Intuitionistic Fuzzy Logic and Intuitionistic Fuzzy Set Theory
by G. Takeuti and S. Titani

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Introduction

This paper is only 15 pages, 2 sections and 6 references. Hopefully it makes sense for people who don't like Set Theory...
Takeuti and Titani have 3 papers in the subject:
JSL1984,
Archive for Mathematical Logic, 1992 "Fuzzy logic and fuzzy set theory",
"Heyting valued universes of intuitionistic set theory", 1981.
Paper had a great impact:
Cited by 163 in Scholar, 4 in Jstor.
Work continuing...

Introduction?...

1965 Zadeh: definition of fuzzy sets

Basic idea: A notion of the membership relation which takes truth values in the closed interval of real numbers [0, 1] .

If an element really belongs to a set, the truth value is 1. If it really does not belong to the set then the truth value is 0, and all other reals are 'degrees' of belonging.

The logical operations implication and negation on [0, 1] used for Zadeh's fuzzy sets look like Łukasiewciz's logic, where $p \to q = min(1, 1 - p + q)$, $\neg p = 1 - p$.

For Łukasiewicz's logic, $P \wedge (P \to Q) \to Q$ is not valid. Translating it to the set version, it follows that the axiom of extensionality does not hold.

Hay(1963) extended Łukasiewicz's logic to a predicate logic and proved a weak completeness theorem: if P is valid then P + P^n is provable for each positive integer n. Hay also showed that one can (without losing consistency) obtain completeness of the system by use of an additional infinitary rule.

Introduction

On the other hand (...)
The closed interval [0, 1] is a complete Heyting algebra (cHa).
Grayson showed that for each cHa Ω the sheaf model $V^{(\Omega)}$ over Ω is a model of intuitionistic set theory ZF$_I$.
Ω-valued set theory is realized in V(Ω) as a model of ZF$_I$.
Denote the cHa [0, 1] by I, where:
$p \to q = \{r \in [0,1] | p \wedge r \leq q\} = 1$ if $p \leq q, q$ if $p \geq q$
and $\neg p = p \to 0 = 1 if p = 0, 0$ if $p > 0$

Call the I-valued logic 'intuitionistic fuzzy logic' (IF) and I-valued set theory 'intuitionistic fuzzy set theory' (ZF$_{IF}$).
This paper develops IF and ZF$_{IF}$.

Summary

Section1 axiomatizes intuitionistic fuzzy logic IF in a language with propositional variables, as suggested by R. Jensen.
Section 1 also proves the consistency and strong completeness of the logic IF: if a sequent $\Gamma \Rightarrow \Delta$ is valid in every I-valued model then $\Gamma \Rightarrow \Delta$ is provable, where Γ is a finite or infinite sequence of formulas and Δ is a sequence of at most one formula.
Section 2 presents the intuitionistic fuzzy set theory $\mathsf{ZF}_{\mathsf{IF}}$ and develops its calculus $\mathsf{ZF}_{\mathsf{IF}}$.
The axioms of $\mathsf{ZF}_{\mathsf{IF}}$ are the axioms of intuitionistic set theory ZF_I, plus Dependent Choice and Double Complement.

Intuitionistic Set Theories?..

Apparently started in early 70s by Myhill and Friedman, IZF.
Keep things as much as possible like in ZFC (Zermelo-Fraenkel with Choice), just change base logic to intuitionism. Must remove Choice and LEM, then check that LEM doesn't come back...(nice article on SEP on constructive and intuitionistic set theories).
What is ZF$_I$? Greyson [3] defines ZF_I, considers set theories expressed in a first-order language with relation symbols $\in$, $=$, and governed by the usual axioms and rules of *intuitionistic* predicate logic.
The usual schemas of Extensionality, Pairing, Unions, Power-set, Infinity and Separation:
(Ext) $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$
(Pair) $\exists z \forall w(w \in z \leftrightarrow w = x \vee w = y)$
(Union) $\exists z \forall w(w \in z \leftrightarrow \exists y \in x w \in y)$
(Power) $\exists z \forall w(w \in z \leftrightarrow \forall y \in w y \in x)$
(Inf) $\exists x(\exists y \in x \wedge \forall y \in x \exists z \in x y \in z)$
(Sep) $\exists z \forall w(w \in z \leftrightarrow w \in x \wedge \varphi)$

Intuitionistic Set Theories?..

