Generated proof tree

April 18, 2018

The following proof tree has been automatically generated by sequent using the ebproof package.

1

$$\frac{\overline{A \Longrightarrow A}\ \text{init}}{\Longrightarrow A \supset A}\ {\supset}R$$

2

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A,B,C \Longrightarrow C}\ \text{init}}{A,B,B \supset C \Longrightarrow C}\ P{\supset}L}{A,B \supset C,A \supset B \Longrightarrow C}\ P{\supset}L}{(A \supset B) \wedge (B \supset C),A \Longrightarrow C}\ \wedge L}{(A \supset B) \wedge (B \supset C) \Longrightarrow A \supset C}\ {\supset}R}{\Longrightarrow (A \supset B) \wedge (B \supset C) \supset A \supset C}\ {\supset}R$$

3

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A,B,C \Longrightarrow C}\ \text{init}}{A,B,B \supset C \Longrightarrow C}\ P{\supset}L}{A,B \supset C,A \supset B \Longrightarrow C}\ P{\supset}L}{(A \supset B) \land (B \supset C),A \Longrightarrow C}\ {\land}L}{(A \supset B) \land (B \supset C) \Longrightarrow A \supset C}\ {\supset}R}{\Longrightarrow (A \supset B) \land (B \supset C) \supset A \supset C}\ {\supset}R$$

4

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B,A,C \Longrightarrow C}\ \text{init}}{B,A,B \supset C \Longrightarrow C}\ P{\supset}L}{B,A,A \supset B \supset C \Longrightarrow C}\ P{\supset}L}{A \supset B \supset C,B \Longrightarrow A \supset C}\ {\supset}R}{A \supset B \supset C \Longrightarrow B \supset A \supset C}\ {\supset}R}{\Longrightarrow (A \supset B \supset C) \supset B \supset A \supset C}\ {\supset}R$$

5

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A,B,C \Longrightarrow C}}{A,B,B\supset C \Longrightarrow C}\ P{\supset}L}{A,B,A\supset B\supset C \Longrightarrow C}\ P{\supset}L}{A\wedge B,A\supset B\supset C \Longrightarrow C}\ \wedge L}{A\supset B\supset C \Longrightarrow A\wedge B\supset C}\ {\supset}R}{\Longrightarrow (A\supset B\supset C)\supset A\wedge B\supset C}\ {\supset}R}{}\ \text{init}
$$

6

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A,B,C \Longrightarrow C}}{A,B,B\supset C \Longrightarrow C}\ P{\supset}L}{A,B,A\supset B\supset C \Longrightarrow C}\ P{\supset}L}{A\wedge B\supset C,A,B \Longrightarrow C}\ \wedge{\supset}L}{A\wedge B\supset C,A \Longrightarrow B\supset C}\ {\supset}R}{A\wedge B\supset C \Longrightarrow A\supset B\supset C}\ {\supset}R}{\Longrightarrow (A\wedge B\supset C)\supset A\supset B\supset C}\ {\supset}R\ \text{init}
$$

7

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\ }{A,B,C \Longrightarrow C}\ \text{init}
}{A,B,B \supset C \Longrightarrow C}\ P{\supset}L
}{A,B,A \supset B \supset C \Longrightarrow C}\ P{\supset}L
}{A \wedge B \supset C, A, B \Longrightarrow C}\ {\wedge}{\supset}L
}{A \wedge B \supset C, A \Longrightarrow B \supset C}\ {\supset}R
}{A \wedge B \supset C \Longrightarrow A \supset B \supset C}\ {\supset}R
}{\Longrightarrow (A \wedge B \supset C) \supset A \supset B \supset C}\ {\supset}R
$$

8

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\ }{C,A,B \Longrightarrow B}\ \text{init}
}{C,A,A \supset B \Longrightarrow B}\ P{\supset}L
}{A \supset B, C, C \supset A \Longrightarrow B}\ P{\supset}L
}{A \supset B, C, C \supset A \Longrightarrow B}\ {\supset}R
}{A \supset B, C \supset A \Longrightarrow C \supset B}\ {\supset}R
}{A \supset B \Longrightarrow (C \supset A) \supset C \supset B}\ {\supset}R
}{\Longrightarrow (A \supset B) \supset (C \supset A) \supset C \supset B}\ {\supset}R
$$

9

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{\dfrac{}{A,C,B\Longrightarrow B}\ \text{init}}{A,C,A\supset B\Longrightarrow B}\ P\supset L
    }{A\wedge C,A\supset B\Longrightarrow B}\ \wedge L
    \qquad
    \cfrac{
      \cfrac{\dfrac{}{A,B,C\Longrightarrow C}\ \text{init}}{A,C,A\supset B\Longrightarrow C}\ P\supset L
    }{A\wedge C,A\supset B\Longrightarrow C}\ \wedge L
  }{
    \cfrac{A\wedge C,A\supset B\Longrightarrow B\wedge C}{
      \cfrac{A\supset B\Longrightarrow A\wedge C\supset B\wedge C}{\Longrightarrow (A\supset B)\supset A\wedge C\supset B\wedge C}\ \supset R
    }\ \supset R
  }\ \wedge R
}{}
$$

10

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{\dfrac{}{A,B,C\Longrightarrow C}\ \text{init}}{C,A,A\supset B\Longrightarrow C}\ P\supset L
    }{C\wedge A,A\supset B\Longrightarrow C}\ \wedge L
    \qquad
    \cfrac{
      \cfrac{\dfrac{}{C,A,B\Longrightarrow B}\ \text{init}}{C,A,A\supset B\Longrightarrow B}\ P\supset L
    }{C\wedge A,A\supset B\Longrightarrow B}\ \wedge L
  }{
    \cfrac{C\wedge A,A\supset B\Longrightarrow C\wedge B}{
      \cfrac{A\supset B\Longrightarrow C\wedge A\supset C\wedge B}{\Longrightarrow (A\supset B)\supset C\wedge A\supset C\wedge B}\ \supset R
    }\ \supset R
  }\ \wedge R
}{}
$$

11

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{}{A,B \Longrightarrow B}\ \text{init}
      }{A,B \Longrightarrow B \vee C}\ \vee R_1
    }{A,A \supset B \Longrightarrow B \vee C}\ P{\supset}L
    \quad
    \cfrac{
      \cfrac{
        \cfrac{}{A \supset B, C \Longrightarrow C}\ \text{init}
      }{A \supset B, C \Longrightarrow B \vee C}\ \vee R_2
    }{}
  }{
    \begin{array}{c}
    A \supset B, A \Longrightarrow B \vee C \quad\ \vee L
    \end{array}
  }
}{}
$$

$$
\cfrac{A \vee C, A \supset B \Longrightarrow B \vee C}{
  \cfrac{A \supset B \Longrightarrow A \vee C \supset B \vee C}{
    \Longrightarrow (A \supset B) \supset A \vee C \supset B \vee C
  }\ \supset R
}\ \supset R
$$

12

$$
\cfrac{
  \cfrac{}{A,B \Longrightarrow B}\ \text{init}
}{A,B \Longrightarrow C \vee B}\ \vee R_2
$$
$$
\cfrac{A,A \supset B \Longrightarrow C \vee B}{A \supset B, A \Longrightarrow C \vee B}\ P{\supset}L,\ \vee L
$$
$$
\cfrac{
  \cfrac{}{A \supset B, C \Longrightarrow C}\ \text{init}
}{A \supset B, C \Longrightarrow C \vee B}\ \vee R_1
$$
$$
\cfrac{C \vee A, A \supset B \Longrightarrow C \vee B}{
  \cfrac{A \supset B \Longrightarrow C \vee A \supset C \vee B}{
    \Longrightarrow (A \supset B) \supset C \vee A \supset C \vee B
  }\ \supset R
}\ \supset R,\ \vee L
$$

13

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\ }{A, \bot \Longrightarrow B}\ \bot L
}{A, A \supset \bot \Longrightarrow B}\ P{\supset}L
}{A \supset \bot \Longrightarrow A \supset B}\ {\supset}R
}{\Longrightarrow (A \supset \bot) \supset A \supset B}\ {\supset}R
$$

14

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\ }{A, \bot \Longrightarrow B}\ \bot L
}{A, A \supset \bot \Longrightarrow B}\ P{\supset}L
}{A \Longrightarrow (A \supset \bot) \supset B}\ {\supset}R
}{\Longrightarrow A \supset (A \supset \bot) \supset B}\ {\supset}R
$$

15

$$
\cfrac{
\cfrac{
\cfrac{\ }{A, B \Longrightarrow B}\ \text{init}
}{B \Longrightarrow A \supset B}\ {\supset}R
}{\Longrightarrow B \supset A \supset B}\ {\supset}R
$$

