THE KONTSEVICH WEIGHT OF A WHEEL WITH SPOKES POINTING OUTWARD

MICHEL VAN DEN BERGH

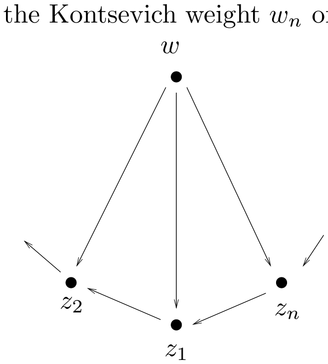
Abstract. This is a companion note to "Hochschild cohomology and Atiyah classes" by Damien Calaque and the author. We compute the Kontsevich weight of a wheel with spokes pointing outward. The result is in terms of modified Bernouilli numbers. Our computation uses Stokes theorem together with some basic properties proved by Kontsevich.

Contents

1. Introduction and statement of the main result

This is a companion note to [2]. We refer to [2, §8] and [4] for unexplained notations and conventions.

Below we will compute the Kontsevich weight w_n of the "opposite" wheel



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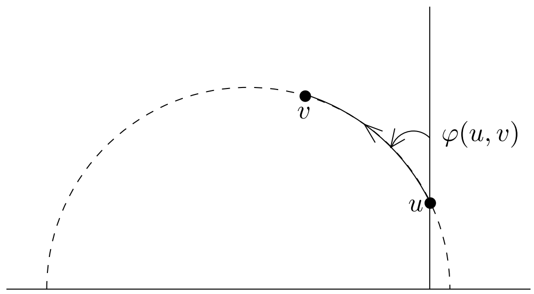
with edge ordering $(z_1, z_2) < \cdots < (z_n, z_1) < (w, z_1) < \cdots < (w, z_n)$.

Thus[1]

$$w_n = \frac{1}{(2\pi)^{2n}} \int_{C_{n+1,0}} d\varphi(z_1, z_2) \cdots d\varphi(z_1, z_n) d\varphi(w, z_1) \cdots d\varphi(w, z_n)$$

where $\varphi(u, v)$ is computed as in the following diagram.



and where $C_{n+1,0}$ is the configuration space of $n+1$ points in the complex upper half plane $\mathcal{H}$. If we normalize things by putting $w = i$ then the integration domain is equal to $\mathcal{H}^n$.

Recall (e.g. [8]) that the modified Bernouilli numbers $ß_n$ are defined by

$$\begin{aligned}\sum_n ß_n x^n &= \frac{1}{2} \log \frac{e^{x/2} - e^{-x/2}}{x} \\ &= \frac{x^2}{48} - \frac{x^4}{5760} + \frac{x^6}{362880} - \frac{x^8}{19353600} + \frac{x^{10}}{958003200} - \frac{691x^{12}}{31384184832000} + \cdots\end{aligned}$$

The main result of this note is (see §9.6)

$$w_n = -(-1)^{n(n-1)/2} n ß_n \tag{1.1}$$

A result similar to (1.1) was announced by Shoikhet in [7, §2.3.1]. Our proof below uses Stokes theorem together with some basic properties proved by Kontsevich. A more conceptual proof would be useful.

The structure of this note is as follows: in §3 we represent our computations graphically. This makes it easy to see which boundary components in Stokes theorem yield non-trivial contributions. We ignore signs as these are not easy to handle graphically.

In §4-§8 we compute the contributions of the relevant boundary components with precise signs. Thanks to some "lucky" cancellations we obtain a system of recursions for computing w_n.

Finally in §9 we solve the recursions.

The reader will notice that some formulas and figures appear more than once in this paper. This has been done to make the three logical parts §3,§4-§8,§9 more or less independently readable. This should make our computations easier to follow.

[1] This is the traditional Kontsevich weight. In [2, §8] we use $W_n = (-1)^{2n(2n-1)/2} w_n = (-1)^n w_n$. Since $w_n = 0$ for n odd we actually have $W_n = w_n$.

2. Acknowledment

The symbolic computations were done using the SAGE [6] interface to the computer algebra package Maxima [5]. Many identities and integrals were tested numerically using the Monte Carlo integration library which is part of GSL [3].

3. Graphical representation of the computation

In this section we give a graphical overview of our computations. We will ignore signs. Hence we don't care about orientations or the ordering of the factors in a product of forms.

3.1. **Graphical language.** In the Kontevich integral we associate to a graph Γ a form ω_Γ on a suitable configuration space $C^+_{p,q}$. The form ω_Γ is the product of $d\varphi(u,v)$, where (u,v) runs over the edges of Γ, divided by $(2\pi)^e$ where e is the number of edges.

Besides the usual Kontsevich graphs we introduce graphs with some new kind of arrows.

An arrow of the form

$$\underset{u}{\bullet}\xrightarrow{\ n\ }\underset{v}{\bullet}$$

corresponds to a factor $d(\varphi(u,v)^n)/(2\pi)^n$ in ω_Γ. For such an arrow we assume that we have made a branch cut such that $\varphi(u,v) \in]0,2\pi[$. I.e. if $\operatorname{Re} u = \operatorname{Re} v$ then $\operatorname{Im} v < \operatorname{Im} u$. Note that we have to specify the branch cut as the value of the integral depends on it.

A bold arrow of the form

$$\underset{u}{\bullet}\boldsymbol{\xrightarrow{\ n\ }}\underset{v}{\bullet}$$

corresponds to a factor $\varphi(u,v)^n/(2\pi)^n$ in ω_Γ. We make the same branch cut as above. If n is not indicated then we assume $n = 1$.

We introduce a third kind of special arrow. A dashed arrow of the form

$$\underset{u}{\bullet}\dashrightarrow\underset{v}{\bullet}$$

indicates a restriction of the integration domain to $\operatorname{Re} v = \operatorname{Re} u + \epsilon$, $\operatorname{Im} v > \operatorname{Im} u$ with ϵ a positive infinitesimal.

Here is the basic computation rule for computing $\int \omega_\Gamma$ for one of the generalized graphs (Rule 1).

$$\int \underset{u}{\bullet}\xrightarrow{\ n\ }\underset{v}{\bullet} = \pm \int_\partial \underset{u}{\bullet}\boldsymbol{\xrightarrow{\ n\ }}\underset{v}{\bullet}$$

$$\int_\partial \underset{u}{\bullet}\boldsymbol{\xrightarrow{\ n\ }}\underset{v}{\bullet} = \left(\int_{\partial_{\text{classical}}} \underset{u}{\bullet}\boldsymbol{\xrightarrow{\ n\ }}\underset{v}{\bullet}\right) \pm \left(\int \underset{u}{\bullet}\dashrightarrow\underset{v}{\bullet}\right)$$

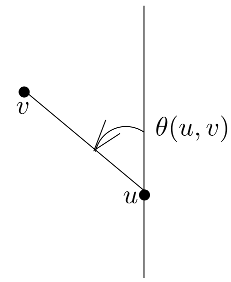
Here ∂ denotes the boundary of the integration domain (including the branch cut). $\partial_{\text{classical}}$ denotes the boundary without the branch cut. $\partial_{\text{classical}}$ can be determined combinatorially by contracting vertices in Γ and moving vertices to the real line.

The difference between ∂ and $\partial_{\text{classical}}$ is given by the two sides of the branch cut. The difference in value of $\varphi(u,v)$ on opposite sides of the branch cut is 2π. This gives the contribution with the dashed arrow.

Here is another obvious computation rule (Rule 2)

$$\begin{tikzcd} \underset{p}{\bullet} \arrow[r, bend left, "n"] \arrow[r, bend right] & \underset{q}{\bullet} \end{tikzcd} \quad = \quad \frac{1}{n+1} \times \underset{p}{\bullet}\xrightarrow{\ n+1\ }\underset{q}{\bullet}$$

Below we will also consider some integrals over the configuration spaces C_p consisting of p points in the complex plane. These are it terms of $\theta(u,v)$ where $\theta(u,v)$ is as in the following diagram

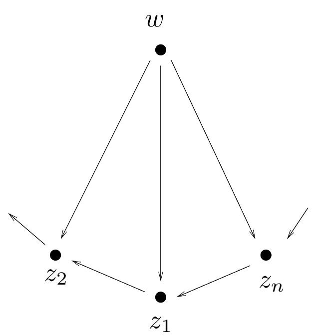


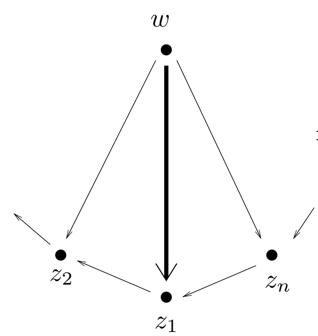
To indicate that we are integrating over some C_p we will put the graph in a box. Otherwise we use the same conventions as above (with $\theta(u,v)$ replacing $\varphi(u,v)$). Since $d\theta(u,v) = d\theta(v,u)$ some arrow indicators are superfluous. We will omit those.

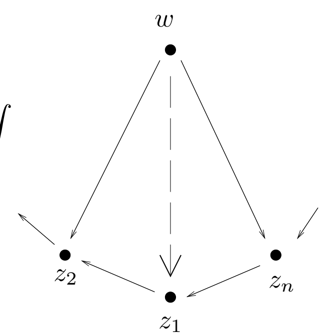
3.2. **Applying Stokes theorem to w_n.** We will now apply this graphical language to computations with wheels. We assume first $n \geq 2$

The first step is (using Rule 1)

(3.1)

$\int_{C_{n+1,0}}$


$= \pm \int_{\partial_{\text{classical}}}$


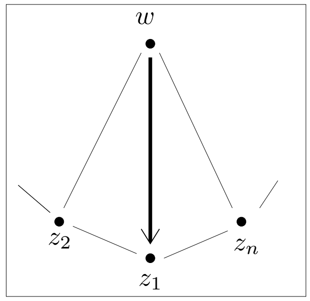
$\pm \int$


We now simplify the integral over $\partial_{\text{classical}}$ by looking at the different components of $\partial_{\text{classical}}$.