Greyson shows axiom of Regularity would bring back LEM so he introduces instead ε-induction,

$$\forall x[\forall y \in x\varphi(y) \to \varphi(x)] \to \forall x\varphi(x)$$

From infinity and ε-induction follows the existence of a set ω of 'natural numbers', a result of Powell.
To complete the system ZF_I Greyson adds the axiom of Replacement (Rep) $\forall y \in x\exists z\varphi \to \exists w\forall y \in x\exists z \in w\varphi$

Dependent Choice? Double Complement?

The axiom of dependent choices, denoted DC, is a weak form of the axiom of choice (AC) which is still sufficient to develop most of real analysis.
The axiom of dependent choice implies the Axiom of countable choice, and is strictly stronger.
The axiom of double complement is

$$\exists u \forall z (z \in u \leftrightarrow \neg\neg z \in x)$$

Yeah, related to Stable sets

$$\{x : \forall u (\neg\neg u \in x \rightarrow u \in x)\}$$

BUT Powell75 needs to be read to make sense of why ZF_I plus Dependent Choice plus Double Complement is the set theory we want to have.
We will just assume it.

Summary

- In intuitionistic set theory, the natural numbers ω and the rationals Q behave as those in classical set theory, but the real numbers R do not
- The calculus in V^I is almost classical. Only the concept of subset is not classical. (for example there exists an infinite and Dedekind finite subset of ω in V^I)
- The calculus is almost classical in $\mathsf{ZF}_{\mathsf{IF}}$ in the sense that it is absolute for Powell's innermodel, which is a model of ZF constructed in $\mathsf{ZF}_{\mathsf{IF}}$.
- Powell constructed the innermodel in [5] to prove the consistency of ZF relative to ZF_I with the axiom of double complement.

The intuitionistic fuzzy logic IF

The language of IF consists of:

- individual free variables: $a_0, a_1, \ldots$;
- individual bound variables: $x_0, x_1, \ldots$;
- individual constants: $c_0, ..., c_i, \ldots (i < u)$;
- propositional variables: $z_0, z_1, , \ldots$
- propositional constants:$p_0, , \ldots p_j, , \ldots (j < i)$;
- predicate constants: $R_0(*, \ldots, *), \ldots, R_k(*, \ldots, *), \ldots$
- logical symbols: $\wedge, \vee, \rightarrow, \neg, \exists, \forall$

The intuitionistic fuzzy logic IF

The proof system for IF consists of intuitionistic logic (Gentzen's LJ) plus the following axiom schemata:

1.$\Rightarrow (A \to B) \vee ((A \to B) \to B)$;

2. $(A \to B) \to B \Rightarrow (B \to A) \vee B$;

3. $(A \wedge B) \to C \Rightarrow (A \to C) \vee (B \to C)$;

4. $A \to (B \vee C) \Rightarrow (A \to B) \vee (A \to C)$;

5. $\forall x(C \vee A(x)) \Rightarrow C \vee \forall x A(x)$, where x does not occur in C;

6. $\forall x A(x) \to C \Rightarrow \exists x(A(x) \to D) \vee (D \to C)$, where x not in D;

Extra Inference Rule:.

$\Gamma \Rightarrow A \vee (C \to z) \vee (z \to B)$ then $\Gamma \Rightarrow A \vee (C \to B)$

where z does not occur in the lower sequent.

This inference rule (called density of rule TT) asserts that the truth value set is dense.

The intuitionistic fuzzy logic IF

This section has 3 theorems.
Theorem 1: The following (19) sequents are provable in IF, where in (6)-(9), (11), (12) x does not occur in C and in (10), (13) x does not occur in D.
Theorem 2 If $\Gamma \Rightarrow A$ is provable in IF, then $\Gamma \Rightarrow A$ is valid. (they prove validity of the density inference rule)
Theorem 3 (main theorem: strong completeness) Let IF be a system of intuitionistic fuzzy logic with countably many constants. If a sequent $\Sigma \Rightarrow \Delta$ is valid then $\Sigma \Rightarrow \Delta$ is provable, where Σ may be infinite.