16

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{A,B,\bot \Longrightarrow \bot}{A,B,B\supset\bot \Longrightarrow \bot}\;P\supset L
}{B\supset\bot,A,A\supset B \Longrightarrow \bot}\;P\supset L
}{A\supset B,B\supset\bot \Longrightarrow A\supset\bot}\;\supset R
}{A\supset B \Longrightarrow (B\supset\bot)\supset A\supset\bot}\;\supset R
}{\Longrightarrow (A\supset B)\supset(B\supset\bot)\supset A\supset\bot}\;\supset R
}{}\;\bot L
$$

17

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{B,A,\bot \Longrightarrow \bot}{B,A,B\supset\bot \Longrightarrow \bot}\;P\supset L
}{B,A,A\supset B\supset\bot \Longrightarrow \bot}\;P\supset L
}{A\supset B\supset\bot,B \Longrightarrow A\supset\bot}\;\supset R
}{A\supset B\supset\bot \Longrightarrow B\supset A\supset\bot}\;\supset R
}{\Longrightarrow (A\supset B\supset\bot)\supset B\supset A\supset\bot}\;\supset R
}{}\;\bot L
$$

18

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\ \ }{A, A, B \Longrightarrow B}\ \text{init}
}{A, B, B \supset A \Longrightarrow B}\ P{\supset}L
}{A, B \supset A, A \supset B \Longrightarrow B}\ P{\supset}L
}{(A \supset B) \wedge (B \supset A), A \Longrightarrow B}\ \wedge L
}{(A \supset B) \wedge (B \supset A) \Longrightarrow A \supset B}\ {\supset}R
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\ \ }{B, B, A \Longrightarrow A}\ \text{init}
}{B, A, A \supset B \Longrightarrow A}\ P{\supset}L
}{B, A \supset B, B \supset A \Longrightarrow A}\ P{\supset}L
}{(A \supset B) \wedge (B \supset A), B \Longrightarrow A}\ \wedge L
}{(A \supset B) \wedge (B \supset A) \Longrightarrow B \supset A}\ {\supset}R
$$

$$
\dfrac{(A \supset B) \wedge (B \supset A) \Longrightarrow (A \supset B) \wedge (B \supset A)}{\Longrightarrow (A \supset B) \wedge (B \supset A) \supset (A \supset B) \wedge (B \supset A)}\ {\supset}R \qquad \wedge R
$$

19

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\ \ }{A, A, B \Longrightarrow B}\ \text{init}
}{A, B, B \supset A \Longrightarrow B}\ P{\supset}L
}{A, B \supset A, A \supset B \Longrightarrow B}\ P{\supset}L
}{(A \supset B) \wedge (B \supset A), A \Longrightarrow B}\ \wedge L
}{(A \supset B) \wedge (B \supset A) \Longrightarrow A \supset B}\ {\supset}R
}{\Longrightarrow (A \supset B) \wedge (B \supset A) \supset A \supset B}\ {\supset}R
$$

10

20

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{B, B, A \Longrightarrow A}\ \text{init}
}{B, A \supset B \Longrightarrow A}\ P{\supset}L
}{B, A \supset B, B \supset A \Longrightarrow A}\ P{\supset}L
}{(A \supset B) \wedge (B \supset A), B \Longrightarrow A}\ \wedge L
}{(A \supset B) \wedge (B \supset A) \Longrightarrow B \supset A}\ {\supset}R
}{\Longrightarrow (A \supset B) \wedge (B \supset A) \supset B \supset A}\ {\supset}R
$$

21

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A, A, B \Longrightarrow B}\ \text{init}
}{A, B \supset A \Longrightarrow B}\ P{\supset}L
}{A, B \supset A, A \supset B \Longrightarrow B}\ P{\supset}L
}{(A \supset B) \wedge (B \supset A), A \Longrightarrow B}\ \wedge L
}{((A \supset B) \wedge (B \supset A)) \wedge A \Longrightarrow B}\ \wedge L
}{\Longrightarrow ((A \supset B) \wedge (B \supset A)) \wedge A \supset B}\ {\supset}R
$$

22

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\ }{B,B,B,A \Longrightarrow A}\ \text{init}
}{B,B,A,A \supset B \Longrightarrow A}\ P{\supset}L
}{B,B,A \supset B, B \supset A \Longrightarrow A}\ P{\supset}L
}{(A \supset B) \wedge (B \supset A), B, B \Longrightarrow A}\ \wedge L
}{((A \supset B) \wedge (B \supset A)) \wedge B, B \Longrightarrow A}\ \wedge L
}{((A \supset B) \wedge (B \supset A)) \wedge B \Longrightarrow B \supset A}\ {\supset}R
}{\Longrightarrow ((A \supset B) \wedge (B \supset A)) \wedge B \supset A}\ {\supset}R
$$

23

$$
\dfrac{
\dfrac{\dfrac{\ }{A \Longrightarrow A}\ \text{init}}{\Longrightarrow A \supset A}\ {\supset}R
\qquad
\dfrac{\dfrac{\ }{A \Longrightarrow A}\ \text{init}}{\Longrightarrow A \supset A}\ {\supset}R
}{\Longrightarrow (A \supset A) \wedge (A \supset A)}\ \wedge R
$$

12

24

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\quad}{A,B,B,A,C \Longrightarrow C}\ \text{init}}{A,B,C,B,B \supset A \Longrightarrow C}\ P{\supset}L}{A,B \supset A,B,C,C \supset B \Longrightarrow C}\ P{\supset}L}{A,C \supset B,B \supset A,B,B \supset C \Longrightarrow C}\ P{\supset}L}{A,B \supset C,C \supset B,B \supset A,A \supset B \Longrightarrow C}\ P{\supset}L}{(A \supset B) \land (B \supset A),A,B \supset C,C,C \supset B \Longrightarrow C}\ \land L}{(A \supset B) \land (B \supset A),(B \supset C) \land (C \supset B),A \Longrightarrow C}\ \land L}{((A \supset B) \land (B \supset A)) \land (B \supset C) \land (C \supset B),A \Longrightarrow C}\ {\supset}R}{((A \supset B) \land (B \supset A)) \land (B \supset C) \land (C \supset B) \Longrightarrow A \supset C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\quad}{C,B,C,B,A \Longrightarrow A}\ \text{init}}{C,B,C,A,A \supset B \Longrightarrow A}\ P{\supset}L}{C,A \supset B,C,B \supset A \Longrightarrow A}\ P{\supset}L}{C,A \supset B,B \supset A,B,B \supset C \Longrightarrow A}\ P{\supset}L}{C,B \supset C,A \supset B,B \supset A,C \supset B \Longrightarrow A}\ P{\supset}L}{(A \supset B) \land (B \supset A),C,B \supset C,C,C \supset B \Longrightarrow A}\ \land L}{(A \supset B) \land (B \supset A),(B \supset C) \land (C \supset B),C \Longrightarrow A}\ \land L}{((A \supset B) \land (B \supset A)) \land (B \supset C) \land (C \supset B),C \Longrightarrow A}\ \land L}{((A \supset B) \land (B \supset A)) \land (B \supset C) \land (C \supset B) \Longrightarrow C \supset A}\ {\supset}R$$

$$\dfrac{((A \supset B) \land (B \supset A)) \land (B \supset C) \land (C \supset B) \Longrightarrow (A \supset C) \land (C \supset A)}{\Longrightarrow ((A \supset B) \land (B \supset A)) \land (B \supset C) \land (C \supset B) \supset (A \supset C) \land (C \supset A)}\ \land R$$

13

25

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \overline{A,B,C,\bot \Longrightarrow \bot}\ \bot L
      }{A,B,C,C\supset\bot \Longrightarrow \bot}\ P\supset L
    }{C\supset\bot,A,B,B\supset C \Longrightarrow \bot}\ P\supset L
  }{C\supset\bot,A,B\supset C,(B\supset\bot)\supset\bot \Longrightarrow \bot}\ \bot\supset L
}{
  C\supset\bot,(B\supset\bot)\supset\bot,A,A\supset B\supset C \Longrightarrow \bot
}\ \supset\supset L
$$

$$
\dfrac{
  \overline{C\supset\bot,A,B\supset C,\bot \Longrightarrow \bot}\ \bot L
}{
  C\supset\bot,A,B\supset C,(B\supset\bot)\supset\bot \Longrightarrow \bot
}\ \bot\supset L
$$

$$
\dfrac{C\supset\bot,A\supset B\supset C,(B\supset\bot)\supset\bot,A \Longrightarrow \bot}{
  C\supset\bot,(B\supset\bot)\supset\bot,A,A\supset B\supset C \Longrightarrow \bot
}\ P\supset L
$$

$$
\dfrac{
  C\supset\bot,A\supset B\supset C,(B\supset\bot)\supset\bot,\bot \Longrightarrow \bot
}{
  \bot\supset L
}
$$

$$
\dfrac{
  \dfrac{
    \overline{C\supset\bot,A\supset B\supset C,(B\supset\bot)\supset\bot,\top \Longrightarrow \bot}\ \bot L
  }{
    C\supset\bot,A\supset B\supset C,(B\supset\bot)\supset\bot,(A\supset\bot)\supset\bot \Longrightarrow \bot
  }\ \supset\supset L
}{
  \dfrac{
    \dfrac{
      \dfrac{
        ((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot),C\supset\bot,A\supset B\supset C \Longrightarrow \bot
      }{
        (A\supset B\supset C)\wedge((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot),C\supset\bot \Longrightarrow \bot
      }\ \wedge L
    }{
      (A\supset B\supset C)\wedge((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot) \Longrightarrow (C\supset\bot)\supset\bot
    }\ \supset R
  }{
    \Longrightarrow (A\supset B\supset C)\wedge((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot)\supset(C\supset\bot)\supset\bot
  }\ \supset R
}\ \wedge L
$$

