Encyclopaedia
of
Proof
Systems

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Part I
Proof Systems

Hilbert and Ackermann's Calculus (1928)

Axioms:

a) $X \lor X \to X$

b) $X \to X \lor Y$

c) $X \lor Y \to Y \lor X$

d) $(X \to Y) \to (Z \lor X \to Z \lor Y)$

e) $(x)F(x) \to F(y)$

f) $F(y) \to (Ex)F(x)$

Rules:

α. Substitution for object, propositional, and predicate variables

β. Generalization rules, i.e.,

$$\frac{\mathfrak{A} \to \mathfrak{B}(x)}{\mathfrak{A} \to (x)\mathfrak{B}(x)} \qquad \frac{\mathfrak{B}(x) \to \mathfrak{A}}{(Ex)\mathfrak{B}(x) \to \mathfrak{A}}$$

where x must not occur in $\mathfrak{A}$.

γ. Modus ponens, i.e.,

$$\frac{\mathfrak{A} \quad \mathfrak{A} \to \mathfrak{B}}{\mathfrak{A}}$$

Clarifications: As in {**??**}, the system is formulated in a language that distinguishes between propositional and predicate constants and variables, but theorems with free variables are now allowed. Fraktur letters are used as schematic metavariables. The substitution rule allows the replacement of individual variables by variables or constants, propositional variables by any formula, and predicate variables with n arguments x_1, $\ldots$, x_n by a formula in which $x_1, \ldots, x_n$ occur free.

History: The system was published in [**HilbertAckermann1928**]. It was the systems for which the problem of providing a completeness proof was first raised.

Remarks: This system was proved complete in [**Godel1930**].

Entry 1 by: Richard Zach

Intuitionistic Natural Deduction NJ (1935)

$$\frac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A}\&\mathfrak{B}}\; UE \qquad \frac{\mathfrak{A}\&\mathfrak{B}}{\mathfrak{A}}\; UB \qquad \frac{\mathfrak{A}\&\mathfrak{B}}{\mathfrak{B}}\; UB$$

$$\frac{\mathfrak{A}}{\mathfrak{A}\vee\mathfrak{B}}\; OE \qquad \frac{\mathfrak{B}}{\mathfrak{A}\vee\mathfrak{B}}\; OE \qquad \frac{\mathfrak{A}\vee\mathfrak{B} \quad \overset{[\mathfrak{A}]}{\underset{\mathfrak{C}}{\vdots}} \quad \overset{[\mathfrak{B}]}{\underset{\mathfrak{C}}{\vdots}}}{\mathfrak{C}}\; OB$$

$$\frac{\mathfrak{F}\mathfrak{a}}{\forall x\mathfrak{F}x}\; AE \qquad \frac{\forall x\mathfrak{F}x}{\mathfrak{F}\mathfrak{a}}\; AB \qquad \frac{\mathfrak{F}\mathfrak{a}}{\exists x\mathfrak{F}x}\; EE \qquad \frac{\exists x\mathfrak{F}x \quad \overset{[\mathfrak{F}\mathfrak{a}]}{\underset{\mathfrak{C}}{\vdots}}}{\mathfrak{C}}\; EB$$

$$\frac{\overset{[A]}{\underset{B}{\vdots}}}{\mathfrak{A}\supset\mathfrak{B}}\; FE \qquad \frac{\mathfrak{A} \quad \mathfrak{A}\supset\mathfrak{B}}{\mathfrak{B}}\; FB \qquad \frac{\overset{[\mathfrak{A}]}{\underset{\lambda}{\vdots}}}{\neg\mathfrak{A}}\; NE \qquad \frac{\mathfrak{A} \quad \neg\mathfrak{A}}{\lambda}\; NB \qquad \frac{\lambda}{\mathfrak{D}}$$

The eigenvariable $\mathfrak{a}$ of an *AE* must not occur in the formula designated in the schema by $\forall x\mathfrak{F}x$; nor in any assumption formula upon which that formula depends. The eigenvariable $\mathfrak{a}$ of an *EB* must not occur in the formula designated in the schema by $\exists x\mathfrak{F}x$; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae designated by $\mathfrak{F}\mathring{a}$.

Clarifications: The names of the rules are those originally given by Gentzen [**Gentzen1935**]:
U = und (and), O = oder (or), A = all, E = es-gibt (exists), F = folgt (follows),
N = nicht (not), E = Einführung (introduction), B = Beseitigung (elimination).

History: The main novelty introduced by Gentzen in this proof system is its *assumption* handling mechanism, which allows formal proofs to reflect more naturally the logical reasoning involved in mathematical proofs.

Remarks: In [**Gentzen1935**], completeness of **NJ** is proven by showing how to translate proofs in the Hilbert-style calculus **LHJ** to **NJ**-proofs, and soundness is proven by showing how to translate **NJ**-proofs to **LJ**-proofs {4}.

Classical Sequent Calculus LK   (1935)

$$\frac{}{A \vdash A}$$

$$\frac{\Gamma \vdash \Lambda, A \quad A, \Delta \vdash \Theta}{\Gamma, \Delta \vdash \Lambda, \Theta} \; cut$$

$$\frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} \; w_l$$

$$\frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, A} \; w_r$$

$$\frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta} \; e_l \qquad \frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} \; c_l$$

$$\frac{\Gamma \vdash \Theta, B, A, \Delta}{\Gamma \vdash \Theta, A, B, \Delta} \; e_r \qquad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A} \; c_r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta} \; \neg_l$$

$$\frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A} \; \neg_r$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta} \; \wedge_l$$

$$\frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B} \; \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta} \; \vee_l$$

$$\frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2} \; \vee_r$$

$$\frac{\Gamma \vdash \Lambda, A \quad B, \Delta \vdash \Theta}{A \to B, \Gamma, \Delta \vdash \Lambda, \Theta} \; \to_l$$

$$\frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \to B} \; \to_r$$

$$\frac{A[\alpha], \Gamma \vdash \Theta}{\exists x. A[x], \Gamma \vdash \Theta} \; \exists_l \qquad \frac{A[t], \Gamma \vdash \Theta}{\forall x. A[x], \Gamma \vdash \Theta} \; \forall_l \qquad \frac{\Gamma \vdash \Theta, A[\alpha]}{\Gamma \vdash \Theta, \forall x. A[x]} \; \forall_r \qquad \frac{\Gamma \vdash \Theta, A[t]}{\Gamma \vdash \Theta, \exists x. A[x]} \; \exists_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

History: This is a modern presentation of Gentzen's original **LK** calculus [**lk:Gentzen1935**], using modern notations and rule names.

Remarks: LK is complete relative to **NK** (i.e. **NJ** {2} with the axiom of excluded middle) and sound relative to a Hilbert-style calculus **LHK** [**lk:Gentzen1935a**]. Cut is eliminable (*Hauptsatz* [**lk:Gentzen1935**]), and hence classical predicate logic is consistent. Any *prenex* cut-free proof may be further transformed into a shape with only propositional inferences above and only quantifier and structural inferences below a *midsequent* [**lk:Gentzen1935a**].

Entry 3 by: Martin Riener

Intuitionistic Sequent Calculus LJ (1935)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash \Theta}{\Gamma, \Delta \vdash \Theta} \; cut$$

$$\frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} \; w_l \qquad \frac{\Gamma \vdash}{\Gamma \vdash A} \; w_r$$

$$\frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta} \; e_l \qquad \frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} \; c_l$$

$$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash} \; \neg_l \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \; \neg_r$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta} \; \wedge_l \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \; \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta} \; \vee_l \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2} \; \vee_r$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash \Theta}{A \rightarrow B, \Gamma, \Delta \vdash \Theta} \; \rightarrow_l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \; \rightarrow_r$$

$$\frac{A[\alpha], \Gamma \vdash \Theta}{\exists x. A[x], \Gamma \vdash \Theta} \; \exists_l \qquad \frac{\Gamma \vdash A[t]}{\Gamma \vdash \exists x. A[x]} \; \exists_r$$

$$\frac{A[t], \Gamma \vdash \Theta}{\forall x. A[x], \Gamma \vdash \Theta} \; \forall_l \qquad \frac{\Gamma \vdash A[\alpha]}{\Gamma \vdash \forall x. A[x]} \; \forall_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

Clarifications: Gentzen introduced the sequent calculi **LK** {3} and **LJ** for classical and intuitionistic logics respectively. The rules in both systems have the same shape, but in **LJ** they may have at most one formula in the succedent (right side of ⊢). This restriction is equivalent to forbidding the axiom of excluded middle in natural deduction.

Remarks: The cut rule is eliminable (*Hauptsatz* [**Gentzen1935**]), and hence intuitionistic predicate logic is consistent and its propositional fragment is decidable [**Gentzen1935a**]. **LJ** is complete relative to **NJ** {2} and sound relative to the Hilbert-style calculus **LHJ** [**Gentzen1935a**].

Entry 4 by: Giselle Reis

Epsilon Calculus   (1923, 1939)

Epsilon calculus is first-order predicate calculus extended by the epsilon-operator and the critical axiom. Terms t and Formulas A, B of epsilon calculus are defined as follows.

$$t ::= a \mid x \mid f(t_0, \ldots, t_{n-1}) \mid \varepsilon_x A, \quad A, B ::= P(t_0, \ldots, t_{n-1}) \mid \neg A \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \exists x.A \mid \forall x.A,$$

where a, x, f, and P range over free variables, bound variables, function symbols, and predicate symbols, respectively. Each f, P has an arbitrary arity n. The critical axiom is, for any formula $A(x)$, given as follows.

$$A(t) \rightarrow A(\varepsilon_x A(x)).$$

Clarifications: *Epsilon calculus* is an extension of classical first-order predicate calculus [**HilbertBernays1939**, **MoserZach06**, **AvigadZach13**]. The symbol ε is called the *epsilon-operator*, which constructs a term by quantifying a bound variable in a formula. A formula A with occurrences of a variable x which is not quantified is written as $A(x)$, and $A(t)$ denotes a formula obtained by replacing the corresponding x by a term t in A. The existential and universal quantifiers are definable due to the epsilon-operator.

$$\exists x.A(x) := A(\varepsilon_x A(x)), \qquad\qquad \forall x.A(x) := A(\varepsilon_x \neg A(x)).$$

Pure epsilon calculus is elementary calculus extended by the epsilon-operator and the critical axiom.

History: Epsilon calculus is due to Hilbert. He formulated the prototype of epsilon calculus [**Hilbert1923**] by means of the *tau-operator* and the axiom $A(\tau_x A(x)) \rightarrow A(t)$ instead of the epsilon-operator and the critical axiom. The formulation based on the epsilon-operator first appeared in Ackermann's dissertation [**Ackermann1924**] under the supervision of Hilbert. Hilbert and Bernays gave a comprehensive account of epsilon calculus and its applications [**HilbertBernays1939**].

Remarks: *First epsilon theorem* states that if there is a proof in epsilon calculus of an $\exists, \forall, \varepsilon$-free formula, this formula is provable in elementary calculus. *Second epsilon theorem* states that if there is a proof in epsilon calculus of an ε-free formula, this formula is provable in predicate calculus. By means of epsilon calculus Hilbert and Bernays gave the first correct proof of *Herbrand's theorem* [**HilbertBernays1939**, **MoserZach06**].

