Equivalence of Logics:

the categorical proof theory perspective

Valeria de Paiva,

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Problem we want to solve

- Q: When should two logics L and L' be called equivalent?
- If we **Assume*** logics come with 'their' own (unique) category of (categorical) models. L → Mod(L)

  L'→ Mod(L')
- Could say L equiv L' if Mod(L) equivalent to Mod(L')
- Then Q transformed to Q': When should two (classes of) categorical models be called equivalent?
- Our methodology: Categorical Proof Theory
- Results for intuitionistic linear logic

Outline

- Categorical Proof Theory (CPT)
- Successes & Challenges of CPT
- Intuitionistic Linear Logic
- Results for Intuitionistic Linear Logic
- Future Directions?

Categorical Logic

Use of Category Theory, a subfield of Algebra, in Logic.

Two main strands:

Categorical Model Theory

Categorical Proof Theory

Both are called **Categorical Semantics.** Leads to

- categorical semantics of programming languages
- categorical semantics of specification, security, concurrency…
- (at large) functional programming, language design, interactive theorem proving, etc.

Categorical Proof Theory

Categorical proof theory models derivations/proofs, not whether theorems are true or not

Proofs definitely first-class citizens

How? Uses extended Curry-Howard isomorphism

Why is it good? Modeling derivations useful in linguistics, functional programming, etc

Why is this important? Widespread use of logic in CS means more than jobs for logicians, means new important problems to solve with our favorite tools.

Why there is little impact on Logic itself?

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Successes and Challenges of CPT

- Successes
  - Models for the untyped lambda-calculus
  - Typed programming language & Typed polymorphism
  - Dependent Type Theory
  - Operational Semantics & Full abstraction results
  - Game Semantics
- Challenges
  - Proof theory of Classical Logic
  - Proof theory of Modal Logics
  - Effect full computation, mobile computing, etc

One Big Success

- For intuitionistic logic IL have extended Curry-Howard isomorphism
- For IL have a unique most general class of categorical models, Cartesian Closed Categories
- Can prove soundness and completeness of categorical models with respect to term calculus
- Can prove other models are instances of most general model CCC
- Back to original problem…

Back to Problem we want to solve

- Q: When should two logics L and L′ be called equivalent?
- **Assume** logics come with 'their' own (unique) class of categorical models L → Mod(L)

  L′ → Mod(L′)
- Say L equiv L′ if Mod(L) equivalent to Mod(L′)
- transformed Q into Q′: When should two (classes of) categorical models be called equivalent?

(this is the problem we were originally trying to solve for linear log)

Equivalence of Logics: Our Ideal Solution

If both our logics L and L′ are **like** IL
Construct category of theories of L, L′: Th(L)
Construct category of models of L, L′: Mod(L)

Prove: Categorical equivalence between Th(L) and Mod(L) called "internal language criterion"

Define: semantics of L=class of models <u>uniquely</u> identified by internal language as most general

Then: L and L′ are equiv iff Th(L) equiv Th(L′)
L and L′ equiv => Mod(L) equiv Mod(L′)

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Problem with our Ideal Solution

Which logics are like IL? For which logics can do the steps below?

Must: Construct category of theories of L Th(L)
Construct category of models of L Mod(L)

Prove: Categorical equivalence between Th(L) and Mod(L) or "internal language criterion"

I warned you: intuitions from Linear Logic

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Intuitionistic Linear Logic (ILL)

Linear Logic interesting case for a semantics of proofs

- Curry-Howard correspondence well-studied
- Categorical modeling of !-free fragment uncontroversial
- But ! is the way to recover classical logical expressivity, must deal with it
- Challenges:
    - Three versions of ND for intuitionistic linear logic: ILL, LNL, DILL
    - Three notions of categorical model for intuitionistic linear logic
    - In which sense are they equivalent? Which is best? Why?

Back to "Problem we want to solve"

- Had **"Assume** logics come with 'their' own (unique) class of categorical models. $L \rightarrow Mod(L)$

  $L' \rightarrow Mod(L')$"

- Assumption above is not valid: intuitionistic linear logic comes with three (equivalent?!) classes of models

- Q': When should two (classes of) categorical models be called equivalent?

- Results for intuitionistic linear logic

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Results: Categorical Models of Linear Logic

- The uncontroversial !-free fragment
- System ILL
- System LNL
- System DILL
- Summing up

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Categorical Models for !-free Linear Logic

- Call **RLL** the fragment of the logic with only linear implication, tensor and unit I

- **RLL** is modeled by symmetric monoidal closed categories or smccs

- An smcc is just like a ccc, except that as we have tensor products instead of cartesian products, we do not have projections or diagonals

  → The logic we're modeling does not satisfy A|- A &A or A&B |-A

- Symmetric monoidal closed categories form a category SMC

- **RLL** is sound and complete with respect to smccs (Szabo 1978)

- **Theorem**: **RLL** satisfies the Internal Language Criterion for SMC,

SMC = Mod (RLL) equiv Th(RLL)

(Maietti et al 01, Mackie et al 93?)

