground,
- IMLA stems from the hope that philosophers, mathematical logicians and computer scientists would share information and tools about intuitionistic modal logics and modal type theories,
- if they knew about each others' work...

# IMLA Meetings

Workshops and invited speakers

- FLoC1999, Trento, Italy, (Pfenning)
- FLoC2002, Copenhagen, Denmark, (Scott and Sambin)
- LiCS2005, Chicago, USA, (Walker, Venema and Tait)
- LiCS2008, Pittsburgh, USA, (Pfenning, Brauner)
- 14th LMPS in Nancy, France, 2011 (Mendler, Logan, Strassburger, Pereira)
- UNILOG 2013, Rio de Janeiro, Brazil. (Gurevich, Vigano and Bellin)
- ESSLLI2017, Toulouse, France

# IMLA Publications

- Fairtlough, Mendler, Moggi, Modalities in Type Theory, MSCS, (2001)
- de Paiva, Goré, Mendler, Modalities in constructive logics and type theories, J of Log and Comp (2004)
- de Paiva, Pientka (eds.) IMLA 2008, Inf. Comput. (2011)
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- N. Alechina, V. de Paiva (eds.) IMLA2011, J. of Log and Comp, (2015)
- S. Artemov and V. de Paiva, J. Applied Logic, 2021

# What’s the state of play?

- IMLA’s goal: functional programmers talking to philosophical logicians and vice-versa
- Not attained, so far
- Communities still largely talking past each other
- Incremental work on intuitionistic modal logics continues, as well as some of the big research programmes that started it
- Does it make sense to try to change this?

# What did I expect twenty years ago?

- Fully worked out Curry-Howard for a collection of intuitionistic modal logics
- Fully worked out design space for intuitionistic modal logic, for classical logic and how to move from intuitionistic modal to classic modal
- Full range of applications of modal type systems
- Fully worked out dualities for desirable systems
- Collections of implementations for proof search/proof normalization

# Why did I think it would be easy?

Some early successes. Systems: CS4, Lax Log, CK

- CS4: On an Intuitionistic Modal Logic (with Bierman, Studia Logica 2000)
- Lax Logic: Computational Types from a Logical Perspective (with Benton, Bierman, JFP 1998)
- CK: Basic Constructive Modal Logic. (with Bellin and Ritter, M4M 2001)
- Kripke semantics for CK (with Mendler 2005), Scheele's PhD CKn, 2014

# Constructive S4 (CS4)

- Better behaved modal system, used by Gödel and Girard
- CS4 motivation is category theory, because of proofs, not simply provability
- Usual intuitionistic axioms plus MP, Nec rules and

## Modal Axioms

$\Box(A \to B) \to (\Box A \to \Box B)$

$\Box A \to A$

$\Box A \to \Box\Box A$

$\Box(A \to \Diamond B) \to (\Diamond A \to \Diamond B)$

$A \to \Diamond A$

# CS4 Sequent Calculus

S4 modal sequent rules first discussed in 1957 by Ohnishi and Matsumoto:

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

$$\frac{\Box\Gamma, A \vdash B}{\Box\Gamma, \Diamond A \vdash B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}$$

Cut-elimination works, for classical and intuitionistic basis.

# CS4 Natural Deduction

Natural Deduction (ND) more complicated
The rule

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

Abramsky's "Computational Interpretation of Linear Logic" (1993) leads to calculus that does **not** satisfy substitution.

# CS4 Natural Deduction

Given proofs $\dfrac{\dfrac{\Box A_1 \quad \Box A_2}{B}}{\Box B}$ and $\dfrac{C \to \Box A_1 \quad C}{\Box A_1}$,

we should be able to substitute

$$\dfrac{\dfrac{C \to \Box A_1 \quad C}{\Box A_1} \qquad \Box A_2}{B}$$

But a **problem**, as the promotion rule is not applicable. anymore.
Not a cut-elimination/normalization problem, a problem before one can even think of normalization

# Natural Deduction for CS4

- Benton, Bierman, de Paiva, Hyland solved the problem for ILL, TLCA 1993.
- Bierman and de Paiva (Amsterdam 1992, journal 2000) same solution for modal logic.

The solution builds in the substitutions into the rule as

$$\frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k \quad \Box A_1, \ldots, \Box A_k \vdash B}{\Gamma \vdash \Box B}(\Box I)$$

# Natural Deduction for CS4

Prawitz uses a notion of “essentially modal subformula” to guarantee substitutivity in his monograph (1965).
Several ’improvements’.
Two solutions: isn’t this enough?