(1) A group of vertices moves to the real line. Since every vertex has a least one normal outgoing arrow this means that to obtain a non-zero result all vertices should move to the real line which is excluded. So components of this type contribute nothing.
(2) A group of vertices S comes together. Assume first $w \in S$. If $z_i \in S$, $i < n$ then to avoid double arrows after contraction we should also have $z_{i+1} \in S$. If $z_1 \in S$ this means we have to contract all vertices. I.e. our the integral

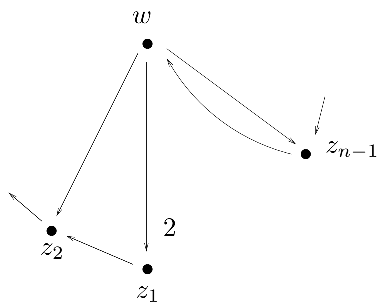
over this component becomes an integral over C_{n+1}

$$\int_{C_{n+1}}$$



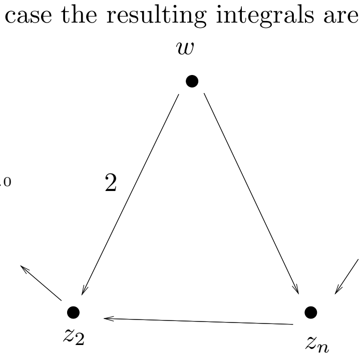
(3) Assume $z_1 \notin S$. If $z_i \in S$ for $i < n$ then we contract more than one edge and the resulting integral is zero by [4, Lemma 6.6]. Thus $S = \{w, z_n\}$. The resulting integral is (using Rule 2)

(3.2) $$\frac{1}{2}\int_{C_{n,0}}$$



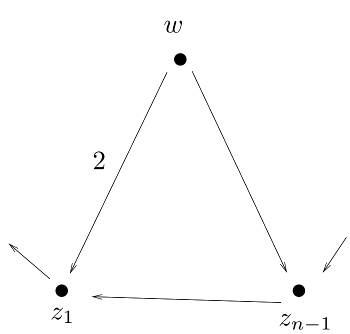
(4) Now assume $w \notin S$. This means that we contract an edge (z_i, z_{i+1}) or (z_n, z_1) (if we contract more edges then the integral is zero by [4, Lemma 6.6]). The only contractions which do not create double edges are (z_1, z_2) and (z_n, z_1). In that case the resulting integrals are (using Rule 2)

(3.3) $$\frac{1}{2}\int_{C_{n.0}}$$



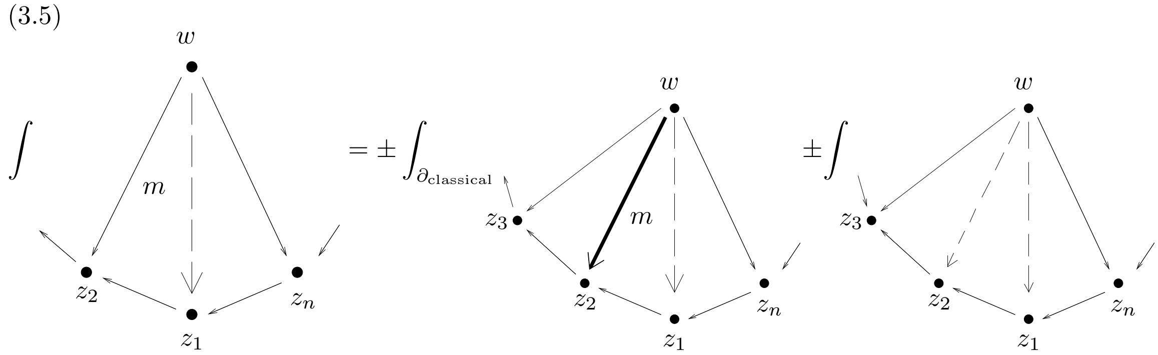
and

(3.4) $$\frac{1}{2}\int_{C_{n,0}}$$



3.3. **The boundary component associated to the branch cut.** We need to compute the integral associated to the graph with the dashed arrow in (3.1). In order to derive a recursion relation we work more generally, starting with the following identity.

(3.5)

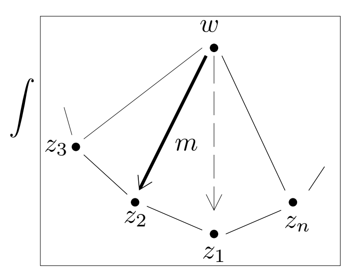


However the integral containing the two dashed arrows is zero since if z_1, z_2 move on the same vertical line then $d\varphi(z_1, z_2) = 0$.

It remains to determine the integral over $\partial_{\mathrm{classical}}$. We consider the various boundary components. We now assume $n \geq 3$.

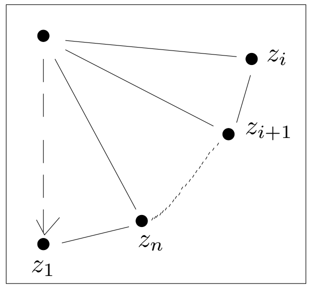
(1) A group of vertices moves to the real line. Since every vertex has a least one normal outgoing arrow (using $n \geq 2$) this means that to obtain a non-zero result all vertices should move to the real line which is excluded. So no contribution from these components.

(2) A group of vertices S comes together. Assume first $w \in S$. If $z_i \in S$, $i \neq 1, n$ then to avoid double arrows after contraction we should also have $z_{i+1} \in S$. However if $z_2 \in S$ of $z_n \in S$ then $z_1 \in S$ for otherwise after contraction a solid and a dashed arrow coincide and thus the result is zero. Thus we either contract all vertices or we contract $w, z_i, \ldots, z_n, z_1$ for $i \geq 3$ or we contract (w, z_1). If we contract all vertices then the integral becomes



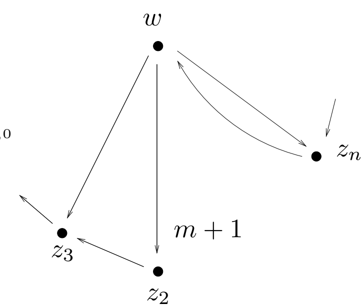
(3) Assume we contract $w, z_i, \dots, z_n, z_1$ for $3 \leq i \leq n$. Then the resulting integral is *a multiple of*

(3.6) $\int$



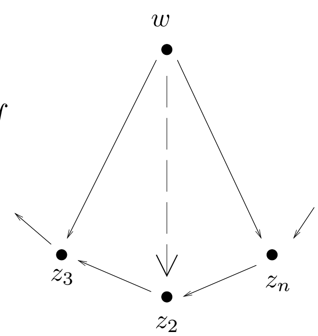
(4) It remains to consider the case $S = \{z_1, w\}$. In that case we get using Rule 2:

$$\frac{1}{m+1} \int_{C_{n,0}}$$



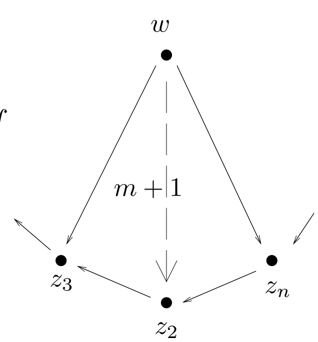
(5) Now assume $w \notin S$. This means that we contract an edge (z_i, z_{i+1}) or (z_n, z_1) (if we contract more edges then the integral is zero by [4, Lemma 6.6]). The only two possibilities which do not give zero immediately are (z_1, z_2) and (z_2, z_3). The resulting integrals are

$$\frac{1}{2} \int$$



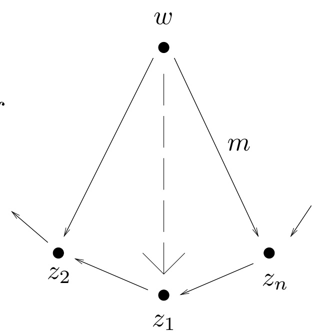
and (using Rule 2)

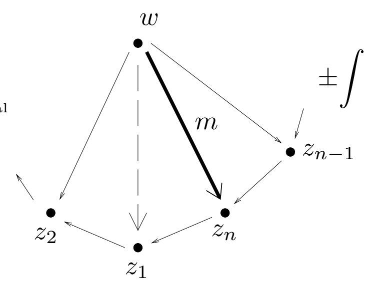
$$\frac{1}{m+1} \int$$

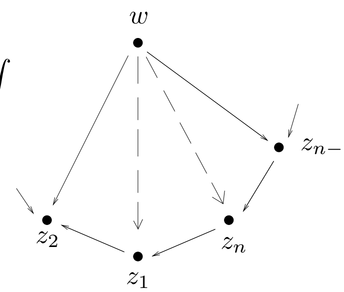


The factor $\frac{1}{2}$ in the first integral comes from the fact that in this boundary component we have $\varphi(w, z_2) = 0$ if z_2 is to the left of z_1.

As will be explained below we need a second method for attacking the term with the dashed arrow in (3.1). Therefore we use

$\int$


$= \pm \int_{\partial_{\text{classical}}}$


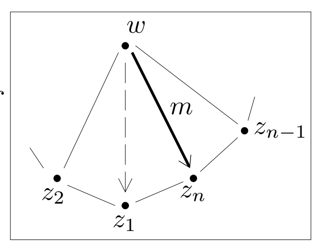
$\pm \int$


Again the integral containing the two dashed arrows is zero. Likewise is the integral over the boundary components where some vertices move to the real line. We discuss the other boundary components of $\partial_{\text{classical}}$ assuming $n \geq 3$.

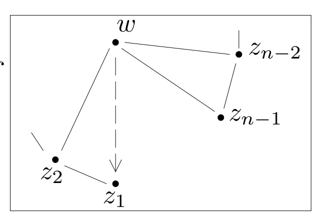
(1) A group of vertices S comes together. Assume first $w \in S$. If $z_i \in S$, $1 \leq i < n-1$ then to avoid double arrows after contraction we should also have $z_{i+1} \in S$. However if $z_n \in S$ of $z_2 \in S$ then $z_1 \in S$ for otherwise after contraction a solid and a dashed arrow coincide and thus the result is zero.

It follows that the possibilities for contraction are $\{z_1, \ldots, z_n, w\}$, $\{z_1, \ldots, z_{n-1}, w\}$ or $\{z_i, \ldots, z_{n-1}, w\}$ for $i \geq 3$. In the last case we must have $i = n-1$ by [4, Lemma 6.6].

