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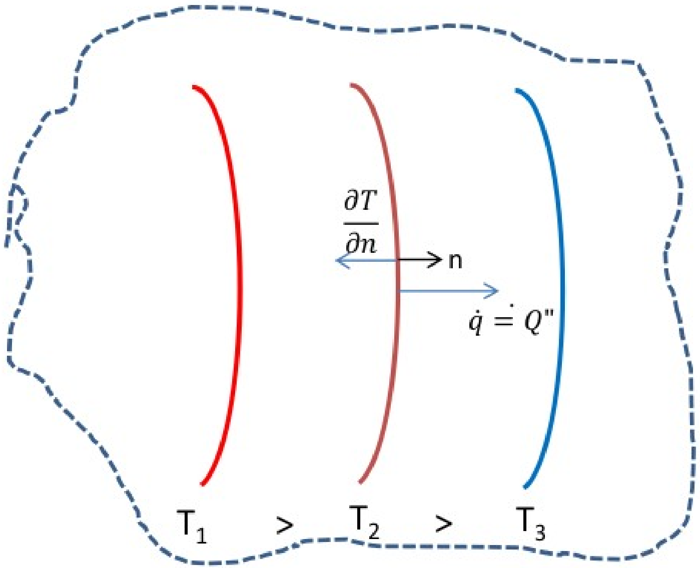
Numerical Heat Transfer
for Applications

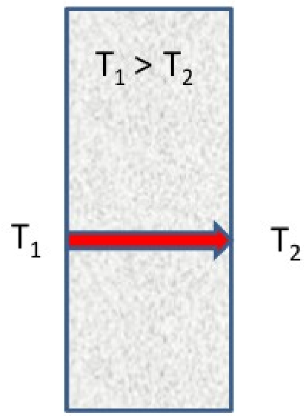
# Basics of Heat Transfer

Dr Valerio D'Alessandro

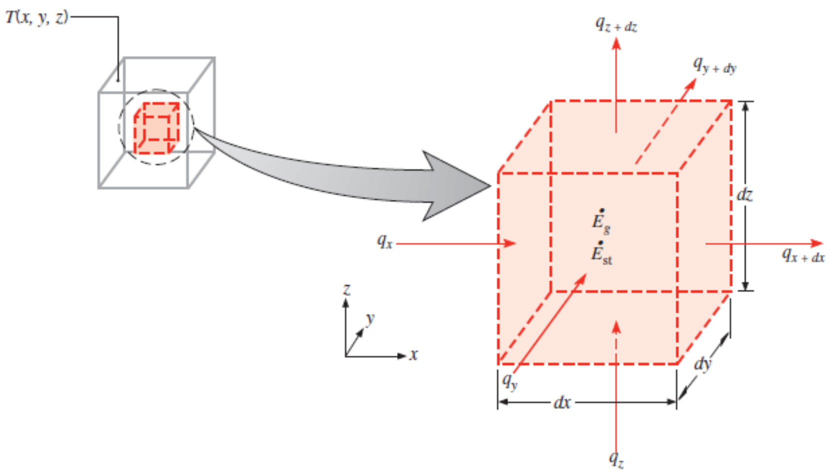
# Heat conduction equation

FOURIER LAW $q = -\lambda \frac{dT}{dx}$ $\Rightarrow$ $\mathbf{q} = -\lambda \nabla T$



# Heat conduction equation




Starting from a differential control volume centered in a point P is possible to obtain the heat conduction equation. Specifically, we obtain our model using the conservation of energy taking into account the heat flux in three directions. Also internal heat generations are considered.

