The Ecumenical Perspective in Logic
Lecture 4

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Plan: *Lecture 4 - Additional Topics*

- *Back to Translations*
- *Do we still have Glivenko's Theorems?*
- *Questions, Problems, Perspectives!!!*

What is really new in the ecumenical perspective? What does it bring to the table beyond translations between logics?

Translations and the ecumenical perspective

It is undeniable that there is a relation between translations and the ecumenical perspective. Prawitz himself observes in his 2015 paper that

Translations and the ecumenical perspective

Highly relevant to these discussions are the well-known translations of classical predicate logic into intuitionistic predicate logic, first discovered by Gentzen and Gödel.[...]. These translations will not be dealt with here. The emphasis will instead be on meaning-theoretical considerations, but they can be seen to some extent as spelling out the philosophical significance of the fact that classical logic can be translated into intuitionistic logic.

Translations and the ecumenical perspective

Already in 1992 Krauss had observed, as Prawitz did later in 2015, that

Of course, our constructive refinement of classical logic is related to the Gödel-Gentzen Negative Translation.

According to Krauss, the constructive interpretation of classical logic he proposes **is not** the Gödel-Gentzen negative because

It is not difficult to see that [his interpretation] interpolates the Gödel-Gentzen negative translation.

But what does it mean to say that Krauss' constructive interpretation interpolates *the Gödel-Gentzen negative translation? According to Krauss, we can define two functions* $[\cdot]^\circ$ *and* $[\cdot]^-$ *such that, if* g *corresponds to the Gödel-Gentzen translation, the following diagram commutes:*

Translations and the ecumenical perspective

$$
\begin{tikzcd}
 & {[\Gamma]^{\circ}} \arrow[dr, "{[\cdot]^{-}}"] & \\
\Gamma \arrow[ur, "{[\cdot]^{\circ}}"] \arrow[rr, "g"] & & {[[\Gamma]^{\circ}]^{-}}
\end{tikzcd}
$$

The idea is that the function $[\cdot]^{\circ}$ *eliminates occurrences of classical operators by their "intuitionistic interpretations". The function* $[\cdot]^{\circ}$ *defined here is adapted to the meaning of the operators given by the rules of Prawitz' system, hence it is different from the one defined by Krauss:*

Translations and the ecumenical perspective

- $[p]^\circ = p$, if p is a propositional variable.
- $[\bot]^\circ = \bot$
- $[\neg A]^\circ = \neg[A]^\circ$
- $[A \wedge B]^\circ = [A]^\circ \wedge [B]^\circ$
- $[A \vee_i B]^\circ = [A]^\circ \vee_i [B]^\circ$
- $[A \rightarrow_i B]^\circ = [A]^\circ \rightarrow_i [B]^\circ$
- $[A \vee_c B]^\circ = \neg(\neg[A]^\circ \wedge \neg[B]^\circ)$
- $[A \rightarrow_c B]^\circ = \neg([A]^\circ \wedge \neg[B]^\circ)$

Translations and the ecumenical perspective

On the other hand, the function $[\cdot]^-$ places double negations in front of propositional variables:

- $[p]^- = \neg\neg p$, if p is a propositional variable;
- $[\bot]^- = \bot$
- $[\neg A]^- = \neg[A]^-$
- $[A \star B]^- = [A]^- \star [B]^-$ for $\star \in \{\wedge, \vee_i, \rightarrow_i, \vee_c, \rightarrow_c\}$

Translations and the ecumenical perspective

Lemma 1

Let A be a formula in the language of $\mathcal{NE}$. Hence

(a) $[A]^\circ$ *is an intuitionistic formula.*

(b) $\vdash_{\mathcal{NE}} (A \leftrightarrow_i [A]^\circ)$

The proof of (a) is trivial, since the only operators that appear in A° are intuitionistic.

(b) is proved by structural induction over A.

Translations and the ecumenical perspective

Since the co-domain of $[\cdot]^\circ$ is the intuitionistic fragment of $\mathcal{NE}$., from now on, we will abuse the notation: whenever convenient we will identify $\to_i, \vee_i$ with IL's implication and disjunction symbols $\to, \vee$.