The intuitionistic fuzzy logic IF

THEOREM 1.1. *The following sequents are provable in IF, where in* (6)–(9), (11), (12) *x does not occur in C and in* (10), (13) *x does not occur in D.*

(1) $\Rightarrow (A \to B) \vee (B \to A).$

(2) $A \to B \vee C \Leftrightarrow (A \to B) \vee (A \to C).$

(3) $A \to B \wedge C \Leftrightarrow (A \to B) \wedge (A \to C).$

(4) $(A \wedge B) \to C \Leftrightarrow (A \to C) \vee (B \to C).$

(5) $(A \vee B) \to C \Leftrightarrow (A \to C) \wedge (B \to C).$

(6) $\forall x(C \vee A(x)) \Leftrightarrow C \vee \forall x A(x).$

(7) $\exists x(C \wedge A(x)) \Leftrightarrow C \wedge \exists x A(x).$

(8) $\forall x(A(x) \to C) \Leftrightarrow \exists x A(x) \to C.$

(9) $\exists x(A(x) \to C) \Rightarrow \forall x A(x) \to C.$

(10) $\forall x A(x) \to C \Rightarrow \exists x(A(x) \to D) \vee (D \to C).$

(11) $\forall x(C \to A(x)) \Leftrightarrow C \to \forall x A(x).$

(12) $\exists x(C \to A(x)) \Rightarrow C \to \exists x A(x).$

(13) $C \to \exists x A(x) \Rightarrow \exists x(D \to A(x)) \vee (C \to D).$

(14) $\Rightarrow \neg A \vee \neg\neg A.$

(15) $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B.$

(16) $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B.$

(17) $\neg\neg(A \to B) \Leftrightarrow \neg\neg A \to \neg\neg B.$

(18) $\neg\neg \forall x A(x) \Rightarrow \forall x \neg\neg A(x).$

(19) $\neg\neg \exists x A(x) \Leftrightarrow \exists x \neg\neg A(x) \Leftrightarrow \neg \forall x \neg A(x).$

The intuitionistic fuzzy logic IF

The proof of the theorem 3 in TT84 is long and not very enlightening.

Luckily three years later Takano in the TSUKUBA J . MATH came out with "Another proof of the completeness of the Intuitionistic Fuzzy Logic", which is short and sweet.

Takano reminds us that in 1969 Horn had introduced and proved (weak) complete a system, H, which was LJ with two extra axioms:
$\forall x(C \vee A(x)) \Rightarrow C \vee \forall x A(x)$, where x does not occur in C;
$\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)$
He also reminds us that Ono had characterized the system H by means of Kripke models
Now since the first axiom above is number 5 in the system TT and since the second axiom above (sometimes called D for Dummett) is provable in LJ from the first two extra axioms in TT, Takano proves the following:

The intuitionistic fuzzy logic IF

Theorem [Takano, Takeuti and Titani] Call TT the system described by Takeuti and Titani in JSL84 and TT^- the system without the density rule. Suppose that the language concerned is countable. The following properties (a)-(d) of a sequent $\Sigma \Rightarrow A$ are equivalent, where Σ may be infinite:
(a) $\Sigma \Rightarrow A$ is valid.
(b) $\Sigma \Rightarrow A$ is provable in H.
(c) $\Sigma \Rightarrow A$ is provable in TT^-.
(d) $\Sigma \Rightarrow A$ is provable in TT.
The proof is simple, algebraic and connects to the structure of the cHa [0,1] as "a countable linearly ordered structure with the distinct maximal and minimal elements".
Takano also mentions that he doesn't know how to do a syntactical proof of (d)$\rightarrow$(c). But we now have an easy proof of TT's theorem 3 and a connection to linearly ordered kripke models...

