Dialectica Models of the Lambek Calculus Revisited

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Introduction

This note recalls a Dialectica model of the Lambek Calculus presented by the first author in the Amsterdam Colloquium 1991. We approach the Lambek Calculus from the perspective of Linear Logic. In that earlier work we took for granted the syntax and only worried about the exciting possibilities of new models of Linear Logic-like systems.

Twenty five years later we find that the work is still interesting and that it might inform some of the most recent work on word vectors. But the Amsterdam Colloquium proceedings were never published and not even the author had a copy of the paper. So we have decided to revisit some of the old work, this time using the new tools that have been developed for type theory and proof systems in the time that elapsed. Thus, we implemented the calculus in Agda and we use `Ott` [4] to check that we do not have silly mistakes in our term systems. The goal is to see if our new implementations can shed new light on some of the issues that remained open on the applicability and fit of the systems to their intended uses.

Historical Overview

The Syntactic Calculus was first introduced by Jim Lambek in 1958 [3], now known as the Lambek Calculus, and is an explanation of the mathematics of sentence structure. After a long period of ostracism, around 1980 the Lambek Calculus was taken up by logicians interested in Computational Linguistics, especially the ones in the area of Categorial Grammar.

The work on Categorial Grammar was given a serious impulse by the advent of Girard's Linear Logic at the end of the 1980s. Girard showed that there is a full embedding, preserving proofs, of Intuitionistic Logic into Linear Logic with a modality "!", which meant that one could consider several systems of resource logics. These refined resource logics were applied to several areas of Computer Science.

The Lambek calculus has seen a significant number of works written about it, quite apart from a number of monographs that deal with logical and linguistic aspects of the generalized type-logical approach. For general background on the type-logical approach, there is a wealth of information in the monographs of Moortgat, Morril, Carpenter and Steedman. For a shorter introduction, see Moortgat's chapter on the Handbook of Logic in Language [**?**].

Type Logical Grammar situates the type-logical approach within the framework of Montague's Universal Grammar and presents detailed linguistic analyses for a substantive fragment of syntactic and semantic phenomena in the grammar of English. Type Logical Semantics offers a general introduction to natural language semantics studied from a type-logical perspective.

This meant that a series of systems, implemented or not, were devised that used the Lambek Calculus or variants of Linear Logic. These systems can be as expressive as Intuitionistic Logic and the claim is that they are more precise i.e. they make finer distinctions. From the beginning it was clear that the Lambek Calculus is the multiplicative fragment of non-commutative Intuitionistic Linear Logic. Hence several interesting questions, considered for Linear Logic, could also be asked of the Lambek Calculus. One of them, posed by Morrill et al is whether we can extend the Lambek calculus with a modality that does for the structural rule of *(exchange)* what the modality *of course* '!' does for the rules of *(weakening)* and *(contraction)*. A very preliminary proposal, which answers this question affirmatively, is set forward in this paper. The 'answer' was provided in semantical terms in the first version of this work. Here we provide also the more syntactic description, building on work of Galatos and others.

We first recall Linear Logic and provide the transformations to show that the Lambek Calculus L really is the multiplicative fragment of non-commutative Intuitionistic Linear Logic. Then we describe the usual String Semantics for the Lambek Calculus L and generalize it, using a categorical perspective in the second section. The third section recalls our Dialectica model for the Lambek Calculus. Finally, in the fourth section we discuss modalities and some untidiness of the Curry-Howard correspondence for the fragments of Linear Logic in question.

1 The Lambek Calculus

The Lambek Calculus, formerly the Syntactic Calculus, is due to Jim Lambek [3]. His idea was to capture the logical structure of sentences, and to do this he introduced a substructural logic with an operator denoting concatenation, $A \otimes B$, and two implications relating the order of phrases, $A \leftharpoonup B$ and $A \rightharpoonup B$, where the former is a phrase of type A when followed by a phrase of type B, and the latter is a phrase of type B when proceeded by a phrase of type A.

It turns out that the Lambek Calculus can be presented as a non-commutative intuitionistic multiplicative linear logic. The syntax of formulas and contexts of the logic are as follows:

$$
\begin{array}{ll}
\text{(formulas)} & A, B, C ::= I \mid A \otimes B \mid A \leftharpoonup B \mid A \rightharpoonup B \mid !A \mid \kappa A \\
\text{(contexts)} & \Gamma ::= A \mid \Gamma_1, \Gamma_2
\end{array}
$$

$$\frac{}{A \vdash A}\ \textsc{ax} \qquad \frac{\Gamma_1 \vdash A \qquad A, \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash B}\ \textsc{cut} \qquad \frac{\Gamma \vdash A}{\Gamma, I \vdash A}\ \textsc{unit}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \mathrm{T}\textsc{l} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \mathrm{T}\textsc{r}$$

