HOMOLOGICAL PROPERTIES OF SKLYANIN ALGEBRAS

JOHN TATE AND MICHEL VAN DEN BERGH

ABSTRACT. To a pair consisting of an elliptic curve and a point on it, Odeskii and Feigin associate certain quadratic algebras ("Sklyanin algebras"), having the Hilbert series of a polynomial algebra.

In this paper we show that Sklyanin algebras have good homological properties and we obtain some information about their so-called linear modules. We also show how the construction by Odeskii and Feigin may be generalized so as to yield other "Sklyanin-type" algebras.

CONTENTS

1991 *Mathematics Subject Classification.* Primary 16E10, 16W50; Secondary 14H52.

Key words and phrases. Sklyanin algebras, quadratic algebras, homological algebra, elliptic curves.

The first author thanks the Ecole Normale Supérieure for its hospitality, and the Ministère de la Recherche et de la Technologie for financial support, during the time when much of his work on this paper was done.

The second author wishes to thank the Institut des Hautes Etudes Scientifiques for its hospitality and financial support during the time when most of this paper was written.

1. Introduction

Let F be a field and let $(E, \sigma, \mathcal{L})$ be a triple where E/F is a one-dimensional Cohen-Macaulay scheme of arithmetic genus 1 embedded in $\mathbb{P}^2$, σ is an automorphism of E and $\mathcal{L} = \mathcal{O}_E(1)$. Assume that in addition $\mathcal{L}^{(1-\sigma)^2} \cong \mathcal{O}_E$. Then in [2] so-called "3-dimensional regular algebras" $A(E, \sigma, \mathcal{L})$ were constructed. These are quadratic algebras $F \oplus A_1 \oplus A_2 \oplus \cdots$ with Hilbert series $1/(1-t)^3$, having good homological properties. To be more precise, in a somewhat different terminology, it was shown that Theorems 1.1, 1.2 and 1.4 below hold for $A(E, \sigma, \mathcal{L})$ [2][3].

In a completely independent development [15][16] Odeskii and Feigin constructed certain graded algebras $A(E, \sigma, \mathcal{L})$ (in their notation : $Q_{r,1}(E, \tau)$) where E is now a *smooth* elliptic curve over F, σ is a *translation* and $\mathcal{L}$ is a line bundle on E of degree $r \geq 3$, which, to some extent, generalize to higher dimensions the algebras constructed in [2].

We now fix the meaning of $(E, \sigma, \mathcal{L})$ as in the previous paragraph and we put $A = A(E, \sigma, \mathcal{L})$. This algebra is defined in §4.1.

If σ is generic then Odeskii and Feigin show that A is a flat deformation of a polynomial algebra, and hence automatically inherits all good properties of such algebras [16, Thm. 3.2 and the discussion thereafter].

When σ is not generic, A is harder to understand. If $r = 4$ then it is shown in [9][10][19][21][22] that Theorems 1.1, 1.2 and 1.4 hold for A for every translation σ, essentially by generalizing and simplifying the methods of [2][3].

The basic method consists in showing that if $r \leq 4$ then there is a regular sequence of central elements in A which defines a quotient algebra $B = B(E, \sigma, \mathcal{L})$ which is easy to analyze. Then the good properties of B may be lifted to A (this is not always easy, especially for Theorem 1.2). Unfortunately Hilbert series considerations show that this approach must fail if $r > 4$.

In this paper we introduce a new method (inspired by [16, Thm. 3.2]) to study A which makes virtually no use of the ring B (the exception is given by the proof of Proposition 4.3.1). This allows us to prove the following theorems for arbitrary r.

Theorem 1.1. Let $A = A(E, \sigma, \mathcal{L})$. Then

1. A has the Hilbert-series of a polynomial algebra in r variables.
2. A is Koszul.
3. A is Noetherian.
4. A satisfies the Auslander condition.
5. A is Cohen-Macaulay.

Theorem 1.2. Let $A = A(E, \sigma, \mathcal{L})$. Then, if σ has finite order, A is finite over its center.

For the definition of "Cohen-Macaulay" and the "Auslander condition" see [9].

Note that, in case $r = 4$, Theorem 1.2 covers part of [19] and [22]. However, in addition, [19] contains a classification of the simple A-modules, and [22] gives a detailed description of the center of A.

From Theorem 1.1 we obtain the following corollary :

Corollary 1.3. Let $A = A(E, \sigma, \mathcal{L})$. Then

1. A has global dimension r.
2. A is a domain.

(3) A is a maximal order.
(4) A is regular in the sense of [1].

Proof. (1) follows immediately from 1.1(1) and 1.1(2); (2) follows from [9]; (3) follows from [23] and (4) is immediate from 1.1(5). □

Let us now outline how we prove Theorems 1.1 and 1.2.

Let $I = \{1, \dots, l\}$ for $0 \leq l \leq r$ and let V be a finite dimensional vector space of dimension r. Denote by $\mathcal{U}_I$ the multiplicative free commutative monoid with basis $u_1, \dots, u_l$.

Our basic notion is that of a *module of I-type*, i.e., a module M over the tensor algebra TV with a decomposition $M = \oplus_{\mu \in \mathcal{U}_I} M_\mu$ (an *I-structure*) as a direct sum of one-dimensional subspaces M_μ such that $VM_\mu = \sum_{i \in I} M_{\mu u_i}$.

A fundamental result (Theorem 3.4) states that if $A = TV/(R)$ is a quadratic algebra, with $\dim R = r(r-1)/2$ and if A has a module M of I-type with $I = \{1, \dots, r\}$ then $A \cong M$. This immediately implies that A has Hilbert series $1/(1-t)^r$, and furthermore it implies that A is Koszul (corollary 3.2).

Modules of I-type may be constructed for $A = A(E, \sigma, \mathcal{L})$ (Theorem 4.2.3, compare also with [16, Thm. 3.2] if $F = \mathbb{C}$) which proves 1.1(1) and 1.1(2).

Then, if σ has finite order, we use the I-structure on A to construct a commutative subring of A, over which A is finite as a module. This implies that A is Noetherian and PI. Then 1.1(1) and 1.1(2) together with [25, Cor. 1.2] prove Theorem 1.2 and also 1.1(3), 1.1(4) and 1.1(5) in the finite order case.

Finally we then prove 1.1(3), 1.1(4) and 1.1(5) in general, using reduction to finite characteristic and the powerful results of Stafford and Zhang in [25].

As a byproduct of our methods we obtain information about the *linear* modules of A. Recall that a graded A-module M is linear of dimension l if it is generated in degree zero and has the Hilbert series of a polynomial algebra in l variables. Detailed understanding of linear modules was very important in the study of three and four-dimensional Sklyanin algebras. A complete classification of linear A-modules in all dimensions has meanwhile been obtained by Joanna Staniszkis in her thesis [26].

If M is of I-type then M is linear. We use this fact to associate linear modules to effective divisors on E. To be more precise let $\mathrm{Div}_l(E)$ be the set of effective divisors on E of degree l and let $D \in \mathrm{Div}_l(E)$ with $\mathcal{O}(D) \ncong \mathcal{L}$. Then $W = H^0(E, \mathcal{L}(-D)) \subset H^0(E, \mathcal{L}) = A_1$ is a subvector space of dimension $r-l$ and we put $M(D) = A/AW$.

If D is sufficiently generic then $M(D)$ carries an I-structure (Proposition 5.1) and hence $M(D)$ is linear. We use semi-continuity to extend this to all D and we prove ($\pi = \sigma^{r-2}$, $\theta = \sigma^{-2}$)

Theorem 1.4. Let $D \in \mathrm{Div}_l(E)$, $\mathcal{O}(D) \ncong \mathcal{L}$. Then $M(D)$ is linear of dimension l. Furthermore if D is of the form $D' + (z_l)$, $D' \in \mathrm{Div}_{l-1}(E)$ and z_l is a rational point of E then there is an exact sequence of graded A-modules

$$0 \to M((\pi^{-1} z_l) + \theta^{-1}(D'))(-1) \to M(D) \to M(D') \to 0 \tag{1.1}$$

where $M(D) \to M(D')$ is the obvious map.

Corollary 1.5. Let $D \in \mathrm{Div}_l(E)$, $\mathcal{O}(D) \ncong \mathcal{L}$. Then the minimal resolution of $M(D)$ is linear and has rank $\binom{r-l}{i}$ in degree i. In particular $\mathrm{pd}_A M(D) = r - l$ and hence $M(D)$ is Cohen-Macaulay.

Proof. This follows by induction from the existence of the exact sequences (1.1). □

This generalizes results in [10]. Note however that in general not all linear A-modules are of the form $M(D)$. See [26].

In the last section of this paper we show that the construction of $A(E,\sigma,\mathcal{L})$ by Odeskii and Feigin may be generalized to cases where σ is not a translation. In particular, we construct algebras of types B, H and E in higher dimensions (see [2, §4.13]). For these new algebras Theorems 1.1, 1.2 and 1.4 hold verbatim.

Finally in an appendix we present an additional result on Sklyanin algebras, which is proved using similar methods, and which we believe is interesting.

We show that A may be embedded as a graded F-algebra in a certain iterated Öre extension Z_r of the field of rational functions $F(E^r)$ on E^r. Furthermore the extension Z_r/A is faithfully flat on both sides. We indicate how this result might be used to give an alternative proof of Theorem 1.1.

The algebras $Q_{r,1}(E,\tau) = A(E,\sigma,\mathcal{L})$ discussed above form part of a family $Q_{r,k}(E,\tau)$, $1 \le k \le r-1$, also defined by Odeskii and Feigin in [15][16]. In [14] Odeskii also constructs a morphism from $Q_{r,1}(E,\tau)$ to an iterated Öre extension and in [17] this is generalized to $Q_{r,k}(E,\tau)$. These constructions are essentially over $\mathbb{C}$ since, instead of working over $F(E)$, one works over the field of meromorphic functions on $\mathbb{C}^*$ (which is an analytic covering of E). An advantage of this approach is that some formulas may be written more elegantly.

The work in [17] strongly suggests that the algebras $Q_{r,k}(E,\tau)$ are of I-type, and hence are amenable to the techniques presented in this paper.

2. Modules of *I*-type

2.1. An exact sequence. Let $I = \{1,\dots,l\}$ be a finite set and denote by $\mathcal{U}_I$ the free abelian monoid generated by $(u_i)_{i\in I}$ (in other words $\mathcal{U}_I \cong \mathbb{N}^I$). The composition law in $\mathcal{U}_I$ will be written multiplicatively, i.e. the elements of $\mathcal{U}_I$ are monomials $\mu = u^a = \prod_{i\in I} u_i^{a_i}$.

Let Γ be an abelian group and put $\Gamma(I) = \mathrm{Maps}(\mathcal{U}_I,\Gamma)$. For $n \ge 0$ define C^n to be the group of alternating functions $c_{i_1\cdots i_n}$ on I^n with values in $\Gamma(I)$.

Consider the sequence

$$0 \to \Gamma \xrightarrow{\epsilon} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \cdots \tag{2.1}$$

defined as follows :

(1) $\epsilon(\gamma)$ is the constant function on $\mathcal{U}_I$ with value γ.
(2) If $c \in C^n$ then

$$(\delta c)_{i_0\cdots i_n}(\mu) = \prod_{\nu=0}^{n} \left(\frac{c_{i_0\cdots\hat{i}_\nu\cdots i_n}(\mu u_i)}{c_{i_0\cdots\hat{i}_\nu\cdots i_n}(\mu)} \right)^{(-)^\nu}$$

for $(i_0\cdots i_n) \in I^{n+1}$.

Proposition 2.1.1. (2.1) is exact.

Proof. We will show that (2.1) is isomorphic to the augmented cochain complex of a contractable regular CW-complex.

Let $\Delta \subset \mathbb{R}^I$ be the "octant" $\{(x_i)_{i\in I} \mid x_i \geq 0 \text{ for all } i\}$. For pairs (μ, S), where $\mu \in \mathcal{U}_I$ and $S \subset I$ we define

$$\sigma_{\mu,S} = \{(x_i)_{i\in I} \subset \Delta \mid x_i = a_i(\mu) \text{ if } i \notin S \text{ and } a_i(\mu) < x_i < a_i(\mu)+1 \text{ if } i \in S\}$$

Here $a_i(\mu)$ is the exponent of u_i in μ.

