UNIVERSITÀ
POLITECNICA
DELLE MARCHE

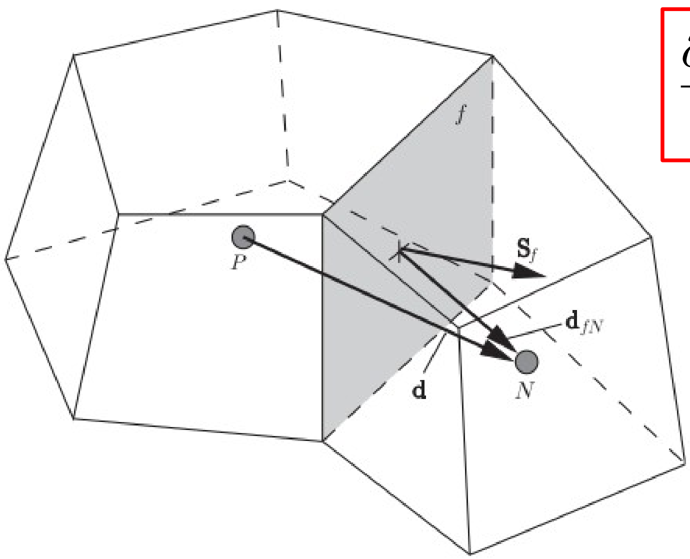
Numerical Heat Transfer
for Applications

FVM for unsteady state heat conduction

Dr Valerio D'Alessandro

Finite volume discretization of Fourier equation

Fourier equation is discretized its integral form over each control volume:



$$\boxed{\frac{\partial T}{\partial t} - \alpha \nabla^2 T = 0} \Longrightarrow \int_{V_P} \frac{\partial T}{\partial t} dV - \int_{V_P} \alpha \nabla^2 T dV = 0$$

$$\int_{V_P} \frac{\partial T}{\partial t} dV - \int_{V_P} \alpha \nabla \cdot (\nabla T)\, dV = 0$$

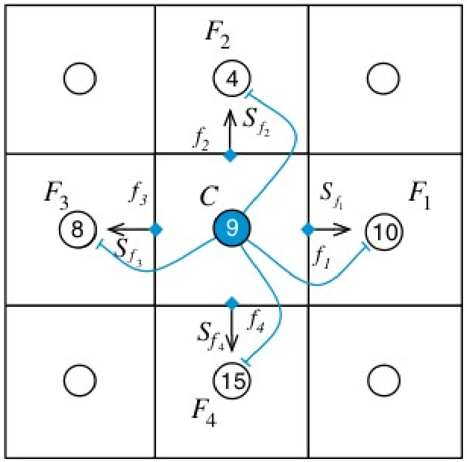
$$\int_{V_P} \frac{\partial T}{\partial t} dV - \int_{\partial V_P} \alpha \nabla T \cdot d\mathbf{S} = 0$$

$$\int_{V_P} (\mathbf{x} - \mathbf{x}_P)\, dV = \mathbf{0}$$

$$\boxed{V_P \frac{dT_P}{dt} - \sum_f \alpha \left(\nabla T\right)_f \cdot \mathbf{S}_f = 0}$$

Finite volume discretization of Fourier equation

$$\sum_f (\nabla T)_f \cdot \mathbf{S}_f = 0 \quad \Rightarrow \quad a_C T_C + \sum_{nb} a_{nb} T_{nb} = b_C$$



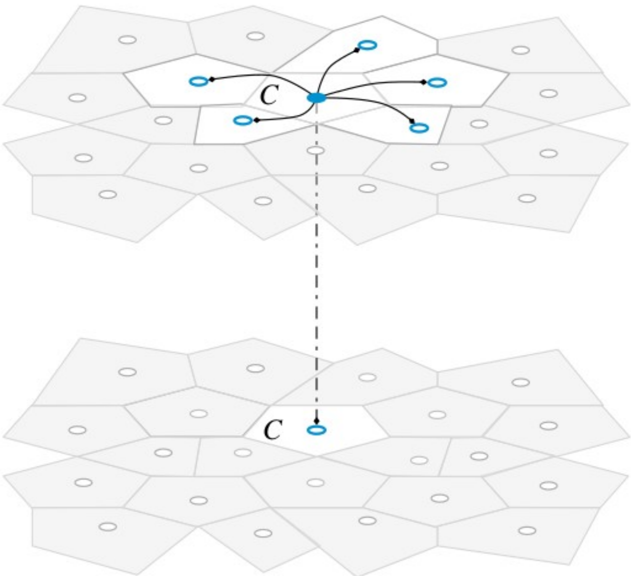
$$a_9 T_9 + a_{10} T_{10} + a_4 T_4 + a_8 T_8 + a_{15} T_{15} = 0$$