Lemma 2

$\Gamma \vdash_{\mathcal{NE}} A$ *if and only if* $[\Gamma]^\circ \vdash_{\mathsf{IL}} [A]^\circ$.

Proof.

By Lemma **??** (b), $\Gamma \vdash_{\mathcal{NE}} A$ implies that $[\Gamma]^\circ \vdash_{\mathcal{NE}} [A]^\circ$. By the normalization theorem for $\mathcal{NE}$. and Lemma **??** (a), the only rules that are used in the derivation of $[A]^\circ$ from $[\Gamma]^\circ$ are rules for the intuitionistic operators. We can immediately conclude that this is a derivation in IL. The other direction is direct, since $\mathcal{NE}$. is a conservative extension of IL. □

Lemma 3

$\Gamma \vdash_{\mathsf{CL}} A$ if and only if $[\Gamma]^{\circ} \vdash_{\mathsf{CL}} [A]^{\circ}$.

Proof.

First of all, observe that $\vdash_{\mathsf{cp}} A \leftrightarrow_i [A]^{\circ}$. In fact, if A is intuitionistic, then this holds trivially since $A = [A]^{\circ}$. If A's main connective is classical, we use the equivalences in Lemma **??**, which are proved strictly in $\mathcal{NE}$., *i.e.* the the rule $\bot_c$ is never applied. The main result then follows easily. $\square$

Observe that cp collapses the ecumenical system into classical logic, since $\vdash_{\mathsf{CL}} A \vee_i \neg A$. However, there is no proof of this formula if we restrict the application of $\bot_c$ to the atomic case. This can be achieved if we restrict the ecumenical formulas to the $\vee_i$-free ecumenical fragment.

Lemma 4

Let Γ, A *be* $\vee_i$*-free. If* Π *is a derivation of* $[\Gamma]^\circ \vdash_{\mathsf{CL}} [A]^\circ$*, then every application of the classical reductio* $\bot_c$ *in* Π *can be restricted to the atomic case, i.e., with an atomic conclusion.*

But the function $[\cdot]^\circ$ is more than a simple device to eliminate classical operators from the ecumenical language. In fact, according to Krauss, it is a *constructive* interpretation of classical propositional reasoning in the *theory* of *stable atomic formulas* with ILp.
Using the terminology introduced by Krauss, let STAT be defined as the set $\{(\neg\neg p \rightarrow_i p) : p\ atomic\}$. We can then prove that:

Theorem 5

Let Γ and A be classical. Then $\Gamma \vdash_{\mathsf{CL}} A$ if and only if $\mathsf{STAT} + [\Gamma]^\circ \vdash_{\mathcal{NE}} [A]^\circ$.[a]

[a] This is an abuse of notation: while STAT may be an infinite set, only the finite subset of axioms involving the atomic subformulas of Γ, A is added to the context.

Let us now consider the Gödel-Gentzen translation g adapted to Prawitz' system:

- $[p]^g = \neg\neg p$, if p is a propositional variable.
- $[A \wedge B]^g = [A]^g \wedge [B]^g$
- $[A \vee_i B]^g = [A]^g \vee_i [B]^g$
- $[A \rightarrow_i B]^g = [A]^g \rightarrow_i [B]^g$
- $[A \vee_c B]^g = \neg(\neg[A]^g \wedge \neg[B]^g)$
- $[A \rightarrow_c B]^g = \neg([A]^g \wedge \neg[B]^g)$

The next result highlights the fact that the provability of the translations collapses in CL.

Lemma 6

$\Gamma \vdash_{\mathsf{CL}} A$ *if and only if* $[\Gamma]^g \vdash_{\mathsf{CL}} [A]^g$. *Hence,* $[\Gamma]^\circ \vdash_{\mathsf{CL}} [A]^\circ$ *if and only if* $[\Gamma]^g \vdash_{\mathsf{CL}} [A]^g$.

Proof.

