Exercises, Problems, Questions, Suggestions, References, ,...

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1 *Some questions (and possible answers!)*

1. Why do we have just one negation?
   – Possible answer 1: $\rightarrow_c$ and $\rightarrow_i$ are interderivable.
   – Possible answer 2: In the case of propositional logic, if can prove $\neg A$, then we can prove $\neg A$ constructively (Glivenko 2!).
   – Maybe we need a stronger equivalence relation: computational isomorphism is a good candidate!
2. Why don't we have Popper's collapse?
   – Prawitz: "It has sometimes been held that a deductive system that contains both classical and intuitionistic constants is impossible, because the different constants would collapse. Popper (1948) was the first to observe that in a system containing the following rules for classical and intuitionistic negation (now formulated without $\perp$):

$$\frac{\begin{matrix}[A] & & [A] \\ B & & \neg_i B\end{matrix}}{\neg_i A} \qquad \frac{A \qquad \neg_i A}{B} \qquad \frac{\begin{matrix}[A] & & [A] \\ B & & \neg_c B\end{matrix}}{\neg_c A} \qquad \frac{\begin{matrix}[\neg_c A] & & [\neg_c A] \\ B & & \neg_c B\end{matrix}}{A}$$

   where A and B may be arbitrary sentences, the two negations collapse into one, that is, $\neg_i$ and $\neg_c$ become derivable from each other".
   – Interesting exercises: [1] show that if these rules $\neg_i A \vdash \neg_c A$ and $\neg_c A \vdash \neg_i A$; [2] show that we do have a total colapse of intuitionistic logic and classical logic with these four rules.
   – Problem: as we have just seen, in $\mathcal{NE}$ we have $\neg_i A \vdash \neg_c A$ and $\neg_c A \vdash \neg_i A$ too. How come??? How can we avoid the total colapse???
3. Why classical implication does not satisfy modus ponens?
   – Possible answer: the assertability conditions for $\rightarrow_c$ are weaker than the assertability conditions for $\rightarrow_i$. We can assert $(A \rightarrow_c B)$ when $\{\neg A, \neg B\} \vdash \perp$, but in this case $\{(A \rightarrow_c B), A\} \nvdash B$.
   – Remember that the usual introduction and elimination rules for implication in Natural Deduction do not completely characterize classical implication (Peirce's formula!).
4. Why classical conjunction does not have projections?
   – The answer is similar: weaker assertability conditions. We can assert $(A \wedge_c B)$ if we have $\{\neg A\} \vdash \perp$ and $\{\neg B\} \vdash \perp$.

5. Why the classical universal quantifier does not satisfy instantiation?
   - The answer is similar: weaker assertability conditions. We can assert $\forall_c x A(x)$ if we have $\{\exists_i x \neg A(x)\} \vdash \perp$.
6. What about "conservative results"?
   - Let A be a formula in the language of IL and let A^* be the result of replacing every operator α in A by α_i. Then, we can easily show that if $\Gamma \vdash_{IL} A$, then $\Gamma^* \vdash_{\mathcal{NE}} A^*$.
   - We saw that this does not hold in the case of CL: we have that $\{p, (p \to q)\} \vdash_{CL} q$, but we don't have $\{p, (p \to_c q)\} \vdash_{\mathcal{NE}} q$. But we saw that for the translation T_c we have (the weaker result):

     Theorem 1. *If* $\Gamma \vdash_{CL} A$, *then* $T_c[\Gamma] \vdash_{\mathcal{NE}} \neg\neg T_c[A]$

     As a direct corollary we obtain:

     Corollary 1. *If* $\vdash_{CL} A$, *then* $\vdash_{\mathcal{NE}} \neg\neg T_c[A]$
7. What about "pure" systems?
   - We saw that all systems we have discussed in Lecture 2 were "impure": the rules for the classical operators make use of negation!
   - Can we have pure systems? Yes!!! We can use Girard's notion of stoup, Restall's notion of alternatives, or Murzi's higher level rules (with $\perp$ as a punctuation mark.

Greg Restall's technique of alternatives uses the *store* rule:

$$\frac{A \qquad \cancel{A}}{\perp}\uparrow$$

The rules $\vee_c$-elimination and $\to_c$-elimination are the same as in Prawitz' system. The rules for $\vee_c$ and $\to_c$ in the system $E - alt$ are:

$\vee_c$-Introduction

$$\frac{\begin{array}{c}[\cancel{A}]^n \qquad [\cancel{B}]^m \\ \Pi \\ \perp\end{array}}{(A \vee_c B)}\, n, m$$

$\to_c$-Introduction

$$\frac{\begin{array}{c}[A]^n \qquad [\cancel{B}]^m \\ \Pi \\ \perp\end{array}}{(A \to_c B)}\, n, m$$

2 *What must we do?*

1. Real Ecumenical Mathematics!!!
   - This was the original motivation of Peter Krauss. We know that if all operators have a constructive "reading", the axiom of choice is a theorem in Martin-Löf Type Theory. But what would happen if we have *hybrid* readings of these same operators?

