The Ecumenical Perspective in Logic
Lecture 3

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What about semantics?

Is there an Ecumenical semantics for Prawitz' system $\mathcal{NE}$?

Two semantics

- A Kripke semantics for the propositional fragment $\mathcal{NE}_{\mathcal{P}}$ of Prawitz' system $\mathcal{NE}$; and
- A Proof-theoretical semantics (PtS) for for the propositional fragment ECI_P of Barroso-Nascimento's system ECI.

A Kripke semantics for the propositional fragment $\mathcal{NE}_{\mathcal{P}}$ of Prawitz' system $\mathcal{NE}$

We have seen that Prawitz's motivation was logical-philosophical: to propose a rule-based semantics for classical logic in terms of assertibility conditions.

But if the system is really ecumenical, it should also allow for a semantics based on truth-conditions (so that the classical logician would be satisfied!)

Definition 1

An ecumenical Kripke model is a pair $\langle \mathcal{F}, e \rangle$ where $\mathcal{F} = \langle W, \leq \rangle$ is an intuitionistic Kripke frame and e is a intuitionistic valuation, i.e., e is a mapping associating with each propositional variable p and element of $U_{\leq}$, where $U_{\leq}$ is the set of all uppersets of $\langle W, \leq \rangle$, i.e. all sets $U \subseteq W$ such that if $\omega \in U$ and $\omega \leq \nu$, then also $\nu \in U_{\leq}$.

Let $\mathcal{M} = \langle \mathcal{F}, e \rangle$ an ecumenical Kripke model and $\omega \in W$. By induction in the construction of a formula φ we define the relation $(\mathcal{M}, \omega) \models \varphi$ as follows:

$$
\begin{array}{lll}
(\mathcal{M}, \omega) \models p & \text{iff} & \omega \in e(p); \\
(\mathcal{M}, \nu) \not\models \bot & & \\
(\mathcal{M}, \omega) \models \neg\psi & \text{iff} & \text{for all } \nu \in W \text{ such that } \omega \leq \nu : (\mathcal{M}, \nu) \not\models \psi; \\
(\mathcal{M}, \omega) \models \varphi \wedge \psi & \text{iff} & (\mathcal{M}, \omega) \models \varphi \text{ and } (\mathcal{M}, \omega) \models \psi; \\
(\mathcal{M}, \omega) \models \varphi \vee_i \psi & \text{iff} & (\mathcal{M}, \omega) \models \varphi \text{ or } (\mathcal{M}, \omega) \models \psi; \\
(\mathcal{M}, \omega) \models \varphi \vee_c \psi & \text{iff} & (\mathcal{M}, \omega) \models \neg(\neg\varphi \wedge \neg\psi); \\
(\mathcal{M}, \omega) \models \varphi \rightarrow_i \psi & \text{iff} & \text{for all } \nu \in W \text{ such that } \omega \leq \nu : (\mathcal{M}, \nu) \models \varphi \text{ implies } (\mathcal{M}, \nu) \models \psi; \\
(\mathcal{M}, \omega) \models \varphi \rightarrow_c \psi & \text{iff} & (\mathcal{M}, \omega) \models \neg(\varphi \wedge \neg\psi);
\end{array}
$$

As usual we define:

- $\Gamma \models_{(\mathcal{M},\omega)} A$ iff $\models_{(\mathcal{M},\omega)} \Gamma \Rightarrow \models_{(\mathcal{M},\omega)} A$;
- $\Gamma \models A$ iff $\forall \mathcal{M}$ and $\forall \omega \in \mathcal{M}$, $\Gamma \models_{(\mathcal{M},\omega)} A$.

As in the case of intuitionistic models we can prove general monotonicity (and this will be very important later!!!):

Lemma 2

If $\models_{(\mathcal{M},\omega)} A$ and $\omega \leq \nu$, then $\models_{(\mathcal{M},\nu)} A$.