Entry 5 by: Kenji Miyamoto, Georg Moser

Kleene's Classical G3 System   (1952)

$$\overline{A, \Gamma \vdash \Theta, A}$$

$$\frac{A \to B, \Gamma \vdash \Theta, A \quad B, A \to B, \Gamma \vdash \Theta}{A \to B, \Gamma \vdash \Theta} \to\vdash \qquad \frac{A, \Gamma \vdash \Theta, A \to B, B}{\Gamma \vdash \Theta, A \to B} \vdash\to$$

$$\frac{A, A \vee B, \Gamma \vdash \Theta \quad B, A \vee B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta} \vee\vdash \qquad \frac{\Gamma \vdash \Theta, A \vee B, A}{\Gamma \vdash \Theta, A \vee B} \vdash\vee_1 \quad \frac{\Gamma \vdash \Theta, A \vee B, B}{\Gamma \vdash \Theta, A \vee B} \vdash\vee_2$$

$$\frac{A, A \wedge B, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta} \wedge\vdash_1 \quad \frac{B, A \wedge B, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta} \wedge\vdash_2 \qquad \frac{\Gamma \vdash \Theta, A \wedge B, A \quad \Gamma \vdash \Theta, A \wedge B, B}{\Gamma \vdash \Theta, A \wedge B} \vdash\wedge$$

$$\frac{\neg A, \Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta} \neg\vdash \qquad \frac{A, \Gamma \vdash \Theta, \neg A}{\Gamma \vdash \Theta, \neg A} \vdash\neg$$

$$\frac{A(t), \forall x A(x), \Gamma \vdash \Theta}{\forall x A(x), \Gamma \vdash \Theta} \forall\vdash \qquad \frac{\Gamma \vdash \Theta, \forall x A(x), A(b)}{\Gamma \vdash \Theta, \forall x A(x)} \vdash\forall$$

$$\frac{A(b), \exists x A(x), \Gamma \vdash \Theta}{\exists x A(x), \Gamma \vdash \Theta} \exists\vdash \qquad \frac{\Gamma \vdash \Theta, \exists x A(x), A(t)}{\Gamma \vdash \Theta, \exists x A(x)} \vdash\exists$$

The term t is free for x in $A(x)$.
The variable b is free for x in $A(x)$ and (unless b is x) does not occur in $\Gamma, \Theta, A(x)$.

Clarifications: A, B are formulae; Γ, Θ are finite (possibly empty) sequences of formulae; x is a variable; $A(x)$ is a formula. In applications of the rules every sequent $\Gamma \vdash \Theta$ can be replaced with a *cognate* one, i.e., a sequent $\Gamma' \vdash \Theta'$ such that the sets of formulae occurring in Γ and Γ' resp. Θ and Θ' are the same.

History: Kleene's systems, introduced in his 1952 monograph, were the staple of generations of logicians, who learned about sequent calculus from his textbooks [**Kleene:1952**] and [**Kleene:1967**].

Remarks: Based on Gentzen's sequent calculus **LK** {3} (called classical **G1** in [**Kleene:1952**]). Seems to be the first system (with {7}) in which admissibility of contraction is obtained by copying the principal formulae into the premisses (accordingly, this is sometimes called *Kleene's Method*). Used together with its single-conclusion version for intuitionistic logic {7} to uniformly obtain decidability of propositional classical and intuitionistic logics via backwards proof search in [**Kleene:1952**].

Entry 6 by: Björn Lellmann, Valeria de Paiva

Kleene's Intuitionistic G3 System (1952)

$$\overline{A, \Gamma \vdash A}$$

$$\frac{A \to B, \Gamma \vdash A \quad B, A \to B, \Gamma \vdash \Theta}{A \to B, \Gamma \vdash \Theta} \ \to\vdash \qquad\qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B} \ \vdash\to$$

$$\frac{A, A \lor B, \Gamma \vdash \Theta \quad B, A \lor B, \Gamma \vdash \Theta}{A \lor B, \Gamma \vdash \Theta} \ \lor\vdash \qquad\qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \lor B} \ \vdash\lor_1 \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \lor B} \ \vdash\lor_2$$

$$\frac{A, A \land B, \Gamma \vdash \Theta}{A \land B, \Gamma \vdash \Theta} \ \land\vdash_1 \quad \frac{B, A \land B, \Gamma \vdash \Theta}{A \land B, \Gamma \vdash \Theta} \ \land\vdash_2 \qquad\qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \ \vdash\land$$

$$\frac{\neg A, \Gamma \vdash A}{\neg A, \Gamma \vdash \Theta} \ \neg\vdash \qquad\qquad \frac{A, \Gamma \vdash \neg A}{\Gamma \vdash \neg A} \ \vdash\neg$$

$$\frac{A(t), \forall x A(x), \Gamma \vdash \Theta}{\forall x A(x), \Gamma \vdash \Theta} \ \forall\vdash \qquad\qquad \frac{\Gamma \vdash A(b)}{\Gamma \vdash \forall x A(x)} \ \vdash\forall$$

$$\frac{A(b), \exists x A(x), \Gamma \vdash \Theta}{\exists x A(x), \Gamma \vdash \Theta} \ \exists\vdash \qquad\qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)} \ \vdash\exists$$

The term t is free for x in $A(x)$.
The variable b is free for x in $A(x)$ and (unless b is x) does not occur in $\Gamma, \Theta, A(x)$.

Clarifications: A, B are formulae; Γ and Θ are a finite (possibly empty) sequences of formulae with Θ containing at most one formula; x is a variable; $A(x)$ is a formula. In applications of the rules every sequent $\Gamma \vdash \Theta$ can be replaced with a *cognate* one, i.e., a sequent $\Gamma' \vdash \Theta'$ such that the sets of formulae occurring in Γ and Γ' resp. Θ and Θ' are the same (respecting the restriction to at most one formula on the right hand side).

History: Kleene's systems, introduced in his 1952 monograph, were the staple of generations of logicians, who learned about sequent calculus from his textbooks [**Kleene:1952**] and [**Kleene:1967**].

Remarks: Based on Gentzen's sequent calculus **LJ** {4} (corresponding to intuitionistic **G1** in [**Kleene:1952**]). Seems to be the first system (with {6}) in which admissibility of contraction is obtained by copying the principal formulae into the premises (accordingly, this is sometimes called *Kleene's Method*). Used together with its multi-conclusion version for classical logic {6} to uniformly obtain decidability of propositional classical and intuitionistic logics via backwards proof search in [**Kleene:1952**].

Multi-Conclusion Sequent Calculus LJ' (1954)

$$\overline{A \vdash A}$$

$$\frac{\Gamma \vdash \Theta, A \quad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda} \; cut$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta} \; \wedge_l \qquad \frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B} \; \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta} \; \vee_l \qquad \frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2} \; \vee_r$$

$$\frac{\Gamma \vdash \Theta, A \quad B, \Delta \vdash \Lambda}{A \to B, \Gamma, \Delta \vdash \Theta, \Lambda} \; \to_l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B} \; \to_r$$

$$\frac{A\alpha, \Gamma \vdash \Theta}{\exists x.Ax, \Gamma \vdash \Theta} \; \exists_l \; \frac{\Gamma \vdash \Theta, At}{\Gamma \vdash \Theta, \exists x.Ax} \; \exists_r \; \frac{At, \Gamma \vdash \Theta}{\forall x.Ax, \Gamma \vdash \Theta} \; \forall_l \; \frac{\Gamma \vdash A\alpha}{\Gamma \vdash \forall x.Ax} \; \forall_r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta} \; \neg_l \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \; \neg_r \qquad \frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta} \; e_l \; \frac{\Gamma \vdash \Theta, B, A, \Lambda}{\Gamma \vdash \Theta, A, B, \Lambda} \; e_r$$

$$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} \; c_l \qquad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A} \; c_r \qquad \frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} \; w_l \qquad \frac{\Gamma \vdash}{\Gamma \vdash A} \; w_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

Clarifications: While **LJ** {4} is defined by restricting **LK** {3} to single conclusion, in **LJ′** only the rules $\neg_r$, $\to_r$ and $\forall_r$ have this restriction.

History: LJ′ was proposed in [**Maehara1954**] and used to prove the completeness of **LJ** {4} in [**takeuti87**]. It also appears in [**dragalin88**] (as GHPC) and [**dummett77**] (as L').

Remarks: LJ′ is equivalent to **LJ**, and this is established by translating sequents of the form $\Gamma \vdash A_1, ..., A_n$ into sequents of the form $\Gamma \vdash A_1 \vee ... \vee A_n$. Cut can be eliminated by using a combination of the rewriting rules for cut-elimination in **LJ** and **LK** and permutation of inferences, as shown by Schellinx [**Schellinx91**] and Reis [**GisellePhD**].

Entry 8 by: Giselle Reis, Valeria de Paiva

Lambek Calculus (1958)

$$\frac{}{A \vdash A}\ ax \qquad\qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2, A, \Gamma_3 \vdash C}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash C}\ cut$$

$$\frac{}{\cdot \vdash I}\ I_r \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, I, \Gamma_2 \vdash A}\ I_l \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes_r \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \otimes B, \Gamma_2 \vdash C}\ \otimes_l$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightharpoonup B}\ \rightharpoonup_r \quad \frac{\Gamma_1 \vdash A \quad \Gamma_2, B, \Gamma_3 \vdash C}{\Gamma_1, A \rightharpoonup B, \Gamma_2, \Gamma_3 \vdash C}\ \rightharpoonup_l \quad \frac{A, \Gamma \vdash B}{\Gamma \vdash B \leftharpoonup A}\ \leftharpoonup_r \quad \frac{\Gamma_1 \vdash A \quad \Gamma_2, B, \Gamma_3 \vdash C}{\Gamma_1, B \leftharpoonup A, \Gamma_2, \Gamma_3 \vdash C}\ \leftharpoonup_l$$

Clarifications: The Lambek Calculus described here was introduced by Joachim Lambek to study sentence structure in 1958 [**lambek1958**]. Actually the calculus Lambek first introduced, despite being motivated by algebraic considerations as we are told in [**lambek1988**], had no constant corresponding to the unit of the tensor product I. The Lambek calculus can be seen as the logic one obtains from Gentzen's Intuitionistic Propositional Logic (LJ) {4} if we remove the structural rules of contraction, weakening and commutation. Lambek also introduced another calculus [**lambek1961**] where even the associativity of the tensor is not valid.

History: The system now known as the basic Lambek Calculus was introduced in 1958 by Joachim Lambek as the "Syntactic Calculus" [**lambek1958**]. Lambek's motivation was to "to obtain an effective rule (or algorithm) for distinguishing sentences from non-sentences, which works not only for the formal languages of interest to the mathematical logician, but also for natural languages [. . .]", as explained by Moortgat in [**moortgat2010**]. After a long period of ostracism, around the middle 1980s the Syntactic Calculus, now called the Lambek Calculus was taken up by logicians interested in Computational Linguistics, especially van Benthem, Buszkowski and Moortgat. They realized that a computational semantics for categorical derivations along the lines of the Curry-Howard proofs-as-programs interpretation would provide us with a "parsing-as-deduction" paradigm and a powerful tool to study "logical" derivational semantics. Around the same time, the introduction of Linear Logic {12}, by Jean-Yves Girard also gave a new impulse to the work in Categorical Grammars. This was because of Linear Logic's insight that even if you had a very weak proof system, you could introduce structural rules in a controlled fashion and hence obtain more expressive systems, by the use of the so called modalities. Since no expressivity is lost in this process, this opened the way for various types of experiments, trying to make sure that the logical system could cope with more phenomena from the language, see discussion of examples in [**moortgat2010**].

Entry 9 by: Harley Eades III, Valeria de Paiva

Expansion trees, *eigenvariables*, and the function $\mathsf{Sh}(-)$ (read *shallow formula of*), that maps an expansion tree to a formula, are defined as follows:

1. If A is $\top$ (true), $\bot$ (false), or a literal, then A is an expansion tree with top node A, and $\mathsf{Sh}(A) = A$.

2. If E is an expansion tree with $\mathsf{Sh}(E) = [y/x]A$ and y is not an eigenvariable of any node in E, then $E' = \forall x.\, A +^y E$ is an expansion tree with top node $\forall x.\, A$ and $\mathsf{Sh}(E') = \forall x.\, A$. The variable y is called an *eigenvariable* of (the top node of) E'. The set of eigenvariables of all nodes in an expansion tree is called the *eigenvariables of* the tree.

3. If $\{t_1, \ldots, t_n\}$ (with $n \geq 0$) is a set of terms and $E_1, \ldots, E_n$ are expansion trees with pairwise disjoint eigenvariable sets and with $\mathsf{Sh}(E_i) = [t_i/x]A$ for $i \in \{1, \ldots, n\}$, then $E' = \exists x.\, A +^{t_1} E_1 \ldots +^{t_n} E_n$ is an expansion tree with top node $\exists x.\, A$ and $\mathsf{Sh}(E') = \exists x.\, A$. The terms $t_1, \ldots, t_n$ are known as the *expansion terms* of (the top node of) E'.

4. If E_1 and E_2 are expansion trees that share no eigenvariables and $\circ \in \{\wedge, \vee\}$, then $E_1 \circ E_2$ is an expansion tree with top node $\circ$ and $\mathsf{Sh}(E_1 \circ E_2) = \mathsf{Sh}(E_1) \circ \mathsf{Sh}(E_2)$.