Clearly $\sigma_{\mu,S}$ is and $|S|$-dimensional open cube in Δ. Furthermore Δ is the disjoint union of the $\sigma_{\mu,S}$ and these define a regular CW-complex on Δ. One verifies that $\sigma_{\mu',S'} \subset \overline{\sigma}_{\mu,S}$ if and only if

$$S' \subset S \qquad \text{and} \qquad \mu \mid \mu' \mid \mu \cdot \prod_{i\in S\setminus S'} u_i \tag{2.2}$$

Now suppose that (μ, S), (μ', S') satisfy (2.2) and $|S| = |S'|+1$; say $S = \{i_0, \dots, i_n\}$, $S' = \{i_0, \dots, \hat{i}_\nu, \dots, i_n\}$ where $i_0 < \cdots < i_n$. Put $\sigma = \sigma_{\mu,S}$, $\sigma' = \sigma_{\mu',S'}$. Then we define the incidence numbers $\alpha_{\sigma,\sigma'}$ as

$$\alpha_{\sigma,\sigma'} = \begin{cases} (-)^\nu & \text{if } \mu = \mu' u_{i_\nu} \\ (-)^{\nu+1} & \text{if } \mu = \mu' \end{cases}$$

It is easily seen that the $\alpha_{\sigma,\sigma'}$ satisfy the axioms of [11, IV 7.2].

Let Δ_n be the set of n-cells in Δ and let $C^n(\Delta) = \mathrm{Maps}(\Delta_n \to \Gamma)$ be the group of n-cochains with values in Γ. The differential $\delta : C^n(\Delta) \to C^{n+1}(\Delta)$ is given by

$$\delta(\phi)(\sigma) = \prod_{\substack{\sigma' \subset \overline{\sigma} \\ \sigma' \in \Delta_n}} \phi(\sigma')^{\alpha_{\sigma,\sigma'}}$$

and the augmentation $\Gamma \xrightarrow{\epsilon} C^0(\Delta)$ maps $\gamma \in \Gamma$ to the constant function on Δ_0 with value γ. The augmented cochain complex

$$0 \to \Gamma \xrightarrow{\epsilon} C^0(\Delta) \xrightarrow{\delta} C^1(\Delta) \xrightarrow{\delta} \cdots$$

is exact since Δ is contractable.

Now we define a map $\theta : C^n(\Delta) \to C^n$ as follows

$$(\theta(\phi))_{i_1\cdots i_n}(\mu) = \phi(\sigma_{\mu,\{i_1,\dots,i_n\}})$$

where we have assumed $i_1 < \cdots < i_n$. θ is clearly an isomorphism and one verifies that θ commutes with ϵ and δ. Hence (2.1) is also exact. $\square$

Actually, we will need the acyclicity of (2.1) only at places C^0, C^1, C^2 and it easy to give a direct proof of this and of lemma 2.1.2 below without appealing to topology. For the benefit of the reader we give explicitly the first three differentials

If $b \in C^0$ then

$$(\delta b)_i(\mu) = \frac{b(\mu u_i)}{b(\mu)}$$

If $c \in C^1$ then

$$(\delta c)_{ij}(\mu) = \frac{c_j(\mu u_i) c_i(\mu)}{c_j(\mu) c_i(\mu u_j)}$$

If $d \in C^2$ then

$$(\delta d)_{ijk}(\mu) = \frac{d_{jk}(\mu u_i) d_{ki}(\mu u_j) d_{ij}(\mu u_k)}{d_{jk}(\mu) d_{ki}(\mu) d_{ij}(\mu)}$$

(we have used that $d_{ki} = d_{ik}^{-1}$).

It is sometimes convenient to know an explicit normalization of the elements of $C^1/\operatorname{im}\delta$. This is the content of the following lemma :

Lemma 2.1.2. Any element of $C^1/\operatorname{im}\delta$ is uniquely represented by an element $c \in C^1$ such that $c_i(u_i^{\gamma_i} \cdots u_l^{\gamma_l}) = 1$ i.e. $c_i(\mu) = 1$ for μ not divisible by u_j, $j < i$.

Proof. It suffices to prove the analogous statement for $C^1(\Delta)/\operatorname{im}\delta$. Let $\Delta^1 \subset \Delta$ be the 1-skeleton on Δ, i.e. the union of all 0 and 1-cells. So Δ^1 is a graph. Then $C^1(\Delta)/\operatorname{im}\delta = H^1(\Delta^1, \Gamma)$.

It is well-known that any element of H^1 of a graph may be uniquely represented by a cocycle, which takes the value 1 on the edges belonging to a spanning tree. In our situation we take for a spanning tree the one where the 1-cells are given by $\sigma_{\mu,\{i\}}$, where μ is not divisible by u_j, $j < i$. $\square$

2.2. Modules of I-type. Below, F will be a base field and V will be a vector space over F of dimension r. $T = TV$ will be the tensor algebra of V. A T-module M together with a decomposition $M = \oplus_{\mu \in \mathcal{U}_I} M_\mu$ as a direct sum of one-dimensional subspaces, indexed by monomials $\mu \in \mathcal{U}_I$, is of *I-type* if

$$VM_\mu = \sum_{i \in I} M_{\mu u_i}$$

The $(M_\mu)_{\mu \in \mathcal{U}_I}$ are said to define an *I-structure* on M. Of course M may be $\mathbb{Z}$-graded via $M_d = \sum_{\deg \mu = d} M_\mu$. Note also that the existence of a module of I-type implies that $l = |I| \le r$.

If we choose for each μ a basis element $v(\mu)$ of M_μ then the action of an $x \in V$ on this basis is described by

$$x \cdot v(\mu) = \sum_{i \in I} \lambda_{\mu,i}(x)\, v(\mu u_i) \tag{2.3}$$

where for each μ, $(\lambda_{\mu,i})_{i \in I}$ is an independent set of linear maps $\lambda_{\mu,i} : V \longrightarrow F$.

Conversely, if we are given for each μ an independent family of elements $\lambda_{\mu,i} \in V^*$, indexed by $i \in I$, then (2.3) defines a T-module $M = \bigoplus_\mu Fv(\mu)$ of I-type.

Given such M and such a basis $(v(\mu))_{\mu \in \mathcal{U}_I}$ there is for each pair (μ, i) an element $x_{\mu,i}$ in V (unique if $r = l$) such that $x_{\mu,i} v(\mu) = v(\mu u_i)$ and in particular $x_{\mu,i} M_\mu = M_{\mu u_i}$.

Hence $M = Tv(1)$ is generated by one element. Also for each μ, the subspace M_μ of M generates a submodule TM_μ which is of I-type via $(TM_\mu)_{\mu'} = M_{\mu\mu'}$.

For an integer $m \ge 0$ we call the T-module $M_{\le m} = M/\sum_{\deg \mu > m} M_\mu$ the *m-truncation* of M.

2.3. Twisting. Recall the exact sequence (2.1). We now take for our abelian group Γ the multiplicative group F^* of the ground field F.

So C^n is the group of alternating functions $c = c_{i_1\cdots i_n}$ on I^n with values in the group of maps $\mathcal{U}_I \to F^*$. Elements of C^n are called n-cochains and they are called a cocycle or a coboundary if they lie in $\ker\delta = \operatorname{im}\delta$.

Let c be a 1-cochain and M a module of I-type with basis $(v(\mu))_{\mu\in\mathcal{U}_I}$, defined by (2.3). We define a new module of I-type $M(c)$ with basis $v_c(\mu)$ by

$$x \cdot v_c(\mu) = \sum_{i\in I}(\lambda_{\mu,i}(x)/c_i(\mu))\, v_c(\mu u_i) \tag{2.4}$$

It is easy to see that $M(c)$ with its I-structure is, up to isomorphism, independent of the choice of basis $(v(\mu))_\mu$ for the I-structure on M.

Proposition 2.3.1. $M(c) \cong M$ if and only if c is a cocycle.

Proof. Suppose that $c = \delta b$, i.e.

$$c_i(\mu) = \frac{b(\mu u_i)}{b(\mu)}$$

Then (2.4) becomes

$$x \cdot \frac{v_c(\mu)}{b(\mu)} = \sum_{i\in I} \lambda_{\mu,i}(x)\frac{v_c(\mu u_i)}{b(\mu u_i)}$$

So $v(\mu) \mapsto v_c(\mu)/b(\mu)$ gives an isomorphism of T-modules $M \cong M(c)$.

To show the converse, assume that $\phi : M \to M(c)$ is an isomorphism. We first claim that $\phi(M_\mu) = M(c)_\mu$, which we prove by induction on μ. The claim is clearly true for $\mu = 1$. Suppose that it is true for some $\mu \in \mathcal{U}_I$. Let $i \in I$ and let $x_{\mu,i}v(\mu) = v(\mu u_i)$, i.e. $\lambda_{\mu,j}(x_{\mu,i}) = \delta_{j,i}$. Then (2.4) shows that $x_{\mu,i}v_c(\mu) = v_c(\mu u_i)/c_i(\mu)$. Hence $\phi(M_{\mu u_i}) = \phi(x_{\mu,i}M_\mu) = x_{\mu,i}\phi(M_\mu) \overset{\text{ind.}}{=} x_{\mu,i}M(c)_\mu = M(c)_{\mu u_i}$.

By the claim we just proved we have $\phi(v(\mu)) = v_c(\mu)/b(\mu)$ for some 0-cochain b. Then (2.4) shows that $c = \delta b$. $\square$

Remark 2.3.2. The isomorphism $M \cong M(c)$ is unique up to a constant factor.

Let M be as above. For $l \times l$-matrices $(h_{ij}(\mu))_{i,j\in I}$ with entries in F^*, indexed by $\mu \in \mathcal{U}_I$ we define a series of T-modules $(N(h,\mu))_{\mu\in\mathcal{U}_I}$ as follows :

$N(h,\mu)$ has an F-basis consisting of elements $[\nu]$, indexed by monomials ν of degree ≤ 2 and for $x \in V$

$$\begin{aligned} x\cdot[1] &= \sum_i \lambda_{\mu,i}(x)[u_i] \\ x\cdot[u_i] &= \sum_j \lambda_{\mu u_i,j}(x)h_{ij}(\mu)[u_iu_j] \\ x\cdot[u_iu_j] &= 0 \end{aligned} \tag{2.5}$$

Proposition 2.3.3. Let M and h be as above. Let c be a 1-cochain and $\mu \in \mathcal{U}_I$. Then the 2-truncation $T(M(c)_\mu)_{\leq 2}$ is isomorphic to $N(h,\mu)$, if and only if

$$\frac{h_{ji}(\mu)}{h_{ij}(\mu)} = (\delta c)_{ij}(\mu)$$

Proof. For each $\mu \in \mathcal{U}_I$ we want to find isomorphisms

$$\phi_\mu : T(M(c)_\mu)_{\leq 2} \to N(h,\mu)$$

As in the proof of Proposition 2.3.1 one shows that ϕ_μ must respect the given bases of $T(M(c)_\mu)_{\leq 2}$ and $N(h,\mu)$, up to a scalar multiple. That is, ϕ_μ must have the form

$$\begin{aligned} v_c(\mu) &\mapsto \alpha(\mu)[1] \\ v_c(\mu u_i) &\mapsto \beta_i(\mu)[u_i] \\ v_c(\mu u_i u_j) &\mapsto \gamma_{ij}(\mu)[u_i u_j] \end{aligned} \tag{2.6}$$

where $\gamma_{ij}(\mu) = \gamma_{ji}(\mu)$. We may normalize ϕ_μ in such a way that $\alpha(\mu) = 1$.

Expressing the fact that (2.6) is a morphism of T-modules then yields

$$\begin{aligned} \beta_i(\mu) &= c_i(\mu) \\ \gamma_{ij}(\mu) &= h_{ij}(\mu) c_j(\mu u_i) c_i(\mu) \end{aligned}$$

Expressing the fact that $\gamma_{ij}(\mu)$ must be symmetric in i and j yields

$$\frac{h_{ji}(\mu)}{h_{ij}(\mu)} = \frac{c_j(\mu u_i) c_i(\mu)}{c_j(\mu) c_i(\mu u_j)} = (\delta c)_{ij}(\mu) \quad \square$$

Corollary 2.3.4. Let M and h be as above. Put $d_{ij}(\mu) = h_{ji}(\mu)/h_{ij}(\mu)$. Then there exists a 1-cochain c such that the module $M(c)$ has the property that for each μ, $T(M(c)_\mu)_{\leq 2}$ is isomorphic to $N(h,\mu)$ as defined by (2.5), if and only if $\delta d = 0$. Moreover such a module $M(c)$ is unique up to isomorphism.