26

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\;}{A,B,\bot \Longrightarrow \bot}\,\bot L
      }{A,B,B\supset\bot \Longrightarrow \bot}\,P{\supset}L
    }{B\supset\bot,A,A\supset B \Longrightarrow \bot}\,P{\supset}L
  }{\bot\supset\bot,B\supset\bot,A\supset B,A \Longrightarrow \bot}\,\bot{\supset}L
}{B\supset\bot,A\supset B,(A\supset\bot)\supset\bot \Longrightarrow \bot}\,\bot{\supset}L
$$

$$
\dfrac{
  \bot\supset\bot,(A\supset\bot)\supset\bot,B\supset\bot,A\supset B \Longrightarrow \bot
}{
  B\supset\bot,A\supset B,\bot \Longrightarrow \bot
}\,\bot L
$$

$$
\dfrac{
  \dfrac{\;}{B\supset\bot,A\supset B,\bot \Longrightarrow \bot}\,\bot L
}{
  (A\supset\bot)\supset\bot,B\supset\bot,((A\supset B)\supset\bot)\supset\bot \Longrightarrow \bot
}\,{\supset}{\supset}L
$$

$$
\dfrac{
  \dfrac{\;}{(A\supset\bot)\supset\bot,B\supset\bot,\top,\bot \Longrightarrow \bot}\,\bot L
}{
  (A\supset\bot)\supset\bot,B\supset\bot,((A\supset B)\supset\bot)\supset\bot \Longrightarrow \bot
}\,{\supset}{\supset}L
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      (A\supset\bot)\supset\bot,B\supset\bot,((A\supset B)\supset\bot)\supset\bot \Longrightarrow \bot
    }{
      ((A\supset B)\supset\bot)\supset\bot,(A\supset\bot)\supset\bot \Longrightarrow (B\supset\bot)\supset\bot
    }\,{\supset}R
  }{
    ((A\supset B)\supset\bot)\supset\bot \Longrightarrow ((A\supset\bot)\supset\bot)\supset(B\supset\bot)\supset\bot
  }\,{\supset}R
}{
  \Longrightarrow (((A\supset B)\supset\bot)\supset\bot)\supset((A\supset\bot)\supset\bot)\supset(B\supset\bot)\supset\bot
}\,{\supset}R
$$

27

$$\overline{A,B,C,\bot \Longrightarrow \bot}\ \bot L$$
$$\overline{A,B,C,C \supset \bot \Longrightarrow \bot}\ P\supset L$$
$$\overline{C \supset \bot, A, B \supset C, A \supset B \Longrightarrow \bot}\ P\supset L$$
$$\overline{C \supset \bot, A, B \supset C, A \supset B \Longrightarrow \bot}\ P\supset L$$
$$\overline{C \supset \bot, A, A \supset B, B \supset C \Longrightarrow \bot}\ \bot L$$
$$\overline{\bot \supset \bot, C \supset \bot, A, A \supset B, B \supset C \Longrightarrow \bot}\ \bot\supset L$$
$$\overline{C \supset \bot, A, A \supset B, ((B \supset C)\supset\bot)\supset\bot, C \supset \bot \Longrightarrow \bot}\ \supset\supset L$$
$$\overline{(B \supset C)\supset\bot, C \supset \bot, A, ((A \supset B)\supset\bot)\supset\bot \Longrightarrow \bot}\ \bot\supset L$$

$$\overline{((B \supset C)\supset\bot)\supset\bot, C \supset \bot, A, \bot \Longrightarrow C}\ \bot L$$
$$\overline{((B \supset C)\supset\bot)\supset\bot, C \supset \bot, A, ((A \supset B)\supset\bot)\supset\bot \Longrightarrow C}\ \supset\supset L$$

$$\overline{((A \supset B)\supset\bot)\supset\bot,((B \supset C)\supset\bot)\supset\bot,(A \supset C)\supset\bot \Longrightarrow \top}$$
$$\overline{\top \Longrightarrow \top \supset (\top \supset \bot),(A \supset C)\supset\bot \Longrightarrow \top}\ \lor L$$
$$\top \supset \top \supset (\top \supset \bot),(A \supset C)\supset\bot \Longrightarrow \top$$
$$(((A \supset B)\supset\bot)\supset\bot)\land(((B \supset C)\supset\bot)\supset\bot),(A \supset C)\supset\bot \Longrightarrow ((A \supset C)\supset\bot)\supset\bot$$
$$\Longrightarrow (((A \supset B)\supset\bot)\supset\bot)\land(((B \supset C)\supset\bot)\supset\bot) \supset (\top \supset \bot) \supset \top \supset (\top \supset \bot) \supset ((A \supset C)\supset\bot)\supset\bot$$

28

$$\dfrac{\dfrac{}{A,B,\bot \Longrightarrow \bot}\;\bot L}{A,B,B\supset\bot \Longrightarrow \bot}\;P\supset L$$
$$\dfrac{}{\bot\supset\bot,A,B\supset\bot,B \Longrightarrow \bot}\;\bot\supset L$$

$$A,B\supset\bot,(\ldots$$
$$(B\supset\bot)\supset\bot,\ldots$$
$$\bot\supset\bot,A\supset B\supset\bot$$

Middle derivation:

$$\dfrac{\dfrac{\dfrac{}{A,B,\bot \Longrightarrow \bot}\;\bot L}{A,B,B\supset\bot \Longrightarrow \bot}\;P\supset L}{\dfrac{\bot\supset\bot,B\supset\bot,A,B \Longrightarrow \bot}{\bot\supset\bot,A\wedge B,B\supset\bot \Longrightarrow \bot}\;\wedge L}\;\bot\supset L \qquad \dfrac{}{B\supset\bot,\bot \Longrightarrow \bot}\;\bot L}{\dfrac{B\supset\bot,(A\wedge B\supset\bot)\supset\bot \Longrightarrow \bot\supset\bot}{(A\wedge B\supset\bot)\supset\bot \Longrightarrow (B\supset\bot)\supset\bot}\;\supset R}\;\supset\supset L$$

Left–bottom derivation:

$$\dfrac{\dfrac{\dfrac{}{A,B,\bot \Longrightarrow \bot}\;\bot L}{A,B,A\supset\bot \Longrightarrow \bot}\;P\supset L}{\dfrac{\bot\supset\bot,A\supset\bot,A,B \Longrightarrow \bot}{\bot\supset\bot,A\wedge B,A\supset\bot \Longrightarrow \bot}\;\wedge L}\;\bot\supset L \qquad \dfrac{}{A\supset\bot,\bot \Longrightarrow \bot}\;\bot L}{\dfrac{A\supset\bot,(A\wedge B\supset\bot)\supset\bot \Longrightarrow \bot\supset\bot}{(A\wedge B\supset\bot)\supset\bot \Longrightarrow (A\supset\bot)\supset\bot}\;\supset R}\;\supset\supset L$$

Combined conclusion:

$$\dfrac{(A\wedge B\supset\bot)\supset\bot \Longrightarrow ((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot)}{\Longrightarrow ((A\wedge B\supset\bot)\supset\bot)\supset(((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot))}\;\supset R \quad \wedge R$$

$$\Longrightarrow (((A\wedge B\supset\bot)\supset\bot)\supset((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot))\vee(((A\supset\bot)\supset\bot)\wedge((B\supset\bot)\supset\bot))\vee\ldots$$

17

29

Right derivation:

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\dfrac{
\dfrac{
\dfrac{\ }{A,B,A,C \Longrightarrow C}\ \text{init}
}{A,B,C,B\supset A \Longrightarrow C}\ P{\supset}L
}{A,B\supset A,B,B\supset C \Longrightarrow C}\ P{\supset}L
}{B\supset C,A,B\supset A,A\supset B \Longrightarrow C}\ P{\supset}L
}{(A\supset B)\wedge(B\supset A),B\supset C,A \Longrightarrow C}\ \wedge L
}{(A\supset B)\wedge(B\supset A),B\supset C \Longrightarrow A\supset C}\ {\supset}R
}{(A\supset B)\wedge(B\supset A) \Longrightarrow (B\supset C)\supset A\supset C}\ {\supset}R
}{(A\supset B)\wedge(B\supset A)\supset B\supset C)\wedge((B\supset C)\supset A\supset C)}\ \wedge R
$$

Left derivation:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\ }{B,A,B,C \Longrightarrow C}\ \text{init}
}{B,A,C,A\supset B \Longrightarrow C}\ P{\supset}L
}{B,A\supset B,A,A\supset C \Longrightarrow C}\ P{\supset}L
}{A\supset C,B,A\supset B,B\supset A \Longrightarrow C}\ P{\supset}L
}{(A\supset B)\wedge(B\supset A),A\supset C,B \Longrightarrow C}\ \wedge L
}{(A\supset B)\wedge(B\supset A),A\supset C \Longrightarrow B\supset C}\ {\supset}R
}{(A\supset B)\wedge(B\supset A) \Longrightarrow (A\supset C)\supset B\supset C}\ {\supset}R
}{\Longrightarrow (A\supset B)\wedge(B\supset A)\supset ((A\supset C)\supset B\supset C)\wedge((B\supset C)\supset A\supset C)}\ {\supset}R
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\quad}{C,A,C,A,B \Longrightarrow B}\ \text{init}}{C,A,C,B,B\supset A \Longrightarrow B}\ P\supset L}{C,A,C,B\supset A,A\supset B \Longrightarrow B}\ P\supset L}{(A\supset B)\wedge(B\supset A),C,A,C \Longrightarrow B}\ \wedge L}{(A\supset B)\wedge(B\supset A),C,A \Longrightarrow C\supset B}\ \supset R}{(A\supset B)\wedge(B\supset A),C \Longrightarrow A\supset C\supset B}\ \supset R}{(A\supset B)\wedge(B\supset A) \Longrightarrow C\supset A\supset C\supset B}\ \supset R
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\quad}{C,B,C,B,A \Longrightarrow A}\ \text{init}}{C,B,C,A,A\supset B \Longrightarrow A}\ P\supset L}{C,B,C,A\supset B,B\supset A \Longrightarrow A}\ P\supset L}{(A\supset B)\wedge(B\supset A),C,B,C \Longrightarrow A}\ \wedge L}{(A\supset B)\wedge(B\supset A),C,B \Longrightarrow C\supset A}\ \supset R}{(A\supset B)\wedge(B\supset A),C \Longrightarrow B\supset C\supset A}\ \supset R}{(A\supset B)\wedge(B\supset A) \Longrightarrow C\supset B\supset C\supset A}\ \supset R
$$

$$
\dfrac{(A\supset B)\wedge(B\supset A) \Longrightarrow (C\supset A\supset C\supset B)\wedge(C\supset B\supset C\supset A)}{\Longrightarrow (A\supset B)\wedge(B\supset A)\supset(C\supset A\supset C\supset B)\wedge(C\supset B\supset C\supset A)}\ \supset R
$$

with $\wedge R$ joining the two derivations above.

31

Proof tree (upper right branch):

$$
\dfrac{
  \dfrac{
    \dfrac{B, A \supset B, B \supset A, C \Longrightarrow C}{(A \supset B) \land (B \supset A), B, C \Longrightarrow C}\ \land L
  }{(A \supset B) \land (B \supset A), B \land C \Longrightarrow C}\ \land L
}{\ }\ \text{init}
$$

Proof tree (middle, conclusion $\Longrightarrow A$):

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{B, C, B, A \Longrightarrow A}{B, C, A \supset B, A \Longrightarrow A}\ P \supset L
    }{B, C, A \supset B, B \supset A \Longrightarrow A}\ P \supset L
  }{(A \supset B) \land (B \supset A), B, C \Longrightarrow A}\ \land L
}{(A \supset B) \land (B \supset A), B \land C \Longrightarrow A}\ \land L
\quad \text{init}
$$

$$
\dfrac{(A \supset B) \land (B \supset A), B \land C \Longrightarrow A \land C}{(A \supset B) \land (B \supset A) \Longrightarrow B \land C \supset A \land C}\ \supset R
$$
$$\land R$$

Proof tree (lower middle, conclusion $\Longrightarrow C$):

$$
\dfrac{
  \dfrac{A, A \supset B, B \supset A, C \Longrightarrow C}{(A \supset B) \land (B \supset A), A, C \Longrightarrow C}\ \land L
}{(A \supset B) \land (B \supset A), A \land C \Longrightarrow C}\ \land L
\quad \text{init}
$$

$$
\dfrac{(A \supset B) \land (B \supset A), A \land C \Longrightarrow B \land C}{(A \supset B) \land (B \supset A) \Longrightarrow A \land C \supset B \land C}\ \supset R
$$
$$\land R$$

$$
\dfrac{(A \supset B) \land (B \supset A) \Longrightarrow (A \land C \supset B \land C) \land (B \land C \supset A \land C)}{\Longrightarrow (A \supset B) \land (B \supset A) \supset (A \land C \supset B \land C) \land (B \land C \supset A \land C)}\ \supset R
$$

Proof tree (lower left, conclusion $\Longrightarrow B$):

$$
\dfrac{
  \dfrac{
    \dfrac{A, C, A, B \Longrightarrow B}{A, C, B \supset A, A \Longrightarrow B}\ P \supset L
  }{A, C, A \supset B, B \supset A \Longrightarrow B}\ P \supset L
}{ }
\quad \text{init}
$$

$$
\dfrac{(A \supset B) \land (B \supset A), A, C \Longrightarrow B}{(A \supset B) \land (B \supset A), A \land C \Longrightarrow B}\ \land L
$$
$$\land L$$

32

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{C,B,B,A \Longrightarrow A}\ \text{init}}{C,B,A,A\supset B \Longrightarrow A}\ P\supset L}{C,B,A\supset B,B\supset A \Longrightarrow A}\ P\supset L}{(A\supset B)\wedge(B\supset A),C,B \Longrightarrow A}\ \wedge L}{(A\supset B)\wedge(B\supset A),C\wedge B \Longrightarrow A}\ \wedge L \qquad \dfrac{\dfrac{\overline{B,A\supset B,B\supset A,C \Longrightarrow C}\ \text{init}}{(A\supset B)\wedge(B\supset A),C,B \Longrightarrow C}\ \wedge L}{(A\supset B)\wedge(B\supset A),C\wedge B \Longrightarrow C}\ \wedge L}{\dfrac{(A\supset B)\wedge(B\supset A),C\wedge B \Longrightarrow C\wedge A}{(A\supset B)\wedge(B\supset A) \Longrightarrow C\wedge B\supset C\wedge A}\ \supset R}\ \wedge R
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{C,A,A,B \Longrightarrow B}\ \text{init}}{C,A,B,B\supset A \Longrightarrow B}\ P\supset L}{C,A,B\supset A,A\supset B \Longrightarrow B}\ P\supset L}{(A\supset B)\wedge(B\supset A),C,A \Longrightarrow B}\ \wedge L}{(A\supset B)\wedge(B\supset A),C\wedge A \Longrightarrow B}\ \wedge L \qquad \dfrac{\dfrac{\overline{A,A\supset B,B\supset A,C \Longrightarrow C}\ \text{init}}{(A\supset B)\wedge(B\supset A),C,A \Longrightarrow C}\ \wedge L}{(A\supset B)\wedge(B\supset A),C\wedge A \Longrightarrow C}\ \wedge L}{\dfrac{(A\supset B)\wedge(B\supset A),C\wedge A \Longrightarrow C\wedge B}{(A\supset B)\wedge(B\supset A) \Longrightarrow C\wedge A\supset C\wedge B}\ \supset R}\ \wedge R
$$

$$
\dfrac{(A\supset B)\wedge(B\supset A) \Longrightarrow (C\wedge A\supset C\wedge B)\wedge(C\wedge B\supset C\wedge A)}{\Longrightarrow (A\supset B)\wedge(B\supset A) \supset (C\wedge A\supset C\wedge B)\wedge(C\wedge B\supset C\wedge A)}\ \supset R
$$

21

$$
\infer[\wedge R]
{(A\supset B)\wedge(B\supset A)\Longrightarrow(A\vee C\supset B\vee C)\wedge(B\vee C\supset A\vee C)}
{}
$$