In the expansion tree $\forall x.\, A +^x E$ (resp. in $\exists x.\, A +^{t_1} E_1 \ldots +^{t_n} E_n$), we say that x (resp. t_i) *labels* the top node of E (resp. E_i, for any $i \in \{1, \ldots, n\}$). A term t *dominates* a node in an expansion tree if it labels a parent node of that node in the tree.

For an expansion tree E, the quantifier-free formula $\mathsf{Dp}(E)$, called the *deep formula of* E, is defined as:

- $\mathsf{Dp}(E) = E$ if E is $\top$, $\bot$, or a literal;
- $\mathsf{Dp}(E_1 \circ E_2) = \mathsf{Dp}(E_1) \circ \mathsf{Dp}(E_2)$ for $\circ \in \{\wedge, \vee\}$;
- $\mathsf{Dp}(\forall x.\, A +^y E) = \mathsf{Dp}(E)$; and
- $\mathsf{Dp}(\exists x.\, A +^{t_1} E_1 \cdots +^{t_n} E_n) = \mathsf{Dp}(E_1) \vee \ldots \vee \mathsf{Dp}(E_n)$ if $n > 0$, and $\mathsf{Dp}(\exists x.\, A) = \bot$.

Let $\mathcal{E}$ be an expansion tree and let $<^0_{\mathcal{E}}$ be the binary relation on the occurrences of expansion terms in $\mathcal{E}$ defined by $t <^0_{\mathcal{E}} s$ if there is an x which is free in s and which is the eigenvariable of a node dominated by t. Then $<_{\mathcal{E}}$, the transitive closure of $<^0_{\mathcal{E}}$, is called the *dependency relation* of $\mathcal{E}$.

An expansion tree $\mathcal{E}$ is said to be an *expansion proof* if $<_{\mathcal{E}}$ is acyclic and $\mathsf{Dp}(\mathcal{E})$ is a tautology; in particular, $\mathcal{E}$ is an *expansion proof of* $\mathsf{Sh}(\mathcal{E})$.

Clarifications: The soundness and completeness theorem for expansion trees is the following. A formula B is a theorem of first-order logic if and only if there is an expansion proof Q such that $\mathsf{Sh}(Q) = B$.

History: Expansion trees and proofs [**miller87sl**, **miller83**] generalize Herbrand's disjunctions and Gentzen's mid-sequents to the non-prenex case. They were originally defined for higher-order classical logic and used to prove soundness of skolemization and a generalization of Herbrand's theorem for this logic. Expansion trees are an early example of a matrix-based proof system emphasizing parallelism in a manner similar to that found in proof nets {**??**}. That parallelism is explicitly analyzed in [**chaudhuri14jlc**] using a multi-focused version of LKF {**??**}.

Intuitionistic Linear Logic (ILL) (1987)

STRUCTURAL			

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \quad A,\Delta \vdash B}{\Gamma,\Delta \vdash B} \ (cut) \qquad \frac{\Gamma,A,B,\Delta \vdash C}{\Gamma,B,A,\Delta \vdash C}$$

MULTIPLICATIVE

$$\frac{}{\vdash 1} \qquad \frac{\Gamma \vdash A}{\Gamma,1 \vdash A} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma,\Delta \vdash A \otimes B} \qquad \frac{\Gamma,A,B \vdash C}{\Gamma,A \otimes B \vdash C}$$

$$\frac{\Gamma,A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Gamma \vdash A \quad B,\Delta \vdash C}{\Gamma,A \multimap B,\Delta \vdash C}$$

ADDITIVE

$$\frac{}{\Gamma \vdash \top} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \qquad \frac{\Gamma,A_i \vdash B}{\Gamma,A_1 \& A_2 \vdash B}$$

$$\frac{}{\Gamma,0 \vdash A} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2} \qquad \frac{\Gamma,A \vdash C \quad \Gamma,B \vdash C}{\Gamma,A \oplus B \vdash C}$$

EXPONENTIAL

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \qquad \frac{\Gamma \vdash B}{\Gamma,!A \vdash B} \qquad \frac{\Gamma,A \vdash B}{\Gamma,!A \vdash B} \qquad \frac{\Gamma,!A,!A \vdash B}{\Gamma,!A \vdash B}$$

Clarifications: Succedents are single formulas. Antecedents are ordered list of formulas. If Γ is the list $A_1,\ldots,A_n$ of formulas, $!\Gamma$ denotes the list $!A_1,\ldots,!A_n$. First order quantifiers can be added with rules similar to **LJ** {4}. Conversely, removing the exponential rules leads to the intuitionistic multiplicative additive linear logic (IMALL). And by further removing the additive rules, the intuitionistic multiplicative linear logic (IMLL) [**mints1977closed**] is obtained.

History: Introduced by Girard and Lafont in [**lafont1987tapsoft**] as intuitionistic variant of **LL** {12}. **ILL** has multiple applications in categorical logic.

Remarks: Enjoys cut elimination [**lafont1987tapsoft**].

Linear Sequent Calculus LL   (1987)

$$\frac{}{\vdash A^\perp, A} \qquad \frac{\vdash \Gamma, A \qquad \vdash \varDelta, A^\perp}{\vdash \Gamma, \varDelta} \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \varDelta, B}{\vdash \Gamma, \varDelta, A \otimes B} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \,\bindnasrepma\, B} \qquad \frac{}{\vdash 1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \qquad \frac{}{\vdash \Gamma, \top}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}$$

$$\begin{array}{ll} & (A \otimes B)^\perp = A^\perp \,\bindnasrepma\, B^\perp \quad 1^\perp = \bot \\ (X^\perp)^\perp = X & (A \,\bindnasrepma\, B)^\perp = A^\perp \otimes B^\perp \quad \bot^\perp = 1 \\ (!A)^\perp = ?(A^\perp) & (A \oplus B)^\perp = A^\perp \,\&\, B^\perp \quad 0^\perp = \top \\ (?A)^\perp = !(A^\perp) & (A \,\&\, B)^\perp = A^\perp \oplus B^\perp \quad \top^\perp = 0 \end{array}$$

Γ and $\varDelta$ are lists of formulas.
σ is a permutation.

Clarifications: If $\Gamma = A_1, \ldots, A_n$ then $?\Gamma = ?A_1, \ldots, ?A_n$. Negation is not a connective. It is defined using De Morgan's laws so that $(A^\perp)^\perp = A$. The linear implication can be defined as $A \multimap B = A^\perp \,\bindnasrepma\, B$.

History: Linear Logic and its sequent calculus **LL** [**ll**] come from the analysis of intuitionistic logic through Girard's decomposition of the intuitionistic implication into the linear implication: $A \to B = !A \multimap B$.

Remarks: Cut elimination holds. **LL** is sound and complete with respect to phase semantics [**ll**]. **LL** is not decidable [**Lincoln1992**]. Sequent calculi **LK** {3} and **LJ** {4} can be translated into **LL**.

Entry 12 by: Olivier Laurent

Pure Type Systems (1989)

$$\frac{}{\vdash c : s} \; axiom \; ((c : s) \in \mathcal{A}) \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \; start \; (x \notin \Gamma)$$

$$\frac{\Gamma \vdash M : B \quad \Gamma \vdash A : s}{\Gamma, x : A \vdash M : B} \; weakening \; (x \notin \Gamma) \qquad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.B : s_3} \; product \; ((s_1, s_2, s_3) \in \mathcal{R})$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.B : s}{\Gamma \vdash \lambda x : A.M : \Pi x : A.B} \; abstraction \qquad \frac{\Gamma \vdash M : \Pi x : A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M \, N : B[x := N]} \; application$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s \quad A \equiv_\beta B}{\Gamma \vdash M : B} \; conversion$$

Clarifications: *Pure type systems* (PTS) are a general class of typed λ calculus. They represent logical systems through the Curry-Howard correspondence and the "propositions as types" interpretation. The syntax is given by the grammar:

$$\mathcal{T} ::= \mathcal{V} \mid C \mid \Pi \mathcal{V} : \mathcal{T}.\mathcal{T} \mid \lambda \mathcal{V} : \mathcal{T}.\mathcal{T} \mid \mathcal{T} \, \mathcal{T}$$

where $\mathcal{V}$ is a set of variables and C is a set of constants. A PTS is parameterized by a *specification* $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ where $\mathcal{S} \subseteq C$ is the set of *sorts*, $\mathcal{A} \subseteq C \times \mathcal{S}$ is the set of *axioms*, and $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{S}$ is the set of *rules*.

History: Pure type systems were independently introduced by Berardi and Terlouw as a generalization of systems of the λ cube, and further developed and popularized by Barendregt, Geuvers, Nederhof [**barendregt˙introduction˙1991**, **geuvers˙modular˙1991**, **barendregt˙lambda˙1992**, **geuvers˙logics˙1993**]. Many important systems can be expressed as PTSs, including simply typed λ calculus ($\lambda{\rightarrow}$), $\lambda\Pi$ calculus {17} (λP), system F {**??**} ($\lambda 2$), and the calculus of constructions (λC):

$$\mathcal{S} = \{*, \Box\} \qquad \mathcal{A} = \{(*, \Box)\} \qquad \mathcal{R}_{\rightarrow} = \{(*, *, *)\}$$

$$\mathcal{R}_P = \mathcal{R}_{\rightarrow} \cup \{(*, \Box, \Box)\} \qquad \mathcal{R}_2 = \mathcal{R}_{\rightarrow} \cup \{(\Box, *, *)\} \qquad \mathcal{R}_C = \mathcal{R}_P \cup \mathcal{R}_2 \cup \{(\Box, \Box, \Box)\}$$

as well as intuitionistic higher-order logic (λHOL). Pure type systems form the basis of many proof assistants such as Automath, Lego, Coq, Agda, and Matita.

Remarks: Soundness and decidability of type checking in PTSs are closely related to *strong normalization* (SN), i.e. the property that all well-typed terms terminate. Not all pure type systems are SN. Examples of PTSs that are *not* SN (and are therefore inconsistent) are Girard's system U and the universal PTS $\lambda *$:

$$\mathcal{S} = \{*\} \qquad \mathcal{A} = \{(*, *)\} \qquad \mathcal{R} = \{(*, *, *)\}$$

Entry 13 by: Ali Assaf

Full Intuitionistic Linear Logic (FILL) (1990)

$$\dfrac{}{x : A \vdash x : A} \; Ax \qquad\qquad \dfrac{\Gamma \vdash t : A \mid \Delta \quad \Gamma', y : A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]\Delta'} \; Cut$$

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, x : \top \vdash \mathsf{let}\, x\, \mathsf{be}\, * \,\mathsf{in}\, \Delta} \; \top_L \qquad\qquad \dfrac{}{\cdot \vdash * : \top} \; \top_R$$

$$\dfrac{}{x : \bot \vdash \cdot} \; \bot_L \qquad\qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \circ \, : \bot \mid \Delta} \; \bot_R$$

$$\dfrac{\Gamma, x : A, y : B \vdash \Delta}{\Gamma, z : A \otimes B \vdash \mathsf{let}\, z\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, \Delta} \; \otimes_L \qquad \dfrac{\Gamma \vdash t_1 : A \mid \Delta \quad \Gamma' \vdash t_2 : B \mid \Delta'}{\Gamma, \Gamma' \vdash t_1 \otimes t_2 : A \otimes B \mid \Delta \mid \Delta'} \; \otimes_R$$

$$\dfrac{\Gamma \vdash t : A \mid \Delta \quad \Gamma', x : B \vdash t_i : C_i}{\Gamma, y : A \multimap B, \Gamma' \vdash [y\,t/x]t_i : C_i \mid \Delta} \; \multimap_L \qquad \dfrac{\Gamma, x : A \vdash t : B \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.t : A \multimap B \mid \Delta} \; \multimap_R$$

$$\dfrac{\Gamma, x : A \vdash t_i : C_i \quad \Gamma', y : B \vdash t_j : D_j}{\Gamma, \Gamma', z : A \,⅋\, B \vdash \mathsf{let\text{-}pat}\, z\, (x \,⅋\, -)\, t_i : C_i \mid \mathsf{let\text{-}pat}\, z\, (- \,⅋\, y)\, t_j : D_j} \; ⅋_L$$

$$\dfrac{\Gamma \vdash \Delta \mid t_1 : A \mid t_2 : B \mid \Delta'}{\Gamma \vdash \Delta \mid t_1 \,⅋\, t_2 : A \,⅋\, B \mid \Delta'} \; ⅋_R$$

Clarifications: Both the left-hand and right-hand sides of sequents above are multisets of formulas, denoted Γ and Δ. The terms annotating formulas are standard terms used in the simply typed λ-calculus. Capture avoiding substitution is denoted by $[t/x]t'$, and uniformly replaces every occurrence of x in t' with t. The definition of the let-pattern function used in the rule $⅋_L$ is defined as follows:

$$\mathsf{let\text{-}pat}\, z\, (x \,⅋\, -)\, t = t \quad \mathsf{let\text{-}pat}\, z\, (- \,⅋\, y)\, t = t \quad \mathsf{let\text{-}pat}\, z\, p\, t = \mathsf{let}\, z\, \mathsf{be}\, p\, \mathsf{in}\, t$$
$$\text{where } x \notin \mathsf{FV}(t) \qquad \text{where } y \notin \mathsf{FV}(t)$$

We denote vectors of terms (resp. types) by t_i (resp. A_j). The function $\mathsf{FV}(\Delta)$ constructs the set of all free variables in each term found in Δ.