Proof. This follows from the exactness of (2.1) and Propositions 2.3.1 and 2.3.3. $\square$

3. Algebras of I-type

In this section we suppose that $l = |I| = \dim V = r$. Assume that $A = \oplus_{\mu \in \mathcal{U}_I} A_\mu$ is a T-module of I-type and fix an element $1 \in A_0$. Then we say that A is an *algebra of I-type* if the annihilator of 1 in T is a twosided ideal in T, i.e. A carries a canonical algebra structure with unit element 1.

Assume now that A is an algebra of I-type. Clearly A has the Hilbert series of a polynomial algebra in r-variables. Since for $\mu \in \mathcal{U}_I$, $Tv(\mu) = Av(\mu)$ is also of I-type, it follows that $v(\mu)$ is a right non-zero divisor. Hence if $\mu, \mu' \in \mathcal{U}_I$, $\mu' \mid \mu$, there is a unique element $y_{\mu',\mu}$ of A with the property that $y_{\mu',\mu} v(\mu') = v(\mu)$.

Let $\Lambda \subset \mathcal{U}_I$ the set of square free monomials. For $\mu', \mu \in \Lambda$, $\mu = u_1 \cdots u_{i_s}$, $\mu' = u_{i_1} \cdots \hat{u}_{i_\nu} \cdots u_{i_s}$, $i_1 < \cdots < i_s$, we put $\alpha_{\mu,\mu'} = (-1)^\nu$.

Proposition 3.1. Let A be an algebra of I-type and $r = |I|$. Then the minimal resolution of F as a left A-module has the following form :

$$0 \to Av(u_1 \cdots u_r) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus_{\substack{\mu \in \Lambda \\ \deg \mu = 2}} Av(\mu) \xrightarrow{\delta} \bigoplus_{\substack{\mu \in \Lambda \\ \deg \mu = 1}} Av(\mu) \xrightarrow{\delta} Av(1) \tag{3.1}$$

Here $\delta = \oplus_{\mu',\mu} \alpha_{\mu',\mu} i_{\mu,\mu'}$ where $i_{\mu,\mu'}$ is the inclusion $Av(\mu) \hookrightarrow Av(\mu')$ if $\mu' \mid \mu$.

Or in terms of generators

$$\delta(v(\mu)) = \bigoplus_{\substack{\mu'|\mu \\ \deg \mu' = \deg \mu - 1}} \alpha_{\mu,\mu'} y_{\mu',\mu} v(\mu')$$

Proof. Recall that for $\mu \in \mathcal{U}_I$, $Av(\mu)$ has a basis $v(\mu')$, $\mu \mid \mu'$.

Denote the complex (3.1) by $K^{\cdot}$. For $\mu \in \mathcal{U}_I$ let $K^{\cdot}_\mu$ be the following complex (the $[\mu']$ are symbols indexed by $\mu' \in \Lambda$)

$$\cdots \xrightarrow{\delta} \bigoplus_{\substack{\mu'|\mu \\ \deg \mu' = p}} F[\mu'] \xrightarrow{\delta} \bigoplus_{\substack{\mu'|\mu \\ \deg \mu' = p-1}} F[\mu'] \xrightarrow{\delta} \cdots$$

where

$$\delta[\mu'] = \bigoplus_{\substack{\mu''|\mu' \\ \deg \mu'' = \deg \mu' - 1}} \alpha_{\mu',\mu''}[\mu'']$$

Then $\bigoplus_{\mu \in \mathcal{U}_I} K^{\cdot}_\mu \cong K^{\cdot}$ as complexes where $[\mu']$ in K_μ corresponds to $v(\mu)$ in $K^{\cdot}$, considered as an element of $Av(\mu')$.

Now it is easy to see that $K^{\cdot}_\mu$ is the reduced chain complex of the abstract simplicial complex

$$\Lambda_\mu = \{\mu' \in \Lambda \mid \mu' \text{ divides } \mu\}$$

Hence $K^{\cdot}_\mu$ is acyclic except if $\mu = 1$. Translating this back means that (3.1) is acyclic except in degree 0, where its homology is F. $\square$

Corollary 3.2. *A as above.*

(1) *A is quadratic with r generators $x_{1,i}$, $i \in I$ and $r(r-1)/2$ quadratic relations $x_{u_i,j}x_{1,i} = x_{u_j,i}x_{1,j}$, $i,j \in I$, $i \neq j$ where $x_{\mu,i} \in A_1 = V$ is the unique element such that $x_{\mu,i}v(\mu) = v(\mu u_i)$.*
(2) *A is Koszul.*
(3) *A has global dimension r.*

Proof. These are all immediate consequences of the minimal resolution of F. For (3) one uses that

$$\operatorname{gl\,dim} A = \operatorname{pd}_A F$$

(see [5, Theorem 1] for the ungraded case). $\square$

In view of corollary 3.2 it is natural to ask whether an algebra of I-type automatically has other good properties. We don't know this, but the following (counter) example is encouraging.

Example 3.3. Let $A = k\langle x, y\rangle/(yx)$. It is known that A is Koszul, but otherwise it has very bad properties. For example A is not Noetherian. Here we show that it is impossible to put an I-structure on A.

The element y in A_1 has the property that $yA_n = Fy^{n+1}$ is one-dimensional for each $n \geq 0$. On the other hand, if A has an I-structure then $\dim zA_n \geq n/2$ for each non-zero z in A_1, because the elements $zv(u_1^{(n-2i)}u_2^{2i})$, $0 \leq i \leq n/2$ are linearly independent.

Now we give a criterion for recognizing algebras of I-type.

Theorem 3.4. Suppose that A is an algebra with r generators and $r(r-1)/2$ relations in degree 2. Suppose that A has a module M of I-type with $l = |I| = r$. Then M is isomorphic to A. In particular A is itself of I-type.

Proof. Let $T = TA_1$, so $A = T/(R)$ with $R \subset A_1 \otimes A_1 = T_2$. Let $M = \oplus_{\mu \in \mathcal{U}_I} Fv(\mu)$ be the I-structure. For each $\mu \in \mathcal{U}_I$, $T_2 v(\mu) = \sum_{i \le j} Fv(u_i u_j)$ (direct sum) and $Rv(\mu) = 0$. A dimension count shows that for each μ

$$R = \{t \in T_2 \mid tv(\mu) = 0\}$$

Let $\tilde{M}$ be the linear space with basis $v(\tilde{\mu})$ indexed by monomials in r *non-commuting* generators $\tilde{u}_i$, $i \in I$. Make $\tilde{M}$ into a T-module via ($x \in A_1$)

$$x \cdot v(\tilde{\mu}) = \sum_{i \in I} \lambda_{\mu,i}(x)\, v(\tilde{\mu}\tilde{u}_i)$$

where the $\lambda_{\mu,i}(x)$ are defined by

$$x \cdot v(\mu) = \sum_{i \in I} \lambda_{\mu,i}(x)\, v(\mu u_i) \quad \text{ in } M$$

and where μ is deduced from $\tilde{\mu} = \tilde{u}_{i_1} \cdots \tilde{u}_{i_n}$ by putting $\mu = u_{i_1} \cdots u_{i_n}$.

Then $Tv(\tilde{1}) = \tilde{M}$ and, comparing dimensions, it follows that $t \mapsto tv(\tilde{1})$ is a bijection $T \cong \tilde{M}$.

Consider the commutative diagram

$$\begin{array}{ccccc} T & \xrightarrow{t \mapsto tv(\tilde{1})} & \tilde{M} & & v(\tilde{\mu}) \\ {\scriptstyle \text{can. map}}\big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \psi} & & \big\downarrow \\ A & \xrightarrow{a \mapsto av(1)} & M & & v(\mu) \end{array}$$

We want to show that $\ker \phi$ maps onto $\ker \psi$ since then $\ker \phi \cong \ker \psi$ and hence $A \cong M$, since ϕ and ψ are surjective.

What is $\ker \psi$? It is spanned by things of the form

$$v(\tilde{\mu}\tilde{u}_i\tilde{u}_j\tilde{\mu}') - v(\tilde{\mu}\tilde{u}_j\tilde{u}_i\tilde{\mu}') \tag{3.2}$$

as one sees by thinking of $\ker(F\langle \tilde{u}_1, \ldots, \tilde{u}_n \rangle \to F[u_1, \ldots, u_n])$.

Now express each of the basis elements $v(\tilde{\mu})$ as follows :

$$v(\tilde{u}_{i_1}\tilde{u}_{i_2} \ldots \tilde{u}_{i_n}) = x_{u_{i_1} \cdots u_{i_{n-1}}, i_n} \cdots x_{u_{i_1}, i_2} x_{1, i_1} v(\tilde{1})$$

where $x_{\mu,i}$ is the unique element of A_1 with the property that $x_{\mu,i} v(\mu) = v(\mu u_i)$. Then (3.2) is of the form

$$(\zeta x_{\mu u_i, j} x_{\mu,i} \zeta' - \zeta x_{\mu u_j, i} x_{\mu, j} \zeta') v(\tilde{1}) = \zeta (x_{\mu u_i, j} x_{\mu, i} - x_{\mu u_j, i} x_{\mu, j}) \zeta' v(\tilde{1})$$

Now $x_{\mu u_i, j} x_{\mu, i} - x_{\mu u_j, i} x_{\mu, j}$ kills $v(\mu)$ and hence is in R. Therefore $\zeta x_{\mu u_i, j} x_{\mu, i} \zeta' - \zeta x_{\mu u_j, i} x_{\mu, j} \zeta'$ is in $\ker \phi$. $\square$

4. Sklyanin algebras

4.1. Construction. Let E be some fixed elliptic curve over F, let $\sigma : E \to E$ be a translation and let $\mathcal{L}$ be an invertible sheaf on E of degree $r \geq 3$. Put $V = H^0(E, \mathcal{L})$, a vector space of dimension r.

If π, ρ are automorphisms of E then on $E \times E$ denote by $[\pi, \rho]$ the map $(p, q) \mapsto (\pi(p), \rho(q))$. Let $\mathcal{L} \boxtimes \mathcal{L}$ denote $\mathrm{pr}_1^* \mathcal{L} \otimes_{\mathcal{O}_{E\times E}} \mathrm{pr}_2^* \mathcal{L}$, where $\mathrm{pr}_i : E \times E \to E$ denotes projection on the i'th factor. Clearly $H^0(E \times E, \mathcal{L} \boxtimes \mathcal{L}) = V \otimes V$. Let $\Delta \subset E \times E$ be the diagonal. Then $\Gamma_\pi = [1, \pi]\Delta$ is the graph of the automorphism π of E.

Define $\mathcal{L}' = (\sigma^{-1})^* \mathcal{L}$, $V' = H^0(E, \mathcal{L}')$, $\pi = \sigma^{r-2}$ and $\theta = \sigma^{-2}$.

The construction of the Sklyanin algebra is based on the fact that there is an isomorphism

$$\phi : [1, \theta^{-1}]^* (\mathcal{L}' \boxtimes \mathcal{L}' \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Delta)) \to \mathcal{L} \boxtimes \mathcal{L} \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Gamma_\pi) \tag{4.1}$$

To prove this we can assume F algebraically closed, and then, by the see-saw principle [13, p. 53], it is enough to check the easily verified fact that the two invertible sheaves in question have isomorphic restrictions to the fibers $p \times E$ and $E \times p$ for closed points p on E.

Now $H^0(E \times E, \mathcal{L}' \boxtimes \mathcal{L}' \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Delta))$ contains the $r(r-1)/2$-dimensional subspace $\bigwedge^2 V'$ spanned by sections of the form $x \otimes y - y \otimes x$ for $x, y \in V'$.

