EQUATION OF CONTINUITY

Source:- Physics of Climate (1992)

This is a quick recap of derivation of the continuity equation.

The principle of conservation of mass for a given element of mass $\delta m = \rho \delta V$ is given by $d(\delta m)/dt = d(\rho \delta V)/dt = 0$, which leads to equation of continuity.

If we expand the previous equation

$$\frac{d(\rho \delta V)}{dt} = \rho \frac{d(\delta V)}{dt} + \delta V \frac{d\rho}{dt} = 0$$

$$\boxed{-\frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{\delta V}\frac{d(\delta V)}{dt}}$$

The RHS can be expressed in terms of wind velocity vector (3D) 'c' as follows

$$\boxed{-\frac{1}{\rho}\frac{d\rho}{dt} = \operatorname{div} c}$$

If we rewrite $\rho^{-1} = \alpha$ (Specific volume)

$$\boxed{\frac{d\alpha}{dt} = \alpha \operatorname{div} c}$$

advection term

Since $\underbrace{\frac{d}{dt}}_{\text{Material derivative}} = \underbrace{\frac{\partial}{\partial t}}_{\text{Eulerian time derivative}} + c\cdot \text{grad}$

The equation of continuity in local form can be written as

$$\frac{d\rho}{dt} = -\rho \operatorname{div} c$$

$$\frac{\partial \rho}{\partial t} + c\cdot \text{grad}\,\rho = -\rho \operatorname{div} c$$

$$\frac{\partial \rho}{\partial t} = -\rho \operatorname{div} c - c\cdot \text{grad}\,\rho$$

using vector identity

$$\operatorname{div} \rho \vec{c} = \rho \operatorname{div} \vec{c} + \vec{c}\cdot \text{grad}\,\rho$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\operatorname{div} \rho c}$$

Hence the above equation gives us the local form of continuity equation in x, y, z, t coordinates.

If hydrostatic equilibrium is assumed.

i.e, $\frac{dp}{dz} = -\rho g$ (Pressure gradient in the vertical balance gravitational force)

we can replace (x, y, z, t) to (x, y, p, t). Here

$$\delta m = \frac{\delta x \delta y \delta p}{g} \quad \text{and}$$

$$\frac{d}{dt}(\delta m) = 0$$

or

$$\frac{1}{\delta m}\frac{d(\delta m)}{dt} = \frac{1}{\delta x \delta y}\frac{d}{dt}(\delta x \delta y) + \frac{1}{\delta p}\frac{d}{dt}(\delta p) = 0$$

or

$$\boxed{\text{div}\, V_h + \frac{\partial \omega^*}{\partial p} = 0}$$

here $V_h \rightarrow$ horizontal component of velocity

$\omega^* \rightarrow \frac{dp}{dt} \approx \omega \frac{\partial p}{\partial z} \approx -\rho g \omega$

(quasi hydrostatic equilibrium)