History: The original formulation of FILL by Valeria de Paiva in her thesis [**dePaiva:1990**] did not satisfy cut-elimination, as shown by Schellinx. Martin Hyland and Valeria de Paiva [**Hyland:1993**] added a term assignment system to cope with the notion of dependency in the right implication rule and obtain cut-elimination. However, there was still a mistake in the par rule in [**Hyland:1993**], which was corrected independently, with different proof methods, by Bierman [**Bierman:1996**], Bellin [**Bellin:1997**], Brauner/dePaiva [**Brauner:1998**], dePaiva/Ritter [**dePaiva:2006**]. The version here is the minimal modification suggested by Bellin, (who used proofnets), but using a traditional term assignment, as described in Eades/dePaiva [**Eades:2015**].

Constructive Classical Logic LC   (1991)

$$\frac{}{\vdash \neg P \, ; P} \qquad \frac{\vdash \Gamma \, ; P \qquad \vdash \Delta, \neg P \, ; \Pi}{\vdash \Gamma, \Delta \, ; \Pi} \qquad \frac{\vdash \Gamma, N \, ; \qquad \vdash \Delta, \neg N \, ; \Pi}{\vdash \Gamma, \Delta \, ; \Pi}$$

$$\frac{\vdash \Gamma \, ; \Pi}{\vdash \sigma(\Gamma) \, ; \Pi} \qquad \frac{\vdash \Gamma \, ; P}{\vdash \Gamma, P \, ;} \qquad \frac{\vdash \Gamma, A, A \, ; \Pi}{\vdash \Gamma, A \, ; \Pi} \qquad \frac{\vdash \Gamma \, ; \Pi}{\vdash \Gamma, A \, ; \Pi}$$

$$\frac{}{\vdash \, ; V} \qquad \frac{}{\vdash \Gamma, \neg F \, ; \Pi}$$

$$\frac{\vdash \Gamma \, ; P \qquad \vdash \Delta \, ; Q}{\vdash \Gamma, \Delta \, ; P \wedge Q} \qquad \frac{\vdash \Gamma, M \, ; \Pi \qquad \vdash \Delta, N \, ; \Pi}{\vdash \Gamma, \Delta, M \wedge N \, ; \Pi}$$

$$\frac{\vdash \Gamma \, ; P \qquad \vdash \Delta, N \, ;}{\vdash \Gamma, \Delta \, ; P \wedge N} \qquad \frac{\vdash \Gamma, M \, ; \qquad \vdash \Delta \, ; Q}{\vdash \Gamma, \Delta \, ; M \wedge Q}$$

$$\frac{\vdash \Gamma, A, B \, ; \Pi}{\vdash \Gamma, A \vee B \, ; \Pi} A \vee B \text{ negative} \qquad \frac{\vdash \Gamma \, ; P}{\vdash \Gamma \, ; P \vee Q} \qquad \frac{\vdash \Gamma \, ; Q}{\vdash \Gamma \, ; P \vee Q}$$

$$\neg\neg X = X \qquad \neg(A \wedge B) = \neg A \vee \neg B \qquad \neg(A \vee B) = \neg A \wedge \neg B$$

Formulas:　　　　　　$A, B ::= \ P \ | \ N$
Positive formulas:　$P, Q ::= \ X \ | \ V \ | \ F \ | \ P \wedge Q \ | \ P \wedge N \ | \ M \wedge Q \ | \ P \vee Q$
Negative formulas:　$M, N ::= \neg X \ | \ \neg V \ | \ \neg F \ | \ M \vee N \ | \ M \vee Q \ | \ P \vee N \ | \ M \wedge N$

Γ and Δ are lists of formulas, and Π consists of 0 or 1 positive formula.
σ is a permutation.

Clarifications: Negation is not a connective. It is defined using De Morgan's laws so that $\neg\neg A = A$. There are two atomic formulas for truth (a positive one V and a negative one $\neg F$) and two atomic formulas for falsity (a positive one F and a negative one $\neg V$). Sequents have the shape $\vdash \Gamma \, ; \Pi$ where Π is called the stoup.

History: **LC** [**lc**] comes from the analysis of classical logic inside the coherent semantics of linear logic [**ll**] together with the use of the focusing property [**focal**].

Remarks: Cut elimination holds. **LK** {3} can be translated into **LC**, but not in a canonical manner. **LC** satisfies constructive properties such as the disjunction property: if $\vdash \, ; P \vee Q$ is provable then $\vdash \, ; P$ or $\vdash \, ; Q$ as well. **LC** admits a denotational semantics through correlation spaces [**lc**] (a variant of coherence spaces [**ll**]).

Classical Natural Deduction ($\lambda\mu$-calculus) (1992)

STRUCTURAL SUBSYSTEM

$$\frac{A^a \in \Gamma}{a : \Gamma \vdash A \mid \Delta} \; Ax$$

$$\frac{c : \Gamma \vdash A^\alpha, \Delta}{\mu\alpha.c : \Gamma \vdash A \mid \Delta} \; Focus \qquad \frac{p : \Gamma \vdash A \mid \Delta \qquad A^\alpha \in \Delta}{[\alpha]p : \Gamma \vdash \Delta} \; Unfocus$$

INTRODUCTION RULES

$$\frac{p : \Gamma \vdash A_1 \wedge A_2 \mid \Delta}{\pi_1(p) : \Gamma \vdash A_i \mid \Delta} \; \wedge_E^i \qquad \frac{p_1 : \Gamma \vdash A_1 \mid \Delta \quad p_2 : \Gamma \vdash A_2 \mid \Delta}{(p_1, p_2) : \Gamma \vdash A_1 \wedge A_2 \mid \Delta} \; \wedge_I$$

$$\frac{p : \Gamma \vdash A_1 \vee A_2 \mid \Delta \quad p_1 : \Gamma, A_1^{a_1} \vdash C \mid \Delta \quad p_2 : \Gamma, A_2^{a_2} \vdash C \mid \Delta}{\text{case } p \text{ of } [a_1 \Rightarrow p_1 \mid a_2 \Rightarrow p_2] : \Gamma \vdash C \mid \Delta} \; \vee_E$$

$$\frac{q : \Gamma \vdash A_i \mid \Delta}{\iota_i(q) : \Gamma \vdash A_1 \vee A_2 \mid \Delta} \; \vee_I^i$$

$$\frac{p : \Gamma \vdash A \rightarrow B \mid \Delta \quad q : \Gamma \vdash A \mid \Delta}{p\,q : \Gamma \vdash B \mid \Delta} \; \rightarrow_E \qquad \frac{p : \Gamma, A^a \vdash B \mid \Delta}{\lambda a.p : \Gamma \vdash A \rightarrow B \mid \Delta} \; \rightarrow_I$$

$$\frac{p : \Gamma \vdash \exists x A \mid \Delta \quad q : \Gamma, A[y/x]^a \vdash C \mid \Delta}{\text{dest } p \text{ as } (y, a) \text{ in } q : \Gamma \vdash C \mid \Delta} \; \exists_E \qquad \frac{p : \Gamma \vdash A[t/x] \mid \Delta}{(t, p) : \Gamma \vdash \exists x A \mid \Delta} \; \exists_I$$

$$\frac{p : \Gamma \vdash \forall x A \mid \Delta}{p\,t : \Gamma \vdash A[t/x] \mid \Delta} \; \forall_E \qquad \frac{p : \Gamma \vdash A[y/x] \mid \Delta}{\lambda y.p : \Gamma \vdash \forall x A \mid \Delta} \; \forall_I$$

$$\frac{p : \Gamma \vdash \bot \mid \Delta}{\text{efq } p : \Gamma \vdash C \mid \Delta} \; \bot_E \qquad \frac{}{\Gamma \vdash () : \top \mid \Delta} \; \top_I$$

Clarifications: There are two kinds of sequents: first $p : \Gamma \vdash A \mid \Delta$ with a distinguished formula on the right for typing the so-called *unnamed* term p, second $c : \Gamma \vdash \Delta$ with no distinguished formula for typing the so-called *named* term c. The syntax of the underlying $\lambda\mu$-calculus is:

$$c \quad ::= [\alpha]p$$
$$p, q ::= a \mid \mu\alpha.c \mid (p, p) \mid \pi_i(p) \mid \iota_i(p) \mid \text{case } p \text{ of } [a_1 \Rightarrow p_1 \mid a_2 \Rightarrow p_2]$$
$$\mid \quad \lambda a.p \mid p\,q \mid \lambda x.p \mid p\,t \mid (t, p) \mid \text{dest } p \text{ as } (x, a) \text{ in } q \mid () \mid \text{efq } p$$

The variables used for referring to assumptions in Γ and to conclusions in Δ range over distinct classes (denoted by Latin and Greek letters respectively). In the rules $\exists_E$ (resp. $\forall_I$), y is assumed fresh in Γ, Δ and $\exists x A$ (resp. $\forall x A$).

History: This system, defined in Parigot [**Parigot92**], highlights that classical logic in natural deduction can be obtained from allowing several conclusions with contraction and weakening on the right of the sequent, as in Gentzen's LK. Additionally, the system assigns to this form of classical reasoning a computational content, based on the μ and bracket operator which provides with a fine-grained decomposition of the operators `call-cc` (from Scheme/ML) or C (from [**FelFriKohDub86**]) that were known at this time to provide computational content to classical logic [**Griffin90**], as well as a decomposition of Prawitz's classical elimination rule of negation [**Prawitz65**].

The original presentation [**Parigot92**] only contains implication as well as first-order and second-order universal quantification à la Curry (i.e. without leaving trace of the quantification in the proof-term, what corresponds to computationally interpreting quantification as an intersection type). The presentation above has quantification à la Church (i.e. with an explicit trace in the proof term) what makes the calculus compatible with several reduction strategies such as both call-by-name or call-by-value (see e.g. [**HerbelinHdR**]). Variants with multiplicative disjunctions can be found in [**Selinger01**] or [**PymRitter01**], or multiplicative conjunctions in [**HerbelinHdR**].

A standard variant originating in [**deGroote94**] uses only one kind of sequents, interpreting $c : \Gamma \vdash \Delta$ as $c : \Gamma \vdash \bot \mid \Delta$ (and hence removing $\bot_E$ and merging the syntactic categories c and p into one). This variant is logically equivalent to the original presentation (in the presence of $\bot$), but not computationally equivalent [**HerbelinSaurin10**].