Theorem 3 (Soundness)

$\Gamma \vdash_{\mathcal{N}\mathcal{E}_{\mathcal{P}}} A \Longrightarrow \Gamma \models A$

The proof of soundness is carried out by induction on the length of derivations. We shall only consider the case $\vee_c$, the other operators being treated in a similar way.

(1) $r[\Pi]$ is $\vee_c$-introduction. The derivation Π has the following form:

$$\dfrac{\begin{matrix}[\neg A] & \Gamma & [\neg B] \\ & \Pi_1 & \\ & \bot & \end{matrix}}{(A \vee_c B)}$$

By the induction hypothesis, we know that $\{\Gamma, \neg A, \neg B\} \models_{E_p} \bot$, and this means that there is no world ω such that $(\mathcal{M}, \omega) \models_{E_p} \Gamma \cup \{\neg A\} \cup \{\neg B\}$. Now, assume that $\Gamma \not\models_{E_p} (A \vee_c B)$, which is equivalent to say that $\Gamma \not\models_{E_p} \neg(\neg A \wedge \neg B)$. According to the definition of semantical consequence, $\Gamma \not\models_{E_p} \neg(\neg A \wedge \neg B)$ if and only if there exists a world ν such that $(\mathcal{M}, \nu) \models_{E_p} \Gamma$ and a world ν' such that $\nu' > \nu$ and $(\mathcal{M}, \nu') \models_{E_p} (\neg A \wedge \neg B)$. But, this leads to a contradiction, because for this world ν', $(\mathcal{M}, \nu') \models_{E_p} \Gamma \cup \{\neg A\} \cup \{\neg B\}$.

In order to prove the completeness theorem for the system $\mathcal{NE}_\mathcal{P}$, we shall define a translation Pr from the language of $\mathcal{NE}_\mathcal{P}$, into the language of intuitionistic propositional logic.

1. $Pr[\varphi] = \varphi$, if φ is a propositional variable.
2. $Pr[\bot] = \bot$
3. $Pr[\neg A] = \neg Pr[A]$
4. $Pr[(A \wedge B)] = (Pr[A] \wedge Pr[B])$.
5. $Pr[(A \vee_i B)] = (Pr[A] \vee_i Pr[B])$.
6. $Pr[(A \rightarrow_i B)] = (Pr[A] \rightarrow_i Pr[B])$.
7. $Pr[(A \vee_c B)] = \neg(\neg Pr[A] \wedge \neg Pr[B])$.
8. $Pr[(A \rightarrow_c B)] = \neg(Pr[A] \wedge \neg Pr[B])$.

Lemma 4

$\Gamma \vdash_{\mathcal{N}\mathcal{E}_{\mathcal{P}}} A$ if and only $\mathrm{Pr}[\Gamma] \vdash_{I_p} Pr[A]$, where $\vdash_{I_p}$ is the usual derivability relation in the propositional fragment of Prawitz' natural deduction intuitionistic system I.

From left to right: By induction over the length of the derivation Π of $\Gamma \vdash_{\mathcal{N}\mathcal{E}_{\mathcal{P}}} A$. We shall only consider the case where the last rule applied in Π is $\to_c$-introduction, the other cases being treated in a similar way. Let Π be the following derivation in $\mathcal{N}\mathcal{E}_{\mathcal{P}}$:

$$\dfrac{\begin{array}{ccc}[A] & [\neg B] & \Gamma \\ & \Pi_1 & \\ & \bot & \end{array}}{A \to_c B}$$

By the induction hypothesis, we have a derivation $Pr[\Pi_1]$ of $Pr[\bot]$ from $Pr[A]$, $Pr[\neg B]$ and $Pr[\Gamma]$. We can then take our derivation $Pr[\Pi]$ in the system I to be:

$$
\dfrac{\dfrac{\begin{array}{c}
\dfrac{[(Pr[A] \wedge \neg Pr[B])]}{Pr[A]} \qquad \dfrac{\dfrac{\dfrac{[(Pr[A] \wedge \neg Pr[B])]}{\neg Pr[B]}}{}}{Pr[\neg B]} \qquad Pr[\Gamma] \\
Pr[\Pi_1] \\
Pr[\bot]
\end{array}}{\neg(Pr[A] \wedge \neg Pr[B])}}{Pr[(A \rightarrow_c B)]}
$$

In order to prove the other direction, we shall prove an auxiliary lemma:

Lemma 5

$\vdash_{\mathcal{NE}_{\mathcal{P}}} (A \leftrightarrow_i Pr[A])$

The proof is carried out by induction on ed[A]. We shall examine two cases, the other cases being treated similarly.