If we contract all vertices then the contribution to the integral is

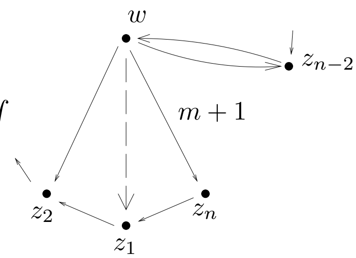
$\int$


(2) If we contract $w, z_1, \ldots, z_{n-1}$ then the resulting integral is *a multiple of*

(3.7) $\int$


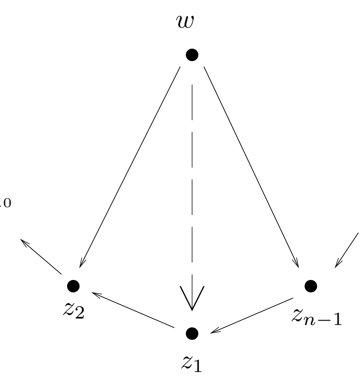
(3) It remains to consider the case $S = \{z_{n-1}, w\}$. In that case we obtain (using Rule 2)

$$\frac{1}{m+1}\int$$



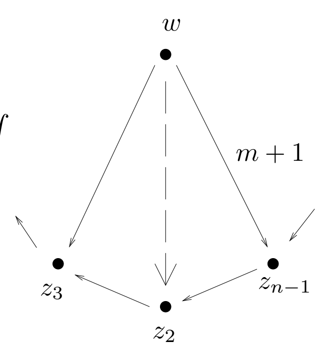
(4) Now assume $w \notin S$. This means that we contract an edge (z_n, z_1) or (z_{n-1}, z_n) (otherwise the result is zero by [4, Lemma 6.6] or by the fact that we create double arrows). The resulting integrals are

$$\frac{1}{2}\int_{C_{n.0}} \tag{3.8}$$



and (using Rule 2)

$$\frac{1}{m+1}\int$$

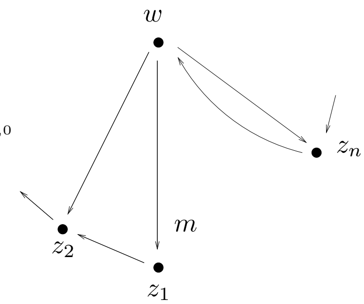


The factor $1/2$ in the first integral comes from the fact that due to our branch cut $\varphi(w, z_n) = 0$ if z_2 is to the left of z_1.

3.4. **Further comments.** We now indicate how the relations we have derived will be used. We still do not specify precise signs.

(1) We first discuss the partial evaluation of w_n using (3.1). One ingredient that enters is (3.2). We define

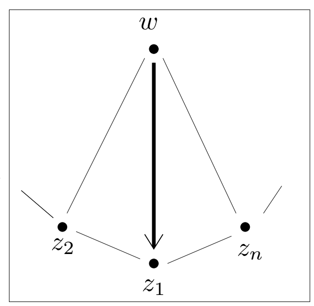
$$\alpha_{n,m} = \int_{C_{n+1,0}}$$



We will evaluate $\alpha_{n,m}$ by subsequently integrating over z_1, z_2, etc.... See §4 below.

(2) Another ingredient that enters is (8.2).

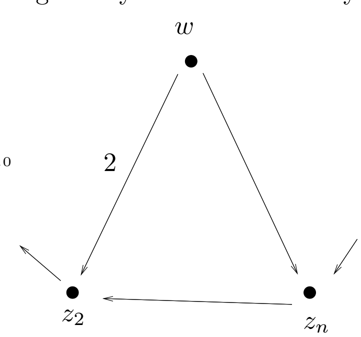
$$\int_{C_{n+1}}$$



We do not know how to evaluate this integral in general but we will show by a symmetry argument that it is zero if n is even (see §8.2).

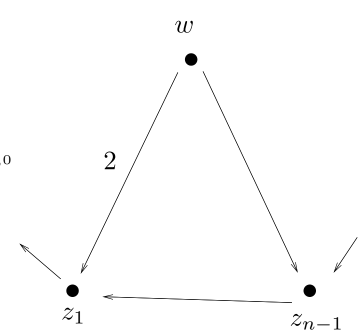
(3) The final ingredient is given by the two identically looking integrals

$$\int_{C_{n.0}}$$



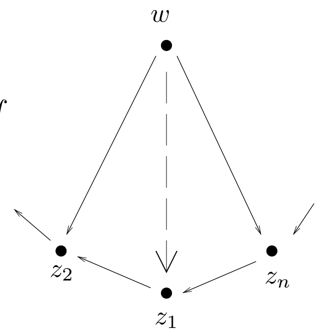
and

$$\int_{C_{n,0}}$$

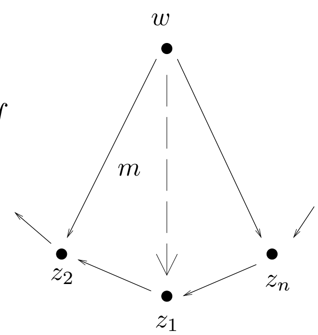


We will show that the signs are such that these two integrals cancel (see §8.3).

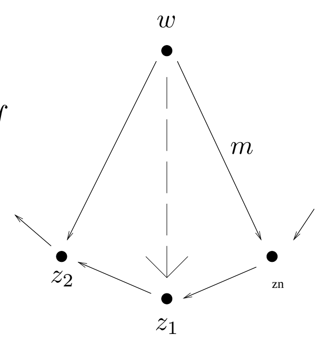
(4) So ultimately we get an expression for w_n in terms of $\alpha_{n-1,m+1}$ (which we know how to compute) and

$\int$ 

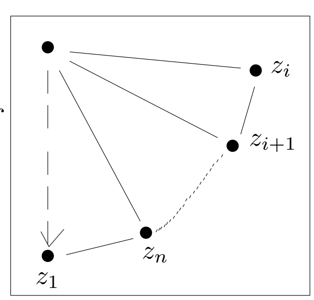
which is a special case of both the folloing integrals

$\beta_{n,m} \stackrel{\text{def}}{=} \int$ 

and

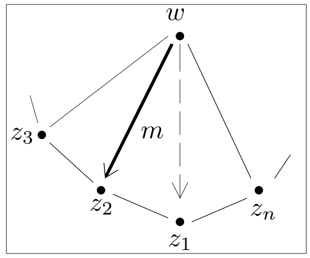
$\bar{\beta}_{n,m} \stackrel{\text{def}}{=} \int$ 

(5) We now discuss $\beta_{n,m}$ using (3.5). We will show below (see §5.2) that

$\int$ 

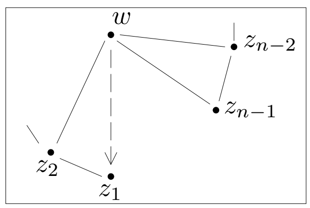
is zero. Hence we obtain an expression for $\beta_{n,m}$ in terms of $\alpha_{n-1,m+1}$, $\beta_{n-1,1}$, $\beta_{n+1,m-1}$ *and*

$$\epsilon_{n,m} \stackrel{\text{def}}{=} \int$$



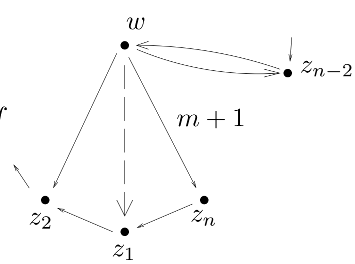
(6) In a similar way we discuss $\bar{\beta}_{n,m}$. As already pointed out we will show that the integral

$$\int$$



is zero (see §5.2 below). We will also show that the integral

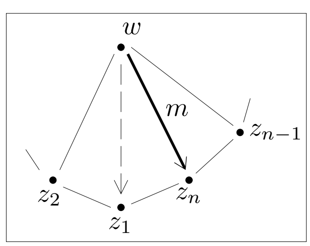
$$\int$$



is zero (see §6.1).

As a result we obtain an expression for $\bar{\beta}_{n,m}$ in terms of $\bar{\beta}_{n-1,m+1}$, $\beta_{n-1,1}$ *and*

$$\bar{\epsilon}_{n,m} \stackrel{\text{def}}{=} \int$$



(7) We will show below by a symmetry argument that $\epsilon_{n,m} = -\bar{\epsilon}_{n,m}$ (see §7.1).
(8) Unfortunately we do not know how to compute $\epsilon_{n,m}$. However we will show below that by considering a suitable linear combination of $\beta_{n,m}$ and $\bar{\beta}_{n,m}$ the unknown quantity $\epsilon_{n,m}$ cancels out in the computation. This will ultimately give us a recursive procedure for computing w_n.

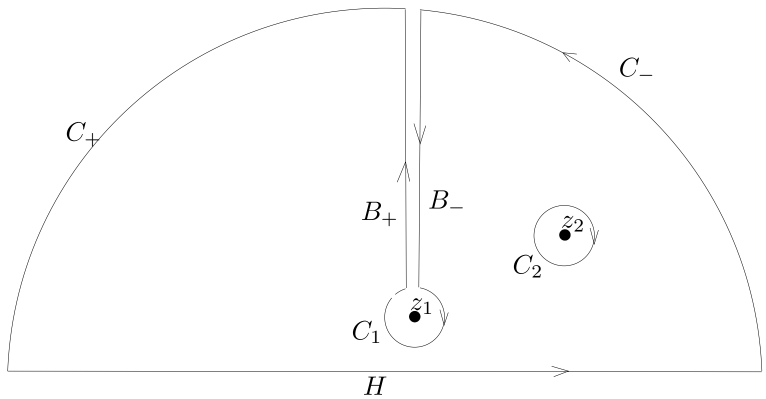
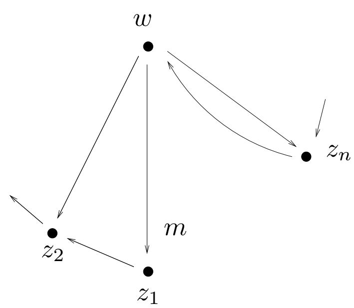


FIGURE 1

4. A RECURSION RELATION FOR $\alpha_{n,m}$

By our definition in §3.4, $\alpha_{n,m}$ is the integral associated to the enhanced graph



Thus with our standard ordering of edges we have

$$\alpha_{n,m} = \frac{1}{(2\pi)^{2n+m-1}} \int_{C_{n+1,0}} d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, w) d\varphi(w, z_1)^m d\varphi(w, z_2) \cdots d\varphi(w, z_n)$$

In this section we prove

$$\alpha_{1,m} = -\left(\frac{1}{2} - \frac{1}{m+1}\right) \tag{4.1}$$

and for $n \geq 2$.

$$\alpha_{n,m} = (-1)^n \left(\frac{1}{2}\alpha_{n-1,2} - \frac{1}{m+1}\alpha_{n-1,m+1}\right) \tag{4.2}$$

We put $\alpha_{0,m} = 1$ such that (4.2) holds for $n \geq 1$.