It is easy to show that $\vdash_{\mathsf{CL}} (A \leftrightarrow_i [A]^g)$. In fact, for p atomic $[p]^g = \neg\neg p$ and $\vdash_{\mathsf{CL}} (p \leftrightarrow \neg\neg p)$. The rest of the proof is similar to Lemma **??**. □

Obviously, the translation $[\cdot]^\circ$ *is not* the translation g, given that the former does not put double-negations in front of propositional variables. But if we have the translation $[\cdot]^-$ that places *double negations* in front of propositional variables, we can immediately see that:

Lemma 7

For every ecumenical formula A, $[A]^g = [[A]^\circ]^-$.

Theorem 8

$[\Gamma]^g \vdash_{\mathsf{IL}} [A]^g$ *if and only if* $[[\Gamma]^\circ]^- \vdash_{\mathsf{IL}} [[A]^\circ]^-$

Proof.

Directly from Lemma **??**. $\square$

It is in this sense that, according to Krauss, the constructive interpretation he proposes interpolates the Gödel-Gentzen translation.

New translations and the ecumenical perspective

As we saw, derivability is preserved if we replace the classical operators by their constructive interpretation given by the function $[\cdot]^\circ$. But what can we say if we want to preserve classical derivability in the ecumenical system? Is it the case that if $\Gamma \vdash_{\mathsf{CL}} A$, then $\Gamma^* \vdash_{\mathcal{NE}} A^*$, where A^* is the result of replacing every occurrence of $\vee$ and $\rightarrow$ in A and in every formula B in Γ by their classical counterparts $\vee_c$ and $\rightarrow_c$?

Clearly full preservation of derivability cannot be obtained, as the following simple example shows:

$$\{p, (p \to q)\} \vdash_{\mathsf{CL}} q, \text{ but } \{p, (p \to_c q)\} \nvdash_{\mathcal{NE}} q$$

In order to examine the relation between *classical derivability* and *ecumenical derivability* more closely, let us define the translation function T_c suggested above from the language of propositional classical logic into the language of the ecumenical system.

1. $T_c[p] = p$
2. $T_c[\bot] = \bot$
3. $T_c[\neg A] = \neg T_c[A]$
4. $T_c[A \wedge B] = T_c[A] \wedge T_c[B]$
5. $T_c[A \vee B] = T_c[A] \vee_c T_c[B]$
6. $T_c[A \rightarrow B] = T_c[A] \rightarrow_c T_c[B]$

Although, as we saw above, we cannot get full preservation of derivability, we can get the following weaker result.

Theorem 9

If $\Gamma \vdash_{\mathsf{CL}} A$, *then* $T_c[\Gamma] \vdash_{\mathcal{NE}} \neg\neg T_c[A]$

As a direct corollary we obtain:

Corollary 10

If $\vdash_{\mathsf{CL}} A$, *then* $\vdash_{\mathcal{N}\mathcal{E}} \neg\neg T_c[A]$

But in Prawitz' ecumenical system we also have:

Lemma 11

If $\vdash_{\mathcal{NE}} \neg\neg T_c[A]$ *then* $\vdash_{\mathcal{NE}} T_c[A]$

From the corollary and lemma we obtain:

Theorem 12

If $\vdash_{\mathsf{CL}} A$, *then* $\vdash_{\mathcal{NE}} T_c[A]$.

Glivenko's Theorems from an ecumenical perspective

The results known as Glivenko's theorems in the area of logic were published in 1929, in French, in the Bulletins de la Classe des Sciences de la Académie Royale de Belgique, under the title Sur quelques points de la logique de M. Brouwer. The first Glivenko theorem establishes that if a formula A *is classically provable in* CPL*, then its double negation is intuitionistically provable in* IPL*:*

Glivenko's Theorems

Theorem 13

If $\vdash_{CPL} A$, *then* $\vdash_{IPL} \neg\neg A$.

The first theorem is a trivial consequence of Seldin's normalization strategy for classical proposition logic.

Interesting remark

- *Seldin's normalization strategy* $\Rightarrow$ *Glivenko*
- *Glivenko* $\Rightarrow$ *Raggio's normalization strategy*

Glivenko's Theorems

The second Glivenko theorem establishes that if a formula $\neg A$ is provable in CPL, then the same formula is provable in IPL:

Theorem 14

If $\vdash_{CPL} \neg A$*, then* $\vdash_{IPL} \neg A$*.*

The second theorem is a trivial consequence of the first theorem: by the first theorem, $\vdash_{CPL} \neg A$ implies $\vdash_{IPL} \neg\neg\neg A$, and $\vdash_{IPL} \neg\neg\neg A$ intuitionistically implies $\vdash_{IPL} \neg A$.

