Dialectica Categories Surprising Application: mapping cardinal invariants

Valeria de Paiva

Rearden Commerce
University of Birmingham

March, 2012

Outline

1. Dialectica Categories
2. Computational Complexity
3. Cardinal characteristics
4. Structure of $\mathcal{PV}$
5. More Dialecticas

Introduction

I'm a logician professionally, but I feel like an outsider in this conference. Some times being an outsider is a good thing.

I reckon that mathematicians should try some more of it, especially with different kinds of mathematics.

> *"There are two ways to do great mathematics. The first way is to be smarter than everybody else. The second way is to be stupider than everybody else – but persistent."* – *Raoul Bott*

A little bit of personal history...

Some twenty years ago I finished my PhD thesis "The Dialectica Categories" in Cambridge. My supervisor was Dr Martin Hyland.

Dialectica categories came from Gödel's Dialectica Interpretation

The interpretation is named after the Swiss journal Dialectica where it appeared in a special volume dedicated to Paul Bernays 70th birthday in 1958.

I was originally trying to provide an internal categorical model of the Dialectica Interpretation. The categories I came up with proved (also) to be a model of Linear Logic...

Dialectica categories are models of Linear Logic

Linear Logic was created by Girard (1987) as a proof-theoretic tool: the dualities of classical logic plus the constructive content of proofs of intuitionistic logic.

Linear Logic: a tool for semantics of Computing.

Dialectica Interpretation: motivations...

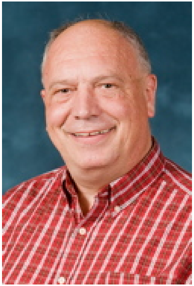
For Gödel (in 1958) the interpretation was a way of proving consistency of arithmetic. Aim: liberalized version of Hilbert's programme – to justify classical systems in terms of notions as intuitively clear as possible.

Since Hilbert's finitist methods are not enough (Gödel's incompleteness theorem and experience with consistency proofs) must admit some abstract notions. G's approach: computable (or primitive recursive) functionals of finite type.

For me (in 1988) an internal way of modelling Dialectica that turned out to produce models of Linear Logic instead of models of Intuitionistic Logic, which were expected...

For Blass (in 1995) a way of connecting work of Votjáš in Set Theory with mine and his own work on Linear Logic and cardinalities of the continuum.

Dialectica categories: useful for proving what...?



Blass (1995) Dialectica categories as a tool for proving inequalities between nearly countable cardinals.
Questions and Answers – A Category Arising in Linear Logic, Complexity Theory, and Set Theory in Advances in Linear Logic (ed. J.-Y. Girard, Y. Lafont, and L. Regnier) London Math. Soc. Lecture Notes 222 (1995).
Also *Propositional connectives and the set theory of the continuum* (1995) and the survey *Nearly Countable Cardinals*.

Questions and Answers: The short story

Blass realized that my category for modelling Linear Logic was also used by Peter Votjáš for set theory, more specifically for proving inequalities between cardinal invariants and wrote *Questions and Answers – A Category Arising in Linear Logic, Complexity Theory, and Set Theory* (1995).
When we discussed the issue in 1994/95 I simply did not read the sections of the paper on Set Theory. or Computational Complexity.

Two years ago I learnt from Samuel about his and Charles work using Blass/Votjáš' ideas and got interested in understanding the cardinal invariants connections.
Late last year we decided that a short visit would be a good way forward.
This is a visit to start a collaboration. Hence I will be talking about old results...

Questions and Answers: The categories GSets and $\mathcal{PV}$

Is this simply a coincidence?

