The Ecumenical Perspective in Logic
Lecture 2

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Plan: *Lecture 2*
ECUMENICAL SYSTEMS

- *PRAWITZ' ECUMENICAL SYSTEM* $\mathcal{NE}$
- *KRAUSS' ECUMENICAL SYSTEM* KrE
- *BARROSO-NASCIMENTO's ECUMENICAL SYSTEM* ECI
- *ECUMENICAL SEQUENT CALCULUS*
- *AN ECUMENICAL TABLEAUX*
- *ECUMENICAL MODALITIES*
- *SOME INTERESTING QUESTIONS*

ECUMENICAL SYSTEMS

PRAWITZ' ECUMENICAL SYSTEM $\mathcal{NE}$

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

We saw in the the first Lecture that Prawitz seems to agree with Quine, when he says:

> *When the classical and intuitionistic codifications attach different meanings to a constant, we need to use different symbols, and I shall use a subscript c for the classical meaning and i for the intuitionistic.*

But, according to Prawitz, to assert that the classical and the intuitionistic codifications attach different meanings to some constant and to recognize the need to use different symbols corresponding to these different meanings is not a kind of trivial acceptance of Quine's position, for, as Prawitz puts it,

> *This does not imply that the classical meanings of these constants cannot be explained in the same general way as the intuitionistic meanings of the logical constants have been explained.*

Important Slogan

Different connectives, but the same semantical fraework

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

In order to explore and develop this idea, Prawitz introduces:

- *a language with some constants that are common for classical and intuitionistic logic ($\bot$, $\neg$, $\wedge$, and $\forall$), constants that are classical ($\vee_c$, $\rightarrow_c$ and $\exists_c$), and constants that are intuitionistic ($\vee_i$, $\rightarrow_i$, and $\exists_i$);*
- *an ecumenical natural deduction system in which, as we said, "the classical and intuitionistic constants can then have a peaceful coexistence in a language that contains both".*

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

The language $\mathcal{L}_{\mathcal{E}}$ of the ecumeninal system $\mathcal{NE}$ is defined as follow:
Alphabet

1. Classical Predicate letters: $P_c, Q_c, R_c, \ldots$
2. Intuitionistic Predicate letters: $P_i, Q_i, R_i, \ldots$
3. Individual variables: x, y, z,...
4. individual parameters: a, b, c, ...
5. logical constants: $\bot$, $\wedge$, $\neg$, $\vee_i$, $\vee_c$, $\rightarrow_i$, $\rightarrow_c$, $\forall$, $\exists_i$ and $\exists_c$.
6. Auxiliary signs: '(' , ')', ',' .

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

The grammar for the language of $\mathcal{L}_{\mathcal{E}}$ is inductively defined in the usual way.
Besides 'pure" classical and "pure" intuitionistic formulas, we can also have "hybrid" formulas in the language, as for example,

- $((\neg p \vee_c q) \rightarrow_i r)$ and
- $\exists_c x(P(x) \rightarrow_i Q(x))$.

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

Intuitionistic rules

$$\frac{A \rightarrow_i B \quad A}{B} \rightarrow_i\text{-elim} \qquad \frac{\begin{array}{c}[A]\\ \Pi\\ B\end{array}}{A \rightarrow_i B} \rightarrow_i\text{-int} \qquad \frac{A \vee_i B \quad \begin{array}{c}[A]\\ \Pi_1\\ C\end{array} \quad \begin{array}{c}[B]\\ \Pi_2\\ C\end{array}}{C} \vee_i\text{-elim}$$

$$\frac{A_j}{A_1 \vee_i A_2} \vee_i\text{-int}_j \qquad \frac{\exists_i x.A \quad \begin{array}{c}[A(a/x)]\\ \Pi\\ B\end{array}}{B} \exists_i\text{-elim} \qquad \frac{A(t/x)}{\exists_i x.A} \exists_i\text{-int}$$