Prawitz' Ecumenical system and Glivenko

It is obvious that we cannot have Glivenko's theorems in Prawitz' ecumenical system, for the plain reason that we do not have two systems, the intuitionistic system and the classical system. However, we can have a kind of internal Glivenko, which establishes Glivenko-type relations between classical operators and intuitionistic operators.

Prawitz' Ecumenical system and Glivenko

However, we can have a kind of internal Glivenko, which establishes Glivenko-type relations between classical operators and intuitionistic operators.

Theorem 15

For any formula C such that the main operator of C is a classical operator we have that $C \vdash_{\mathcal{NE}} \neg\neg C^*$, where C^* is the result of replacing the classical operator by the corresponding intutionistic operator.

Prawitz' Ecumenical system and Glivenko

1. C is $(A \vee_c B)$. We can proof that $(A \vee_c B) \vdash \neg\neg(A \vee_i B)$ as follows:

$$\dfrac{\dfrac{(A \vee_c B) \qquad \dfrac{\dfrac{\dfrac{[A]^1}{(A \vee_i B)} \qquad [\neg(A \vee_i B)]^3}{\bot}}{\neg A}\,1 \qquad \dfrac{\dfrac{\dfrac{[B]^2}{(A \vee_i B)} \qquad [\neg(A \vee_i B)]^3}{\bot}}{\neg B}\,2}{\bot}}{\neg\neg(A \vee_i B)}\,3$$

Prawitz' Ecumenical system and Glivenko

2. C is $(A \to_c B)$. We can proof that $(A \to_c B) \vdash \neg\neg(A \to_i B)$ as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{(A \to_c B) \quad [A]^2 \quad \cfrac{\cfrac{\cfrac{[B]^1}{(A \to_i B)} \quad [\neg(A \to_i B)]^3}{\bot}}{\neg B}\;1}{\bot}}{B}}{(A \to_i B)}\;2 \quad [\neg(A \to_i B)]^3}{\bot}}{\neg\neg(A \to_i B)}\;3$$

3. C is $\exists_c x A(x)$. We can proof that $\exists_c x A(x) \vdash \neg\neg\exists_i x A(x)$ as follows:

$$
\dfrac{\exists_c x A(x) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A(a)]^1}{\exists x A(x)} \qquad [\neg\exists x A(x)]^2}{\bot}}{\neg A(a)}\,1}{\forall x \neg A(x)}}{\dfrac{\bot}{\neg\neg\exists x A(x)}\,2}
$$

Prawitz' Ecumenical system and Glivenko

Theorem 16

For any formula $\neg C$ such that the main operator of C is a classical operator we have that $\neg C \vdash_{\mathcal{NE}} \neg\neg\neg C^*$, where C^* is the result of replacing the classical operator by the corresponding intutionistic operator.

Prawitz' Ecumenical system and Glivenko

We can also have a *Glivenko-type* of result that corresponds to Glivenko's second theorem:

Theorem 17

For any formula C such that the main operator of c is a classical operator we have that $\neg C \vdash_{\mathcal{NE}} \neg C^*$, where C^* is the result of replacing the classical operator by the corresponding intutionistic operator.

Proof:

Prawitz' Ecumenical system and Glivenko

1. C is $(A \vee_c B)$. We can proof that $\neg(A \vee_c B) \vdash \neg(A \vee_i B)$ as follows:

$$\dfrac{\dfrac{[(A \vee_i B)]^3 \qquad \dfrac{\dfrac{[A]^1}{(A \vee_c B)} \qquad \neg(A \vee_c B)}{\bot} \qquad \dfrac{\dfrac{[B]^2}{(A \vee_c B)} \qquad \neg(A \vee_c B)}{\bot}}{\bot}\ 1,2}{\neg(A \vee_i B)}\ 3$$

Barroso-Nascimento's system ECI

Main idea: "a *generalist approach* that consists of adding general rules which allow the introduction of ecumenical versions of any formula. Thus, instead of directly defining assertion conditions for specific ecumenical connectives, the generalist approach aims to define assertion conditions for ecumenical formulas in general, so as we can introduce the remaining ecumenical operators as special cases of the general rule."