(1) A is $\neg B$.

$$\dfrac{\dfrac{\dfrac{[B] \qquad B \leftrightarrow_i Pr[B]}{Pr[B]} \qquad \dfrac{Pr[\neg B]}{\neg Pr[B]}}{\bot}}{\neg B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[Pr[B]] \qquad (B \leftrightarrow_i Pr[B])}{B} \qquad \neg B}{\bot}}{\neg Pr[B]}}{\overline{Pr[\neg B]}}$$

(2) A is $(B \to_c C)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[B] \qquad B \leftrightarrow_i Pr[B]}{Pr[B]} \qquad \dfrac{\dfrac{\dfrac{[\neg C] \qquad \neg C \leftrightarrow_i Pr[\neg C]}{Pr[\neg C]}}{}}{\neg Pr[C]}}{(Pr[B] \wedge \neg Pr[C]} \qquad \dfrac{\dfrac{Pr[(B \to_c C)]}{}}{\neg(Pr[B] \wedge \neg Pr[C])}}{\bot}}{(B \to_c C)}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{(Pr[B] \wedge \neg Pr[C])}{Pr[B]} \qquad (B \leftrightarrow_i Pr[B])}{B} \qquad \dfrac{\dfrac{\dfrac{(Pr[B] \wedge \neg Pr[C])}{\neg Pr[C]}}{\overline{Pr[\neg C]}} \qquad (\neg C \leftrightarrow_i Pr[\neg C])}{\neg C} \qquad (B \to_c C]}{\bot}}{\neg (Pr[B] \wedge \neg Pr[C])}}{\overline{Pr[(B \to_c C)]}}
$$

The direction right to left now follows from the lemma above and the fact that $\mathrm{Pr}[\Gamma] \vdash_{I_p} Pr[A]$ implies that $\mathrm{Pr}[\Gamma] \vdash_{\mathcal{N}\mathcal{E}_\mathcal{P}} Pr[A]$.

Lemma 6

If $\Gamma \models A$, then $\mathrm{Pr}[\Gamma] \models_{I_p} Pr[A]$, where $\models_{I_p}$ is the usual consequence relation of propositional intuitionistic logic.

Proof.

1. Assume that $\Gamma \models A$.
2. Then, $\models (conj[\Gamma] \rightarrow_i A)$
3. By Lemma [6], $\vdash_{\mathcal{NE_P}} (conj[\Gamma] \leftrightarrow_i Pr[(conj[\Gamma])$
4. By the soundness of $\mathcal{NE_P}$, we have $\models_{\mathcal{NE_P}} (conj[\Gamma] \leftrightarrow_i Pr[(conj[\Gamma])$
5. Again, by Lemma [6] and soundness, we have $\models_{\mathcal{NE_P}} (A \leftrightarrow_i Pr[A])$
6. We then obtain $\models (Pr[conj[\Gamma]] \rightarrow_i Pr[A])$
7. From which it follows that $Pr[\Gamma] \models Pr[A]$.
8. Given that all the operators occurring in a formula Pr[B], for any B, are intuitionistic, we finally obtain that $Pr[\Gamma] \models_{I_p} Pr[A]$.

$\square$

Theorem 7 (Completeness)

Let $\Gamma \vDash A$*. Then* $\Gamma \vdash_{\mathcal{N}\mathcal{E}_{\mathcal{P}}} A$

Proof.