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

Classic rules

$$\frac{A \to_c B \quad A \quad \neg B}{\bot} \to_c\text{-elim} \qquad \frac{\begin{matrix}[A, \neg B] \\ \Pi \\ \bot\end{matrix}}{A \to_c B} \to_c\text{-int} \qquad \frac{A \vee_c B \quad \neg A \quad \neg B}{\bot} \vee_c\text{-elim}$$

$$\frac{\begin{matrix}[\neg A, \neg B] \\ \Pi \\ \bot\end{matrix}}{A \vee_c B} \vee_c\text{-int} \qquad \frac{\exists_c x.A \quad \forall x.\neg A}{\bot} \exists_c\text{-elim} \qquad \frac{\begin{matrix}[\forall x.\neg A] \\ \Pi \\ \bot\end{matrix}}{\exists_c x.A} \exists_c\text{-int}$$

$$\frac{P_c(t) \quad \neg P_i(t)}{\bot} P_c\text{-elim} \qquad \frac{\begin{matrix}[\neg P_i(t)] \\ \Pi \\ \bot\end{matrix}}{P_c(t)} P_c\text{-int}$$

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

Neutral rules

$$\frac{A_1 \wedge A_2}{A_j}\ \wedge\text{-elim}_j \qquad \frac{A \quad B}{A \wedge B}\ \wedge\text{-int} \qquad \frac{A \quad \neg A}{\bot}\ \neg\text{-elim} \qquad \frac{\begin{array}{c}[A]\\ \Pi\\ \bot\end{array}}{\neg A}\ \neg\text{-int}$$

$$\frac{\bot}{A}\ \bot{-}elim^* \qquad \frac{\forall x.A}{A(a/x)}\ \forall\text{-elim} \qquad \frac{A(a/x)}{\forall x.A}\ \forall{-}int^\dagger$$

An old idea! - 1978

for these sentences. The intuitionistic meaning of disjunction and existence are in the same way determined by the canonical forms of arguments for $A_1 \vee A_2$ and $\exists x A(x)$, which are indicated by the figures

$$\dfrac{\begin{matrix}\Sigma \\ A_i\end{matrix}}{A_1 \vee A_2} \quad \text{and} \quad \dfrac{\begin{matrix}\Sigma \\ A(t)\end{matrix}}{\exists x A(x)}$$

($i = 1$ or 2), while the classical meaning of disjunction and existence is determined by specifying other canonical forms, which are at the same time the forms of canonical arguments for $\neg(\neg A \wedge \neg B)$ and $\neg\forall x \neg A(x)$ i.e. the forms

$$\dfrac{\begin{matrix}[\neg A][\neg B] \\ \Sigma \\ \bot\end{matrix}}{A \vee B} \qquad \dfrac{\begin{matrix}[\neg A(t)] \\ \Sigma \\ \bot\end{matrix}}{\exists x A(x)}.$$

where $\bot$ stands for falsehood.

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

Example 1: $\vdash_{\mathcal{NE}} (((p \rightarrow_i q) \rightarrow_i p) \rightarrow_c p)$

$$
2,3\;\dfrac{\dfrac{\dfrac{1\;\dfrac{\dfrac{\dfrac{[p]^1 \qquad [\neg p]^2}{\bot}}{q}}{(p \rightarrow_i q)}\rightarrow_i -Int \qquad [((p \rightarrow_i q) \rightarrow_i p)]^3}{p}\rightarrow_i -Elim \qquad [\neg p]^2}{\bot}}{(((p \rightarrow_i q) \rightarrow_i p) \rightarrow_c p)}\rightarrow_c -Int
$$

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

Example 2: $\vdash_{\mathcal{NE}} (p \vee_c \neg p)$

$$\dfrac{\dfrac{[\neg p]^1 \qquad [\neg\neg p]^2}{\bot}}{(p \vee_c \neg p)}\ 1,2$$

Example 3: $\vdash_{\mathcal{NE}} (\neg\forall x\neg A(x) \rightarrow_i \exists_c x A(x))$

$$\dfrac{\dfrac{\dfrac{[\forall x\neg A(x)]^1 \qquad \neg\forall x\neg A(x)}{\bot}}{\exists_c x A(x)}}{(\neg\forall x\neg A(x) \rightarrow_i \exists_c x A(x))}\,1$$

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

Very important: the following result states that logical consequence in $\mathcal{NE}$ is intrinsically intuitionistic.

Theorem 1

$\Gamma \vdash B$ *is provable in* $\mathcal{NE}$ *iff* $\vdash_{\mathcal{NE}} \bigwedge \Gamma \rightarrow_i B$.