Typed LF for Type Theories   (1994)

$$\frac{}{<>\vdash \textbf{valid}} \quad \frac{\Gamma \vdash K \,\textbf{kind} \quad x \notin FV(\Gamma)}{\Gamma, x : K \vdash \textbf{valid}} \quad \frac{\Gamma, x : K, \Gamma' \vdash \textbf{valid}}{\Gamma, x : K, \Gamma' \vdash x : K} \; (1) \qquad \frac{\Gamma \vdash \textbf{valid}}{\Gamma \vdash \textbf{Type kind}} \quad \frac{\Gamma \vdash A : \textbf{Type}}{\Gamma \vdash El(A) \,\textbf{kind}} \; (5)$$

$$\frac{\Gamma \vdash k : K \quad \Gamma \vdash K = K'}{\Gamma \vdash k : K'} \quad \frac{\Gamma \vdash k = k' : K \quad \Gamma \vdash K = K'}{\Gamma \vdash k = k' : K'} \; (2)^* \qquad \frac{\Gamma, x : K, \Gamma' \vdash J \quad \Gamma \vdash k : K}{\Gamma, [k/x]\Gamma' \vdash [k/x]J} \; (3)^{**}$$

$$\frac{\Gamma \vdash K \,\textbf{kind} \quad \Gamma, x : K \vdash K' \,\textbf{kind}}{\Gamma \vdash (x : K)K' \,\textbf{kind}} \quad \frac{\Gamma \vdash K_1 = K_2 \quad \Gamma, x : K_1 \vdash K'_1 = K'_2}{\Gamma \vdash (x : K_1)K'_1 = (x : K_2)K'_2}$$

$$\frac{\Gamma, x : K \vdash k : K'}{\Gamma \vdash [x : K]k : (x : K)K'} \quad \frac{\Gamma \vdash K_1 = K_2 \quad \Gamma, x : K_1 \vdash k_1 = k_2 : K}{\Gamma \vdash [x : K_1]k_1 = [x : K_2]k_2 : (x : K_1)K}$$

$$\frac{\Gamma \vdash f : (x : K)K' \quad \Gamma \vdash k : K}{\Gamma \vdash f(k) : [k/x]K'} \quad \frac{\Gamma \vdash f = f' : (x : K)K' \quad \Gamma \vdash k_1 = k_2 : K}{\Gamma \vdash f(k_1) = f'(k_2) : [k_1/x]K'}$$

$$\frac{\Gamma, x : K \vdash k' : K' \quad \Gamma \vdash k : K}{\Gamma \vdash ([x : K]k')(k) = [k/x]k' : [k/x]K'} \quad \frac{\Gamma \vdash f : (x : K)K' \quad x \notin FV(f)}{\Gamma \vdash [x : K]f(x) = f : (x : K)K'} \; (4)$$

Clarifications: We follow [**Luo:94**]. Terms of **LF** are of the forms **Type**, $El(A)$, $(x : K)K'$ (dependent product), $[x : K]K'$ (abstraction), $f(k)$, and judgements of the forms $\Gamma \vdash \textbf{valid}$ (validity of context), $\Gamma \vdash K \textbf{kind}$, $\Gamma \vdash k : K$, $\Gamma \vdash k = k' : K$, $\Gamma \vdash K = K'$. Rule groups: (1) rules for contexts and assumptions; (2)* equality rules (reflexivity, symmetry and transitivity rules are ommitted); (3)** substitution rules (J denotes the right side of any of the five forms of judgement); (4) rules for dependent product kinds; (5) and the kind **Type**.

History: First defined in [**Luo:94**], ch. 9, **LF** is a typed version of Martin-Löf's logical framework [**NPS:90**]. In difference from Edinburgh LF it may be used to specify type theories. *E.g.*, theories specified in **LF** were used as basis of proof-assistants Lego and Plastic. Later the system was extended to include coercive subtyping [**Luo:99, SolLuo:02, LuoSolXue:13**].

Remarks: The proof-theoretical analysis of **LF** above was used in meta-theoretical studies of larger theories defined on its basis, *e.g.*, UTT (Unifying Theory of dependent Types) that includes inductive schemata, second order logic SOL with impredicative type *Prop* and a hierarchy of predicative universes [**Luo:94**]. H. Goguen defined a typed operational semantics for UTT and proved strong normalization theorem [**HG:94**]. For **LF** with coercive subtyping conservativity results were obtained [**Luo:99, SolLuo:02, LuoSolXue:13**].

$\overline{\lambda}$-calculus (1994)

CUT-FREE SYSTEM

$$\frac{}{\Gamma;\cdot:A \vdash \cdot():A}\ Ax \qquad \frac{\Gamma;\cdot:A \vdash \cdot(l):C \quad (a:A)\in\Gamma}{\Gamma \vdash a(l):C}\ Cont$$

$$\frac{\Gamma \vdash p:A \quad \Gamma;\cdot:B \vdash \cdot(l):C}{\Gamma \mid (p,l):A \to B \vdash C}\ \to_L \qquad \frac{\Gamma,a:A \vdash p:B}{\Gamma \vdash \lambda a.p:A \to B}\ \to_R$$

CUT RULES

$$\frac{\Gamma \vdash p:A \quad \Gamma;\cdot:A \vdash \cdot(l):C}{\Gamma \vdash p(l):C}\ Cut^1_H \qquad \frac{\Gamma;\cdot:A \vdash \cdot(l):B \quad \Gamma;\cdot:B \vdash \cdot(l'):C}{\Gamma;\cdot:A \vdash \cdot(l@l'):C}\ Cut^2_H$$

$$\frac{\Gamma \vdash p:A \quad \Gamma,a:A,\Gamma' \vdash q:C}{\Gamma,\Gamma' \vdash q[p/a]:C}\ Cut^1_M \qquad \frac{\Gamma \vdash p:A \quad \Gamma,a:A,\Gamma';\cdot:B \vdash \cdot(l):C}{\Gamma,\Gamma';\cdot:B \vdash \cdot(l[p/a])C}\ Cut^2_M$$

Clarifications: This calculus can be seen as an organization of the rules of Gentzen's intuitionistic sequent calculus in a way such that: there is computational interpretation of proofs as λ-calculus-like terms; there is a simple one-to-one correspondence between cut-free proofs and normal proofs of natural deduction.

The definition of the calculus is based on two kinds of sequents: the sequents $\Gamma \vdash p:A$ have a focus on the right and are annotated by a program p; the sequents $\Gamma;\cdot:A \vdash \cdot(l):B$ have an extra focussed formula on the left annotated by a placeholder name $\cdot$ while the formula on the right is annotated by a program referring to this placeholder. The syntax of the underlying calculus is:

$$(l),(l') ::= ()\mid (p,l)\mid (l@l')\mid (l[p/a])$$
$$p,q \quad ::= a(l)\mid \lambda a.p \mid p(l)\mid q[p/a]$$

with () and (p,l) denoting lists of arguments, $l@l'$ denoting concatenation of lists, $l[p/a]$ and $p[q/a]$ denoting explicit substitution, $x(l)$ and $p(l)$ denoting cut-free and non cut-free application, respectively. The first two items of each entry characterize the syntax of cut-free proofs.

History: The $\overline{\lambda}$-calculus has been designed in [**Herbelin94**, **HerbelinPhD**]. It can be seen as the direct counterpart for sequent calculus of what λ-calculus is for natural deduction, along the lines of the Curry-Howard correspondence between proofs and programs. The idea of focussing a specific formula of the sequent comes from Girard [**girard91mscs**] which himself credits it to Andreoli [**andreoli92jlc**] (see also {**??**}). With proof annotations removed, the calculus can be seen as the intuitionistic fragment LJT of the subsystem LKT of LK [**danos93wll**], with LKT and LKQ representing two dual ways to add asymmetric focus to LK.

Extensions to other connectives than implication can be given. Extensions to classical logic, namely a computational presentation of LKT, can be obtained by adding the μ and bracket operators of $\lambda\mu$-calculus {16} and by considering instead three kinds of sequents, $\Gamma \vdash p:A \mid \Delta$, or $\Gamma;\cdot:A \vdash \cdot(l):B$, or $c:(\Gamma \vdash \Delta)$ (see [**HerbelinPhD**]). A variant with implicit substitution is possible.

The symmetrization of $\overline{\lambda}$-calculus led to $\mathbf{LK}_{\mu\tilde{\mu}}$ {21}.

Entry 18 by: Hugo Herbelin

Full Intuitionistic Logic (FIL) (1995)

$$\frac{}{A(n) \Rightarrow A/\{n\}} \; ax \qquad\qquad \frac{}{\bot(n) \Rightarrow A_1/\{n\},\dots,A_k/\{n\}} \; \bot\Rightarrow$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A/S \quad A(n), \Gamma_1 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_1 \Rightarrow \Delta_1, \Delta_1^*} \; cut \qquad \frac{\Gamma_1, A(m), B(n), \Gamma_1 \Rightarrow \Delta}{\Gamma_1, B(n), A(m), \Gamma_1 \Rightarrow \Delta} \; perm \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta_1, A/S_1, B/S_1, \Delta_1}{\Gamma \Rightarrow \Delta_1, B/S_1, A/S_1, \Delta_1} \Rightarrow perm \qquad \frac{\Gamma \Rightarrow \Delta}{A(n), \Gamma \Rightarrow \Delta^*} \; weak \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A/\{\}} \Rightarrow weak \qquad \frac{\Gamma, A(n), A(m) \Rightarrow \Delta}{\Gamma, A(k) \Rightarrow \Delta^*} \; cont \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta, A/S_1, A/S_1}{\Gamma \Rightarrow \Delta, A/S_1 \cup S_1} \Rightarrow cont \qquad \frac{\Gamma_1, A(n) \Rightarrow \Delta_1 \quad \Gamma_1, B(m) \Rightarrow \Delta_1}{\Gamma_1, \Gamma_1, (A \vee B)(k) \Rightarrow \Delta_1^*, \Delta_1^*} \; \vee \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta, A/S_1, B/S_1}{\Gamma \Rightarrow \Delta, (A \vee B)/S_1 \cup S_1} \Rightarrow \vee \qquad \frac{\Gamma, A(n), B(m) \Rightarrow \Delta}{\Gamma, (A \wedge B)(k) \Rightarrow \Delta^*} \; \wedge \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta, A/S_1 \quad \Gamma \Rightarrow \Delta, B/S_1}{\Gamma \Rightarrow \Delta, (A \wedge B)/S_1 \cup S_1} \Rightarrow \wedge \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A/S \quad B(n), \Gamma_1 \Rightarrow \Delta_1}{(A \to B)(n), \Gamma_1, \Gamma_1 \Rightarrow \Delta_1, \Delta_1^*} \; \to\Rightarrow$$

$$\frac{\Gamma, A(n) \Rightarrow \Delta, B/S}{\Gamma \Rightarrow \Delta, (A \to B)/S - \{n\}} \Rightarrow\to$$

Clarifications: Sequents are of the form $\Gamma \Rightarrow \Delta$ where Γ is a multiset of pairs of formulas and natural number indicies, and Δ is a multiset of pairs of formulas and sets of natural number indicies. The set of natural number indicies for a particular conclusion, formula on the right, indicates which hypotheses the conclusion depends on. This dependency tracking is used to enforce intuitionism in the rule $\Rightarrow\to$. See [**dePaiva:2005**] for more details.

History: The system FIL was announced in the abstract [**dePaiva:1995**] but only published officially ten years later in [**dePaiva:2005**]. The system was conceived after the remark in the paper describing FILL {14} that intuitionism is about proofs that resemble functions, not about a cardinality constraint in the sequent calculus. The system shows we can use a notion of *dependency between formulae* to enforce the constructive character of derivations. This is similar to an impoverished Curry-Howard term assignment.