**Thermal powerd entering on “x” :** $q_x dydz$

**Thermal power outgoing from “x+dx” :** $\left(q_x + \frac{\partial q_x}{\partial x} dx\right) dydz$

**Resulting power in “x” direction:** $q_x dydz - \left(q_x + \frac{\partial q_x}{\partial x} dx\right) dydz = -\frac{\partial q_x}{\partial x} dxdydz$

## Heat conduction equation

**Resulting power in "x" direction:** $q_x dydz - \left(q_x + \frac{\partial q_x}{\partial x}dx\right)dydz = -\frac{\partial q_x}{\partial x}dxdydz$

**Resulting power in "y" direction:** $q_y dxdz - \left(q_y + \frac{\partial q_y}{\partial y}dy\right)dxdz = -\frac{\partial q_y}{\partial y}dxdydz$

**Resulting power in "z" direction:** $q_z dxdy - \left(q_z + \frac{\partial q_z}{\partial z}dz\right)dxdy = -\frac{\partial q_z}{\partial z}dxdydz$

**Internal heat generation** $Gdxdydz$

## Energy conservation: $\sum \dot{Q}_{in} - \sum \left|\dot{Q}_{out}\right| = \frac{dU}{dt}$

$$\left(\sum \dot{Q}_{in} - \sum \left|\dot{Q}_{out}\right| = \frac{dU}{dt}\right)_x + \left(\sum \dot{Q}_{in} - \sum \left|\dot{Q}_{out}\right| = \frac{dU}{dt}\right)_y + \left(\sum \dot{Q}_{in} - \sum \left|\dot{Q}_{out}\right| = \frac{dU}{dt}\right)_z + \dot{Q}_{g,int} = \frac{dU}{dt}$$

$$-\frac{\partial q_x}{\partial x}dxdydz - \frac{\partial q_y}{\partial y}dxdydz - \frac{\partial q_z}{\partial z}dxdydz + Gdxdydz = \frac{d}{dt}\left(\rho dxdydz c_p T\right)$$

## Heat conduction equation

**Resulting power in "x" direction:** $q_x dydz - \left( q_x + \frac{\partial q_x}{\partial x} dx \right) dydz = -\frac{\partial q_x}{\partial x} dxdydz$

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**Resulting power in "z" direction:** $q_z dxdy - \left( q_z + \frac{\partial q_z}{\partial z} dz \right) dxdy = -\frac{\partial q_z}{\partial z} dxdydz$

**Internal heat generation** $Gdxdydz$

## Energy conservation: $\sum \dot{Q}_{in} - \sum \left| \dot{Q}_{out} \right| = \frac{dU}{dt}$

$$\left( \sum \dot{Q}_{in} - \sum \left| \dot{Q}_{out} \right| = \frac{dU}{dt} \right)_x + \left( \sum \dot{Q}_{in} - \sum \left| \dot{Q}_{out} \right| = \frac{dU}{dt} \right)_y + \left( \sum \dot{Q}_{in} - \sum \left| \dot{Q}_{out} \right| = \frac{dU}{dt} \right)_z + \dot{Q}_{g,int} = \frac{dU}{dt}$$

$$-\frac{\partial q_x}{\partial x} \cancel{dxdydz} - \frac{\partial q_y}{\partial y} \cancel{dxdydz} - \frac{\partial q_z}{\partial z} \cancel{dxdydz} + G\cancel{dxdydz} = \frac{d}{dt} \left( \rho \cancel{dxdydz} c_p T \right)$$

## Heat conduction equation

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} + G = \rho c_p \frac{\partial T}{\partial t} \quad \Rightarrow \quad -\nabla \cdot \mathbf{q} + G = \rho c_p \frac{\partial T}{\partial t}$$

taking into account Fourier law we obtain the final equation:

$$\nabla \cdot (\lambda \nabla T) + G = \rho c_p \frac{\partial T}{\partial t}$$

Considering a uniform medium then the thermophysical properties are not depended from a specific point so the heat conduction equation can be re-written as follows

$$\boxed{\nabla^2 T + \frac{G}{\lambda} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

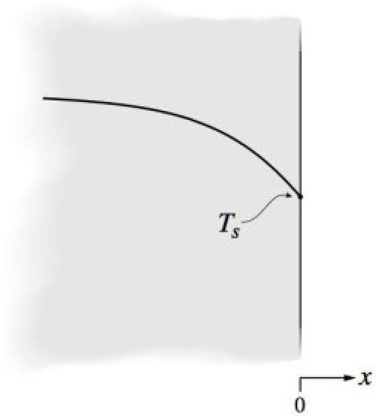
$$\nabla^2 T + \frac{G}{\lambda} = 0$$ Poisson equation

