Oregon Lectures

Logics & Lambda Calculi

Valeria de Paiva
Topos Institute

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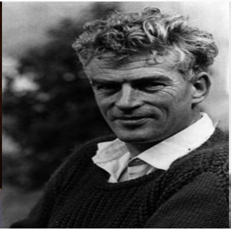
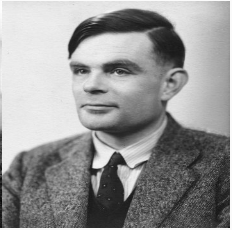
Thanks for the invitation!

Topos Institute

Journey


Background

Outline

1. Math, Logic, Computation
2. Linear and Modal logic
3. Explicit substitutions

Math

A Revolution in Mathematics? What Really Happened a Century Ago and Why It Matters Today

2012

Frank Quinn

The physical sciences all went through "revolutions": wrenching transitions in which methods changed radically and became much more powerful. It is not widely realized, but there was a similar transition in mathematics between about 1890 and 1930. The first section briefly describes the changes that took place and why they qualify as a "revolution", and the second describes turmoil and resistance to the changes at the time.

The mathematical event was different from those in science, however. In science, most of the older material was wrong and discarded, while old mathematics needed precision upgrades but was mostly correct. The sciences were completely transformed while mathematics split, with the core changing profoundly but many applied areas, and mathematical science outside the core, relatively unchanged. The strangest difference is that the scientific revolutions were highly visible, while the significance of the mathematical event is essentially unrecognized. The section "Obscurity" explores factors contributing to this situation and

management [2] might avoid this, but only if the disease is recognized.

The Revolution

This section describes the changes that took place in 1890–1930, drawbacks, objections, and why the change remains almost invisible. In spite of the resistance, it was incredibly successful. Young mathematicians voted with their feet, and, over the strong objections of some of the old guard, most of the community switched within a few generations.

Contemporary Core Methodology

To a first approximation the method of science is "find an explanation and test it thoroughly", while modern core mathematics is "find an explanation without rule violations". The criteria for validity are radically different: science depends on comparison with external reality, while mathematics is internal.

The conventional wisdom is that mathematics has always depended on error-free logical argument, but this is not completely true. It is quite

Algebra & Proofs

> *[...] a fundamental shift occurred in mathematics from about 1880 to 1940–the consideration of a wide variety of mathematical "structures" – groups, fields, lattices, etc.– satisfying some axioms. This approach is so common now that it is almost superfluous to mention it explicitly, but it represented a major conceptual shift in answering the question: What is mathematics?*

The axiomatization of Linear Algebra, Moore, Historia Mathematica, 1995.

Math not about numbers
About structures, connections, and proofs

Algebra & Proofs

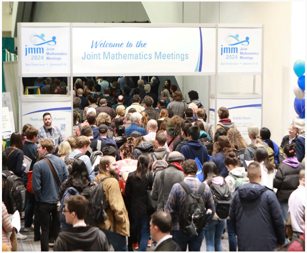

Bourbaki in 1938

> *The axiomatization of algebra was begun by Dedekind and Hilbert. It was then completed in the years following 1920 by Artin, Noether, and their colleagues at Göttingen.*

Bourbaki, Elements of the History of Mathematics, 1960.

Algebra: not solving 8th grade equations

Invisible Revolution?



- no cataclysms in Math
- Euclid (300 BC) still taught at high school
- research in Math? how come?

Shock Still...

- Math invented or discovered?
- foundations: essential or silly?
- unique ’correct’ framework or pluralistic?

