Fibrational Versions of Dialectica Categories

Valeria de Paiva

Cuil, Inc.

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Outline

Introduction: What, why?

(Prove existence and describe) Cartesian Closed Dialectica categories

- Chapter 4 of Bodil Biering's PhD thesis "*Dialectica Interpretations: A Categorical Analysis*", 2008
- Why? Categorical understanding of Gödel's Dialectica Interpretation
- What for?
  - For Gödel, the interpretation was a way of proving consistency of arithmetic, an extension of Hilbert's programme
  - For me (20 years ago) a way of producing models of Linear Logic from a proven way of understanding logic
  - For Biering: a way of unifying categorical structures, brought back to proof theory?...

Biering's Thesis Abstract

- The Dialectica interpretations are remarkable syntactic constructions
- Use these constructions to develop new mathematical structures: Dialectica categories, the Dialectica- and Diller-Nahm triposes, and the Dialectica- and Diller-Nahm toposes
- The mathematical structures created from the functional interpretations provide us with new models for type theories and programming logics
- Studying the mathematical structures we gain new insights into the syntactical constructions.
- Product: Biering et al "Copenhagen interpretation"

Biering's Thesis Table of Contents

A collection of articles, want to discuss chapter 4...

- Introduction
- Topos Theoretic Versions of Dialectica Interpretations
- A Unified View on the Dialectica Triposes
- Cartesian Closed Dialectica Categories
- The Copenhagen interpretation
- (BI Hyperdoctrines and Higher Order Separation Logic)

Functional Interpretations

- Starting with Gödel's Dialectica interpretation a series of "translation functions" between theories
- Avigad and Feferman on the Handbook of Proof Theory:

  > *This approach usually follows Gödel's original example: first, one reduces a classical theory C to a variant I based on intuitionistic logic; then one reduces the theory I to a quantifier-free functional theory F.*

- Examples of functional interpretations:
  - Kleene's realizability
  - Kreisel's modified realizability
  - Kreisel's No-CounterExample interpretation
  - Dialectica interpretation
  - Diller-Nahm interpretation

Gödel's Dialectica Interpretation

For each formula A of HA we associate a formula of the form $A^D = \exists u \forall x A_D(u,x)$ (where A_D is a quantifier-free formula of Gödel's system T) inductively as follows: when A_{at} is an atomic formula, then its interpretation is itself.
Assume we have already defined $A^D = \exists u \forall x . A_D(u,x)$ and $B^D = \exists v \forall y . B_D(v,y)$.
We then define:

- $(A \wedge B)^D = \exists u, v \forall x, y . (A_D \wedge B_D)$
- $(A \rightarrow B)^D = \exists f \colon U \rightarrow V, F \colon U \times X \rightarrow Y, \forall u, y .$ $(\mathrm{A}_D(u, F(u,y)) \rightarrow B_D(fu; y))$
- $(\forall z A)^D(z) = \exists f \colon Z \rightarrow U \forall z, x . A_D(z, f(z), x)$
- $(\exists z A)^D(z) = \exists z, u \forall x . A_D(z, u, x)$

The intuition here is that if u realizes $\exists u \forall x . A_D(u,x)$ then $f(u)$ realizes $\exists v \forall y . B_D(v,y)$ and at the same time, if y is a counterexample to $\exists v \forall y . B_D(v,y)$, then $F(u,y)$ is a counterexample to $\forall x . A_D(u,x)$.

Categorical Dialectica Interpretation

- Main references:
- The ‘Dialectica’ Interpretation and Categories (P. J. Scott, Zeit. fur Math Logik und Grund. der Math. 24, 1978)
- The Dialectica Categories, (de Paiva, AMS vol 92, 1989)
- Proof Theory in the Abstract, (Hyland, APAL 2002)
- Dialectica categories are naturally symmetric monoidal closed, but not cartesian closed categories.
- We would like them to be cartesian closed.
  Why?
  Can we make them Cartesian closed?
  Yes, in different ways.

Plan of Biering's 'Cartesian Closed Dialectica Categories'

- Recall the definitions and basic properties of original dialectica categories
- Discuss three approaches to classes of Cartesian closed Dialectica categories
- Preferred way explained, leads to a generalisation of construction
- Show monads and comonads in generalisation
- Example of non-Girardian comonad that produces weak exponentials
- Example of extensional version of Dialectica
- Conclusions

Original Dialectica Categories (V. de Paiva, 1987)

Suppose that we have a category C, with finite limits, interpreting some type theory.
The category $Dial(C)$ has as objects triples $A = (U, X, \alpha)$, where U, X are objects of C and α is a sub-object of $U \times X$, that is a monic in $Sub(U \times X)$.
A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of maps (f, F) in C, $f \colon U \to V$, $F \colon U \times X \to Y$ such that

$$\alpha(u, F(u, y)) \leq \beta(f(u), y)$$

The predicate α is not symmetric: read (U, X, α) as $\exists u. \forall x. \alpha(u, x)$, a proposition in the image of the Dialectica interpretation. The functionals f and F correspond to the dialectica interpretation of implication.