$$V_P \frac{dT_C}{dt} - \sum_f \alpha (\nabla T)_f \cdot \mathbf{S}_f = 0$$

$$V_P \frac{T_C^{(n+1)} - T_C^{(n)}}{\Delta t} - \underbrace{\sum_f \alpha (\nabla T)_f \cdot \mathbf{S}_f}_{\textbf{Implicit} \text{ or explicit ?}} = 0$$

$$\frac{V_P}{\Delta t} T_C^{(n+1)} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = -b_C + \frac{V_P}{\Delta t} T_C^{(n)}$$

Finite volume discretization of Fourier equation – implicit method



$$V_P \frac{dT_C}{dt} - \sum_f \alpha \left(\nabla T\right)_f \cdot \mathbf{S}_f = 0$$

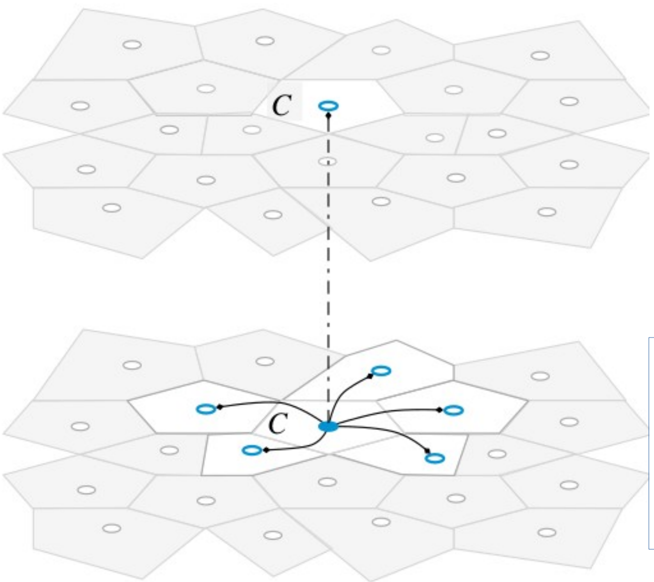
$$\frac{dT_C}{dt} = \frac{T_C^{(n+1)} - T_C^{(n)}}{\Delta t}$$

$$\boxed{\frac{V_P}{\Delta t} T_C^{(n+1)} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = -b_C + \frac{V_P}{\Delta t} T_C^{(n)}}$$

It is possible to use also different time integration schemes rather than backward Euler. A possible alternative is the following one:

$$\frac{dT_C}{dt} = \frac{3T_C^{(n+1)} - 4T_C^{(n)} + T_C^{(n-1)}}{2\Delta t} + O\left(\Delta t^2\right)$$

Finite volume discretization of Fourier equation – explicit method



$$V_P \frac{dT_C}{dt} - \sum_f \alpha \left(\nabla T\right)_f \cdot \mathbf{S}_f = 0$$

$$\frac{dT_C}{dt} = \frac{T_C^{(n+1)} - T_C^{(n)}}{\Delta t}$$

$$\frac{V_P}{\Delta t} T_C^{(n+1)} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = -b_C + \frac{V_P}{\Delta t} T_C^{(n)}$$

$$T_C^{(n+1)} = T_C^{(n)} + \frac{\Delta t}{V_P} \left(a_C T_C^{(n)} + \sum_{nb} a_{nb} T_{nb}^{(n)} + b_C \right)$$

