Lesson 1 - Integrals and primitive

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1 A few reminders

1.1 Fractions and integrals

We are looking for a primitive of $f : x \mapsto \frac{P(x)}{Q(x)}$. How to do it? Let us derive an example. We will consider, $f(x) = \frac{x^3+6x^2+9x}{2(x+1)^2(x+2)}$.

- the first step is to look at the degrees of the polynomials involved in our function $f(x) = \frac{P(x)}{Q(x)}$. Here, $P(x) = x^3 + 6x^2 + 9x$ and $Q(x) = 2(x + 1)^2(x + 2)$. Hence, the degree of P is equal to 3 and the degree of Q is equal to 3. In order to proceed we need the **degree of P to be strictly smaller than the one of Q**. How to do this? It is quite simple. Since the degree of P is larger than the one of Q we can write
$$P(x) = U(x)Q(x) + V(x) \tag{1}$$
with the degree of V strictly smaller than the one of Q. This is the **euclidean division** for the polynomials. How to compute U and V? We know that U will be of degree zero since $U(x)Q(x)$ has the same degree as $P(x)$ and $3 = 0 + 3$. So $U(x) = a$. This yields, recall that the degree of V is strictly smaller than the one of Q, i.e 3,
$$\begin{aligned} x^3 + 6x^2 + 9x &= 2a(x + 1)^2(x + 2) + V(x) \\ x^3 + 6x^2 + 9x &= 2ax^3 + \cdots + V(x) \end{aligned}. \tag{2}$$
Matching the coefficients we got, $a = \frac{1}{2}$. We denote $V(x) = \alpha x^2 + \beta x + \gamma$ and we have
$$\begin{aligned} x^3 + 6x^2 + 9x &= 2a(x + 1)^2(x + 2) + V(x) \\ x^3 + 6x^2 + 9x &= x^3 + 2x^2 + x + 2x^2 + 4x + 2 + V(x). \\ V(x) &= 2x^2 + 4x - 2 = 2(x^2 + 2x - 1) \end{aligned} \tag{3}$$
So we have
$$f(x) = \frac{1}{2} + \frac{x^2 + 2x - 1}{(x + 1)^2(x + 2)} \tag{4}$$
- The part $\frac{1}{2}$ is easy to integrate and yields $\frac{x}{2}$. Now let us focus on $\frac{x^2+2x-1}{(x+1)^2(x+2)}$. We know that
$$\frac{x^2 + 2x - 1}{(x + 1)^2(x + 2)} = \frac{a}{(x + 1)^2} + \frac{b}{x + 1} + \frac{c}{x + 2}. \tag{5}$$
Multiplying by $(x + 1)^2$ and setting $x = -1$ we find that $a = -2$. Multiplying by $x + 2$ and setting $x = -2$ we find that $c = -1$. How to find b?
- As presented during the private lesson, we can set $x = 0$ and see what we obtain
$$\frac{-1}{2} = a + b + \frac{c}{2}. \tag{6}$$
Hence, $b = 2$

Finally, we have
$$f(x) = \frac{1}{2} + \frac{-2}{(x + 1)^2} + \frac{2}{x + 1} - \frac{1}{x + 2} \tag{7}$$
Now we compute the primitive,
$$\int f(x)\mathrm{d}x = \frac{x}{2} + \frac{2}{x + 1} + \log\left(\left|\frac{(x + 1)^2}{x + 2}\right|\right) + C \tag{8}$$

Remark: here it was not possible to use the "limit trick" to compute b, i.e multiplying by x and making x goes to infinity.

1.2 Bioche rule

The rules to compute trigonometric integrals are precised here. We have $f(x) = g(\cos(x), \sin(x), \tan(x))$. We set $F(x) = f(x)\mathrm{d}x$. Suppose:

- $F(x) = F(-x)$ then substitute $\cos(x)$ to x
- $F(x) = F(\pi - x)$ then substitute $\sin(x)$ to x
- $F(x) = F(\pi + x)$ then substitute $\tan(x)$ to x

Remark: don't forget that there is $\mathrm{d}x$ in F!

2 Exercises

Compute the following integrals:

- $\int_0^1 \frac{1}{(x+2)^2(x-1)}\mathrm{d}x$
- $\int_0^1 \frac{1}{1+x^2}\mathrm{d}x$

Compute the primitive of the following functions:

- $f(x) = \frac{1}{\sin(x)\cos(x)}$
- $g(x) = \frac{1}{\tan(x)}$

Let $n \in \mathbb{N}$. We define $J_n = \int_0^{\frac{\pi}{4}} \tan(x)^n \mathrm{dx}$.

1 Give a formula linking J_{n+2} et J_n. We will compute $J_{n+2} + J_n$.

2 Compute J_0, J_1 and J_n for every n.