Original Dialectica Categories

My thesis has four chapters, four main definitions and four main theorems. The first two chapters are about the "original" dialectica categories.

Theorem (V de Paiva, 1987)

If C is a ccc with stable, disjoint coproducts, then Dial(C) has products, tensor products, units and a linear function space $(1, \times, \otimes, I, \rightarrow)$ *and Dial(C) is symmetric monoidal closed.*

This means that $Dial(C)$ models Intuitionistic Linear Logic (ILL) without modalities. How to get modalities? Need to define a special comonad and lots of work to prove theorem 2...

Original Dialectica Categories

$!A$ must satisfy $!A \to !A \otimes !A$, $!A \otimes B \to !A$, $!A \to A$ and $!A \to !!A$, together with several equations relating them.

The point is to define a comonad such that its coalgebras are commutative comonoids and the coalgebra and the comonoid structure interact nicely.

Theorem (V de Paiva, 1987)

Given C a cartesian closed category with free monoids (satisfying certain conditions), we can define a comonad T on $Dial(C)$ such that its Kleisli category $Dial(C)^T$ is cartesian closed.

Define T by saying $A = (U, X, \alpha)$ goes to (U, X^*, α^*) where X^* is the free commutative monoid on X and α^* is the multiset version of α.

Loads of calculations prove that the linear logic modalitiy ! is well-defined and we obtain a full model of ILL and IL, a posteriori of CLL.

Construction generalized in many ways, cf. dePaiva, TAC, 2006.

Fibrational Dialectica (M. Hyland, 2002)

Given C a category with finite limits and a pre-ordered fibration $p\colon E \to C$ with for each I in C, a preordered collection of predicates $E(I) = (E(I), \vdash)$, construct a category $Dial(p)$. The objects A of $Dial(p)$ are triples (U, X, α) where U, X are objects in C and α is in $E(U \times X)$. Maps in $Dial(p)$ are pairs of maps in C,(f, F) in C, $f\colon U \to V$, $F\colon U \times X \to Y$ such that

$$\alpha(u, F(u, y)) \vdash \beta(f(u), y)$$

Theorem (Hyland, 2002)

If C is a CCC and $p\colon E \to C$ is (pre-ordered) fibered cartesian closed then $Dial(p)$ is symmetric monoidal closed.

If, moreover, C has finite, distributive coproducts and $E(0) \cong 1$ and the injections $X \to X + Y$ and $Y \to X + Y$ induce an equivalence $E(X + Y) \equiv E(X) \times E(Y)$ then $Dial(p)$ has finite products.

This is a generalization of the original Dialectica categories, where the fibration is the subobject fibration.

Categorical Logic in slogans

"categorical model theory" or "categorical proof theory"?

- Categorical model theory: model theory, where models are categories, instead of sets.
- Categorical proof theory: models are categories, propositions are objects, derivations are arrows in the category. concept of two different proofs being 'the same'
- Logic connectives correspond to structure in the categories, Lawvere and Lambek in the 60's.
- Textbook: Lambek and Scott "Introduction to higher order categorical logic", 1986.
- Main example: Intuitionistic propositional logic modeled as Cartesian Closed Categories (CCCs).
- Works for first-order intuitionistic logic too, models are hyperdoctrines

Fibrations for Dummies

Fibrations can be used for modeling several kinds of type theories. Too complicated, perhaps?

- Intuition: given a set I, consider the I-indexed family of sets $(X_i)_{i \in I}$ or $X(i)$, for each i in I.
- Want something similar where instead of indexing sets, we have a base category B, creating a "family of objects of a category indexed by objects I in the base category B"
- There are two "equivalent" ways of doing this, using "indexed categories" or "fibrations".
- A fibration is a structure $p\colon E \to B$, where p is a functor and E, B are categories, satisfying certain axioms.
- Think of B as **Sets** and E as Fam(**Sets**) where Fam(**Sets**) is the category where objects are families of sets $\{X_i\}_{i \in I}$, I and X_i are sets. Maps from $f\colon \{X_i\}_{i \in I} \to \{Y_j\}_{j \in J}$ consist of a function $\phi : I \to J$ (a re-indexing function) together with a family of maps $\{f_i\colon X_i \to Y_{\phi(i)}\}_{i \in I}$

