Q.4 $x = (x_1, x_2)$ $z = (z_1, z_2)$

$$K(x,z) = x_1 z_1 + x_1 e^{z_2} + z_1 e^{x_2} + e^{(x_2+z_2)}$$

$$K(x,z) = x_1(z_1 + e^{z_2}) + e^{x_2}(z_1 + e^{z_2})$$

$$K(x,z) = (x_1 + e^{x_2})(z_1 + e^{z_2})$$

According to the above equation we can say that:-

$$K(z,x) = (z_1 + e^{z_2})(x_1 + e^{x_2})$$

Therefore $K(x,z) = K(z,x)$.

This Prove the first Condion of the mercer's law.

Bothe the functions are symmetry.

Now, lets assume $X = (X_1, X_2)$ and $X' = (X_1', X_2')$

According to 2nd Condition in Mercer's Low, the Matrix should be Positive semi-definite for any $X_1 \dots X_n$ (in our Case its 2 dimensional. So:-

$$\begin{bmatrix} K(X,X) & K(X,X') \\ K(X',X) & K(X',X') \end{bmatrix}$$

According to the equation

$$K(X,X) = K((X_1,X_2),(X_1,X_2))$$
$$= (X_1 + e^{X_2})(X_1 + e^{X_2}) = (X_1 + e^{X_2})^2$$

$$K(X,X') = K((X_1,X_2),(X_1',X_2'))$$
$$= (X_1 + e^{X_2})(X_1' + e^{X_2'})$$

$$K(X',X) = (X_1' + e^{X_2'})(X_1 + e^{X_2})$$

$$K(X',X') = (X_1' + e^{X_2'})^2$$

Same as above two function.

So our Matrix is

$$\begin{bmatrix} (x_1 + e^{x_2})^2 & (x_1+e^{x_2})(x_1'+e^{x_2'}) \\ (x_1'+e^{x_2'})(x_1+e^{x_2}) & (x_1'+e^{x_2'})^2 \end{bmatrix}$$

Now In order for a Matrix to be Positive Semi-definite, it should meet the following Properties.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{aligned} M_{trace} &= a + d \geq 0 \\ M_{det} &= a \times d - b \times c \geq 0 \end{aligned}$$

Lets apply the same Properties on our Matrix.

$$M_{trace} = (x_1 + e^{x_2})^2 + (x_1' + e^{x_2'})^2$$

ʌʌ {Since this is the sum of the squared values, it will always be ≥ 0}

$$M_{det} = (x_1+e^{x_2})^2 (x_1'+e^{x_2'})^2 - \left[(x_1+e^{x_2})(x_1'+e^{x_2'})\right] * \left[(x_1'+e^{x_2'})(x_1+e^{x_2})\right]$$

$$= (x_1+e^{x_2})^2 (x_1'+e^{x_2'})^2 - (x_1+e^{x_2})^2 (x_1'+e^{x_2'})^2$$

$$= \underline{\underline{0}}$$

Therefore, our Matrix is Positive Semi-definite Matrix.

Since $K(x,z)$ is meeting both the Conditions of Mercer's law, its a Kernel function. ʌʌ another way to prove the (+) Semi-definite Matrix is

$z^T M z \geq 0$

lets Consider $x_1 + e^{x_2} = a$

and $x_1' + e^{x_2'} = a'$

So our Matrix will be

$$M = \begin{bmatrix} a^2 & a \cdot a' \\ a' \cdot a & a'^2 \end{bmatrix}$$

lets assume $z^T = [1 \ 1]$, So $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$z^T M z = [1, 1] \begin{bmatrix} a^2 & a \cdot a' \\ a' \cdot a & a'^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left[(a^2 + a' \cdot a) \quad (a \cdot a' + a'^2) \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= (a^2 + a' \cdot a) + (a \cdot a' + a'^2)$$

$$= a^2 + 2 a' \cdot a + a'^2$$

$$= (a + a')^2$$

This is a ~~sq~~ squared Value, ~~Can~~ will always be Positive.