Assume that $\Gamma \vDash A$. By lemma [6], $\mathrm{Pr}[\Gamma] \vDash_{I_p} Pr[A]$. By the completeness of the system I_p, we have $\mathrm{Pr}[\Gamma] \vdash_{I_p} Pr[A]$. The result now follows directly from lemma [5]. $\square$

Ecumenical PtS semantics

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Model-theoretic semantics: mathematical structures help on supporting the notion of validity, which is based on a notion of *truth*.

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Intuitionistic disjunction: Kripke structures.

Truth tables

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

What is the meaning of a logical connective?

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Model-theoretic semantics: mathematical structures help on supporting the notion of validity, which is based on a notion of *truth*.

Intuitionistic disjunction: Kripke structures.

Generalizing

w

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

v

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

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Model-theoretic semantics: mathematical structures help on supporting the notion of validity, which is based on a notion of *truth*.

Intuitionistic disjunction: Kripke structures.

v

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Adding relations

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Proof-theoretic semantics (PtS) provides an alternative perspective for the meaning of logical operators compared to the viewpoint offered by model-theoretic semantics. In PtS, the concept of truth *is substituted with that of* proof*, emphasizing the fundamental nature of proofs as a means through which we gain demonstrative knowledge, particularly in mathematical contexts.*

PtS has as philosophical background inferentialism, *according to which inferences establish the meaning of expressions. This makes PtS a superior approach for comprehending reasoning since it ensures that the meaning of logical operators, such as connectives in logics, is defined on the basis of their use in inferences. (Wittgenstein!!)*

- **Logical ecumenism (LE)** aims to provide formal environments in which two or more "rival" logics may peacefully coexist.
- **Proof-theoretic semantics (PtS)** aims to provide accounts of logic in which notions of *proof*, and not of *truth*, are considered the basic units of semantic analysis.
- **Our proposal**: combine both approaches to obtain a environment in which *distinct notions of proof* may peacefully coexist as the basic units of semantic analysis.

Which version of PtS? Which logics?

- **Base-extension Semantics**

- Atomic bases (or bases);
- Extensions;
- Semantic clauses;

- **Classical and Intuitionistic logic**

- A *atomic rule* is a natural deduction rule which concludes an atom using other atoms as premises.

$$\frac{p}{q} \qquad\qquad \frac{r \qquad\qquad s}{t} \qquad\qquad \frac{p \qquad\qquad \begin{array}{c}[q]\\ \vdots \\ r\end{array}}{s} \qquad\qquad \frac{p \qquad\qquad \begin{array}{c}\dfrac{q}{r}\,1 \\ \vdots \\ s\end{array}}{t}\,1$$

- A atomic base S is a set of atomic rules.

- $\Gamma \vdash_S p$ holds if and only if p is derivable from the set of atoms Γ using the rules of S.

- A atomic base S' is an extension of a atomic base S if and only if $S \subseteq S'$.

Base-extension Semantics

Consider a set S with the following rules:

$$\frac{p}{q}\ \text{Rule 1} \qquad\qquad \frac{q \qquad r}{s}\ \text{Rule 2} \qquad\qquad \begin{array}{c}[r]\\ \vdots\\ \dfrac{s}{t}\ \text{Rule 3}\end{array}$$

This is a derivation showing $p \vdash_S t$ in S:

$$\dfrac{\dfrac{p}{q}\ \text{Rule 1} \qquad [r]}{\dfrac{s}{t}\ \text{Rule 3}}\ \text{Rule 2}$$

- If S' contains only rules 1 and 2, then S is an extension of S'.

Semantic clauses

Definition 8 (Sandqvist-style semantic clauses)

1. $\vDash_S p$ iff $\vdash_S p$, for atomic p;
2. $\vDash_S (A \wedge B)$ iff $\vDash_S A$ and $\vDash_S B$;
3. $\vDash_S (A \to B)$ iff $A \vDash_S B$;
4. $\vDash_S \bot$ iff $\vDash_S p$, for all atomic p;
5. $\vDash_S (A \vee B)$ iff $\forall S'(S \subseteq S')$: $A \vDash_{S'} p$ and $B \vDash_{S'} p$ implies $\vDash_{S'} p$, for all atomic p;
6. $\Gamma \vDash_S A$ iff $\forall S'(S \subseteq S')$: if $\vDash_{S'} B$ for all $B \in \Gamma$ then $\vDash_{S'} A$, for non-empty Γ;
7. $\Gamma \vDash A$ iff $\Gamma \vDash_S A$ for all S.