Polarized Linear Sequent Calculus LLP   (2000)

$$\frac{}{\vdash P^\perp, P} \qquad \frac{\vdash \Gamma, P \quad \vdash \Delta, P^\perp, \Pi}{\vdash \Gamma, \Delta, \Pi} \qquad \frac{\vdash \Gamma, \Pi}{\vdash \sigma(\Gamma), \Pi}$$

$$\frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q} \qquad \frac{\vdash \Gamma, N, M, \Pi}{\vdash \Gamma, N \,\invamp\, M, \Pi} \qquad \frac{}{\vdash 1} \qquad \frac{\vdash \Gamma, \Pi}{\vdash \Gamma, \bot, \Pi}$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q} \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q} \qquad \frac{\vdash \Gamma, M, \Pi \quad \vdash \Gamma, N, \Pi}{\vdash \Gamma, M \,\&\, N, \Pi} \qquad \frac{}{\vdash \Gamma, \top, \Pi}$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, ?P} \qquad \frac{\vdash \Gamma, N}{\vdash \Gamma, !N} \qquad \frac{\vdash \Gamma, N, N, \Pi}{\vdash \Gamma, N, \Pi} \qquad \frac{\vdash \Gamma, \Pi}{\vdash \Gamma, N, \Pi}$$

$$
\begin{array}{lll}
(P \otimes Q)^\perp = P^\perp \,\invamp\, Q^\perp & 1^\perp = \bot \\
(!N)^\perp = ?(N^\perp) \quad (P \oplus Q)^\perp = P^\perp \,\&\, Q^\perp & 0^\perp = \top \\
(X^\perp)^\perp = X \quad (N \,\invamp\, M)^\perp = N^\perp \otimes M^\perp & \bot^\perp = 1 \\
(?P)^\perp = !(P^\perp) \quad (N \,\&\, M)^\perp = N^\perp \oplus M^\perp & \top^\perp = 0
\end{array}
$$

Positive formulas: $P, Q ::= X \mid P \otimes Q \mid 1 \mid P \oplus Q \mid 0 \mid !N$
Negative formulas: $N, M ::= X^\perp \mid N \,\invamp\, M \mid \bot \mid N \,\&\, M \mid \top \mid ?P$

Γ and Δ are lists of negative formulas.
Π consists of 0 or 1 positive formula.
σ is a permutation.

Clarifications: Negation is not a connective. It is defined using De Morgan's laws so that $(A^\perp)^\perp = A$. Negative connectives which turn negative formulas into negative formulas ($\invamp$, $\bot$, $\&$ and $\top$) are the reversible connectives of **LL** {12}. Their dual, the positive connectives ($\otimes$, 1, $\oplus$, 0) have the focusing property [**focal**], related here with the "at most one positive formula" property of sequents.

History: LLP [**phdlaurent**] comes from the natural embedding of Girard's **LC** {15} into linear logic {12}. It is obtained by restricting **LL** to polarized formulas and then by generalizing the structural rules (contraction, weakening and context of promotion) to arbitrary negative formulas, not only those starting with a ?-connective.

Remarks: Cut elimination holds. In the categorical models of **LLP**, positive formulas are interpreted as $\otimes$-comonoids while negative formulas are interpreted as $\invamp$-monoids.

$\text{LK}_{\mu\tilde{\mu}}$ (2000)

$$\frac{(a : A) \in \Gamma}{\Gamma \vdash a : A \,|\, \Delta} \; Ax_R \qquad \frac{\Gamma \vdash v : A \,|\, \Delta \quad \Gamma \,|\, e : A \vdash \Delta}{\langle v | e \rangle : (\Gamma \vdash \Delta)} \; Cut \qquad \frac{(\alpha : A) \in \Delta}{\Gamma \,|\, \alpha : A \vdash \Delta} \; Ax_L$$

$$\frac{c : (\Gamma, a : A \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}a.c : A \vdash \Delta} \; Focus_L \qquad \frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c : A \,|\, \Delta} \; Focus_R$$

INTRODUCTION RULES

$$\frac{\Gamma \,|\, e : A_i \vdash \Delta}{\Gamma \,|\, \pi_i \cdot e : A_1 \wedge A_2 \vdash \Delta} \; \wedge_L^i \qquad \frac{\Gamma \vdash v_1 : A_1 \,|\, \Delta \quad \Gamma \vdash v_2 : A_2 \,|\, \Delta}{\Gamma \vdash (v_1, v_2) : A_1 \wedge A_2 \,|\, \Delta} \; \wedge_R$$

$$\frac{\Gamma \,|\, e_1 : A_1 \vdash \Delta \quad \Gamma \,|\, e_2 : A_2 \vdash \Delta}{\Gamma \,|\, [e_1, e_2] : A_1 \vee A_2 \vdash \Delta} \; \vee_L \qquad \frac{\Gamma \vdash v : A_i \,|\, \Delta}{\Gamma \vdash \iota_i(v) : A_1 \vee A_2 \,|\, \Delta} \; \vee_R^i$$

$$\frac{\Gamma \vdash v : A \,|\, \Delta \quad \Gamma \,|\, e : B \vdash \Delta}{\Gamma \,|\, v \cdot e : A \to B \vdash \Delta} \; \to_L \qquad \frac{\Gamma, a : A \vdash v : B \,|\, \Delta}{\Gamma \vdash \lambda a.v : A \to B \,|\, \Delta} \; \to_R$$

$$\frac{\Gamma \,|\, e : A[y] \vdash \Delta}{\Gamma \,|\, \tilde{\lambda}x.e : \exists x\, A[x] \vdash \Delta} \; \exists_L \qquad \frac{\Gamma \vdash v : A[t] \,|\, \Delta}{\Gamma \vdash t \cdot v : \exists x\, A[x] \,|\, \Delta} \; \exists_R$$

$$\frac{\Gamma \,|\, e : A[t] \vdash \Delta}{\Gamma \,|\, t \cdot e : \forall x\, A[x] \vdash \Delta} \; \forall_L \qquad \frac{\Gamma \vdash v : A[y] \,|\, \Delta}{\Gamma \vdash \lambda x.v : \forall x\, A[x] \,|\, \Delta} \; \forall_R$$

$$\frac{}{\Gamma \,|\, [\,] : \bot \vdash \Delta} \; \bot_L \qquad \frac{}{\Gamma \vdash () : \top \,|\, \Delta} \; \top_R$$

Clarifications: There are three kinds of sequents: first $\Gamma \vdash v : A \,|\, \Delta$ with a distinguished formula on the right for typing the program v, second $\Gamma \,|\, e : A \vdash \Delta$ with a distinguished formula on the left for typing the evaluation context e, and finally $c : (\Gamma \vdash \Delta)$ with no distinguished formula for typing command c, i.e. the interaction of a program within an evaluation context. The typing contexts Γ and Δ are lists of named formulas so that a non-ambiguous correspondence with λ-calculus is possible (if it were sets or multisets, there were e.g. no way to distinguish the two distinct proofs of $x : A, x : A \vdash x : A \,|\,$). Weakening rules are implemented implicitly at the level of axioms. Contraction rules are derived, using a cut against an axiom. No exchange rule is needed. Not all cuts are eliminable: only those not involving an axiom rule are. Negation $\neg A$ can be defined as $A \to \bot$. In the rules $\exists_E$ and $\forall_R$, y is assumed fresh in Γ, Δ and $A[x]$. The syntax of the underlying λ-calculus is:

$$c ::= \langle v | e \rangle$$
$$e ::= \alpha \,|\, \tilde{\mu}a.c \,|\, \pi_i \cdot e \,|\, [e, e] \,|\, v \cdot e \,|\, (t, e) \,|\, \tilde{\lambda}x.e \,|\, [\,]$$
$$v ::= a \,|\, \mu\alpha.c \,|\, (v, v) \,|\, \iota_i(v) \,|\, \lambda a.v \,|\, \lambda x.v \,|\, (t, v) \,|\, ()$$

History: The purpose of this system is to provide a λ-calculus-style computational meaning to Gentzen's LK {3} and to highlight how the symmetries of sequent calculus show computationally. Seeing the rules as typing rules, the left/right symmetry is a symmetry between programs and their evaluation contexts. At the level of cut elimination, giving priority to the left-hand side relates to call-by-name evaluation, while giving priority to the right-hand side relates to call-by-value evaluation [**CurienHerbelin00**]. Thanks to the presence of two dual axiom rules and implicit contraction rules, the system supports a tree-like sequent-free presentation, as

originally presented by Gentzen for natural deduction [**HerbelinHdR**] (see {25}). The system can be seen as a symmetric variant of $\bar{\lambda}$-calculus {18}.

The structural subsystem can be adapted to various sequent calculi. Restriction to intuitionistic logic can be obtained by demanding that the right-hand side has exactly one formula.

The presentation of this calculus with conjunctive and disjunctive additive connectives has been studied in [**Wadler03**, **HerbelinHdR**]. A variant with only commands, called $\mathcal{X}$, has been studied in [**BakLenLes05**], based on previous work in [**UrbanPhD**]. Various extensions of the system emphasizing different symmetries can be found in the literature.

Remarks: The system is obviously logically equivalent to Gentzen's **LK** when equipped with the corresponding connectives and observed through the sequents of the form $\Gamma \vdash \Delta$.

Constructive Modal Logic S4 (CS4) (2000)

$$\frac{}{\Delta, A \vdash A} \ ax \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \ cut \qquad \frac{}{\Gamma, \bot \vdash A} \ \bot\,\mathcal{L}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \ \vee\mathcal{L} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \ \vee\mathcal{R} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \ \vee\mathcal{R}$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \ \wedge\mathcal{L} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C} \ \wedge\mathcal{L} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \ \wedge\mathcal{R}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} \ \to\mathcal{L} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \ \to\mathcal{R} \qquad \frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B} \ \Box\mathcal{L}$$

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma, \Delta \vdash \Box A} \ \Box\mathcal{R} \qquad \frac{\Box\Gamma, A \vdash \Diamond B}{\Delta, \Box\Gamma, \Diamond A \vdash \Diamond B} \ \Diamond\mathcal{L} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A} \ \Diamond\mathcal{R}$$

Clarifications: Left contexts, denoted Γ or Δ, are multisets of formulas. Furthermore, if $\Gamma = A_1, \ldots, A_n$, then $\Box\Gamma = \Box A_1, \ldots, \Box A_n$.

History: The intuitionistic system for S4 that we are calling constructive S4 (CS4) here, was originally described by Prawitz in his Natural Deduction book [**prawitznatural**] in 1965. This system differs from what is more widely called now IS4, originally defined by Fisher-Servi [**Fisher-Servi:1981**] and thoroughly studied in Simpson's PhD thesis [**simpson1994phd**] in that it does not satisfy the distribution of possibility over disjunctions, either binary ($\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$) or nullary ($\Diamond\bot \to \bot$). The calculus for CS4 was thoroughly investigated by Bierman and de Paiva in [**bierman2000**].

Entry 22 by: Harley Eades III, Valeria de Paiva

Hybrid Logic (HL)   (2001)

$$\frac{@_a\phi \quad @_a\psi}{@_a(\phi \wedge \psi)} \ (\wedge I) \qquad \frac{@_a(\phi \wedge \psi)}{@_a\phi} \ (\wedge E1) \qquad \frac{@_a(\phi \wedge \psi)}{@_a\psi} \ (\wedge E2)$$

$$[@_a\phi]$$
$$\vdots$$
$$\frac{@_a\psi}{@_a(\phi \to \psi)} \ (\to I) \qquad\qquad \frac{@_a(\phi \to \psi) \quad @_a\phi}{@_a\psi} \ (\to E) \qquad\qquad \frac{[@_a\neg\phi]}{\vdots} $$
$$\frac{@_a\bot}{@_a\phi} \ (\bot 1)^*$$

$$\frac{@_a\phi}{@_c@_a\phi} \ (@I) \qquad\qquad \frac{@_c@_a\phi}{@_a\phi} \ (@E) \qquad\qquad \frac{@_a\bot}{@_c\bot} \ (\bot 2)$$

$$[@_a\Diamond c]$$
$$\vdots$$
$$\frac{@_c\phi}{@_a\Box\phi} \ (\Box I)^\star \qquad\qquad \frac{@_a\Box\phi \quad @_a\Diamond e}{@_e\phi} \ (\Box E)$$

$$\frac{}{@_a a} \ (Ref) \qquad\qquad \frac{@_a c \quad @_a\phi}{@_c\phi} \ (Nom1)^* \qquad\qquad \frac{@_a c \quad @_a\Diamond b}{@_c\Diamond b} \ (Nom2)$$

$*$ ϕ is a propositional symbol (ordinary or a nominal).

$\star$ c does not occur in $@_a\Box\phi$ or in any undischarged assumptions other than the occurrences of $@_a\Diamond c$.

Clarifications: Hybrid logic is an extension of ordinary modal logic which allows explicit reference to individual points in a Kripke model. Formulas of HL are defined by $S \ ::= \ p\,|\,a\,|\,S \wedge S\,|\,S \to S\,|\,\bot\,|\,\Box S\,|\,@_a S$ where p ranges over ordinary propositional symbols and a ranges over nominals (a second sort of propositional symbols that refer to points in the model). As usual, $\neg\phi$ stands for $\phi \to \bot$ and $\Diamond\phi$ stands for $\neg\Box\neg\phi$.