Ex3_FVM – Unsteady state heat conduction

Matlab code

```
format long e


alpha= 1e-2;
dt   = 0.01;
Time = 50;
caseName='cube';
%----------------------------------
m=mesh_data(caseName);
Nc=m.numberOfElements;
Ni=m.numberOfInteriorFaces;
Nb=m.numberOfBElements;
A=zeros(Nc,Nc);
b=zeros(Nc,1);

. . .

T0   = ones(Nc,1);
Tc   = zeros(Nc,1);

A   = -alpha*A;
b   = -alpha*b;

for i=1:Nc
    Vc     =   m.elements(i).volC;
    A(i,i) =  A(i,i) + Vc/dt;
end
```

$$\frac{V_P}{\Delta t}T_C^{(n+1)} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = -b_C + \frac{V_P}{\Delta t}T_C^{(n)}$$

A matrix implemention, as well as B.C. for laplacian term discretization, is extactly the same of the steady state case.
However, it is mandatory to switch **A** and **b** entries sign deriving from spatial terms discretization.

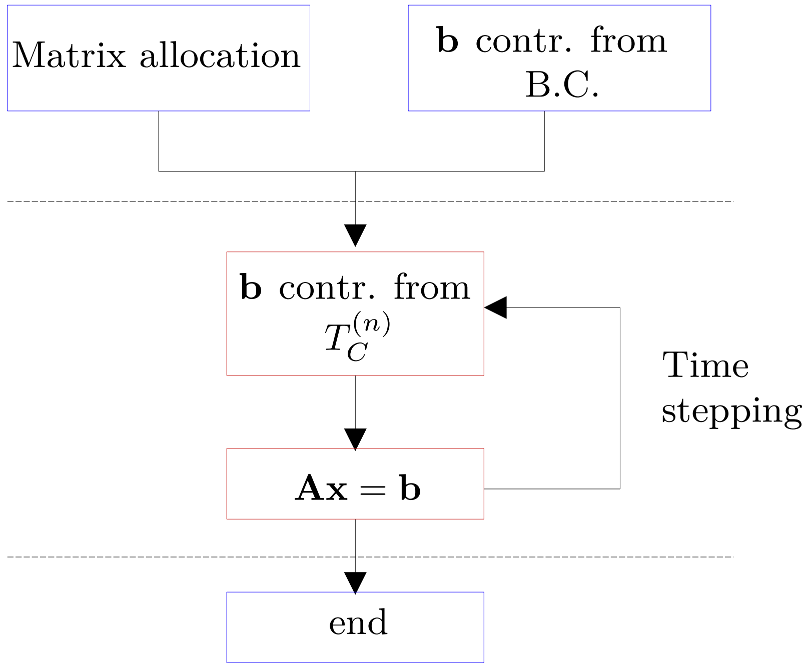
$$\frac{V_P}{\Delta t} \Rightarrow$$ **A** matrix diagonal terms contributions deriving from unsteady term discretization.

Ex3_FVM – Unsteady state heat conduction

Matlab code

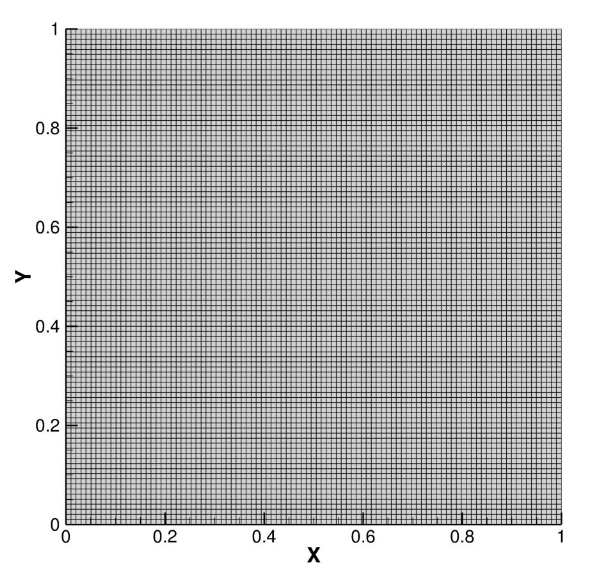
```
bn  = zeros(Nc,1);
flow_time = 0;

for i= 1: (Time/dt)
    for j=1:Nc
        Vc     =   m.elements(j).volC;
        bn(j)  =   b(j) + (Vc/dt)*T0(j);
    end
%
    Tc = bicg(A,b,1e-12,1000);
    Tc = pcg(A,b,1e-12,1000,L, U);
    T0 = Tc;
    bn = zeros(Nc,1);
%
    flow_time = flow_time + dt;
    if   ( mod(i,500) == 0)
           wrtfld(flow_time, m , Tc, 'Tc', caseName)
    end
end
```



$$\frac{V_P}{\Delta t} T_C^{(n+1)} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = -b_C + \frac{V_P}{\Delta t} T_C^{(n)}$$