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \rightharpoonup B \vdash C}\ \mathrm{IR}\textsc{l} \qquad \frac{\Gamma_1 \vdash A \qquad B, \Gamma_2 \vdash C}{\Gamma_1, \Gamma_2, A \leftharpoonup B \vdash C}\ \mathrm{IL}\textsc{l}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightharpoonup B}\ \mathrm{IR}\textsc{r} \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \leftharpoonup B}\ \mathrm{IL}\textsc{r} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \mathrm{C}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \mathrm{W} \qquad \frac{!\Gamma \vdash B}{!\Gamma \vdash !B}\ \mathrm{B}\textsc{r} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \mathrm{B}\textsc{l} \qquad \frac{\kappa\Gamma \vdash B}{\kappa\Gamma \vdash \kappa B}\ \mathrm{E}\textsc{r}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \kappa A \vdash B}\ \mathrm{E}\textsc{l} \qquad \frac{\Gamma, \kappa A, B \vdash C}{\Gamma, B, \kappa A \vdash C}\ \mathrm{E1} \qquad \frac{\Gamma, A, \kappa B \vdash C}{\Gamma, \kappa B, A \vdash C}\ \mathrm{E2}$$

Figure 1: The Lambek Calculus: L

We denote mapping the modalities over an arbitrary context by $!\Gamma$ and $\kappa\Gamma$. The inference rules are defined in Figure 1. Because the operator $A \otimes B$ denotes the type of concatenations the types $A \otimes B$ and $B \otimes A$ are not equivalent, and hence, L is non-commutative which explains why implication must be broken up into two operators $A \leftharpoonup B$ and $A \rightharpoonup B$.

The usual modality, $!A$, known as the of-course modality due to Girard [1] corresponds to a comonad which adds the structural rules for weakening and contraction as an effect to linear logic, but here we add a second modality, κA, which we simply call the exchange modality, and corresponds to a second comonad adding exchange as an effect. Notice that rules E1 and E2 allow any formula under κ to commute with any other formula regardless if it is under κ.

Categorically, one models of-course as a functor endowed with the structure of a comonad with some additional structure. That is, there are maps $\delta_A : !A \longrightarrow !!A$ and $\varepsilon_A : !A \longrightarrow A$ subject to a few coherence diagrams, and maps $c_A : !A \longrightarrow !A \otimes !A$ and $w_A : !A \longrightarrow I$. Using this structure we can interpret the rules of the of-course modality. Consider the rule C, and suppose we have a map $\Gamma \otimes (!A \otimes !A) \xrightarrow{f} B$, then we can obtain a new map $\Gamma \otimes !A \xrightarrow{\mathsf{id}_\Gamma \otimes c_A} \Gamma \otimes (!A \otimes !A) \xrightarrow{f} B$. The rule W is similar, but we start with a map $\Gamma \xrightarrow{f} B$ and then we can define the map $\Gamma \otimes !A \xrightarrow{\mathsf{id}_\Gamma \otimes w_A} \Gamma \otimes I \xrightarrow{\cong} \Gamma \xrightarrow{f} B$. Notice that the previous map exploits the fact that I is the unit for tensor. Now consider the rule Bl, and suppose we have a map $!\Gamma \xrightarrow{f} B$, then we may obtain a second map using the fact that of-course is a functor $!\Gamma \xrightarrow{\delta} !!\Gamma \xrightarrow{!f} !B$. Finally, consider the rule BR and suppose we have a map $\Gamma \otimes A \xrightarrow{f} B$, then we can construct the map $\Gamma \otimes !A \xrightarrow{\mathsf{id}_\Gamma \otimes \varepsilon_A} \Gamma \otimes A \xrightarrow{f} B$. This analysis tells us a few things about interpreting logics into categorical models. Sequents, $\Gamma \vdash B$, are interpreted as morphisms, $\Gamma \xrightarrow{f} B$, where Γ is I if it is empty, or it is the tensor product of the interpretations of its formulas. Then interpreting inference rules amounts to starting with the morphisms corresponding to the premises, and then building a map corresponding to the conclusion. The cut-elimination procedure is defined by a set of equations between derivations, and hence, in the model corresponds to equations between morphisms. The various coherence diagrams relating the structure of the model enforce that these equations hold.

We can similarly interpret the rules for the exchange modality. That is, as a functor endowed with the structure of a second comonad, but also with the maps $e_1 : A \otimes \kappa B \longrightarrow \kappa B \otimes A$ and $e_2 : \kappa A \otimes B \longrightarrow B \otimes \kappa A$. Then, each of the inference rules for the exchange modality can easily be interpreted into the model.