Instead of ecumenical operators, ecumenical formulas, A^i and A^c.

The new rules

$$\dfrac{\begin{array}{c}[\neg A]\\ \Pi\\ \bot\end{array}}{A^c}\ I_C \qquad\qquad \dfrac{A^c \qquad \neg A}{\bot}\ E_C$$

Barroso-Nascimento's system ECI

Glivenko-type of results are somehow trivial in Nascimento's system ECI

Barroso-Nascimento's system ECI

Theorem 18

$A^c \vdash_{ECI} \neg\neg A^i$

Proof:

$$\dfrac{\dfrac{A^c \qquad [\neg A]^1}{\bot}}{\neg\neg A}\,1$$

Barroso-Nascimento's system ECI

Theorem 19

$A \vdash_{ECI} A^c$

Proof:

$$\dfrac{\dfrac{A \qquad [\neg A]^1}{\perp}}{A^c}\, 1$$

Barroso-Nascimento's system ECI

Theorem 20

$\neg A^c \vdash_{ECI} \neg A$

Proof: Directly from the theorem above.

The classical universal quantifier

In Prawitz' ecumenical system, negation, conjunction and the universal quantifier are *neutral* operators, i.e., operators *shared* by the classical and the intuitionistic parties. In the system ECI, we have the possibility of labelling an universal formula with the label/constant c, producing the *classical* formula $(\forall x A(x))^c$. In the ecumenical system proposed by Peter Krauss we have an explicit classical universal quantifier $\forall_c$ with the following introduction and elimination rules (here adapted to a Gentzen-Prawitz style):

The classical universal quantifier

$$\dfrac{\begin{matrix}[\exists_i x \neg A(x)] \\ \Pi \\ \bot\end{matrix}}{\forall_c x A(x)}\ \forall_c\text{-Int} \qquad\qquad \dfrac{\forall_c x A(x)}{\neg\neg A(x/t)}\ \forall_c\text{-Elim}$$

Let us call this variant of Prawitz' system $\mathcal{NE}^*$.

The classical universal quantifier

These rules are *harmonic*:

$$\dfrac{\dfrac{\begin{array}{c}[\exists_i x \neg A(x)]\\ \Pi \\ \bot\end{array}}{\forall_c x A(x)}\,\forall_c\text{-Int}}{\neg\neg A(x/t)}\,\forall_c\text{-Elim} \qquad \text{reduces to} \qquad \dfrac{\begin{array}{c}\dfrac{[\neg A(x/t)]}{\exists_i x \neg A(x)}\\ \Pi \\ \bot\end{array}}{\neg\neg A(x/t)}$$

The classical universal quantifier

As expected, we do not have a Glivenko-type result for the classical universal quantifier: $\vdash_{\mathcal{NE}^*} \forall_c x A(x)$ does not imply $\vdash_{\mathcal{NE}^*} \neg\neg\forall_i x A(x)$ (and $\forall_c x A(x) \nvdash_{\mathcal{NE}^*} \neg\neg\forall_i x A(x)$). It is interesting to observe that we do have a Glivenko-type result for the classical universal quantifier that corresponds to Glivenko's second theorem, $\neg\forall_c x A(x) \vdash_{\mathcal{NE}^*} \neg\forall_i x A(x)$:

The classical universal quantifier

$$\dfrac{\dfrac{\dfrac{\dfrac{[\exists_i x\neg A(x)]^2 \qquad \dfrac{\dfrac{[\forall_i xA(x)]^3}{A(x/a)} \qquad [\neg A(x/a)]^1}{\bot}}{\bot}\,1}{\forall_c xA(x)}\,2 \qquad \neg\forall_c xA(x)}{\bot}}{\neg\forall_i xA(x)}\,3$$

The classical universal quantifier

But as we saw, the Glivenko-type of results are somehow trivial in the system ECI! Even for a universal formula we have that $(\forall x A(x))^c \vdash_{ECI} \neg\neg\forall_i x A(x)$ and $\neg(\forall x A(x))^c \vdash_{ECI} \neg\forall_i x A(x)$! If we now assume that in the formula $A(x)$ we have no occurrences of the label/constant c, we do have something that looks like a full Glivenko's first theorem!