History: This natural deduction system for classical HL was originally suggested in [**Brauner01c**] and developed in [**Brauner01b**]. A natural deduction system for intuitionistic hybrid logic can be found in the entry {24}. These and other proof systems are included in the book [**Brauner11a**], which considers a spectrum of different hybrid logics (propositional, first-order, intensional first-order, and intuitionistic) and different types of proof systems for hybrid logic (natural deduction, Gentzen, tableau, and axiom systems). See [**AC06**] for a general introduction to hybrid logic.

Remarks: The system satisfies normalization, and normal derivations satisfy a version of the subformula property. Completeness is preserved when the system is extended with additional rules corresponding to first-order conditions on Kripke frames expressed by geometric theories.

Entry 23 by: Torben Braüner

Intuitionistic Hybrid Logic (IHL) (2003)

$$\frac{@_aA \quad @_aB}{@_a(A \wedge B)} \wedge I \qquad \frac{@_a(A \wedge B)}{@_aA} \wedge E_1 \qquad \frac{@_a(A \wedge B)}{@_aB} \wedge E_2 \qquad \frac{@_aA}{@_aA \vee B} \vee I_1$$

$$\frac{@_aB}{@_aA \vee B} \vee I_2 \qquad \frac{@_aA \vee B \quad \overset{[@_aA]}{\overset{\vdots}{C}} \quad \overset{[@_aB]}{\overset{\vdots}{C}}}{C} \vee E \qquad \frac{\overset{[@_aA]}{\overset{\vdots}{@_aB}}}{@_a(A \to B)} \to I \qquad \frac{@_a(A \to B) \quad @_aA}{@_aB} \to E$$

$$\frac{@_a \bot}{C} \bot E \qquad \frac{@_aA}{@_c@_aA} @I \qquad \frac{}{a : a} \text{Ref} \qquad \frac{a : c \quad a : A}{c : A} \text{Nom}_1 \qquad \frac{a : c \quad a : \Diamond b}{c : \Diamond b} \text{Nom}_2$$

$$\frac{@_c@_aA}{@_aA} @E \qquad \frac{@_eA \quad @_a\Diamond e}{@_a\Diamond A} \Diamond I \qquad \frac{@_a\Diamond A \quad \overset{[@_cA][@_a\Diamond c]}{\overset{\vdots}{C}}}{C} \Diamond E \qquad \frac{\overset{[@_a\Diamond c]}{\overset{\vdots}{@_cA}}}{@_a\Box A} \Box I \qquad \frac{@_a\Box A \quad @_a\Diamond e}{@_eA} \Box E$$

Clarifications: Formulas of IHL are defined by the following grammar:

$$A, B, C ::= p \mid a \mid A \wedge B \mid A \vee B \mid A \to B \mid \bot \mid \Box A \mid \Diamond A \mid @_aA$$

where a ranges over nominals and p propositional symbols. In the rule $\Diamond E$, c does not occur in $@_a\Diamond A$, in C, or in any undischarged assumptions other than the specified occurrences of $@_cA$ and $@_a\Diamond c$. Furthermore, in $\Box I$, c does not occur in $@_a\Box A$ or in any undischarged assumptions other than the specified occurrences of $@_a\Diamond c$. In the rule Nom_1, A, is any proposition (ordinary or nominal).

History: The Natural Deduction system for IHL was originally suggested in [**braunerdepaiva2003**] and developed in [**braunerdepaiva2006**]. This system adds nominals and satisfaction operators to a version of Intuitionistic Modal Logic described using labelled deduction, in the style of Simpson [**simpson1994**]. Axioms, or a Hilbert-style calculus version of the system, were provided in [**brauner2006**]. Some of the properties of the intuitionistic system, as well as a discussion of some of its applications to type systems in computing, appeared in [**brauner2011**].

LK$_{\mu\tilde{\mu}}$ in sequent-free tree form   (2005)

STRUCTURAL SUBSYSTEM

$$\frac{\vdash A \quad A \vdash}{\vdash} \; Cut$$

$$\begin{array}{cc} [\vdash A] & [A \vdash] \\ \vdots & \vdots \\ \dfrac{\vdash}{A \vdash} \; Focus_L & \dfrac{\vdash}{\vdash A} \; Focus_R \end{array}$$

INTRODUCTION RULES

$$\frac{A_i \vdash}{A_1 \wedge A_2 \vdash} \wedge_L^i \qquad \frac{\vdash A_1 \quad \vdash A_2}{\vdash A_1 \wedge A_2} \wedge_R$$

$$\frac{A_1 \vdash \quad A_2 \vdash}{A_1 \vee A_2 \vdash} \vee_L \qquad \frac{\vdash A_i}{\vdash A_1 \vee A_2} \vee_R^i$$

$$\begin{array}{cc} & [\vdash A] \\ & \vdots \\ \dfrac{\vdash A \quad B \vdash}{A \to B \vdash} \to_L & \dfrac{\vdash B}{\vdash A \to B} \to_R \end{array}$$

$$\frac{A[y] \vdash}{\exists x\, A[x] \vdash} \exists_L \qquad \frac{\vdash A[t]}{\vdash \exists x\, A[x]} \exists_R$$

$$\frac{A[t] \vdash}{\forall x\, A[x] \vdash} \forall_L \qquad \frac{\vdash A[y]}{\vdash \forall x\, A[x]} \forall_R$$

$$\frac{}{\bot \vdash} \bot_L \qquad \frac{}{\vdash \top} \top_R$$

Clarifications: There are three kinds of nodes, $\vdash A$ for asserting formulas, $A \vdash$ for refuting formulas, and $\vdash$ for expressing a contradiction. Negation $\neg A$ can be defined as $A \to \bot$. In the rules $\exists_E$ and $\forall_R$, y is assumed fresh in all the unbracketed assumption formula upon which that the derivation of $A(y)$ depends.

History: The purpose of this system is to show that the original distinction in Gentzen [**Gentzen1935**] between natural deduction presented as a tree of formulas and sequent calculus presented as a tree of sequents is no longer relevant. It is known from at least Howard [**Howard80**] that natural deduction can be presented with sequents. The above formulation shows that systems based on left and right introductions ("sequent-calculus style") can be presented as a sequent-free tree of formulas [**HerbelinHdR**].

The terminology "sequent calculus" seems to have become popular from [**Prawitz65**] followed then e.g. by [**Troelstra73**] who were associating the term "sequents" to Gentzen's LJ and LK systems. The terminology having lost the connection to its etymology, this motivated some authors to use alternative terminologies such as "L" systems [**Munch-Maccagnoni09**].

Remarks: As pointed out e.g. in [**GeuversPhD**] in the context of natural deduction, to obtain a computationally non-degenerate proof-as-program correspondence with a presentation of a calculus as a tree of formulas, the bracketed assumptions have to be annotated with the exact occurrence of the rule which bracketed them. Then, annotation by proof-terms can optionally be added as in {21}.

Entry 25 by: Hugo Herbelin

$\lambda\Pi$-Calculus Modulo (2007)

TYPING RULES FOR TERMS

$$\frac{\Gamma \vdash^{ctx} \Delta}{\Gamma;\Delta \vdash \mathbf{Type} : \mathbf{Kind}} \ \textbf{(Sort)}$$

$$\frac{\Gamma \vdash^{ctx} \Delta \qquad (c : A) \in \Gamma}{\Gamma;\Delta \vdash c : A} \ \textbf{(Constant)} \qquad \frac{\Gamma \vdash^{ctx} \Delta \qquad (x : A) \in \Delta}{\Gamma;\Delta \vdash x : A} \ \textbf{(Variable)}$$

$$\frac{\Gamma;\Delta \vdash t : \Pi x : A.B \qquad \Gamma;\Delta \vdash u : A}{\Gamma;\Delta \vdash tu : B[x/u]} \ \textbf{(Application)}$$

$$\frac{\Gamma;\Delta(x : A) \vdash t : B \qquad \Gamma;\Delta \vdash \Pi x : A.B : s}{\Gamma;\Delta \vdash \lambda x : A.t : \Pi x : A.B} \ \textbf{(Abstraction)}$$

$$\frac{\Gamma;\Delta \vdash A : \mathbf{Type} \qquad \Gamma;\Delta(x : A) \vdash B : s}{\Gamma;\Delta \vdash \Pi x : A.B : s} \ \textbf{(Product)}$$

$$\frac{\Gamma;\Delta \vdash t : A \qquad \Gamma;\Delta \vdash B : s \qquad A \equiv_{\beta\Gamma} B}{\Gamma;\Delta \vdash t : B} \ \textbf{(Conversion)}$$

WELL-FORMEDNESS FOR LOCAL CONTEXTS

$$\frac{\Gamma \ \mathbf{wf}}{\Gamma \vdash^{ctx} \emptyset} \qquad \frac{\Gamma \vdash^{ctx} \Delta \qquad \Gamma;\Delta \vdash U : \mathbf{Type} \qquad x \notin dom(\Delta)}{\Gamma \vdash^{ctx} \Delta(x : U)}$$

WELL-FORMEDNESS RULES FOR GLOBAL CONTEXTS

$$\frac{}{\emptyset \ \mathbf{wf}} \qquad \frac{\Gamma \ \mathbf{wf} \qquad \Gamma;\emptyset \vdash U : \mathbf{Type}}{\Gamma(c : U) \ \mathbf{wf}} \qquad \frac{\Gamma \ \mathbf{wf} \qquad \Gamma;\emptyset \vdash K : \mathbf{Kind}}{\Gamma(C : K) \ \mathbf{wf}}$$

$$\frac{\Gamma \ \mathbf{wf} \qquad \rightarrow_\beta \cup \rightarrow_{\Gamma\Xi} \text{ is confluent} \qquad \begin{array}{c} (\forall i)\Gamma \vdash u_i \hookrightarrow v_i \\ \Xi = (u_1 \hookrightarrow v_1)\dots(u_n \hookrightarrow v_n) \end{array}}{\Gamma\Xi \ \mathbf{wf}}$$

Clarifications: The $\lambda\Pi$-Calculus Modulo is an extension of the λ-Calculus with dependent types and rewrite rules. Computational equivalence is extended from β-equivalence to $\beta\Gamma$-equivalence ($\equiv_{\beta\Gamma}$), the congruence generated by β-reduction and the rewrite rules ($u \hookrightarrow v$) in the global context Γ.

History: The $\lambda\Pi$-Calculus Modulo has been introduced by Cousineau and Dowek [**DBLP:conf/tlca/CousineauD07**] as an expressive logical framework. It has been used to design *shallow encodings* of many logics and calculus such as functional Pure Type Systems [**DBLP:conf/tlca/CousineauD07**], Higher-Order Logic [**AliHOL**], the Calculus of Inductive Constructions [**Coqine**], resolution and superposition [**Resolution**], or the ς-calculus [**RaphaelSigma**]. The well-formedness rules for global contexts were not part of the original type system and have been introduced by Saillard [**ModuloBeta**]. The $\lambda\Pi$-Calculus Modulo is implemented in the proof checker Dedukti [**Dedukti**].

Remarks: Confluence of the rewriting relation $\rightarrow_{\beta\Gamma}$ is required to guarantee subject reduction. This requirement can be weakened to confluence for a notion of rewriting modulo β [**ModuloBeta**]. Decidability of type inference depends on strong normalization.