Then we find an $r(r-1)/2$-dimensional subspace inside

$$H^0(E \times E, \mathcal{L} \boxtimes \mathcal{L} \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Gamma_\pi)) \subset H^0(E \times E, \mathcal{L} \boxtimes \mathcal{L}) = V \otimes V$$

given by

$$R = \phi([1, \theta^{-1}]^* (\textstyle\bigwedge^2 V')) \tag{4.2}$$

According to [16] R is the space of quadratic relations defining the Sklyanin algebra $TV/(R) = A(E, \sigma, \mathcal{L})$.

The following is immediate from the definition.

Proposition 4.1.1. (1) Assume that $\mu : E' \to E$ is an isomorphism between elliptic curves E, E'. Then $\mu^{-1}\sigma\mu$ is a translation and there is a canonical isomorphism $A(E, \sigma, \mathcal{L}) \to A(E', \mu^{-1}\sigma\mu, \mu^*\mathcal{L})$ sending $x \in V = H^0(E, \mathcal{L})$ to $\mu^* x \in H^0(E', \mu^*\mathcal{L})$.

(2) $A(E, \sigma, \mathcal{L})^\circ \cong A(E, \sigma^{-1}, \mathcal{L})$ where $(\ )^\circ$ denotes the opposite ring.

Proof. (1) follows from applying $[\mu, \mu]^*$ to (4.1) whereas (2) follows by applying ω^* to (4.1), where $\omega(z_1, z_2) = (z_2, z_1)$. $\square$

Remark 4.1.2. Proposition 4.1.1 implies in particular that if F is an algebraically closed field then $A(E, \sigma, \mathcal{L})$ does not depend on $\mathcal{L}$, since any two $\mathcal{L}$ will be conjugate under a translation, which then commutes with σ. This explains why $\mathcal{L}$ is missing in the Odeskii-Feigin notation $Q(\tau, E)$ for $A(E, \sigma, \mathcal{L})$. ($\tau$ = "class of $\sigma p - p$" is the element of the Jacobian of E corresponding to σ.)

Furthermore, with F still algebraically closed, one finds as a curiosity that $A(E, \sigma, \mathcal{L})^\circ \cong A(E, \sigma, \mathcal{L})$ since we may apply Proposition 4.1.1(1) with $\mu : x \mapsto -x$.

Let $B(E,\pi,\mathcal{L})$ be the twisted homogeneous coordinate ring associated to the triple $(E,\pi,\mathcal{L})$ (see [2][4]), i.e. the ring of global sections of

$$\mathcal{O}_E \oplus \mathcal{L} \oplus \mathcal{L}\otimes\pi^*\mathcal{L} \oplus \mathcal{L}\otimes\pi^*\mathcal{L}\otimes\pi^{2*}\mathcal{L} \oplus \cdots$$

Then the fact that

$$R \subset H^0(E\times E, \mathcal{L}\boxtimes\mathcal{L}\otimes_{\mathcal{O}_{E\times E}}\mathcal{O}(-\Gamma_\pi))$$

implies that there is a canonical surjective homomorphism

$$A(E,\sigma,\mathcal{L}) \to B(E,\pi,\mathcal{L}) \tag{4.3}$$

which is the identity on $A_1 = B_1$.

The ring $B(E,\pi,\mathcal{L})$ together with the map (4.3) played a fundamental role in several earlier papers on Sklyanin algebras.

4.2. Modules of I-type over a Sklyanin algebra. For technical reasons we assume from now on that $\mathcal{L} = \mathcal{O}(d)$ where d is a divisor on E. If we define $d' = \sigma d$ then $\mathcal{L}' = \mathcal{O}(d')$. We put $A = A(E,\sigma,\mathcal{L})$.

The isomorphism ϕ in (4.1) is now multiplication by a rational function h on $E\times E$ such that

$$d\times E + E\times d - \Gamma_\pi + (h) = d'\times E + E\times\theta(d') - \Gamma_\theta$$

As before let $I = \{1,\dots,l\}$, $l\le r$ and let $z = (z_1,\dots,z_l)\in E^l$ be a rational point. Define an action of $\mathcal{U}_I$ on E^l by

$$(u_j z)_i = \begin{cases} \pi^{-1}z_i & \text{if } i=j \\ \theta^{-1}z_i & \text{otherwise.}\end{cases}$$

Thus, if $\mu = u_1^{a_1}\cdots u_l^{a_l}$ then $(\mu z)_i = \pi^{-a_i}\theta^{-\deg\mu + a_i}z_i$.

We will assume that $(z_1,\dots,z_l)$ satisfies the following conditions

(A) For all $\mu\in\mathcal{U}_I$ and for all $i\ne j$, $(\mu z)_i \ne (\mu z)_j$.
(B) If $l = r$ then for all $\mu\in\mathcal{U}_I$, the divisor given by $((\mu z)_1)+\cdots+((\mu z)_l)$ is not linearly equivalent with d.
(C) For all $\mu\in\mathcal{U}_I$ and for all $i\in\{1,\dots,l\}$, $(\mu z)_i$ does not lie in the support of d, d' or $\theta(d')$.

Remark 4.2.1. (1) Note that points $z_1,\dots,z_l$ with properties (A), (B), (C) do not always exist. However they do exist over an extension of the base field. For example we may replace F by the field of rational functions on the product of l copies of E.
By being more careful one can even find such points over a finite extension of the base field (exercise !). However we will not need this fact.

(2) Condition (C) is less fundamental than (A) and (B) in the sense that by moving d in its linear equivalence class we can always make sure that (C) becomes true.
To prove this we may assume that F is finitely generated over the prime field. It is then sufficient to choose a divisor, in the given class, whose support contains no rational point. To see that this is possible note that d moves in a linear system, so we can view the possible d's as the inverse images, under a map $E\to\mathbb{P}^1$ of the divisors of degree one on $\mathbb{P}^1$. So we

can take d as the inverse image of the difference of a point of degree three and a point of degree two on $\mathbb{P}^1$.

If F is not finite then we can even find an effective d by choosing height functions, and noting that there are many more points of height $< x$ on $\mathbb{P}^1$ than on E so that we can find a rational point on $\mathbb{P}^1$, not in the image of $E(F)$. (Thanks to Mike Artin and Filipe Voloch for these arguments.)

For such $z = (z_1, \dots, z_l)$ we define a module $M_0(z) = \oplus_\mu F v^z(\mu)$ of I-type over the tensor algebra by putting

$$x \cdot v^z(\mu) = \sum_i x((\mu z)_i)\, v^z(\mu u_i)$$

for x in $V = H^0(E, \mathcal{O}(d))$.

In the notation of §2.2 the linear functional $\lambda_{\mu,i} : V \to F$ is the evaluation of the function $x \in V$ at the point $(\mu z)_i$. This makes sense by (C). Furthermore we obtain a module of I-type since, by (A), (B) and the Riemann-Roch theorem on E, evaluations of elements of V in the points $(\mu z)_i$ are linearly independent.

Now define

$$h^z_{ij}(\mu) = \begin{cases} h((u_i \mu z)_j, (\mu z)_i) & \text{for } i \neq j \\ 1 & \text{otherwise} \end{cases}$$

This makes sense, and $h^z_{ij}(\mu) \in F^*$ for all i, j, μ. The zeroes and poles of h are on divisors of the form $(d, d', \theta(d')) \times E$, $E \times (d, d', \theta(d'))$, Γ_π and Γ_θ. By condition (C) $(u_i \mu z)_j$ and $(\mu z)_i$ are disjoint from $d, d', \theta(d')$, and hence we only have to worry about Γ_π and Γ_θ. Assume that $((u_i\mu z)_j, (\mu z)_i) \in \Gamma_\theta$. Then $(\mu z)_i = \theta((u_i \mu z)_j) = (\mu z)_j$, contradicting (A). Assume now that $((u_i \mu z)_j, (\mu z)_i) \in \Gamma_\pi$. Then $(\mu z)_i = \pi((u_i \mu z)_j)$ and hence $(u_i \mu z)_j = \pi^{-1}((\mu z)_i) = (u_i \mu z)_i$ contradicting again (A).

Proposition 4.2.2. For each μ, the module $N(h^z, \mu)$ defined in terms of $M_0(z)$ and $(h^z_{ij}(\mu))_{i,j,\mu}$, as in (2.5), is killed by R.

Proof. Since $h^z_{ij}(\mu) = h^{\mu z}_{ij}(1)$, and since the map $v^{\mu z}(\mu') \to v^z(\mu\mu')$ is an isomorphism of $M_0(\mu z)$ with the submodule of $M_0(z)$ generated by $v^z(\mu)$, it suffices to prove the proposition for the case $\mu = 1$.

Let f be a rational function on $E \times E$ contained in $H^0(E \times E, \mathcal{O}(d \times E + E \times d)) = V \otimes V$. Then an easy verification shows that the action of f on $N(h^z, 1)$ is given by

$$f \cdot [1] = \sum_i f(\pi^{-1} z_i, z_i)[u_i^2] + \sum_{i \neq j} (hf)(\theta^{-1} z_i, z_j)[u_i u_j] \tag{4.4}$$

(It is enough to prove this for $f = x \otimes y$, and for x and y one can use the formulas (2.5).)

We now prove that (4.4) is 0 when $f \in R$. First note that $f(\pi^{-1} z_i, z_i) = 0$ since f vanishes on Γ_π. Second, for $i \neq j$ the coefficient of $[u_i u_j]$ in (4.4) is $(hf)(\theta^{-1} z_i, z_j) + (hf)(\theta^{-1} z_j, z_i)$. To see that this is zero, note that by definition hf is a linear combination of functions of the form $[1, \theta](x \otimes y - y \otimes x)$ with $x, y \in V'$ and $[1, \theta](x \otimes y - y \otimes x)(\theta^{-1} z_i, z_j) = x(\theta^{-1} z_i) y(\theta^{-1} z_j) - y(\theta^{-1} z_i) x(\theta^{-1} z_j)$ which is anti-symmetric in i and j. $\square$

We now want to apply corollary 2.3.4. The last step to show that we can twist $M_0(z)$ by a cocycle c^z to get an A-module $M(z)$ is to prove that $d^z_{ij}(\mu) = h^z_{ji}(\mu)/h^z_{ij}(\mu)$ is a cocycle, because if all 2-truncations of $M(z)$ are annihilated by R, then $M(z)$ itself is annihilated by R and hence is an A-module.

Since $h^{\mu z}_{ij}(\mu') = h^z_{ij}(\mu\mu')$ we have $d^z_{ij}(\mu) = d^{\mu z}_{ij}(1)$ and $(\delta d^z)_{ijk}(\mu) = (\delta d^{\mu z})_{ijk}(1)$.

Hence it suffices to show that $(\delta d^z)_{ijk}(1) = 1$. $d^z_{ij}(1) = g(z_i, z_j)$ where $g(z,w)$ is the rational function on $E \times E$ defined by

$$g(z,w) = \frac{h(\theta^{-1}z, w)}{h(\theta^{-1}w, z)}$$

The divisor of g on $E \times E$ is

$$(d - \theta(d)) \times E + E \times (\theta(d) - d) + \Gamma_{\theta^{-1}\pi} - \Gamma_{\theta\pi^{-1}} \tag{4.5}$$

Furthermore, the rational function $(z,w) \mapsto h(\theta^{-1}z, w)$ has a simple zero on Δ unless $\pi = \theta$, in which case it is regular on Δ. We deduce by a local computation that

$$g \mid \Delta = \begin{cases} -1 & \text{if } \pi \neq \theta \\ 1 & \text{otherwise} \end{cases} \tag{4.6}$$

The cocycle condition is that for all $(i,j,k) \in I^3$, $i \neq j \neq k \neq i$

$$\frac{g(\theta^{-1}z_i, \theta^{-1}z_j)}{g(z_i, z_j)} \cdot \frac{g(\theta^{-1}z_j, \theta^{-1}z_k)}{g(z_j, z_k)} \cdot \frac{g(\theta^{-1}z_k, \theta^{-1}z_i)}{g(z_k, z_i)} = 1 \tag{4.7}$$

Using (4.5) one verifies that (4.7) is a scalar and by restricting to $z_i = z_j = z_k$, using (4.6) we see that this scalar is one.