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ Fourier equation

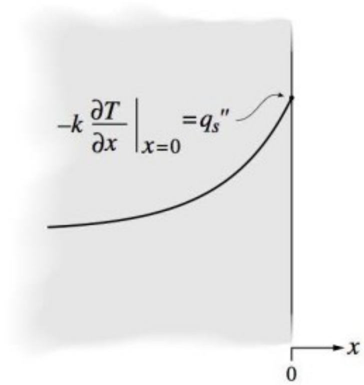
$$\nabla^2 T = 0$$ Laplace equation

# Heat conduction equation

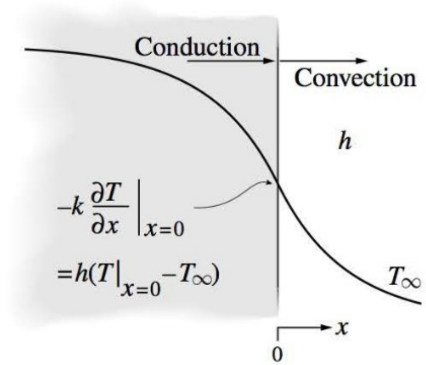
I kind

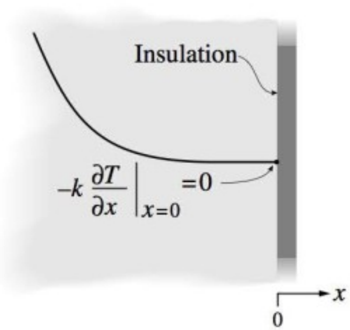



II kind

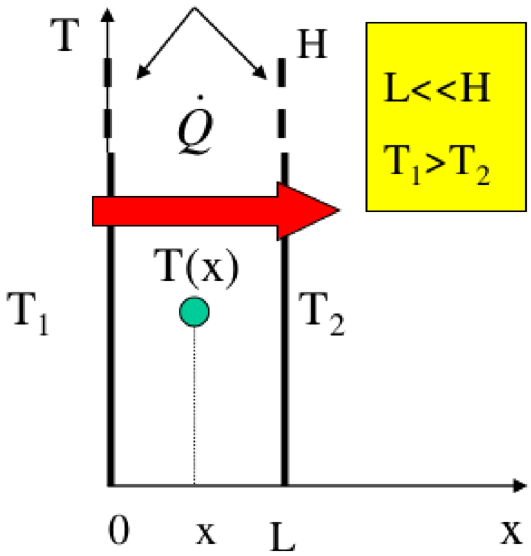



III kind



## Steady state heat conduction a 1-D plate




- Steady state
- No heat generation
- 1-D heat flux
- Isotropic and homogeneous medium

$$\nabla^2 T = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial z^2} = 0$$

- 1-D heat flux

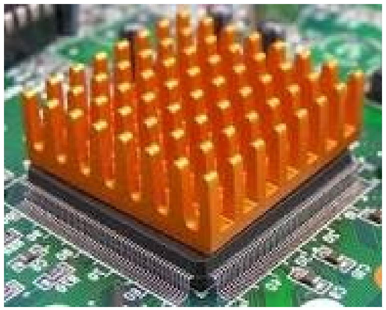
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial z^2} = 0 \Rightarrow \frac{d^2 T}{dx^2} = 0$$

- Boundary conditions: $T(x=0) = T_1 \qquad T(x=L) = T_2$

$$\boxed{T(x) = T_1 - \frac{T_1 - T_2}{L} x}$$

$$\boxed{\dot{Q} = -\lambda A \frac{dT(x)}{dx} = \frac{\lambda A}{L}(T_1 - T_2)}$$