4.1. **Step 1.** We will prove

$$\int_{z \in \mathcal{H} \setminus \{z_1, z_2\}} d\varphi(z_1, z)^m d\varphi(z, z_2) = (2\pi)^m \varphi(z_1, z_2) - 2\pi\varphi(z_1, z_2)^m \tag{4.3}$$

where we have made a branch cut such that $\varphi(z_1, z) \in]0, 2\pi[$.

In Figure 1 we have indicate the integration domain and its boundary. Here

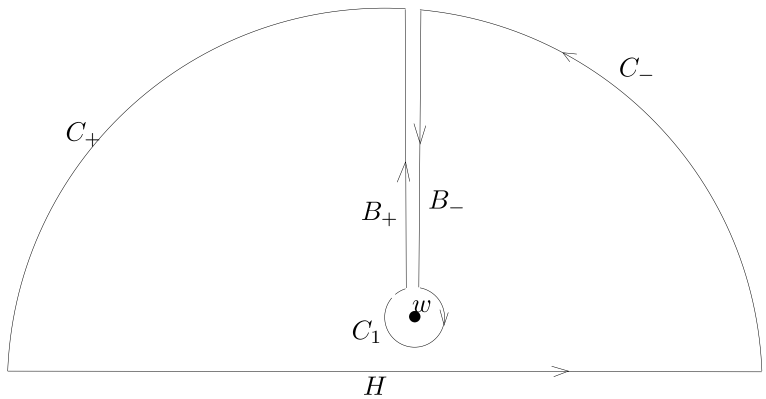


FIGURE 2

$C_+ \cup C_-$ is the circle at infinity, B_+, B_- are infinitesimally close together and C_1, C_2 have infinitesimal radius. Finally H is the real line. Thus we have

$$\int_{z\in\mathbb{C}\setminus\{z_1,z_2\}} d\varphi(z_1,z)^m d\varphi(z,z_2) = \int_{\text{boundary}} \varphi(z_1,z)^m d\varphi(z,z_2)$$

We first compute the integral over $C_\pm \cup B_\pm \cup C_1 \cup H$. On $B_- \cup C_-$ we have that $\varphi(z_1,z)$ is constant and equal to 2π. On $B_+ \cup C_+$ we have that $\varphi(z_1,z) = 0$. On $H \cup C_1$ we have that $d\varphi(z,z_2) = 0$.

Thus we find

$$\int_{B_\pm\cup C_\pm\cup C_1\cup H} \varphi(z_1,z)^m d\varphi(z,z_2) = (2\pi)^m \int_{B_-\cup C_-} d\varphi(z,z_2) = (2\pi)^m\varphi(z_1,z_2)$$

The integral over C_2 is equal to $-2\pi\varphi(z_1,z_2)^m$. This finishes the proof of (4.3).

4.2. **Step 2.** We will prove

$$\int_{z\in\mathcal{H}\setminus\{w\}} d\varphi(w,z)^m d\varphi(z,w) = (2\pi)^{m+1}\left(\frac{1}{2} - \frac{1}{m+1}\right) \tag{4.4}$$

In Figure 2 we have indicated the integration domain and its boundary Thus we have

$$\int_{z\in\mathbb{C}\setminus\{w\}} d\varphi(w,z)^m d\varphi(z,w) = \int_{\text{boundary}} \varphi(w,z)^m d\varphi(z,w)$$

On $B_- \cup C_-$ we have that $\varphi(w,z)$ is constant and equal to 2π. On $B_+ \cup C_+$ we have that $\varphi(w,z) = 0$. On H we have that $d\varphi(z,w) = 0$. Thus

$$\int_{C_\pm\cup B_\pm\cup H} \varphi(w,z)^m d\varphi(z,w) = (2\pi)^m\pi$$

It remains to compute

$$\int_{C_1} \varphi(w,z)^m d\varphi(z,w)$$

We put $z = w + re^{-i\theta}$ for $\theta \in]-\pi/2, 3\pi/2[$. We have $\varphi(w,z) = 3\pi/2 - \theta$ and hence $d\varphi(z,w) = -d\theta$. Thus we get

$$\int_{C_1} \varphi(w,z)^m d\varphi(z,w) = -\int_{-\pi/2}^{3\pi/2} (3\pi/2-\theta)^m d\theta = -\int_0^{2\pi} \alpha^m d\alpha = -\frac{(2\pi)^{m+1}}{m+1}$$

This finishes the proof of (4.4).

4.3. **Step 3.** Now we prove (4.1).

$$\alpha_{1,m} = \frac{1}{(2\pi)^{m+1}} \int d\varphi(z_1, w) d\varphi(w, z_1)^m = -\frac{1}{(2\pi)^{m+1}} \int d\varphi(w, z_1)^m d\varphi(z_1, w) = -\left(\frac{1}{2} - \frac{1}{m+1}\right)$$

by (4.4).

4.4. **Step 4.** Now we prove (4.2).

$$\alpha_{n,m} = (-1)^n \frac{1}{(2\pi)^{2n+m-1}} \int d\varphi(w, z_1)^m d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, w) d\varphi(w, z_2) \cdots d\varphi(w, z_n)$$

Integrating over z_1 and using (4.3) first we find

$$\begin{aligned} \alpha_{n,m} = (-1)^n \frac{1}{(2\pi)^{2n+m-1}} \int & ((2\pi)^m \varphi(w, z_2) - 2\pi \varphi(w, z_2)^m) \times \\ & d\varphi(z_2, z_3) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, w) d\varphi(w, z_2) \cdots d\varphi(w, z_n) \end{aligned}$$

which is equal to

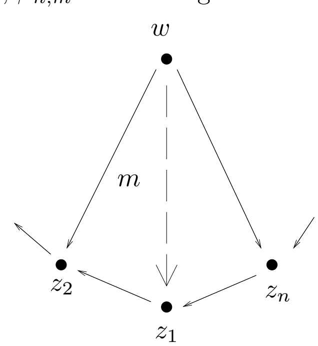
$$\begin{aligned} &(-1)^n \frac{1}{2(2\pi)^{2n-1}} \int d\varphi(z_2, z_3) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, w) d\varphi(w, z_2)^2 \cdots d\varphi(w, z_{n-1}) \\ &-(-1)^n \frac{1}{(m+1)(2\pi)^{2n+m-2}} \int d\varphi(z_2, z_3) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, w) d\varphi(w, z_2)^{m+1} \cdots d\varphi(w, z_n) \end{aligned}$$

So that we indeed find

$$\alpha_{n,m} = (-1)^n \left(\frac{1}{2}\alpha_{n-1,2} - \frac{1}{m+1}\alpha_{n-1,m+1}\right)$$

5. A recursion relation for $\beta_{n,m}$

By our definition in §3.4, $\beta_{n,m}$ is the integral associated to the enhanced graph



Thus with our standard edge ordering

$$\beta_{n,m} = \frac{1}{(2\pi)^{2n+m-2}} \int_{B_-\setminus D} d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2)^m \cdots d\varphi(w, z_n)$$

where

$$B_- = \{\operatorname{Re} z_1 = \operatorname{Re} w+, \operatorname{Im} z_1 > \operatorname{Im} w\} \subset C_{n+1,0}$$

$$D = \{\operatorname{Re} z_2 = \operatorname{Re} w, \operatorname{Im} z_2 > \operatorname{Im} w\} \subset C_{n+1,0}$$

We need to be careful about specifying the orientation of the integration domain. Put $w = i$. If $z_j = x_j + iy_j$ then $C_{n+1,0}$ is oriented by [1]

$$dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n$$

In the neighborhood of B_- we have coordinates $(x_1, y_1, \ldots, x_n, y_n)$ with $x_1 \geq 0$. Therefore if B_- is oriented with its outgoing normal (as is necessary for the application of Stokes theorem) then it is oriented by

$$-dy_1 dx_2 dy_2 \cdots dx_n dy_n \tag{5.1}$$

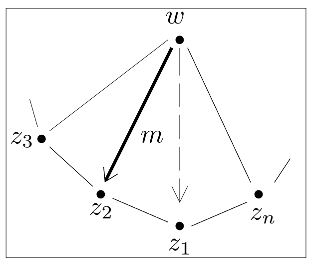
In this section we will prove the following

$$\beta_{2,m} = -\frac{1}{8} + \frac{1}{m+1}\left(\frac{1}{2} - \frac{1}{m+2}\right) \tag{5.2}$$

and for $n \geq 3$
(5.3)

$$\beta_{n,m} = -\frac{1}{m+1}\alpha_{n-1,m+1} + (-1)^{n+1}\frac{1}{2}\beta_{n-1,1} + (-1)^n \frac{1}{m+1}\beta_{n-1,m+1} + (-1)^n \epsilon_{n,m}$$

where $\epsilon_{n,m}$ is the integral associated to the graph



(see §3.4). Thus with our standard orientation on edges

$$\epsilon_{n,m} = \frac{1}{(2\pi)^{2n+m-2}} \int \theta(w, z_2)^m d\theta(z_1, z_2) d\theta(z_2, z_3) \cdots d\theta(z_n, z_1) d\theta(w, z_3) \cdots d\theta(w, z_n)$$

Here we put $w = 0$, $z_1 = i$ and the integral is over the complement of the diagonal in $(\mathbb{C} - \{0, i\})^{n-1}$.

5.1. **Step 1.** We first consider the case $n = 2$. Thus

$$\beta_{2,m} = \frac{1}{(2\pi)^{2+m}} \int_{B_- \setminus D} d\varphi(z_1, z_2) d\varphi(z_2, z_1) d\varphi(w, z_2)^m$$

We first integrate over w. We get

$$\int_{B_- \setminus D} d\varphi(w, z_2)^m = \varphi(z_1, z_2)^m - (2\pi)^m [z_1, z_2]$$

where

$$[z_1, z_2] = \begin{cases} 1 & z_1 \text{ is to the left of } z_2 \\ 0 & \text{otherwise} \end{cases}$$

Thus we must compute

$$\frac{1}{(2\pi)^{2+m}} \int \varphi(z_1, z_2)^m d\varphi(z_1, z_2) d\varphi(z_2, z_1) \tag{5.4}$$

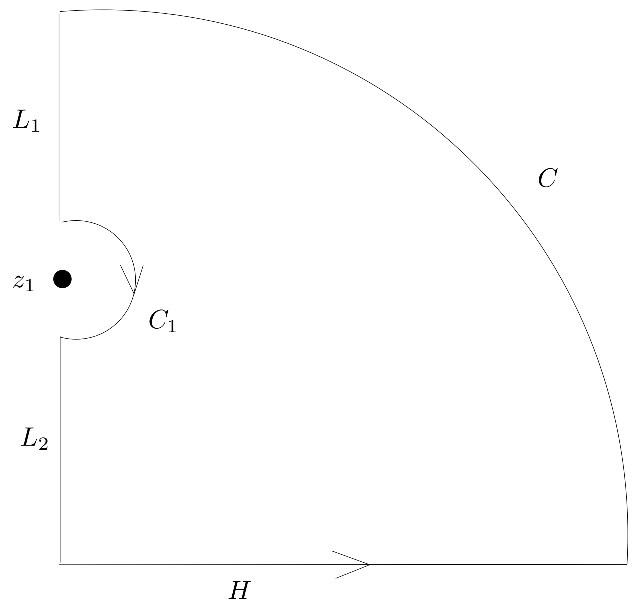


FIGURE 3

and

$$\frac{1}{(2\pi)^2}\int [z_1, z_2] d\varphi(z_1, z_2) d\varphi(z_2, z_1) \tag{5.5}$$

Using Stokes theorem we rewrite (5.5) as a path integral as in Figure 3

$$\frac{1}{(2\pi)^2}\int_{L_1\cup C_1\cup L_2\cup H\cup C} \varphi(z_1, z) d\varphi(z, z_1)$$

If $z \in L_{1,2} \cup H$ then $d\varphi(z, z_1) = 0$. If $z \in C$ then $\varphi(z_1, z) = 2\pi$.

