Section 1: Proof Theory and Categorial Logic
Full Intuitionistic Linear Logic - (Abstract)

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This paper introduces a logic very easily described by a Gentzen-sequent calculus. It is an obvious logic to consider, once one has read about Girard's Linear Logic. This system differs from a 2-sided sequent calculus version of Classical Linear Logic - hereafter CLL essentially only in rule $(\multimap_r)$, where we make the intuitionistic restriction that Δ be a single formula. But the restriction on implication is the important one for Intuitionistic Logic, reason why we call this system Full Intuitionistc Linear Logic - hereafter $FILL$.

Comparing with Takeuti's book on Proof Theory the system $FILL$ probably reminds you of LJ' as LJ' is LK where the rules for implication $(\rightarrow_r)$ on the right and for $(\forall_r)$ for all on the right are modified. If we restrict ourselves to the propositional case, then an obvious analogy gives us the following table:

LK	LJ	LJ'
Classical Lin Logic Girard	Int Lin Logic Gir/Lafont	Full Int Lin Logic this paper

This analogy ought to be enough to justify the interest in the logic above, as this system is the version of LJ' that one obtains, if one takes seriously Girard's motives for getting rid of weakening and contraction.

But our interest in this logic comes from a more concrete motivation: we have a very pretty categorical model for this logic, which keeps all the distinctions that linear logic purports to make. Thus the four constants I,1,0 and $\perp$ are distinct objects in the category and the four connectives of linear logic are distinct bifunctors.

Of course this pretty model of (full Intuitionistic) Linear Logic can be restricted to a model of Classical Linear Logic - keeping all the distinctions - much in the same way as every Heyting algebra can be restricted to a Boolean algebra. Actually we have several models, which can be restricted to the other well-known algebraic models of CLL, eg quantales.

The approach taken here makes *linear implication* an intuitionistic implication, so that both the De Morgan Laws hold for the additive connectives, but not for the multiplicatives. Classical Linear logic builds on its symmetries and if it makes connectives finer-grained, it also makes them redundant as $\otimes$ can be reduced to $\Box$ (Girard's *par*) and & can be reduced to $\oplus$. By contrast, the situation in $FILL$ is that negation is defined in terms of a constant $\perp$ (as it can be in CLL), $\oplus$ can be reduced to & as we have $(A \oplus B)^\perp \cong A^\perp \& B^\perp$, but $\Box$ cannot be reduced to tensor as we only have $A^\perp \Box B^\perp \vdash (A \otimes B)^\perp$. We have between these connectives, the same relationship that we have in Intuitionistic propositional logic, i.e $A^\perp \Box B \vdash A \multimap B$ and $(A \Box B)^\perp \cong A^\perp \otimes B^\perp$, but not $(A \otimes B)^\perp \vdash A^\perp \Box B^\perp$

References

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