Dialectica Categories... and Lax Topological Spaces?

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March, 2012


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Goals

Describe two kinds of dialectica categories

Show they are models of ILL and CLL, respectively

Compare and contrast dialectica categories with Chu spaces

Discuss how we might be able to use these ideas for topological spaces

Functional Interpretations and Gödel's Dialectica

Starting with Gödel's Dialectica interpretation(1958) a series of "translation functions" between theories

Avigad and Feferman on the Handbook of Proof Theory:

> *This approach usually follows Gödel's original example: first, one reduces a classical theory C to a variant I based on intuitionistic logic; then one reduces the theory I to a quantifier-free functional theory F.*

Examples of functional interpretations:

- Kleene's realizability
- Kreisel's modified realizability
- Kreisel's No-CounterExample interpretation
- Dialectica interpretation
- Diller-Nahm interpretation, etc...

Gödel's Dialectica Interpretation

Gödel's Aim: liberalized version of Hilbert's programme – to justify classical systems in terms of notions as intuitively clear as possible.

Gödel's **result:** an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type T
idea: translate every formula A of HA to $A^D = \exists u \forall x. A_D$ where A_D is quantifier-free.
use: If HA proves A then T proves $A_D(t, y)$ where y is a string of variables for functionals of finite type, t a suitable sequence of terms not containing y.

Method of 'functional interpretation' extended and adapted to several other theories systems, cf. Feferman.

Gödel's Dialectica Interpretation

For each formula A of HA we associate a formula of the form $A^D = \exists u \forall x A_D(u,x)$ (where A_D is a quantifier-free formula of Gödel's system T) inductively as follows: when A_{at} is an atomic formula, then its interpretation is itself.

Assume we have already defined $A^D = \exists u \forall x.A_D(u,x)$ and $B^D = \exists v \forall y.B_D(v,y)$.

We then define:

$$(A \wedge B)^D = \exists u, v \forall x, y.(A_D \wedge B_D)$$
$$(A \to B)^D = \exists f\colon U \to V, F\colon U \times X \to Y, \forall u, y.$$
$$(\ \mathsf{A}_D(u, F(u,y)) \to B_D(fu;y))$$
$$(\forall z A)^D(z) = \exists f\colon Z \to U \forall z, x.A_D(z, f(z), x)$$
$$(\exists z A)^D(z) = \exists z, u \forall x.A_D(z,u,x)$$

The intuition here is that if u realizes $\exists u \forall x.A_D(u,x)$ then $f(u)$ realizes $\exists v \forall y.B_D(v,y)$ and at the same time, if y is a counterexample to $\exists v \forall y.B_D(v,y)$, then $F(u,y)$ is a counterexample to $\forall x.A_D(u,x)$.

Gödel's Dialectica Interpretation: logical implication

Most interesting aspect: If $A^D = \exists u \forall x . A_D$ and $B^D = \exists v \forall y . B_D$

$$(A \Rightarrow B)^D = \exists V(u) \exists X(u,y) \forall u \forall y [A_D(u, X(u,y)) \Rightarrow B_D(V(u), y)]$$

This can be explained (Spector'62) by the following equivalences:

$$\begin{aligned}
(A \Rightarrow B)^D &= [\exists u \forall x A_D \Rightarrow \exists v \forall y B_D]^D && (1)\\
&= [\forall u (\forall x A_D \Rightarrow \exists v \forall y B_D)^D] && (2)\\
&= [\forall u \exists v (\forall x A_D \Rightarrow \forall y B_D)]^D && (3)\\
&= [\forall u \exists v \forall y (\forall x A_D \Rightarrow B_D)]^D && (4)\\
&= [\forall u \exists v \forall y \exists x (A_D \Rightarrow B_D)]^D && (5)\\
&= \exists V \exists X \forall u, y (A_D(u, X(u,y)) \Rightarrow B_D(V(u), y)] && (6)
\end{aligned}$$

All steps classically valid. But:
Step (2) (moving $\exists v$ out) is IP (not intuitionistic),
Step (4) uses gener. of Markov's principle (not intuitionistic)
for (6) need to skolemize twice, using AC (not intuitionistic).