Thus

$$\frac{1}{(2\pi)^2}\int_{L_1\cup L_2\cup H\cup C} \varphi(z_1, z) d\varphi(z, z_1) = \frac{1}{(2\pi)}(\pi - 0) = \frac{1}{2}$$

Now we compute

$$\frac{1}{(2\pi)^2}\int_{C_1} \varphi(z_1, z) d\varphi(z, z_1)$$

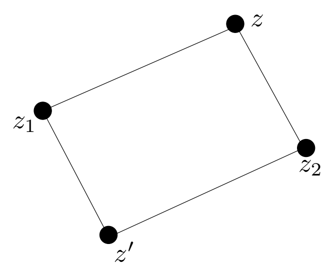
We put $z = z_1 + re^{-i\theta}$ for $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Then $\varphi(z_1, z) = \frac{3}{2}\pi - \theta$ and hence $d\varphi(z, z_1) = -d\theta$.

Thus

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_{C_1} \varphi(z_1, z) d\varphi(z, z_1) &= -\frac{1}{(2\pi)^2}\int_{-\pi/2}^{\pi/2} (\frac{3}{2}\pi - \theta) d\theta \\ &= \frac{1}{(2\pi)^2}\int_{-\pi/2}^{\pi/2} (\theta - \frac{3}{2}\pi) d\theta \\ &= \frac{1}{(2\pi)^2}\int_{-2\pi}^{-\pi} \theta d\theta \\ &= -\frac{3}{8}\end{aligned}$$

and hence
$$\frac{1}{(2\pi)^2}\int [z_1,z_2]d\varphi(z_1,z_2)d\varphi(z_2,z_1)=\frac{1}{2}-\frac{3}{8}=\frac{1}{8}$$
One the other hand by (4.3)
$$\begin{aligned}\frac{1}{(2\pi)^{2+m}}\int \varphi(z_1,z_2)^m d\varphi(z_1,z_2)d\varphi(z_2,z_1) &= \frac{1}{(m+1)(2\pi)^{2+m}}\int\ d\varphi(z_1,z_2)^{m+1}d\varphi(z_2,z_1)\\ &= \frac{1}{m+1}\left(\frac{1}{2}-\frac{1}{m+2}\right)\end{aligned}$$
so that our final formula is
$$\beta_{2,m}=-\frac{1}{8}+\frac{1}{m+1}\left(\frac{1}{2}-\frac{1}{m+2}\right)$$

5.2. **Step 2.** We claim
$$\int_{z\in\mathbb{C}\setminus\{z_1,z_2\}} d\theta(z_1,z)d\theta(z,z_2)=0 \tag{5.6}$$
Since $d\theta(z,z_2)=d\theta(z_2,z)$ we may as well prove
$$\int_{z\in\mathbb{C}\setminus\{z_1,z_2\}} d\theta(z_1,z)d\theta(z_2,z)=0$$
We may assume $z_1\neq z_2$ since otherwise the claim is trivial. Consider the following figure



Then we have $d\theta(z_1,z)d\theta(z_2,z)=-d\theta(z_1,z')d\theta(z_2,z')$. Since the map $z\mapsto z'$ is orientation preserving this proves what we want.

5.3. **Step 3.** Now we assume $n\geq 3$. First we observe
$$\begin{aligned}\beta_{n,m} &= (-1)^n\frac{1}{(2\pi)^{2n+m-2}}\int_{B_-\setminus D} d\varphi(w,z_2)^m d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n)d\varphi(z_n,z_1)d\varphi(w,z_3)\cdots d\varphi(w,z_n)\\ &= (-1)^n\frac{1}{(2\pi)^{2n+m-2}}\int_{\partial(B_-\setminus D)} \varphi(w,z_2)^m d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n)d\varphi(z_n,z_1)d\varphi(w,z_3)\cdots d\varphi(w,z_n)\end{aligned}$$
The components of $\partial(B_-\setminus D)$ which yield non-zero contributions to the integral have been listed in §3.3. Note that the contribution of (3.6) is zero by (5.6) (integrate first over z_i). We now list the contributing integrals with the correct signs.

(1) Consider the boundary component ∂_1 given by contracting all edges. The coordinates in the neighborhood of this component are given by $r,x'_2,y'_2,\ldots,x'_n,y'_n$
$$\begin{aligned}y_1&=1+r\\ x_i&=rx'_i\\ y_i&=1+ry'_i\end{aligned}$$

with $r \geq 0$. In these coordinate the orientation form on B_- is (up to a positive multiple) given by

$$-drdx_2'dy_2' \cdots dx_n'dy_n'$$

and hence ∂_1 is oriented by $dx_2'dy_2' \cdots dx_n'dy_n'$. Whence the contribution to the integral is equal to $(-1)^n \epsilon_{n,m}$.

(2) Now consider the boundary component ∂_2 given by contracting $\{z_1, w\}$. We claim $\partial_2 \cong C_{n,0}$, taking orientations into account. Coordinates in a neihborhood of ∂_2 are given by $r, x_2, y_2, \ldots, x_n, y_n$ where $y_1 = 1 + r$ and $r \geq 0$. Thus the orientation form is $-drdx_2dy_2 \cdots x_n, y_n$ and hence ∂_2 is oriented by $dx_2dy_2 \cdots x_n, y_n$ which is the normal orientation on $C_{n,0}$ [1]. We obtain

$$\begin{aligned}
&(-1)^n \frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_2} \varphi(w, z_2)^m d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_3) \cdots d\varphi(w, z_n) \\
&= (-1)^n \frac{1}{(2\pi)^{2n+m-2}} \int_{C_{n,0}} \varphi(w, z_2)^m d\varphi(w, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_3) \cdots d\varphi(w, z_n) \\
&= (-1)^n \frac{1}{(m+1)(2\pi)^{2n+m-2}} \int_{C_{n,0}} d\varphi(w, z_2)^{m+1} d\varphi(z_2, z_3) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_3) \cdots d\varphi(w, z_n) \\
&= -\frac{1}{(m+1)(2\pi)^{2n+m-2}} \int_{C_{n,0}} d\varphi(z_2, z_3) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2)^{m+1} d\varphi(w, z_3) \cdots d\varphi(w, z_n) \\
&= -\frac{1}{m+1} \alpha_{n-1,m+1}
\end{aligned}$$

(3) Let ∂_3 be obtained by contracting (z_1, z_2). We claim that $\partial_3 = -C_2' \times B_-'$ where B_-' is the analogue of B_- inside $C_{n,0}$ and where C_2' is the part of C_2 where the two points are not on the same vertical line. Thus C_2' and hence ∂_3 has two connected components.

Coordinates in a neighborhood of ∂_3 are given by $r, \theta, y_1, x_3, y_3, \ldots, x_n, y_n$ where

$$\begin{aligned}
x_2 &= r \cos \theta \\
y_2 &= y_1 + r \sin \theta
\end{aligned}$$

and $r \geq 0$ and $\theta \notin \{\pi/2, -\pi/2\}$ (the latter because of the branch cut involving $\varphi(w, z_2)$). Hence we find that the orientation form on B_- is up to a positive factor given by

$$-dy_1 drd\theta dx_3 dy_3 \cdots dx_n dy_n$$

Therefore the orientation form on ∂_3 (using the outward normal) is given by

$$-dy_1 d\theta dx_3 dy_2 \cdots dx_n dy_n = (-d\theta) \times (-dy_1 dx_3 dy_3 \cdots dx_n dy_n)$$

which does indeed represent the orientation on $-C_2' \times B_-'$.

The resulting integral is now

$$(-1)^n \frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_3} \varphi(w,z_2)^m d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_1) d\varphi(w,z_3)\cdots d\varphi(w,z_n)$$
$$= (-1)^{n+1} \frac{1}{(2\pi)^{2n-1}} \int_{C_2'\times B_-'} d\varphi(z_2,z_3)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_2) d\varphi(w,z_3)\cdots d\varphi(w,z_n)$$
$$= (-1)^{n+1}\frac{1}{2}\beta_{n-1,1}$$

where the factor $1/2$ comes from the fact that $\varphi(w,z_2)=0$ if z_2 is to the left of z_1.

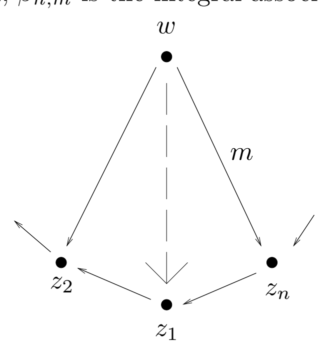
(4) Let ∂_4 be obtained by contracting (z_2,z_3). A similar computation as above shows $\partial_4 \cong -C_2 \times B_-'$. The resulting integral is

$$(-1)^n \frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_4} \varphi(w,z_2)^m d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_1) d\varphi(w,z_3)\cdots d\varphi(w,z_n)$$
$$= (-1)^{n+1} \frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_4} \varphi(w,z_2)^m d\varphi(z_2,z_3) d\varphi(z_1,z_2) d\varphi(z_3,z_4)\cdots d\varphi(z_{n-1},z_n)\times$$
$$d\varphi(z_n,z_1) d\varphi(w,z_3)\cdots d\varphi(w,z_n)$$
$$= (-1)^n \frac{1}{(2\pi)^{2n+m-3}} \int_{B_-'} \varphi(w,z_3)^m d\varphi(z_1,z_3) d\varphi(z_3,z_4)\cdots d\varphi(z_{n-1},z_n)\times$$
$$d\varphi(z_n,z_1) d\varphi(w,z_3)\cdots d\varphi(w,z_n)$$
$$= (-1)^n \frac{1}{(m+1)(2\pi)^{2n+m-3}} \int_{B_-'} d\varphi(z_1,z_3) d\varphi(z_3,z_4)\cdots d\varphi(z_{n-1},z_n)\times$$
$$d\varphi(z_n,z_1) d\varphi(w,z_3)^{m+1}\cdots d\varphi(w,z_n)$$
$$= (-1)^n \frac{1}{m+1}\beta_{n-1,m+1}$$

Combining all contributions we get (5.3).