Fibrations for Dummies 2

- Which certain axioms? Given a functor $p\colon E \to B$ when can we think of each object X in E as a family $(X_i)_{i\in I}$ indexed by $I = pX$ in B?
- There is a functor $p\colon \mathit{Fam}(\mathbf{Sets}) \to (\mathbf{Sets})$ that takes the family $\{X_i\}_{i\in I}$ to I.
- Call a map $f\colon \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$, *vertical* if $pf = 1_I$ – no re-indexing going on.
- Call a map $f\colon \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$, *cartesian* if each f_i is an isomorphism, pure re-indexing, f_is don't do any work.
- Given a family of sets $\{X_i\}_{i\in I}$ any map $\alpha\colon K \to I$ induces a (cartesian) map $f\colon \{X_{\alpha(k)}\}_{k\in K} \to \{X_i\}_{i\in I}$ where each f_i is the identity.
- cartesian maps have a universal property: an arbitrary map f factors through a cartesian map g, when $\phi = pf$ factors through $\alpha = pg$ and the factorization is uniquely determined.

Fibrations for Dummies 3

Definition

A map $g\colon X' \to X$ in E is called **cartesian** if given any map $f\colon Y \to X$, each factorization of $\phi = pf$ through $\alpha = pg$ uniquely determines a factorization of f through g.

[insert picture]

Definition

A functor $p\colon E \to B$ is a **fibration**, if for all X in E and maps $\alpha\colon K \to I = pX$ there exists an object X' and a cartesian map $g\colon X' \to X$ such that $pg = \alpha$.

Cartesian liftings are unique up to iso, so: If $p\colon E \to B$ is a fibration, a **cleavage** for this fibration is a particular choice of cartesian liftings. A fibration equipped with a particular cleavage is called a **cloven fibration**. (usually to show a functor is a fibration, we produce a cleavage)

Proof Theory in the Abstract (Troelstra Fest)

A destilation of the program of ”proof theory in the abstract” as developed since the late 60s. Structure of paper in 2002:

- Background
- Dialectica
- Diller-Nahm
- Classical logic

Highly recommended for philosophy and clarity of explanation of the problems with classical logic.

Proof Theory in the Abstract (Troelstra Fest)

For Dialectica main theorems are:

Theorem (Hyland Thm 2.3, p.8, 2002)

If T is a CCC interpreting some type theory and $p\colon P \to T$ is (pre-ordered) fibered cartesian closed then $Dial = Dial(p)$ as defined is symmetric monoidal closed.

Natural propositional structure in place

Theorem (Hyland Thm 2.5, p.9, 2002)

The fibration $q\colon Dial \to T$ has left and right adjoints to re-indexing along product projections. These satisfy the Beck-Chevalley conditions.

Predicate structure in place.
But to really interpret Dialectica we need conjunctions and disjunctions. Here the maths turns out not be so pretty.

Proof Theory in the Abstract (Troelstra Fest)

For conjunction, to cope with diagonals $A \to A \otimes A$ need 'weak cases definition'. To cope with projections $A \otimes B \to A$ need inhabited types. For disjunction we need a weak coproduct as well as a weak initial object and a codiagonal. The structure can be made to work, but it ain't good category theory. (These problems and way-outs were known from the non-fibrational case).

Theorem (Hyland, Thm 2.6 p.14, 2002)

The poset reflection of our indexed category Dial $\to$ T of proofs is a first-order hyperdoctrine: we get indexed Heyting algebras and good quantification.

Diller-Nahm has better structure!
Hyland suggests an abstract analysis of the interpretation of set theory studied by Burr.

Jacobs, Streicher and de Paiva first version of 2.3 in note in 1995 (Jacob's book exercise). Streicher' note "A Semantic Version of the Diller-Nahm Variant of Gödel's Dialectica Interpretation", 2000.

Cartesian Closed Dialectica Categories?

Several people took up more (categorical) analysis of Dialectica, e.g. Streicher, Rosolini, Birkedal. etc...

The natural structure of $Dial(p)$ is smcc with finite products.
Biering: How can we make it cartesian closed?

- Make the tensor product, a cartesian product
  Hyland's 'hackery' above
- Get a Girardian comonad on the Dialectica category
  Diller-Nahm variants (VdP88), (Strei?), (Burr98)?, ...
- Add enough structure to define a weak function space – without making the tensor a product, (Biering06)

Cloven Dialectica Categories

Hyland defined the category *Dial*(p) for pre-ordered fibrations. Biering generalized it, using fibrations into usual categories, requiring p to be a cloven fibration. This is needed to obtain associativity of composition in *Dial*(p). She defines *Dial*(p) and prove it's a category. With appropriate conditions it has some of the structure we want.