Basics of Categorical Semantics

Model theory using categories, instead of sets or posets.

Two main uses "categorical semantics":
categorical proof theory, and categorical model theory

Categorical proof theory models derivations (proofs) not simply whether theorems are true or not

Categorical Semantics: what?

Based on Curry-Howard Isomorphism:

Natural deduction (ND) proofs	$\Leftrightarrow$	λ-terms
Normalization of ND proof	$\Leftrightarrow$	reduction in λ-calculus.

Originally: constructive prop. logic $\Leftrightarrow$ simply typed λ-calculus.
Applies to other (constructive) logics and λ-calculi, too.

Cat Proof Theory:
λ-calculus types $\Leftrightarrow$ objects in (appropriate) category
λ-calculus terms $\Leftrightarrow_{1-1}$ morphisms in category
Cat. structure models logical connectives (type operators).

Isomorphism: transfers results/tools from one side to the other

λ-calculus basis of functional programming: *logical* view of programming

Where is my category theory?

Categorical Semantics: a picture

$$\begin{tikzcd} & \lambda - calculus \arrow[dl, dash] \arrow[dr, dash] & \\ CartClosedCat \arrow[rr] & & IntPropLogic \end{tikzcd}$$

Framework connecting

$$\begin{tikzcd} & Functional\ Program \arrow[dl, dash] \arrow[dr, dash] & \\ Category\ Theory \arrow[rr] & & Logic \end{tikzcd}$$

lots of important stuff elided

Categorical Logic?

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Well known that we can do categorical models as well as set theoretical ones.

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E.g. Scott's λ-calculus

Categorical Semantics: some gains..

Functional Programming: optimizations that do not compromise semantic foundations (e.g. referential transparency)

Logic: new applications of old theorems (e.g. normalization)

Category Theory: new concepts not from maths

Philosophy: new criteria for identities of proofs

Basics of Linear Logic

A new(ish..) logic taking resource sensitivity into account

Developed mainly from proof theoretic perspective:
— must keep track of where premises / assumptions are used

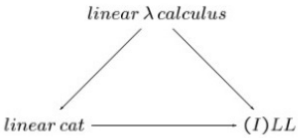
But ability to ignore resource sensitivity whenever you want:

(a) Modal ! allowing duplication and erasing of resources
(b) $A \Rightarrow B \;=\; !A \multimap B$

Where is my category theory?

Categorical Semantics of (Intuitionistic) Linear Logic

Want linear version of Extended Curry-Howard Isomorphism



- Logic intuitionistic
- Already have Intuitionistic Linear Logic (ILL),
  — must find other two sides of triangle...

Linear categories

Model multiplicative-additive fragment is easy

- multiplicative fragment: need a a symmetric monoidal closed category (SMCC)
- additive fragment: need (weak) categorical products and coproducts
- linear negation: need an involution

modalities (exponentials): the problem...

several solutions, depending on linear λ-calculus chosen...

Symmetric Monoidal Closed Categories (SMCC)

A category $\mathcal{L}$ is a symmetric monoidal closed category if

- it has a tensor product $A \otimes B$ for all object A,B of $\mathcal{C}$.
- for each object B of $\mathcal{C}$, the tensor-product functor $\_ \otimes B$ has a right-adjoint, written $B \multimap \_$
- (ie have isomorphism between $(A \otimes B) \to C$ and $A \to (B \multimap C)$)

Recap:

	Categorical product $A \times B$	Tensor product $A \otimes B$
1	Projections onto A and B	Not necessarily
2	For any object C with maps to A and B, there is a *unique* map from C to $A \times B$	Not necessarily

Modelling the Modality !

Objects A in a CCC have duplication $A \to A \times A$ and erasing $A \to 1$ (satisfying equations of a comonoid).

Objects A in a SMCC have no duplication $A \to A \otimes A$ nor erasing $A \to I$, in general. Neither have they projections $A \otimes B \to A, B$.

But objects of the form $!A$ should have duplication and erasing. Moreover they must satisfy S4-Box introduction and elimination rules.