Prawitz' Natural Deduction Ecumenical system $\mathcal{NE}$

Exercise 1: Show:

- $\vdash_{\mathcal{NE}} (((p \rightarrow_i q) \rightarrow_c p) \rightarrow_c p)$
- $\vdash_{\mathcal{NE}} (((p \rightarrow_c q) \rightarrow_c p) \rightarrow_c p)$
- $\vdash_{\mathcal{NE}} (A \rightarrow_i B) \rightarrow_i (A \rightarrow_c B)$
- $\vdash_{\mathcal{NE}} (A \wedge B) \rightarrow_i \neg(\neg A \vee_c \neg B)$
- $\vdash_{\mathcal{NE}} (A \wedge B) \rightarrow_i \neg(A \rightarrow_c \neg B)$
- $\vdash_{\mathcal{NE}} \neg(\neg A \wedge \neg B) \leftrightarrow_i (A \vee_c B)$
- $\vdash_{\mathcal{NE}} \neg(A \wedge \neg B) \leftrightarrow_i (A \rightarrow_c B)$

Normalization

As in the case of classical and intuitionistic logic, the introduction and elimination rules of the system $\mathcal{NE}$ may produce detours. We may introduce a formula by an application of an introduction rule and immediately after use this formula as the major premiss of an elimination rule. Some detours of this kind may be hidden by applications of $\vee_i$-elimination and $\exists_i$-elimination.. We introduce now these detours and the operations whose aim is to eliminate them.

Normalization

Definition 2

A formula A in a derivation Π is a *maximum formula* if and only if

1. A is the major premiss of an application of an α-introduction rule and at the same time the major premiss of an application of an α-elimination rule.
2. A is the conclusion of an application of $\vee_i$-elimination or $\exists_i$-elimination and at the same time major premiss of an application of an elimination rule.

Normalization

As usual, we have some operations, the so-called reductions, whose aim is to locally eliminate maximum formulas. The reductions for $\wedge$, $\neg$, $\forall$ and for the intuitionistic operators are the usual Prawitz-reductions (including the permutative reductions).

Normalization

$\rightarrow_i$-reduction

$$\dfrac{\begin{array}{c}\Pi_1\\ A\end{array} \qquad \dfrac{\begin{array}{c}[A]^n\\ \Pi_2\\ B\end{array}}{(A \rightarrow_i B)}\, n}{B} \qquad \text{reduces to} \qquad \begin{array}{c}\Pi_1\\ [A]\\ \Pi_2\\ B\end{array}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A(t)\end{matrix}}{\exists_i x A(x)} \qquad \begin{matrix}[A(a)]^n\\ \Pi_2\\ C\end{matrix}}{C}\, n \qquad \text{reduces to} \qquad \begin{matrix}\Pi_1\\ [A(t)]\\ \Pi_2[a/t]\\ C\end{matrix}$$

$\vee_i$-permutative reduction

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\(A \vee_i B)\end{matrix} \quad \begin{matrix}[A]\\\Pi_2\\C\end{matrix} \quad \begin{matrix}[B]\\\Pi_3\\C\end{matrix}}{C} \quad \begin{matrix}\Sigma_1\\D_1\end{matrix} \quad \dots \quad \begin{matrix}\Sigma_k\\D_k\end{matrix}}{E}$$

Normalization

reduces to

$$\dfrac{\begin{matrix}\Pi_1\\(A \vee_i B)\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \Pi_2\\ C\end{matrix} \qquad \begin{matrix}\Sigma_1\\ D_1\end{matrix} \qquad \dots \qquad \begin{matrix}\Sigma_k\\ D_k\end{matrix}}{E} \qquad \dfrac{\begin{matrix}[B]\\ \Pi_3\\ C\end{matrix} \qquad \begin{matrix}\Sigma_1\\ D_1\end{matrix} \qquad \dots \qquad \begin{matrix}\Sigma_k\\ D_k\end{matrix}}{E}}{E}$$

Normalization

The new reductions for the classical operators are:

Normalization

$\to_c$-reduction

$$\dfrac{\dfrac{\begin{matrix}[A]^n & [\neg B]^m \\ & \Pi_1 \\ & \bot\end{matrix}}{A \to_c B}\, n, m \qquad \begin{matrix}\Pi_2 \\ A\end{matrix} \qquad \begin{matrix}\Pi_3 \\ \neg B\end{matrix}}{\bot} \qquad \text{reduces to} \qquad \begin{matrix}\Pi_2 & & \Pi_3 \\ [A] & & [\neg B] \\ & \Pi_1 & \\ & \bot & \end{matrix}$$