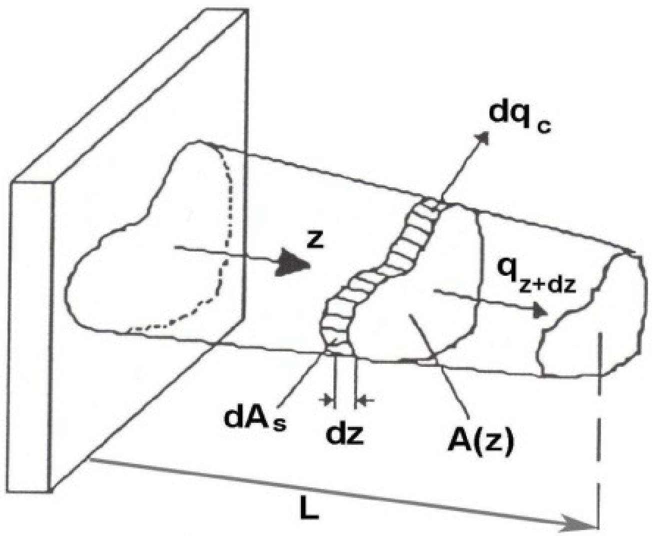
# Electronic devices



| Component family | Make and model | Package size ($cm^2$) | Heat flux ($W\ cm^{-2}$) | Allowable temperature (°C) |
|---|---|---|---|---|
| CPU | Intel Xeon E5 268Wv2 | 23.62 | 6.4 | 72 |
| | Intel Xeon E5 4627v2 | 23.62 | 5.5 | 88 |
| | Intel Xeon Gold 6154 | 42.94 | 4.7 | 82 |
| | Intel Xeon Platinum 8280 | 42.94 | 4.8 | 84 |
| | Intel Xeon Platinum 8260 | 42.94 | 3.8 | 90 |
| | Intel Xeon W3275M | 42.94 | 4.8 | 76 |
| | IBM Power9 02CY227 | 25.00 | 7.6 | 85 |
| | IBM Power9 02CY414 | 25.00 | 6.4 | 85 |
| | AMD Ryzen Threadtripper 3990X | 44.11 | 6.3 | 68 |
| | Intel Core i9 10990XE | 23.62 | 16 | 86 |
| GPU (co-processor) | NVIDIA GM100 | 1.48 | 152 | 75 |
| | NVIDIA GP100 | 6.10 | 49 | 80 |
| | NVIDIA GP102 | 4.71 | 53 | 80 |
| | NVIDIA GP104 | 3.14 | 57 | 80 |
| | NVIDIA GP106 | 2.00 | 60 | 80 |
| | NVIDIA GV100 | 8.15 | 37 | 83 |
| | NVIDIA TU102 | 7.54 | 37 | 85 |
| | NVIDIA TU104 | 5.45 | 46 | 85 |
| | NVIDIA TU106 | 4.45 | 39 | 85 |
| | NVIDIA TU116 | 2.84 | 44 | 85 |
| | NVIDIA GA100 | 8.26 | 48 | 90 |
| | NVIDIA GA102 | 7.00 | 39 | 90 |
| SiC transistors | STM SCTH35N65G2V-7 | 1.19 | 175 | 175 |
| | SLSIC1MO120E0080 | 3.12 | 181 | 175 |
| | STM SCTW100N120G2AG | 2.46 | 73 | 175 |
| | NTBG040N120SC1 | 9.10 | 195 | 175 |
| | NVH4L020N120SC1 | 2.11 | 121 | 175 |

Sarkar et al. Review of jet impingement cooling of electronic devices: Emerging role of surface engineering, International Journal of Heat and Mass Transfer, Vol 206, 123888, 2023.

## Fin equation




External surfaces are typically adopted for thermal management of high power electronic equipment.
The enegrgy conservation can be re-written as follows:

$$\dot{q}\left(z\right) = \dot{q}\left(z + dz\right) + \dot{q}_c$$

$$\dot{q}\left(z\right) = \left(\dot{q}\left(z\right) + \frac{d\dot{q}}{dz}dz\right) + d\dot{q}_c$$

$$-\frac{d\dot{q}}{dz}dz = hpdz\left(T\left(z\right) - T_\infty\right)$$

$$\boxed{\frac{d^2T}{dz^2} = \frac{hp}{\lambda A}\left(T\left(z\right) - T_\infty\right)}$$