S4-axioms give you proofs $!A \to A$ and $!A \to !!A$, uniformly for any A object A in $\mathcal{C}$. Hence want a functor (unary operator) on category, $! \colon \mathcal{C} \to \mathcal{C}$ such that there are such natural transformations. Such a structure already in category theory, a *comonad*.

Modelling the Modality !

An object $!A$ must have four maps:

$$\epsilon\colon !A \to A, \;\; \delta\colon !A \to !!A, \;\; \mathsf{er}\colon !A \to I, \;\; \mathsf{dupl}\colon !A \to !A \otimes !A$$

To make identity of proofs sensible, must have lots of equations. The ones relating to duplicating and erasing say $!A$ is a comonoid with respect to tensor; the ones relating to S4-axioms say ! is a *comonad*.

First idea (Seely'87): Logical equivalence $!(A\&B) \equiv !A \otimes !B$
Introduce a comonad $(!, \delta, \epsilon)$ defining ! such that

$$!(A \times B) \cong !A \otimes !B$$

Say this comonad takes additive comonoid structure
$(A, \;\; A \to A \times A, \;\; A \to 1)$ (if it exists) to desired comonoid
$(!A, \;\; \mathsf{dupl}\colon !A \to !A \otimes !A, \;\; \mathsf{er}\colon !A \to I)$.

Categorical Semantics of LL: problem

Modelling uses isomorphism $!(A \& B) \cong !A \otimes !B$

But can't model multiplicatives & modalities without additives.

So, better idea:
If we don't have additive conjunction, ask for morphisms

$$m_{A,B} \colon \; !(A \otimes B) \to !A \otimes !B$$

$$m_I \colon !I \to I$$

i.e. *monoidal* comonad to deal with contexts

Must also ask for (coalgebra) conditions relating comonad structure to comonoid structure.
Hence:

Categorical Semantics of LL: solution

A *linear category* comprises
— an SMCC $\mathcal{C}$, (with products and coproducts),
— a symmetric monoidal comonad $(!, \epsilon, \delta, m_I, m_{A,B})$
(a) For every free co-algebra $(!A, \delta)$ there are nat. transfns.
$\mathsf{er}_A : !A \to I$ and $\mathsf{dupl}_A : !A \to !A \otimes !A$, forming a commutative comonoid, which are coalgebra maps.
(b) Every map of free coalgebras is also a map of comonoids.
Bierman 1995 (based on Benton et al'93)

long definition, lots of diagrams to check..
Prove $A \Rightarrow B \cong !A \multimap B$ by constructing CCC out of SMCC plus comonad !. Extended Curry-Howard works!!

Classical linear category = linear category with involution, $()^\perp : \mathcal{C}^{op} \to \mathcal{C}$, to model negation, same as $*$-autonomous category, with (weak) products and coproducts plus a symm monoidal comonad satisfying (a) and (b) above.

Alternative Categorical Semantics

(Benton 1995) A model of LNL (Linear/NonLinear Logic) consists of a SMC category $\mathbf{L}$ together with a cartesian closed category $\mathbf{C}$ and a monoidal adjunction between these categories.

Modelling two logics, intuitionistic and linear at the same time. Logics on (more) equal footing..

First definition builds the cartesian closed category out of the monoidal comonad, this def asks for the adjunction

Easier to remember, easier to say, same diagrams to check..

Extended Curry-Howard works (Barber's thesis)

(reduction calculus (+explicit subst) needed for implementation, Wollic'98)

classical version: *-autonomous cat, plus monoidal adjunction, plus products

Categorical Semantics: Back to Dialectica

Have a definition of what a categorical model of ILL ought to be, can we find some concrete ones?

Dialectica categories (Boulder'87) one such model..

Assume C is cartesian closed category

DC objects are relations between U and X in C,
monics $A\alpha U \times X$ written as $(U\alpha X)$.
DC maps are pairs of maps of C
$f\colon U \to V$, $F\colon U \times Y \to X$
making a certain pullback condition hold.
Condition in Sets: $u\alpha F(u,y) \Rightarrow f(u)\beta y$.