Normalization

$\vee_c$-*reduction*

$$
\dfrac{\dfrac{\begin{array}{c}[\neg A] \qquad [\neg B] \\ \Pi_1 \\ \bot\end{array}}{A \vee_c B} \qquad \begin{array}{c}\Pi_2 \\ \neg A\end{array} \qquad \begin{array}{c}\Pi_3 \\ \neg B\end{array}}{\bot}
\quad \text{reduces to} \quad
\begin{array}{c}\Pi_2 \qquad \Pi_3 \\ [\neg A] \qquad [\neg B] \\ \Pi_1 \\ \bot\end{array}
$$

Normalization

$\forall_c$-reduction

$$\dfrac{\dfrac{\begin{matrix}[\forall x \neg A(x)]^n \\ \Pi_1 \\ \bot\end{matrix}}{\exists_c x A(x)}\,n \qquad \begin{matrix}\Pi_2 \\ \forall x \neg A(x)]^n\end{matrix}}{\bot}$$

reduces to

$$\begin{matrix}\Pi_2 \\ [\forall x A(x)] \\ \Pi_1 \\ \bot\end{matrix}$$

Normalization

Definition 3

The *ecumenical degree* of A, ed(A), and is defined as follows:

1. ed(ϕ) $=$ 0, for any propositional variable ϕ.
2. ed($\neg A$) $=$ ed(A) $+$ 1
3. ed(A $\square$ B) $=$ ed(A) $+$ ed(B) $+$ 1, if $\square$ is $\wedge$ or an intuitionistic operator.
4. ed(A $\vee_c$ B) $=$ ed(A) $+$ ed(B) $+$ 4
5. ed(A $\rightarrow_c$ B) $=$ ed(A) $+$ ed(B) $+$ 3
6. ed($\forall x A(x)$) $= ed(A(x)) + 1$
7. ed($\exists x A(x)$) $= ed(A(x)) + 3$

Normalization

Definition 4

The ecumenical degree of a derivation Π, $ed(\Pi)$, is defined as max{ed[A] : A is maximum formula in Π}.

Definition 5

A derivation Π is *normal* if and only if $ed[\Pi] = 0$.

Normalization

Definition 6

A derivation Π is called critical if and only if

1. Π ends with an application of an elimination rule α.
2. The major premiss A of α is a maximum formula.
3. $\text{ed}[\Pi] = \text{ed}[\text{A}]$.
4. For every proper subderivation Π' of Π, $\text{ed}[\Pi'] < \text{ed}[\text{A}] = \text{ed}[\Pi]$.

Normalization

Lemma 7

Let Π_1 / A and A/Π_2 be two derivations in $\mathcal{NE}$ such that ed(Π_1) = n1 and ed(Π_2) = n2. Then, ed[(Π_1/[A]/Π_2)] = max{ed[A], n1, n2}.

Proof.

Induction on the length oh Π_2. □

Lemma 8

If Π reduces to Π', then ed[Π'] $\leq$ ed[Π].

Proof.

Directly from the form of the reductions and the definition of the complexity measure "ed". □

Normalization

Lemma 9

Let Π be a critical derivation of $\Gamma \vdash A$. Then, Π reduces to a derivation Π' of $\Delta \subseteq \Gamma \vdash A$ such that d(Π') $< d(\Pi)$.

Proof: By induction on the length of Π.

Lemma 10

Let Π be a derivation of $\Gamma \vdash A$ such that ed$[\Pi] > 0$. Then, Π reduces to a derivation Π' of $\Delta \subseteq \Gamma \vdash A$ such that ed[Π'] $<$ ed[Π].

Proof.

Directly from lemma [3] using induction on the length of Π. $\square$

Normalization

Theorem 11

Let Π be a derivation of $\Gamma \vdash A$. Then, Π reduces to a normal derivation Π' of $\Delta \subseteq \Gamma \vdash A$.

Proof.

Directly from lemma [4] by induction on the degree of Π. $\square$

Peter Krauss's Ecumenical system

Peter Krauss's Ecumenical system

> *The basic idea rests on the observation that classical mathematics uses the logical operators*
>
> *and, or, if-then, for all, there exists*
>
> *with two different meanings. One meaning is constructive and is determined by the rules of intuitionistic logic. The other meaning involves proofs by contradiction and becomes non-constructive through the elimination of double negation. Such proofs are usually called indirect. These two meanings can only be separated as long as one abstains from eliminating double negations. With elimination of double negation the two meanings fuse and the proof loses its constructive validity.*

(Peter Krauss, 1992)

Peter Krauss's Ecumenical system

In this section we introduce a formal system of first-order logic which permits us to distinguish between two different meanings of the logical operators and, or, if-then, for all, there exists. The logical operator 'not' will be used with one meaning only.

