Dialectica and Chu Constructions: Cousins?

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Objectives

- Compare and contrast Dialectica and Chu constructions
  - Dialectica and Chu constructions produce categories, which are models of Linear Logic, given a category C and an object K in C.
  - Resulting categories have same objects but structure/properties are very different
- Describe a mild generalisation of dialectica categories
  - Construct a dialectica category out of a smcc as in Chu.

Outline

- Categorical Semantics of Linear Logic

- Chu and Dialectica Constructions

- Generalizing Dialectica: $\mathsf{Dial}_L(\mathbf{C})$

- Conclusions

Categorical Semantics of Linear Logic

- Categorical semantics models derivations (proofs) not simply whether theorems are true or not.

- Usually via extended Curry-Howard isomorphism

- Linear Logic is interesting case:

  - To model multiplicative-additive fragment is uncontroversial: need a smcc with products and coproducts

  - To model linear negation: need an involution

  - To model modalities/exponentials: need *linear exponential comonad (and monad)*

Modelling the Modality !

- Equations for S4-like axioms say ! is a *comonad*.
  Duplication and erasing equations say $!A$ is *comonoid* wrt tensor.
  Comonad is *monoidal*: $m_{A,B}$: $!(A \otimes B) \to !A \otimes !B$ $\;m_I$: $!I \to I$
  plus (coalgebra) conditions relating comonad and comonoid structures.

- A *linear category* comprises
  — an SMCC $\mathcal{C}$, (with products and coproducts),
  — a symmetric monoidal comonad $(!, \epsilon, \delta, m_I, m_{A,B})$
  such that:
  (a) For every free co-algebra $(!A, \delta)$ the nat. transfns.
  er_A: $!A \to I$ and dupl_A: $!A \to !A \otimes !A$, are coalgebra maps, and a comm. comonoid.
  (b) Every map of free coalgebras is also a map of comonoids.

- A comonad as above is called a *linear exponential comonad*.

- Using linear category, we prove $A \Rightarrow B \cong (!A \multimap B)$ by constructing ccc out of smcc plus comonad !.
  Extended Curry-Howard works!
  Linear category: what is needed to model proofs of Intuitionistic Linear Logic.

- To model classical linear logic, just add the involution, making the smcc, *-autonomous.

- Summing up:
  A model of ILL is a smcc, with finite products and equipped with a *linear exponential comonad*.
  A model of CLL is a *-autonomous category, with finite products and equipped with a linear exponential comonad.

Now for some concrete models...

Chu Construction

If C is smcc with pullbacks, K any object of C:
$\mathrm{Chu}_K\mathrm{C}$ objects are (generalised) relations between U and X or maps of the form $U \otimes X \xrightarrow{\alpha} K$.

$\mathrm{Chu}_K\mathrm{C}$ maps are pairs of maps of C $f\colon U \to V$, $F\colon Y \to X$ such that $u\alpha F(y) = f(u)\beta y$.

Chu Spaces: Chu construction when C is **Sets** and K is 2 or $\{\mathsf{true}, \mathsf{false}\}$. Equality can be read as logical bi-implication, $u\alpha F(y) \Leftrightarrow f(u)\beta y$

$$\begin{tikzcd}
U \times Y \arrow[r, "U \times F"] \arrow[d, "f \times Y"'] & U \times X \arrow[d, "\alpha"] \\
V \times Y \arrow[r, "\beta"'] & 2
\end{tikzcd}$$

Dialectica-like Construction GC

If C is ccc (hence smcc) and $(K, \leq)$ is object with a partial order: GC objects are generalised relations between U and X in C, or $U \times X \to K$. (initially monics $A \overset{\alpha}{\hookrightarrow} U \times X$)

GC maps are pairs of maps of C $f\colon U \to V$, $F\colon Y \to X$, such that $u \alpha F(y) \Rightarrow f(u) \beta y$. (initially making a certain pullback condition hold)

Dialectica Spaces Dialectica-like construction when C is **Sets**, K is 2= $\{\mathsf{true}, \mathsf{false}\}$, condition reads "If $u \alpha F(u, y)$ then $f(u) \beta y$".

$$\begin{tikzcd} U \times Y \arrow[r, "U \times F"] \arrow[d, "f \times Y"'] \arrow[dr, phantom, "\Leftarrow"] & U \times X \arrow[d, "\alpha"] \\ V \times Y \arrow[r, "\beta"'] & 2 \end{tikzcd}$$

(Initial construction harder to present, but a bit more powerful.)