Theorem (Biering, 2008)

Let $p\colon E \to T$ be a cloven fibration. If T has finite, distributive coproducts and products, and the injections $X \to X + Y$ and $Y \to X + Y$ induce an equivalence $\mu\colon E(X) \times E(Y) \equiv E(X + Y)$, natural in X and Y, then Dial(p) has binary products. Moreover, if $E(0) \cong 1$, then Dial(p) has a terminal object.

The proof is a direct generalization of Hyland's proof.

How can one make the category *Dial*(p) above cartesian closed? It has products and a terminal object, but no function spaces. Biering will define 'weak' function spaces for a particular example of fibration, the codomain fibration.

Codomain Dialectica Categories?

What is the codomain fibration?
For any category C, there is a category $Arr(C)$, whose objects are the arrows of C, say $\alpha\colon X \to I$.
A morphism in $Arr(C)$ from $\alpha\colon X \to I$ to $\beta\colon Y \to J$ consists of a pair of morphisms, $f\colon X \to Y$ and $g\colon I \to J$.
There is a functor $cod\colon Arr(C) \to C$ which sends an object in $Arr(C)$, say $\alpha\colon X \to I$ to its codomain I, and a morphism, the square, $(f\colon X \to Y, g\colon I \to J)$ to g, the 'lower' edge.
The functor $cod\colon Arr(C) \to C$ is a fibration.

Theorem (Biering, Prop 3.7 p6, 2008)

Let C be a category with finite limits and finite coproducts, and assume that the coproducts are stable and disjoint, then $Dial(cod(C))$ has finite products.

Can we make $Dial(cod(C))$ cartesian closed? Almost...

A comonad in *Dial*(p)

Definition

Let C be a category with finite products and stable, disjoint coproducts. The functor $-+1\colon C \to C$ together with families of maps $\iota\colon X \to X+1$ and $\mu\colon (X+1)+1 \to X+1$ is a monad on C. Define a comonad $L+$ on the subobject fibration $\mathit{Dial}(\mathit{Sub}(C))$ using the monad $-+1$ as follows. Let α be a subobject of $U \times X$ in C , then $L+(\alpha, U, X) = (\alpha+, U, (X+1))$ where $\alpha+$ is reindexing of α along the arrow $U \times (X+1) \cong U \times X + U \to (U \times X) + 1$.

This means that $\alpha^+(u,x) = \alpha(u,x)$, if x is in X, $\top$ if x is in 1. This comonad simply makes the second coordinate well-pointed.

A comonad in *Dial*(p)

The comonad $L+$ just defined is not Girardian, that is we do not have the isomorphism

$$!(A \times B) \cong !A \otimes !B$$

Had this iso being satisfied, then the Kleisli category of the comonad would be cartesian closed. As it is, the best we can do is to have weak function spaces, ie is a retraction

$$C(A \times B, C) \trianglelefteq C(A, [B, C])$$

where the weak function space is given by
$B \to C = (W^V \times (1+Y)^{V \times Z)}, V \times Z, \gamma)$.

Theorem (Biering, 2008, 4.7, p 12)

Let C be a cartesian closed category with finite limits, and stable, disjoint coproducts, which is locally cartesian closed. Then the Dialectica-Kleisli category, $Dial_L + (cod(C))$, which we denote by $Dial+$, has finite products and weak function spaces.

Same is true for $Dial_L + (Sub(C))$.
Main result, proof takes 6 pages, 'souped up' proof of dial thm

Discussion

Examples of fibrations that meet the conditions of Theorem 4.7:

- $cod(PER) \to PER$
- $cod(Set) \to Set$
- the codomain fibration $cod(C) \to C$ for a topos C and
- the subobject fibration $Sub(C) \to C$

Biering decribes example where the fibration is $Fam(PER) \to PER$ (a per is a partial equivalence relation), the product as as well as the weak function space in $Dial^{+}(Fam(PER))$. But calculating there is unwieldy.

Discussion

- Biering concludes: Two new variants of Dialectica categories.
- First, make proof theory out of *Dial*$^+$. One needs to extend Gödel's T with stable, disjoint coproducts and subset types and interpret implication by new weak function space. Advantage: don't need atomic formulas to be decidable.
- Second, make proof theory of *Dial*(p). A type-theoretic version of Dialectica. Instead of formulas over Heyting arithmetic (original dialectica) have dependent types over some type system and Dialectica turns the dependent type system into a lambda-calculus without eta-rule. Advantage?
- Further work: Other comonads for Dialectica? Oliva's work. Do PER model better? Didn't describe structure for generalized dialectica categories, apart from products.

Conclusions

- A worked out example of “proof theory in the abstract”
- dialectica categories *Dial*(C) (1988)
- dialectica pre-ordered fibration *Dial*(p) (2002)
- Dialectica-Kleisli category $Dial^{+}$ (2008)
- More to come Oliva, Hofstra, Triffonov, etc..?

THANK YOU!

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