Hence we have proved

Theorem 4.2.3. Assume that $z = (z_1, \dots, z_l)$ is a rational point of E^l, satisfying conditions (A), (B) and (C). Then there exists an A-module of I-type $M(z) = \oplus_{\mu \in \mathcal{U}_I} Fv(\mu)$ with action for $x \in V = A_1 = H^0(E, \mathcal{O}(d))$

$$x \cdot v(\mu) = \sum_{i=1}^{l} \frac{x((\mu z)_i)}{c_i(\mu)} v(\mu u_i) \tag{4.8}$$

where the $c_i(\mu)$ are non-zero constants, satisfying

$$\frac{c_j(\mu u_i) c_i(\mu)}{c_i(\mu u_j) c_j(\mu)} = \frac{h((u_i \mu z)_j, (\mu z)_i)}{h((u_j \mu z)_i, (\mu z)_j)}$$

for $i \neq j$, where h is a rational function on $E \times E$ with divisor

$$(d' - d) \times E + E \times (\theta(d') - d) - \Gamma_\theta + \Gamma_\pi$$

Furthermore, up to isomorphism, $M(z)$, with its I-structure, is independent of the choice of c.

Proof of (1) and (2) of Theorem 1.1. We may without loss of generality extend the base field so that there exists a rational point $z = (z_1, \dots, z_r)$ in E^r satisfying (A), (B) and (C). Then A is of I-type by Theorem 4.2.3 and Theorem 3.4 and hence has the right Hilbert series. Furthermore A is Koszul by corollary 3.2. □

Remark 4.2.4. (1) To stress the dependency on z let us write (4.8) as

$$x \cdot v^z(\mu) = \sum_{i=1}^{l} \frac{x((\mu z)_i)}{c_i^z(\mu)} v^z(\mu u_i)$$

Let $\mu_0 \in \mathcal{U}_I$. Then the submodule of $M(z)$ generated by $v^z(\mu_0)$ is spanned by $v^z(\mu_0\mu)$, $\mu \in \mathcal{U}_I$ and is isomorphic to $M(\mu_0 z)$ via $v^{\mu_0 z}(\mu) \mapsto v^z(\mu_0\mu)$ if we take $c_i^{\mu_0 z}(\mu) = c_i^z(\mu\mu_0)$.

(2) If $z' = (z_1, \dots, z_{l-1})$ then there is an exact sequence of the form

$$0 \to M(u_l z)(-1) \to M(z) \to M(z') \to 0$$

This follows from (1), since obviously $M(z)/Av(u_l) = M(z')$ and $Av(u_l) \cong M(u_l z)$.

4.3. The case in which σ is of finite order. Assume that there is a rational point $z = (z_1, \dots, z_r)$ in E^r satisfying (A), (B), (C). Then by Theorem 3.4 $A \cong M(z)$, i.e. A has a basis $(v(\mu))_{\mu\in\mathcal{U}_I}$ on which A acts as given by (4.8). Put $A_\mu(z) = Fv(\mu)$. By remark 4.2.4(1), the map $a \mapsto av(\mu_0)$ is an isomorphism of the left module $M(\mu_0 z)$ onto the submodule of $M(z)$ generated by $v(\mu_0)$. Hence

$$A_\mu(\mu_0 z)A_{\mu_0}(z) = A_{\mu\mu_0}(z)$$

for all $\mu, \mu_0 \in \mathcal{U}_I$.

Assume now that σ is of finite order, say $\sigma^s = 1$, $s > 0$. Then $\theta^s = \pi^s = 1$, so $\mu^s z = z$ for all $\mu \in \mathcal{U}_I$. Put $\Omega_i = v(u_i^s)$.

Proposition 4.3.1. (1) The $(\Omega_i)_{i=1,\dots,r}$ commute.

(2) A is finite free of rank s^r as a right module over the subring generated by $(\Omega_i)_{i=1,\dots,r}$, which is itself a polynomial ring in r-variables. The elements $v(u_1^{a_1} \cdots u_r^{a_r})$ where $0 \le a_i \le s-1$ for $1 \le i \le r$ form a basis.

Proof. Since $\mu_i^s z = z$ for $z \in E^r$ we have $A_\mu(z)A_{u_i^s}(z) = A_{\mu u_i^s}(z)$ for all μ in $\mathcal{U}_I$. Hence

$$A_\mu(z)\Omega_i = A_{\mu u_i^s}(z)$$

Therefore A is generated, as a right module over the subring $F[\Omega_1, \dots, \Omega_r]$, by the elements $v(u_1^{a_1} \cdots u_r^{a_r})$ where $0 \le a_i \le s-1$ for $1 \le i \le r$. Also

$$\Omega_i\Omega_j = \lambda\Omega_j\Omega_i \tag{4.9}$$

for some $\lambda \in F^*$, because $\Omega_i\Omega_j$ and $\Omega_j\Omega_i$ are bases for the one-dimensional space $A_{u_i^s u_j^s}(z)$.

We take the image of (4.9) in $B = B(E, \pi, \mathcal{L})$, which was defined in §4.1. Since

$$\Omega_i = x_{u_i^{s-1},i} \cdots x_{u_i,i}$$

is a product of non-zero elements in $A_1 = B_1$, and since, in this case, B is a domain, the images of $\Omega_i\Omega_j$ and $\Omega_j\Omega_i$ are non-zero. Furthermore the elements of B_s commute pairwise. Hence the images of Ω_i and Ω_j commute in B and therefore $\lambda = 1$.

Hilbert series considerations (using Theorem 1.1(1) which has already been proved) prove the other claims in the proposition. □

Proof of Theorem 1.2. We may extend the ground field, so that we may assume that there is a rational point $z = (z_1, \ldots, z_r)$ satisfying conditions (A),(B) and (C). Then A is finitely generated as a right module over a commutative subring by Proposition 4.3.1. Hence A is right Noetherian and satisfies a polynomial identity. Since by Proposition 4.1.1 A° is also a Sklyanin algebra, A is also left Noetherian. Therefore A is FBN [12, Cor. 13.66(iii)].

(1) and (2) of Theorem 1.1, which have already been proved above, imply that A has finite global dimension. We then conclude, using [25, Cor. 1.2], that A is finite over its center. $\square$

4.4. The general case. To prove (3), (4) and (5) of Theorem 1.1 we will use reduction mod p. To this end we will have to construct Sklyanin algebras over a commutative base ring, say S. For our purposes it suffices to take S a domain, which is what we will do.

We define a family of triples $\mathcal{T}_S = (E_S, \sigma_S, \mathcal{L}_S)$, parametrized by $\operatorname{Spec} S$ to be a smooth family E_S of elliptic curves over S, together with an S-automorphism σ_S of E_S, which is a translation in each geometric fiber, and an invertible sheaf $\mathcal{L}_S$ on E_S which is of degree r in each geometric fiber.

Put also $V_S = H^0(E, \mathcal{L}_S)$, $\mathcal{L}'_S = \sigma_S^{*-1}\mathcal{L}_S$, $V'_S = H^0(E, \mathcal{L}'_S)$, $\pi_S = \sigma_S^{r-2}$ and $\theta_S = \sigma_S^{-2}$.

Lemma 4.4.1. (1) V_S (and hence V'_S) is a projective S-module of rank r, compatible with base change.

(2) There exists a unique invertible S-module M_S and a map ϕ_S, unique up to a unit of S, which defines an isomorphism of $\mathcal{O}_{E_S} \times \mathcal{O}_{E_S}$-modules

$$\phi_S : [1, \theta_S^{-1}]^*(\mathcal{L}'_S \boxtimes \mathcal{L}'_S \otimes \mathcal{O}(-\Delta)) \otimes M_S \to \mathcal{L}_S \boxtimes \mathcal{L}_S \otimes \mathcal{O}(-\Gamma_{\pi_S})$$

Proof. (1) follows from [7, III 7.8] or [8, p. 288 cor 12.9] because $H^1(E_k, \mathcal{L}_k) = 0$ for every geometric point $S \to k$ of $\operatorname{Spec} S$.

To prove (2) we note that $[1, \theta_S^{-1}]^*(\mathcal{L}'_S \boxtimes \mathcal{L}'_S \otimes \mathcal{O}(-\Delta))$ and $\mathcal{L}_S \boxtimes \mathcal{L}_S \otimes \mathcal{O}(-\Gamma_{\pi_S})$ are invertible sheaves on $E_S \times E_S$ which are linearly equivalent in each geometric fiber. This was shown in §4.1.

By the representability of the Picard functor [6] this implies the existence and uniqueness of ϕ_S and M_S. $\square$

We consider $\bigwedge^2 V'_S$ as sitting in $H^0(E \times E, \mathcal{L}'_S \boxtimes \mathcal{L}'_S \otimes \mathcal{O}(-\Delta))$ and use this to define a projective S-module R_S as

$$R_S = \phi_S([1, \theta_S^{-1}]^*(\bigwedge^2 V'_S) \otimes_S M_S)$$

Let TV_S be the tensor algebra of V_S over S. Then we define the Sklyanin algebra $A_S = A(E_S, \sigma_S, \mathcal{L}_S)$ associated to the triple $\mathcal{T}_S$ as $TV_S/(R_S)$.

By construction, if $S \to S'$ is a ring homomorphism then the algebra $S' \otimes_S A_S$ is isomorphic to $A_{S'}$ which is associated to the triple $\mathcal{T}_{S'} = S' \otimes_S \mathcal{T}_S$.

In particular this is true for each geometric point $S \to k$ of $\operatorname{Spec} S$. Since by Theorem 1.1(1) all A_k have the same Hilbert series, this implies that A_S is flat over S.

Proof of (3), (4) and (5) of Theorem 1.1. Let $\mathcal{T}_F = (E, \sigma, \mathcal{L})$ be the triple defined over the base field F we have been using in this paper. So $A = A_F$. Then there is a subring S of F finitely generated over $\mathbb{Z}$ and a family of triples $\mathcal{T}_S = (E_S, \sigma_S, \mathcal{L}_S)$ such that $F \otimes_S \mathcal{T}_S = \mathcal{T}_F$. Choose a maximal ideal m in S. Then S/m is finite and in particular $\sigma_{S/m} = S/m \otimes_S \sigma_S$ has finite order. Choose a valuation ring $(\overline{S}, \overline{m})$ in F dominating (S, m). Then $\sigma_{\overline{S}/\overline{m}}$ still has finite order, and hence by Theorem 1.2 $A_{\overline{S}/\overline{m}}$ is Noetherian. Therefore by [2, lemma 8.9] A itself is Noetherian. This finishes the proof of Theorem 1.1(3).

Now we prove 1.1(4) and 1.1(5). For this we heavily use the results of Stafford and Zhang in [25]. We will show that A has the property stated below, which is equivalent with 1.1(4) and 1.1(5) together, as we will show in lemma 4.4.2 below.

Claim Let M be a finitely generated graded left or right A-module. Then $j(M) \geq r - \operatorname{gk}\dim M$ and $\operatorname{gk}\dim \operatorname{Ext}^i_A(M, A) \leq r - i$, where the Ext's are taken on the appropriate sides.

Here $j(M)$ is the smallest j such that $\operatorname{Ext}^j(M, A) \neq 0$.

By the fact that A is Noetherian, M has a finite presentation and this implies that both M and A are defined over a subfield of F, finitely generated over the prime field. Without loss of generality we may, and we will, replace F by this subfield.

Now inside F we may find a finitely generated subring S such that M and A are defined over S. I.e. there is a Sklyanin algebra A_S over S and a finitely generated A_S-module M_S such that $A = F \otimes_S A_S$, $M = F \otimes_S M_S$. Choose a maximal ideal m in S. We will now replace the pair (S, m) by a *discrete* valuation ring in F dominating (S, m). This is possible since F is now finitely generated [8, II Ex. 4.11].

Having done this replacement we may assume that M_S is flat over S, by dividing out S-torsion.

Now by combining 1.1(1), 1.1(2), 1.2 together with [25, Cor. 3.13, lemma 6.1(ii), lemma 4.4] it follows that $A_{S/m}$ is Cohen-Macaulay and Auslander-Gorenstein, and hence by lemma 4.4.2 it satisfies the Claim. We will now lift this to A. Let π be the uniformizing element of S. I.e. $A_{S/m} = A_S/\pi A_S$.