Reformulating DC

C is $\mathbf{Sets}$: objects are maps $A = U \times X\alpha 2$, and $B = V \times Y\beta 2$. morphisms are pairs of maps $f\colon U \to V$, $F\colon U \times Y \to X$ such that $\alpha(u, F(u,y)) \leq \beta(f(u), y)$:

$$\begin{tikzcd}
U \times Y \arrow[r, "\langle \pi_1, F \rangle"] \arrow[d, "f \times id_Y"'] & U \times X \arrow[d, "\alpha"] \\
V \times Y \arrow[r, "\beta"'] & 2
\end{tikzcd}$$

Usually write maps as

$$\begin{tikzcd}
U \arrow[d, "f"'] & \alpha & X \\
 & \Downarrow & \\
V & \beta & Y \arrow[uu, "F"']
\end{tikzcd}
\qquad \forall u \in U, \forall y \in Y \quad \alpha(u, F(u,y)) \leq \beta(f(u), y)$$

Relationship to Dialectica

Objects of DC represent essentially A^D.
Maps some kind of normalisation class of proofs.
abstractly: maps are realisation of the formula $A^D \to B^D$.
Expected: cat DC is ccc, as Gödel int. as constructive as possible.
Surprise: DC is a model of ILL, not IL.
But (very neat result!)
Diller-Nahm variant gives CCC.

Note: reformulation makes basic predicates A_D decidable, as in Gödel's work, but first version is more general.

Structure of DC

DC is a SMCC with categorical products:

$$A \oslash B = (U \times V \alpha \oslash \beta X \times Y)$$
$$A \multimap B = (V^U \times X^{U \times Y} \alpha \multimap \beta U \times Y)$$
$$A \& B = (U \times V \alpha \beta X + Y)$$

only weak coproducts:

$$A + B = (U + V \alpha + \beta X^U \times Y^V)$$

Morever

$$!A = (U \alpha^* X^*)$$

is cofree comonoid on A.

Conjunction: neat detail

$A \not\vdash A \oslash A$ as need a DC map (Δ, δ) such that

$$\begin{tikzcd}
U \arrow[d, "\Delta"'] \arrow[r, phantom, "\alpha"] & X \\
U \times U \arrow[r, phantom, "\alpha \oslash \alpha"] & X \times X. \arrow[u, "\delta"']
\end{tikzcd}$$

$\forall u \in U, \forall x, x' \in X \times X \quad \alpha(u, \delta(u, x, x')) \Rightarrow (\alpha \oslash \alpha)(\Delta(u), (x, x'))$.
But $(\alpha \oslash \alpha)(\Delta(u), (x, x')) = \alpha(u, x) \wedge \alpha(u, x')$
hence unless α decidable can't do anything...
If α decidable define:
$\delta(u, x, x') = x'$ if $\alpha(u, x)$
$\delta(u, x, x') = x$ otherwise. Same logical argument..

Dialectica categories DC: summary

DC pluses:
structure needed to model ILL,
including non-trivial modality !
related to Diller-Nahm variant
conjunction explained
independent confirmation of LL's worth;
Dialectica is interesting (and mysterious).
DC minuses:
accounting of recursion?
mathematics involved;
terse style – AMS vol.92

Dialectica model GC

Girard's suggestion '87

C cartesian closed category, say **Sets**.

GC objects are relations between U and X, in C as before.
GC maps are pairs of maps of C, $f\colon U \to V$, $F\colon Y \to X$ such that $u\alpha F(y) \Rightarrow f(u)\beta y$.

As before, can 'reformulate' and say A is map $U \times X \to 2$

Structure of GC

GC is $*$-autonomous with products and coproducts. Involution is given by implication into $\perp$, unit for extra monoidal structure $\invamp$.

$$\begin{array}{rcl} A \multimap B & = & (V^U \times X^Y \alpha \multimap \beta U \times Y) \\ A \otimes B & = & (U \times V \alpha \otimes \beta X^V \times Y^U) \\ A \invamp B & = & (V^X \times U^Y \alpha \invamp \beta X \times Y) \\ A \& B & = & (U \times V \alpha . \beta X + Y) \\ A \oplus B & = & (U + V \alpha . \beta X \times Y) \end{array}$$