Left derivation:

$$
\infer[\wedge L]
{(A\supset B)\wedge(B\supset A),A\Longrightarrow B\vee C}
{\infer[\vee L]
{A,A\supset B,B\supset A\Longrightarrow B\vee C}
{\infer[P\supset L]
{A,B\supset A,A\supset B\Longrightarrow B\vee C}
{\infer[\vee L]
{A,B\supset A,B\Longrightarrow B\vee C}
{\infer[P\supset L]
{A,B,B\supset A\Longrightarrow B\vee C}
{\infer[\vee R_1]
{A,B,A\Longrightarrow B\vee C}
{\infer[\text{init}]{A,A,B\Longrightarrow B}{}}}}}}
$$

$$
\infer[\wedge L]
{(A\supset B)\wedge(B\supset A),C\Longrightarrow B\vee C}
{\infer[\vee R_2]
{C,A\supset B,B\supset A\Longrightarrow B\vee C}
{\infer[\text{init}]{A\supset B,B\supset A,C\Longrightarrow C}{}}}
$$

$$
\infer[\supset R]
{(A\supset B)\wedge(B\supset A)\Longrightarrow A\vee C\supset B\vee C}
{\infer[\vee L]
{(A\supset B)\wedge(B\supset A),A\vee C\Longrightarrow B\vee C}
{}}
$$

Right derivation:

$$
\infer[\wedge L]
{(A\supset B)\wedge(B\supset A),B\Longrightarrow A\vee C}
{\infer[\vee L]
{B,A\supset B,B\supset A\Longrightarrow A\vee C}
{\infer[P\supset L]
{B,A\supset B,B\supset A\Longrightarrow A\vee C}
{\infer[\vee L]
{B,A,A\supset B\Longrightarrow A\vee C}
{\infer[P\supset L]
{B,A\supset B,A\Longrightarrow A\vee C}
{\infer[\vee R_1]
{B,A,B\Longrightarrow A\vee C}
{\infer[\text{init}]{B,B,A\Longrightarrow A}{}}}}}}
$$

$$
\infer[\wedge L]
{C,A\supset B,B\supset A\Longrightarrow A\vee C}
{\infer[\vee R]
{A\supset B,B\supset A,C\Longrightarrow C}
{\infer[\text{init}]{A\supset B,B\supset A,C\Longrightarrow C}{}}}
$$

$$
\infer[\supset R]
{(A\supset B)\wedge(B\supset A)\Longrightarrow B\vee C\supset A\vee C}
{(A\supset B)\wedge(B\supset A),B\vee C\Longrightarrow A\vee C}
$$

$$
\infer[\supset R]
{\Longrightarrow(A\supset B)\wedge(B\supset A)\supset(A\vee C\supset B\vee C)\wedge(B\vee C\supset A\vee C)}
{(A\supset B)\wedge(B\supset A)\Longrightarrow(A\vee C\supset B\vee C)\wedge(B\vee C\supset A\vee C)}
$$

34

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\overline{B,B,A \Longrightarrow A}\ \text{init}}{B,A,B \Longrightarrow C \vee A}\ \vee R_2
}{B,A,A \supset B \Longrightarrow C \vee A}\ {\supset}
}{B,A \supset B, A \Longrightarrow C \vee A}\ \vee L
}{B,A \supset B, B \supset A \Longrightarrow C \vee A}
}{B,A \supset B, B \supset A \Longrightarrow C \vee A}
}{(A \supset B) \wedge (B \supset A), B \Longrightarrow C \vee A}
}{}
$$

$$
\dfrac{
\dfrac{
\dfrac{\overline{A \supset B, B \supset A, C \Longrightarrow C}\ \text{init}}{C, A \supset B, B \supset A \Longrightarrow C \vee A}\ \vee R_1
}{(A \supset B) \wedge (B \supset A), C \Longrightarrow C \vee A}\ \wedge L
\qquad \cdots
}{
\dfrac{(A \supset B) \wedge (B \supset A), C \vee B \Longrightarrow C \vee A}{(A \supset B) \wedge (B \supset A) \Longrightarrow C \vee B \supset C \vee A}\ \supset R
}\ \wedge R
$$

$$
\dfrac{
\dfrac{
\dfrac{\overline{A,A,B \Longrightarrow B}\ \text{init}}{A,B,A \Longrightarrow C \vee B}\ \vee R_2
}{A,B,B \supset A \Longrightarrow C \vee B}\ {\supset}L
\;\;
\dfrac{
\dfrac{A,B \supset A,A \supset B \Longrightarrow C \vee B}{A,A \supset B, B \supset A \Longrightarrow C \vee B}\ \vee L
}{}
}{
\dfrac{(A \supset B) \wedge (B \supset A), A \Longrightarrow C \vee B}{(A \supset B) \wedge (B \supset A), C \vee A \Longrightarrow C \vee B}\ \supset R
}\ \vee L
$$

$$
\dfrac{
\dfrac{
\dfrac{\overline{A \supset B, B \supset A, C \Longrightarrow C}\ \text{init}}{C, A \supset B, B \supset A \Longrightarrow C \vee B}\ \vee R_1
}{(A \supset B) \wedge (B \supset A), C \Longrightarrow C \vee B}\ \wedge L
}{
(A \supset B) \wedge (B \supset A) \Longrightarrow C \vee A \supset C \vee B
}
$$

$$
\dfrac{
(A \supset B) \wedge (B \supset A) \Longrightarrow (C \vee A \supset C \vee B) \wedge (C \vee B \supset C \vee A)
}{
\Longrightarrow (A \supset B) \wedge (B \supset A) \supset (C \vee A \supset C \vee B) \wedge (C \vee B \supset C \vee A)
}\ \supset R
$$

23

35

Left derivation:

```
                              ⊥L
        ――――――――――――――――――――――――
        B,A⊃B,A,⊥ ⟹ ⊥
        ―――――――――――――――――― P⊃L
        B,A⊃B,A,A⊃⊥ ⟹ ⊥
        ―――――――――――――――――――― P⊃L
        A⊃⊥,B,A⊃B,B⊃A ⟹ ⊥
        ――――――――――――――――――――――――― ∧L
        (A⊃B)∧(B⊃A),A⊃⊥,B ⟹ ⊥
        ――――――――――――――――――――――――― ⊃R
        (A⊃B)∧(B⊃A),A⊃⊥ ⟹ B⊃⊥
        ―――――――――――――――――――――――――――― ⊃R
        (A⊃B)∧(B⊃A) ⟹ (A⊃⊥)⊃B⊃⊥
```

Right derivation:

```
                              ⊥L
        ――――――――――――――――――――――――
        A,B⊃A,B,⊥ ⟹ ⊥
        ―――――――――――――――――― P⊃L
        A,B⊃A,B,B⊃⊥ ⟹ ⊥
        ―――――――――――――――――――― P⊃L
        B⊃⊥,A,B⊃A,A⊃B ⟹ ⊥
        ――――――――――――――――――――――――― ∧L
        (A⊃B)∧(B⊃A),B⊃⊥,A ⟹ ⊥
        ――――――――――――――――――――――――― ⊃R
        (A⊃B)∧(B⊃A),B⊃⊥ ⟹ A⊃⊥
        ―――――――――――――――――――――――――――― ⊃R
        (A⊃B)∧(B⊃A) ⟹ (B⊃⊥)⊃A⊃⊥
```

Conclusion:

```
(A⊃B)∧(B⊃A) ⟹ ((A⊃⊥)⊃B⊃⊥)∧((B⊃⊥)⊃A⊃⊥)
――――――――――――――――――――――――――――――――――――――――――――――― ∧R
  ⟹ (A⊃B)∧(B⊃A)⊃((A⊃⊥)⊃B⊃⊥)∧((B⊃⊥)⊃A⊃⊥)                ⊃R
```

36

```
            init                         init
C,B,A ⟹ A                     A,C,B ⟹ B
――――――――― ∧L                  ――――――――― ∧L
A∧B,C ⟹ A                     B∧C,A ⟹ B
――――――――― ∧L                  ――――――――― ∧L
(A∧B)∧C ⟹ A                   A∧B∧C ⟹ A∧B
```

```
            init
A,B,C ⟹ C
――――――――― ∧L
A∧B,C ⟹ C
――――――――――――― ∧R
(A∧B)∧C ⟹ B∧C
```

```
(A∧B)∧C ⟹ A∧B∧C
――――――――――――――――――――― ⊃R
⟹ (A∧B)∧C⊃A∧B∧C
```

```
            init                         init
B,C,A ⟹ A                     A,B,C ⟹ C
――――――――― ∧L                  ――――――――― ∧L
B∧C,A ⟹ A                     B∧C,A ⟹ C
――――――――― ∧L                  ――――――――― ∧L
A∧B∧C ⟹ A                     A∧B∧C ⟹ C
```

```
            init
A,C,B ⟹ B
――――――――― ∧L
B∧C,A ⟹ B
――――――――――――――― ∧R
A∧B∧C ⟹ A∧B
```

```
A∧B∧C ⟹ (A∧B)∧C
――――――――――――――――――――― ⊃R
⟹ A∧B∧C⊃(A∧B)∧C
```

```
⟹ ((A∧B)∧C⊃A∧B∧C)∧(A∧B∧C⊃(A∧B)∧C)   ∧R
```