Takano's easy proof:

Want to show (a) implies (b) where (a) $\Sigma \Rightarrow A$ is valid.
(b) $\Sigma \Rightarrow A$ is provable in H.
Method: Assume $\Sigma \Rightarrow A$ is unprovable in H and then construct a model $\langle \mathcal{A}, [[.]] \rangle$ in which $\Sigma \Rightarrow A$ is not valid. Assume (WLOG) infinitely many variables that do not occur in $\Sigma \Rightarrow A$ and that $\Delta = A$.
Let Let $\mathcal{A}$ and $\mathcal{F}$ be the sets of all terms and all formulas, respectively.
Four propositions.

Takano's easy proof:

Prop1. There exists a set $\mathcal{G}$ of formulas which satisfiesthe following conditions (l)-(3): (1) $\Sigma \subseteq \mathcal{G}$ and $A \notin \mathcal{G}$. (2) If $\mathcal{G}$ satisfies a disjunction $B_1 \vee B_2 \vee B_v$, then $B_i \in \mathcal{G}$ for some i. (3) If $B(t) \in \mathcal{G}$ for every t in $\mathcal{A}$, then $\forall x B(x)$ is an element of $\mathcal{G}$. (inductively build up the sets from Σ and A).
Prop2. $\langle \mathcal{F}/, \leq \rangle$ is a countable linearly ordered structure with the distinct maximal element $|A \rightarrow A|$ and the minimal element $\neg(A \rightarrow A)$.
Prop3. The following properties hold in $\langle \mathcal{F}/, \leq \rangle$:
$|B \wedge C| = min(|B|, |C|)$, blah, etc... (describe the Heyting algebra...)
Prop4.(Horn69) If $\langle L, \leq \rangle$ is a countable linearly ordered structure with the distinct maximal and minimal elements, then there exists a monomorphism on$\langle L, \leq \rangle$ to $\langle [0, 1], \leq \rangle$, which preserves the maximal and the minimal elements as well as all existing supremums and infimums in $\langle L, \leq \rangle$. Hence there exists such a monomorphism on $\langle L, \leq \rangle$ to $\langle [0, 1], \leq \rangle$.

Takano's easy proof:

Almost fits in one slide, but not quite..

Put 2 and 4 together to construct a monomorphism h, and put $[[B]] = h(|B|)$ for every B in $\mathcal{F}$ and we obtain a model $\langle \mathcal{A}, [[.]] \rangle$ by prop3. This gives a contradiction, as for every B we have $[[B]] = 1 \leftrightarrow |B| = 1 \leftrightarrow B \in \mathcal{G}$. So in this model $B \in \Sigma$ implies $B \in \mathcal{G} \leftrightarrow [[B]] = 1$, while A is not in $\mathcal{G}$, so $[[A]] \neq 1$, hence $\Sigma \Rightarrow A$ is not valid.

The intuitionistic fuzzy logic IF

Horn, 1969: It is known that the theorems of the intuitionist predicate calculus are exactly those formulas which are valid in every Heyting algebra. The simplest kind of Heyting algebra is a linearly ordered set. This paper concerns the question of determining all formulas which are valid in every linearly ordered Heyting algebra.

Baaz and Zach, 2000 [Use Avron's hypersequents to prove cut elimination and "density elimination" of IF] The logic is known to be axiomatizable, but no deduction system amenable to proof-theoretic, and hence, computational treatment, has been known.

Ciabattoni and Metcalfe, 2005 Density elimination by substitutions is introduced as a uniform method for removing applications of the Takeuti-Titani density rule from proofs in first- order hypersequent calculi.

Intuitionistic fuzzy set theory ZF_{IF}

The logic of ZF_{IF} is the intuitionistic fuzzy logic IF whose constants are predicate constants $\in$, $=$ and E

The axioms of ZF_{IF} are those of intuitionistic set theory ZF_{I} together with the axiom of Dependent Choices (DC) and Double Complement $(\neg\neg)$