The classical universal quantifier

But we know that Glivenko's first theorem does not extend to universal formulas! *What's the trick here?* In order to understand the real meaning of the Glivenko-type of results we can prove in ECI we will have a look at some relations between the behavior of the classical operator $\forall_c$ and the behavior of the label/constant c applied to a universal formula.

The classical universal quantifier

We can emulate an application of the I_c rule of ECI with conclusion $\forall_c x A(x)$ by an application of the $\forall_c$-Introduction rule (the suscript i will be kept):
Assume that we have a derivation Π of $\perp$ from $\neg\forall_i x A(x)$:

$$\begin{array}{c} \neg\forall_i x A(x) \\ \Pi \\ \perp \end{array}$$

The classical universal quantifier

We can now construct the following derivation with the use of $\forall_c$-Introduction:

$$\dfrac{\dfrac{[\exists_i x \neg A(x)]^3 \qquad \dfrac{\dfrac{[\forall_i x A(x)]^2}{A(x/a)} \qquad [\neg A(x/a)]^1}{\bot}}{\bot}\ 1}{[\neg \forall_i x A(x)]}\ 2$$

$$\Pi$$

$$3\ \dfrac{\bot}{\forall_c x A(x)}\ \forall_c\text{-Int}$$

The classical universal quantifier

We can also emulate an application of the $\forall_c$-Elimination rule by an application of the E_c rule of ECI:
Assume we have a derivation Π of $\forall_c x A(x)$:

$$\begin{array}{c} \Pi \\ \forall_c x A(x) \end{array}$$

The classical universal quantifier

We can now construct the following derivation with the use of the rule Ec of ECI:

$$
\dfrac{\dfrac{\begin{array}{c}\Pi \\ \forall_c x A(x)\end{array} \qquad \dfrac{\dfrac{\dfrac{[\forall_i x A(x)]^1}{A(x/t)} \qquad [\neg A(x/t)]^2}{\bot}}{\neg\forall_i x A(x)}\,1}{\bot}\,E_c}{\neg\neg A(x/t)}\,2
$$

The classical universal quantifier

But we cannot emulate the rule E_c of ECI by means of the rule $\forall_c$-Elimination, and the rule $\forall_c$-Introduction by means of the rule I_c of ECI, and this means that the deductive behavior of the operator is different from the deductive behavior of the label/constant.

The classical universal quantifier

In a certain sense, a formula $\forall_c x A(x)$ *can be interpreted as* $\neg\exists x \neg A(x)$*, whereas a formula* $(\forall x A(x)^c)$ *can be interpreted as* $\neg\neg\forall x A(x)$*, and these interpretations are not intuitionistically equivalent!*

The classical universal quantifier

And now I think we can understand in which sense the Glivenko-type results are trivial in ECI: what the theorem $A^c \vdash_{ECI} \neg\neg A^i$ says can be interpreted simply as $\neg\neg A \vdash_{ECI} \neg\neg A$, and in the particular case of universal formulas, as $\neg\neg\forall x A(x) \vdash_{ECI} \neg\neg\forall x A(x)$. Mistery solved!

Summation

Summation

1. *As we have seen, there are clear connections between the ecumenical perspective and translations, but these connections cannot been understood as a reduction of the former to the latter. In particular, Prawitz' ecumenical proposal* is not *a double-negation translation.*

Summation

2. *Translations clearly elucidate the behavior of the classical operators and how they interact with their intuitionistic counterparts. If one still wants to think of translations with respect to the ecumenical perspective, one should think of translations of derivations instead of translations between languages: the ecumenical perspective helps us to identify places where we have to be classical, and obviously we do not have to be classical at all times and everywhere.*

Summation

3. In a certain sense, we can describe the two approaches in the following way:
   - Prawitz' system: *classical, intuitionistic and neutral operators*;
   - The system ECI: distinguish a classical behavior from an intuitionistic behavior of a formula.