The second dialectica category in my thesis GSets (for Girard's sets) is the **dual** of GT the Galois-Tukey connections category in Votjáš work.
Blass calls this category $\mathcal{PV}$.
The objects of $\mathcal{PV}$ are triples $A = (U, X, \alpha)$ where U, X are sets and $\alpha \subseteq U \times X$ is a binary relation, which we usually write as $u\alpha x$ or $\alpha(u, x)$.
Blass writes it as (A_-, A^+, A) but I get confused by the plus and minus signs.
Two conditions on objects in $\mathcal{PV}$ (not the case in GSets):
1. U, X are sets of cardinality at most $|\mathbb{R}|$.
2. The condition in Moore, Hrusák and Dzamonja (MHD) holds:

$$\forall u \in U \exists x \in X \text{ such that } \alpha(u, x)$$

and

$$\forall x \in X \exists u \in U \text{ such that } \neg\alpha(u, x)$$

Questions and Answers: The category $\mathcal{PV}$

In Category Theory morphisms are more important than objects.

Given objects $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$
a map from A to B in GSets is a pair of functions $f \colon U \to V$ and $F \colon Y \to X$ such that $\alpha(u, Fy)$ implies $\beta(fu, y)$.

Usually I write maps as

$$\begin{tikzcd} U \arrow[d, "f"'] & \alpha \arrow[d, phantom, "\Downarrow"] & X \\ V & \beta & Y \arrow[u, "F"'] \end{tikzcd} \qquad \forall u \in U, \forall y \in Y \quad \alpha(u, Fy) \text{ implies } \beta(fu, y)$$

But a map in $\mathcal{PV}$ satisfies the dual condition that is $\beta(fu, y) \to \alpha(u, Fy)$.
Trust set-theorists to create morphisms $A \to B$ where the relations go in the opposite direction!...

Can we give some intuition for these morphisms?

Blass makes the case for thinking of problems in computational complexity.
Intuitively an object of GSets or $\mathcal{PV}$

$$(U, X, \alpha)$$

can be seen as representing a problem.
The elements of U are instances of the problem, while the elements of X are possible answers to the problem instances.
The relation α say whether the answer is correct for that instance of the problem or not.

Which problems?

A **decision problem** is given by a set of instances of the problem, together with a set of positive instances.
The problem is to determine, given an instance whether it is a positive one or not.
Examples:
1. Instances are graphs and positive instances are 3-colorable graphs;
2. Instances are Boolean formulas and positive instances are satisfiable formulas.
A **many-one reduction** from one decision problem to another is a map sending instances of the former to instances of the latter, in such a way that an instance of the former is positive if and only its image is positive.
An algorithm computing a reduction plus an algorithm solving the latter decision problem can be combined in an algorithm solving the former.

Computational Complexity 101

A **search problem** is given by a set of instances of the problem, a set of witnesses and a binary relation between them.
The problem is to find, given an instance, some witness related to it.
The 3-way colorability decision problem (given a graph is it 3-way colorable?) can be transformed into the 3-way coloring search problem: Given a graph find me one 3-way coloring of it.
Examples:
1. Instances are graphs, witnesses are 3-valued functions on the vertices of the graph and the binary relation relates a graph to its proper 3-way colorings
2. Instances are Boolean formulas, witnesses are truth assignments and the binary relation is the satisfiability relation.
Note that we can think of an object of $\mathcal{PV}$ as a *search problem*, the set of instances is the set U, the set of witnesses the set X.

Computational Complexity 101

A **many-one reduction** from one **search** problem to another is a map sending instances of the former to instances of the latter, in such a way that an instance of the former is positive if and only its image is positive. An algorithm computing a reduction plus an algorithm solving the latter decision problem can be combined in an algorithm solving the former.

We can think of morphisms in $\mathcal{PV}$ as reductions of problems.
If this intuition is useful, great, if not, carry on thinking simply of sets and a relation and complicated notion of how to map triples to other triples.

Questions and Answers

Objects in $\mathcal{PV}$ are only certain objects of GSets, as they have to satisfy the two extra conditions. What happens to the structure of the category when we restrict ourselves to this subcategory?

Fact: GSets has products and coproducts, as well as a terminal and an initial object.