Intuitionistic fuzzy set theory ZF_{IF}

Equality Axioms: $u = u$; $u = v \Rightarrow v = u$; $u = v \wedge v = w \Rightarrow u = w$;
$u = v \wedge A(u) \rightarrow A(v)$;
$(Eu \vee Ev \rightarrow u = v) \rightarrow u = v$
Other Axioms:
Extensionality. $\forall z(z \in x \leftrightarrow z \in y) \wedge (Ex \leftrightarrow Ey) \rightarrow x = y$.
$\in$-strictness. $x \in u \rightarrow Ex \wedge Eu$.
$\in$-induction. $\forall x(\forall y \in x(\varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$.
Separation. $\forall x \exists u \forall z(z \in u \leftrightarrow z \in x \wedge \varphi(z))$.
Collection. $\forall x[\forall y \in x \exists z \varphi(y, z) \rightarrow \exists u \forall y \in x \exists z \in u(\varphi(y, z)]$.
Pair. $\forall x, y \exists u \forall z[z \in u(z = x) \vee (z = y)]$.
Union. $\forall x \exists u z[z \in u \exists y \in x(z \in y)]$.
Power. $\forall x \exists u \forall z[z \in u \forall y \in z(y \in x)]$.
Infinity.$\exists x[\exists y(y \in x) \wedge \forall y \in x \exists z \in x(y \in z)]$.
DepChoice. $\forall x \in u \exists y \in u \varphi(x, y) \rightarrow \forall z \in u \exists f[f : \omega \rightarrow u \wedge f(\omega) = z \wedge \forall x \in \omega \varphi(f(x), f(x + 1))]$
DNeg $\exists u \forall z[\neg\neg z \in x \leftrightarrow z \in u]$.

Intuitionistic fuzzy set theory $\mathsf{ZF}_I F$

Grayson proved the following three facts about ZF_I:
(I) The principle of ω-induction:

$$\varphi(0) \wedge \forall y[\varphi(y) \rightarrow \varphi(y+1)] \Rightarrow \forall y \in \omega \varphi(y)$$

and the trichotomy of elements of ω:

$$\forall x, y \in \omega(x \in y \vee x = y \vee y \in x)$$

The arithmetic sum, product and exponent are given by the usual recursive definitions. The set of integers Z and its order and arithmetic are constructed as usual, and similarly for the rationals Q.
(II) The real numbers R (Dedekind reals) are defined as pairs of subsets of Q, <L, U>, such that 5 properties hold.
(III) DC implies the axiom of Countable Choice:
CAC: $\forall x \in \omega \exists y \in u \varphi(x, y) \rightarrow \exists f[f : \omega \rightarrow u \wedge \forall x \in \omega \varphi(x, f(x))]$,
CAC implies the fact that every Dedekind real number is a Cauchy real (a limit of a Cauchy sequence of rational numbers), i.e.
$\forall x \in R \exists f : \omega \rightarrow Q \forall n(|x - f(n)| < 1/n)$

Sheaf models...

From Fact III they say it follows that:
Theorem 2.1 Every Dedekind real number is Cauchy real in ZF$_{IF}$.

Provide a definition of a model of ZF$_{IF}$, V^I, following Grayson:
Set $V_0^I = \emptyset$.
Assuming that V_α^I has been constructed and each element of V_α^I is a pair $\langle |u|, Eu \rangle$, where $|u|$ is the characteristic function, $\mathcal{D}(u) \to I$, and $Eu \in I$ is the truth value of "there exists u", define:
$V_{\alpha+1}^I = \{\langle |u|, Eu \rangle | \mathcal{D}(u) \subseteq V_\alpha^I, |u| \colon \mathcal{D}(u) \to I, \forall t \in \mathcal{D}(u)(|u|(t) \leq Eu \wedge Et), Eu \in I\}$;
if β is a limit ordinal, $V_\beta^I = \bigcup\limits_{\alpha<\beta} V_\alpha^I$; and $V^I = \cup_{\alpha \in On} V_\alpha^I$.

Then it is known that V^I is a model of ZF$_I$, and V is embedded in V^I with the mapping $\check{}\colon V \to V^I$, given by $\mathcal{D}(\check{u}) = \{\check{t} | t \in u\}$, $\check{u}(\check{t}) = 1$ for $t \in u$, $E\check{u} = 1$.
(Need to read 'Heyting valued universes of intuitionistic set theory'...)