Original Dialectica Construction DC

If C is ccc and $(K, \leq)$ object with a partial order:
DC objects are generalised relations between U and X in C, or $U \times X \to K$. (initially monics $A \overset{\alpha}{\hookrightarrow} U \times X$)

DC maps are pairs of maps of C $f\colon U \to V$, $F\colon U \times Y \to X$, such that $u\alpha F(u,y) \Rightarrow f(u)\beta y$. (initially making a certain pullback condition hold)

Dialectica construction when C is **Sets**, K is 2 or $\{\mathsf{true}, \mathsf{false}\}$ and condition reads "If $u\alpha F(u,y)$ then $f(u)\beta y$" is an internal version of Gödel's **Dialectica** Interpretation.

$$\begin{tikzcd}
U \times Y \arrow[r, "\langle \pi_1, F \rangle"] \arrow[d, "f \times Y"'] \arrow[dr, phantom, "\Leftarrow"] & U \times X \arrow[d, "\alpha"] \\
V \times Y \arrow[r, "\beta"'] & 2
\end{tikzcd}$$

Comparing Dialectica and Chu Constructions

Both Chu and Dialectica constructions use a given category $\mathbf{C}$ and an object K of $\mathbf{C}$ to construct a new category. Would like to compare:

- constructions & requirements on the constructions
- structure of category obtained
- game-theoretical understanding of the categories
- higher-order categorical understanding

Comparison complicated by the existence of *two* different Dialectica constructions **GC** and **DC**.

Comparing Constructions

All have the same objects (generalised relations $U \otimes X \to K$), but maps are different in all three cases.

GC and $\mathbf{Chu}_K\mathbf{C}$ are closer: every map in Chu(A, B) is a map in GC(A, B), but **GC** has more maps between these objects.

Both **GC** and $\mathbf{Chu}_K\mathbf{C}$ model classical linear logic, while **DC** only models intuitionistic linear logic.

Requirements:
Chu: any object K, Dialectica: K needs (logical) structure.
Chu: needs **C** smcc & pullbacks, Dialectica: needs **C** ccc.

reason for generalization at the end

Logical Structure of Dialectica Spaces

GSets is autonomous (smcc) with products and coproducts:

$$
\begin{aligned}
A \& B &= (U \times V \xleftarrow{\alpha.\beta} X + Y) \\
A \oplus B &= (U + V \xleftarrow{\alpha.\beta} X \times Y)
\end{aligned}
$$

Negation is given by $A^{\perp} = A \multimap \perp$ where $\perp$ is unit for $\parr$.

$$
\begin{aligned}
A \multimap B &= (V^U \times X^Y \xleftarrow{\alpha \multimap \beta} U \times Y) \\
A \otimes B &= (U \times V \xleftarrow{\alpha \otimes \beta} X^V \times Y^U) \\
A \parr B &= (V^X \times U^Y \xleftarrow{\alpha \parr \beta} X \times Y)
\end{aligned}
$$

Moreover, GC has Exponentials

$$
\begin{aligned}
!A &= (U \xleftarrow{!\alpha} (X^*)^U) \\
?A &= ((U^*)^X \xleftarrow{?\alpha} X)
\end{aligned}
$$

Logical Structure of Chu Spaces

$\mathrm{Chu}_K\mathbf{Sets}$ is $*$-autonomous with same products and coproducts.

$$A\&B \;=\; (U \times V \xleftarrow{\;\alpha.\beta\;} X + Y)$$

$$A \oplus B \;=\; (U + V \xleftarrow{\;\alpha.\beta\;} X \times Y)$$

Involution is given by transpose of the matrix.

$$A^{\perp} \;=\; (X \xleftarrow{\;\overline{\alpha}\;} U)$$

$$A \multimap B \;=\; (\mathcal{P}_1 \xleftarrow{\;\alpha \multimap \beta\;} U \times Y)$$

$$A \otimes B \;=\; (U \times V \xleftarrow{\;\alpha \otimes \beta\;} \mathcal{P}_2)$$

where $\mathcal{P}_1, \mathcal{P}_2$ are suitable pullbacks in $V^U \times X^Y$ and $X^V \times Y^U$.

'Par' obtained from involution, ie $A \parr B = (A^{\perp} \otimes B^{\perp})^{\perp}$.

Logical Structure of Chu Spaces: Exponentials

Originally (Barr'79) Chu$_K$C a model of CLL without exponentials.

Barr'90: ! in **ChuC** if **C** is smcc, acessible and complete and $\perp$ is an internal cogenerator (thm 4.8)

Barr'91: ! for the subcategory of separated objects of $\mathbf{Chu}_\perp C$, when **C** cocomplete & complete ccc and $\perp$ is internal cogenerator (thm 9.2). In particular this holds for **Sets** and 2. (Also !'s if **C** compact closed.)

L&S'91: ! based on Dialectica's exponentials. It's not *cofree*.

Logical Structure of DC

DC is a smcc with categorical products:

$$A \oslash B = (U \times V \overset{\alpha \oslash \beta}{\nleftarrow} X \times Y)$$

$$A \multimap B = (V^U \times X^{U \times Y} \overset{\alpha \multimap \beta}{\nleftarrow} U \times Y)$$

$$A \& B = (U \times V \overset{\alpha\beta}{\nleftarrow} X + Y)$$

only weak coproducts:

$$A + B = (U + V \overset{\alpha + \beta}{\nleftarrow} X^U \times Y^V)$$

Morever

$$!A = (U \overset{\alpha^*}{\nleftarrow} X^*)$$

is *cofree* comonoid on A.