$$T\left(x = 0\right) = T_b$$

$$\left.\frac{dT}{dx}\right|_{x=L} = 0$$

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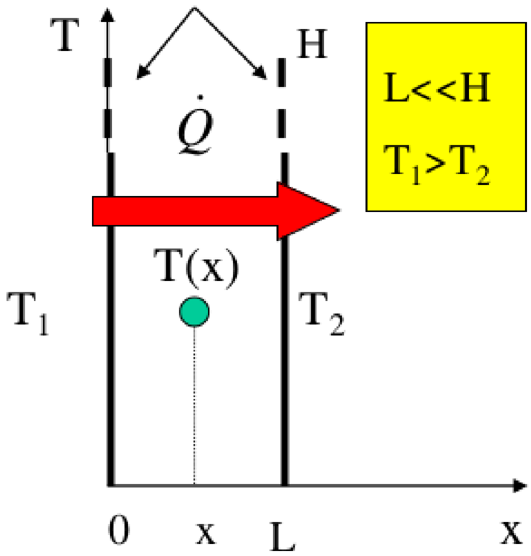
$$\frac{\theta\left(x\right)}{\theta_b} = \frac{1 - \cosh\left(mL\left(1 - \frac{z}{L}\right)\right)}{\cosh\left(mL\right)} \qquad \theta\left(x\right) = T\left(x\right) - T_\infty \qquad m^2 = \frac{hp}{\lambda A}$$

## Steady state heat conduction a 1-D plate




- Steady state
- No heat generation
- 1-D heat flux
- Isotropic and homogeneous medium

$$\nabla^2 T = 0 \quad \Rightarrow \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial z^2} = 0$$

- 1-D heat flux

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial z^2} = 0 \quad \Rightarrow \quad \frac{d^2 T}{dx^2} = 0$$

- Boundary conditions: $T(x = 0) = T_1 \qquad T(x = L) = T_2$

$$\boxed{T(x) = T_1 - \frac{T_1 - T_2}{L} x}$$

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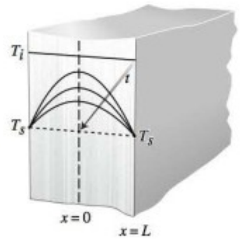
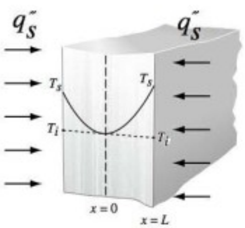
# Unsteady heat conductiuon

| Geometry | Temperature Profile[1] |
|---|---|

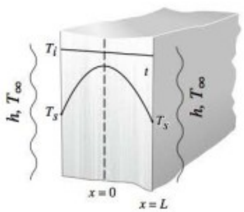



$$\frac{T-T_s}{T_i-T_s}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2L}e^{-\left(\frac{(2n+1)\pi}{2}\right)^2\frac{\alpha t}{L^2}}$$




$$T = T_i + 2A\alpha t + Ax^2 - \frac{AL^2}{3} - \frac{4AL^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}e^{-(n\pi)^2\frac{\alpha t}{L^2}}$$

where $A = q_s''/(2kL)$




$$\frac{T-T_\infty}{T_i-T_\infty} = \sum_{n=1}^{\infty}\frac{2(\lambda_n^2+(h^2/k^2))}{h/k+L(\lambda_n^2+(h^2/k^2))} \times \sin(\lambda_n L)\cos(\lambda_n x)e^{-\lambda_n^2\alpha t}$$