37

$$
\dfrac{
  \dfrac{
    \dfrac{\overline{A \Longrightarrow A}\ \text{init}}{A \Longrightarrow A \vee (B \vee C)}\ \vee R_1
  }{A \vee B \Longrightarrow A \vee (B \vee C)}\ \vee R_1
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\overline{B \Longrightarrow B}\ \text{init}}{B \Longrightarrow B \vee C}\ \vee R_1
    }{B \Longrightarrow A \vee (B \vee C)}\ \vee R_2
  }{A \vee B \Longrightarrow A \vee (B \vee C)}\ \vee L
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\overline{C \Longrightarrow C}\ \text{init}}{C \Longrightarrow B \vee C}\ \vee R_2
    }{C \Longrightarrow A \vee (B \vee C)}\ \vee R_2
  }{}\
}{}
$$

$$
\dfrac{
  \dfrac{(A \vee B) \vee C \Longrightarrow A \vee (B \vee C)}{\Longrightarrow (A \vee B) \vee C \supset A \vee (B \vee C)}\ \supset R
}{}\ \vee L
$$

$$
\dfrac{
  \dfrac{\overline{A \Longrightarrow A}\ \text{init}}{A \Longrightarrow A \vee B}\ \vee R_1
}{A \Longrightarrow (A \vee B) \vee C}\ \vee R_1
\qquad
\dfrac{
  \dfrac{\overline{B \Longrightarrow B}\ \text{init}}{B \Longrightarrow A \vee B}\ \vee R_2
}{B \Longrightarrow (A \vee B) \vee C}\ \vee R_1
\qquad
\dfrac{\overline{C \Longrightarrow C}\ \text{init}}{C \Longrightarrow (A \vee B) \vee C}\ \vee R_1
$$

$$
\dfrac{B \vee C \Longrightarrow (A \vee B) \vee C}{\dfrac{A \vee (B \vee C) \Longrightarrow (A \vee B) \vee C}{\Longrightarrow A \vee (B \vee C) \supset (A \vee B) \vee C}\ \supset R}\ \vee L
$$

$$
\Longrightarrow ((A \vee B) \vee C \supset A \vee (B \vee C)) \wedge (A \vee (B \vee C) \supset (A \vee B) \vee C)\quad \wedge R
$$

38

$$
\dfrac{
  \dfrac{\overline{A, B \Longrightarrow B}\ \text{init}}{A \wedge B \Longrightarrow B}\ \wedge L
  \qquad
  \dfrac{\overline{B, A \Longrightarrow A}\ \text{init}}{A \wedge B \Longrightarrow A}\ \wedge L
}{
  \dfrac{A \wedge B \Longrightarrow B \wedge A}{\Longrightarrow A \wedge B \supset B \wedge A}\ \supset R
}\ \wedge R
$$

$$
\dfrac{
  \dfrac{\overline{B, A \Longrightarrow A}\ \text{init}}{B \wedge A \Longrightarrow A}\ \wedge L
  \qquad
  \dfrac{\overline{A, B \Longrightarrow B}\ \text{init}}{B \wedge A \Longrightarrow B}\ \wedge L
}{
  \dfrac{B \wedge A \Longrightarrow A \wedge B}{\Longrightarrow B \wedge A \supset A \wedge B}\ \supset R
}\ \wedge R
$$

$$
\Longrightarrow (A \wedge B \supset B \wedge A) \wedge (B \wedge A \supset A \wedge B)
$$

25

$$
\cfrac{
  \cfrac{\dfrac{\ }{A \Longrightarrow A}\ \text{init}}{A \Longrightarrow B \vee A}\ \vee R_2
  \qquad
  \cfrac{\dfrac{\ }{B \Longrightarrow B}\ \text{init}}{B \Longrightarrow B \vee A}\ \vee R_1
}{
  \cfrac{\cfrac{A \vee B \Longrightarrow B \vee A}{\Longrightarrow A \vee B \supset B \vee A}\ \supset R}{}
}\ \vee L
$$

$$
\cfrac{
  \cfrac{\dfrac{\ }{B \Longrightarrow B}\ \text{init}}{B \Longrightarrow A \vee B}\ \vee R_2
  \qquad
  \cfrac{\dfrac{\ }{A \Longrightarrow A}\ \text{init}}{A \Longrightarrow A \vee B}\ \vee R_1
}{
  \cfrac{B \vee A \Longrightarrow A \vee B}{\Longrightarrow B \vee A \supset A \vee B}\ \supset R
}\ \vee L
$$

$$
\Longrightarrow (A \vee B \supset B \vee A) \wedge (B \vee A \supset A \vee B) \qquad \wedge R
$$

$$
\cfrac{
  \cfrac{\dfrac{\ }{B,A \Longrightarrow A}\ \text{init} \qquad \dfrac{\ }{A,B \Longrightarrow B}\ \text{init}}{A,B \Longrightarrow A \wedge B}\ \wedge R
}{A,B \Longrightarrow A \wedge B \vee A \wedge C}\ \vee R_1
$$

$$
\cfrac{
  \cfrac{\dfrac{\ }{C,A \Longrightarrow A}\ \text{init} \qquad \dfrac{\ }{A,C \Longrightarrow C}\ \text{init}}{A,C \Longrightarrow A \wedge C}\ \wedge R
}{A,C \Longrightarrow A \wedge B \vee A \wedge C}\ \vee R_2
$$

$$
\cfrac{B \vee C, A \Longrightarrow A \wedge B \vee A \wedge C}{
  \cfrac{A \wedge (B \vee C) \Longrightarrow A \wedge B \vee A \wedge C}{\Longrightarrow A \wedge (B \vee C) \supset A \wedge B \vee A \wedge C}\ \supset R
}\ \wedge L
$$
(with $\vee L$)

$$
\cfrac{\dfrac{\ }{B,A \Longrightarrow A}\ \text{init}}{A \wedge B \Longrightarrow A}\ \wedge L
\qquad
\cfrac{\dfrac{\ }{C,A \Longrightarrow A}\ \text{init}}{A \wedge C \Longrightarrow A}\ \wedge L
$$

$$
A \wedge B \vee A \wedge C \Longrightarrow A \qquad \vee L
$$

$$
\cfrac{\dfrac{\ }{A,B \Longrightarrow B}\ \text{init}}{A,B \Longrightarrow B \vee C}\ \vee R_1
\quad\Longrightarrow\quad
\cfrac{A \wedge B \Longrightarrow B \vee C}{\ }\ \wedge L
$$

$$
\cfrac{\dfrac{\ }{A,C \Longrightarrow C}\ \text{init}}{A,C \Longrightarrow B \vee C}\ \vee R_2
\quad\Longrightarrow\quad
\cfrac{A \wedge C \Longrightarrow B \vee C}{\ }\ \wedge L
$$

$$
A \wedge B \vee A \wedge C \Longrightarrow B \vee C \qquad \vee L
$$

$$
\cfrac{A \wedge B \vee A \wedge C \Longrightarrow A \wedge (B \vee C)}{\Longrightarrow A \wedge B \vee A \wedge C \supset A \wedge (B \vee C)}\ \supset R
\qquad \wedge R
$$

$$
\Longrightarrow (A \wedge (B \vee C) \supset A \wedge B \vee A \wedge C) \wedge (A \wedge B \vee A \wedge C \supset A \wedge (B \vee C)) \qquad \wedge R
$$