Revolution Now?
NYT: A.I. Is Coming for Mathematics (2023)
Prospects for Formal Mathematics, Hausdorff Institute, Bonn, May-Aug 2024

Category Theory

There's an underlying unity of mathematical concepts/theories

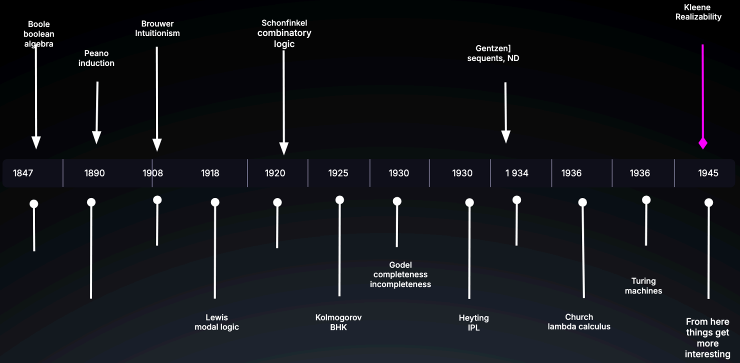
More important than the mathematical concepts themselves is how they relate to each other

Topological spaces come with continuous maps, while vector spaces come with linear transformations

Morphisms: how structures transform into others in the (most reasonable) way to organize the mathematical edifice.

Logic

(brief) Logic History timeline



Computation

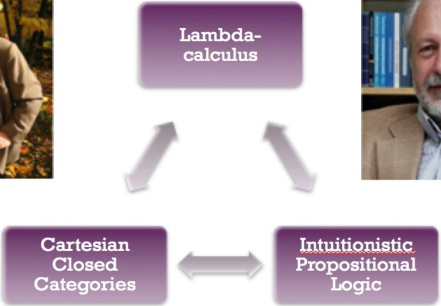
Curry-Howard Correspondence

1847– 1920 – 1936 – 1969 – now?

+ Curry-Howard Correspondence

1963

1965



What is this?

- a fundamental result connecting Logics, Programming Languages and Categories
- Each one of the arrows connects two different fields
- Original Curry-Howard ties logic and type theory. Category Theory is a late addition. BUT..
- More in Pientka, Komel and North lectures. Also in Wadler's *Proofs as*

How did it come about?

- Mathematics in turmoil because of paradoxes in set theory.
- **Hilbert's program** to provide secure foundations for all mathematics
- How? Formalization!
- Base math on *finitistic methods*
- goal: Prove *consistency of Arithmetic*

Hilbert's Wish List

- **Consistent**: no contradiction can be obtained in the formalism
- **Complete**: all true math statements can be proven
- **Conservative**: any result about "real objects" obtained using "ideal objects" (such as uncountable sets) can be proved without ideal objects.
- **Decidable**: an algorithm for deciding the truth or falsity of any math statement.

Failure of Hilbert's Program?

Gödel's Incompleteness Theorems (1931)

Hilbert's program impossible, if interpreted in the most direct way.

THEN

- use more powerful methods, Gentzen
- Proof Theory, to know what can be proved with what

War Time Proofs

- Gödel (1933, 1942, 1958)
- Liberalized version of Hilbert's program (?) – justify classical systems in terms of notions as intuitively clear as possible.
- Computable (or primitive recursive) functionals of finite type (System T), using the Dialectica Interpretation. (later?)

War Time Proofs

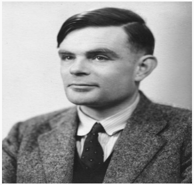
To prove the consistency of Arithmetic G. Gentzen (Hilbert's assistant) invented his systems of

NATURAL DEDUCTION

SEQUENT CALCULUS (1934)

These are the main proof systems used nowadays by provers (humans and machines)

Programs?



- Alonzo Church: lambda calculus (1936) a term for each machine computable function
- Haskell Curry: combinators and Combinatory Logic (1930) (also Schoenfinkel 1920)
- Church Thesis: λ-definability, recursive functions, Turing machines, all equivalent

Curry-Howard for Implication

Natural deduction rules for implication (without λ-terms)

$$\frac{A \to B \quad A}{B} \qquad \frac{\begin{array}{c}[A] \\ \vdots\ \pi \\ B\end{array}}{A \to B}$$

Curry-Howard for Implication

Natural deduction rules for implication (with λ-terms)

$$\frac{M : A \to B \quad N : A}{M(N) : B}$$

$$\frac{\begin{array}{c}[x : A] \\ \vdots\ \pi \\ M : B\end{array}}{\lambda x.M : A \to B}$$

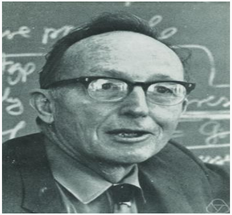
function application

abstraction

Proofs as programs

- Lambda calculus as **universal programming language**
- Effects, parallel programming, distributed computing, others are active research
- How much can we extend it?
- A cornucopia of new logics/program constructs based on the correspondence between proofs and programs.