Summation

IMPORTANT: *If we restrict the two approaches to the propositional fragment, it is (apparently!) indifferent if we use ECI or Prawitz' system (with the addition of $\wedge_c$). But this is not true when we add the universal quantifier and the question now of which approach corresponds more faithfully to a classical universal quantifier is everything but negligible.*

Final remarks

3. Conjecture: Defining a translation based on the classical-mark c in the case of CPL will induce a result similar to Seldin's normalization strategy.

$$\dfrac{\begin{matrix}[\neg A]\\ \Sigma\\ \bot\end{matrix}}{A} \quad \text{is translated into} \quad \dfrac{\begin{matrix}[\neg A]\\ \Sigma\\ \bot\end{matrix}}{A^c}$$

Final remarks

4. I have not presented here, but we also have an extension of Prawitz' ecumenical system wth a classical conjunction.
5. Let Pr^* be the system obtained from Prawitz's ecumenical system by adding the rules for the classical universal quantifier. Because of the "imbalance" shown between Pr^* system and Nascimento's ECI system, we believe we can prove that:

1. $\{(\forall x A(x))^c, \neg\forall x A(x)\} \vdash_{ECI} B$, but $\{\forall_c x A(x), \neg\forall_i x A(x)\} \nvdash_{Pr^*} B$
2. $\{(\exists x \neg A(x) \to B), \neg B\} \vdash_{Pr^*} \forall_c x A(x)$, but $\{(\exists x \neg A(x) \to B), \neg B\} \nvdash_{ECI} \forall_c x A(x)$

Lots of things to be done ....

Questions, Problems, Perspectives!!!

Some questions (and possible answers!)

1. Why do we have just one negation?
   - Possible answer 1: $\to_c$ and $\to_i$ are interderivable.
   - Possible answer 2: In the case of propositional logic, if can prove $\neg A$, then we can prove $\neg A$ constructively (Glivenko 2!).
   - Maybe we need a stronger equivalence relation: computational isomorphism is a good candidate!
2. Why don't we have Popper's collapse?
   - Prawitz: "It has sometimes been held that a deductive system that contains both classical and intuitionistic constants is impossible, because the different constants would collapse. Popper (1948) was the first to observe that in a system containing the following rules for classical and intuitionistic negation (now formulated withou $\perp$):

$$\frac{\begin{array}{cc}[A] & [A] \\ B & \neg_i B\end{array}}{\neg_i A} \qquad \frac{A \qquad \neg_i A}{B} \qquad \frac{\begin{array}{cc}[A] & [A] \\ B & \neg_c B\end{array}}{\neg_c A} \qquad \frac{\begin{array}{cc}[\neg_c A] & [\neg_c A] \\ B & \neg_c B\end{array}}{A}$$

where A and B may be arbitrary sentences, the two negations collapse into one, that is, $\neg_i$ and $\neg_c$ become derivable from each other".

- Interesting exercises: [1] show that if these rules $\neg_i A \vdash \neg_c A$ and $\neg_c A \vdash \neg_i A$; [2] show that we do have a total colapse of intuitionistic logic and classical logic with these four rules.
- Problem: as we have just seen, in $\mathcal{NE}$ we have $\neg_i A \vdash \neg_c A$ and $\neg_c A \vdash \neg_i A$ too. How come??? How can we avoid the total colapse???

3. Why classical implication does not satisfy modus ponens?
   - Possible answer: the assertability conditions for $\rightarrow_c$ are weaker than the assertability conditions for $\rightarrow_i$. We can assert $(A \rightarrow_c B)$ when $\{\neg A, \neg B\} \vdash \bot$, but in this case $\{(A \rightarrow_c B), A\} \nvdash B$.
   - Remember that the usual introduction and elimination rules for implication in Natural Deduction do not completely characterize classical implication (Peirce's formula!).
4. Why classical conjunction does not have projections?
   - The answer is similar: weaker assertability conditions. We can assert $(A \wedge_c B)$ if we have $\{\neg A\} \vdash \bot$ and $\{\neg B\} \vdash \bot$.
5. Why the classical universal quantifier does not satisfy instantiation?
   - The answer is similar: weaker assertability conditions. We can assert $\forall_c x A(x)$ if we have $\{\exists_i x \neg A(x)\} \vdash \bot$.

