Lorenzen Games for Full Intuitionistic Linear Logic

Valeria de Paiva

Rearden Commerce

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Outline

Introduction

Classical logic is based on truth values.

Intuitionistic logic is based on proofs.

Lorenzen proposed semantics for both logics based on dialogues/games.

> *Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.*

Goethe

Linear Logic

Girard introduced Linear Logic in 1987, idea comes from semantics of Programming Languages.
Many models for Linear Logic, especially categorical ones, e.g. Dialectica categories (de Paiva 1989).
Blass was the first to propose a semantics for Linear Logic based on dialogues or games (1992).
This started a whole new area of research: Games for semantics of programming languages – some people call them Linear Logic Games.

How do they compare to Hintikka Games, Lorenzen Dialogues/Dialogical games, Ehrenfeucht-Fraïssé games, Economic games, etc..?

Linear Logic

A logic where premises in derivations are resources that need to be accounted for.

Using a sequent calculus to describe Linear Logic, two structural rules are missing: weakening and contraction.

Similar to Relevant logic, but better:

Linear Logic comes with a translation to Intuitionistic Logic that allows you to account for resources when you want to, but be able to prove anything that traditional logic does.
(Relevant logic bans weakening for philosophical reasons. If you ban weakening and contraction your logic is very weak, it does not prove much. LL allows you to recover the expressive power via special modalities, the exponentials.)

Linear Logic Games

Blass (1992) A game semantics for linear logic, (1994) Some Semantical Aspects of Linear Logic

A game (or dialogue) semantics in the style of Lorenzen (1959) for Girard's linear logic (1987).
Lorenzen: the (constructive) meaning of a proposition φ specified by telling how to conduct a debate between a proponent $\mathbf{P}$ who asserts φ and an opponent $\mathbf{O}$ who denies φ.
Propositions are interpreted as games, connectives as operations on games, and validity as existence of a winning strategy for $\mathbf{P}$.
Affine logic (linear logic plus weakening) is sound for this interpretation.
Obtain: a completeness theorem for the additive fragment of affine logic, but completeness fails for the multiplicative fragment.

This work is seminal, spawned a whole area of research, because the semantics is intuitive and the applications seem endless. BUT a big problem: strategies do not compose associatively, no category?...

Lorenzen Dialogues

Basic idea: To understand a (logical) sentence is to know the rules for attacking and defending it in a debate.
Dialogue, debate and game are used synonymously.
The meaning of a propositional connective is given by explaining how to debate a compound formula, assuming that one knows how to debate its constituents.
A game (or an argument, dialogue or protocol) consists of two players, one (the Proponent or Player) seeking to establish the truth of a formula under consideration (trying to prove it) while the other (the Opponent) disputing it, trying to prove it false.
The two players alternate, attacking and defending their positions.

The essence of the semantics consists of rules of the debate between the players.

Lorenzen Dialogues Intuitions

To attack a conjunction, $\mathbf{0}$ may select either conjunct, and $\mathbf{P}$ must then defend that conjunct.

To attack a disjunction, $\mathbf{0}$ may demand that $\mathbf{P}$ select and defend one of the disjuncts.

To attack a negation $\neg A$, $\mathbf{0}$ may assert and defend A, with $\mathbf{P}$ now playing the role of opponent of A.

To attack an implication $A \to B$ $\mathbf{0}$ may assert A; then $\mathbf{P}$ may either attack A, or assert and defend B.

The simplicity of this description of the connectives is deceptive, for Lorenzen needs supplementary rules to obtain a game semantics for *constructive* logic.

Lorenzen games were developed by Lorenz, Felscher and Rahman and co-authors, who established a collection of games for specific (non-classical) logics, including Linear Logic.

This general framework was named Dialogic.

Lorenzen Dialogues

Language L: standard first-order logic connectives $\wedge, \vee, \rightarrow, \neg$

Two labels, $\mathbf{O}$ (Opponent) and $\mathbf{P}$ (Proponent).

Special force symbols: ?... and !... , where ? stands for attack (or question) and ! stands for defense.