41

$$
\dfrac{
  \dfrac{
    \dfrac{\overline{A \Longrightarrow A}\ \text{init}}{A \Longrightarrow A \vee B}\ \vee R_1
    \qquad
    \dfrac{\dfrac{\overline{C,B \Longrightarrow B}\ \text{init}}{B,C \Longrightarrow A \vee B}\ \vee R_2}{\dfrac{B \wedge C \Longrightarrow A \vee B}{\ }\ \wedge L}\ \vee L
  }{A \vee B \wedge C \Longrightarrow A \vee B}
  \qquad
  \dfrac{
    \dfrac{\dfrac{\overline{B,C \Longrightarrow C}\ \text{init}}{B,C \Longrightarrow A \vee C}\ \vee R_2}{B \wedge C \Longrightarrow A \vee C}\ \wedge L \quad \dfrac{\overline{A \Longrightarrow A}\ \text{init}}{A \Longrightarrow A \vee C}\ \vee R_1
  }{A \vee B \wedge C \Longrightarrow A \vee C}\ \vee L
}{
  \dfrac{A \vee B \wedge C \Longrightarrow (A \vee B) \wedge (A \vee C)}{\Longrightarrow A \vee B \wedge C \supset (A \vee B) \wedge (A \vee C)}\ \supset R
}\ \wedge R
$$

$$
\dfrac{
  \dfrac{
    \dfrac{\overline{A,A \Longrightarrow A}\ \text{init}}{A,A \Longrightarrow A \vee B \wedge C}\ \vee R_1
    \qquad
    \dfrac{\overline{B,A \Longrightarrow A}\ \text{init}}{A,B \Longrightarrow A \vee B \wedge C}\ \vee R_1
  }{A \vee B, A \Longrightarrow A \vee B \wedge C}\ \vee L
  \qquad \cdots
}{
  \dfrac{A \vee B, A \vee C \Longrightarrow A \vee B \wedge C}{\dfrac{(A \vee B) \wedge (A \vee C) \Longrightarrow A \vee B \wedge C}{\Longrightarrow (A \vee B) \wedge (A \vee C) \supset A \vee B \wedge C}}
}
$$

$$
\Longrightarrow (A \vee B \wedge C \supset (A \vee B) \wedge (A \vee C)) \wedge ((A \vee B) \wedge (A \vee C) \supset A \vee B \wedge C)
$$

42

$$
\dfrac{
  \dfrac{\dfrac{\overline{A,A \Longrightarrow A}\ \text{init}}{A \wedge A \Longrightarrow A}\ \wedge L}{\Longrightarrow A \wedge A \supset A}\ \supset R
  \qquad
  \dfrac{\dfrac{\overline{A \Longrightarrow A}\ \text{init} \quad \overline{A \Longrightarrow A}\ \text{init}}{A \Longrightarrow A \wedge A}\ \wedge R}{\Longrightarrow A \supset A \wedge A}\ \supset R
}{\Longrightarrow (A \wedge A \supset A) \wedge (A \supset A \wedge A)}\ \wedge R
$$

27

43

$$
\cfrac{
\cfrac{
\cfrac{\cfrac{}{A \Longrightarrow A}\text{init} \quad \cfrac{}{A \Longrightarrow A}\text{init}}{A \lor A \Longrightarrow A}\lor L
}{\Longrightarrow A \lor A \supset A}\supset R
\quad
\cfrac{
\cfrac{\cfrac{}{A \Longrightarrow A}\text{init}}{A \Longrightarrow A \lor A}\lor R_1
}{\Longrightarrow A \supset A \lor A}\supset R
}{\Longrightarrow (A \lor A \supset A) \land (A \supset A \lor A)}\land R
$$

44

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\cfrac{}{A \Longrightarrow A}\text{init} \quad \cfrac{}{B,A \Longrightarrow A}\text{init}}{A \lor B, A \Longrightarrow A}\lor L
}{A \land (A \lor B) \Longrightarrow A}\land L
}{\Longrightarrow A \land (A \lor B) \supset A}\supset R
\quad
\cfrac{
\cfrac{\cfrac{}{A \Longrightarrow A}\text{init}}{A \Longrightarrow A \land (A \lor B)}
}{\Longrightarrow A \supset A \land (A \lor B)}\supset R
}{\Longrightarrow (A \land (A \lor B) \supset A) \land (A \supset A \land (A \lor B))}\land R
$$

45

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\cfrac{}{B,A \Longrightarrow A}\text{init}}{A \land B \Longrightarrow A}\land L
}{A \lor A \land B \Longrightarrow A}\lor L
}{\Longrightarrow A \lor A \land B \supset A}\supset R
\quad
\cfrac{
\cfrac{\cfrac{}{A \Longrightarrow A}\text{init}}{A \Longrightarrow A \lor A \land B}\lor R_1
}{\Longrightarrow A \supset A \lor A \land B}\supset R
}{\Longrightarrow (A \lor A \land B \supset A) \land (A \supset A \lor A \land B)}\land R
$$

28

46

$$\frac{\dfrac{\overline{A, A, B \Longrightarrow B}\ \text{init}}{A, A \Longrightarrow B \supset B}\supset R}{A \Longrightarrow A \supset B \supset B}\supset R$$

$$\frac{\overline{A, A, B \Longrightarrow B}\ \text{init}}{A, B \Longrightarrow A \supset B}\supset R$$
$$\frac{}{A \Longrightarrow B \supset A \supset B}\supset R$$
$$\frac{A \Longrightarrow (A \supset B \supset B) \land (B \supset A \supset B)}{\Longrightarrow A \supset (A \supset B \supset B) \land (B \supset A \supset B)}\begin{array}{l}\land R\\[2pt]\supset R\end{array}$$

47

$$\frac{\dfrac{\overline{A, B, B \Longrightarrow B}\ \text{init}}{B, A \Longrightarrow B \supset B}\supset R}{B \Longrightarrow A \supset B \supset B}\supset R$$

$$\frac{\overline{B, A, B \Longrightarrow B}\ \text{init}}{B, B \Longrightarrow A \supset B}\supset R$$
$$\frac{}{B \Longrightarrow B \supset A \supset B}\supset R$$
$$\frac{B \Longrightarrow (A \supset B \supset B) \land (B \supset A \supset B)}{\Longrightarrow B \supset (A \supset B \supset B) \land (B \supset A \supset B)}\begin{array}{l}\land R\\[2pt]\supset R\end{array}$$

48

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{A,B,A,\bot \Longrightarrow \bot}\ \bot L}
{A,B,A,A\supset\bot \Longrightarrow \bot}\ P\supset L}
{A\supset\bot,A,B \Longrightarrow A\supset\bot}\ \supset R}
{A\supset\bot,A \Longrightarrow B\supset A\supset\bot}\ \supset R}
{A\supset\bot \Longrightarrow A\supset B\supset A\supset\bot}\ \supset R
\qquad
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{A\supset\bot,A,\bot \Longrightarrow B}\ \bot L}
{A\supset\bot,A,A\supset\bot \Longrightarrow B}\ P\supset L}
{A\supset\bot,A\supset\bot \Longrightarrow A\supset B}\ \supset R}
{A\supset\bot \Longrightarrow (A\supset\bot)\supset A\supset B}\ \supset R
}
{\cfrac{A\supset\bot \Longrightarrow (A\supset B\supset A\supset\bot)\wedge((A\supset\bot)\supset A\supset B)}
{\Longrightarrow (A\supset\bot)\supset(A\supset B\supset A\supset\bot)\wedge((A\supset\bot)\supset A\supset B)}\ \supset R}\ \wedge R
$$

49

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{A,B,A,\bot \Longrightarrow \bot}\ \bot L}
{A,B,A,B\supset\bot \Longrightarrow \bot}\ P\supset L}
{B\supset\bot,A,B \Longrightarrow A\supset\bot}\ \supset R}
{B\supset\bot,A \Longrightarrow B\supset A\supset\bot}\ \supset R}
{B\supset\bot \Longrightarrow A\supset B\supset A\supset\bot}\ \supset R
\qquad
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{B\supset\bot,A,\bot \Longrightarrow B}\ \bot L}
{B\supset\bot,A,A\supset\bot \Longrightarrow B}\ P\supset L}
{B\supset\bot,A\supset\bot \Longrightarrow A\supset B}\ \supset R}
{B\supset\bot \Longrightarrow (A\supset\bot)\supset A\supset B}\ \supset R
}
{\cfrac{B\supset\bot \Longrightarrow (A\supset B\supset A\supset\bot)\wedge((A\supset\bot)\supset A\supset B)}
{\Longrightarrow (B\supset\bot)\supset(A\supset B\supset A\supset\bot)\wedge((A\supset\bot)\supset A\supset B)}\ \supset R}\ \wedge R
$$

50

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{B,B,A \Longrightarrow A}{A\land B, B \Longrightarrow A}\ \land L
}{B \Longrightarrow A\land B \supset A}\ \supset R
\qquad
\cfrac{
\cfrac{
B,A \Longrightarrow A \ \text{init}
\qquad
\cfrac{A,B \Longrightarrow B}{B,A \Longrightarrow A\land B}\ \land R
}{B,A \Longrightarrow A\land B}
}{B \Longrightarrow A \supset A\land B}\ \supset R
}{B \Longrightarrow (A\land B \supset A)\land(A\supset A\land B)}\ \land R
}{\Longrightarrow B \supset (A\land B \supset A)\land(A\supset A\land B)}\ \supset R
$$

$B,B,A \Longrightarrow A$ init, $B,A \Longrightarrow A$ init, $A,B \Longrightarrow B$ init