Treatments of absurdity

The absurdity constant $\bot$ can be either atomic or non-atomic.

- **Sandqvist's clause**

  $\bot$ is not atomic.

  $\Vdash_S \bot$ iff $\Vdash_S p$ holds for all atomic p.

- **Explosion requirement**

  $\bot$ is atomic;

  For every base S and every atom p, there is a rule in S concluding p from $\bot$.

Treatments of absurdity

- **Problematic definition**

  $\bot$ is not atomic;

  $\not\models_S \bot$, for all S.

- **Corollary**. $\models_S \neg\neg p$, for all p.

> *"This fact, that any atom 'a' is validated in some extension of any atomic base, might be considered a fault of validity-based proof-theoretic semantics, since it speaks against the intuitionistic idea of negation $\neg A$ as expressing that A can never be verified. We do not deal with this issue here."*

Failure of Completeness for Proof-theoretic Semantics (Schroeder-Heister; De Campos Sanz; Piecha, 2015).

Treatments of absurdity

- **Consistency requirement**

  $\bot$ is atomic;

  $\nvdash_S \bot$, for all S.

- Only consistent bases and consistent extensions of bases are considered.

- Our results only hold if we use the consistency requirement.

How does this solve the problem?

- Consider a base S with the following rule:

$$\frac{p}{\perp}$$

- If there was any extension S' of S capable of producing a deduction Π showing $\vdash_{S'} p$ we would be able to apply the rule at the end of Π and obtain a deduction Π' showing $\vdash_{S'} \perp$, so S' would be inconsistent. Since all bases are required to be consistent there can be no such S'.
- More generally: if $p \vdash_S \perp$ then for all $S \subseteq S'$ we have $\nvdash_{S'} p$, hence we can show $\vDash_S \neg p$ and $\nvDash_S \neg\neg p$.

Intuitionistic and classical proofs

- **What counts as a intuitionistic proof?**

- There is a proof of A if and only if it is possible to construct A.

What counts as a classical proof?

- There is a proof of A if and only if by assuming $A \vdash \bot$ one can prove a contradiction (*Reductio ad absurdum*).
- There is a proof of A if and only if $A \nvdash \bot$.

Since from a construction of A and the assumption $A \vdash \bot$ one can prove a contradiction, the existence of a intuitionistic proof of A implies the existence of a classical proof of A.

Intuitionistic and classical proofs

Those notions can be transposed to atomic bases as follows:

- **Intuitionistic proofs in the atomic base S:**

  $\vDash_S p^i$ iff $\vdash_S p$.

- **Classical proofs in the atomic base S:**

  $\vDash_S p^c$ iff $p \nvdash_S \bot$.

"If arbitrarily chosen axioms together with everything which follows from them do not contradict one another, then they are true, and the things defined by the axioms exist. For me that is the criterion of truth and existence"
Letter from Hilbert to Frege, in " Gottlob Frege: Philosophical and Mathematical Correspondence 1980"

Extension to non-atomic formulas

Theorem. $\vDash_S p^i$ implies $\vDash_S p^c$.

Proof. Assume $\vDash_S p^i$. Then, $\vdash_S p$.

By assuming $p \vdash_S \bot$ we can show $\vdash_S \bot$, violating the consistency requirement. Hence $p \nvdash_S \bot$, and so $\vDash_S p^c$. $\square$

- We can use semantic clauses in order to extend the definition of validity to non-atomic formulas A^i and A^c.
- It becomes possible to prove $A^i \vDash A^c$ using the same strategy after the extension.

Monotonicity

One caveat: intuitionistic proofs are *monotonic*, but classical proofs are not.

- $\vdash_S p$ and $S \subseteq S'$ implies $\vdash_S p$;
- It is not the case that $p \nvdash_S \bot$ and $S \subseteq S'$ implies $p \nvdash_{S'} \bot$.