6. What about "conservative results"?

- Let A be a formula in the language of IL and let A^* be the result of replacing every operator α in A by α_i. Then, we can easily show that if $\Gamma \vdash_{IL} A$, then $\Gamma^* \vdash_{\mathcal{NE}} A^*$.
- We saw that this does not hold in the case of CL: we have that $\{p, (p \to q)\} \vdash_{CL} q$, but we don't have $\{p, (p \to_c q)\} \vdash_{\mathcal{NE}} q$. But we saw that for the translation T_c we have (the weaker result):

Theorem 21

If $\Gamma \vdash_{\mathsf{CL}} A$*, then* $T_c[\Gamma] \vdash_{\mathcal{NE}} \neg\neg T_c[A]$

As a direct corollary we obtain:

Corollary 22

If $\vdash_{\mathsf{CL}} A$*, then* $\vdash_{\mathcal{NE}} \neg\neg T_c[A]$

7. What about "pure" systems?
   - We saw that all systems we have discussed in Lecture 2 were "impure": the rules for the classical operators make use of negation!
   - Can we have pure systems? Yes!!! We can use Girard's notion of stoup, Restall's notion of alternatives, or Murzi's higher level rules (with $\perp$ as a punctuation mark.

Greg Restall's technique of alternatives uses the *store* rule:

$$\frac{A \qquad \cancel{A}}{\bot}\uparrow$$

The rules $\vee_c$-elimination and $\rightarrow_c$-elimination are the same as in Prawitz' system. The rules for $\vee_c$ and $\rightarrow_c$ in the system $E - alt$ are:

$\vee_c$-Introduction $\qquad\qquad$ $\rightarrow_c$-Introduction

$$\begin{array}{c} [\cancel{A}]^n \qquad [\cancel{B}]^m \\ \Pi \\ \dfrac{\bot}{(A \vee_c B)}\, n, m \end{array} \qquad\qquad \begin{array}{c} [A]^n \qquad [\cancel{B}]^m \\ \Pi \\ \dfrac{\bot}{(A \rightarrow_c B)}\, n, m \end{array}$$

What must we do?

1. Real Ecumenical Mathematics!!!
    - This was the original motivation of Peter Krauss. We know that if all operators have a constructive "reading", the axiom of choice is a theorem in Martin-Löf Type Theory. But what would happen if we have *hybrid* readings of these same operators?
    - And what if we consider other principles, like the principle of well-ordering?
2. Applications in Computer Science!!!
3. First-order PtS.
    - We showed a PtS semantics for the propositional fragment of Barroso-Nascimento's system ECI. This kind of semantics must be extended to:
        - The full system ECI;
        - Prawitz' system $\mathcal{NE}$;
        - Krauss' system KrE.

4. New ecumenical codifications.
   - We have showned several ecumenical systems ($\mathcal{NE}$, ECI, KrE) for classical and intuitionistic logic. What about other logics?
     - Barroso-Nascimento has an ecumenical system for intuitionistic and minimal logic;
     - Sernada and Rasga have an ecumenical system for intuitionistic logic and classical $S4$;
     - Quite recently (April 2025), Rasga and Sernadas showed how to systematically connect translations to ecumenical systems and propose an ecumenical system for classical logic and Jaskowski's paraconsistent logic ("From translations to non-collapsing logic combinations").

5. Comparision with other mechanisms of combining logics?
   - The area of Combining Logics is a well-established field, with its distinctive techniques (Fibring, Possible-Translations,...). A certainly interesting question is how the ecumenical perspective relates to the field and techniques of Combining Logics.
6. The proof theory of Krauss' system.
   - Maybe it is just manual labour, but the proof-theory of Krauss' system is not done!
   - By the way, a full scale proof-theory for $\mathcal{NE}$ is not done either! (strong normalization, separability, unicity,...).