Category Theory

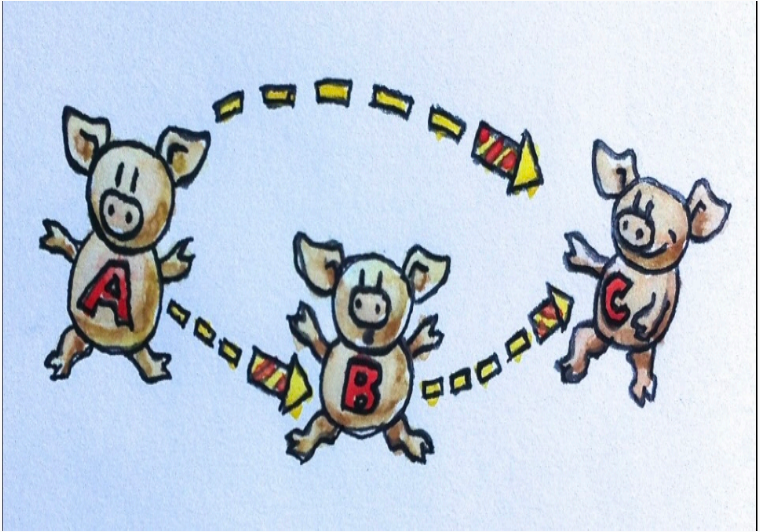


- a general mathematical theory of structures and systems of structures
- Eilenberg & MacLane (1945): cats for functors & natural transformations
- initially for algebraic topology & homological algebra
- some history: SEP, Jean-Pierre Marquis 2019, de Paiva & Rodin 2013

Category Theory

- Types: formulae/objects in a category
- Terms/programs: proofs/morphisms in **appropriate** category
- Logical constructors: categorical constructions
- Most important: Reduction is proof normalization (William Tait)

A glimpse of the basics



Bartosz Milewski (2015)

A glimpse of the basics

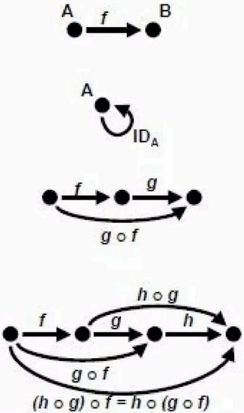
Definition of a Category

- **A category consists of:**
  - → a class of *objects*
  - → a class of *morphisms* ("arrows")
  - → for each morphism, f, one object as the *domain* of f and one object as the *codomain* of f.
  - → for each object, A, an *identity morphism* which has domain A and codomain A. ("ID_A")
  - → for each pair of morphisms $f{:}A \to B$ and $g{:}B \to C$, (i.e. cod(f)=dom(g)), a *composite morphism*, $g \circ f{:}\ A \to C$

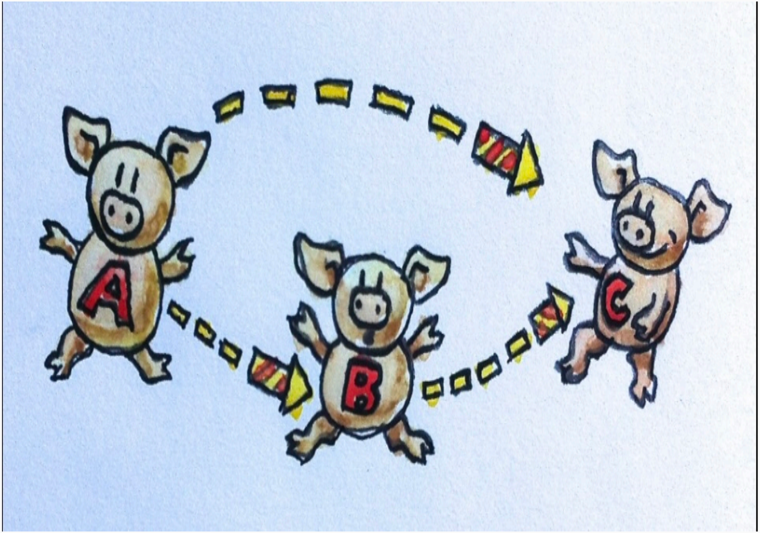
- **With these rules:**
  - → *Identity composition:* For each morphism $f{:}A \to B$,