Given objects of $\mathcal{PV}$, $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$
the product $A \times B$ in GSets is the object $(U \times V, X + Y, \text{choice})$
where choice: $U \times V \times (X + Y) \to 2$ is the relation sending
$(u, v, (x, 0))$ to $\alpha(u, x)$ and $(u, v, (y, 1))$ to $\beta(v, y)$.

The terminal object is $T = (1, 0, e)$ where e is the empty relation,
$e \colon 1 \times 0 \to 2$ on the empty set.

Questions and Answers

Similarly the coproduct of A and B in GSets is the object
$(U+V, X\times Y, \text{choice})$
where choice: $U+V\times(X\times Y)\to 2$ is the relation sending
$((u,0),x,y)$ to $\alpha(u,x)$ and $((v,1),x,y)$ to $\beta(v,y)$

The initial object is $0=(0,1,e)$ where $e\colon 0\times 1\to 2$ is the empty relation.

Now if the basic sets all U,V,X,Y have cardinality up to $|\mathbb{R}|$ then (co-)products will do the same.
But the MHD condition is a different story.

The structure of $\mathcal{PV}$

Note that neither T or 0 are objects in $\mathcal{PV}$,
as they don't satisfy the MHD condition.

$\forall u \in U, \exists x \in X$ such that $\alpha(u,x)$ and $\forall x \in X \exists u \in U$ such that $\neg\alpha(u,x)$

To satisfy the MHD condition neither U nor X can be empty.

Also the object $I = (1, 1, \mathrm{id})$ is not an object of $\mathcal{PV}$, as it satisfies the first half of the MHD condition, but not the second. And the object $\perp = (1, 1, \neg\mathrm{id})$ satisfies neither of the halves.
The constants of Linear Logic do not fare too well in $\mathcal{PV}$.

The structure of $\mathcal{PV}$

Back to morphisms of $\mathcal{PV}$, the use that is made of the category in applications is simply of the pre-order induced by the morphisms.

It is somewhat perverse that here, in contrast to usual categorical logic,

$$A \leq B \iff \text{There is a morphism from } B \text{ to } A$$

Examples of objects in $\mathcal{PV}$

1. The object $(\mathbb{N}, \mathbb{N}, =)$ where n is related to m iff $n = m$.
To show MHD is satisfied we need to know that $\forall n \in \mathbb{N} \exists m \in \mathbb{N}(n = m)$, can take $m = n$. But also that $\forall m \in \mathbb{N} \exists k \in \mathbb{N}$ such that $\neg(m = k)$. Here we can take $k = suc(m)$.

2. The object $(\mathbb{N}, \mathbb{N}, \leq)$ where n is related to m iff $n \leq m$.
3. The object $(\mathbb{R}, \mathbb{R}, =)$ where r_1 and r_2 are related iff $r_1 = r_2$, same argument as 1 but equality of reals is logically much more complicated.
4. The objects $(2, 2, =)$ and $(2, 2, \neq)$ with usual equality.

What about GSets?

GSets is a category for categorical logic, we have:

$$A \leq B \iff \text{There is a morphism from } A \text{ to } B$$

Examples of objects in GSets:
"Truth-value" constants of Linear Logic as discussed T, 0, $\perp$ and I.

All the $\mathcal{PV}$ objects are in GSets. Components of objects such as U, X are not bound above by the cardinality of $\mathbb{R}$.

Also the object 2 of Sets plays an important role in GSets, as our relations α are maps into 2, but the objects of the form $(2, 2, \alpha)$ have played no major role in GSets so far.

What have Set Theorists done with $\mathcal{PV}$?

Cardinal Characteristics of the Continuum

Blass: "One of Set Theory's first and most important contributions to mathematics was the distinction between different infinite cardinalities, especially countable infinity and non-countable one."

Write $\mathbb{N}$ for the natural numbers and ω for the cardinality of $\mathbb{N}$.

