UNPUBLISHED RESULTS BY AIDAN SCHOFIELD[1]

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Let X be a smooth projective curve over a field k of genus g. Let $\mathcal{F} \in \operatorname{coh}(X)$ be such that $\operatorname{Supp} \mathcal{F} = X$. Put

$$\mathcal{F}^{\perp} = \{\mathcal{E} \in \operatorname{Qch}(X) \mid \operatorname{Hom}_X(\mathcal{F}, \mathcal{E}) = \operatorname{Ext}^1(\mathcal{F}, \mathcal{E}) = 0\}$$

It is easy to see that $\mathcal{F}^{\perp}$ is an abelian subcategory of $\operatorname{Qch}(X)$ closed under direct sums. So it is particular a Grothendieck category.

Proposition 0.1. *The inclusion $\mathcal{F}^{\perp} \subset \operatorname{Qch}(X)$ has a left adjoint.*

Proof. Let $\mathcal{U} \in \operatorname{Qch}(X)$. We must construct $\mathcal{U}' \in \mathcal{F}^{\perp}$ such that

$$\operatorname{Hom}_X(\mathcal{U}, \mathcal{E}) = \operatorname{Hom}_X(\mathcal{U}', \mathcal{E}) \tag{0.1}$$

for all $\mathcal{E} \in \mathcal{F}^{\perp}$.

We will first show that there exists a morphism $\mathcal{U} \to \mathcal{U}'$ inducing an isomorphism after applying $\operatorname{Hom}_X(-, \mathcal{E})$, $\mathcal{E} \in \mathcal{F}^{\perp}$ and satisfying either of the following two properties

(1) $\operatorname{Hom}_X(\mathcal{F}, \mathcal{U}') = 0$;
(2) $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{U}') = 0$

This is sufficient since we may then construct a series of maps

$$\mathcal{U} \to \mathcal{U}_0 \to \mathcal{U}_1 \to \mathcal{U}_2 \to$$

such that all maps induce isomorphisms after applying $\operatorname{Hom}_X(-, \mathcal{E})$ and such that $\operatorname{Hom}_X(\mathcal{F}, \mathcal{U}_{2k}) = 0$, $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{U}_{2k+1}) = 0$. We then take $\mathcal{U}' = \varinjlim_n \mathcal{U}_n$.

To satisfy (1) we construct a series of maps

$$\mathcal{U} \to \mathcal{U}_0 \to \mathcal{U}_1 \to \mathcal{U}_2 \to$$

such that

$$\mathcal{U}_{k+1} = \operatorname{coker}(\operatorname{Hom}(\mathcal{F}, \mathcal{U}_k) \otimes_k \mathcal{F} \to \mathcal{U}_k)$$

and take $\mathcal{U}' = \varinjlim_k \mathcal{U}_k$.

To satisfy (2) we construct a series of maps

$$\mathcal{U} \to \mathcal{U}_0 \to \mathcal{U}_1 \to \mathcal{U}_2 \to$$

such that $\mathcal{U}_{k+1}$ is the universal extension

$$0 \to \mathcal{U}_k \to \mathcal{U}_{k+1} \to \mathcal{F}^{\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{U}_k)} \to 0$$

and take $\mathcal{U}' = \varinjlim_k \mathcal{U}_k$. $\square$

Below we denote the left adjoint to $\mathcal{F}^{\perp} \to \operatorname{Qch}(X)$ by L. Let $p \in X$. There exists an epimorphism $\phi : \mathcal{F} \to \mathcal{O}_p$. Put $\mathcal{F}' = \ker \phi$, $\mathcal{P} \overset{\text{def}}{=} L(\mathcal{F}')$.

