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MSE derived from adiabatic form of first law of thermodynamics & Kirchoff Law.

First law of thermodynamics

$$dQ = dU + pdV$$

For adiabatic transformations

$dQ = 0$

$$\boxed{dU = -pdV}$$

The Kirchhoff's equation is written as

$$\boxed{\frac{dL_{i,ii}}{dT} = C_{pii} - C_{pi}}$$

$L_{i,ii} \equiv \underbrace{K_{ii} - K_i}$ (Latent heat pertaining to phase transition ii & i)

diff between the heat content/ Specific enthalpies of two phases. ($K \equiv u + pv$)

u → Specific Internal energy

$$L_{i,ii} \equiv K_{ii} - K_i$$

$$dL_{i,ii} = dT\left[\frac{\partial K_{ii}}{\partial T}\Big|_p - \frac{\partial K_i}{\partial T}\Big|_p\right] + dP\left[\frac{\partial K_{ii}}{\partial P}\Big|_T - \frac{\partial K_i}{\partial P}\Big|_T\right]$$

$$\equiv C_p = \frac{\partial Q}{\partial T}\Big|_p$$

$$dL_{i,ii} = dT\left[c_{pii} - c_{pi}\right] + dp\left[\frac{\partial v_{ii}}{\partial p}\Big|_T - \frac{\partial v_i}{\partial p}\Big|_T + \frac{\partial}{\partial p}(p\alpha)_{ii}\Big|_T + \frac{\partial(p\alpha_i)}{\partial p}\Big|_T\right]$$

$$= dT\left[c_{pii} - c_{pi}\right] + dp\left[\alpha_{ii} - \alpha_i + p\frac{\partial \alpha_{ii}}{\partial p}\Big|_T + p\frac{\partial \alpha_i}{\partial p}\Big|_T\right]$$

For liquid water & ice, α & $\partial\alpha/\partial p$ are small. Hence

$$\boxed{dL_{i,ii} = dT\left[c_{pii} - c_{pi}\right]}$$

From First law of thermodynamics

$$dQ = dK - \alpha dp$$

$$0 = dK - \alpha dp$$

$$dK = \alpha dp$$

given $K = (c_{pd} + r_t c_l)T + L_v r$

$$dK = (c_{pd} + r_t c_l)dT + d(L_v r)$$

$$\boxed{(c_{pd} + r_t c_l)dT + d(L_v r) - \alpha_d dp = 0}$$

Also,

$$\alpha = \alpha_d\left[\frac{1 + r_l(\alpha_l/\alpha_d) + r_i(\alpha_i/\alpha_d)}{1 + r_T}\right]$$

where α_l & α_i -> Specific vol of liquid water & ice.

$r_T = r + r_l + r_i$ (Net water mix ratio)

Since α_l & $\alpha_i \ll \alpha_d$ & r_l & $r_i \approx 10^{-3}$

$$\alpha = \frac{\alpha_d}{1 + r_T} \Rightarrow \boxed{\alpha(1 + r_T) = \alpha_d}$$

Now For strictly hydrostatic changes

$$\alpha\, dp = -g\, dz$$

using this in main equ,

$$\boxed{(c_{pd} + r_t c_l)\, dT + d(L_v r) + (1 + r_T)\, g\, dz = 0 \equiv dh}$$

If r_T is also conserved, the above can be integrated to yield,

$$\boxed{h \equiv (c_{pd} + r_t c_l) T + L_v r + (1 + r_t) g z}$$

'h' is conserved for adiabatic, saturated or unsaturated transformations in which mass is conserved & in which the pressure change is <u>strictly hydrostatic</u>.

The dry static energy h_d conserved under hydrostatic unsaturated transformations

$$\boxed{h_d \equiv (c_{pd} + r c_{pv})T + (1+r)gz}$$

The static energies h & h_d are closely related to the entropy variables θ_e & θ respectively.

$$k = k_d + r_v k_v + r_l k_l$$

↳ (Total enthalpy per unit mass of dry air)

$$k = k_d + L_v r + k_l r_t$$

$$k = (c_{pd} + r_t c_l)T + L_v r$$

↳ Moist enthalpy

$$(C_{pd} + r_t C_l) \ln \theta_e = S + R_d \ln P_0$$

Equivalent Pot temp

Ref. pres

$$\theta_e = T\left(\frac{P_0}{P_d}\right)^{\frac{R_d}{(C_{pd} + C_l r_t)}} \cdot H^{-\frac{r R_v}{(C_{pd} + C_l r_t)}}$$

$$\cdot \exp\left[\frac{L_v r}{(C_{pd} + r_t C_l) T}\right]$$

when $r = 0$, $\theta_e = \theta$

WETBULB

Since the process is isobaric, the total enthalpy of air-water system is kept constant

$$(C_{pd} + r_t C_l) T_w + L_v(T_w) r(T_w)$$
$$= (C_{pd} + r_t C_l) T + L_v(T) r$$

$$\Rightarrow (C_{pd} + r_t C_l)(T_w - T) + L_v(T_w) r(T_w)$$
$$- L_v(T) r = 0.$$

$$r_t + x = r(T_w)$$

'x' is the amount of liquid necessary to saturate

$$\Rightarrow \boxed{(C_{pd} + r(T_w) C_l)(T - T_w) = L_v(T_w) r(T_w) - L_v(T) r}$$

Surface energy balance:-

$$\boxed{MSE = h \equiv (c_{pd} + r_t c_l)T + L_v r + (1+r)gz}$$

$(J\,kg^{-1})$.

$$\boxed{\frac{d(MSE)}{dt} = \frac{\partial}{\partial t}(MSE) + u_h \cdot \nabla_h (MSE) + w \cdot \frac{\partial MSE}{\partial z}}$$

$(J\,kg^{-1}s^{-1}$ or $W\,kg^{-1})$

Source : Atmospheric Convection,Kerry A. Emanuel,1994, Chapter 4, 110 - 126.