Similarly $\mathbb{R}$ for the reals and 2^ω for their cardinality.

All the cardinal characteristics considered will be smaller or equal to the cardinality of the reals.

They are of little interest if the Continuum Hypothesis holds, as then there are no cardinalities between the cardinality of the integers ω and the cardinality of the reals 2^ω.

But if the continuum hypothesis does NOT hold there are many interesting connections (in general in the form of inequalities) between various characteristics that Votjáš discusses.

Cardinals from Analysis

I recall the main definitions that Blass uses in Questions and Answers and his main "theorem":

- If X and Y are two subsets of $\mathbb{N}$ we say that X **splits** Y if both $X \cap Y$ and $Y \setminus X$ are infinite.
- The **splitting number** s is the smallest cardinality of any family $\mathcal{S}$ of subsets of $\mathbb{N}$ such that every infinite subset of $\mathbb{N}$ is split by some element of $\mathcal{S}$.

Recall the **Bolzano-Weierstrass Theorem:** Any bounded sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, $(x_n)_{n\in A}$.
One can extend the theorem to say:
For any countably bounded many sequences of real numbers $\mathbf{x}_k = (x_{kn})_{n\in\mathbb{N}}$ there is a single infinite set $A \subseteq \mathbb{N}$ such that the subsequences indexed by A, $(x_{kn})_{n\in A}$ all converge.
If one tries to extend the result for uncountably many sequences s above is the first cardinal for which the analogous result fail.

Cardinals from Analysis

- If f and g are functions $\mathbb{N} \to \mathbb{N}$, we say that f **dominates** g if for all except finitely many n's in $\mathbb{N}$, $f(n) \leq g(n)$.
- The **dominating number** d is the smallest cardinality of any family $\mathcal{D}$ contained in $\mathbb{N}^{\mathbb{N}}$ such that every g in $\mathbb{N}^{\mathbb{N}}$ is dominated by some f in $\mathcal{D}$.
- The **bounding number** b is the smallest cardinality of any family $\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$ such that no single g dominates all the members of $\mathcal{B}$.

Connecting Cardinals to the Category

Blass "theorems" : We have the following inequalities

$$\omega \leq s \leq d \leq 2^{\omega}$$

$$\omega \leq b \leq r \leq r_{\sigma} \leq 2^{\omega}$$

$$b \leq d$$

The proofs of these inequalities use the category of Galois-Tukey connections and the idea of a "norm" of an object of $\mathcal{PV}$.
(and a tiny bit of structure of the category...)

The structure of the category $\mathcal{PV}$

Given an object $A = (U, X, \alpha)$ of $\mathcal{PV}$ its "norm" $||A||$ is the smallest cardinality of any set $Z \subseteq X$ sufficient to contain at least one correct answer for every question in U.

Blass again: “It is an *empirical fact* that proofs between cardinal characteristics of the continuum usually proceed by representing the characteristics as norms of objects in $\mathcal{PV}$ and then exhibiting explicit morphisms between those objects.”

One of the aims of our proposed collaboration is to explain this empirical fact, preferably using categorical tools.

Haven't done it, yet. So will try to explain some of the easy instances.

The structure of $\mathcal{PV}$

Proposition[Rangel] The object $(\mathbb{R}, \mathbb{R}, =)$ is maximal amongst objects of $\mathcal{PV}$.

Given any object $A = (U, X, \alpha)$ of $\mathcal{PV}$ we know both U and X have cardinality small than $|\mathbb{R}|$. In particular this means that there is an injective function $\varphi \colon U \to \mathbb{R}$. (let ψ be its left inverse, i.e $\psi(\varphi u) = u$)

Since α is a relation $\alpha \subseteq U \times X$ over non-empty sets, if one accepts the **Axiom of Choice**, then for each such α there is a map $f : U \to X$ such that for all u in U, $u \alpha f(u)$.