    $f \circ ID_A = f$ and $ID_B \circ f = f$
  - → *Associativity:* For each set of morphisms $f{:}A \to B$, $g{:}B \to C$, $h{:}C \to D$,

    $(h \circ g) \circ f = h \circ (g \circ f)$



A glimpse of the basics



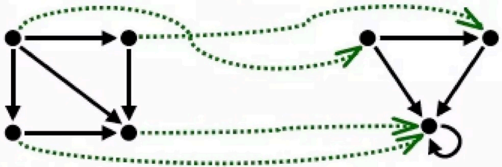
A glimpse of the basics

Functors

- **Definition of functor:**
  - → Consider the category in which the objects are categories and the morphisms are mappings between categories. The morphisms in such a category are known as *functors*.
  - → Given two categories, C and D, a functor F:C→D maps each morphism of C onto a morphism of D, such that:

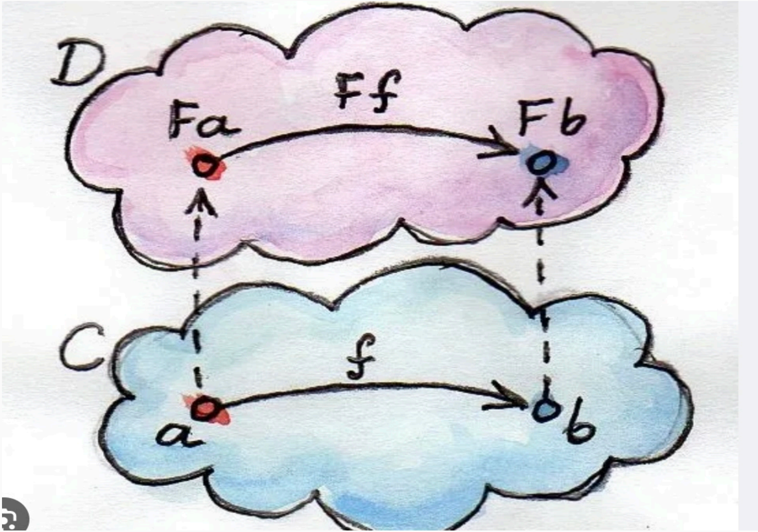
    F preserves identities - i.e. if x is a C-identity, then F(x) is a D-identity

    F preserves composition - i.e $F(f \circ g) = F(f) \circ F(g)$



- **Example functor**
  - → *From* the category of topological spaces and continuous maps *to* the category of sets of points and functions between them

A glimpse of the basics



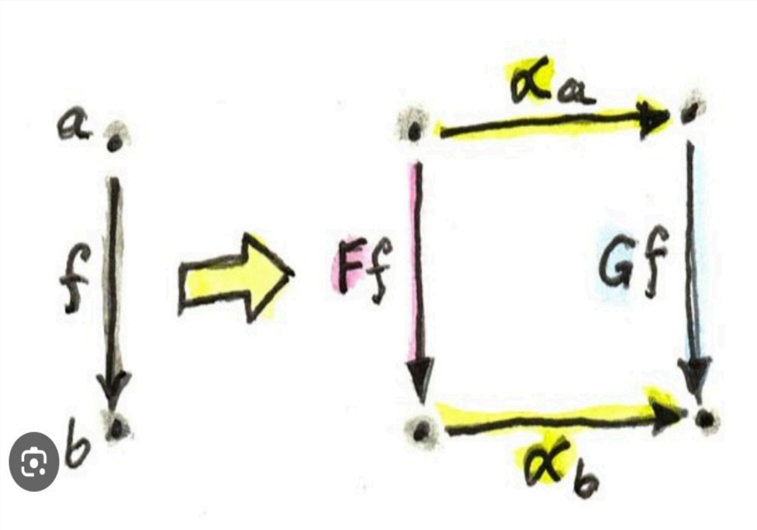
A glimpse of the basics

Natural Transformations

Given functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\alpha\colon F \to G$ consists of morphisms $\alpha_A\colon F(A) \to G(A)$ such that, for any A in $\mathbf{C}$ and for any $f\colon A \to B$ in $\mathbf{C}$ we have:

$$
\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\alpha_B"'] & G(B)
\end{tikzcd}
$$

A glimpse of the basics



You need many exercises to really get this!