Non-monotonicity leads to weird semantic behavior!

Example: $A^c \nvDash A^c \vee B$ holds for Sandqvist's entailment relation.

Weak and strong ecumenical semantics

Two possible solutions:

- **Weak ecumenical semantics**
  - We define a additional notion of logical entailment in order to induce monotonic behavior.
- **Strong ecumenical semantics**
  - We redefine the notion of classical proof in order to make it monotonic.

Weak ecumenical semantics

- **Local entailment**

  $\Gamma \vDash^L_S A$ iff $\forall S'(S \subseteq S')$: $\vDash^L_{S'} B$ for all $B \in \Gamma$ implies $\vDash^L_{S'} A$.

- **Global entailment**

  $\Gamma \vDash^G_S A$ iff $\forall S'(S \subseteq S')$: if for all S'' such that $S' \subseteq S''$ we have $\vDash^L_{S''} B$ for all $B \in \Gamma$, then $\vDash^L_{S''} A$ for all S'' such that $S' \subseteq S''$.

Theorem. If monotonicity holds for A and all formulas in Γ, $\Gamma \vDash^L_S A$ iff $\Gamma \vDash^G_S A$.

Local entailment + Monotonicity = Global entailment!

Weak ecumenical semantics

Definition 9 (Weak semantics - Unchanged clauses)

1. $\models^L_S p^i$ iff $\vdash_S p$, for atomic p;
2. $\models^L_S p^c$ iff $p \nvdash_S \bot$, for atomic p;
3. $\models^L_S A^c$ iff $A \not\models^L_S \bot$, for non-atomic A;
4. $\models^L_S (A \wedge B)^i$ iff $\models^L_S A$ and $\models^L_S B$;
5. $\Gamma \models^L_S A$ iff $\forall S'(S \subseteq S')$: if $\models^L_{S'} B$ for all $B \in \Gamma$ then $\models^L_{S'} A$;

Weak ecumenical semantics

Definition 10 (Weak semantics - Changed clauses)

1. $\vDash^L_S (A \to B)^i$ iff $A \vDash^G_S B$;
2. $\vDash^L_S (A \vee B)^i$ iff $\forall S' (S \subseteq S')$: $A \vDash^L_{S'} p^i$ and $B \vDash^L_{S'} p^i$ implies $\vDash^L_{S'} p^i$, for all atomic p;
3. $\Gamma \vDash^G_S A$ iff $\forall S' (S \subseteq S')$: if for all S'' such that $S' \subseteq S''$ we have $\vDash^L_{S''} B$ for all $B \in \Gamma$, then $\vDash^L_{S''} A$ for all S'' such that $S' \subseteq S''$.
4. $\Gamma \vDash A$ iff $\Gamma \vDash^G_S A$ for all S.

Results

To improve notation, we omit the superscript of operators whenever it is *i*. Superscript of formulas A are still written, unless they are irrelevant in that context.

- Classical validities are locally weaker than intuitionistic double negations:

  $A^c \not\models_S^L \neg\neg A^i$ and $\neg\neg A^i \models_S^L A^c$

- Classical validities and intuitionistic double negations are interdemonstrable:

  $A^c \models_S^G \neg\neg A^i$ and $\neg\neg A^i \models_S^G A^c$

Classical proof + Monotonicity = Intuitionistic proof of double negation!

Results

- $\perp^i \vDash \perp^c$ and vice-versa.

  There is no extension of S in which $\perp$ is derivable and no extension of S in which $\perp \nvDash \perp$.

- Every formula containing only intuitionistic subformulas is monotonic. Formulas containing classical subformulas are not always monotonic.

Results

- The following are all equivalent (for atomic p):

- $p^i \vDash_S^L \bot$;
- $p^i \vDash_S^G \bot$.
- $\vDash_S^L \neg p^i$;
- $p \vdash_S \bot$;

- $A \nvDash_S^L \bot$ iff there is some $S \subseteq S'$ such that $\vDash_{S'}^L A$
- For all S and all A, either $\vDash_S^L \neg A^i$ or $\vDash_S^L A^c$.