Moreover, GC has Exponentials

$$!A = (U !\alpha (X^*)^U)$$

$$?A = ((U^*)^X ?\alpha X)$$

Dialectica-like categories GC

GC pluses:
the best model of LL ?..
no collapsing of unities
maths easier for modality-free fragment
modalities adaptable
should think of Sets!
GC minuses:
MIX rule
C must be ccc

Chu construction $\mathsf{Chu}_K\mathsf{C}$

Suppose C is cartesian closed category, (but could be smcc with pullbacks) and K any object of C.
$\mathsf{Chu}_K\mathsf{C}$ construction in Barr's book, 1972.
$\mathsf{Chu}_K\mathsf{C}$ objects are (generalised) relations between U and X or maps of the form $U \times X \alpha K$.
$\mathsf{Chu}_K\mathsf{C}$ maps are pairs of maps of C $f\colon U \to V$, $F\colon Y \to X$ such that

$$u\alpha F(y) = f(u)\beta y$$

To make comparison with GC transparent read equality as

$$u\alpha F(y) \Leftrightarrow f(u)\beta y)$$

Comparing GC and $\mathrm{Chu}_K\mathrm{C}$

$\mathrm{Chu}_K\mathrm{C}$ pluses: smcc instead of ccc
topological spaces, plus other representabilities (see Pratt)
$\mathrm{Chu}_K\mathrm{C}$ minuses:
No K can make I and $\perp$ different!!
modalities Barr(1990) harder, also more pullbacks.
Lafont and Streicher (LICS'91), Chu_K**Sets** is GAME_K, but with modality based on GC
Structure:

$$A \multimap_{Chu} B = (\mathcal{L}_1 \alpha \multimap_C \beta U \times Y)$$

where $\mathcal{L}_1 = pullback \subset V^U \times X^Y$
similarly for par and tensor..

A mild generalisation of GC

Categories Dial$_L$C, where C is smcc with products and L a closed poset (or lineale).
Objects of Dial$_L$C:
generalised relations between U and X or maps of the form

$$U \otimes X \alpha L$$

Dial$_L$C maps are pairs of maps of C $f\colon U \to V$, $F\colon Y \to X$
such that $u\alpha F(y) \Rightarrow f(u)\beta y$
(Now implication inherited from L).

Vicker's Topological Systems

A triple (U, α, X), where X is a *frame*, U is a set, and $\alpha : U \times X \to 2$ a binary relation, is called by Vicker's a **topological system** if

when S is a finite subset of X, then

$$\alpha(u, \bigwedge S) \leq \alpha(u, x) \text{ for all } x \in S.$$

when S is any subset of X, then

$$\alpha(u, \bigvee S) \leq \alpha(u, x) \text{ for some } x \in S.$$

$\alpha(u, \top) = 1$ and $\alpha(u, \bot) = 0$ for all $u \in U$.

A Frame, did you say?

A partially ordered set $(X, \leq)$ is a *frame* if and only if

Every subset of X has a join;

if S is a finite subset subset of X it has a meet;

Binary meets distribute over joins:
$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$

Frames are sometimes called locales. They confuse me a lot.

Maps of Topological Systems

If (U, α, X) and (V, β, Y) are **topological systems**, a map consists of $f : U \to V$ and $F \to X$ such that $u\alpha Fy = fu\beta y$.
As in Chu spaces this condition can be broken down into two halves: $u\alpha Fy$ implies $fu\beta y$ and $fu\beta y$ implies $u\alpha Fy$.
If we only have one half, we are back into a dialectica-like situation.

Call this the category of *lax* topological systems, $\mathbf{TopSys}_{\mathcal{L}}$.
What can we prove about $\mathbf{TopSys}_{\mathcal{L}}$?
We know it has an SMCC structure.
Anything else? Would topologists be interested?

Further Work?

Presented 2 kinds of dialectica categories and four theorems about them, explaining where they came from

Hinted at some others...

Not enough work done on these dialectica categories all together

Iteration,(co-)recursion not touched, spent three days thinking about it...

We're thinking about instances of the construction appropriate for cardinal invariants

Also about doing primitive recursion categorically

Work on logical extensions, model theory should be attempted (van Benthem little paper)

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