2 Algebraic Semantics

In Lambek's original paper [3] introducing his calculus L, albeit without modalities, he introduced an algebraic semantics that is now called the String Semantics for L. The semantics begins with a non-empty set of expressions denoted V^+, and then by modeling formulas of L by subsets of V^+ it proceeds by defining operations on these subsets, which correspond to the logical connectives of L. Each operation is defined as follows (using our notation for the logical connectives):

$$\begin{aligned} A \otimes B &= \{xy \in \mathrm{V}^+ \mid x \in A \text{ and } y \in B\} \\ A \leftharpoonup B &= \{x \in \mathrm{V}^+ \mid \text{for all } y \in B, xy \in A\} \\ A \rightharpoonup B &= \{y \in \mathrm{V}^+ \mid \text{for all } x \in A, xy \in B\} \end{aligned}$$

Let $\mathcal{N} = \mathcal{P}(\mathrm{V}^+)$ be the powerset of V^+. Then we can view each of the above definitions as binary operations with type $\mathcal{N} \times \mathcal{N} \longrightarrow \mathcal{N}$. In fact, $\mathcal{N}$ has a natural order induced by set containment, and concatenation, $A \otimes B$, gives $\mathcal{N}$ a non-commutative monoidal structure, where the unit $I = \{\epsilon\}$ is the set containing the empty sequence:

(associativity)	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$
(unit)	$A \otimes I = A = I \otimes A$
(non-commutativity)	$A \otimes B \neq B \otimes A$, in general

There happens to be a more general structure underlying the previous semantics. We now make this structure explicit using the tools developed by Hyland and de Paiva [2]. The ordering on $\mathcal{N}$ induces a poset $(\mathcal{N}, \subseteq)$, but even more so, $\mathcal{N}$ is also a monoid $(\mathcal{N}, \otimes, I)$, but that is not all, these two structures are compatible, that is, given $A \subseteq B$ the following hold:

$$A \otimes C \subseteq B \otimes C, \text{ for all } C \in \mathcal{N}$$
$$C \otimes A \subseteq C \otimes B, \text{ for all } C \in \mathcal{N}$$

Abstracting this structure out yields what is call an ordered non-commutative monoid.

Definition 1. *An **ordered non-commutative monoid**, $(M, \leq, \circ, e)$, is a poset $(M, \leq)$ with a given compatible monoidal structure $(M, \circ, e)$. That is, a set M equipped with a binary relation, $\leq: M \times M \longrightarrow 2$, satisfying:*

$$\begin{array}{rl} \textit{(reflexivity)} & a \leq a \textit{ for all } a \in M \\ \textit{(transitivity)} & a \leq b \textit{ and } b \leq c, \textit{ implies that } a \leq c \textit{ for all } a, b, c \in M \\ \textit{(antisymmetry)} & a \leq b \textit{ and } b \leq a, \textit{ implies that } a = b \end{array}$$

together with a monoidal multiplication $\circ : M \times M \longrightarrow M$ and a distinguished object $e \in M$ satisfying the following:

$$\begin{array}{rl} \textit{(associativity)} & a \circ (b \circ c) = (a \circ b) \circ c \\ \textit{(identity)} & e \circ a = a = a \circ e \end{array}$$

The structures are compatible in the sense that, if $a \leq b$, then the following hold:

$$a \circ c \leq b \circ c \textit{ for any } c \in M$$
$$c \circ a \leq c \circ b \textit{ for any } c \in M$$

It is easy to see that the previous definition accounts for all of the structure we have described so far, and thus, we may conclude that $(\mathcal{N}, \subseteq, \otimes, I)$ is an ordered non-commutative monoid, however, this definition is not able to model the implication operations $A \leftharpoonup B$ and $A \rightharpoonup B$. To do this we need to understand how implication relates to the ordered non-commutative monoid structure. Notice that the following hold:

$$A \otimes (A \rightharpoonup B) \subseteq B$$
$$(A \leftharpoonup B) \otimes A \subseteq B$$

Furthermore, there are no larger objects of $\mathcal{N}$ with these properties. Abstracting this results in the notion of a biclosed poset.

Definition 2. *Suppose $(M, \leq, \circ, e)$ is an ordered non-commutative monoid. If there exists a largest $x \in M$ such that $a \circ x \leq b$ for any $a, b \in M$, then we denote x by $a \rightharpoonup b$ and called it the **left-pseudocomplement** of a w.r.t b. Additionally, if there exists a largest $x \in M$ such that $x \circ a \leq b$ for any $a, b \in M$, then we denote x by $a \leftharpoonup b$ and called it the **right-pseudocomplement** of a w.r.t b.*

*A **biclosed poset**, $(M, \leq, \circ, e, \rightharpoonup, \leftharpoonup)$, is an ordered non-commutative monoid, $(M, \leq, \circ, e)$, such that $a \rightharpoonup b$ and $a \leftharpoonup b$ exist for any $a, b \in M$.*

At this point we have everything we need to model the Lambek Calculus L without modalities.

Lemma 3. *Any biclosed poset* $(M, \leq, \circ, e, \rightharpoonup, \leftharpoonup)$ *is a model for the Lambek Calculus L without modalities.*

Proof. First suppose we have an assignment $(-)^0$ which assigns to each formula of L an element of M. Then if $\Gamma \vdash A$ holds we show that $(\Gamma)^0 \leq (A)^0$. This proof can easily be completed by induction on the form $\Gamma \vdash A$. $\square$

3 Dialectica Lambek Spaces

4 MultiModalities

5 Conclusion

References

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