6. A recursion relation for $\bar{\beta}_{n,m}$

By our definition in §3.4, $\bar{\beta}_{n,m}$ is the integral associated to the enhanced graph



Thus with our standard edge ordering

$$\bar{\beta}_{n,m} = \frac{1}{(2\pi)^{2n+m-2}} \int_{B_-\setminus D} d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_n)^m$$

where

$$B_- = \{\operatorname{Re} z_1 = \operatorname{Re} w+, \operatorname{Im} z_1 > \operatorname{Im} w\} \subset C_{n+1,0}$$

$$D = \{\operatorname{Re} z_n = \operatorname{Re} w, \operatorname{Im} z_2 > \operatorname{Im} w\} \subset C_{n+1,0}$$

and where B_- is oriented with the outgoing normal (see (5.1)). Clearly we have

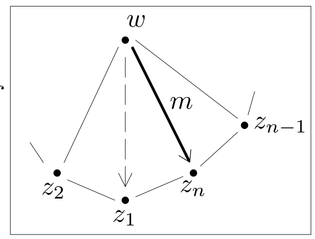
$$\bar{\beta}_{2,m} = \beta_{2,m}$$

$$\bar{\beta}_{n,1} = \beta_{n,1}$$

In this section we will prove for $n \geq 3$

$$\bar{\beta}_{n,m} = (-1)^n \frac{1}{2} \beta_{n-1,1} + (-1)^{n+1} \frac{1}{m+1} \bar{\beta}_{n-1,m+1} + \bar{\epsilon}_{n,m} \tag{6.1}$$

where $\bar{\epsilon}_{n,m}$ is the integral associated to the graph

$\int$ 

(see §3.4). Thus with our standard orientation on edges

$$\bar{\epsilon}_{n,m} = \frac{1}{(2\pi)^{2n+m-2}} \int \theta(w, z_n)^m d\theta(z_1, z_2) d\theta(z_2, z_3) \cdots d\theta(z_n, z_1) d\theta(w, z_2) \cdots d\theta(w, z_{n-1})$$

Here we put $w = 0$, $z_1 = i$ and the integral is over the complement of the diagonal in $(\mathbb{C} - \{0, i\})^{n-1}$.

6.1. **Proof.** We assume $n \geq 3$. We first observe

$$\begin{aligned}
\bar{\beta}_{n,m} &= \frac{1}{(2\pi)^{2n+m-2}} \int_{B_- \setminus D} d\varphi(w, z_n)^m d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1}) \\
&= \frac{1}{(2\pi)^{2n+m-2}} \int_{\partial(B_- \setminus D)} \varphi(w, z_n)^m d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1})
\end{aligned}$$

The components of $\partial(B_- \setminus D)$ which yield non-zero contributions to the integral have been listed in §3.3. Note that the contribution of (3.7) is zero by (5.6) (integrate first over z_{n-1}). We now list the contributing integrals with the correct signs.

(1) Consider the boundary component ∂_1 given by contracting all edges. Then a computation similar to §5.3(1) yieds that the contribution is $\bar{\epsilon}_{n,m}$.
(2) Now consider the boundary component ∂_2 given by contracting (z_{n-1}, w). It this case we get an integral which contains as a subintegral (up to a scalar factor)

$$\int_{z_n \in \mathcal{H} \setminus \{w, z_1\}} d\varphi(w, z_n)^{m+1} d\varphi(z_n, z_1)$$

Using (4.3) this is equal to

$$(2\pi)^{m+1} \varphi(w, z_1) - 2\pi \varphi(w, z_1)^{m+1} = (2\pi)^{m+1} \times 2\pi - 2\pi \times (2\pi)^{m+1} = 0$$

So we are lucky that there is no contribution in this case.

(3) Let ∂_3 be obtained by contracting (z_1, z_n). A similar computation as in §5.33) shows $\partial_3 = -C_2' \times B'_-$. The resulting integral is

$$\begin{aligned}
&\frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_3} \varphi(w, z_n)^m d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1}) \\
&= (-1)^{n-1} \frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_3} \varphi(w, z_n)^m d\varphi(z_n, z_1) d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1}) \\
&= (-1)^n \frac{1}{2} \frac{1}{(2\pi)^{2n-2}} \int_{B'_-} d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1}) \\
&= (-1)^n \frac{1}{2} \bar{\beta}_{n-1,1} = (-1)^n \frac{1}{2} \beta_{n-1,1}
\end{aligned}$$

where the factor $1/2$ comes from the fact that $\varphi(w, z_n) = 0$ if z_2 is to the left of z_1.

(4) Let ∂_4 be obtained by contracting (z_{n-1}, z_n). A similar computation as in §5.33) shows $\partial_4 \cong -C_2 \times B'_-$. The resulting integral is

$$\begin{aligned}
&\frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_4} \varphi(w, z_n)^m d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1}) \\
&= (-1)^n \frac{1}{(2\pi)^{2n+m-2}} \int_{\partial_4} \varphi(w, z_n)^m d\varphi(z_{n-1}, z_n) d\varphi(z_1, z_2) \cdots d\varphi(z_{n-2}, z_{n-1}) \times \\
&\qquad d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1}) \\
&= (-1)^{n+1} \frac{1}{(2\pi)^{2n+m-3}} \int_{B'_-} \varphi(w, z_{n-1})^m d\varphi(z_1, z_2) \cdots d\varphi(z_{n-2}, z_{n-1}) \times \\
&\qquad d\varphi(z_{n-1}, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1}) \\
&= (-1)^{n+1} \frac{1}{(m+1)(2\pi)^{2n+m-3}} \int_{B'_-} d\varphi(z_1, z_2) \cdots d\varphi(z_{n-2}, z_{n-1}) \times \\
&\qquad d\varphi(z_{n-1}, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_{n-1})^{m+1} \\
&= (-1)^{n+1} \frac{1}{m+1} \bar{\beta}_{n-1,m+1}
\end{aligned}$$

Combining all contributions we get (6.1).

7. A recursion not involving unknown quantities.

Put

$$\hat{\beta}_{n,m} = \frac{1}{2}(\beta_{n,m} + (-1)^n \bar{\beta}_{n,m})$$

We claim

$$\hat{\beta}_{n,m} = -\frac{1}{2(m+1)} \alpha_{n-1,m+1} + (-1)^{n+1} \frac{1}{2} \hat{\beta}_{n-1,1} + (-1)^n \frac{1}{m+1} \hat{\beta}_{n-1,m+1} \tag{7.1}$$

with initial condition

$$\hat{\beta}_{2,m} = -\frac{1}{8} + \frac{1}{m+1} \left(\frac{1}{2} - \frac{1}{m+2} \right) \tag{7.2}$$

7.1. **Step 1.** We claim first that

$$\epsilon_{n,m} = -\bar{\epsilon}_{n,m} \tag{7.3}$$

We have

$$\epsilon_{n,m} = \frac{1}{(2\pi)^{2n+m-2}} \int \theta(w,z_2)^m d\theta(z_1,z_2)d\theta(z_2,z_3)\cdots d\theta(z_n,z_1)d\theta(w,z_3)\cdots d\theta(w,z_n)$$

Here we put $w = 0$, $z_1 = i$ and the integral is over a suitable open subset of $\mathbb{C}^{2n-1}$.

We apply the permutation $(2n)(3\,n-1)\cdots$ to $z_1,\ldots,z_n$. This does not change the orientation. We find

$$\begin{aligned}
\epsilon_{n,m} &= \frac{1}{(2\pi)^{2n+m-2}} \int \theta(w,z_n)^m d\theta(z_1,z_n)d\theta(z_n,z_{n-1})\cdots d\theta(z_2,z_1)d\theta(w,z_{n-1})\cdots d\theta(w,z_2)\\
&= \frac{1}{(2\pi)^{2n+m-2}} \int \theta(w,z_n)^m d\theta(z_n,z_1)d\theta(z_{n-1},z_n)\cdots d\theta(z_1,z_2)d\theta(w,z_{n-1})\cdots d\theta(w,z_2)\\
&= (-1)^s\bar{\epsilon}_{n,m}
\end{aligned}$$

The sign $(-1)^s$ comes from the fact that we have invert the order of the 1-forms in $d\varphi(z_n,z_1)d\varphi(z_{n-1}z_n)\cdots d\varphi(z_1,z_2)$ and in $d\varphi(w,z_{n-1})\cdots d\varphi(w,z_2)$. Thus

$$s = \frac{n(n-1)}{2} + \frac{(n-2)(n-3)}{2} = n^2 - 3n + 3$$

which is always an odd number. This proves (7.3).

7.2. **Step 2.** Formula (7.2) is obvious from (5.2) and the fact that $\hat{\beta}_{2,m} = \beta_{2,m}$. Formula (7.1) follows from (5.3) and (6.1) together with the following computation

$$\begin{aligned}
\hat{\beta}_{n,m} &= \frac{1}{2}\left(-\frac{1}{m+1}\alpha_{n-1,m+1} + (-1)^{n+1}\frac{1}{2}\beta_{n-1,1} + \frac{1}{2}\beta_{n-1,1} + (-1)^n\frac{1}{m+1}\beta_{n-1,m+1} - \frac{1}{m+1}\bar{\beta}_{n-1,m+1}\right)\\
&= -\frac{1}{2(m+1)}\alpha_{n-1,m+1} + (-1)^{n+1}\frac{1}{2}\hat{\beta}_{n-1,1} + (-1)^n\frac{1}{m+1}\hat{\beta}_{n-1,m+1}
\end{aligned}$$

8. A formula for w_n

In this section we prove the following fact

$$w_n = \begin{cases} \hat{\beta}_{n,1} - \frac{1}{2}\alpha_{n-1,2} & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases} \tag{8.1}$$

8.1. **Step 1.** We assume n odd. Put $w = i$. We identify $C_{n,0}$ with an open subset of $\mathcal{H}^n$. The orientation is derived from the standard orientation on $\mathcal{H}^n$ (according to [1]).