Need a map $\Phi \colon \mathbb{R} \to X$ such that

$$\begin{tikzcd} U \arrow[d, "\varphi"'] \arrow[r, phantom, "\alpha"] & X \\ \mathbb{R} \arrow[ru, phantom, "\Uparrow"] \arrow[r, phantom, "="] & \mathbb{R} \arrow[u, "\Phi"'] \end{tikzcd}$$

$$\forall u \in U, \forall r \in \mathbb{R} \ \ (\varphi u = r) \to \alpha(u, \Phi r)$$

The structure of $\mathcal{PV}$

Axiom of Choice is essential

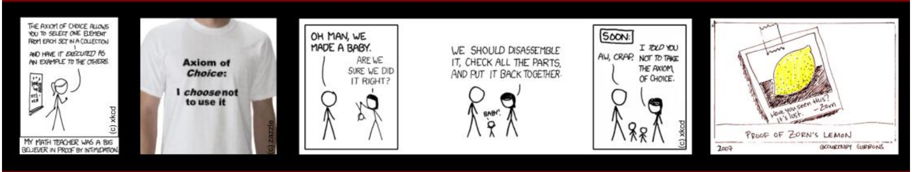
Given cardinality fn $\varphi \colon U \to \mathbb{R}$, let ψ be its left inverse, $\psi(\varphi u) = u$, for all $u \in U$. Need a map $\Phi \colon \mathbb{R} \to X$ such that

$$\begin{tikzcd} U \arrow[d, "\varphi"'] & \alpha \arrow[d, phantom, "\Uparrow"] & X \\ \mathbb{R} & = & \mathbb{R} \arrow[u, "\Phi"'] \end{tikzcd} \qquad \forall u \in U, \forall r \in \mathbb{R} \ \ (\varphi u = r) \to \alpha(u, \Phi r)$$

Let u and r be such that $\varphi u = r$ and **define** Φ as $\psi \circ f$.
Since $\varphi u = r$ can apply Φ to both sides to obtain $\Phi(\varphi(u)) = \Phi(r)$.
Substituting Φ's definition get $f(\psi(\varphi u))) = \Phi(r)$. As ψ is left inverse of φ ($\psi(\varphi u) = u$) get $fu = \Phi(r)$.
Now the definition of f says for all u in U $u\alpha fu$, which is $u\alpha\Phi r$ holds, as desired. (Note that we did not need the cardinality function for X.)

Fun meeting somewhere else

Is the axiom of choice a bit of a joke?



Why is Banach-Tarski not a paradox? Or is it, after all, as weird as you thought?

Come hear Imre Leader and Thomas Forster debate the right way to use -- or not the use -- the Axiom of Choice.

28th February at 18:30 at MR4, in the CMS.

The structure of $\mathcal{PV}$

Proposition[Rangel] The object $(\mathbb{R}, \mathbb{R}, \neq)$ is minimal amongst objects of $\mathcal{PV}$.

This time we use the cardinality function $\varphi\colon X \to \mathbb{R}$ for X. We want a map in $\mathcal{PV}$ of the shape:

$$\begin{array}{ccc} \mathbb{R} & \neq & \mathbb{R} \\ \Phi \big\downarrow & \Uparrow & \big\uparrow \varphi \\ U & \alpha & X \end{array} \qquad \forall r \in \mathbb{R}, \forall x \in X \quad \alpha(\Phi r, x) \to (r \neq \varphi x)$$

Now using **Choice** again, given the relation $\alpha \subseteq U \times X$ we can fix a function $g\colon X \to U$ such that for any x in X $g(x)$ is such that $\neg g(x) \alpha x$.