# Natural Deduction for CS4

Usual Intuitionistic ND rules plus:

$$\frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k \quad \Box A_1, \ldots, \Box A_k \vdash B}{\Gamma \vdash \Box B}(\Box I)$$

$$\frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k, \Gamma \vdash \Diamond B \quad A_1 \ldots A_k, B \vdash \Diamond C}{\Gamma \vdash \Diamond C}(\Diamond E)$$

# CS4: Properties/Theorems

- Axioms satisfy Deduction Thm, are equivalent to sequents,
- Sequents satisfy cut-elimination, sub-formula property
- ND is equivalent to sequents
- ND satisfies normalization, ND assigns $\lambda$-terms CH equivalent
- Categorical model: monoidal comonad plus box-strong monad
- Everything you kind of wanted for Currry-Howard?

# CS4: Properties/Theorems

Issue with Prawitz formulation: idempotency of comonad not warranted...

Problems with CS4 system:

- Impurity of rules?
- Commuting conversions, eek!
- what about other modal logics?

see ”On an Intuitionistic Modal Logic”, Studia Logica, Bierman, de Paiva, 2000

# Dual Intuitionistic and Modal Logic

- Following LL can define a dual system for $\Box$-only modal logic.
- DIML, after Barber and Plotkin's DILL, in Ghani et al, ICALP 1998.

$$\Gamma, x\colon A, \Gamma' | \Delta \vdash x_M \colon A \quad \Gamma | \Delta, x\colon A, \Delta' \vdash x_I \colon A$$

$$\frac{\Gamma | \_ \vdash t\colon A}{\Gamma | \Delta \vdash \Box t\colon \Box A} \qquad \frac{\Gamma | \Delta \vdash t_i \colon \Box A_i \Gamma, x_i \colon A_i | \Delta \vdash u \colon}{\Gamma | \Delta \vdash \mathtt{let}\ t_1, \ldots, t_n \text{ be } \Box x_1, \ldots, \Box x_r}$$

- Less 'impurity' on rules, less commuting conversions, but what about $\Diamond$? what about other modal systems?

# Lax Logic

- Motivation: Moggi's computational lambda calculus, an intuitionistic modal metalanguage for denotational semantics for programming language features: non-termination, differing evaluation strategies, non-determinism, side-effects are examples.
- USing Curry-Howard 'backwards' to get the logic: intuitionistic modal logic with a degenerate possibility, Curry 1952

# Lax Modality Axioms

## Modal Axioms

$A \rightarrow \lozenge A$

$\lozenge\lozenge A \rightarrow \lozenge A$

$(A \rightarrow B) \rightarrow (\lozenge A \rightarrow \lozenge B)$

# System Lax

- Also called CL-logic (for computational lambda calculus)
- Definition: the logic CH-equivalent to a strong monad in a CCC
- Semantic distinction: computations and values.
- If $A$ models values of a type, then $T(A)$ is the object that models computations of the type $A$
- T is a curious possibility-like modality, Curry 1952, rediscovered by Fairtlough and Mendler, Propositional Lax Logic, Information and Computation, 1997

# Lax logic Properties

- Axioms, sequents, and ND are equivalent
- Deduction theorem holds, as does substitution and subject reduction
- The term calculus associated is strongly normalizing
- The reduction system given is confluent
- Cut elimination holds (Curry 1952)
- Lax logic (PLL) categorical models as expected
- Lax logic (PLL) Kripke models as expected
- Fairtlough and Mendler application: hardware correctness, up to constraints

# Constructive K: more difficult

- Constructive K comes from proof-theoretical intuitions provided by Natural Deduction formulations of logic
- Already CS4 does not satisfy distribution of possibiliity over disjunction: $\Diamond(A \vee B) \cong \Diamond A \vee \Diamond B$ and $\Diamond\bot \cong \bot$
- Sequent rules not as symmetric as in constructive S4, harder to model

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \frac{\Gamma, A \vdash B}{\Box\Gamma, \Diamond A \vdash \Diamond B}$$

- Note: only one rule for each connective, also $\Diamond$ depends on $\Box$.

# Constructive K Properties

- Dual-context only for Box fragment
- For Box-fragment, OK. Have subject reduction, normalization and confluence for associated lambda-calculus.
- Have categorical models, but too unconstrained?
- Kripke semantics OK
- No syntax in CK style for Diamonds that I know
- No ideas for uniformity of systems
- More work necessary

# Alternatives

- Simpson 1994 PhD: very robust system for geometric ND theories for intuitionistic modal logic
- Justified by translation into intuitionistic first-order, recovers many of the systems in the literature
- Strong normalization and confluence proved for all the systems
- Normalization establishes completeness of cut-free sequent calculi and decidability of some of the systems
- Decidable systems satisfy ”finite model property”

# Alternatives

- Arnon Avron (1996) Hypersequents (based on Pottinger and Mints)
- Martini and Masini 2-sequents (1996)
- Dosen’s higher-order sequents (1985)
- Display calculus (Belnap 1982, Kracht, Gore’, survey by Wansing 2002)
- multiple-sequent (more than one kind of sequent arrow) Indrejcazk (1998)
- labelled sequent calculus Negri (2005)
- Nested sequents: Bruennler (2009), Hein, Stewart and Stouppa, Strassburger et al,

