COMPUTING OPERATING CHARACTERISTICS FOR RANDOM WALKS

Consider a random walk starting at $x = 0$ between lines $x = a$, $x = b$, $a < 0 < b$ with increments Y having distribution $F(y)$. To calculate the operating characteristics (ignoring overshoots) we should proceed as follows. Consider

$$\theta(s) := \int e^{sy} dF(y)$$

Then $\theta(0) = 1$ and $\theta''(s) > 0$ and under reasonable conditions $\lim_{s\to\pm\infty} \theta(s) = +\infty$. Hence the equation

$$\int e^{sy} dF(y) = 1 \tag{1}$$

has either a double root and 0, and no other real root, or a unique real root $\neq 0$. The case of a double root occurs when $\mu = \theta'(0) = 0$ where $\mu = E(Y)$. I.e. when $E(Y) = 0$. This will be considered as a limiting case. See below.

For now we assume there is a root $h \neq 0$. We have

$$h < 0 \iff \mu > 0$$

The probability for crossing the line $x = b$ first is approximately

$$\boxed{p_b \cong \frac{1 - e^{ha}}{e^{hb} - e^{ha}}} \tag{2}$$

The probability of crossing $x = a$ first is $p_a := 1 - p_b$. I.e.

$$\boxed{p_a \cong \frac{e^{hb} - 1}{e^{hb} - e^{ha}}}$$

An approximate formula for the expected duration is

$$\boxed{E = \frac{p_a a + p_b b}{\mu} \cong -\frac{1}{\mu} \frac{-ae^{hb} + be^{ha} - (b - a)}{(e^{hb} - e^{ha})}} \tag{3}$$

Unfortunately the above formulas may be numerically unstable since they depend for example on the evaluation of $e^x - 1$ where x may be very close to 0 leading to catastrofic cancellation (this will happen if μ is very small). Therefore we introduce functions ϕ_1, ϕ_2 via

$$e^x = 1 + x + x\phi_1(x)$$

and

$$e^x = 1 + x + \frac{x^2}{2} + x^2\phi_2(x) \tag{4}$$

It it easy to evaluate $\phi_1(x)$, $\phi_2(x)$ robustly for small x using Taylor series. Substituting (4) in (1) and rescaling $h = \mu e$ we must solve

$$\int \left(1 + \mu e y + \mu^2 e^2 \frac{y^2}{2} + \mu^2 e^2 y^2 \phi_2(\mu e y)\right) dF(y) = 1$$

or

$$\int \left(y + \mu e \frac{y^2}{2} + \mu e y^2 \phi_2(\mu e y) \right) dF(y) = 0$$

which is equivalent to (for $m_2 = \int y^2 dF(y)$):

$$1 + e \left(\frac{m_2}{2} + \int y^2 \phi_2(\mu e y) dF(y) \right) = 0 \tag{5}$$

This equation can be solved efficiently using Newton's method.

Remark. Note that (5) makes perfect sense for $\mu = 0$ in which case we simply find

$$e = -\frac{2}{m_2}$$

as a solution. This is actually a good approximation for the solution in general if μ is small, in which case we find

$$h \cong -2\frac{\mu}{m_2} \cong -2\frac{\mu}{\sigma^2} \tag{6}$$

where σ is the standard deviation of Y. Applying the formulas (2,3) with h as in (6) is the so-called "Brownian approximation".

Now we evaluate (2) robustly. We calculate

$$\begin{aligned} p_b &= \frac{1 - e^{ha}}{e^{hb} - e^{ha}} \\ &= \frac{-a - a\phi_1(ha)}{b + b\phi_1(hb) - a - a\phi_1(ha)} \end{aligned}$$

Similarly for (3)

$$\begin{aligned} E &= -\frac{1}{\mu} \frac{-a(1 + hb + \frac{(hb)^2}{2} + (hb)^2\phi_2(hb)) + b(1 + ha + \frac{(ha)^2}{2} + (ha)^2\phi_2(ha)) - (b-a)}{(1 + hb + hb\phi_1(hb)) - (1 + ha + ha\phi_1(ha))} \\ &= -\frac{1}{\mu} \frac{-a(hb + \frac{(hb)^2}{2} + (hb)^2\phi_2(hb)) + b(ha + \frac{(ha)^2}{2} + (ha)^2\phi_2(ha))}{(hb + hb\phi_1(hb)) - (ha + ha\phi_1(ha))} \\ &= -\frac{1}{\mu} \frac{-ab(h + \frac{h^2 b}{2} + h^2 b\phi_2(hb)) + ba(h + \frac{h^2 a}{2} + h^2 a\phi_2(ha))}{(hb + hb\phi_1(hb)) - (ha + ha\phi_1(ha))} \\ &= -\frac{hab}{\mu} \frac{-(\frac{b}{2} + b\phi_2(hb)) + (\frac{a}{2} + a\phi_2(ha))}{(b + b\phi_1(hb)) - (a + a\phi_1(ha))} \\ &= eab \frac{(\frac{b}{2} + b\phi_2(hb)) - (\frac{a}{2} + a\phi_2(ha))}{(b + b\phi_1(hb)) - (a + a\phi_1(ha))} \end{aligned}$$

Remark. It is well known how to deduce (3) from (2). We may give a heuristic proof of (2) as follows.

Let $g(z)$ be the probability that the above random walk starts in $x = z$ and ends on the line $x = b$. Then $g(z)$ is determined by the equation

$$g(z) = \int g(z + y) dF(y) \quad \text{for } a \le z \le b \tag{7}$$

with boundary conditions

$$\begin{aligned} g(z) &= 0 \quad \text{for } z \le a \\ g(b) &= 1 \quad \text{for } z \ge b \end{aligned} \tag{8}$$

Clearly (2) is equivalent to

$$g(z) = \frac{1 - e^{h(a-z)}}{e^{h(b-z)} - e^{h(a-z)}} \tag{9}$$

It is clear that the righthand side of (9) does *not* satisfy (8). However it satisfies (8) for $z = a$ and $z = b$. Since we are looking for an approximate solution, let's be satisfied with that.

We now show that (9) satisfies for all z in fact (7). We calculate

$$\begin{aligned}
\int g(z+y)dF(y) &= \int \frac{1 - e^{h(a-z-y)}}{e^{h(b-z-y)} - e^{h(a-z-y)}} dF(y) \\
&= \int \frac{e^{hy} - e^{h(a-z)}}{e^{h(b-z)} - e^{h(a-z)}} dF(y) \\
&= \frac{1}{e^{h(b-z)} - e^{h(a-z)}} \left(\int e^{hy} f(y)dy - e^{h(a-z)} \int dF(y) \right) \\
&= g(z)
\end{aligned}$$