We consider the map $\alpha : \mathcal{H} \to \mathcal{H} : x + iy \mapsto -x + iy$ and we extend α diagonally to a map $\mathcal{H}^n \to \mathcal{H}^n$ also denoted by α. α multiplies the orientation by $(-1)^n$.

Hence we have

$$w_n = (-1)^n \frac{1}{(2\pi)^{2n}} \int_{C_{n,0}} d\varphi(\alpha(z_1),\alpha(z_2))\cdots d\varphi(\alpha(z_1),\alpha(z_n))d\varphi(w,\alpha(z_1))\cdots d\varphi(w,\alpha(z_n))$$

(since $\alpha(w) = w$).

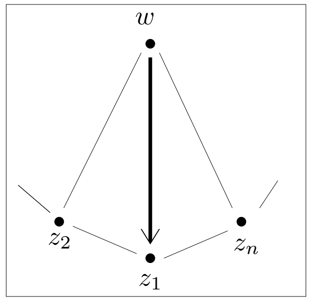
It is now easy to see that $\varphi(\alpha(u),\alpha(v)) = 2\pi - \varphi(u,v)$ and hence $d\varphi(\alpha(u),\alpha(v)) = -d\varphi(u,v)$. Thus

$$w_n = (-1)^n(-1)^{2n}w_n = (-1)^n w_n$$

Since n is odd this implies $w_n = 0$.

8.2. **Step 2.** Consider the following enhanced graph

(8.2)



We claim that its corresponding integral

$$\delta = \frac{1}{(2\pi)^{2n}} \int_{C_{n+1}} \theta(w, z_1) d\theta(z_1, z_2) \cdots d\theta(z_n, z_1) d\theta(w, z_2) \cdots d\theta(w, z_n)$$

is zero when n is even.

It's easy to see that permuting the points does not change the orientation on C_{n+1}. We apply the permutation $(2n)(3\,n-1)\cdots$ for $z_1, z_2, \ldots, z_n$. Hence we also have

$$\begin{aligned}\delta &= \frac{1}{(2\pi)^{2n}} \int_{C_{n+1}} \theta(w, z_1) d\theta(z_1, z_n) d\theta(z_n z_{n-1}) \cdots d\theta(z_2, z_1) d\theta(w, z_n) \cdots d\theta(w, z_2) \\ &= \frac{1}{(2\pi)^{2n}} \int_{C_{n+1}} \theta(w, z_1) d\theta(z_n, z_1) d\theta(z_{n-1} z_n) \cdots d\theta(z_1, z_2) d\theta(w, z_n) \cdots d\theta(w, z_2)\end{aligned}$$

To put this back in standard form we have to invert the order of the 1-forms in $d\theta(z_n, z_1) d\theta(z_{n-1} z_n) \cdots d\theta(z_1, z_2)$ and in $d\theta(w, z_n) \cdots d\theta(w, z_2)$. This introduces a cumulative sign of $(-1)^s$ where

$$s = \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} = (n-1)^2$$

Thus

$$\delta = -(-1)^n \delta$$

and hence if n is even then $\delta = 0$.

8.3. **Step 3.** Now we assume n even. As a pedagogical device we keep signs of the form $(-1)^n$ below. We first observe

$$\begin{aligned}w_n &= (-1)^n \frac{1}{(2\pi)^{2n}} \int d\varphi(w, z_1) d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_n) \\ &= (-1)^n \frac{1}{(2\pi)^{2n}} \int_{\text{boundary}} \varphi(w, z_1) d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_n)\end{aligned}$$

The boundary consists of two parts. One parts consists of both sides of the branch cut $B_\pm$.

$$\begin{aligned}B_- &= \{\operatorname{Re} z_1 = \operatorname{Re} w+, \operatorname{Im} z_1 > \operatorname{Im} w\} \subset C_{n+1,0} \\ B_+ &= \{\operatorname{Re} z_1 = \operatorname{Re} w-, \operatorname{Im} z_1 > \operatorname{Im} w\} \subset C_{n+1,0}\end{aligned}$$

The values of $\varphi(w, -)$ on both sides of the branch cut differ by 2π. Hence the contribution of this part of the boundary is equal to

$$(-1)^n \frac{1}{(2\pi)^{2n-1}} \int_{B_-} \varphi(w, z_1) d\varphi(z_1, z_2) \cdots d\varphi(z_{n-1}, z_n) d\varphi(z_n, z_1) d\varphi(w, z_2) \cdots d\varphi(w, z_n)$$

which is nothing but

$$(-1)^n \beta_{n,1}$$

In §3.2 we have listed the "classical" parts of the boundary which contribute to the integral. Here we give the contributions with the precise signs. Note that the contribution of (8.2) is zero by Step 2.

(1) Let ∂_1 be the boundary component given by contracting (w, z_n). According to [1] we have $\partial_1 \cong -C_2 \times C_{n,0}$. The contribution to the integral is

$$\begin{aligned}
&(-1)^n \frac{1}{(2\pi)^{2n}} \int_{\partial_1} \varphi(w,z_1) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_{n-1}) d\varphi(w,z_n) \\
&(-1)^n \frac{1}{(2\pi)^{2n}} \int_{\partial_1} \varphi(w,z_1) d\varphi(w,z_n) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_{n-1}) \\
&(-1)^{n+1} \frac{1}{(2\pi)^{2n-1}} \int_{C_n} \varphi(w,z_1) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},w) d\varphi(w,z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_{n-1}) \\
&(-1)^{n+1} \frac{1}{2(2\pi)^{2n-2}} \int_{C_n} d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},w) d\varphi(w,z_1)^2 d\varphi(w,z_2)\cdots d\varphi(w,z_{n-1}) \\
&\qquad = (-1)^{n+1} \frac{1}{2} \alpha_{n-1,2}
\end{aligned}$$

(2) Let $\partial_2 \cong -C_2 \times C_{n,0}$ be obtained by contracting (z_1, z_2). We obtain

$$\begin{aligned}
&(-1)^n \frac{1}{(2\pi)^{2n}} \int \varphi(w,z_1) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_n) \\
&= (-1)^{n+1} \frac{1}{(2\pi)^{2n-1}} \int_{C_{n,0}} \varphi(w,z_2) d\varphi(z_2,z_3)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_2) d\varphi(w,z_2)\cdots d\varphi(w,z_n) \\
&= (-1)^{n+1} \frac{1}{(2)(2\pi)^{2n-1}} \int d\varphi(z_2,z_3)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_2) d\varphi(w,z_2)^2\cdots d\varphi(w,z_n)
\end{aligned}$$

(3) Let $\partial_3 \cong -C_2 \times C_{n,0}$ be obtained by contracting (z_n, z_1). We obtain

$$\begin{aligned}
&(-1)^n \frac{1}{(2\pi)^{2n}} \int_{\partial_4} \varphi(w,z_1) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(z_n,z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_n) \\
&- \frac{1}{(2\pi)^{2n}} \int_{\partial_4} \varphi(w,z_1) d\varphi(z_n,z_1) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_n) d\varphi(w,z_2)\cdots d\varphi(w,z_n) \\
&= \frac{1}{(2\pi)^{2n-1}} \int_{C_{n,0}} \varphi(w,z_1) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_{n-1}) d\varphi(w,z_1) \\
&= (-1)^{n-2} \frac{1}{(2\pi)^{2n-1}} \int_{C_{n,0}} \varphi(w,z_1) d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_1) d\varphi(w,z_1) d\varphi(w,z_2)\cdots d\varphi(w,z_{n-1}) \\
&= (-1)^{n-2} \frac{1}{(2)(2\pi)^{2n-1}} \int_{C_{n,0}} d\varphi(z_1,z_2)\cdots d\varphi(z_{n-1},z_1) d\varphi(w,z_1)^2 d\varphi(w,z_2)\cdots d\varphi(w,z_{n-1})
\end{aligned}$$

Hence the contributions of ∂_2 and ∂_3 cancel. The contributions of ∂_1 and B_- yield (8.1) in case n is even.

9. Solving the recursion

9.1. **The problem.** For the benefit of the reader we restate the recursion relations we have derived. See (4.1)(4.2)(7.1)(7.2)(8.1).

$$\alpha_{0,m} = 1$$

$$\alpha_{n,m} = (-1)^n \left(\frac{1}{2}\alpha_{n-1,2} - \frac{1}{m+1}\alpha_{n-1,m+1}\right) \qquad \text{(for } n \geq 1\text{)}$$

$$\hat\beta_{2,m} = -\frac{1}{8} + \frac{1}{m+1}\left(\frac{1}{2} - \frac{1}{m+2}\right)$$

$$\hat\beta_{n,m} = -\frac{1}{2(m+1)}\alpha_{n-1,m+1} + (-1)^{n+1}\frac{1}{2}\hat\beta_{n-1,1} + (-1)^n\frac{1}{m+1}\hat\beta_{n-1,m+1} \qquad \text{(for } n \geq 3\text{)}$$

$$w_n = \hat\beta_{n,1} - \frac{1}{2}\alpha_{n-1,2} \qquad \text{(for } n \geq 2 \text{ even)}$$

9.2. **Eliminating some signs and fractions.** We simplify the equations by putting

$$\tilde\alpha_{n,m} = (-1)^{\frac{n(n+1)}{2}}\frac{1}{m!}\alpha_{n,m}$$

Then we have

$$\tilde\alpha_{0,m} = \frac{1}{m!}$$

and

$$\tilde\alpha_{n,m} = \frac{1}{m!}\tilde\alpha_{n-1,2} - \tilde\alpha_{n-1,m+1} \tag{9.1}$$

Similarly we put

$$\tilde\beta_{n,m} = (-1)^{\frac{n(n+1)}{2}}\frac{\hat\beta_{n,m}}{m!}$$

so that we get.

$$\tilde\beta_{2,m} = \frac{1}{8m!} - \frac{1}{2(m+1)!} + \frac{1}{(m+2)!}$$

and

$$\tilde\beta_{n,m} = -(-1)^n\frac{1}{2}\tilde\alpha_{n-1,m+1} - \frac{1}{2m!}\tilde\beta_{n-1,1} + \tilde\beta_{n-1,m+1} \tag{9.2}$$