51

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{A,B \Longrightarrow B \quad B,B \Longrightarrow B}{A\lor B, B \Longrightarrow B}\ \lor L
}{B \Longrightarrow A\lor B \supset B}\ \supset R
\qquad
\cfrac{
\cfrac{B,B \Longrightarrow B}{B,B \Longrightarrow A\lor B}\ \lor R_2
}{B \Longrightarrow B \supset A\lor B}\ \supset R
}{B \Longrightarrow (A\lor B \supset B)\land(B\supset A\lor B)}\ \land R
}{\Longrightarrow B \supset (A\lor B \supset B)\land(B\supset A\lor B)}\ \supset R
$$

$A,B \Longrightarrow B$ init, $B,B \Longrightarrow B$ init

31

52

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{A,B,\bot \Longrightarrow B}\;\bot L}{A,B,B\supset\bot\Longrightarrow B}\;P\supset L
}{A\land B,B\supset\bot\Longrightarrow B}\;\land L
}{B\supset\bot\Longrightarrow A\land B\supset B}\;\supset R
\qquad
\cfrac{
\cfrac{\;
\cfrac{\overline{B,\bot \Longrightarrow A}\;\bot L}{B,B\supset\bot\Longrightarrow A}\;P\supset L
\qquad
\cfrac{\overline{B,\bot \Longrightarrow B}\;\bot L}{B,B\supset\bot\Longrightarrow B}\;P\supset L
}{B\supset\bot,B\Longrightarrow A\land B}\;\land R
}{B\supset\bot\Longrightarrow B\supset A\land B}\;\supset R
}{
\cfrac{B\supset\bot\Longrightarrow(A\land B\supset B)\land(B\supset A\land B)}{\Longrightarrow(B\supset\bot)\supset(A\land B\supset B)\land(B\supset A\land B)}\;\supset R
}\;\land R
$$

53

$$
\cfrac{
\cfrac{
\cfrac{\overline{B\supset\bot,A\Longrightarrow A}\;init
\qquad
\cfrac{\overline{B,\bot \Longrightarrow A}\;\bot L}{B,B\supset\bot\Longrightarrow A}\;P\supset L
}{A\lor B,B\supset\bot\Longrightarrow A}\;\lor L
}{B\supset\bot\Longrightarrow A\lor B\supset A}\;\supset R
\qquad
\cfrac{
\cfrac{\overline{B\supset\bot,A\Longrightarrow A}\;init}{B\supset\bot,A\Longrightarrow A\lor B}\;\lor R_1
}{B\supset\bot\Longrightarrow A\supset A\lor B}\;\supset R
}{
\cfrac{B\supset\bot\Longrightarrow(A\lor B\supset A)\land(A\supset A\lor B)}{\Longrightarrow(B\supset\bot)\supset(A\lor B\supset A)\land(A\supset A\lor B)}\;\supset R
}\;\land R
$$

32

54

$$\frac{\dfrac{\dfrac{\dfrac{}{A, \bot \Longrightarrow \bot}\,\bot L}{A, A\supset\bot \Longrightarrow \bot}\,P\supset L}{A \Longrightarrow (A\supset\bot)\supset\bot}\,\supset R}{\Longrightarrow A \supset (A\supset\bot)\supset\bot}\,\supset R$$

55

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, \bot \Longrightarrow \bot}\,\bot L}{A, A\supset\bot \Longrightarrow \bot}\,P\supset L}{\bot\supset\bot, A, A\supset\bot \Longrightarrow \bot}\,\bot\supset L \qquad \dfrac{}{A, \bot \Longrightarrow \bot}\,\bot L}{A, ((A\supset\bot)\supset\bot)\supset\bot \Longrightarrow \bot\supset\bot}\,\supset\supset L}{\dfrac{((A\supset\bot)\supset\bot)\supset\bot \Longrightarrow A\supset\bot\supset\bot}{\Longrightarrow (((A\supset\bot)\supset\bot)\supset\bot)\supset A\supset\bot\supset\bot}\,\supset R}\,\supset R$$

33

56

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{A,\bot \Longrightarrow \bot}{}\ {\bot L}
        }{A, A\supset\bot \Longrightarrow \bot}\ {P\supset L}
      }{\bot\supset\bot, A\supset\bot, A \Longrightarrow \bot}\ {\bot\supset L}
      \qquad
      (A\supset\bot)\supset\bot, A \Longrightarrow A\ \ \text{init}
    }{A\supset\bot, (A\supset\bot)\supset\bot, A \Longrightarrow A}\ {\supset\supset L}
    \qquad
    \cfrac{A\supset\bot, \bot \Longrightarrow A}{}\ {\bot L}
  }{A\vee(A\supset\bot), (A\supset\bot)\supset\bot \Longrightarrow A}\ {\vee L}
}{
  \cfrac{A\vee(A\supset\bot) \Longrightarrow ((A\supset\bot)\supset\bot)\supset A}{\Longrightarrow A\vee(A\supset\bot)\supset(((A\supset\bot)\supset\bot)\supset A)}\ {\supset R}
}\ {\supset R}
$$

57

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{A,\bot \Longrightarrow \bot}{}\ {\bot L}
    }{A, A\supset\bot \Longrightarrow \bot}\ {P\supset L}
  }{A\wedge(A\supset\bot) \Longrightarrow \bot}\ {\wedge L}
}{\Longrightarrow A\wedge(A\supset\bot)\supset\bot}\ {\supset R}
$$

58

$$
\cfrac{
\cfrac{}{A \Longrightarrow A}\ \text{init}
\qquad
\cfrac{
\cfrac{
\cfrac{
\cfrac{}{B,\bot \Longrightarrow A}\ \bot L
}{B,B\supset\bot \Longrightarrow A}\ P{\supset}L
}{B\land(B\supset\bot) \Longrightarrow A}\ \land L
}{A\lor B\land(B\supset\bot) \Longrightarrow A}\ \lor L
}{\Longrightarrow A\lor B\land(B\supset\bot)\supset A}\ \supset R
$$

$$
\cfrac{
\cfrac{
\cfrac{}{A \Longrightarrow A}\ \text{init}
}{A \Longrightarrow A\lor B\land(B\supset\bot)}\ \lor R_1
}{\Longrightarrow A\supset A\lor B\land(B\supset\bot)}\ \supset R
$$

$$
\Longrightarrow (A\lor B\land(B\supset\bot)\supset A)\land(A\supset A\lor B\land(B\supset\bot))\ \land R
$$

59

$$
\cfrac{
\cfrac{
\cfrac{}{A,B\supset\bot,B \Longrightarrow B}\ \text{init}
}{B\land(B\supset\bot),A \Longrightarrow B}\ \land L
}{A\land B\land(B\supset\bot) \Longrightarrow B}\ \land L
$$

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{}{B,A,B,\bot \Longrightarrow \bot}\ \bot L
}{B,A,B,B\supset\bot \Longrightarrow \bot}\ P{\supset}L
}{B\land(B\supset\bot),B,A \Longrightarrow \bot}\ \land L
}{A\land B\land(B\supset\bot),B \Longrightarrow \bot}\ \land L
}{A\land B\land(B\supset\bot) \Longrightarrow B\supset\bot}\ \supset R
$$

$$
\cfrac{A\land B\land(B\supset\bot) \Longrightarrow B\land(B\supset\bot)}{A\land B\land(B\supset\bot) \Longrightarrow B\land(B\supset\bot)}\ \land R
$$

$$
\cfrac{A\land B\land(B\supset\bot) \Longrightarrow B\land(B\supset\bot)}{\Longrightarrow A\land B\land(B\supset\bot)\supset B\land(B\supset\bot)}\ \supset R
$$

$$
\cfrac{
\cfrac{
\cfrac{}{B,B,\bot \Longrightarrow \bot}\ \bot L
}{B,B,B\supset\bot \Longrightarrow \bot}\ P{\supset}L
}{B\land(B\supset\bot),B \Longrightarrow \bot}\ \land L
$$

$$
\cfrac{B\land(B\supset\bot) \Longrightarrow B\supset\bot}{B\land(B\supset\bot) \Longrightarrow B\supset\bot}\ \supset R
$$

$$
\cfrac{
\cfrac{}{B\supset\bot,B \Longrightarrow B}\ \text{init}
}{B\land(B\supset\bot) \Longrightarrow B}\ \land L
$$

$$
\cfrac{B\land(B\supset\bot) \Longrightarrow B\land(B\supset\bot)}{B\land(B\supset\bot) \Longrightarrow A\land B\land(B\supset\bot)}\ \land R
$$

$$
\cfrac{B\land(B\supset\bot) \Longrightarrow A\land B\land(B\supset\bot)}{\Longrightarrow B\land(B\supset\bot)\supset A\land B\land(B\supset\bot)}\ \supset R
$$

$$
\Longrightarrow (A\land B\land(B\supset\bot)\supset B\land(B\supset\bot))\land(B\land(B\supset\bot)\supset A\land B\land(B\supset\bot))\ \land R
$$