Ex3_FVM – Unsteady state heat conduction in 1-D configuration



$$\frac{\partial T}{\partial t} - \alpha \nabla^2 T = 0$$

$$T(x=0,t) = 1$$

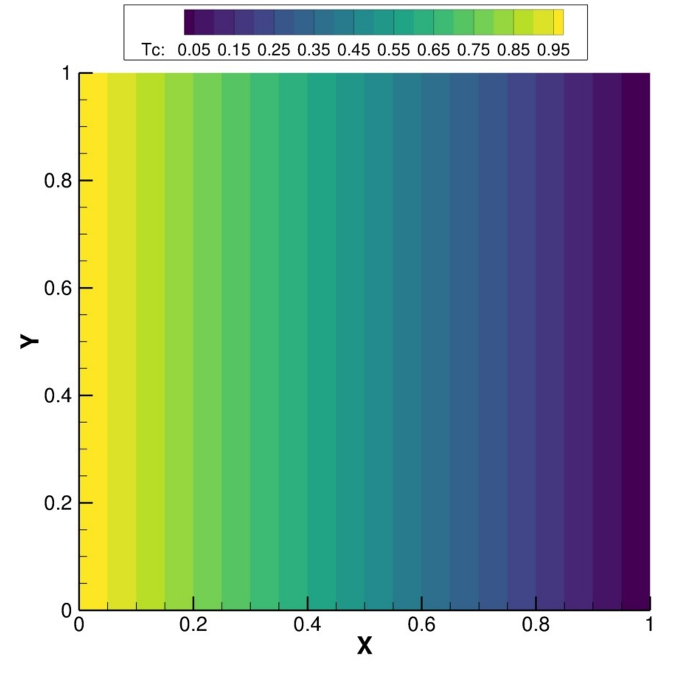
$$T(x=1,t) = 0$$

$$\partial_n T\,(y=0,t) = 0$$

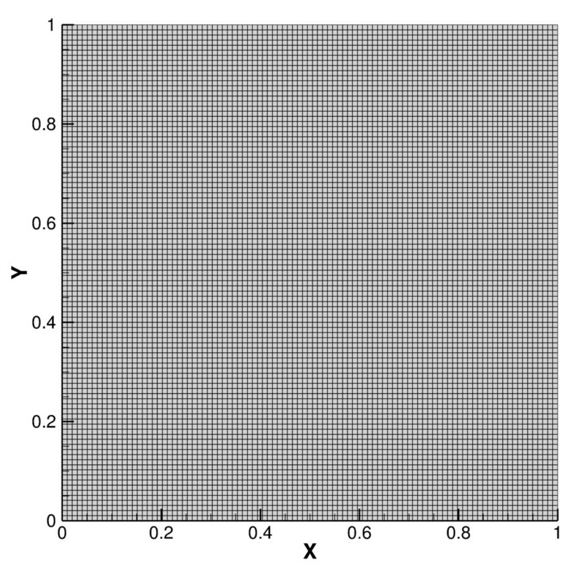
$$\partial_n T\,(y=1,t) = 0$$

$$T(x,t=0) = 1$$

$$T(x, t >> 1) = 1 - \frac{x}{L}$$



Ex3_FVM – Unsteady state heat conduction in 1-D configuration



$$\frac{\partial T}{\partial t} - \alpha \nabla^2 T = 0$$

$$T(x = 0, t) = 1$$

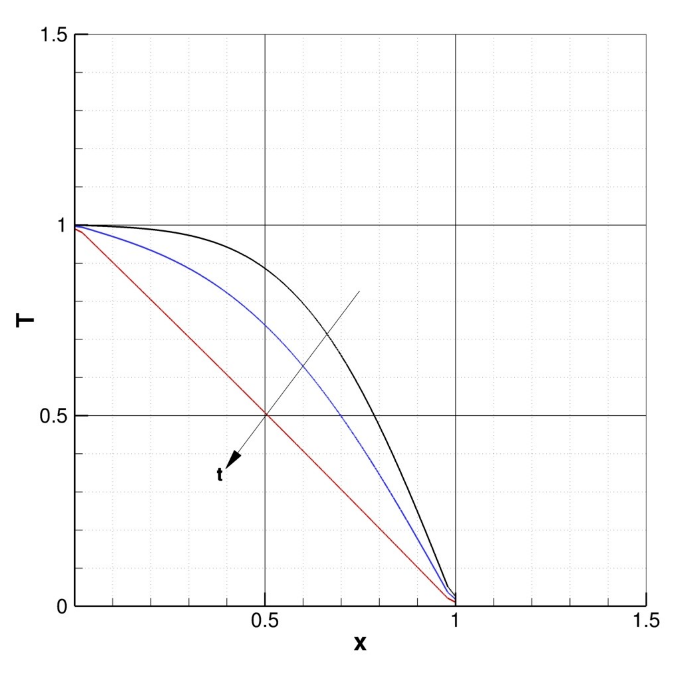
$$T(x = 1, t) = 0$$

$$\partial_n T\,(y = 0, t) = 0$$

$$\partial_n T\,(y = 1, t) = 0$$

$$T(x, t = 0) = 1$$

$$\Rightarrow \quad T(x, t >> 1) = 1 - \frac{x}{L}$$



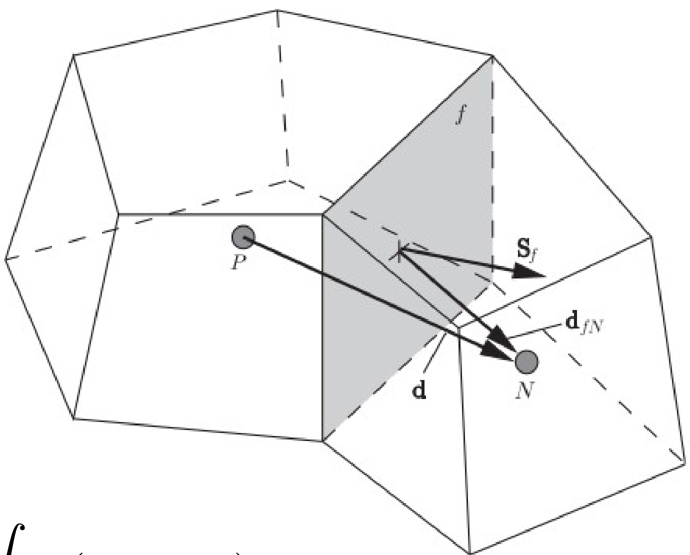
Finite volume discretization of general heat conduction equation

General heat conduction equation is discretized its integral form over each control volume:

$$\rho c \frac{\partial T}{\partial t} - \lambda \nabla^2 T = \rho_b c_{p,b} \omega_b \left(T_{a,0} - T\right) + G_L$$

$$G_L = \alpha I\left(x\right) \quad I\left(x\right) = I_0 e^{-\alpha x} \quad g_p = \frac{\rho_b c_{p,b} \omega_b}{\rho c}$$

$$\frac{\partial T}{\partial t} - \alpha \nabla^2 T + g_p T = g_p T_{a,0} + \frac{1}{\rho c} G_L$$



$$\int_{V_P} \left(\mathbf{x} - \mathbf{x}_P\right) dV = \mathbf{0}$$

$$\int_{V_P} \left(\frac{\partial T}{\partial t} - \alpha \nabla^2 T + g_p T\right) dV = \int_{V_P} \left(g_p T_{a,0} + \frac{1}{\rho c} G_L\right) dV$$

Finite volume discretization of general heat conduction equation

The FVM discretization of generala heat conduction equation differes from the previous case only for the source terms.