9.3. **Computing $\tilde\alpha$.** Iterating (9.1) we find

$$\begin{aligned}
\tilde\alpha_{n,2} &= \frac{1}{2!}\tilde\alpha_{n-1,2} - \tilde\alpha_{n-1,3}\\
&= \frac{1}{2!}\tilde\alpha_{n-1,2} - \frac{1}{3!}\tilde\alpha_{n-2,2} + \tilde\alpha_{n-2,4}\\
&= \left(\frac{1}{2!}\tilde\alpha_{n-1,2} - \frac{1}{3!}\tilde\alpha_{n-2,2} + \cdots + (-1)^{n+1}\frac{1}{(n+1)!}\tilde\alpha_{0,2}\right) + (-1)^n\tilde\alpha_{0,n+2}
\end{aligned}$$

Thus if we put

$$A_2 = \sum_{n\geq 0}\tilde\alpha_{n,2}x^n$$

we get

$$A_2 = \left(\frac{x}{2!} - \frac{x^2}{3!} + \cdots\right)A_2 + \sum_{n\geq 0}(-1)^n\frac{x^n}{(n+2)!}$$

Here

$$\frac{x}{2!} - \frac{x^2}{3!} + \cdots = \frac{e^{-x} - 1 + x}{x}$$

and

$$\sum_{n\geq 0}(-1)^n\frac{x^n}{(n+2)!}=\frac{e^{-x}-1+x}{x^2}$$

from which we deduce

$$A_2=-\frac{1}{x}+\frac{1}{1-e^{-x}}$$

For use below we record

$$A_2' \stackrel{\mathrm{def}}{=} A_2-A_2(0)=-\frac{1}{x}+\frac{1+e^{-x}}{2(1-e^{-x})}$$

Knowing $\tilde{\alpha}_{2,n}$ we can compute arbitrary $\tilde{\alpha}_{m,n}$ via

(9.3)

$$\begin{aligned}\tilde{\alpha}_{n,m} &= \frac{1}{m!}\tilde{\alpha}_{n-1,2}-\tilde{\alpha}_{n-1,m+1}\\ &= \frac{1}{m!}\tilde{\alpha}_{n-1,2}-\frac{1}{(m+1)!}\tilde{\alpha}_{n-2,2}+\cdots+(-1)^{n-1}\frac{1}{(n+m-1)!}\tilde{\alpha}_{0,2}+(-1)^n\frac{1}{(m+n)!}\end{aligned}$$

We will not write down a closed expression for $\tilde{\alpha}_{n,m}$ as we won't need it.

9.4. **Computing a sum.** For use below we compute the sum for $n\geq 2$

$$s_n=(-1)^{n-1}\tilde{\alpha}_{n-1,2}+(-1)^{n-2}\tilde{\alpha}_{n-2,3}+\cdots+\tilde{\alpha}_{2,n-1}$$

where an empty sum is intepreted as 0. Using (9.3) we get

$$\tilde{\alpha}_{2,n-1}=\frac{1}{(n-1)!}\tilde{\alpha}_{1,2}-\frac{1}{n!}\tilde{\alpha}_{0,2}+\frac{1}{(n+1)!}$$

$$\tilde{\alpha}_{3,n-2}=\frac{1}{(n-2)!}\tilde{\alpha}_{2,2}-\frac{1}{(n-1)!}\tilde{\alpha}_{1,2}+\frac{1}{n!}\tilde{\alpha}_{0,2}-\frac{1}{(n+1)!}$$

and

$$\tilde{\alpha}_{n-1,2}=\frac{1}{2!}\tilde{\alpha}_{n-2,2}-\frac{1}{3!}\tilde{\alpha}_{n-3,2}+\cdots+(-1)^{n-2}\frac{1}{n!}\tilde{\alpha}_{0,2}+(-1)^{n-1}\frac{1}{(n+1)!}$$

so that we get

$$s_n=(-1)^{n-1}\frac{1}{2!}\tilde{\alpha}_{n-2,2}+(-1)^{n-2}\frac{2}{3!}\tilde{\alpha}_{n-3,2}+\cdots+(n-2)\frac{1}{(n-1)!}\tilde{\alpha}_{1,2}-(n-2)\frac{1}{n!}\tilde{\alpha}_{0,2}+(n-2)\frac{1}{(n+1)!}$$

Put

$$T=\frac{x^2}{2!}-\frac{2x^3}{3!}+\cdots=-xe^{-x}-e^{-x}+1$$

Then we get

$$s_n=(-1)^{n-1}(TA_2)[x^n]+\frac{1}{n!}\tilde{\alpha}_{0,2}+\frac{n-2}{(n+1)!}$$

where $(-)[x^n]$ denotes the coefficient of x^n. We will regard this formula as a definition if $n=0,1$. We obtain

$$\sum_{n\geq 0}(-1)^n s_n x^n=-TA_2+\sum_{n\geq 0}(-1)^n\tilde{\alpha}_{0,2}\frac{x^n}{n!}+\sum_{n\geq 0}(-1)^n\frac{n-2}{(n+1)!}x^n$$

We have

$$\sum_{n\geq 0}(-1)^n\tilde{\alpha}_{0,2}\frac{x^n}{n!}=\frac{1}{2}e^{-x}$$

and

$$\sum_{n\geq 0}(-1)^n\frac{n-2}{(n+1)!}x^n=\sum_{n\geq 0}(-1)^n\frac{n+1}{(n+1)!}x^n-3\sum_{n\geq 0}(-1)^n\frac{1}{(n+1)!}x^n$$
$$=e^{-x}+3\frac{e^{-x}-1}{x}$$

A computation with a computer algebra package now yields

$$\sum_{n\geq 0}(-1)^n s_n x^n=\frac{-(x+4)e^{-2x}+(2x^2+3x+8)e^{-x}-(2x+4)}{2(1-e^{-x})x}$$

Hence

$$S\stackrel{\text{def}}{=}\sum_{n\geq 0}s_n x^n=\frac{(x-4)e^{2x}+(2x^2-3x+8)e^x+(2x-4)}{2(e^x-1)x}$$

We have $S=-3/2+O(x^4)$. For use below we record

$$S'\stackrel{\text{def}}{=}\sum_{n\geq 2}s_n x^n=S+\frac{3}{2}=\frac{(x-4)e^{2x}+(2x^2+8)e^x-x-4}{2(e^x-1)x}$$

9.5. **Computing $\tilde{\beta}_{n,1}$.** We will only compute $\tilde{\beta}_{n,1}$ since this the only thing we need. Iterating (9.2) we get for $n\geq 3$

$$\begin{aligned}\tilde{\beta}_{n,1}&=(-1)^{n-1}\frac{1}{2}\tilde{\alpha}_{n-1,2}-\frac{1}{2\cdot 1!}\tilde{\beta}_{n-1,1}+\tilde{\beta}_{n-1,2}\\&=\frac{1}{2}\Bigl((-1)^{n-1}\tilde{\alpha}_{n-1,2}+(-1)^{n-2}\tilde{\alpha}_{n-2,3}+\cdots+\tilde{\alpha}_{2,n-1}\Bigr)-\\&\qquad\frac{1}{2}\left(\frac{1}{1!}\tilde{\beta}_{n-1,1}+\frac{1}{2!}\tilde{\beta}_{n-2,1}+\cdots+\frac{1}{(n-2)!}\tilde{\beta}_{2,1}\right)+\tilde{\beta}_{2,n-1}\\&=\frac{s_n}{2}-\frac{1}{2}\left(\frac{1}{1!}\tilde{\beta}_{n-1,1}+\frac{1}{2!}\tilde{\beta}_{n-2,1}+\cdots+\frac{1}{(n-2)!}\tilde{\beta}_{2,1}\right)+\tilde{\beta}_{2,n-1}\end{aligned}$$

Since $s_2=0$ this identity holds for $n=2$ if we interprete an empty sum as zero.

If we put

$$B_1=\sum_{n\geq 2}\tilde{\beta}_{n,1}x^n$$

then we have

$$B_1=\frac{1}{2}S'-\frac{1}{2}\left(\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)B_1+\sum_{n\geq 2}x^n\left(\frac{1}{8(n-1)!}-\frac{1}{2n!}+\frac{1}{(n+1)!}\right)$$

We have

$$\frac{x}{1!}+\frac{x^2}{2!}+\cdots=e^x-1$$
$$\sum_{n\geq 2}\frac{x^n}{(n-1)!}=x(e^x-1)$$
$$\sum_{n\geq 2}\frac{x^n}{n!}=e^x-1-x$$
$$\sum_{n\geq 2}\frac{x^n}{(n+1)!}=\frac{1}{x}\left(e^x-1-x-\frac{x^2}{2}\right)$$

and hence

$$\sum_{n\geq 2} x^n \left(\frac{1}{8(n-1)!} - \frac{1}{2n!} + \frac{1}{(n+1)!}\right) = \frac{1}{8}x(e^x-1) - \frac{1}{2}(e^x-1-x) + \frac{1}{x}\left(e^x - 1 - x - \frac{x^2}{2}\right)$$
$$= \left(\frac{x}{8} - \frac{1}{2} + \frac{1}{x}\right)e^x - \frac{1}{x} - \frac{1}{2} - \frac{x}{8}$$

Thus we get

$$\left(1 + \frac{1}{2}(e^x-1)\right)B_1 = \frac{1}{2}S' + \left(\frac{x}{8} - \frac{1}{2} + \frac{1}{x}\right)e^x - \frac{1}{x} - \frac{1}{2} - \frac{x}{8}$$

Invoking once again a computer algebra package we find

$$B_1 = \frac{(x-2)e^x + x + 2}{4(e^x-1)}$$

9.6. **Computing w_n.** If n is even we have

$$(-1)^{\frac{n(n-1)}{2}} w_n = \tilde{\beta}_{n,1} - \tilde{\alpha}_{n-1,2}$$

Since A_2', B_1 are respectively an odd and an even function of x, $B_1(0) = 0$ and $w_n = 0$ for n odd we obtain

$$\sum_{n\geq 2} (-1)^{\frac{n(n-1)}{2}} w_n x^n = B_1 - xA_2'$$

and using a computer algebra package we find

$$\sum_{n\geq 2} (-1)^{\frac{n(n-1)}{2}} w_n x^n = \frac{(x+2)e^{-x} + x - 2}{4(e^{-x}-1)} \tag{9.4}$$

The derivative of

$$\frac{1}{2}\log \frac{e^{x/2} - e^{-x/2}}{x}$$

is equal to

$$-\frac{(x+2)e^{-x} + x - 2}{4(e^{-x}-1)x}$$

Dividing (9.4) by x and integrating finished the proof of (1.1).

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