(Peter Krauss, 1992)

Rules for classical conjunction

$$\dfrac{\begin{matrix} [\neg A]^1 \\ \Pi_1 \\ \bot \end{matrix} \qquad \begin{matrix} [\neg B]^2 \\ \Pi_2 \\ \bot \end{matrix}}{(A \wedge_c B)}\, 1,2 \qquad \dfrac{(A \wedge_c B) \qquad \neg A}{\bot} \qquad \dfrac{(A \wedge_c B) \qquad \neg B}{\bot}$$

Rules for the universal quantifier

$$\dfrac{\begin{array}{c}[\exists_i x \neg A(x)] \\ \Pi \\ \bot\end{array}}{\forall_c x A(x)}\ \forall_c\text{-Int} \qquad \dfrac{\forall_c x A(x)}{\neg\neg A(x/t)}\ \forall_c\text{-Elim}$$

Example: $(\forall_i x \neg\neg A(x) \to_i \neg\neg\forall_c x A(x))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\exists_i x \neg A(x)]^2 \qquad \dfrac{\dfrac{[\forall_i x \neg\neg A(x)]^4}{\neg\neg A(a)} \qquad [\neg A(a)]^1}{\bot}}{\bot}\,1}{\forall_c x A(x)}\,2 \qquad [\neg\forall_c x A(x)]^3}{\bot}}{\dfrac{\neg\neg\forall_c x A(x)}{(\forall_i x \neg\neg A(x) \to_i \neg\neg\forall_c x A(x))}\,4}\,3
$$

Exercise 2: Show that these expressions are equivalent.

- $\forall_i x(A \rightarrow_i \exists_c y B)$
- $\forall_i x(A \rightarrow_c \exists_i y B)$
- $\forall_c x(A \rightarrow_i \exists_i y B)$

The rules for $\wedge_c$ and $\forall_c$ are harmonic!.

$$\dfrac{\dfrac{\begin{matrix}[\neg A]^1 \\ \Pi_1 \\ \bot\end{matrix} \qquad \begin{matrix}[\neg B]^2 \\ \Pi_2 \\ \bot\end{matrix}}{(A \wedge_c B)}\,1,2 \qquad \begin{matrix}\Pi_3 \\ \neg A\end{matrix}}{\bot}$$

reduces to

$$\begin{matrix}\Pi_3 \\ [\neg A] \\ \Pi_1 \\ \bot\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\exists_i x \neg A(x)]^1 \\ \Pi \\ \bot\end{matrix}}{\forall_c x A(x)}\,1}{\neg\neg A(x/t)}$$

reduces to

$$\dfrac{\begin{matrix}\dfrac{[\neg A(x/t)]^1}{[\exists_i x \neg A(x)]} \\ \Pi \\ \bot\end{matrix}}{\neg\neg A(x/t)}\,1$$

Interesting remarks!

- $\wedge_c$ *does not have PROJECTIONS!!*
- $\forall_c$ *does not have INSTANTIATIONS!!*

Barroso-Nascimento's system ECI

Barroso-Nascimento's system ECI

Main idea: "a *generalist approach* that consists of adding general rules which allow the introduction of ecumenical versions of any formula. Thus, instead of directly defining assertion conditions for specific ecumenical connectives, the generalist approach aims to define assertion conditions for ecumenical formulas in general, so as we can introduce the remaining ecumenical operators as special cases of the general rule."

Instead of ecumenical operators, ecumenical formulas, A^i and A^c.

The new rules

$$\dfrac{\begin{array}{c}[\neg A]\\ \Pi \\ \bot\end{array}}{A^c}\,I_C \qquad\qquad \dfrac{A^c \qquad \neg A}{\bot}\,E_C$$

The new rules are harmonic

$$\dfrac{\dfrac{\begin{matrix}[\neg A] \\ \Pi \\ \bot\end{matrix}}{A^c}\, I_C \qquad \begin{matrix}\Pi_2 \\ \neg A\end{matrix}}{\bot}\, E_C \qquad \text{reduces to} \qquad \begin{matrix}\Pi_2 \\ \neg A \\ \Pi_1 \\ \bot\end{matrix}$$