From the exact sequences

$$0 \to M_S \xrightarrow{\pi} M_S \to M_S/\pi M_S \to 0$$

and the canonical isomorphism

$$\operatorname{Ext}^i_{A_S}(M_S/\pi M_S, A_S) \cong \operatorname{Ext}^{i-1}_{A_S/\pi A_S}(M_S/\pi M_S, A_S/\pi A_S)$$

which exists because of the fact that π is a non-zero divisor in A_S [18, Thm. 9.37], we obtain exact sequences

$$\operatorname{Ext}^i_{A_S}(M_S, A_S) \xrightarrow{\pi} \operatorname{Ext}^i_{A_S}(M_S, A_S) \to \operatorname{Ext}^i_{A_S/\pi A_S}(M_S/\pi M_S, A_S/\pi A_S)$$

By [2, lemma 8.9] A_S is Noetherian and hence $\operatorname{Ext}^i_{A_S}(M_S, A_S)$ is finitely generated as an A_S module. Therefore $\operatorname{Ext}^i_{A_S}(M_S, A_S)_n$ is finitely generated as S-module for every n.

By Nakayama's lemma this implies that

$$\begin{aligned}\dim_F \operatorname{Ext}^i_A(M,A)_n &= \dim_F F\otimes_S \operatorname{Ext}^i_{A_S}(M_S,A_S)_n \\ &\le \dim_{S/m} \operatorname{Ext}^i_{A_S/\pi A_S}(M_S/\pi M_S, A_S/\pi A_S)_n\end{aligned}$$

I.e. $\operatorname{gk}\dim \operatorname{Ext}^i_A(M,A) \le \operatorname{gk}\dim \operatorname{Ext}^i_{A_S/\pi A_S}(M_S/\pi M_S, A_S/\pi A_S)$, which shows what we want. $\square$

Lemma 4.4.2. Assume that $A = F\oplus A_1\oplus\cdots$ is a Noetherian graded F-algebra of finite global dimension and of gk-dimension r. Then A is Auslander-Gorenstein and Cohen-Macaulay if and only if A has the property stated in the Claim contained in the above proof.

Proof. Note that by [25, lemma 4.4] we only have to be concerned with the graded versions of the Auslander-Gorenstein and Cohen-Macaulay property.

Assume that A has the property stated in the Claim and let M be a finitely generated graded, say left, A-module of $\operatorname{gk}\dim u$.

By Hilbert series considerations we find that $\operatorname{gk}\dim \operatorname{Ext}^{r-u}_A(M,A) = u$ (using the fact that by hypothesis the other Ext's have smaller gk-dimension). Hence $j(M) = r-u = r - \operatorname{gk}\dim M$ which proves that A is Cohen-Macaulay.

Now let $N\subset \operatorname{Ext}^i(M,A)$ be a graded submodule. Then $\operatorname{gk}\dim N \le r-i$. Hence if $j<i$ then $j < r-\operatorname{gk}\dim N$ and therefore $\operatorname{Ext}^j(N,A)=0$. Hence A satisfies the Auslander condition.

To prove the opposite implication, assume that A is Cohen-Macaulay and Auslander-Gorenstein and let M be as above. The fact that $j(M)\ge r-u$ follows trivially from Cohen-Macaulayness. To prove the other part of the Claim, assume that there exists an i such that $\operatorname{gk}\dim \operatorname{Ext}^i_A(M,A) = r-j > r-i$. Then again by Cohen-Macaulayness $\operatorname{Ext}^j_A(\operatorname{Ext}^i_A(M,A))\neq 0$, but this contradicts the Auslander condition. $\square$

5. Linear modules

Let $(E,\sigma,\mathcal{L})$ be as before and put $A = A(E,\sigma,\mathcal{L})$. We will usually tacitly assume that $\mathcal{L} = \mathcal{O}(d)$, where d is a divisor on E, so as to be able to evaluate elements of $V = H^0(E,\mathcal{L})$ at rational points of E. Put $0\le l\le r = \dim V$ and $I = \{1,\dots,l\}$.

Denote by $\operatorname{Div}_l(E)$ the set of effective divisors on E of degree l. A *linear A-module of dimension l* will be a graded A module, generated in degree 0 and having the Hilbert series of a polynomial ring in l variables. In particular if M is of I-type then M is linear,

In this section we will show that one can naturally associate such linear modules to elements of $\operatorname{Div}_l(E)$. Let $D\in \operatorname{Div}_l(E)$ and assume that $\mathcal{O}(D)\not\cong\mathcal{L}$. Then $W_D = H^0(E,\mathcal{L}(-D))\subset H^0(E,\mathcal{L}) = A_1$ has dimension $r-l$ and we define $M(D) = A/AW_D$.

Assume that $D = (z_1)+\cdots+(z_l)$ where $z = (z_1,\dots,z_l)$ is a rational point of E^l, satisfying (A), (B), (C). Then in $M(z)$ (as defined in Theorem 4.2.3) we clearly have $W_D v(1) = 0$. I.e. there is a canonical surjective map $M(D)\to M(z)$.

Proposition 5.1. If D, z are as above then $M(D)\to M(z)$ is an isomorphism.

Proof. It suffices to prove that $M(D)$ has the correct Hilbert series. Without loss of generality we may extend the base field and hence we may assume that there exist rational points $z_{l+1}, \dots, z_r$ such that $z' = (z_1, \dots, z_l, z_{l+1}, \dots, z_r)$ still satisfies (A), (B), (C).

By Theorem 3.4 $A \cong M(z')$ and hence $A \cong \bigoplus_{\mu \in \mathcal{U}_I} Fv(\mu)$, where A acts on the basis $(v(\mu))_{\mu \in \mathcal{U}_I}$ as in (4.8). Then it is easy to see that the $v(u_j)$ for $l+1 \leq j \leq r$ form a basis for W_D. Therefore $M(D) = A/AW_D$ has a basis consisting of the images of those $v(\mu)$ for which μ is not divisible by u_j for $l+1 \leq j \leq r$. I.e. $M(D)$ has the right Hilbert series. $\square$

If $D = (z_1) + \cdots + (z_l)$ where $z = (z_1, \dots, z_l)$ is a rational point of E^l then we will sometimes write $M(z)$ for $M(D)$. In view of Proposition 5.1, no confusion can result from this.

We also write W_z for W_D.

Lemma 5.2. Let $z = (z_1, \dots, z_l)$ be a rational point of E^l such that $(z_1)+\cdots+(z_l)$ is not linearly equivalent with d. Then for $n \in \mathbb{N}$, $\dim M(z)_n \geq \binom{l+n-1}{n}$ with equality if $n \leq 2$.

Proof. If z is generic then by Proposition 5.1 $\dim M(z)_n = \binom{l+n-1}{n}$. Hence by semicontinuity we have $\dim M(z)_n \geq \binom{l+n-1}{n}$. This proves the first part of the lemma.

Equality for $n = 0, 1$ is obvious. Hence we have only to show that $\dim M(z)_2 \leq \binom{l+1}{2} = \frac{l(l+1)}{2}$.

By extending the base field we may assume that there exists a rational point $y = (y_1, \dots, y_r) \in E^r$ satisfying (A), (B), (C), with the additional requirement that the divisors $(z_1) + \cdots + (z_l) + (\theta^{-1}y_{l+1}) + \cdots + (\theta^{-1}y_r)$ and $(z_1) + \cdots + (z_l) + (\theta^{-1}y_{l+1}) + \cdots + (\pi^{-1}y_k) + \cdots + (\theta^{-1}y_r)$ for $l+1 \leq k \leq r$, are not linearly equivalent with d.

Let $A \cong \oplus_\mu Fv(\mu)$ be the I-structure on A associated to y.

We choose a basis $(x_k)_{l+1 \leq k \leq r}$ for W_z as follows : $x_k \in W_z$ has the property $x_k(\theta^{-1}y_j) = \delta_{jk}$ for $l+1 \leq j \leq r$. This is possible by our choice of y.

We will show that $(x_k v(u_i))_{k \geq i}$ are linearly independent elements of A_2. Since there are $r(r+1)/2 - l(l+1)/2$ such elements, this shows that $\dim W_z A_1 \geq r(r+1)/2 - l(l+1)/2$. Now by working in the opposite algebra, and using Proposition 4.1.1(2) the same property holds for $A_1 W_z$. Hence $\dim M(z)_2 = \dim(A/AW_z)_2 \leq l(l+1)/2$.

A basis for A_2 is given by $v(u_i u_j) = v(u_j u_i)$, $1 \leq i \leq j \leq r$. We totally order this basis lexicographycally as follows : for $i \leq j$, $i' \leq j'$, $v(u_i u_j) < v(u_{i'} u_{j'})$ if either $j < j'$ or $j = j'$ and $i < i'$. We apply (4.8) in our case. If $k \neq i$ then

$$x_k v(u_i) = \sum_{\substack{j=1 \\ j \neq i}}^{l} \frac{x_k(\theta^{-1}y_j)}{c_j(u_i)} v(u_i u_j) + \frac{x_k(\pi^{-1}y_i)}{c_i(u_i)} v(u_i^2) + \frac{1}{c_k(u_i)} v(u_i u_k)$$

and

$$x_k v(u_k) = \sum_{j=1}^{l} \frac{x_k(\theta^{-1}y_j)}{c_j(u_k)} v(u_j u_k) + \frac{x_k(\pi^{-1}y_k)}{c_k(u_k)} v(u_k^2)$$

By our choice of y the coefficient of $v(u_k^2)$ in the last equation is non-zero. Therefore these equations show that for $l < k \le r$ and $1 \le i \le k$,

$$x_k v(u_i) = c_{i,k} v(u_i u_k) + (\text{linear combination of } v(u_{i'} u_{j'})\text{'s which are} < v(u_i u_k))$$

with $c_{i,k} \neq 0$. Hence those elements are linearly independent, as claimed. $\square$

Proof of Theorem 1.4. We may extend the base field, so that we may assume that $D = (z_1) + \cdots + (z_l)$ where $z = (z_1, \ldots, z_l)$ is a rational point of E^l. Put $z' = (z_1, \ldots, z_{l-1})$, $D' = (z_1) + \cdots + (z_{l-1})$.

Then the divisor $(\pi^{-1} z_l) + \theta^{-1} D'$ is the one associated to the point $u_l z$ for the action of $\mathcal{U}_I$ on E^l defined in §4.2.

We first show that at least there are exact sequences of the form

$$M(u_l z)(-1) \to M(z) \to M(z') \to 0 \tag{5.1}$$

This is equivalent with the claim that $W_{u_l z} W_{z'} \subset A_1 W_z$, or, put more fancily using lemma 5.2, that the image of the natural map

$$W_{u_l z} \otimes W_{z'} \oplus A_1 \otimes W_z \to A_2 \tag{5.2}$$

should have dimension $r(r+1)/2 - l(l+1)/2$.

Let $U \subset E^l$ be the open set consisting of the points $z = (z_1, \ldots, z_l)$ such that $(z_1) + \cdots + (z_l)$ is not linearly equivalent with d and let $\mathcal{W}$ resp. $\mathcal{W}'$ be the subvector bundles of $A_1 \otimes \mathcal{O}_U$ which associate to $z \in U$, W_z resp. $W_{z'}$.

Then (5.2) is obtained by specializing the natural map

$$\phi : u_l^* \mathcal{W} \otimes_{\mathcal{O}_U} \mathcal{W}' \oplus A_1 \otimes_F \mathcal{W} \to A_2 \otimes_F \mathcal{O}_U$$

at z.

Now if z is generic then, by remark 4.2.4, sequences of the form (5.1) exist and hence $\dim \operatorname{im} \phi_z \le r(r+1)/2 - l(l+1)/2$. Then by semi-continuity the same holds for arbitrary z.

We now prove that $M(z)$ is linear of dimension l by ascending induction on l, the case $l = 0$, $M(z) = k$, being clear. Suppose $0 < l \le r$ and that we have shown that $M(z')$ is linear of dimension $l-1$ for all rational points $z' = (z_1, \ldots, z_{l-1})$ in E^{l-1}.

Fix n and suppose that we have shown that $\dim M(z)_j = \binom{l+j-1}{j}$ for all rational points $z = (z_1, \ldots, z_l)$ and for all $j \le n-1$. By lemma 5.2 this holds for $n \le 3$.