$$\int_{V_P} \left(\frac{\partial T}{\partial t} - \alpha \nabla^2 T + g_p T \right) dV = \int_{V_P} \left(g_p T_{a,0} + \frac{1}{\rho c} G_L \right) dV$$

$$\Downarrow$$

$$V_P \frac{T_C^{(n+1)} - T_C^{(n)}}{\Delta t} - \sum_f \alpha \left(\nabla T\right)_f \cdot \mathbf{S}_f + V_P g_P T_C = V_P \left(g_p T_{a,0} + \frac{1}{\rho c} G_L \right)$$

Backward Euler and an implcit time integration strategy is adopted:

$$V_P \frac{T_C^{(n+1)} - T_C^{(n)}}{\Delta t} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} + V_P g_P T_C^{(n+1)} = V_P \left(g_p T_{a,0} + \frac{1}{\rho c} G_L \right)$$

$$\frac{V_P}{\Delta t} T_C^{(n+1)} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = -b_C + \frac{V_P}{\Delta t} T_C^{(n)}$$

Fourier equation

Finite volume discretization of general heat conduction equation

$$V_P \frac{T_C^{(n+1)} - T_C^{(n)}}{\Delta t} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} + V_P g_P T_C^{(n+1)} = V_P \left(g_p T_{a,0} + \frac{1}{\rho c} G_L \right)$$

$$\frac{V_P}{\Delta t} T_C^{(n+1)} - a_C T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = -b_C + \frac{V_P}{\Delta t} T_C^{(n)}$$

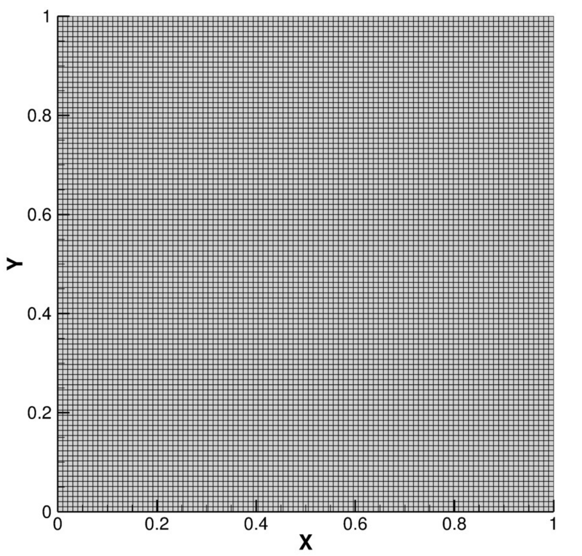
Fourier equation

$$\boxed{\frac{V_P}{\Delta t} T_C^{(n+1)} + \tilde{a}_c T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = \tilde{b}_C + \frac{V_P}{\Delta t} T_C^{(n)}}$$

$$\tilde{a}_c = -a_c + V_P g_P$$

$$\tilde{b}_C = -b_C + V_P \left(g_p T_{a,0} + \frac{1}{\rho c} G_L \right)$$

Ex4_FVM – Heat conduction in a biological tissue under laser heating



$$\rho c\frac{\partial T}{\partial t} - \lambda\nabla^2 T = c_{p,b}w_b\left(T_{a,0} - T\right) + G_L$$