- And what if we consider other principles, like the principle of well-ordering?

2. Applications in Computer Science!!!
3. First-order PtS.
   - We showed a PtS semantics for the propositional fragment of Barroso-Nascimento's system ECI. This kind of semantics must be extended to:
     - The full system ECI;
     - Prawitz' system $\mathcal{NE}$;
     - Krauss' system KrE.
4. Kripke semantics for the full system $\mathcal{NE}$.
5. Ecumenical Categorical semantics.
6. New ecumenical codifications.
   - We have showned several ecumenical systems ($\mathcal{NE}$, ECI, KrE) for classical and intuitionistic logic. What about other logics?
     - Barroso-Nascimento has an ecumenical system for intuitionistic and minimal logic;
     - Sernada and Rasga have an ecumenical system for intuitionistic logic and classical $S4$;
     - Quite recently (April 2025), Rasga and Sernadas showed how to systematically connect translations to ecumenical systems and propose an ecumenical system for classical logic and Jaskowski's paraconsistent logic ("From translations to non-collapsing logic combinations").
7. Comparision with other mechanisms of combining logics?
   - The area of Combining Logics is a well-established field, with its distinctive techniques (Fibring, Possible-Translations,...). A certainly interesting question is how the ecumenical perspective relates to the field and techniques of Combining Logics.
8. The proof theory of Krauss' system.
   - Maybe it is just manual labour, but the proof-theory of Krauss' system is not done!
   - By the way, a full scale proof-theory for $\mathcal{NE}$ is not done either! (strong normalization, separability, unicity,...).
9. Ecumenical Algebraic Semantics (on the way!! - Marcelo Coniglio + Elaine Pimentel).

3 *Exercises*

- Exercise 1: Show:
  - $\vdash_{\mathcal{NE}} (((p \rightarrow_i q) \rightarrow_c p) \rightarrow_c p)$
  - $\vdash_{\mathcal{NE}} (((p \rightarrow_c q) \rightarrow_c p) \rightarrow_c p)$
  - $\vdash_{\mathcal{NE}} (A \rightarrow_i B) \rightarrow_i (A \rightarrow_c B)$
  - $\vdash_{\mathcal{NE}} (A \wedge B) \rightarrow_i \neg(\neg A \vee_c \neg B)$
  - $\vdash_{\mathcal{NE}} (A \wedge B) \rightarrow_i \neg(A \rightarrow_c \neg B)$

  - $\vdash_{\mathcal{NE}} \neg(\neg A \wedge \neg B) \leftrightarrow_i (A \vee_c B)$
  - $\vdash_{\mathcal{NE}} \neg(A \wedge \neg B) \leftrightarrow_i (A \rightarrow_c B)$
- Exercise 2: Show that these expressions are equivalent in Krauss' system:
  - $\forall_i x(A \rightarrow_i \exists_c y B)$
  - $\forall_i x(A \rightarrow_c \exists_i y B)$
  - $\forall_c x(A \rightarrow_i \exists_i y B)$
- Exercise 3: Show that $\vdash_{\mathcal{NE}} (A \rightarrow_c \bot) \leftrightarrow_i (A \rightarrow_i \bot)$.
- Exercise 4:

  Definition 1. *A formula B is called externally classical (denoted by B^c) if and only if B is $\bot$, a classical predicate letter, or its main operator is classical (that is: $\rightarrow_c, \vee_c, \exists_c$). A formula C is* classical *if it is built from classical atomic predicates using only the connectives: $\rightarrow_c, \vee_c, \exists_c, \neg, \wedge, \forall$, and the unit $\bot$.*

  For externally classical formulas prove the following theorems:
  - $\vdash_{\mathcal{NE}} (A \rightarrow_c B^c) \rightarrow_i (A \rightarrow_i B^c)$.
  - $\vdash_{\mathcal{NE}} (A \wedge (A \rightarrow_c B^c)) \rightarrow_i B^c$.
  - $\vdash_{\mathcal{NE}} \neg\neg B^c \rightarrow_i B^c$.
  - $\vdash_{\mathcal{NE}} \neg\exists_c x.\neg B^c \rightarrow_i \forall x.B^c$.
- Exercise 5: Show that if A is *externally classical* , then $\Gamma, \neg A \vdash \bot$ implies $\Gamma \vdash A$.
- Exercise 6: Prove the normalization theorem for Krauss' system KrE.
- Exercise 7: Continue the definition of the ecumenical sequent calculus and prove that it satisfies cut-elimination.
- Exercise 8: Prove the same theorems of Exercise 1 in the tableaux system.
- Exercise 9: The tableaux system presented during Lecture 2 is a Fitting-style tableaux. Define an ecumenical Priest-style tableaux.

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