Barroso-Nascimento's system ECI

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^1}{(A \vee \neg A)} \qquad [\neg(A \vee \neg A)]^2}{\bot}}{\neg A}\,1}{(A \vee \neg A)} \qquad [\neg(A \vee \neg A)]^2}{\bot}}{(A \vee \neg A)^c}\,2$$

Ecumenical Sequent Calculus

Ecumenical Sequent Calculus

Initial Sequent and Structural Rules

$$\frac{}{A, \Gamma \Rightarrow A}\ \text{init} \qquad \frac{}{\bot, \Gamma \Rightarrow A}$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A} \qquad \frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}$$

Ecumenical Sequent Calculus

Classical rules

$$\frac{A, \Gamma \Rightarrow \bot \qquad B, \Gamma \Rightarrow \bot}{(A \vee_c B), \Gamma \Rightarrow \bot} \qquad \frac{\neg A, \neg B, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow (A \vee_c B)}$$

$$\frac{(A \rightarrow_c B), \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \bot}{(A \rightarrow_c B), \Gamma \Rightarrow \bot} \qquad \frac{A, \neg B, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow (A \rightarrow_c B)}$$

$$\frac{A[y/x], \Gamma \Rightarrow \bot}{\exists_c x A(x), \Gamma \Rightarrow \bot} \qquad \frac{\Gamma, \forall x \neg A(x) \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x A(x)}$$

Ecumenical Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow \bot}}{\neg p, p \Rightarrow q}}{\neg p \Rightarrow (p \to_i q)} \qquad p \Rightarrow p}{\neg p, ((p \to_i q) \to_i p) \Rightarrow p}}{\neg p, \neg p, ((p \to_i q) \to_i p) \Rightarrow \bot}}{\neg p, ((p \to_i q) \to_i p) \Rightarrow \bot}}{\Rightarrow (((p \to_i q) \to_i p) \to_c p)}$$

Ecumenical Tableaux

(Renato R. Leme, Giorgio Venturi, Bruno Lopes, Marcelo Coniglio)

Ecumenical Tableaux

$$\frac{S, T(A \wedge B)}{S, TA, TB} \qquad \frac{S, F(A \vee_c B)}{S, T(\neg A), T(\neg B)}$$

$$\frac{S, T(\neg A)}{S, FA} \qquad \frac{S, F(A \to_c B)}{S, TA, T(\neg B)}$$

Ecumenical Tableaux

$$\frac{S, F(A \wedge B)}{S, FA | S, FB} \qquad \frac{S, T(A \to_i B)}{S, FA | S, TB} \qquad \frac{S, T(A \vee_i B)}{S, TA | S, TB}$$

$$\frac{S, T(A \to_c B)}{S, FA | S, F(\neg B)} \qquad \frac{S, T(A \vee_c B)}{S, F(\neg A) | S, F(\neg B)}$$

An example

$$(0,0)$$
$$F(p \vee_c \neg p)$$
$$(1,0)$$
$$T\neg p$$
$$(2,0)$$
$$T\neg\neg p$$
$$(7,0)$$
$$Fp$$
$$(18,2)$$
$$F\neg p$$
$$(47,18)$$
$$Tp$$
$$END$$

Ecumenical Modal Systems

Ecumenical Modal Systems

The semantics of modal logics is often determined by means of *Kripke models*. Here, we will follow the approach where a modal logic is characterized by the respective interpretation of the modal model in the meta-theory (called *meta-logical characterization*).
Formally, given a variable x, we recall the standard translation $trm[\cdot]_x$ from modal formulas into first-order formulas with at most one free variable, x, as follows:

- if p is atomic, then $trm[p]_x = p(x)$;
- $trm[\bot]_x = \bot$;
- for any binary connective $\star$, $trm[(A \star B)]_x = (trm[A]_x \star trm[B]_x)$;
- for the modal connectives

$$\begin{array}{rcl} trm[\Box A]_x & = & \forall y(R(x,y) \rightarrow trm[A]_y) \\ trm[\Diamond A]_x & = & \exists y(R(x,y) \wedge trm[A]_y) \end{array}$$

  where $R(x,y)$ is a binary predicate.

Ecumenical Modal Systems

Such a translation has as underlying motivation the interpretation of alethic modalities in a Kripke model $\mathcal{M} = (W, R, V)$:

- $\mathcal{M}, w \models \Box A$ iff for all v such that w R v, $\mathcal{M}, v \models A$.
- $\mathcal{M}, w \models \Diamond A$ iff there exists v such that w R v and $\mathcal{M}, v \models A$.