By lemma 5.2

$$\dim M(z)_n \ge \binom{l+n-1}{n}$$

On the other hand by (5.1)

$$\begin{aligned} \dim M(z)_n &\le \dim M(u_l z)_{n-1} + \dim M(z')_n \\ &= \binom{l+n-2}{n-1} + \binom{l+n-2}{n} \qquad \text{(induction)} \\ &= \binom{l+n-1}{n} \end{aligned}$$

This shows that $M(z)$ is linear.

Furthermore by checking Hilbert series one sees that (5.1) is exact on the left which implies the existence of (1.1). □

6. Some examples

The attentive reader will have noted that we have not really used the actual form of (θ, π) as given in §4.1. Indeed, the only thing we have used is the fact that θ and π commute and the existence of ϕ in (4.1). In particular, nothing prevents us from considering other types of automorphisms (θ, π).

The case where (θ, π) are translations turns out to be a natural generalization to higher dimension of the so-called type A algebras, as defined in [2, §4.13]. We will show below that for odd r, there are no other algebras of this type than the ones we have discussed. For even r there are others, but, possibly after a ground field extension, they are twists of the ones we have treated.

We also show that the algebras of type B, H and E in which π is, respectively, a complex multiplication by a primitive 2^{nd}, 4^{th} and 3^{d} root of unity, have generalizations to higher dimensions, if suitable points of order 4, 2 and 3 are rational so that the equations (6.5)(6.6) have solutions β, γ, with $\beta \neq 0$.

Curiously, it turns out that the algebras of types B and H are twists of type A's. The same holds for type E if $(r, 3) = 1$. However if $3 \mid r$ then a type E algebra is very likely not a twist of a type A.

Let E be a smooth elliptic curve and let $\mathcal{L}$ be a line bundle on E of degree $r \geq 3$. Let (θ, π) be commuting automorphisms of E such that $\mathcal{L}'$ and ϕ exist in (4.1). Put $V = H^0(E, \mathcal{L})$. Then we define $A(E, \pi, \theta, \mathcal{L}) = TV/(R)$ where R is given by (4.2). There is no ambiguity in this notation since, if $\mathcal{L}'$ exists, it must be unique.

Proposition 4.1.1 remains valid. For completeness we restate it in the current notation.

Proposition 6.1. (1) Assume that $\mu : E' \to E$ is an isomorphism between elliptic curves E, E'. Then there is a canonical isomorphism $A(E, \pi, \theta, \mathcal{L}) \to A(E', \mu^{-1}\pi\mu, \mu^{-1}\theta\mu, \mu^*\mathcal{L})$ sending $x \in V = H^0(E, \mathcal{L})$ to $\mu^* x \in H^0(E', \mu^*\mathcal{L})$.
(2) $A(E, \pi, \theta, \mathcal{L})^\circ \cong A(E, \pi^{-1}, \theta^{-1}, \mathcal{L})$

Proposition 6.1 implies in particular that if $\pi\mu = \mu\pi$, $\theta\mu = \mu\theta$, and if there is an isomorphism $\psi : \mu^*\mathcal{L} \to \mathcal{L}$ then $x \mapsto \psi(\mu^* x)$ defines an automorphism of $A(E, \pi, \theta, \mathcal{L})$. With a slight abuse of notations we denote this algebra automorphism also by μ.

It is somewhat subtle whether every automorphism of $A(E, \pi, \theta, \mathcal{L})$ is of this type since this depends on the fact whether the "point variety" of $A(E, \pi, \theta, \mathcal{L})$ is E or something bigger (e.g. take for π and θ the identity). See [3] for the three dimensional case. For the purposes of this section we will call an automorphism of the type μ *geometric*. That is, it is associated to the geometric data $(E, \pi, \theta, \mathcal{L})$. It follows from the description of the point variety in [20][21][26] that, for an algebra of type $A(E, \sigma, \mathcal{L})$, every automorphism is geometric if $\sigma^r \neq 1$.

Proposition 6.2. Let μ be as above. Then $A(E, \pi, \theta, \mathcal{L})_\mu \cong A(E, \pi\mu, \theta\mu, \mathcal{L})$ where $(\ )_\mu$ denotes twisting by the automorphism μ (as defined in [3, §8]).

Proof. If we identify $\mu^*\mathcal{L}$ with $\mathcal{L}$ then the relations of $A(E,\pi,\theta,\mathcal{L})_\mu$ are given by $[1,\mu^{-1}]^*(R)$. We construct a map by composing :

$$\begin{aligned}\phi_\mu : [1,\mu^{-1}\theta^{-1}]^*(\mathcal{L}'\boxtimes\mathcal{L}'\otimes\mathcal{O}(-\Delta)) \xrightarrow{[1,\mu^{-1}]^*\phi} [1,\mu^{-1}]^*&(\mathcal{L}\boxtimes\mathcal{L}\otimes\mathcal{O}(-\Gamma_\pi))\\ &\cong \mathcal{L}\boxtimes\mathcal{L}\otimes\mathcal{O}(-\Gamma_{\pi\mu})\end{aligned}$$

Then $\phi_\mu([1,\mu^{-1}\theta^{-1}]^*(\bigwedge^2 V))$ is precisely $[1,\mu^{-1}]^*(R)$, as it should. □

Explicit conditions on π, θ and $\mathcal{L}$ for the algebra $A(E,\pi,\theta,\mathcal{L})$ to be defined are given by the following lemma.

Lemma 6.3. Let E be a smooth curve of genus 1. Let $\mathcal{L}$ be a line bundle on E of degree $r\geq 3$ and let λ be its class in $\operatorname{Pic} E$. Let θ and π be automorphisms of E. Then θ and π commute and $\mathcal{L}'$ and ϕ as in (4.1) exist iff there exists a rational point β on the Jacobian $\operatorname{Pic}^0 E$ such that, denoting translation by β by τ_β, the following conditions hold :

$$\theta = \tau_\beta\pi \tag{6.1}$$

$$\tau_\beta\pi = \pi\tau_\beta \tag{6.2}$$

$$2\beta = \theta\lambda - \lambda. \tag{6.3}$$

When that is the case, (4.1) holds if the class of $\mathcal{L}'$ is $\lambda-\beta$.

Proof. Straightforward exercise using the see-saw principle. □

Now if in (6.1)(6.2)(6.3) $\beta = 0$, i.e. $\theta=\pi$, then we can take $\mathcal{L}'=\mathcal{L}$ and $\theta^*\mathcal{L}$ is isomorphic to $\mathcal{L}$. Hence there is an automorphism θ_V of V, which induces θ on E. This θ_V extends to an automorphism of the commutative polynomial algebra SV, and it is easy to see that the algebra $A(E,\theta,\theta,\mathcal{L})$ is the twist of SV by θ_V.

If $A(E,\pi,\theta,\mathcal{L})$ is defined then either both π and θ are translations, or neither is. Let us call $A(E,\pi,\theta,\mathcal{L})$ a type A algebra when they are translations.

Proposition 6.4. A type A algebra is given by $\pi=\tau_{\gamma-\beta}$, $\theta=\tau_\gamma$ where β and γ are rational points on $\operatorname{Pic}^0 E$ such that $2\beta = r\gamma$. The line bundle $\mathcal{L}$ is arbitrary.

Proof. (6.2) is automatic and if $\theta=\tau_\gamma$ then $\theta\lambda-\lambda = r\gamma$. □

Corollary 6.5. The algebras $A(E,\sigma,\mathcal{L})$ discussed in this paper are the only odd dimensional algebras of type A.

Proof. If $r=2m+1$ take $\sigma=\tau_{m\gamma-\beta}$. □

If $r=2m$, things are more complicated. Then $\alpha=\beta-m\gamma$ is a point of order 2. If $\alpha=0$ then A can be said to be of type $A(E,\sigma,\mathcal{L})$ with $\sigma^2=\tau_{-\gamma}$. This makes sense, because for r even, $A(E,\sigma,\mathcal{L})$ depends only on σ^2. If $\alpha\neq 0$ then we can reduce to the case $\alpha=0$ by twisting in the sense of Proposition 6.2 with $\mu=\tau_\omega$ where ω is any solution of $m\omega=\alpha$. This can be done rationally by taking $\omega=\alpha$ if m is odd, i.e. if r is congruent to 2 mod 4, but may require a ground field extension if $4\mid m$.

To simplify the discussion of the case in which π is not a translation, we assume that π has a rational fixed point o, which we take as origin to identify E with $\operatorname{Pic}^0 E$. Then π is a complex multiplication by a primitive m'th root of unity ζ,

$m = 2, 3, 4, 6$. Suppose that the class of $\mathcal{L}$ is $r[o] + \gamma$, where $[o]$ is the class of the divisor (o).

Then the conditions (6.1)(6.2)(6.3) become

$$\theta(z) = \zeta z + \beta \tag{6.4}$$

$$(1 - \zeta)\beta = 0 \tag{6.5}$$

$$(r - 2)\beta = (1 - \zeta)\gamma \tag{6.6}$$

If the order of ζ is 6 then $1 - \zeta$ is a unit and hence, by (6.5), $\beta = 0$. Therefore, by the discussion above, $A(E, \pi, \theta, \mathcal{L})$ is a twist of a symmetric algebra.

Suppose we have a solution r, ζ, β, γ to (6.5) and (6.6). Is the resulting algebra

$$A(E, z \mapsto \zeta z, z \mapsto \zeta z + \beta, \mathcal{O}_E((r - 1)(o) + (\gamma)))$$

a twist of a type A algebra by a geometric automorphism ? By Proposition 6.2 we should look for automorphisms μ of the form $z \mapsto \zeta^{-1} z + \eta$ commuting with π and θ and fixing the class $r[o] + \gamma$.

These conditions translate into

$$(1 - \zeta)\eta = 0 \tag{6.7}$$

$$(1 - \zeta)\gamma = -r\zeta\eta \tag{6.8}$$

If the order of ζ is not 3 then $(1 - \zeta)$ divides 2, so by (6.5), $2\beta = 0$ and we take $\eta = -\zeta^{-1}\beta$. Hence suppose ζ is of order 3.

If $3 \nmid r$ then $r \equiv \pm 1 \mod 1 - \zeta$ and hence by (6.8) $\eta = \pm\zeta^{-1}(1 - \zeta)\gamma$. (6.5)(6.6) imply that (6.7) is indeed satisfied.

So assume $3 \mid r$. Then $r\eta = 0$ and hence (6.7)(6.8) have no solution iff $(1-\zeta)\gamma \neq 0$. Assuming this we find by (6.6) that $\beta = (1-\zeta)\gamma$, which satisfies (6.5) iff $3\gamma = 0$.

Summarizing, we find that if $3 \mid r$ then we obtain a generalization of type E by taking $j(E) = 0$, $\pi(z) = \zeta z$, $\theta(z) = \zeta z + \beta$, $\mathcal{L} = \mathcal{O}((r - 1)(o) + (\gamma))$, $\beta = (1 - \zeta)\gamma$ and γ a point of order 3 of E, not fixed by π.

Remark 6.6. Given a triple $(E, \pi, \mathcal{L})$, $\deg \mathcal{L} = 4$, π a translation, T. Stafford shows in [24] that it is possible to associate a 1-parameter family of "regular" algebras to the geometric data $(E, \pi, \mathcal{L})$. I.e. quadratic algebras having the Hilbert series of a polynomial ring in 4 variables, mapping surjectively to $B(E, \pi, \mathcal{L})$.

These algebras cannot be constructed by the Odeskii-Feigin method nor with the slight generalization we have given above.

Nevertheless T. Stafford shows with essentially the same method as in [21] (which in turn is a generalization of [2]) that Theorem 1.1 is valid for these algebra. It would therefore be interesting to know whether they are of I-type, and whether an analogue of Theorem 1.4 is true for them.

Appendix A. An additional result on Sklyanin algebras

Let $(E, \sigma, \mathcal{L})$ be as before and put $A = A(E, \sigma, \mathcal{L})$. We assume again $\mathcal{L} = \mathcal{O}(d)$ to be able to evaluate sections of $\mathcal{L}$. As before $I = \{1, \dots, l\}$ where $0 \leq l \leq r$.

