THE GENERALIZED MAXIMUM LIKELIHOOD RATIO FOR THE EXPECTATION VALUE OF A DISTRIBUTION

MICHEL VAN DEN BERGH

1. Exact results

Assume given real numbers

$$a_1 < a_2 < \cdots < a_N$$

and a probability distribution

$$P : \{a_1, \dots, a_N\} \to \mathbb{R} : a_i \mapsto p_i$$

Assume a sample taken from $\{a_1, \dots, a_N\}$ according to P has sample distribution $(\hat{p}_i)_{i=1,\dots,N}$. We want to compute the corresponding MLE for the true distribution $(p_i)_{i=1,\dots,N}$, subject to the condition that the latter's expectation value is s. I.e. $\sum_i p_i a_i = s$.

For simplicity we will assume

$$a_1 < s < a_N, \forall i : \hat{p}_i \neq 0 \tag{1.1}$$

Proposition 1.1. *The ML distribution is unique. It is given by*

$$p_i = \frac{\hat{p}_i}{1 + \theta(a_i - s)} \tag{1.2}$$

where θ is the unique root of the equation

$$\sum_i \frac{\hat{p}_i(a_i - s)}{1 + \theta(a_i - s)} = 0 \tag{1.3}$$

in the interval $[-1/(a_N - s), 1/(s - a_1)]$.

Proof. We have to maximize the objective function

$$\mathrm{LLR}((p_i)_i) = \sum_i \hat{p}_i \log p_i$$

subject to the constraints

$$\begin{aligned} \sum_i p_i &= 1 \\ \sum_i a_i p_i &= s \\ p_i &> 0 \end{aligned} \tag{1.4}$$

The objective function is continuous on (1.4) and approaches $-\infty$ on the boundary. So it has at least one maximum. To prove that it has a unique maximum it suffices to prove that there is a unique extremal value.

Using Lagrange multipliers we have to determine the extremal values of

$$\sum_i \hat{p}_i \log p_i - \lambda(\sum_i p_i - 1) - \theta(\sum_i p_i a_i - s)$$

We obtain

$$(\lambda + \theta a_i) p_i = \hat{p}_i$$

and hence by (1.1) $\lambda + \theta a_i \neq 0$ so that

$$p_i = \frac{\hat{p}_i}{\lambda + \theta a_i} \tag{1.5}$$

where λ, θ must satisfy

$$\sum_i \frac{\hat{p}_i}{\lambda + \theta a_i} = 1 \tag{1.6}$$

$$\sum_i \frac{\hat{p}_i a_i}{\lambda + \theta a_i} = s \tag{1.7}$$

Evaluating λ(1.6)+θ(1.7) we find

$$\lambda + \theta s = 1$$

and hence $\lambda = 1 - \theta s$ and we immediately obtain (1.2) from (1.5).

If is clear that (1.6)(1.7) imply (1.3). Assume (1.3) holds. Then

$$\begin{aligned} 1 &= \sum_i \hat{p}_i \\ &= \sum_i \frac{\hat{p}_i(1 + (a_i - s)\theta)}{1 + (a_i - s)\theta} \\ &= \sum_i \frac{\hat{p}_i}{1 + (a_i - s)\theta} + \theta \sum_i \frac{\hat{p}_i(a_i - s)}{1 + (a_i - s)\theta} \end{aligned}$$

so that (1.6) holds. On the other hand we also have

$$\sum_i \frac{\hat{p}_i(1 + (a_i - s)\theta)}{1 + (a_i - s)\theta} = (1 - s\theta) \sum_i \frac{\hat{p}_i}{1 + (a_i - s)\theta} + \theta \sum_i \frac{\hat{p}_i a_i}{1 + (a_i - s)\theta}$$

We conclude that (1.7) holds, unless perhaps if $\theta = 0$. If $\theta = 0$ then $\lambda = 1$ and (1.7) is equivalent to

$$\hat{\mu} := \sum_i \hat{p}_i a_i = s$$

which also follows from (1.3).