The structure of $\mathcal{PV}$

Axiom of Choice is essential

Given cardinality fn $\varphi\colon X \to \mathbb{R}$, let $\psi\colon \mathbb{R} \to X$ be its left inverse, $\psi(\varphi x) = x$, for all $x \in X$. Need a map $\Phi\colon \mathbb{R} \to U$ such that

$$\begin{array}{ccccc} \mathbb{R} & \neq & \mathbb{R} & & \\ \Phi\big\downarrow & \Uparrow & \big\uparrow\varphi & & \forall r \in \mathbb{R}, \forall x \in X \quad \alpha(\Phi r, x) \to \neg(\varphi x = r) \\ U & \alpha & X & & \end{array}$$

Exactly the same argument goes through. Let r and x be such that $\varphi x = r$ and **define** Φ as $\psi \circ g$.
Since $\varphi x = r$ can apply Φ to both sides to obtain $\Phi(\varphi(x)) = \Phi(r)$.
Substituting Φ's definition get $g(\psi(\varphi x))) = \Phi(r)$. As ψ is left inverse of φ, $(\psi(\varphi x) = x)$ get $gx = \Phi(r)$.
But the definition of g says for all x in X $\neg gx\alpha x$, which is $\neg\alpha(\Phi r, x)$ holds. Not quite the desired, unless you're happy with RAA.

More structure of GSets

Given objects A and B of GSets we can consider their tensor products.
Actually two different notions of tensor products were considered in GSets, but only one has an associated internal-hom.
(Blass considered also a mixture of the two tensor products of GSets, that he calls a sequential tensor product.)
Having a tensor product with associated internal hom means that we have an equation like:

$$A \otimes B \to C \Longleftrightarrow A \to (B \to C)$$

Can we do the same for $\mathcal{PV}$? Would it be useful?
The point is to check that the extra conditions on $\mathcal{PV}$ objects are satisfied.

Tensor Products in GSets

Given objects A and B of GSets we can consider a preliminary tensor product, which simply takes products in each coordinate. Write this as $A \oslash B = (U \times V, X \times Y, \alpha \times \beta)$ This is an intuitive construction to perform, but it does not provide us with an adjunction.
To "internalize" the notion of map between problems, we need to consider the collection of all maps from U to V, V^U, the collection of all maps from Y to X, X^Y and we need to make sure that a pair $f\colon U \to V$ and $F\colon Y \to X$ in that set, satisfies our dialectica (or co-dialectica) condition:

$$\forall u \in U, y \in Y, \alpha(u, Fy) \leq \beta(fu, y) \text{ (respectively } \geq)$$

This give us an object $(V^U \times X^Y, U \times Y, \mathrm{eval})$ where eval: $V^U \times X^Y \times (U \times Y) \to 2$ is the map that evaluates f, F on the pair u, y and checks the implication between relations.

More structure in GSets

By reverse engineering from the desired adjunction, we obtain the 'right' tensor product in the category.
The tensor product of A and B is the object $(U \times V, Y^U \times X^V, \text{prod})$, where $\text{prod}\colon (U \times V) \times (Y^U \times X^V) \to 2$ is the relation that first evaluates a pair (φ, ψ) in $Y^U \times X^V$ on pairs (u, v) and then checks the (co)-dialectica condition.
Blass discusses a mixture of the two tensor products, which hasn't showed up in the work on Linear Logic, but which was apparently useful in Set Theory.

An easy theorem of GSets/$\mathcal{PV}$...

Because it's fun, let us calculate that the reverse engineering worked...

$$A \otimes B \to C \text{ if and only if } A \to [B \to C]$$

$$\begin{tikzcd}
U \times V \arrow[d, "f"'] & \alpha \otimes \beta \arrow[d, phantom, "\Downarrow"] & X^V \times Y^U \\
W & \gamma & Z \arrow[u, "{(g_1, g_2)}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d] & \alpha \arrow[d, phantom, "\Downarrow"] & X \\
W^V \times Y^Z & \beta \to \gamma & V \times Z \arrow[u]
\end{tikzcd}$$

More Original Dialectica Categories

My thesis has four chapters, four main definitions and four main theorems. The first two chapters are about the "original" dialectica categories.