Entry 26 by: Ronan Saillard

Propagation rules:

$$\frac{X[\mathcal{S} \Rightarrow (A, \mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}]}{X[\mathcal{S}, A \Rightarrow (\mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}]} \; pl_1 \qquad \frac{X[(\mathcal{S} \Rightarrow \mathcal{T}, A), \mathcal{S}' \Rightarrow \mathcal{T}']}{X[(\mathcal{S} \Rightarrow \mathcal{T}), \mathcal{S}' \Rightarrow A, \mathcal{T}']} \; pr_1$$

$$\frac{X[\mathcal{S}, A, (\mathcal{S}' \Rightarrow \mathcal{T}') \Rightarrow \mathcal{T}]}{X[\mathcal{S}, (\mathcal{S}', A \Rightarrow \mathcal{T}') \Rightarrow \mathcal{T}]} \; pl_2 \qquad \frac{X[\mathcal{S} \Rightarrow \mathcal{T}, A, (\mathcal{S}' \Rightarrow \mathcal{T}')]}{X[\mathcal{S} \Rightarrow \mathcal{T}, (\mathcal{S}' \Rightarrow \mathcal{T}', A)]} \; pr_2$$

Identity and logical rules: In branching rules, $X[\,] \in X_1[\,] \bullet X_2[\,]$, $\mathcal{S} \in \mathcal{S}_1 \bullet \mathcal{S}_2$ and $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$.

$$\frac{X[\,], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\mathcal{U}, p \Rightarrow p, \mathcal{V}]} \; id^d \qquad \frac{X[\,], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\bot, \mathcal{U} \Rightarrow \mathcal{V}]} \; \bot_l^d \qquad \frac{X[\mathcal{S} \Rightarrow \mathcal{T}]}{X[\mathcal{S} \Rightarrow \mathcal{T}, \bot]} \; \bot_r^d$$

$$\frac{X[\mathcal{S} \Rightarrow \mathcal{T}]}{X[\mathcal{S}, I \Rightarrow \mathcal{T}]} \; I_l^d \qquad \frac{X[\,], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\mathcal{U} \Rightarrow I, \mathcal{V}]} \; I_r^d$$

$$\frac{X[\mathcal{S}, A, B \Rightarrow \mathcal{T}]}{X[\mathcal{S}, A \otimes B \Rightarrow \mathcal{T}]} \; \otimes_l^d \qquad \frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2 \Rightarrow B, \mathcal{T}_2]}{X[\mathcal{S} \Rightarrow A \otimes B, \mathcal{T}]} \; \otimes_r^d$$

$$\frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S}, A \multimap B \Rightarrow \mathcal{T}]} \; \multimap_l^d \qquad \frac{X[\mathcal{S} \Rightarrow \mathcal{T}, (A \Rightarrow B)]}{X[\mathcal{S} \Rightarrow \mathcal{T}, A \multimap B]} \; \multimap_r^d$$

$$\frac{X_1[\mathcal{S}_1, A \Rightarrow \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S}, A \bindnasrepma B \Rightarrow \mathcal{T}]} \; \bindnasrepma_l^d \qquad \frac{X[\mathcal{S} \Rightarrow A, B, \mathcal{T}]}{X[\mathcal{S} \Rightarrow A \bindnasrepma B, \mathcal{T}]} \; \bindnasrepma_r^d$$

$$\frac{X[\mathcal{S}, (A \Rightarrow B) \Rightarrow \mathcal{T}]}{X[\mathcal{S}, A \prec B \Rightarrow \mathcal{T}]} \; \prec_l^d \qquad \frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S} \Rightarrow A \prec B, \mathcal{T}]} \; \prec_r^d$$

Clarifications: Following Kashima [**DBLP:journals/sLogica/Kashima94**], nested sequents are defined as below where A_i and B_j are formulae [**DBLP:conf/csl/CloustonDGT13**]:

$$S \quad T ::= S_1, \ldots, S_k, A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n, T_1, \ldots, T_l$$

Γ and Δ are multisets of formulae and P, Q, S, T, X, Y, etc., are nested sequents, and $\mathcal{S}, \mathcal{X}$, etc., are multisets of nested sequents and formulae.

Inference rules in BiILL$_{dn}$ are applied in a *context*, i.e., a nested sequent with a hole [ ]. Notice that BiILL$_{dn}$ contain no structural rules. The branching rules require operations to merge contexts and nested sequents, which are explained below. The zero-premise rules require that certain sequents or contexts are *hollow*, i.e., containing no occurrences of formulae.

The *merge set* $X_1 \bullet X_2$ of two sequents X_1 and X_2 is defined as:

$$\begin{aligned} X_1 \bullet X_2 = \{ \, &(\Gamma_1, \Gamma_2, Y_1, \ldots, Y_m \Rightarrow \Delta_1, \Delta_2, Z_1, \ldots, Z_n) \mid \\ &X_1 = (\Gamma_1, P_1, \ldots, P_m \Rightarrow \Delta_1, Q_1, \ldots, Q_n) \text{ and} \\ &X_2 = (\Gamma_2, S_1, \ldots, S_m \Rightarrow \Delta_2, T_1, \ldots, T_n) \text{ and} \\ &Y_i \in P_i \bullet S_i \text{ for } 1 \leq i \leq m \text{ and } Z_j \in Q_j \bullet T_j \text{ for } 1 \leq j \leq n \, \} \end{aligned}$$

When $X \in X_1 \bullet X_2$, we say that X_1 and X_2 are a *partition* of X.

The merge set $X_1[\,] \bullet X_2[\,]$ of two contexts $X_1[\,]$ and $X_2[\,]$ is defined in [**DBLP:conf/ifipTCS/DawsonCGT14**]. If $X[\,] = X_1[\,] \bullet X_2[\,]$ we say $X_1[\,]$ and $X_2[\,]$ are a *partition* of $X[\,]$. The notion of a merge set between multisets of formulae and sequents is as follows. Given $\mathcal{X} = \Gamma \cup \{X_1, \ldots, X_n\}$ and $\mathcal{Y} = \Delta \cup \{Y_1, \ldots, Y_n\}$ their merge set contains all multisets of the form: $\Gamma \cup \Delta \cup \{Z_1, \ldots, Z_n\}$ where $Z_i \in X_i \bullet Y_i$.

History: The sequent calculus arose from an attempt to give a display calculus for full intuitionistic linear logic (FILL). As usual for display calculi, a detour is necessary through an extension of FILL with an

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"exclusion" connective $\prec$ which forms an adjunction with $\invamp$. The resulting logic is called Bi-intuitionistic Linear Logic (BiILL). Although sound and complete for BiILL, the resulting display calculus is bad for backward proof search. Following Kashima [**DBLP:journals/sLogica/Kashima94**], Alwen Tiu first obtained a shallow nested sequent calculus for BiILL, and then refined that into a deep nested sequent calculus for BiILL. The proof of cut-elimination for the shallow calculus, and the equivalence of the shallow and deep calculi requires over 615 different cases!

Remarks: The calculus shown is for Bi-Intuitionistic Linear Logic [**DBLP:conf/csl/CloustonDGT13**]. It is sound and complete. The soundness w.r.t. the categorical semantics is via the shallow nested sequent calculus for BiILL. A nested sequent is a (nested) *FILL-sequent* if it has no nesting of sequents on the left of $\Rightarrow$, and no occurrences of $\prec$ at all. Only in the deep nested sequent calculus is it obvious that a derivation of a FILL-sequent encounters FILL-sequents only. The deep sequent calculus enjoys the subformula property and terminating backward-proof search. The validity problem for FILL is co-NP complete and BiILL is conservative over FILL [**DBLP:conf/csl/CloustonDGT13**]. All of these proofs were eventually formalised in Isabelle [**DBLP:conf/ifipTCS/DawsonCGT14**] by Jeremy Dawson. As far as is known, it is the only sequent calculus for FILL ({14}) that does not require (type) annotations.

Contextual Natural Deduction (2013)

$$\overline{\Gamma, a : A \vdash a : A}$$

$$\frac{\Gamma, a : A \vdash b : C_\pi[B]}{\Gamma \vdash \lambda_\pi a^A.b : C_\pi[A \to B]} \to_I (\pi)$$

$$\frac{\Gamma \vdash f : C_{\pi_1}^1[A \to B] \quad \Gamma \vdash x : C_{\pi_2}^2[A]}{\Gamma \vdash (f\ x)_{(\pi_1;\pi_2)}^{\rightarrow} : C_{\pi_1}^1[C_{\pi_2}^2[B]]} \to_{\overrightarrow{E}} (\pi_1;\pi_2)$$

$$\frac{\Gamma \vdash f : C_{\pi_1}^1[A \to B] \quad \Gamma \vdash x : C_{\pi_2}^2[A]}{\Gamma \vdash (f\ x)_{(\pi_1;\pi_2)}^{\leftarrow} : C_{\pi_1}^2[C_{\pi_2}^1[B]]} \to_{\overleftarrow{E}} (\pi_1;\pi_2)$$

π, π_1 and π_2 must be positive positions. a is allowed to occur in b only if π is strongly positive.

Clarifications: $C_\pi[F]$ denotes a formula with F occurring in the hole of a *context* $C_\pi[]$. π is the position of the hole. It is: *positive* iff it is in the left side of an even number of implications; *strongly positive* iff this number is zero.

History: Contextual Natural Deduction [**ContextualND**] combines the idea of deep inference with Gentzen's natural deduction {2}.

Remarks: Soundness and completeness w.r.t. minimal logic are proven [**ContextualND**] by providing translations between **ND^c** and the minimal fragment of **NJ** {2}. **ND^c** proofs can be quadratically shorter than proofs in the minimal fragment of **NJ**.

Entry 28 by: Bruno Woltzenlogel Paleo

$$\frac{\Gamma_1, A \vdash F \quad \Gamma_2, B \vdash F}{\Gamma_1, \Gamma_2, A \vee B \vdash F} \vee_l \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee_r^1 \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee_r^2 \quad \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \neg_l \quad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A} \neg_r$$

$$\frac{\Gamma, A, B \vdash F}{\Gamma, A \wedge B \vdash F} \wedge_l \quad \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \wedge_r \quad \frac{\Gamma_1 \vdash A \quad \Gamma_2, B \vdash F}{\Gamma_1, \Gamma_2, A \rightarrow B \vdash F} \rightarrow_l \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_r$$

$$\frac{\Gamma \vdash F}{\Gamma, A \vdash F} w_l \quad \frac{\Gamma \vdash}{\Gamma \vdash A} w_r \quad \frac{\Gamma, A, A \vdash F}{\Gamma, A \vdash F} c_l \quad \frac{\Gamma_1 \vdash A \quad \Gamma_2, A \vdash F}{\Gamma_1, \Gamma_2 \vdash F} cut \quad \frac{}{A[\nu_x^l \, F] \vdash A[\nu_x^l \, F]} a$$

$$\frac{\Gamma \vdash A[\alpha]}{\Gamma \vdash \forall x. A[x]} \forall_r \quad \frac{\Gamma, A[\alpha] \vdash F}{\Gamma, \exists x. A[x] \vdash F} \exists_l \quad \frac{\Gamma, A[t] \vdash F}{\Gamma, \forall x. A[x] \vdash F} \forall_l' \quad \frac{\Gamma \vdash A[t]}{\Gamma \vdash \exists x. A[x]} \exists_r'$$

ν denotes the binders ε or τ. The term t must be accessible in the conclusion sequent (*accessibility condition*). Accessible occurrences of t or any of its ε-subterms in Γ and F must have a constant as a label (*label condition*). l is a constant in a (*initial condition*).

Clarifications: A term t is *accessible* in a formula F iff at least one of the following two conditions hold:

- for any top-level (i.e., not nested inside another ε-term) ε-term $\nu_x \, G$ in t, it is the case that $F[\nu_x \, G \leadsto x]$ is a sub-formula of G ($\leadsto$ denotes term rewriting); or
- t contains a nested ε-term $\nu_y \, H$ such that $\nu_y \, H$ is accessible in F and $t[\nu_y \, H \leadsto y]$ is accessible in $F[\nu_y \, H \leadsto y]$.

t is accessible in a sequent S iff all top-level ε-terms in t are accessible in some formula occurring in S. ε-terms replace strong quantifiers in formulas (i.e., the ones that require eigenvariables). Intuitively, term accessibility corresponds to the availability of the eigenvariable in a proof of the non-epsilonized formula. Labels on ε-terms make proof epsilonization injective and allow proofs to be de-epsilonized.

History: Skolemization is known to be unsound in intuitionistic logic. *Epsilonization* is similar to skolemization, but replaces strongly quantified variables by ε-terms, instead of Skolem terms. **LJ⋆** was introduced in [**Reis2016**] as a sequent calculus for intuitionistic logic where *epsilonization* is sound: if the epsilonization of a sequent S is derivable in **LJ⋆**, then S is derivable in **LJ**. This is achieved by restricting the rules $\forall_l$, $\exists_r$ and initial from **LJ** {4} to take into account information available in ε-terms.

Remarks: **LJ⋆** is sound and complete with respect to **LJ** {4} for ε-free formulas. A procedure for de-epsilonizing **LJ⋆** proofs, resulting in valid **LJ** proofs, is defined in [**Reis2016**].

Entry 29 by: Giselle Reis, Bruno Woltzenlogel Paleo

Part II
Indexes