[1] Notes by Michel Van den Bergh

Proposition 0.2. *The object $\mathcal{P}$ is a small projective generator for the category $\mathcal{F}^\perp$. If $\mathcal{E} \in \mathcal{F}^\perp$ then $\operatorname{Hom}_X(\mathcal{P}, \mathcal{E})$ is finite dimensional if and only if $\mathcal{E}$ is coherent.*

Proof. If $\mathcal{E} \in \mathcal{F}^\perp$ we have $\operatorname{Hom}_X(\mathcal{O}_p, \mathcal{E}) = 0$ and hence $\operatorname{Ext}^1_X(\mathcal{O}_p, -)$ is an exact fuctor on $\mathcal{F}^\perp$. Thus for $\mathcal{E} \in \mathcal{F}^\perp$

$$\operatorname{Ext}^1_X(\mathcal{O}_p, \mathcal{E}) = \operatorname{Hom}_X(\mathcal{F}', \mathcal{E}) = \operatorname{Hom}_X(L(\mathcal{F}'), \mathcal{E}) = \operatorname{Hom}_X(\mathcal{P}, \mathcal{F}) \tag{0.2}$$

Thus $\mathcal{P}$ is a projective object in $\mathcal{F}^\perp$. It is small since (0.2) shows that $\operatorname{Hom}_X(\mathcal{P}, -)$ commutes with direct sums. Futhermore we find $\mathcal{P}^\perp \cap \mathcal{F}^\perp = (\mathcal{F} \oplus \mathcal{O}_p)^\perp$ and since $\mathcal{F} \oplus \mathcal{O}_p$ is a compact generator of $D(\operatorname{Qch}(X))$ we deduce $= (\mathcal{F} \oplus \mathcal{O}_p)^\perp = 0$. This implies that $\mathcal{P}$ is a generator for $\mathcal{F}^\perp$.

It is clear from (0.2) that if $\mathcal{E}$ is coherent then $\operatorname{Hom}_X(\mathcal{P}, \mathcal{E})$ is finite dimensional. We will now prove the converse. Assume $\mathcal{E} \in \mathcal{F}^\perp$ and $s = \dim \operatorname{Ext}^1_X(\mathcal{O}_p, \mathcal{E}) < \infty$. Assume that $\mathcal{E}$ is not coherent. If $\mathcal{E}' \subset \mathcal{E}$ is a coherent subobject then we claim that the maximal submodule of $\mathcal{E}/\mathcal{E}'$ supported in p has finite length. Assume this is not the case. Then we can construct an infinite chain

$$\mathcal{E}' = \mathcal{E}'_0 \subsetneq \mathcal{E}'_1 \subsetneq \mathcal{E}'_2 \subsetneq \cdots \subset \mathcal{E}$$

of coherent submodules such that $\mathcal{E}'_{i+1}/\mathcal{E}'_i$ has support in p. We then have that $\deg \mathcal{E}'_i$ is unbounded whereas its $\operatorname{rk} \mathcal{E}'_i$ is constant. Put $(r, d) = (\operatorname{rk} \mathcal{F}, \deg \mathcal{F})$ and $(r', d'_i) = (\operatorname{rk} \mathcal{E}'_i, \deg \mathcal{E}'_i)$. A straightforward computation using the Rieman-Roch theorem shows

$$\chi(\mathcal{F}, \mathcal{E}'_i) = \dim \operatorname{Hom}_X(\mathcal{F}, \mathcal{E}'_i) - \dim \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{E}'_i) = rr'(1-g) + rd'_i - dr'$$

Since $\operatorname{Hom}_X(\mathcal{F}, \mathcal{E}'_i) = 0$ we have $\chi(\mathcal{F}, \mathcal{E}'_i) \leq 0$. On the other hand since d'_i is unbounded we may assume $\chi(\mathcal{F}, \mathcal{E}'_i) > 0$ which is a contradition.

We now construct a chain of coherent subobjects

$$0 = \mathcal{E}''_0 \subsetneq \mathcal{E}''_1 \subsetneq \mathcal{E}''_2 \subsetneq \cdots \subset \mathcal{E}$$

such that $\operatorname{rk} \mathcal{E}''_{i+1} > \operatorname{rk} \mathcal{E}''_i$ and such that $\mathcal{E}/\mathcal{E}''_i$ has no subobject supported in p. This is possible by the previous discussion. This chain cannot stop since $\mathcal{E}$ is not coherent. Put $r''_i = \operatorname{rk} \mathcal{E}''_i = \dim \operatorname{Ext}^1_X(\mathcal{O}_p, \mathcal{E}''_i)$. Then it is clear that for all i we have $r''_i < r''_{i+1} \leq s$, which is impossible. □