# Recent Alternatives

- Fitch-style systems of several kinds, e.g. Borghuis 1994
- Judgmental modal logic, Pfenning, Davies 2001, 1995,
- Contextual Modal Type Theory, Nanevski, Pfenning, Pientka 2008
- Adjoint logic, Pfenning, Reed 2009, 2018
- Kavvos, Dual-context calculi for modal logic, 2017
- Shulman, Licata, Riley, A Fibrational Framework for Substructural and Modal Logics, 2017
- MTT Gratzer, 2019, 2023

# What I wanted

- Constructive modal logics with axioms, sequents and natural deduction formulations, proved equivalent
- Cut-elimination, finite model property, (strong) normalization, confluence, and decidability
- Algebraic, Kripke, and categorical semantics
- Translating proofs more than simply theorems
- A broad view of constructive and/or modality
- If possible, limitative results

# Simpson's desiderata

- IML is a conservative extension of IPL.
- IML contains all substitution instances of theorems of IPL and is closed under modus ponens.
- If $A \vee B$ is a theorem of IML either $A$ is a theorem or $B$ is a theorem too. (Disjunction Property)
- Box $\Box$ and Diamond $\Diamond$ are independent in IML
- Adding excluded middle to IML yields a standard classical modal logic
- (Intuitionistic) Meaning of the modalities, wrt IML is sound and complete

# Avron’s desiderata

- Should be able to handle a great diversity of logics, esp. the traditional ones
- Should be independent of any particular semantics
- Structures should not be too complicated, yield a “real” subformula property
- Rules of inference should have a small fixed number of premises, and a local nature of application
- should give us a better understanding of the logics and the differences between them

# Diamond over Disjunction

- distribution of possibility over disjunction binary and nullary: CS4 vs. IS4 (Simpson)

$$\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$$
$$\Diamond\bot \to \bot$$

- Distribution is canonical for classical modal logics
- Many constructive systems don't satisfy it
- Should it be required for constructive ones or not?
- Consequence: adding excluded middle gives you back classical modal logic (or not)?

# Labelled vs. unlabelled

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \text{ vs. } \frac{\Gamma \quad [xRy] \vdash y : A}{\Gamma \vdash x : \Box A}$$

- Introduction rule for $\Box$ says if $A$ holds at every world $y$ visible from $x$ then $\Box A$ holds at $x$.
- Simpson two kinds of hypotheses: $x : A$ means that the modal formula $A$ is true in the world $x$; $xRy$, which says that world $y$ is accessible from world $x$
- How reasonable is it to have your proposed semantics as part of your syntax?
- Proof-theoretic properties achieved, but no categorical semantics

# Modularity of framework

> *The framework of ordinary sequents is not capable of handling all interesting logics. There are logics with nice, simple semantics and obvious interest for which no decent, cut-free formulation seems to exist... Larger, but still satisfactory frameworks should, therefore, be sought. Avron (1996)*

Can we modify systems minimally and obtain other modal logics?

# IK and CK cubes

Cut-elimination for cubes below, syntax works, but very complicated?
Curry-Howard for CK cube, OK?

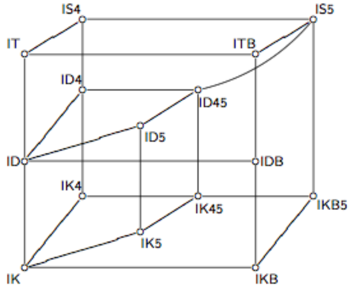
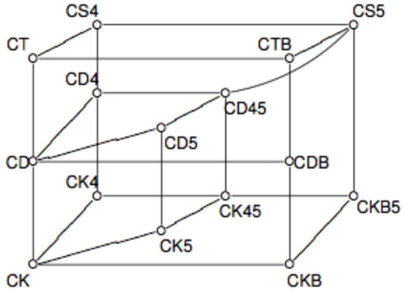



**Fig. 2.** The intuitionistic “modal cube”



# Schools of Intuitionistic Modal logic

- Justification Logics, Artemov et al
- Judgemental Modal Logic, Adjoint logic, Pfenning et al,
- Simpson framework, Negri sequent calculus
- Avron hyper-sequents, Dosen's higher-order sequents, Belnap display calculus, Bruennler/Strassbuerger, Poggiolesi, Pimentel and Lellmann, "nested sequents"
- all the modal type theories

# Conclusions

- (tried) Panorama of Curry-Howard for constructive modal logics
- Plenty of recent work on pure syntax from Pfenning, Strassburger, Bruennler, Marin, Das, Pimentel, and many others
- Many applications of the ideas of constructive modal logic
- Many interesting papers on specific applications, especially homotopy type theory, etc
- Is an overarching framework even possible? Desirable?

# Thanks!

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