Proof Theory using Categories...

Category: a collection of objects and of morphisms, satisfying obvious laws

Functors: the natural notion of morphism between categories

Natural transformations: the natural notion of morphisms between functors

Constructors: products, sums, limits, duals...

Adjunctions: an abstract version of equality

How does this relate to logic?

Where are the theorems?

A long time coming:

Curry and Feys, Schoenfinkel, Howard (1969, published in 1980)

Proof Theory using Categories

Motivation

Logic

Formula A

Sequent $\Gamma \vdash A$

Cut elimination rules, e.g.

$$\dfrac{\dfrac{\Gamma \vdash A}{\Gamma \vdash\, !A}\,\bar{d} \qquad \dfrac{A \vdash B}{!A \vdash B}\,d}{\Gamma \vdash B}\,\text{cut}$$

$$\rightsquigarrow \qquad \dfrac{\Gamma \vdash A \qquad A \vdash B}{\Gamma \vdash B}\,\text{cut}$$

equations between rewritings.

Bi Category theory

Object A

Map $\Gamma \xrightarrow{a} A$

2-cells
$$\begin{tikzcd} \Gamma \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$$

Equations between maps

$$\begin{tikzcd}
\Gamma \arrow[r, "a"] \arrow[ddr, bend right, "a"'] & A \arrow[r, "\bar{d}"] \arrow[dr, bend right, "1_A"'] & !A \arrow[d, "d"] \\
& & A \arrow[dd, "b"] \arrow[ul, phantom, "\Leftarrow"] \\
& A \arrow[dr, bend right, "b"'] \arrow[uu, phantom, "\Leftarrow"] & \\
& & B
\end{tikzcd}$$

equations between 2-cells

thanks Nicola Gambino!

Why Categories?

- Model derivations/proofs, not whether theorems are true or not
- Why is it good? Modeling derivations useful in linguistics, functional programming, compilers
- Why is it important? Solve the problem where it's easier and transport solution
- Also CS brings new important problems to solve with our favorite tools.

Why so little impact on maths or logic? Why so little impact on industry?

Logics

- Intuitionistic Logic
- System F
- Dependent type theory
- Linear Logics
- Modal logics
- Classical logic
- ecumenical logics
- Higher-order logics, etc

Many Curry-Howard Correspondences

many more!