$$T(x=0,t) = 37$$

$$\partial_n T\left(x=1,t\right) = 0$$

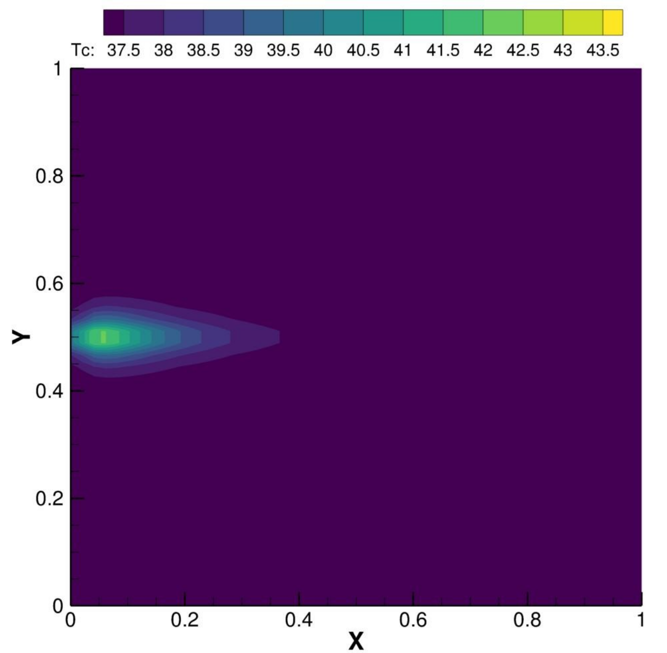
$$\partial_n T\left(y=0,t\right) = 0$$

$$\partial_n T\left(y=1,t\right) = 0$$

$$T(\mathbf{x},t=0) = 37$$

- LASER source is active only for 10 s. It is applied at (x,y)=(0,0.5). the spot diameter is 1cm, while LASER intensity is 15 MW/m².
- Tissue properties are standard. (Take a look to the code on gitHub to obtain their values.)

LASER source



Ex4_FVM – Heat conduction in a biological tissue under laser heating

Matlab code

```
alpha = 1.26e-3;
rhot  = 1050;
cpb   = 3770;
%
cpt   = 3340;
wb    = 0.5;
Ta0   = 37;
Tini  = 37;
om    = 0.05;
caseName='cube';
%----------------------------------
dt       = 0.01;
Time     = 1;
writeInt = 100;
%
%
%----------------------------------
%
m=mesh_data(caseName);
Nc=m.numberOfElements;
Ni=m.numberOfInteriorFaces;
Nb=m.numberOfBElements;
A=zeros(Nc,Nc);
b=zeros(Nc,1);
```

$$\frac{V_P}{\Delta t}T_C^{(n+1)} + \tilde{a}_c T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = \tilde{b}_C + \frac{V_P}{\Delta t}T_C^{(n)}$$

$$\tilde{a}_c = -a_c + V_P g_P \qquad \tilde{b}_C = -b_C + V_P\left(g_p T_{a,0} + \frac{1}{\rho c} G_L\right)$$

- **A** matrix implementation, as well as **b** vector one, for laplacian and unsteady terms are exactly the same of the Fourier equation case.
- Pennes and Beer law terms can be implemented summing their contribution on **b** and **A** diagonal terms deriving from the Fourier equation discretization.

Ex4_FVM – Heat conduction in a biological tissue under laser heating

Matlab code

```
%-----------------------------------
% Initial condition
%
T0   = zeros(Nc,1);
for i = 1:Nc
    T0(i) = T0(i) + Tini;
end
wrtfld(0, m , T0, 'Tc', caseName);

%-----------------------------------
% A and b allocation
%
Tc   = zeros(Nc,1);
%
A   = -alpha*A;
b   = -alpha*b;
for i=1:Nc
    Vc     =  m.elements(i).volC;
    A(i,i) =  A(i,i) + Vc/dt;
end

%-----------------------------------
```

$$\frac{V_P}{\Delta t} T_C^{(n+1)} + \tilde{a}_c T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = \tilde{b}_C + \frac{V_P}{\Delta t} T_C^{(n)}$$

$$\tilde{a}_c = -a_c + V_P g_P \qquad \tilde{b}_C = -b_C + V_P \left(g_p T_{a,0} + \frac{1}{\rho c} G_L \right)$$

- **A** matrix implementation, as well as **b** vector one, for laplacian and unsteady terms are exactly the same of the Fourier equation case.

- **Pennes** and Beer law terms can be implemented summing their contribution on **b** and **A** diagonal terms deriving from the Fourier equation discretization.