Results

- The following also hold:

  $\models A^c \vee \neg A^i$

  $\models ((A^i \to B) \to A^i) \to A^c$

- However:

  $(A \vee B)^i, (A \to C)^i, (B \to C)^i \not\models C$

Final comments on Weak Semantics

- When those definitions are used, local validity seems to induce classical behavior, whereas global validity induces intuitionistic behavior.
- Weak ecumenical semantics induce classical behavior in a interesting way, but we lose some important validities.
- It also seems to have no simple syntactical characterization!

Recap

- In strong ecumenical semantics, classical proofs are monotonic by definition:

  $\vDash_S p^c$ if and only if $\forall S'(S \subseteq S') : p \nvdash_{S'} \bot$

- Local entailment + Monotonicity = Global entailment.
- Since all proofs are now monotonic, global and local notions of entailment collapse.

Strong ecumenical semantics

Definition 11 (Strong semantics)

1. $\vDash_S p^i$ iff $\vdash_S p$, for atomic p;
2. $\vDash_S p^c$ iff $\forall S'(S \subseteq S')$: $p \nvdash_{S'} \bot$, for atomic p;
3. $\vDash_S A^c$ iff $\forall S'(S \subseteq S')$: $A^i \nvDash_{S'} \bot$, for non-atomic A;
4. $\vDash_S (A \wedge B)^i$ iff $\vDash_S A$ and $\vDash_S B$;
5. $\vDash_S (A \to B)^i$ iff $A \vDash_S B$;
6. $\vDash_S (A \vee B)^i$ iff $\forall S'(S \subseteq S')$: $A \vDash_{S'} p^i$ and $B \vDash_{S'} p^i$ implies $\vDash_{S'} p^i$, for all atomic p;
7. $\Gamma \vDash_S A$ iff $\forall S'(S \subseteq S')$: if $\vDash_{S'} B$ for all $B \in \Gamma$ then $\vDash_{S'} A$;
8. $\Gamma \vDash A$ iff $\Gamma \vDash_S A$ for all S.

Soundness and Completeness

- Let NEb be the calculus obtained by adding the following rules to intuitionistic natural deduction:

$$\dfrac{\begin{array}{c}[\neg A^i]\\ \Pi\\ \bot\end{array}}{A^c}\ A^c\text{-int} \qquad \dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ A^c\end{array}\qquad\begin{array}{c}\Gamma_2\\ \Pi_2\\ \neg A^i\end{array}}{\bot}\ A^c\text{ - elim}$$

- Then:

  $\Gamma \vDash A$ iff $\Gamma \vdash_{\mathsf{NEb}} A$

- By removing the new rule we get soundness and completeness w.r.t. intuitionistic logic.

Sandqvist-style proofs of completeness and the consistency requirement

- Sandqvist uses a interesting strategy to prove completeness for base-extension semantics.
- α is a function mapping formulas A to unique atoms p^A (atoms are mapped to themselves);
- Given any mapping α, $\mathfrak{A}$ is a simulation base containing only atomic rules which mimic the behavior of non-atomic rules:

$$\frac{p^A \qquad p^B}{p^{A\wedge B}} \qquad \frac{p^{A\wedge B}}{p^A} \qquad \frac{p^{A\wedge B}}{p^B}$$

Sandqvist-style proofs of completeness and the consistency requirement

- Give any consequence $A^1, ..., A^n \vDash A^{n+1}$, we use a function α mapping all subformulas of the formulas $A^1, ..., A^n$ and of A^{n+1} to atoms and build a simulation base $\mathfrak{A}$.
- We them use this mapping to show that

  $A^1, ..., A^n \vDash A^{n+1}$

  implies that there is a deduction Π showing

  $p^{A^1}, ..., p^{A^n} \vdash_{\mathfrak{A}} p^{A^{n+1}}$

  and we extract a natural deduction derivation Π' showing $A^1, ..., A^n \vdash_{\mathsf{Int}} A^{n+1}$ from the atomic deduction Π by replacing every occurrence of p^A in it for A.