Categorical Models for Linear Logic II

- Call **ILL** the term calculus for the logic given by (Benton, Bierman, Hyland and de Paiva 1993)
- **ILL** is modeled by linear categories (Bierman), symmetric monoidal closed categories with a *linear exponential comonad*.
- The linear exponential comonad equips each object of the category with maps er:!A →I, dupl:!A→!A!A, eps:!A→A, delta:!A→!!A,

  → Objects !A have weakening, contraction, promotion, dereliction
- Linear categories form a category LIN
- **ILL** is sound and complete with respect to LIN (Bierman 1994)
- **Theorem**: **ILL** satisfies the Internal Language Criterion for LIN,

  LIN = Mod (ILL) equiv Th(ILL) (Maietti et al 01)

Categorical Models for Linear Logic III

- Call **LNL** the term calculus for the logic given by (Benton, 1995)
- **LNL** is modeled by a symmetric monoidal adjunction (F-|G) between an smcc and a cartesian **closed** category.
- The monoidal adjunction relates two worlds: the linear and the cartesian/intuitionistic one and makes the definition of categorical model much shorter
- Benton adjunctions form a category ADJ_LNL
- **LNL** is sound and complete with respect to ADJ_LNL (Benton 1994)
- **Theorem**: **LNL** satisfies the Internal Language Criterion for ADJ_B,

  ADJ_LNL = Mod (LNL) equiv to Th(LNL)   (Maietti et al 01)

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Categorical Models for Linear Logic IV

- Call **DILL** the term calculus for the logic given by (Barber, 1997)
- **DILL** is modeled by a symmetric monoidal adjunction (F-|G) between an smcc and a cartesian (**not necessarily closed**) category.
- The monoidal adjunction relates two worlds: the linear and the cartesian/intuitionistic one and makes the definition of categorical model much shorter
- These adjunctions form a category ADJ
- **DILL** is sound and complete with respect to ADJ (Barber1997)
- But NO **Theorem**:

DILL does not satisfy the Internal Language Criterion for ADJ,

ADJ = Mod (DILL) NOT equiv Th(DILL) (Maietti et al 01)

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But Problem Results

Pure type theory tells us

THEOREM: The category of theories of ILL Th(ILL) is equivalent to the category of theories of DILL, Th(ILL) equiv Th(DILL)

Pure category theory tells us
THEOREM*: The category LIN (of linear categories) is isomorphic to a full **sub**category of ADJ (symmetric monoidal adjunctions between a smmc and a cartesian category).

Hence: Two logics whose categories of theories are equivalent, but whose classes of models are not??!!

Solution of problem with Linear Logic

Carve out from ADJ the categories for which DILL is the internal language really

THEOREM: The category ADJ_DILL is the **sub**category of ADJ (symmetric monoidal adjunctions between a smmc and a cartesian category) corresponding to the theories of DILL Th(DILL).

[ADJ_DILL defined via finite product sym mon adjunctions (Hyland)]

Now:

LIN=Mod(ILL) equiv Th(ILL) equiv Th(DILL) equiv Mod(DILL)=ADJ_DILL

Add products and can relate LNL too:

LIN equiv ADJ_DILL both full subcategories of ADJ_LNL

using generic CT theorems about Eilenberg-Moore adjunctions

Back to Problem we want to solve

- Q: When should two logics L and L′ be called equivalent?
- Say L equiv L′ if Mod(L) equiv to Mod(L′) &

  L,L′ models satisfy the internal language criterion
- Our ideal solution works for linear logic now
- Recall

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Our Ideal Solution (Again)

If both our logics L and L′ are like IL, ILL
Construct category of theories of L, L′: Th(L)
Construct category of models of L, L′: Mod(L)

Prove: Categorical equivalence between Th(L) and Mod(L) or "internal language criterion"

Define: semantics of L=class of models <u>uniquely</u> identified by internal language as most general

Then: L and L′ are equiv iff Th(L) equiv Th(L′)

Hence DILL=ILL same logic, but LNL not

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How Far does Our Ideal Solution go?

If embracing fully Categorical Proof Theory must have **semantics of proofs** for logics L, L'

> Research Program: for **which** logics can we have a semantics of proofs?

But more (or less?) interesting, if do not have semantics of proofs, can still prove internal language-like criterion and have notion of equivalence of logics: Maietti's submission to the Contest

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Conclusions

Proposed a more stringent criterion than soundness and completeness for categorical modeling of logic

Used this criterion to classify models of intuitionistic linear logic

Showed that in this framework have a sensible notion of equivalence of logics

Suggested a similar criterion could be used without semantics of proofs, only with categories of theories and models, cf. Maietti's contribution.

Need to work out applicability of criterion in both cases.

Thank you!

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