Curry-Howard Correspondence

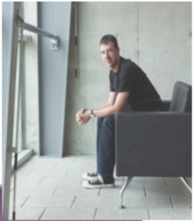


Linear Lambda-Calculus

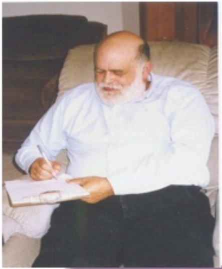

Linear Categories

Linear Logic

Modal (S4) Curry-Howard Correspondence



Modal S4 Lambda-Calculus



CCC+monoidal comonad

Constructive S4 Modal Logic

Lambek calculus Curry-Howard Correspondence

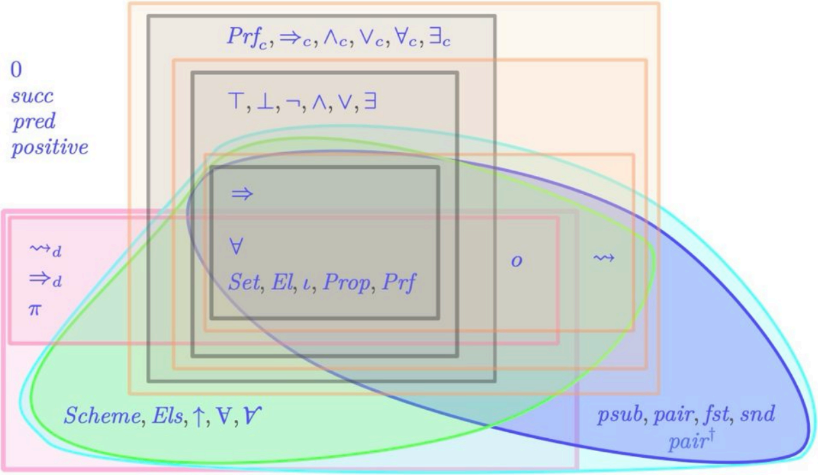
2-sided Linear Lambda-Calculus

Monoidal bi-closed Categories

Lambek Calculus

Preliminary conclusions

- We have seen how the Curry-Howard correspondence comes from different fields/problems
- Tomorrow we will discuss the extensions of Curry Howard, in particular for linear and modal logics
- It should be clear that the several logical systems hinted at are not exhaustive


$Prf_c, \Rightarrow_c, \wedge_c, \vee_c, \forall_c, \exists_c$
0
$succ$
$pred$
$positive$
$\top, \bot, \neg, \wedge, \vee, \exists$
$\Rightarrow$
$\forall$
$Set, El, \iota, Prop, Prf$
$\rightsquigarrow_d$
$\Rightarrow_d$
π
o
$\rightsquigarrow$
$Scheme, Els, \uparrow, \forall, \forall$
$psub, pair, fst, snd$
$pair^\dagger$

Deducteam Dowek et al, LambdaPi based proposal

Thanks!

Digression: Gödel Dialectica and ACT

- **Goal** Prove HA consistent. How?
- **Idea:** Translate every formula A of HA to

$$A^D = \exists u \forall x A_D$$

  where A_D quantifier-free.
- Translation defined by clauses on the connectives.
- 'Easy' to prove the theorem desired, but hard to see **why** it works.

Gödel Dialectica

Theorem (Gödel 1958): if HA proves A, then System T proves quantifier-free $A_D(t, x)$, where x are functionals of finite type, and t a suitable sequence of terms (not containing x).

Proof by induction on length of derivations, Troelstra 1973.

How intuitive are the functionals of finite type?

An internal **categorical model of Gödel's Dialectica interpretation in my phd thesis.**

Categorical Dialectica

Given C with finite limits, build a new category $\mathfrak{Dial}(\mathrm{C})$, with objects $A = (U, X, \alpha)$ where α is a subobject of $U \times X$ in C; this object represents the formula

$$\exists u \forall x \alpha(u, x).$$

A map from $\exists u \forall x \alpha(u, x)$ to $\exists v \forall y \beta(v, y)$ can be thought of as a pair $(f : U \to V, F : U \times Y \to X)$ of terms/maps, subject to the entailment condition

$$\alpha(u, F(u, y)) \vdash \beta(f(u), y).$$

Surprise! A model of Linear Logic, instead of Constructive Logic

Dialectica categories

- Justifies Linear Logic in terms of Gödel's proof-theoretic tool. and conversely.
- Keep the differences that Girard wanted to make.
- Justifies Harper's Trinitarism, connections to programming and using CT as syntax guidance.
- Loads of applications, lenses, games, automata, etc.

Dialectica categories timeline

- 1940 Gödel lecture at Yale
- 1958 published in Dialectica
- 1988 first categorical interpretation
- 2008 fibrational generalization (Biering)
- 2011 modern version (Hofstra)
- 2018 dependent type theory (von Glehm, Moss)

Recent work with Trotta and Spadetto where the assumptions in Gödel's argument (hacks?) are used (2022, 2023)

Applications of Dialectica

- Concurrency theory, Petri nets and others (1990’s, di Lavore& Leal to apear)
- linear functional programming (2000’s, 2014-2018 Linear Haskell)
- partial compilers (Budiu and Plotkin, 2013)
- Lenses, BX-transformations, *Lenses for Philosophers*, Hedges 2017, several others
- Automated Differentiation, Pedrot and Kerjean 2022, LICS2024, etc.

Applied Category Theory?