Sandqvist-style proofs of completeness and the consistency requirement

- But only consistent bases are admissible in the new semantics, and so we must show that every possible simulation base $\mathfrak{A}$ is consistent.
- How?

- The answer: atomic normalization!

$$\dfrac{\dfrac{\begin{matrix}\Pi \\ p^A\end{matrix} \qquad \begin{matrix}\Pi' \\ p^B\end{matrix}}{p^{A\wedge B}}}{\begin{matrix}p^A \\ \Pi''\end{matrix}} \quad \Rightarrow \quad \begin{matrix}\Pi \\ p^A \\ \Pi''\end{matrix}$$

- By adapting natural deduction notions to atomic bases we can show that, if there is a deduction showing $\Gamma^{At} \vdash_{\mathfrak{A}} p$ then there is a normal deduction showing $\Gamma^{At} \vdash_{\mathfrak{A}} p$.

- **Corollary**: $\nvdash_{\mathfrak{A}} p^{\perp}$, and since $p^{\perp} = \perp$, we have $\nvdash_{\mathfrak{A}} \perp$.

Sandqvist-style proofs of completeness and the consistency requirement

- In the completeness proof, we can prove the inductive step constructively for all formulas A^i , but the inductive step for A^c seems to requires classical reasoning.
- $\vDash_S A^c$ iff, for all $S \subseteq S'$, there is a $S' \subseteq S''$ such that $\vDash_{S''} A$.
- Classical reasoning is needed because A^c is defined in terms of non-entailment.

Adaptations for $\bot$

- The proof also requires a final adaptation to deal with the new treatment of $\bot$.
- Let $\mathfrak{U}$ be a simulation base and $\Gamma \vDash_{\mathfrak{U}} A$ a fixed consequence. Let $\mathfrak{U}'$ be the base obtained by adding a rule concluding p^B (for the mapping α) from no premises for every $B \in \Gamma$.Let Γ^{At} be the set of formulas to which the formulas of Γ are mapped by α.
  We divide the proof in two cases:

Adaptations for $\perp$

Case 1. $\mathfrak{U}'$ is consistent. Then we can show $\vdash_{\mathfrak{U}'} p^A$ (Sandqvist's Lemma). By replacing all instances of the rules added to $\mathfrak{U}'$ with conclusion p^B by assumptions with shape p^B we obtain a deduction showing $\Gamma^{At} \vdash_{\mathfrak{U}} p^A$, and by replacing every atom in this deduction by its superscript we obtain a deduction showing $\Gamma \vdash_{NEb} A$.

Case 2. $\mathfrak{U}'$ is inconsistent. Then we can show $\vdash_{\mathfrak{U}'} \perp$. By replacing all instances of the rules added to $\mathfrak{U}'$ with conclusion p^B by assumptions with shape p^B we obtain a deduction showing $\Gamma^{At} \vdash_{\mathfrak{U}} \perp$, and by replacing every atom in this deduction by its superscript we obtain a deduction showing $\Gamma \vdash_{NEb} \perp$. We conclude by including an application of EFQ at the end of this deduction to obtain one showing $\Gamma \vdash_{NEb} A$.

Some conclusions

- Proof-theoretical ecumenism can be achieved by combining classical notions of proof with intuitionistic notions of proof.
- Monotonicity is of great relevance in the ecumenical context.
- Weak ecumenical semantics: interesting semantic behaviour but complex syntactic characterization.
- Strong ecumenical semantics: interesting semantic behavior and simple syntactic characterization.

Lot's of things to be done

1. *Kripke semantics for the full system $\mathcal{NE}$.*
2. *PtS semantics for $\mathcal{NE}_{\mathcal{P}}$.*
3. *First-order PtS semantics.*
4. *Ecumenical Algebraic Semantics (on the way!! - Marcelo Coniglio + Elaine Pimentel)*
5. *Categorical Semantics (Valeria de Paiva??)*