Ex4_FVM – Heat conduction in a biological tissue under laser heating

Matlab code

```
%--------------------------------
% Initial condition
%
T0   = zeros(Nc,1);
for i = 1:Nc
    T0(i) = T0(i) + Tini;
end
wrtfld(0, m , T0, 'Tc', caseName);

. . .

%--------------------------------
% Beer law term
%
yL    = 0.5;
gamma = 8;
I0    = 1.5e7;
bl    = zeros(Nc,1);
for i=1:Nc
    Vc     =  m.elements(i).volC;
    if ( abs( m.elements(i).Cc(2) - yL ) < 0.01 )
        bl(i) = gamma*I0*exp(-
gamma*m.elements(i).Cc(1))/(rhot*cpt);
        bl(i) = bl(i)*Vc;
    end
end
%--------------------------------
```

$$\tilde{a}_c = -a_c + V_P g_P$$

$$\tilde{b}_C = -b_C + V_P\left(g_p T_{a,0} + \frac{1}{\rho c} G_L\right)$$

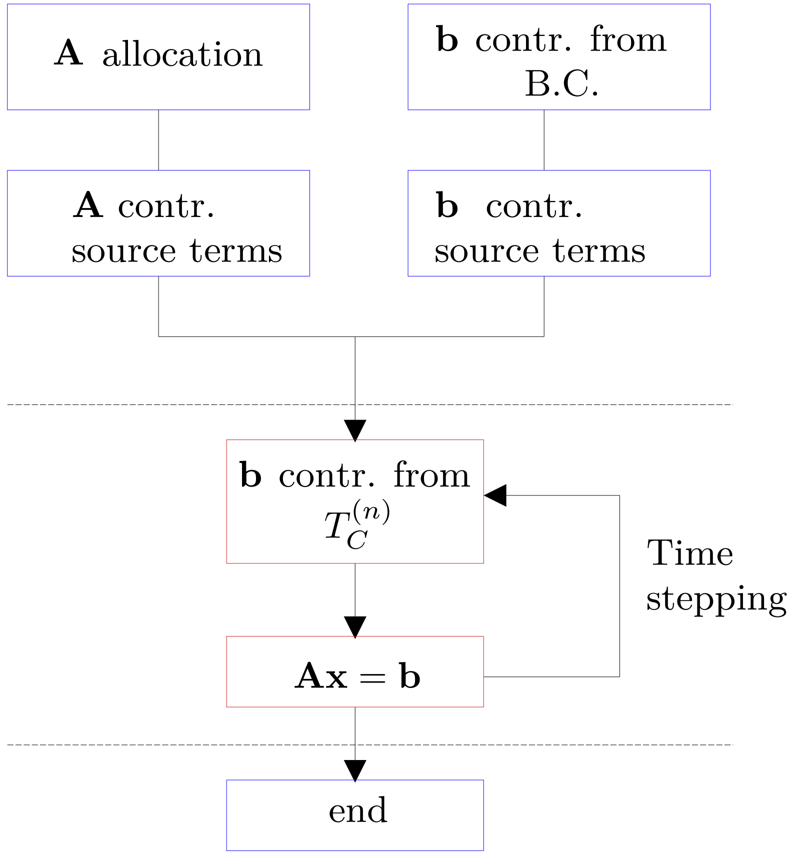
$$G_L = \alpha I\left(x\right) \qquad I\left(x\right) = I_0 e^{-\alpha x}$$

- **A** matrix implementation, as well as **b** vector one, for laplacian and unsteady terms are exactly the same of the Fourier equation case.
- Pennes and **Beer law** terms can be implemented summing their contribution on **b** and **A** diagonal terms deriving from the Fourier equation discretization.

Ex4_FVM – Heat conduction in a biological tissue under laser heating

Matlab code

```
bn  = zeros(Nc,1);
flow_time = 0;
for i= 1: (Time/dt)
    for j=1:Nc
        Vc     =   m.elements(j).volC;
        bn(j)  =   b(j) -
                   bl(j)*min( sign(flow_time-10),0)
                 + (Vc/dt)*T0(j);
    end
    %
    Tc = bicg(A,bn,1e-12,1000);
    %Tc = pcg(A,bn,1e-12,1000,L, U);
    T0 = Tc;
    bn = zeros(Nc,1);
    %
    flow_time = flow_time + dt;
    if   ( mod(i,writeInt) == 0)
           wrtfld(flow_time, m , Tc , 'Tc', caseName)
    end
end
```



$$\frac{V_P}{\Delta t} T_C^{(n+1)} + \tilde{a}_c T_C^{(n+1)} - \sum_{nb} a_{nb} T_{nb}^{(n+1)} = \tilde{b}_C + \frac{V_P}{\Delta t} T_C^{(n)}$$