Comparing Logical Structure

Dialectica GC	Chu
$(K, \leq)$ lineale	any K
$A^\perp = A \multimap \perp = (X \xleftarrow{\alpha^\perp} U)$	$A^\perp = (X \xleftarrow{\overline{\alpha}} U)$
$(V^U \times X^Y \xleftarrow{\alpha \multimap \beta} U \times Y)$	$(\mathcal{P}_1 \xleftarrow{\alpha \multimap_C \beta} U \times Y)$
$(U \times V \xleftarrow{\alpha \otimes \beta} X^V \times Y^U)$	$(U \times V \xleftarrow{\alpha \otimes_C \beta} \mathcal{P}_2)$
$(V^X \times U^Y \xleftarrow{\alpha \parr \beta} X \times Y)$	$(\mathcal{P}_3 \xleftarrow{\alpha \parr_C \beta} X \times Y)$
same products, coproducts	same products, coproducts
!, ?	alternatives...
FILL, CLL	CLL
too many maps?	too many pullbacks!

Comparison: Game-theoretical understanding

- Dialectica and Chu have a 'superpower game' reading (cf. Blass). Chu_K**Sets** rediscovered by Lafont/Streicher and named GAME_K.

- Recent work by Hyland and Schalk reinterprets Chu, Dialectica and Totality Spaces (Loader) as abstract games from **Rel**.

  - self-dualization $\mathbf{C} \to \mathbf{C} \times \mathbf{C}^{op}$
  - comonoid indexing $\mathbf{C} \to \mathbf{C}_K$
  - Glueing $\mathbf{C} \to G(\mathbf{C})$
  - orthogonality: tight and loose

Their work will supercede generalization in next slides, when (if?) written.

Comparison: higher-order categorical understanding

Pavlovic's insight: $\mathbf{Chu}_K\mathbf{C} = \mathbf{C}/(.)^{\perp}$ a comma category also works for Dialectica, $\mathbf{GC} = (\mathbf{C}/(.)^{\perp})_{\mathsf{lax}}$.

Pavlovic proves the existence of adjunctions $\mathsf{I} \dashv \mathsf{D} \colon \mathsf{CAT} \to \mathsf{C}\vec{\mathsf{A}}\mathsf{T}$, $\mathsf{i} \dashv \mathsf{C} \colon \mathsf{DU} \to \mathsf{SA}$, and $\mathsf{i} \dashv \mathsf{Chu} \colon \mathsf{AU}_* \to \mathsf{AU}_{\perp}$, and that the 'inclusion' functors above are all comonadic. The first adjunction goes through easily in the lax setting, but not the others. In particular $(\mathbf{C}/(.)^{\perp})^{op}_{\mathsf{lax}}$ is another self-adjunction, but not a duality.

This work doesn't accomodate exponentials, as presented. This tends to be the difficult part.

Koslowski/Barr: generalize $\mathbf{C}$ to a bicategory...

Generalizing Dialectica

Categories $\mathsf{Dial}_L\mathsf{C}$, where C is smcc with products and L a lineale (a closed poset).

Objects of $\mathsf{Dial}_L\mathsf{C}$:
generalised relations between U and X or maps of the form

$$U \otimes X \stackrel{\alpha}{\to} L$$

$\mathsf{Dial}_L\mathsf{C}$ maps are pairs of maps of C $f: U \to V$, $F: Y \to X$
such that $u\alpha F(y) \Rightarrow f(u)\beta y$
(Now implication inherited from L).

Structure of Dial$_L$C

Dial$_L$C is a smcc, with internal-hom and tensor product given by

$$[A, B] = ([U, V] \times [Y, X] \xleftarrow{\alpha \multimap \beta} U \otimes Y)$$

$$A \otimes B = (U \otimes V \xleftarrow{\alpha \otimes \beta} [V, X] \times [U, Y])$$

It has intuitionistic 'par':

$$A \invamp B = ([X, V] \times [Y, U] \xleftarrow{\alpha \invamp \beta} X \otimes Y)$$

Modalities:

$$!A = (U_* \xleftarrow{!\alpha} (X^*)^{U_*})$$

$$?A = ((U^*)^{X_*} \xleftarrow{?\alpha} X_*)$$

Products and coproducts as before

Results on Dial$_L$C

- Dial$_L$C is SMCC, with involution and products.

- Dial$_L$C has distinct objects for logical unities, I, $\perp$, 1 and 0.

- Dial$_L$C does not satisfy MIX.

- Hard part is to prove modalities work.

- Applications:
  general Petri nets, for C=Sets and L=Nat
  Lambek calculus, for C=Sets and L=matrices

Future Work

- Presented 3 kinds of dialectica categories, hinted at two others

- Not enough work done on Dialectica. Examples:
  - Iteration, recursion not touched, some is known to work
  - Some of the connections to games still to sort out?
  - Generalisations Koslowski style to be worked out?
  - Domain theoretic connections (Lamarche) work?
  - Work on model theory (van Benthem, Feferman) to be done

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