where $\lambda_n$ is given by $\lambda_n \tan(\lambda_n L) = h/k$

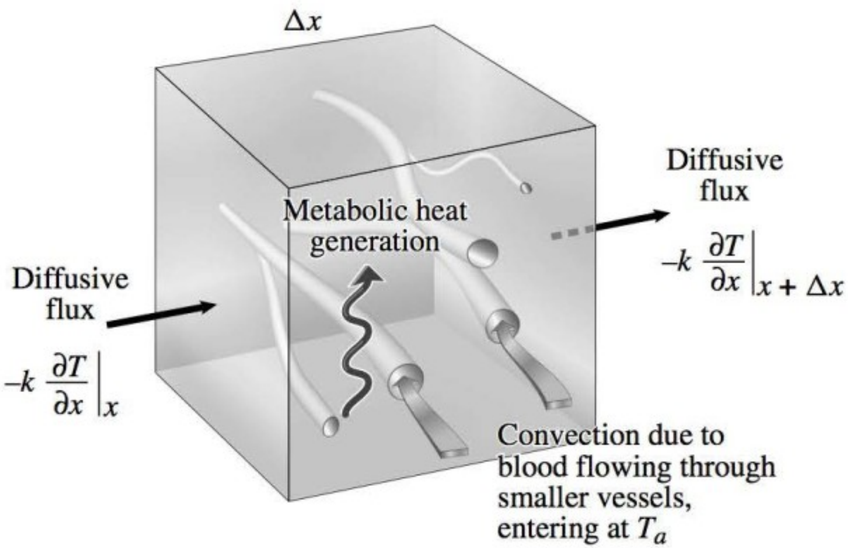
[1] The constant heat flux solution is taken from Carslaw and Jaeger (1992), page 112.

Heat conduction in equation has a limited range of anlytical solutions. Also 1D unsteady problems can be faced in a restricted number of configurations through analytical methods.

Therefore, it is very easy to understand the relevance of numerical methods for similar problems.

$$Fo = \frac{\alpha t}{L^2}$$

# Heat transfer in biological tissues – Pennes equation




Pennes model relies on continuum model approach for the sake of modelling the heat transfer in a biological tissue.
Specifically, the heat transfer rate is proportional to the temperature difference between arterial blood and venous one.

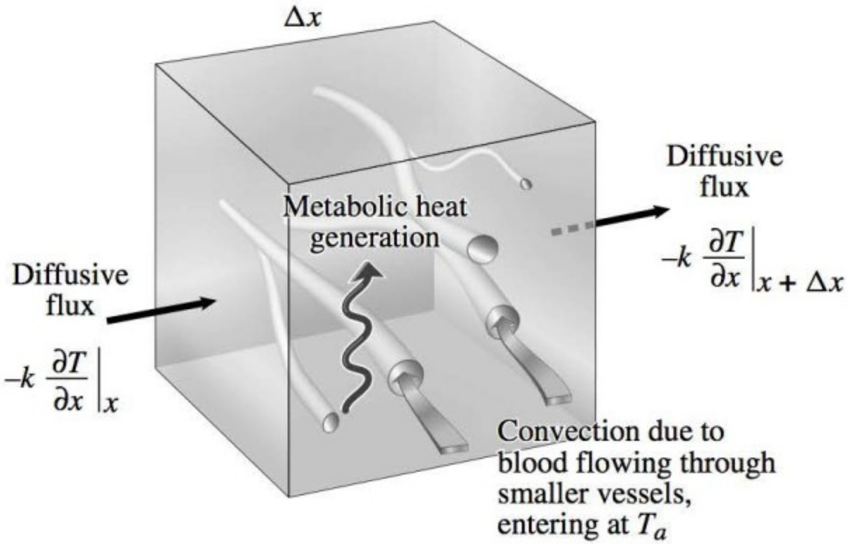
$$\dot{Q} = \dot{m}_b c_{p,b} \left(T_a - T_v\right)$$

The blood perfusione terms is taken into account in the heat conduction equation as follows:

$$G = \frac{\dot{Q}}{V} = \rho_b \underbrace{\left(\frac{\dot{V}_b}{V}\right)}_{\omega_b} c_{p,b} \left(T_a - T_v\right)$$

$$\nabla \cdot \left(\lambda_t \nabla T_t\right) + G = \rho_t c_{p,t} \frac{\partial T_t}{\partial t} \Rightarrow \boxed{\nabla \cdot \left(\lambda_t \nabla T_t\right) + \rho_b c_{p,b} \omega_b \left(T_a - T_v\right) = \rho_t c_{p,t} \frac{\partial T_t}{\partial t}}$$

## Heat transfer in biological tissues – Pennes equation




$$\boxed{\nabla \cdot (\lambda_t \nabla T_t) + \rho_b c_{p,b} \omega_b (T_a - T_v) = \rho_t c_{p,t} \frac{\partial T_t}{\partial t}}$$