The set of rules in dialogic is divided into *particle* rules and *structural* rules.
Particle rules describe the way a formula can be attacked and defended, according to its main connective.
Structural rules specify the general organization of the game.
The difference between classical and intuitionistic logic is in the structural rules.

Lorenzen Dialogues: Structural Rules for Classical Logic

[S1] Proponent $\mathbf{P}$ may assert an atomic formula only after it has been asserted by opponent $\mathbf{O}$.
[S4] A Proponent $\mathbf{P}$-assertion may be attacked at most once.
[S5] Opponent $\mathbf{O}$ can react only upon the *immediately* preceding proponent $\mathbf{P}$-statement.

Lorenzen Dialogues:
Structural Rules for Intuitionistic Logic

[S1] $\mathbf{P}$ may assert an atomic formula only after it has been asserted by $\mathbf{O}$.
[S2] If p is an $\mathbf{P}$-position, and if at round $n-1$ there are several open attacks made by $\mathbf{O}$, then only the latest of them may be answered at n (and the same with $\mathbf{P}$ and $\mathbf{O}$ reversed).
[S3] An attack may be answered at most once.
[S4] A $\mathbf{P}$-assertion may be attacked at most once.
[S5] $\mathbf{O}$ can react only upon the immediately preceding $\mathbf{P}$-statement.
The problem with the structural rules is that it is not clear which modifications can be made to them without changing the set of provable formulas.

Lorenzen Games for Linear Logic

Rahman's definitions of Lorenzen games for Linear logic start from the sequent formulation of the logic.
From the dialogical point of view, assumptions are the Opponent's concessions, while conclusions are the Proponent's claims.
In Linear Logic each occurrence of one formula in a proof must be taken as a distinct resource for the inference process: we must use all and each formula that has been asserted throughout the dialogue. And we cannot use one round more than once.
Linear dialogues are contextual, the flow of information within the proof is constrained by an explicit structure of *contexts*, ordered by a relation of subordination. How contexts are split is essential information for the games.

Lorenzen Games for (multiplicative) Linear Logic

Rules for connectives (particle rules):
Linear implication $\multimap$ and linear negation $()^{\perp}$ are the 'same' as for Intuitionistic and Classical Logic: to attack linear implication, one must assert the antecedent, hoping that the opponent cannot use it to prove the consequent. The defence against such an attack then consists of a proof of the consequent.
The only way to attack the assertion $A^{\perp}$ is to assert A, and be prepared to defend this assertion. Thus there is no proper defence against such an attack, but it may be possible to counterattack the assertion of A.

Lorenzen Games for Linear Logic

Rules for connectives (Particle rules):
Multiplicative conjunction: Tensor
The rule for tensor introduction shows that context splitting for tensor occurs when it is asserted by the Proponent.
so the dialogical particle rule will let the challenger (Opponent) choose the context where the dialogue will proceed.

Lorenzen Games for Linear Logic

Rules for connectives (Particle rules):
Multiplicative Disjunction: Par
The multiplicative disjunction par will generate a splitting of contexts when asserted by the Opponent, thus the particle rule will let the defender choose the context.
Important to notice that while Rahman deals exclusively with classical linear logic, he does mention that an intuitionistic structural rule could be used instead.
This is what we do.

Full Intuitionistic Linear Logic

Full Intuitionistic Linear Logic (FILL): a variant of Linear Logic due to Hyland and de Paiva APAL 1993.
Independent multiplicative (and additive connectives), i.e. independent multiplicative disjunction, conjunction and implication.
motivation: category theory (Dialectica categories)

Games for FILL

Games particle rules for tensor, par and linear implication identical to Rahman's for Linear Logic.
But we use Rahman's intuitionistic structural rules.
Soundness and completeness should follow as for Rahman and Keiff.
Calculations still to be checked...

Conclusions

Games for Linear Logic in two traditions:
Blass-style and Lorenzen-style games (as introduced by Rahman, Keiff).
Emphasis is given to the interpretation of linear implication, (instead of involutive negation) plus structural intuitionistic rule.
These Lorenzen games give us the ability to model full intuitionistic linear logic easily.
Preliminary: still need to prove our soundness and completeness
More importantly: need to make sure that strategies are compositional, to make sure we can produce **categories** of Lorenzen games.