$R(x, y)$ then represents the *accessibility relation* R in a Kripke frame. This intuition can be made formal based on the one-to-one correspondence between classical/intuitionistic translations and Kripke modal models.

Ecumenical Modal Systems

The object-modal logic OL is then characterized in the first-order meta-logic ML as $\vdash_{OL} A \quad$ *iff* $\quad \vdash_{ML} \forall x.trmAx$
Hence, if ML is classical logic (CL), the former definition characterizes the classical modal logic K*, while if it is intuitionistic logic (IL), then it characterizes the intuitionistic modal logic* IK*.*
We can adopt now adopt an ecumenical meta-theory (given by the system $\mathcal{NE}$*), hence characterizing what we will define as the ecumenical modal logic* $\mathcal{EK}$*.*

Ecumenical Modal Systems

The ecumenical translation $trme[\cdot]_x$ from propositional ecumenical formulas into $\mathcal{NE}$ is defined in the same way as the modal translation $trm[\cdot]_x$. For the case of modal connectives, as in the case of FOL, the interpretation of ecumenical consequence should be essentially *intuitionistic*. This implies that the box modality ($\Box$) is a *neutral connective*. The diamond ($\Diamond$), on the other hand, has two possible interpretations: classical and intuitionistic, since its leading connective is an existential quantifier. Hence we should have the ecumenical modalities: $\Box, \Diamond_i, \Diamond_c$, determined by the translations

- $trme[\Box A]_x = \forall y(R(x,y) \rightarrow_i trmeAy)$
- $trme[\Diamond_i A]_x = \exists_i y(R(x,y) \wedge trme[A]_y)$
- $trme[\Diamond_c A]_x = \exists_c y(R(x,y) \wedge trme[A]_y)$

Ecumenical Modal Systems

The basic idea behind labeled proof systems for modal logic is to internalize elements of the associated Kripke semantics (namely, the worlds of a Kripke structure and the accessibility relation between them) into the syntax.

Labeled modal formulas are either *labeled formulas* of the form $x : A$ or *relational atoms* of the form xRy, where x, y range over a set of variables and A is a modal formula. *Labeled sequents* have the form $\Gamma \vdash x : A$, where Γ is a multiset containing labeled modal formulas. The labeled ecumenical system $\mathcal{EK}$ is presented in the next slide.

Ecumenical Modal Systems

$$\frac{}{x : p_i, \Gamma \Rightarrow x : p_i} \quad \frac{}{x : \bot, \Gamma \Rightarrow z : C}\ \bot_L \quad \frac{\Gamma \Rightarrow y : \bot}{\Gamma \Rightarrow x : A}$$

$$\frac{x : p_i, \Gamma \Rightarrow z : \bot}{x : p_c, \Gamma \Rightarrow z : \bot} \quad \frac{x : p_i, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : p_c}$$

$$\frac{x : A, x : B, \Gamma \Rightarrow z : C}{x : A \wedge B, \Gamma \Rightarrow z : C} \quad \frac{\Gamma \Rightarrow x : A \quad \Gamma \Rightarrow x : B}{\Gamma \Rightarrow x : A \wedge B} \quad \frac{x : \neg A, \Gamma \Rightarrow z : A}{x : \neg A, \Gamma \Rightarrow z : \bot}\ \neg L$$

Ecumenical Modal Systems

$$\frac{x:A,\Gamma\Rightarrow x:\bot}{\Gamma\Rightarrow x:\neg A}\quad \frac{x:A,\Gamma\Rightarrow z:C\quad x:B,\Gamma\Rightarrow z:C}{x:A\vee_i B,\Gamma\Rightarrow z:C}\quad \frac{\Gamma\Rightarrow x:A_j}{\Gamma\Rightarrow x:A_1\vee_i A_2}$$

$$\frac{x:A,\Gamma\Rightarrow z:\bot\quad x:B,\Gamma\Rightarrow z:\bot}{x:A\vee_c B,\Gamma\Rightarrow z:\bot}\quad \frac{\Gamma,x:\neg A,x:\neg B\Rightarrow x:\bot}{\Gamma\Rightarrow x:A\vee_c B}$$

$$\frac{x:A\rightarrow_i B,\Gamma\Rightarrow x:A\quad x:B,\Gamma\Rightarrow z:C}{x:A\rightarrow_i B,\Gamma\Rightarrow z:C}\quad \frac{x:A,\Gamma\Rightarrow x:B}{\Gamma\Rightarrow x:A\rightarrow_i B}$$

$$\frac{x:A\rightarrow_c B,\Gamma\Rightarrow x:A\quad x:B,\Gamma\Rightarrow z:\bot}{x:A\rightarrow_c B,\Gamma\Rightarrow z:\bot}\quad \frac{x:A,x:\neg B,\Gamma\Rightarrow x:\bot}{\Gamma\Rightarrow x:A\rightarrow_c B}$$