Theorem (V de Paiva, 1987)

If C is a ccc with stable, disjoint coproducts, then Dial(C) has products, tensor products, units and a linear function space $(1, \times, \otimes, I, \to)$ *and Dial(C) is symmetric monoidal closed.*

This means that $Dial(C)$ models **Intuitionistic Linear Logic** (ILL) without modalities. How to get modalities? Need to define a special comonad and lots of work to prove theorem 2...

Original Dialectica Categories

$!A$ must satisfy $!A \to !A \otimes !A$, $!A \otimes B \to !A$, $!A \to A$ and $!A \to !!A$, together with several equations relating them.
The point is to define a comonad such that its coalgebras are commutative comonoids and the coalgebra and the comonoid structure interact nicely.

Theorem (V de Paiva, 1987)

Given C a cartesian closed category with free monoids (satisfying certain conditions), we can define a comonad T on $Dial(C)$ such that its Kleisli category $Dial(C)^T$ is cartesian closed.

Define T by saying $A = (U, X, \alpha)$ goes to (U, X^*, α^*) where X^* is the free commutative monoid on X and α^* is the multiset version of α.
Loads of calculations prove that the linear logic modalitiy ! is well-defined and we obtain a full model of ILL and IL, a posteriori of CLL.
Construction generalized in many ways, cf. dePaiva, TAC, 2006.

What is the point of (these) Dialectica categories?

First, the construction ends up as a model of Linear Logic, instead of constructive logic.This allows us to see where the assumptions in Godel's argument are used (Dialectica still a bit mysterious...)

It justifies linear logic in terms of a more traditional logic tool and conversely explains the more traditional work in terms of a 'modern' (linear, resource conscious) decomposition of concerns.

Theorems(87/89): Dialectica categories provide models of linear logic as well as an internal representation of the dialectica interpretation. Modeling the exponential ! is hard, first model to do it. Still (one of) the best ones.

Dialectica categories: 20 years later...

It is pretty: produces models of Intuitionistic and classical linear logic and special connectives that allow us to get back to usual logic.

Extended it in a number of directions:

a robust proof can be pushed in many ways...

used in CS as a model of Petri nets (more than 3 phds),
it has a non-commutative version for Lambek calculus (linguistics),
it has been used as a model of state (with Correa and Hausler, Reddy ind.)
Also in Categorical Logic: generic models (with Schalk04) of Linear Logic,
Dialectica interp of Linear Logic and Games (Shirahata and Oliveira)
fibrational versions (B. Biering and P. Hofstra).
Most recently and exciting:
Formalization of partial compilers: correctness of MapReduce in MS .net frameworks via DryadLINQ, "The Compiler Forest", M. Budiu, J. Galenson, G. Plotkin 2011.

Conclusions

Introduced you to dialectica categories GSets/$\mathcal{PV}$.
Hinted at Blass and Votjáš use of them for mapping cardinal invariants.
Uses for Categorical Proof Theory very different from uses in Set Theory for cardinal invariants.
Showed one easy, but essential , theorem in categorical logic.
But haven't even started looking at Parametrized Diamond Principles...
Haven't even started talking about "lax topological systems" in the sense of Vickers, a different connection to Topology.

Believe it is not a simple coincidence that dialectica categories are useful in these disparate areas.
We're starting our collaboration, so hopefully real "new" theorems will come up.

More Conclusions...

Working in interdisciplinary areas is hard, but rewarding.
The frontier between logic, computing, linguistics and categories is a fun place to be.
Mathematics teaches you a way of thinking, more than specific theorems.
Fall in love with your ideas and enjoy talking to many about them...

Thanks Samuel, Marcelo and all locals for this lovely meeting.
Thanks to Charles and Samuel for mentioning my work in connection to their stuff on parametrized Diamond principles and MHD's work...
and thanks Samuel, Andreas and Thierry for all the effort to bring me here.