Hence we have to solve (1.3) for θ. Moreover the fact that $p \geq 0$ leads to the additional constraint

$$\hat{p}_i > 0 \Rightarrow 1 + \theta(a_i - s) > 0$$

So we should have

$$\begin{aligned} \theta &> -\frac{1}{a_i - s} & &\text{if } s < a_i \text{ and } \hat{p}_i > 0 \\ \theta &< \frac{1}{s - a_i} & &\text{if } s > a_i \text{ and } \hat{p}_i > 0 \end{aligned}$$

By (1.1) this is equivalent to

$$\theta \in \left] -\frac{1}{a_N - s}, \frac{1}{s - a_1} \right[$$

One verifies that on this interval the left hand side of (1.3) is strictly descending and goes from $+\infty$ to $-\infty$. Hence (1.3) has a unique solution. □

Remark 1.2. It is easy to see that Proposition 1.1 is still true under the weaker hypothesis $\hat{p}_1 > 0$, $\hat{p}_N > 0$. Moreover if there are i, j such that $a_i < s < a_j$ and $\hat{p}_i \neq 0$, $\hat{p}_j \neq 0$ then a suitable analogue of Proposition 1.1 still holds (θ must be in the interval between the poles of (1.3) which contains zero). If such i, j do not exist then the description of the ML distribution is different. When (1.1) does not hold it is easier in practice to deform $\hat{p}_i$ a little bit so that it becomes true. One may think of this as introducing a very weak prior.

Remark 1.3. (1.3) can be trivially solved numerically. For example using Newton's method.

2. Approximate results

Proposition 2.1. *Let* LLR *be the generalized log-likelihood ratio for* $\mu = \mu_0$ *versus* $\mu = \mu_1$*, divided by the sample size. Then we have*

$$\mathrm{LLR} \cong \frac{1}{2} \log \left(\frac{\sum_i \hat{p}_i (\mu_0 - a_i)^2}{\sum_i \hat{p}_i (\mu_1 - a_i)^2} \right) \tag{2.1}$$

Proof. Let $\theta = \theta(s)$ be the solution to (1.3). By (1.2) the corresponding maximal log-likelihood value (divided by the sample size) is given by

$$\mathrm{LL}(s) := -\sum_i \hat{p}_i \log(1 + \theta(s)(a_i - s)) \tag{2.2}$$

Developing the left hand side of (1.3) in a Taylor series in θ and keeping only the first order term we get

$$\sum_i \hat{p}_i (a_i - s) - \theta \sum_i \hat{p}_i (a_i - s)^2 = \hat{\mu} - s - \theta \sum_i \hat{p}_i (a_i - s)^2$$

so that we get

$$\theta(s) \cong \frac{\hat{\mu} - s}{\sum_i \hat{p}_i (a_i - s)^2} \tag{2.3}$$

This is the only approximation we make in the proof. From (2.2) we obtain

$$\begin{aligned}
\mathrm{LLR} &= -\sum_i \hat{p}_i \int_{\mu_0}^{\mu_1} \frac{d}{ds} \log(1 + \theta(s)(a_i - s))\, ds \\
&= -\sum_i \hat{p}_i \int_{\mu_0}^{\mu_1} \frac{\theta'(s)(a_i - s) - \theta(s)}{1 + \theta(s)(a_i - s)}\, ds \\
&= \int_{\mu_0}^{\mu_1} \theta(s)\, ds && \text{by (1.3)(1.6)} \\
&\cong \int_{\mu_0}^{\mu_1} \frac{\hat{\mu} - s}{\sum_i \hat{p}_i (a_i - s)^2}\, ds && \text{by (2.3)}
\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{d}{ds}\log\left(\sum_i \hat{p}_i(s-a_i)^2)\right) &= 2\frac{\sum \hat{p}_i(s-a_i)}{\sum_i \hat{p}_i(s-a_i)^2}\\ &= 2\frac{s-\hat{\mu}}{\sum_i \hat{p}_i(s-a_i)^2}\end{aligned}$$

so that we find

$$\begin{aligned}\mathrm{LLR} &\cong -\frac{1}{2}\int_{\mu_0}^{\mu_1}\frac{d}{ds}\log\left(\sum_i \hat{p}_i(s-a_i)^2\right) ds\\ &= \frac{1}{2}\log\left(\sum_i \hat{p}_i(\mu_0-a_i)^2\right) - \frac{1}{2}\log\left(\sum_i \hat{p}_i(\mu_1-a_i)^2\right)\end{aligned}$$

finishing the proof. □

Remark 2.2. Experiments show that the approximation (2.1) is very accurate.