Ecumenical Modal Systems

$$\frac{xRy, y : A, x : \Box A, \Gamma \Rightarrow z : C}{xRy, x : \Box A, \Gamma \Rightarrow z : C}\ \Box L \qquad \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A}\ \Box R \qquad \frac{xRy, y : A, \Gamma \Rightarrow z : C}{x : \Diamond_i A, \Gamma \Rightarrow z : C}\ \Diamond_i L$$

$$\frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond_i A}\ \Diamond_i R \qquad \frac{xRy, y : A, \Gamma \Rightarrow z : \bot}{x : \Diamond_c A, \Gamma \Rightarrow z : \bot}\ \Diamond_c L \qquad \frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A}\ \Diamond_c R$$

We can show in $\mathcal{EK}$:
$\Diamond_c A \equiv_i \neg\Box\neg A$

$$\cfrac{\cfrac{x : \Box\neg A \Rightarrow x : \Box\neg A}{x : \Box\neg A, \neg\Box\neg A \Rightarrow x : \bot}}{x : \neg\Box\neg A \Rightarrow x : \Diamond_c A}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{xRy, y : A \Rightarrow y : a}{xRy, y : A, y : \neg A \Rightarrow y : \bot}}{xRy, y : A, x : \Box\neg A \Rightarrow y : \bot}}{xRy, y : A, x : \Box\neg A \Rightarrow z : \bot}}{x : \Diamond_c A, x : \Box\neg A \Rightarrow z : \bot}}{x : \Diamond_c A, x : \Box\neg A \Rightarrow x : \bot}}{x : \Diamond_c A, \Rightarrow x : \neg\Box\neg A}$$

Some interesting questions and properties

Some interesting questions

1. One or two negations?

We saw that in Prawitz' system $\mathcal{NE}$ and in Krauss' system we have just ***one negation*** *(negation is a* ***neutral operator****) and that although the system defined by Gilles Dowek begins with two negations, at some point in the paper (p.232), Dowek concludes that the system can work with just one negation.The problem now is: why do we have just one negation, given that we have two implications and that the negation of A could be understood as* ***"A implies $\perp$"****?*

Some interesting questions

2. *Strange Classical Operators!*

- *Classical implication* $\to_c$ *does not satisfy the rule of modus ponens.*
- *Classical conjunction* $\wedge_c$ *does not satisfy projections*
- *Classical universal quantifier* $\forall_c$ *does not satisfy instantiation.*

Some interesting questions

2. Purity

We saw that the classical rule are formulated with the aid of occurrences of negations. Is is possible to define rules for the classical operators that are PURE, i. e., that does not use ocurrences of negation?

Exercise:

Definition 12

A formula B is called *externally classical* (denoted by B^c) if and only if B is $\perp$, a classical predicate letter, or its main operator is classical (that is: $\rightarrow_c, \vee_c, \exists_c$). A formula C is *classical* if it is built from classical atomic predicates using only the connectives: $\rightarrow_c, \vee_c, \exists_c, \neg, \wedge, \forall$, and the unit $\perp$.

For externally classical formulas we can now prove the following theorems:

1. $\vdash_{\mathcal{NE}} (A \rightarrow_c B^c) \rightarrow_i (A \rightarrow_i B^c)$.
2. $\vdash_{\mathcal{NE}} (A \wedge (A \rightarrow_c B^c)) \rightarrow_i B^c$.
3. $\vdash_{\mathcal{NE}} \neg\neg B^c \rightarrow_i B^c$.
4. $\vdash_{\mathcal{NE}} \neg\exists_c x.\neg B^c \rightarrow_i \forall x.B^c$.

Some interesting questions

Exercise:

Show that if A is externally classical , then $\Gamma, \neg A \vdash \bot$ implies $\Gamma \vdash A$.

Exercise:

Prove the normalization theorem for Krauss' system KrE.