References

Categorical Semantics of Linear Logic for All, Manuscript.
Dialectica and Chu constructions: Cousins?Theory and Applications of Categories, Vol. 17, 2006, No. 7, pp 127-152.
A Dialectica Model of State. (with Correa and Haeusler). In CATS'96, Melbourne, Australia, Jan 1996.
Full Intuitionistic Linear Logic (extended abstract). (with Martin Hyland). Annals of Pure and Applied Logic, 64(3), pp.273-291, 1993.
Thesis TR: The Dialectica Categories

http://www.cl.cam.ac.uk/techreports/UCAM-CL-TR-213.html

A Dialectica-like Model of Linear Logic.In Proceedings of CTCS, Manchester, UK, September 1989. LNCS 389
The Dialectica Categories. In Proc of Categories in Computer Science and Logic, Boulder, CO, 1987. Contemporary Mathematics, vol 92, American Mathematical Society, 1989

all available from `http://www.cs.bham.ac.uk/~vdp/publications/papers.html`

Functional Interpretations and Gödel's Dialectica

Starting with Gödel's Dialectica interpretation(1958) a series of "translation functions" between theories

Avigad and Feferman on the Handbook of Proof Theory:

> *This approach usually follows Gödel's original example: first, one reduces a classical theory C to a variant I based on intuitionistic logic; then one reduces the theory I to a quantifier-free functional theory F.*

Examples of functional interpretations:

- Kleene's realizability
- Kreisel's modified realizability
- Kreisel's No-CounterExample interpretation
- Dialectica interpretation
- Diller-Nahm interpretation, etc...

Gödel's Dialectica Interpretation

For each formula A of HA we associate a formula of the form $A^D = \exists u \forall x A_D(u,x)$ (where A_D is a quantifier-free formula of Gödel's system T) inductively as follows: when A_{at} is an atomic formula, then its interpretation is itself.
Assume we have already defined $A^D = \exists u \forall x. A_D(u,x)$ and $B^D = \exists v \forall y. B_D(v,y)$.
We then define:

$$(A \wedge B)^D = \exists u, v \forall x, y.(A_D \wedge B_D)$$
$$(A \rightarrow B)^D = \exists f\colon U \rightarrow V, F\colon U \times X \rightarrow Y, \forall u, y.$$
$$(\ \mathsf{A}_D(u, F(u,y)) \rightarrow B_D(fu; y))$$
$$(\forall z A)^D(z) = \exists f\colon Z \rightarrow U \forall z, x. A_D(z, f(z), x)$$
$$(\exists z A)^D(z) = \exists z, u \forall x. A_D(z, u, x)$$

The intuition here is that if u realizes $\exists u \forall x. A_D(u,x)$ then $f(u)$ realizes $\exists v \forall y. B_D(v,y)$ and at the same time, if y is a counterexample to $\exists v \forall y. B_D(v,y)$, then $F(u,y)$ is a counterexample to $\forall x. A_D(u,x)$.

Where is my category theory?

Categorical Semantics: a picture

$$\begin{tikzcd} & \lambda - calculus \arrow[dl, dash] \arrow[dr, dash] & \\ CartClosedCat \arrow[rr] & & IntPropLogic \end{tikzcd}$$

Framework connecting

$$\begin{tikzcd} & Functional\ Program \arrow[dl, dash] \arrow[dr, dash] & \\ Category\ Theory \arrow[rr] & & Logic \end{tikzcd}$$

lots of important stuff elided

Where is my category theory?

Categorical Semantics of (Intuitionistic) Linear Logic

Want linear version of Extended Curry-Howard Isomorphism

$$\begin{tikzcd} & \textit{linear}\,\lambda\,\textit{calculus} \arrow[dl, no head] \arrow[dr, no head] & \\ \textit{linear cat} \arrow[rr] & & (I)LL \end{tikzcd}$$

- Logic intuitionistic
- Already have Intuitionistic Linear Logic (ILL),
  — must find other two sides of triangle...