Venous blood temperature depends from the thermal equilibrium degree with the tissue:

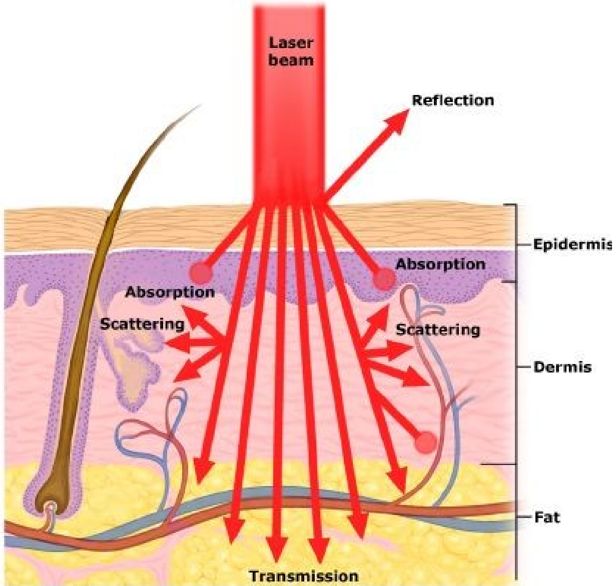
$$T_v = T_t + k' (T_a - T_t)$$

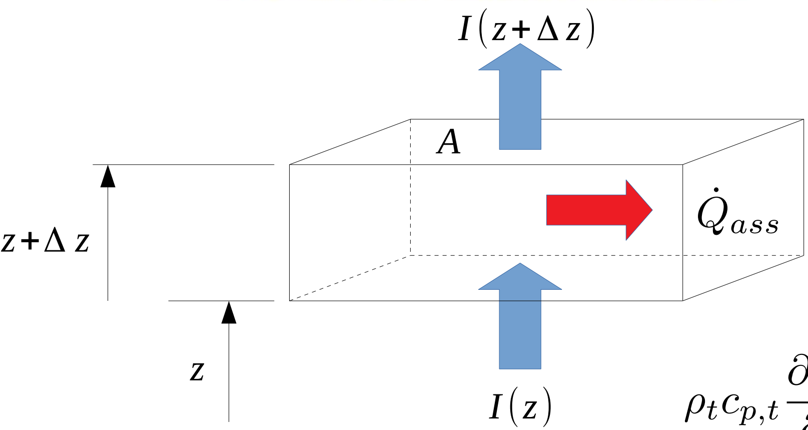
Pennes model adopts the following hypotesis:
(i) arterial temperature, $T_a$ , is uniform;
(ii) k' → 0, i.e. $T_v = T_t$;

$$G = \rho_b \omega_b c_{p,b} (T_{a,0} - T_t)$$

$$\nabla \cdot (\lambda_t \nabla T_t) + G = \rho_t c_{p,t} \frac{\partial T_t}{\partial t} \Rightarrow \boxed{\nabla \cdot (\lambda_t \nabla T_t) + \rho_b c_{p,b} \omega_b (T_{a,0} - T_t) = \rho_t c_{p,t} \frac{\partial T_t}{\partial t}}$$

## LASER heating of a biological tissue




A biological tissue irradiated by LASER light put in evindence a standard interaction:

- A portion of the LASER radiation is reflected
- Another portion is adsorbed in two different ways: scattering and pure adsorbtion

$$G_L = \frac{\dot{Q}_{ass}}{V} = \frac{A \cdot I(z) - A \cdot I(z+\Delta z)}{A \Delta z}$$

