Dialectica Models of the Lambek Calculus Revisited

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Introduction

This note recalls a Dialectica model of the Lambek Calculus presented by the first author in the Amsterdam Colloquium 1991. We approach the Lambek Calculus from the perspective of Linear Logic. In that earlier work we took for granted the syntax and only worried about the exciting possibilities of new models of Linear Logic-like systems.

Twenty five years later we find that the work is still interesting and that it might inform some of the most recent work on word vectors. But the Amsterdam Colloquium proceedings were never published and not even the author had a copy of the paper. So we have decided to revisit some of the old work, this time using the new tools that have been developed for type theory and proof systems in the time that elapsed. Thus, we implemented the calculus in Agda and we use `Ott` [1] to check that we do not have silly mistakes in our term systems. The goal is to see if our new implementations can shed new light on some of the issues that remained open on the applicability and fit of the systems to their intended uses.

Historical Overview

The Syntactic Calculus was first introduced by Jim Lambek in 1958, now known as the Lambek Calculus, and is an explanation of the mathematics of sentence structure. After a long period of ostracism, around 1980 the Lambek Calculus was taken up by logicians interested in Computational Linguistics, especially the ones in the area of Categorial Grammar.

The work on Categorial Grammar was given a serious impulse by the advent of Girard's Linear Logic at the end of the 1980s. Girard showed that there is a full embedding, preserving proofs, of Intuitionistic Logic into Linear Logic with a modality "!", which meant that one could consider several systems of resource logics. These refined resource logics were applied to several areas of Computer Science.

The Lambek calculus has seen a significant number of works written about it, quite apart from a number of monographs that deal with logical and linguistic aspects of the generalized type-logical approach. For general background on the type-logical approach, there is a wealth of information in the monographs of Moortgat, Morril, Carpenter and Steedman [**?**]. For a shorter introduction, see Moortgat's chapter on the Handbook of Logic in Language [**?**].

Type Logical Grammar situates the type-logical approach within the framework of Montague's Universal Grammar and presents detailed linguistic analyses for a substantive fragment of syntactic and semantic phenomena in the grammar of English. Type Logical Semantics offers a general introduction to natural language semantics studied from a type-logical perspective.

This meant that a series of systems, implemented or not, were devised that used the Lambek Calculus or variants of Linear Logic. These systems can be as expressive as Intuitionistic Logic and the claim is that they are more precise i.e. they make finer distinctions. From the beginning it was clear that the Lambek Calculus is the multiplicative fragment of non-commutative Intuitionistic Linear Logic. Hence several interesting questions, considered for Linear Logic, could also be asked of the Lambek Calculus. One of them, posed by Morrill et al is whether we can extend the Lambek calculus with a modality that does for the structural rule of *(exchange)* what the modality *of course* '!' does for the rules of *(weakening)* and *(contraction)*. A very preliminary proposal, which answers this question affirmatively, is set forward in this paper. The 'answer' was provided in semantical terms in the first version of this work. Here we provide also the more syntactic description, building on work of Galatos and others.

We first recall Linear Logic and provide the transformations to show that the Lambek Calculus L really is the multiplicative fragment of non-commutative Intuitionistic Linear Logic. Then we describe the usual String Semantics for the Lambek Calculus L and generalize it, using a categorical perspective in the second section. The third section recalls our Dialectica model for the Lambek Calculus. Finally, in the fourth section we discuss modalities and some untidiness of the Curry-Howard correspondence for the fragments of Linear Logic in question.

1 The Lambek Calculus

The Lambek Calculus, formerly the Syntactic Calculus, is due to Jim Lambek [**?**]. His idea was to capture the logical structure of sentences, and to do this he introduced a substructural logic with an operator denoting concatenation, $A \otimes B$, and two implications relating the order of phrases, $A \leftharpoonup B$ and $A \rightharpoonup B$, where the former is a phrase of type A when followed by a phrase of type B, and the latter is a phrase of type B when proceeded by a phrase of type A.

It turns out that the Lambek Calculus can be presented as a non-commutative intuitionistic multiplicative linear logic. The syntax of formulas and contexts of the logic are as follows:

$$\begin{array}{ll} \text{(formulas)} & A, B, C ::= I \mid A \otimes B \mid A \leftharpoonup B \mid A \rightharpoonup B \\ \text{(contexts)} & \Gamma ::= A \mid \Gamma_1, \Gamma_2 \end{array}$$

$$\frac{}{A \vdash A}\ \textsc{ax} \qquad \frac{\Gamma_1 \vdash A \qquad A, \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash B}\ \textsc{cut} \qquad \frac{\Gamma \vdash A}{\Gamma, I \vdash A}\ \textsc{unit}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \mathrm{T}\textsc{l} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \mathrm{T}\textsc{r}$$

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \rightharpoonup B \vdash C}\ \mathrm{IR}\textsc{l} \qquad \frac{\Gamma_1 \vdash A \qquad B, \Gamma_2 \vdash C}{\Gamma_1, \Gamma_2, A \leftharpoonup B \vdash C}\ \mathrm{IL}\textsc{l}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightharpoonup B}\ \mathrm{IR}\textsc{r}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \leftharpoonup B}\ \mathrm{IL}\textsc{r}$$

Figure 1: The Lambek Calculus: L

The inference rules are defined in Figure 1. Because the operator $A \otimes B$ denotes the type of concatenations the types $A \otimes B$ and $B \otimes A$ are not equivalent, and hence, L is non-commutative which explains why implication must be broken up into two operators $A \leftharpoonup B$ and $A \rightharpoonup B$.

2 Algebraic Semantics

3 Dialectica Lambek Spaces

4 MultiModalities

5 Conclusion

References

[1] P. Sewell, F. Nardelli, S. Owens, G. Peskine, T. Ridge, S. Sarkar, and R. Strnisa. Ott: Effective tool support for the working semanticist. In *Journal of Functional Programming*, volume 20, pages 71–122, 2010.