In this section we construct F-algebra homomorphisms ϕ_l from A to certain iterated Ore extensions of $F(E^l)$ denoted by Z_l. We prove in particular that ϕ_r is injective and that in this way Z_r becomes a left and right faithfully flat extension of A. This result might be used to give a descent theoretic proof of Theorem 1.1, i.e. a proof that does not use reduction to finite characteristic.

We use freely the notations introduced in §4.1 and §4.2.

First a general result. Let K be a field, $(\sigma_i)_{i\in I}$ commuting automorphisms of K and $(\lambda_{ij})_{i,j\in I}$ elements of K with the property $\lambda_{ij}\lambda_{ji}=1$ and $\lambda_{ii}=1$.

Define

$$Z_l = K[t_1, \dots, t_l]$$

with relations

$$\begin{aligned} t_j f &= \sigma_j(f) t_j \\ t_j t_i &= t_i t_j \lambda_{ij} \end{aligned} \tag{A.1}$$

where $f \in K$, $i, j \in I$, $i \neq j$.

Proposition A.1. Suppose that $\forall (i,j,k) \in I^3$, $i \neq j \neq k \neq i$,

$$\frac{\sigma_i^{-1}(\lambda_{jk})}{\lambda_{jk}} \cdot \frac{\sigma_j^{-1}(\lambda_{ki})}{\lambda_{ki}} \cdot \frac{\sigma_k^{-1}(\lambda_{ij})}{\lambda_{ij}} = 1. \tag{A.2}$$

Then (A.2) holds for all $(i,j,k) \in I^3$ and the automorphisms σ_k can be extended to automorphisms of Z_l in such a way that $\sigma_k(t_i) = t_i\sigma(\lambda_{ik})$. Moreover, Z_l is an iterated Öre extension of K; more precisely, we have $Z_j = Z_{j-1}[t_j\,;\,\sigma_j|Z_{j-1}]$, for $j = 1, 2, \cdots, l$. Consequently, Z_l has the Hilbert series of a polynomial ring in l variables and the ordered monomials $t_1^{a_1} t_2^{a_2} \cdots t_l^{a_l}$ are left and right linearly independent over K.

Proof. To check that the indicated extension of σ_k is well defined one uses (A.2) (together with the fact that the σ's commute and the commutativity of the ground field K) to show that it preserves the relations defining Z_l, given by (A.1). Then one observes that for each j, those relations, for $i < j$ define the indicated Öre extension. □

Now fix l. We define $\overline{F} = F(E^l)$, $\overline{A} = \overline{F} \otimes_F A$, $(\overline{E}, \overline{\sigma}, \overline{\mathcal{L}}) = \overline{F} \otimes_F (E, \sigma, \mathcal{L})$, $\overline{V} = \overline{F} \otimes_F V$.

Let $w = (w_1, \dots, w_l) \in \overline{E}^l$ be the rational point corresponding to the generic point of E^l. We will identify rational functions on E^l with their value at w (after extension of the base field to $\overline{F}$).

As before we use the convention that if σ acts on a space X then σ acts on a rational function f on X by $f \circ \sigma^{-1}$.

We keep the definition of Z_l above, but now we specialize to $K = \overline{F}$, $\sigma_i = u_i$ and $\lambda_{ij} = g(w_j, w_i)$ where u_i and g are as defined in §4.2. Note that (A.2) is true because of (4.7).

Theorem A.2. There exists a (uniquely determined) F-algebra homomorphism $\phi_l : A \to Z_l$ with the property that for $x \in V$

$$\phi_l(x) = \sum_{i=1}^{l} t_i x(w_i) \tag{A.3}$$

Proof. (A.3) obviously extends to a map $\phi_l : TV \to Z_l$. We have to show that $\phi_l(R) = 0$.

First let $f \in V \otimes V$ be arbitrary. Then an easy computation shows

$$\phi_l(f) = \sum_i t_i^2 f(\pi^{-1} w_i, w_i) + \sum_{i \neq j} t_i t_j f(\theta^{-1} w_i, w_j)$$

(it is sufficient to verify this for $f = x \otimes y$).

Now let $f \in R$. Then $f(\pi^{-1} w_i, w_i) = 0$ since f vanishes on Γ_π.

Furthermore f is a linear combination of functions of the form $h^{-1}[1, \theta](x \otimes y - y \otimes x)$ with $x, y \in V'$ and h, V' as defined in §4. Then

$$f(\theta^{-1} w_i, w_j) = h(\theta^{-1} w_i, w_j)^{-1}(x(\theta^{-1} w_i)y(\theta^{-1} w_j) - y(\theta^{-1} w_i)x(\theta^{-1} w_j))$$

and hence

$$\sum_{i \neq j} t_i t_j f(\theta^{-1} w_i, w_j) = \sum_{j > i} t_i t_j (h(\theta^{-1} w_i, w_j)^{-1} - g(w_j, w_i) h(\theta^{-1} w_j, w_i)^{-1}) \times$$
$$(x(\theta^{-1} w_i)y(\theta^{-1} w_j) - y(\theta^{-1} w_i)x(\theta^{-1} w_j)) = 0 \quad \square$$

We now use ϕ_l to define a left $\overline{A}$-module structure on Z_l by

$$(a \otimes f) \cdot \zeta = \phi_l(a)\zeta f \tag{A.4}$$

where $a \in A$, $f \in \overline{F}$, $\zeta \in Z_l$.

Taking $\zeta = 1$ one obtains a map of left $\overline{A}$-modules

$$\overline{\phi}_l : \overline{A} \to Z_l : a \otimes f \mapsto \phi_l(a) f$$

Theorem A.3. For the left $\overline{A}$-module structure defined by (A.4), Z_l becomes an $\overline{A}$-module of I-type, with I-structure given by the monomials in $t_1, \ldots, t_l$.

Proof. We have to show that

$$\overline{V} t_1^{a_1} \cdots t_l^{a_l} = \bigoplus_{i=1}^{l} \overline{F} t_1^{a_1} \cdots t_i^{a_i+1} \cdots t_l^{a_l}$$

which is equivalent with

$$\phi_l(V) t_1^{a_1} \cdots t_l^{a_l} \overline{F} = \bigoplus_{i=1}^{l} \overline{F} t_1^{a_1} \cdots t_i^{a_i+1} \cdots t_l^{a_l}$$

which then reduces to

$$\phi_l(V)\overline{F} = \bigoplus_{i=1}^{l} \overline{F} t_i \tag{A.5}$$

Since $l \leq r$, and since even if $l = r$ the divisor $(w_1) + \cdots + (w_l)$ is not linearly equivalent to $\overline{d}$, it follows from the Riemann-Roch theorem on $\overline{E}$ that there exist functions in $\overline{V} = H^0(\overline{E}, \mathcal{O}_{\overline{E}}(\overline{d}))$ which vanish at all but one of the points w_j, but not on all of them. Hence there exist functions $\overline{x}_i \in \overline{V}$ such that

$$\overline{x}_i(w_j) = \delta_{ij} \tag{A.6}$$

$\overline{x}_i$ may be written as $\sum_k x_{ik} \otimes f_k$, $x_{ik} \in V$, $f_k \in \overline{F}$ and then (A.6) implies $\sum x_{ik}(w_j) f_k = \delta_{ij}$ Hence $\sum_k \phi_l(x_{ik}) f_k = \sum_{k,j} t_j x_{ik}(w_j) f_k = t_i$ Therefore (A.5) is true. □

Corollary A.4. (1) For $l \leq r$, $\overline{\phi}_l$ is surjective and $\overline{\phi}_r$ is an isomorphism.

(2) ϕ_r is an injection and, via ϕ_r, Z_r becomes a left and right faithfully flat extension of A.

Proof. The fact that $\overline{\phi}_r$ is an isomorphism follows from looking at Hilbert series (or by applying Theorem 3.4 to the quadratic algebra $\overline{A}$).

Since $Z_r \cong A \otimes_F \overline{F}$ as left A-modules, Z_r/A is obviously faithfully flat on the left. Right faithfully flatness may be proved by considering A°. □

Remark A.5. (1) Corollary A.4 yields an alternative construction of the algebra $A = A(E, \sigma, \mathcal{L})$. It is the subalgebra of Z_r, generated by $\phi_r(V)$.

(2) One may regard Z_l as a generic linear A-module of dimension l. For some mysterious reason it also carries a right A-structure.
Since Z_l carries an I-structure one may obtain A-modules of I-type using suitable specializations. This can be used as an alternative method for constructing modules of I-type.

(3) ϕ_l may be injective even when $l < r$. For example if $r = 3$ then ϕ_2 is injective.

(4) When suitably interpreted, Z_r has the properties listed in Theorem 1.1. This may be proved inductively, using the fact that Z_r is an iterated Öre extension of $\overline{F}$ (Proposition A.1 above). It is then very likely possible to descend these properties to A using faithfully flat descent. This would yield another proof of Theorem 1.1 which does not use reduction to finite characteristic.

(5) Morphisms from a Sklyanin algebra to an iterated Öre extension have also been constructed by Odeskii and Feigin in [14] and [17]. These constructions are essentially over $\mathbb{C}$ since, instead of working over $F(E)$, one works over the field of meromorphic functions on $\mathbb{C}^*$ (which is an analytic covering of E). An advantage of this approach is that some formulas may be written more elegantly.

References

1. M. Artin and W. Schelter, *Graded algebras of global dimension 3*, Adv. in Math. **66** (1987), 171–216.
2. M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, vol. 1, Birkhäuser, 1990, pp. 33–85.
3. ———, *Modules over regular algebras of dimension 3*, Invent. Math. **106** (1991), 335–388.
4. M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra **188** (1990), 249–271.
5. M. Auslander, *On the dimension of modules and algebras III*, Nagoya Math. J. **9** (1955), 67–77.
6. A. Grothendieck, Bourbaki Exposé 234.
7. A. Grothendieck and J. Dieudonné, *Elements de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math., 1960-1967.
8. R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
9. T. Levasseur, *Some properties of non-commutative regular rings*, Glasgow Math. J. **34** (1992), 277–300.

10. T. Levasseur and S. P. Smith, *Modules over the 4-dimensional Sklyanin algebra*, Bull. Soc. Math. France **121** (1993), 35–90.
11. W. S. Massey, *Singular homology theory*, Springer-Verlag, 1980.
12. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Wiley-Interscience, Chichester, 1977.
13. D. Mumford, *Abelian varieties*, Oxford University Press, Oxford, 1970.
14. A. V. Odeskii, *Rational degeneration of elliptic quadratic algebras*, Internat. J. of Modern Phys. A **7** (1992), 773–779, Suppl. 1B.
15. A. V. Odeskii and B. L. Feigin, *Elliptic Sklyanin algebras*, Functional Anal. Appl. **23** (1989), no. 3, 207–214.
16. ———, *Sklyanin algebras associated with an elliptic curve*, preprint Institute for Theoretical Physics, Kiev, 1989.
17. ———, *Constructions of Sklyanin elliptic algebras and quantum matrices*, Functional Anal. Appl. **27** (1993), no. 1, 37–45.
18. J. J. Rotman, *An introduction to homological algebra*, Academic Press, Inc, San Diego, 1979.
19. S. P. Smith, *The 4-dimensional Sklyanin algebra at points of finite order*, preprint, 1992.
20. ———, *Point modules over Sklyanin algebras*, Math. Z. **215** (1994), 169–177.
21. S. P. Smith and J. T. Stafford, *Regularity of the 4-dimensional Sklyanin algebra*, Compositio Math. **83** (1992), 259–289.
22. S. P. Smith and J. Tate, *The center of the 3-dimensional and 4-dimensional Sklyanin algebras*, to appear in K-theory, 1992.
23. J. T. Stafford, *Auslander-regular algebras and maximal orders*, to appear in J. London Math. Soc. (2), 1991.
24. ———, *Regularity of algebras related to the Sklyanin algebras*, Trans. Amer. Math. Soc. **341** (1994), no. 2, 895–916.
25. J. T. Stafford and J. J. Zhang, *Homological properties of (graded) Noetherian PI rings*, to appear in J. Algebra, 1992.
26. J. Staniszkis, *Linear modules over Sklyanin algebras*, preprint, 1993.

Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA
E-mail address: tate@math.utexas.edu

Limburgs Universitair Centrum, Departement WNI, Universitaire Campus, 3590 Diepenbeek, Belgium
E-mail address: vdbergh@luc.ac.be