$$\Delta z \to 0$$

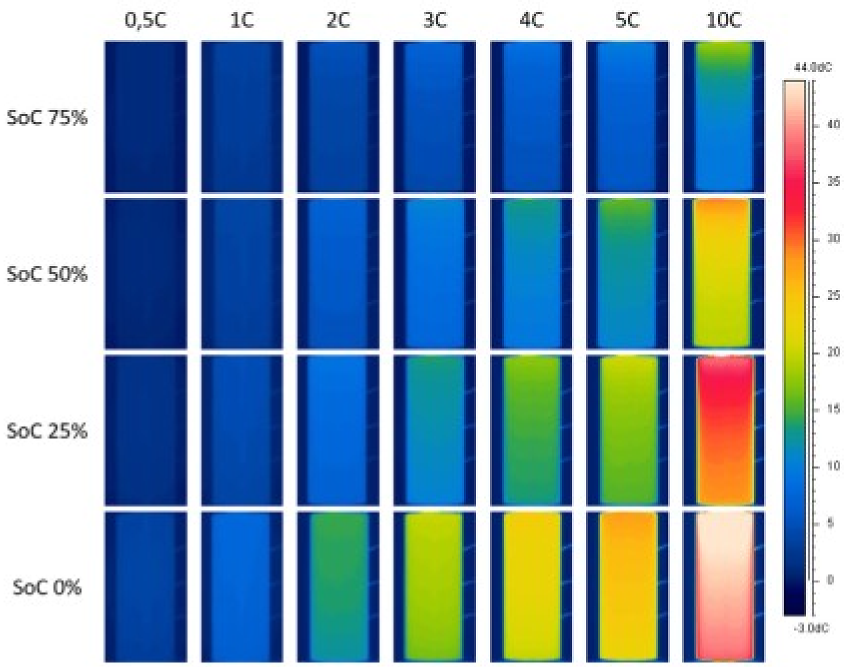
$$G_L = -\frac{\partial I}{\partial z} \qquad \boxed{I(z) = I_0 e^{-\gamma z}}$$

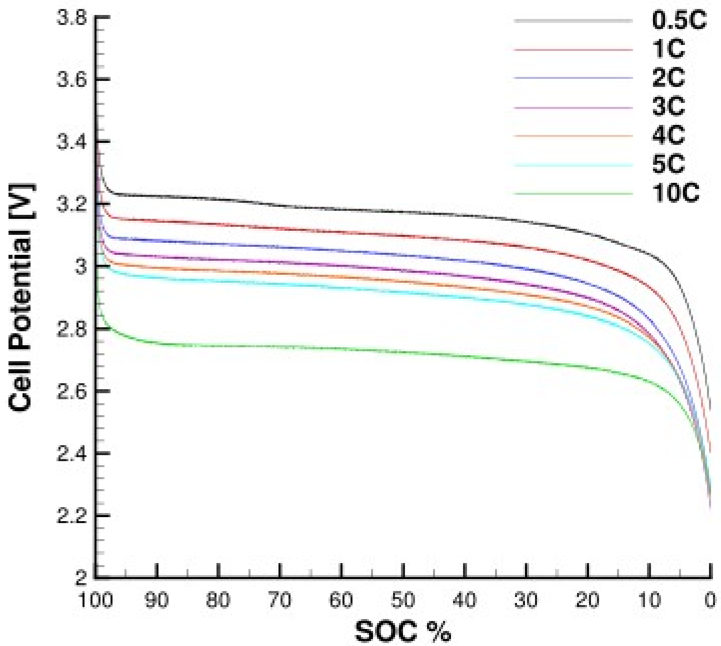
$$\boxed{G_L = -\frac{\partial}{\partial z}\left(I_0 e^{-\gamma z}\right) = \gamma I(z) = \alpha I(z)}$$

$$\rho_t c_{p,t} \frac{\partial T_t}{\partial t} = \nabla \cdot (\lambda_t \nabla T_t) + \rho_b c_{p,b} \omega_b (T_{a,0} - T_t) + \dot{Q}_m + G_L$$

## Li-Ion Cells

Li-Ion cells are strongly subject to heating phenomena during their operative conditions. Many models are available in literature in oder to take into account also electrochemical phenomena inside the cell itself. The most simplified model is the Bernardi one reported below in the slide.




$$\nabla \cdot (\lambda \nabla T) + G = \rho c_p \frac{\partial T}{\partial t}$$

$$\dot{Q}_c = I\left(U_{oc} - V\right) - IT\frac{\partial U_{oc}}{\partial T}$$