- "Applied Category Theory in Chemistry, Computing, and Social Networks" MRC 2022
- David Corfield "Philosophy and Innovation", nLab 2024
- Bradley, Tai-Danae (2018) "What is Applied Category Theory?"
- and many more

Category theory and its uses

Corfield, 2024

Category theory and its uses Applications (with approximate dates)

- Mathematics (from 1940s)
- Logic (from 1960s)
- Computing (from 1970s)
- Physics (from 1980s)
- *Applied Category Theory* (from the 2010s)

ACT Philosophy perspective

Corfield, 2024

- Initially a convenient language: ‘standard’ mathematician does not approve of CT
- Now not only a language?

  We would like to say that our proofs are proofs by “formal nonsense” and in particular analysis-free. (Clausen and Scholze)

ACT Philosophy perspective

Corfield, 2024

In Physics:

- Topological quantum field theory (1980s)
- String diagrams as morphisms in monoidal categories for quantum mechanics (2000s)
- Higher gauge theory, 2000s
- Modal and linear homotopy type theory
- The Quantum Monadology, Schreiber

https://ncatlab.org/davidcorfield/files/NUL.pdf

ACT Baez et al: How?

> *Society is increasingly complex and connected through the internet and social media, planetary climate and ecological challenges, transnational organization and global supply chains. To navigate and thrive in this networked world, we rely on scientific advances to help us manage this complexity by enabling robust communication, cooperation, and collaboration.*

”Applied Category Theory in Chemistry, Computing, and Social Networks” Notices of AMS (Baez, Cicala, Cho, Otter, de Paiva) 2022

ACT: themes

- compositionality
- functorial semantics,
- implementing these structures into user-friendly software.

a bird’s eye view of themes
logic/programming languages only one of these

ACT: more themes

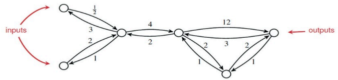


Figure 1. Open Markov process.

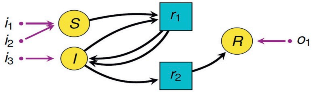


Figure 2. Open SIR model as a Petri net.

- Markov processes
- chemical reaction networks
- Epidemiology
- Petri nets
- Stock and flow diagrams

MRC: outcomes

- Baez’s group: X. Li, S. Libkind, N. D. Osgood, E. Patterson, ”Compositional modeling with stock and flow diagrams”, 5th International Conference ACT, EPTCS 380 (2022), 77–96.
- J. C. Baez, X. Li, S. Libkind, N. D. Osgood and E. Redekopp, ”A categorical framework for modeling with stock and flow diagrams”, (to appear)
- ”Agent-Based Models (Parts 1,2)” blog post, J.C. Baez

MRC: outcomes

- de Paiva’s group: J. Dorta, S. Jarvis, and Nelson Niu. ”Monoidal structures on generalized polynomial categories”. May 2023. ACT 2023, arXiv:2305.05655
- ”On a fibrational construction for optics, lenses, and Dialectica categories”, M. Capucci, B. Gavranović, A. Malik, F. Rios, J. Weinberger, to appear in ACT2024.
- ”The dependent Goedel fibration”, D. Trotta, J. Weinberger, V de Paiva, abstract CT2023 Jonathan Weinberger

Much more ACT

- People applying category theory for:
- Causality, probabilistic reasoning, statistics, learning theory, deep neural networks, dynamical systems, information theory, database theory, natural language processing, cognition, consciousness, active inference, systems biology, genomics, epidemiology, chemical reaction networks, neuroscience, complex networks, game theory, robotics, quantum computing,...

Much more ACT

- higher category theory
- Categorical probability theory
- Categorical differentiation
- category theory for AI? geometric deep learning, categorical deep learning
- Deep Learning for Theorem Proving `https://arxiv.org/pdf/2404.09939`

Conclusions

- Applied Category Theory now!
- (still) underappreciated (categorical) Curry-Howard correspondence
- Important for interdisciplinary work: Math, Logic and Programming
- favorite example: Dialectica categories, Gödel fibrations and doctrines, rediscovered over and over
- Plenty of other examples/applications to develop

Thanks!

Some References

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