Theorems in section 2..

Let V be a model of ZFC in which we construct a model V^I of ZF$_{IF}$. We denote the sets of natural numbers, rationals and reals in V by ω, Q and R, respectively.

Theorems in section 2..

THEOREM 2.5. V^I is a model of ZF$_{IF}$.
Proof: Since it is clear that $\langle V^I, I, [[.]]\rangle$ is a model of IF, it suffices to show that $[[DC]] = [[\neg\neg]] = 1$, which they show by showing that $[[DC]] = 1$ and that $[[\neg\neg]] = 1$ separately.

Theorems in section 2..

The following theorem shows that the real numbers in V^I are classical.
Theorem 2.6$[[R = \check{R}]] = 1.$
THEN they can do usual definitions like:
Definition For $u \in R$ and $\varepsilon \in Q^+$, a neighborhood $B(u, \varepsilon)$ of u is defined by $B(u, \varepsilon) = \{x \in R$ such that $|x - u| < \varepsilon\}$.
and many others like:
A is an *open* set if all its elements have open neighborhoods contained in A, or:
A is open if $\forall x \in A \exists \varepsilon \in Q^+$ such that $(B(x, \varepsilon) \subseteq A)$,
The closure of A, $\overline{A} = \{x \in R | \exists f : \omega \to A(lim_{n \to \infty}, f(n) = x)\}$, etc..

Traditional analysis theorems in V^I

in page 15 we have: A formula A is said to be decidable if $A \vee \neg A$, i.e.,$\neg\neg A \leftrightarrow A$.

All traditional development of continuous and differentiable functions so that we obtain the classical forms of Rolle's theorem, the mean value theorem, and Taylor's theorem.

ROLLE'S THEOREM. Let $a, b \in R$, $f : [a, b] \to R$ continuous, and f differentiable on (a, b). Let $f(a) = f(b) = 0$. Then there is $c \in (a, b)$ such that $f'(c) = 0$.

MEAN VALUE THEOREM. Let $a, b \in R$ and$f : [a, b] \to R$ and f be a differentiable function on the interval (a, b). Then there is $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

TAYLOR'S THEOREM. Let U be a closed interval of R, $a, b \in U$, and let $f : U \to R$ be $n + 1$ times differentiable on U. Then there exists $c \in R$ with $min(a, b) < c < max(a, b)$ such that

$$f(b) = \Sigma_{k=0}^{n} f^{(k)}(a)(b-a)^k + f^{(n+1)}(c)(b-c)^n(b-a).$$

Traditional analysis theorems in V^I

Finally

Theorem 2.10 If $g\colon \omega \to R$ and $\alpha \in R$, then the formula $lim_{n\to\infty} g(n) = \alpha$ is decidable.

Integrals are also decidable and the functions e^x, $logx$, $sinx$ and $cosx$ can be defined via their powerseries and integral, so:

The calculus without using the concept of subset is classical in V^I, in the sense that for each sentence A in the calculus we have $A \leftrightarrow \neg\neg A \leftrightarrow A^V$

It follows that the calculus without using the concept of subset is classical in ZF$_{IF}$, in the sense that for each A in the calculus, $A \leftrightarrow A^S$.

but what does this mean?...

Open Problems

If a sentence A holds in V, then A^S is valid in V^I.
Let T be the set $\{A^S | A \text{ holds in } V\}$.
Then, is every valid formula in V^I provable from T in ZF_{IF}?

What happened since?..

About the logic we saw already:
Takano, 87, simplified proof,
CSL2000 (Baaz and Zach: Hypersequent and the Proof Theory of Intuitionistic Fuzzy Logic), syntactic cut-elimination plus density elimination
Ciabattoni and Metcalfe: syntactic density elimination...
About the set theory?....
A natural interpretation of fuzzy sets and fuzzy relations M. Shimoda 2002? 18 cit
Fuzzy sets Zadeh 1965 Cited by 27153.
a different notion of "intuitionistic fuzzy sets" by Atanassov and co-workers long controversy
'Krassimir Atanassov has published about 370 papers in journals, 140 conference reports, 20 monographs.' says wikipedia...